Stanilov–Tsankov–Videv Theory*

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Abstract. We survey some recent results concerning Stanilov–Tsankov–Videv theory, conformal Osserman geometry, and Walker geometry which relate