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Translation to Bundle Operators*

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Abstract. We give explicit formulas for conformally invariant operators with leading term an m-th power of Laplacian on the product of spheres with the natural pseudo-Riemannian product metric for all m.

Key words: conformally invariant operators; pseudo-Riemannian product of shperes; Fefferman–Graham ambient space; intertwining operator of the conformal group $\mathrm{O}(p+1,q+1)$

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1 Introduction

Conformally invariant operators have been one of the major subjects in mathematics and physics. Getting explicit formulas of such operators on many manifolds is potentially important. One use of spectral data, among other things, would be in application to Polyakov formulas in even dimensions for the quotient of functional determinants of operators since the precise form of these Polyakov formulas only depends on some constants that appear in the spectral asymptotics of the operators in question [3].

In 1987, Branson [1] showed explicit formulas of invariant operators on functions and differential forms over the double cover $S^1 \times S^{n-1}$ of the n dimensional compactified Minkowski space. And lately, Branson and Hong [5, 10, 9] gave explicit determinant quotient formulas of operators on spinors and twistors including the Dirac and Rarita Schwinger operators over $S^1 \times S^{n-1}$. Gover [7] recently exhibited explicit formulas of invariant operators with leading term a power of Laplacian on functions over conformally Einstein manifolds.

In this paper, we show explicit formulas of invariant operators with leading term a power of Laplacian on functions over general product of spheres, $S^p \times S^q$ with the natural pseudo-Riemannian metric.

2 Yamabe and Paneitz operators

Consider $S^p \times S^q$ with the natural signature (p, q) metric (p minus signs), with $p + q = n$. We view this as imbedded in the natural way in $\mathbb{R}^{n+2}$, which carries a signature $(p+1, q+1)$ metric [1, 13] denoting this manifold with metric by $\mathbb{R}^{p+1,q+1}$. We consider the radial vector fields

$$s\partial_s = S = x^a\partial_a \quad \text{in the ambient } \mathbb{R}^{p+1},$$
$$r\partial_r = R = x^b\partial_b \quad \text{in the ambient } \mathbb{R}^{q+1}.$$

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The d'Alembertian $\square$ on $\mathbb{R}^{p+1,q+1}$ is

$$\begin{aligned}\square &= \triangle_{\mathbb{R}^{q+1}} - \triangle_{\mathbb{R}^{p+1}} = -\partial_r^2 - \frac{q}{r}\partial_r + r^{-2}\triangle_{S^q} + \partial_s^2 + \frac{p}{s}\partial_s - s^{-2}\triangle_{S^p} \\ &= r^{-2}\left(-R^2 - (q-1)R + \triangle_{S^q}\right) - s^{-2}\left(-S^2 - (p-1)S + \triangle_{S^p}\right), \end{aligned} \tag{1}$$

where $\Delta = -g^{ab}\nabla_a\nabla_b$.

It is well known that the following process is conformally invariant [6, 11]:

- Take a function f on $S^p \times S^q$, and extend it to a function F having

  $$XF = \left(m - \frac{n}{2}\right)F, \tag{2}$$

  where

  $$X := R + S.$$

- Compute $\square^m F$.
- Restrict to $S^p \times S^q$.

More precisely, if we view f as an $(m - n/2)$-density on the product of spheres, perform the process above, and view the restricted function as a $-(m + n/2)$-density, we get a conformally invariant operator

$$\mathcal{E}[m - n/2] \to \mathcal{E}[-m - n/2],$$

where $\mathcal{E}[\omega]$ is the bundle of conformal densities of degree ω [8]:

$$f \in \mathcal{E}[\omega] \iff \hat{f} = \Omega^\omega f \text{ under } \hat{g} = \Omega^2 g, \quad \Omega \text{ is a positive smooth function.}$$

In fact, this happens in the more general setting of the Fefferman–Graham ambient space for a pseudo-Riemannian conformal manifold $(M, [g])$, provided the dimension is odd, $2m \le n$, or the Fefferman–Graham obstruction tensor vanishes [6]. In particular, this happens with no restriction on (n, m) whenever $[g]$ is a flat conformal structure and this is the case in our situation. In particular, using only invariance under conformal changes implemented by diffeomorphisms, in our situation we get an intertwining operator A for two representations of the conformal group $\mathrm{O}(p+1, q+1)$ [1, 4]:

$$Au_{m-n/2} = u_{-m-n/2}A.$$

We begin with a function f having homogeneity u in the radial (S) direction in $\mathbb{R}^{p+1}$, and homogeneity v in the radial (R) direction in $\mathbb{R}^{q+1}$.

The (u, v) homogeneity extension is a special case of the extension scheme (2) as long as

$$\omega := u + v = m - n/2. \tag{3}$$

To illustrate our method, we work out the Yamabe operator $(m = 1)$ and the Paneitz operator $(m = 2)$ cases.

Let $Y := R - S$. On $S^p \times S^q$, $r = s = 1$ and we have

$$\begin{aligned}\square f\,|_{S^p\times S^q} &= \left\{-R^2 + S^2 - (q-1)R + (p-1)S + \triangle_{S^q} - \triangle_{S^p}\right\} f \\ &= \left\{ \begin{pmatrix} -XY \\ \text{or} \\ -YX \end{pmatrix} + X\underbrace{\left\{-\frac{q-1}{2} + \frac{p-1}{2}\right\}}_{-\frac{q-p}{2}} + Y\underbrace{\left\{-\frac{q-1}{2} - \frac{p-1}{2}\right\}}_{-\frac{n}{2}+1} + \square_{S^p\times S^q} \right\} f\end{aligned}$$

$$= -\omega Y f - \frac{q-p}{2}\omega f + \left(-\frac{n}{2}+1\right) Y f + \Box_{S^p\times S^q} f, \qquad \text{since} \quad Xf = \omega f.$$

Thus, if we choose $\omega = 1 - \frac{n}{2}$, we get

$$\boxminus|_{S^p\times S^q} = \Box_{S^p\times S^q} - \frac{q-p}{2}\left(1-\frac{n}{2}\right).$$

The scalar curvature on $S^p \times S^q$ is

$$q(q-1) - p(p-1) = (q+p)(q-p) - (q-p) = (n-1)(q-p)$$

and we get the Yamabe operator

$$\boxminus|_{S^p\times S^q} = \Box_{S^p\times S^q} + \frac{n-2}{4(n-1)}\text{Scal}.$$

Now we look at $\boxminus^2$ in the ambient space. Since R, S, $\triangle_{S^p}$, and $\triangle_{S^q}$ all commute,

$$\begin{aligned}
\boxminus^2 &= r^{-4}\triangle_{S^q}^2 + s^{-4}\triangle_{S^p}^2 - 2r^{-2}s^{-2}\triangle_{S^q}\triangle_{S^p} \\
&\quad + r^{-2}\triangle_{S^q}\left(r^{-2}\{-R^2-(q-1)R\} + s^{-2}\{S^2+(p-1)S\}\right) \\
&\quad + \left(r^{-2}\{-R^2-(q-1)R\} + s^{-2}\{S^2+(p-1)S\}\right)\left(r^{-2}\triangle_{S^q}\right) \\
&\quad - s^{-2}\triangle_{S^p}\left(r^{-2}\{-R^2-(q-1)R\} + s^{-2}\{S^2+(p-1)S\}\right) \\
&\quad - \left(r^{-2}\{-R^2-(q-1)R\} + s^{-2}\{S^2+(p-1)S\}\right)\left(s^{-2}\triangle_{S^p}\right) \\
&\quad + r^{-2}\{-R^2-(q-1)R\}\left(r^{-2}\{-R^2-(q-1)R\}\right) \\
&\quad + s^{-2}\{S^2+(p-1)S\}\left(s^{-2}\{S^2+(p-1)S\}\right) \\
&\quad + 2r^{-2}s^{-2}\{-R^2-(q-1)R\}\{S^2+(p-1)S\}.
\end{aligned}$$

Since $r^{-2}\{-R^2-(q-1)R\}\{r^{-2}(-R^2-(q-1)R)\}$ equals

$$\begin{aligned}
& r^{-2}R\left(r^{-2}R\{R^2+(q-1)R\} - 2r^{-2}\{R^2+(q-1)R\}\right) \\
&\quad + r^{-2}(q-1)\left(r^{-2}R\{R^2+(q-1)R\} - 2r^{-2}\{R^2+(q-1)R\}\right) \\
&\quad = r^{-4}\left((R^2-4R+4)\{R^2+(q-1)R\} + (q-1)(R-2)\{R^2+(q-1)R\}\right) \\
&\quad = r^{-4}\{R^2+(q-1)R\}\{R^2+(q-1)R\} + (-4R+4-2(q-1))\{R^2+(q-1)R\}
\end{aligned}$$

and $\left(r^{-2}\{-R^2-(q-1)R\}\right)\left(r^{-2}\triangle_{S^q}\right)$ equals

$$r^{-4}\triangle_{S^q}\left(\{-R^2-(q-1)R\} + 4R - 4 + 2(q-1)\right),$$

on $S^p \times S^q$,

$$\begin{aligned}
\boxminus^2 &= \triangle_{S^q}^2 + \triangle_{S^p}^2 - 2\triangle_{S^q}\triangle_{S^p} \\
&\quad + 2\triangle_{S^q}\left(\{-R^2-(q-1)R\} + \{S^2+(p-1)S\} + 2R - 2 + (q-1)\right) \\
&\quad - 2\triangle_{S^p}\left(\{-R^2-(q-1)R\} + \{S^2+(p-1)S\} - 2S + 2 - (p-1)\right) \\
&\quad + \{-R^2-(q-1)R\}^2 + \{S^2+(p-1)S\}^2 + 2\{-R^2-(q-1)R\}\{S^2+(p-1)S\} \\
&\quad + (4R-4+2(q-1))\{-R^2-(q-1)R\} + (-4S+4-2(p-1))\{S^2+(p-1)S\}.
\end{aligned}$$

Let

$$A := \{-R^2-(q-1)R\}, \qquad B := \{S^2+(p-1)S\},$$

$$C := 2R - 2 + (q-1), \qquad D := -2S + 2 - (p-1).$$

Then,

$$\begin{aligned}\Box^2 |_{S^p \times S^q} = {} & \triangle_{S^q}^2 + \triangle_{S^p}^2 - 2\triangle_{S^q}\triangle_{S^p} + 2\triangle_{S^q}(A+B+C) - 2\triangle_{S^p}(A+B+D) \\ & + (A+B+C)(A+B+D) + (A-B)(C-D) - CD.\end{aligned}$$

Note that, on $\mathcal{E}[\omega]$,

$$\begin{aligned} A+B+C &= \left(-\omega - \frac{n}{2} + 2\right) Y - \left(\frac{q-p}{2} - 1\right)\omega + (q-3), \\ A+B+D &= \left(-\omega - \frac{n}{2} + 2\right) Y - \left(\frac{q-p}{2} + 1\right)\omega - (p-3). \end{aligned}$$

Note also that

$$\begin{aligned}(A-B)(C-D) - CD = {} & Y^2\left(-\omega - \frac{n}{2} + 2\right) + Y\left\{(q-p)\left(-\omega - \frac{n}{2} + 2\right)\right\} \\ & + \left\{\omega^2 + n\omega - 2\omega + (\omega + q - 3)(\omega + p - 3)\right\}.\end{aligned}$$

Since $\omega = 2 - \frac{n}{2}$,

$$\begin{aligned} &\left(\frac{q-p}{2} - 1\right)\omega - (q-3) = 1 - \frac{n-2}{4(n-1)}\mathrm{Scal}, \\ &\left(\frac{q-p}{2} + 1\right)\omega + (p-3) = -1 - \frac{n-2}{4(n-1)}\mathrm{Scal}, \qquad \text{and} \\ &\omega^2 + n\omega - 2\omega + (\omega + q - 3)(\omega + p - 3) = n - \frac{n^2}{2} + pq + 1, \end{aligned}$$

we have

$$\begin{aligned}\Box^2 |_{S^p \times S^q} = {} & \triangle_{S^q}^2 + \triangle_{S^p}^2 - 2\triangle_{S^q}\triangle_{S^p} - 2\left\{1 - \frac{n-2}{4(n-1)}\mathrm{Scal}\right\}\triangle_{S^q} \\ & + 2\left\{-1 - \frac{n-2}{4(n-1)}\mathrm{Scal}\right\}\triangle_{S^p} + \left(\frac{n-2}{4(n-1)}\mathrm{Scal}\right)^2 + n - \frac{n^2}{2} + pq.\end{aligned}$$

We claim that this is the Paneitz operator [3]

$$P = \triangle^2 + \delta T d + \frac{n-4}{2}Q,$$

where

$$\begin{aligned} J &= \mathrm{Scal}/(2(n-1)), \qquad & V &= (\rho - Jg)/(n-2), \\ T &= (n-2)J - 4V\cdot, \qquad & Q &= \frac{n}{2}J^2 - 2|V|^2 + \triangle J. \end{aligned}$$

Since $\mathrm{Scal} = (n-1)(q-p)$ and $J = \frac{q-p}{2}$,

$$\begin{aligned} \delta J d &= \frac{q-p}{2}(\triangle_{S^q} - \triangle_{S^p}), \\ \delta V d &= \frac{1}{n-2}\left\{(p-1)\triangle_{S^p} + (q-1)\triangle_{S^q}\right\} - \frac{q-p}{2(n-2)}(\triangle_{S^q} - \triangle_{S^p}). \end{aligned}$$

Thus

$$\begin{aligned}
\delta T d &= (n-2)\frac{q-p}{2}(\triangle_{S^q} - \triangle_{S^p}) + \frac{1}{n-2}\{-4(p-1) - 2(q-p)\}\triangle_{S^p} \\
&\quad + \frac{1}{n-2}\{-4(q-1) + 2(q-p)\}\triangle_{S^q} \\
&= 2\left(\frac{n-2}{4(n-1)}\mathrm{Scal}\right)(\triangle_{S^q} - \triangle_{S^p}) - 2(\triangle_{S^q} + \triangle_{S^p}).
\end{aligned}$$

On the other hand, since $|V|^2 = \frac{n}{4}$,

$$\begin{aligned}
\frac{n-4}{2}Q = \frac{n-4}{2}\left(\frac{n}{2}\left(\frac{q-p}{2}\right)^2 - \frac{n}{2}\right) &= \frac{n(n-4)}{(n-2)^2}\left(\frac{n-2}{4(n-1)}\mathrm{Scal}\right)^2 - \frac{n(n-4)}{4} \\
&= \left(\frac{n-2}{4(n-1)}\mathrm{Scal}\right)^2 - \frac{4}{(n-2)^2}\left(\frac{n-2}{4(n-1)}\mathrm{Scal}\right)^2 - \frac{n(n-4)}{4} \\
&= \left(\frac{n-2}{4(n-1)}\mathrm{Scal}\right)^2 + n - \frac{n^2}{2} + pq
\end{aligned}$$

and the claim follows.

3 Higher order operators

Let

$$C := \sqrt{\Delta_{S^q} + \left(\frac{q-1}{2}\right)^2}, \qquad B := \sqrt{\Delta_{S^p} + \left(\frac{p-1}{2}\right)^2},$$

so that C and B are nonnegative operators with

$$\Delta_{S^q} = C^2 - \left(\frac{q-1}{2}\right)^2, \qquad \Delta_{S^p} = B^2 - \left(\frac{p-1}{2}\right)^2.$$

The eigenvalue list for $\triangle_{S^q}$ [12, 2] is

$$j(q-1+j), \qquad j = 0, 1, 2, \ldots,$$

so the eigenvalue list for C is

$$j + \frac{q-1}{2}, \qquad j = 0, 1, 2, \ldots. \tag{4}$$

Similarly, the eigenvalue list for B is

$$k + \frac{p-1}{2}, \qquad k = 0, 1, 2, \ldots. \tag{5}$$

Applying $\Box^m$, we get (with $k = m - \ell$)

$$\begin{aligned}
\Box^m f = \sum_{\ell=0}^{m}(-1)^k\binom{m}{\ell}s^{-2\ell}r^{-2k}\left(C + \frac{q-1}{2} + v\right)\cdots\left(C + \frac{q-1}{2} - 2(\ell-1) + v\right)\bullet \\
\left(C - \frac{q-1}{2} - v\right)\cdots\left(C - \frac{q-1}{2} + 2(\ell-1) - v\right)\bullet
\end{aligned}$$

$$\left(B+\frac{p-1}{2}+u\right)\cdots\left(B+\frac{p-1}{2}-2(k-1)+u\right)\bullet$$
$$\left(B-\frac{p-1}{2}-u\right)\cdots\left(B-\frac{p-1}{2}+2(k-1)-u\right)f,$$

where in each $\cdot$, we move in increments or decrements of 2. These increments and decrements are determined by the homogeneity drops implemented by the s^{-2} and r^{-2} factors in (1). To restrict to $S^p\times S^q$, we just set $s=r=1$.

As a result, with

$$Q:=\frac{q-1}{2}+v,\qquad P:=\frac{p-1}{2}+u,$$

as long as we have the correct weight condition (equivalent to (3))

$$P+Q=m-1,$$

then the operator

$$\begin{aligned}A_{2m}(C,B,Q):=\sum_{\ell=0}^{m}(-1)^k\binom{m}{\ell}&(C+Q)\cdots(C+Q-2(\ell-1))\bullet\\&(C-Q)\cdots(C-Q+2(\ell-1))\bullet\\&(B+P)\cdots(B+P-2(k-1))\bullet\\&(B-P)\cdots(B-P+2(k-1)),\qquad k+\ell=m,\qquad P+Q=m-1,\end{aligned}$$

intrinsically defined on $S^p\times S^q$, intertwines $u_{m-n/2}$ and $u_{-m-n/2}$.

Note that the dependence of $A_{2m}(C,B,Q)$ is only on (m,C,B,Q), since (u,P) is determined by (m,C,B,Q). The notation suggests substituting numerical values for C and B, a procedure justified by the eigenvalue lists (4), (5). These numerical values are nonnegative real numbers, and depending on the parities of q and p, they are either integral or properly half-integral.

We *claim* that

Proposition 1.

$$\begin{aligned}A_{2m}(C,B,Q)=&(C+B+m-1)\cdots(C+B-m+1)\\&\times(C-B+m-1)\cdots(C-B-m+1):=G_{2m}(C,B),\end{aligned}\tag{6}$$

where the decrements are by 2 *units each time.*

In particular, we are claiming that the left-hand side of (6) is independent of Q.

The operator $G_{2m}(C,B)$ is in fact a differential operator since

$$\begin{cases}(C+B)(C-B)\displaystyle\prod_{l=1}^{(m-1)/2}[C+(B+2l)]\,[C-(B+2l)]\\ \qquad\times[C+(B-2l)]\,[C-(B-2l)], \qquad m\text{ odd},\\ \displaystyle\prod_{l=1}^{m/2}[C+(B+(2l-1))]\,[C-(B+(2l-1))]\\ \qquad\times[C+(B-(2l-1))]\,[C-(B-(2l-1))], \qquad m\text{ even},\end{cases}$$

$$=\begin{cases}(C^2-B^2)\displaystyle\prod_{l=1}^{(m-1)/2}\left[C^4-2(B^2+(2l)^2)C^2+(B^2-(2l)^2)^2\right], \qquad m\text{ odd},\\ \displaystyle\prod_{l=1}^{m/2}\left[C^4-2(B^2+(2l-1)^2)C^2+(B^2-(2l-1)^2)^2\right], \qquad m\text{ even}.\end{cases}$$

Recently, Gover [7] showed that on conformally Einstein manifolds, the operators are of the form

$$\Box_m = \prod_{l=1}^{m}(\Delta - c_l \mathrm{Sc}),$$

where $c_l = (n+2l-2)(n-2l)/(4n(n-1))$, Sc is the scalar curvature and $\Delta = \nabla^a\nabla_a$.

To get the formula (6) in case of sphere S^n, we set

$$C := \sqrt{\Delta + \left(\frac{n-1}{2}\right)^2}, \qquad B := \frac{1}{2}.$$

And the formula simplifies to

$$\begin{aligned} G_{2m}(C,1/2) &= \prod_{l=1}^{m}\left(C - \frac{2l-1}{2}\right)\left(C + \frac{2l-1}{2}\right) \\ &= \prod_{l=1}^{m}\left(\Delta + \underbrace{\frac{(n+2l-2)(n-2l)}{4n(n-1)}}_{c_l}\underbrace{n(n-1)}_{\mathrm{Sc}}\right). \end{aligned}$$

The "+" sign in the above is due to our convention $\Delta = -\nabla^a\nabla_a$ so the two formulas $\Box_m$ and $G_{2m}(C,1/2)$ agree.

We will now prove the equality in (6).

Because of the eigenvalue lists (4), (5), to prove this in the case in which q and p are odd, it is sufficient to prove the identity (6) with C and B replaced by nonnegative integers. This will hold, in turn, if it holds for $q = p = 1$, the explicit mention of the dimensions having disappeared in (6).

To prove (6) for $q = p = 1$, note that each expression is polynomial in (C, B, v) for fixed m, and that the highest degree terms in (C, B) add up to $(C^2 - B^2)^m$ for each expression. Thus it will be enough to prove that the right-hand side of (6) is the unique (up to constant multiples) intertwinor $u_{m-1} \to u_{-m-1}$ in the case $q = p = 1$.

By K =SO(2)×SO(2) invariance, an intertwinor A must take an eigenvalue on each

$$\varphi_{j,f} := e^{\sqrt{-1}ft}e^{\sqrt{-1}j\rho}$$

for ρ and t the usual angular parameters on the positive-metric S^1 and the negative-metric S^1 respectively, and f and j integers.

The prototypical conformal vector field [1] is

$$T = \cos(\rho)\sin(t)\partial_t + \cos(t)\sin(\rho)\partial_\rho,$$

with conformal factor

$$\omega = \cos(\rho)\cos(t).$$

The representation U_{-r} of the Lie algebra $\mathfrak{so}(2,2)$ has

$$\begin{aligned} U_{-r}(T)\varphi_{j,f} = \frac{1}{4}\big\{&(f+j+r)\varphi_{j+1,f+1} + (f-j+r)\varphi_{j-1,f+1} \\ &+ (-f+j+r)\varphi_{j+1,f-1} + (-f-j+r)\varphi_{j-1,f-1}\big\}. \end{aligned} \tag{7}$$

Consider the operator

$$P(\varepsilon,\delta):\varphi_{j,f}\mapsto\varphi_{j+\varepsilon,f+\delta}$$

for $\varepsilon,\delta\in\{\pm1\}$, and the operators

$$J:\varphi_{j,f}\mapsto j\varphi_{j,f},\qquad F:\varphi_{j,f}\mapsto f\varphi_{j,f}.$$

Note that another expression for $G_{2m}(C,B)$ is as $G_{2m}(J,F)$, since

$$G_{2m}(J,F)=G_{2m}(J,-F)=G_{2m}(-J,F).$$

We have

$$JP(\varepsilon,\delta)=P(\varepsilon,\delta)(J+\varepsilon),\qquad FP(\varepsilon,\delta)=P(\varepsilon,\delta)(F+\delta).$$

By (7),

$$\begin{aligned}U_{\pm m-1}(T)=\frac{1}{4}\big\{&P(1,1)(J+F+1\mp m)+P(-1,1)(-J+F+1\mp m)\\&+P(1,-1)(J-F+1\mp m)+P(-1,-1)(-J-F+1\mp m)\big\}.\end{aligned}$$

With this we may compute that

$$\begin{aligned}4G_{2m}(J,F)U_{m-1}(T)&=G_{2m}(J,F)\big\{P(1,1)(J+F+1-m)+P(-1,1)(-J+F+1-m)\\&\quad+P(1,-1)(J-F+1-m)+P(-1,-1)(-J-F+1-m)\big\}\\&=P(1,1)\big\{(J+F+m+1)\cdots(J+F-m+3)\bullet\\&\quad(J-F+m-1)\cdots(J-F-m+1)\big\}(J+F-m+1)\\&\quad+P(-1,1)\big\{(J+F+m-1)\cdots(J+F-m+1)\bullet\\&\quad(J-F+m-3)\cdots(J-F-m-1)\big\}(-J+F-m+1)\\&\quad+P(1,-1)\big\{(J+F+m-1)\cdots(J+F-m+1)\bullet\\&\quad(J-F+m+1)\cdots(J-F-m+3)\big\}(J-F-m+1)\\&\quad+P(-1,-1)\big\{(J+F+m-3)\cdots(J+F-m-1)\bullet\\&\quad(J-F+m-1)\cdots(J-F-m+1)\big\}(-J-F-m+1),\end{aligned}$$

whereas

$$\begin{aligned}4U_{-m-1}(T)G_{2m}(J,F)&=\big\{P(1,1)(J+F+m+1)+P(-1,1)(-J+F+m+1)\\&\quad+P(1,-1)(J-F+m+1)+P(-1,-1)(-J-F+m+1)\big\}\bullet\\&\quad(J+F+m-1)\cdots(J+F-m+1)(J-F+m-1)\cdots(J-F-m+1).\end{aligned}$$

The right-hand sides of the two preceding displays agree, so we have an intertwining operator.

As a corollary, the claim (6) follows, so that $G_{2m}(C,B)$ is an intertwinor whenever qp is odd. In fact, by polynomial continuation from positive integral values, the identity (6) holds whenever any complex values are substituted for C and B. In particular, we can substitute proper half-integers, and thus remove the condition that qp be odd.

It would be good to have a proof which avoids a dimensional continuation argument. We present in the following appendix a proof which uses only an elementary combinatorial argument.

A Appendix

Here we use induction on the order of the operator. We will do:

- Express $A_{2(m+1)}(C,B,Q)$ in terms of $(Q-1)$ in all terms containing B.
- Compute and see

$$\begin{aligned}&A_{2m}(C-1,B,Q-1)\{A_2(C+m,B,Q+m)+A_2(C+m,B,Q-m)\}\\&\qquad=A_{2m}(C-1,B,Q-1)(C+m-B)(C+m+B)+A_{2(m+1)}(C,B,Q).\end{aligned}$$

- Since the above simply says

$$\begin{aligned}&2A_{2m}(C-1,B,Q-1)(C+m-B)(C+m+B)\\&\qquad=A_{2m}(C-1,B,Q-1)(C+m-B)(C+m+B)+A_{2(m+1)}(C,B,Q),\end{aligned}$$

  conclude

$$A_{2(m+1)}(C,B,Q)=A_{2m}(C-1,B,Q-1)(C+m-B)(C+m+B).$$

To go on, note first that $A_{2(m+1)}(C,B,Q)$ in terms of $(Q-1)$ in all terms containing B is

$$\begin{aligned}&\sum_{\ell=0}^{m+1}(-1)^{m+1-\ell}\binom{m+1}{\ell}\\&\quad\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1)+2(\ell-1))\\&\quad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1)-2(\ell-1))\\&\quad\bullet(C-Q)\cdots(C-Q+2(\ell-1))\\&\quad\bullet(C+Q)\cdots(C+Q-2(\ell-1))\,.\end{aligned}\tag{8}$$

$A_{2m}(C-1,B,Q-1)$ can be written

$$\begin{aligned}&\sum_{\ell=0}^{m}(-1)^{m-\ell}\binom{m}{\ell}\\&\quad\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1)+2\ell)\\&\quad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1)-2\ell)\\&\quad\bullet(C-Q)\cdots(C-Q+2(\ell-1))\\&\quad\bullet(C+Q-2)\cdots(C+Q-2\ell)\,.\end{aligned}\tag{9}$$

Define R_B and R_C to express $A_{2m}(C-1,B,Q-1)$ as

$$\begin{aligned}&(-1)^m\binom{m}{0}\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1))\\&\quad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1))+R_B\end{aligned}$$

or

$$\begin{aligned}&R_C+(-1)^0\binom{m}{m}\bullet(C-Q)\cdots(C-Q+2(m-1))\\&\quad\bullet(C+Q-2)\cdots(C+Q-2m)\,.\end{aligned}$$

We can write

$$\begin{aligned}A_2(C+m,B,Q+m) &= (C+m+B)(C+m-B)\\ &= -(B-(Q+m))(B+(Q+m)) + (C-Q)(C+Q+2m)\end{aligned}$$

and

$$\begin{aligned}A_2(C+m,B,Q-m) &= (C+m+B)(C+m-B)\\ &= -(B-(Q-m))(B+(Q-m)) + (C-Q+2m)(C+Q).\end{aligned}$$

The first product $A_{2m}(C-1,B,Q-1)A_2(C+m,B,Q+m)$ becomes

$$\begin{aligned}&(-1)^m\binom{m}{0}\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1))\\ &\quad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1))\\ &\quad\bullet\{-(B-(Q+m))(B+(Q+m))+(C-Q)(C+Q+2m)\}\\ &\quad+R_B\bullet(C+m+B)(C+m-B),\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&(-1)^{m+1}\binom{m+1}{0}\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1))\,(B-(Q+m))\\ &\quad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1))\,(B+(Q+m))\\ &\quad+(-1)^m\binom{m}{0}\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1))\\ &\quad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1))\\ &\quad\bullet(C-Q)(C+Q+2m)+R_B\bullet(C+m+B)(C+m-B).\end{aligned}$$

The second product $A_{2m}(C-1,B,Q-1)A_2(C+m,B,Q-m)$ is

$$\begin{aligned}&R_C\bullet(C+m+B)(C+m-B)+(-1)^0\binom{m}{m}\bullet(C-Q)\cdots(C-Q+2(m-1))\\ &\quad\bullet(C+Q-2)\cdots(C+Q-2m)\\ &\quad\bullet\{-(B-(Q-m))(B+(Q-m))+(C-Q+2m)(C+Q)\}\\ &\quad=R_C\bullet(C+m+B)(C+m-B)+(-1)^1\binom{m}{m}\bullet(C-Q)\cdots(C-Q+2(m-1))\\ &\quad\bullet(C+Q-2)\cdots(C+Q-2m)\bullet(B-(Q-m))(B+(Q-m))\\ &\quad+(-1)^0\binom{m+1}{m+1}\bullet(C-Q)\cdots(C-Q+2m)\bullet(C+Q)\cdots(C+Q-2m)\,.\end{aligned}$$

So by adding up the above two products, we get

$$\begin{aligned}&(-1)^{m+1}\binom{m+1}{0}\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1))\,(B-(Q+m))\\ &\quad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1))\,(B+(Q+m))\\ &\quad+(-1)^0\binom{m+1}{m+1}\bullet(C-Q)\cdots(C-Q+2m)\bullet(C+Q)\cdots(C+Q-2m)\\ &\quad+(-1)^m\binom{m}{0}\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1))\end{aligned}$$

$$\begin{aligned}
&\bullet (B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1)) \bullet (C-Q)(C+Q+2m)\\
&+(-1)^1\binom{m}{m} \bullet (C-Q)\cdots(C-Q+2(m-1)) \bullet (C+Q-2)\cdots(C+Q-2m)\\
&\bullet (B-(Q-m))(B+(Q-m))\\
&+(R_B+R_C)\bullet(C+m+B)(C+m-B).
\end{aligned}$$

Note that R_B is missing the first term and R_C is missing the last term of (9). So we have

$$\begin{aligned}
&(R_B+R_C)\bullet(C+m+B)(C+m-B)\\
&\quad= A_{2m}(C-1,B,Q-1)(C+m+B)(C+m-B)\\
&\quad+\sum_{\ell=1}^{m-1}(-1)^{m-\ell}\binom{m}{\ell}\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1)+2\ell)\\
&\quad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1)-2\ell)\\
&\quad\bullet(C-Q)\cdots(C-Q+2(\ell-1))\bullet(C+Q-2)\cdots(C+Q-2\ell)\\
&\quad\bullet(C+m+B)(C+m-B).
\end{aligned}$$

Therefore, $A_{2m}(C-1,B,Q-1)(C+m+B)(C+m-B)$ equals (see (8))

1st term of $A_{2(m+1)}(C,B)$ + $(m+2)$nd term of $A_{2(m+1)}(C,B)$

$$\begin{aligned}
&+(-1)^m\binom{m}{0}\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1))\\
&\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1))\\
&\bullet(C-Q)(C+Q+2m) \qquad (10)
\end{aligned}$$

$$\begin{aligned}
&+(-1)^1\binom{m}{m}\bullet(C-Q)\cdots(C-Q+2(m-1))\\
&\bullet(C+Q-2)\cdots(C+Q-2m)\bullet(B-(Q-m))(B+(Q-m)) \qquad (11)
\end{aligned}$$

$$\begin{aligned}
&+\sum_{\ell=1}^{m-1}(-1)^{m-\ell}\binom{m}{\ell}\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1)+2\ell)\\
&\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1)-2\ell)\\
&\bullet(C-Q)\cdots(C-Q+2(\ell-1))\bullet(C+Q-2)\cdots(C+Q-2\ell)\\
&\bullet(C+m+B)(C+m-B). \qquad (12)
\end{aligned}$$

So we want to show (10) + (11) + (12) is exactly the other term in $A_{2(m+1)}(C,B)$.

Since, for any Q_ℓ,

$$(C+m+B)(C+m-B)=\{-(B-Q_\ell)(B+Q_\ell)+(C+m-Q_\ell)(C+m+Q_\ell)\},$$

(12) becomes

$$\begin{aligned}
&\sum_{\ell=1}^{m-1}(-1)^{m+1-\ell}\binom{m}{\ell}\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1)+2\ell)\\
&\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1)-2\ell)\\
&\bullet(C-Q)\cdots(C-Q+2(\ell-1))\bullet(C+Q-2)\cdots(C+Q-2\ell)\bullet(B-Q_\ell)(B+Q_\ell)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\ell=1}^{m-1}(-1)^{m-\ell}\binom{m}{\ell}\bullet(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1)+2\ell)\\
&\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1)-2\ell)\\
&\bullet(C-Q)\cdots(C-Q+2(\ell-1))\bullet(C+Q-2)\cdots(C+Q-2\ell)\\
&\bullet(C+m-Q_\ell)(C+m+Q_\ell)\\
&=:\sum_{\ell=1}^{m-1}(-1)^{m+1-\ell}\binom{m}{\ell}E_\ell+\sum_{\ell=1}^{m-1}(-1)^{m-\ell}\binom{m}{\ell}F_\ell\\
&=\sum_{\ell=2}^{m-2}(-1)^{m+1-\ell}\left(\binom{m}{\ell}E_\ell+\binom{m}{\ell-1}F_{\ell-1}\right)\\
&+(-1)^m\binom{m}{1}E_1+(-1)^1\binom{m}{m-1}F_{m-1}.
\end{aligned}
$$

But $\left(\binom{m}{\ell}E_\ell+\binom{m}{\ell-1}F_{\ell-1}\right)$ becomes

$$
\begin{aligned}
&\binom{m}{\ell}(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1)+2(\ell-1))\\
&\quad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1)-2(\ell-1))\\
&\quad\bullet(C-Q)\cdots(C-Q+2(\ell-1))\bullet(C+Q-2)\cdots(C+Q-2\ell)\\
&\quad+\binom{m}{\ell-1}(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1)+2(\ell-1))\\
&\quad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1)-2(\ell-1))\\
&\quad\bullet(C-Q)\cdots(C-Q+2((\ell-1)-1))\bullet(C+Q-2)\cdots(C+Q-2(\ell-1))\\
&\quad\bullet(C+m-Q_{\ell-1})(C+m+Q_{\ell-1})
\end{aligned}
$$

which is, upon choosing Q_ℓ to be $Q-2\ell+m$,

$$
\begin{aligned}
&(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1)+2(\ell-1))\\
&\quad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1)-2(\ell-1))\\
&\quad\bullet(C-Q)\cdots(C-Q+2(\ell-1))\bullet(C+Q-2)\cdots(C+Q-2(\ell-1))
\end{aligned}
$$

times

$$
\binom{m}{\ell}(C+Q-2\ell)+\binom{m}{\ell-1}(C+Q-2(\ell-1-m)),
$$

since

$$
\begin{aligned}
&B-Q_\ell=B-(Q-1)-(m-1)+2(\ell-1),\\
&B+Q_\ell=B+(Q-1)+(m-1)-2(\ell-1),\\
&C+m-Q_\ell=C-Q+2(\ell-1)\qquad\text{and}\\
&C+m+Q_\ell=C+Q+2(\ell-1-m).
\end{aligned}
$$

Note also that

$$
\binom{m}{\ell}(C+Q-2\ell)+\binom{m}{\ell-1}(C+Q-2(\ell-1-m))=\binom{m+1}{l}(C+Q).
$$

Thus

$$\left(\binom{m}{\ell}E_\ell + \binom{m}{\ell-1}F_{\ell-1}\right) = \text{ the } (\ell+1)\text{st term of } A_{2(m+1)}(C,B,Q).$$

Finally we note that (10) and $(-1)^m\binom{m}{1}E_1$ add up to

$$\begin{aligned}&(-1)^m\,(B-(Q-1)+(m-1))\cdots(B-(Q-1)-(m-1))\\ &\qquad\bullet(B+(Q-1)-(m-1))\cdots(B+(Q-1)+(m-1))\bullet(C-Q)\end{aligned}$$

times

$$\binom{m}{0}(C+Q+2m)+\binom{m}{1}(C+Q-2),$$

since

$$B-Q_1 = B-(Q-1)-(m-1) \qquad \text{and} \qquad B+Q_1 = B+(Q-1)+(m-1).$$

This is the 2nd term of $A_{2(m+1)}(C,B,Q)$, since

$$\binom{m}{0}(C+Q+2m)+\binom{m}{1}(C+Q-2) = \binom{m+1}{1}(C+Q)$$

Similarly, (11) and $(-1)^1\binom{m}{m-1}F_{m-1}$ add up to

$$\begin{aligned}&(-1)^m\,(C-Q)\cdots(C-Q+2(m-1))\bullet(C+Q-2)\cdots(C+Q-2(m-1))\\ &\qquad\bullet(B-(Q-1)+(m-1))(B+(Q-1)-(m-1))\end{aligned}$$

times

$$\binom{m}{m}(C+Q-2m)+\binom{m}{m-1}(C+Q+2),$$

since

$$\begin{aligned}&C+m-Q_{m-1} = C-Q+2(m-1),\\ &C+m+Q_{m-1} = C+Q+2,\\ &B-(Q-m) = B-(Q-1)+(m-1) \qquad \text{and}\\ &B+(Q-m) = B+(Q-1)-(m-1).\end{aligned}$$

This is the $(m+1)$st term of $A_{2(m+1)}(C,B,Q)$, since

$$\binom{m}{m}(C+Q-2m)+\binom{m}{m-1}(C+Q+2) = \binom{m+1}{m}(C+Q).$$

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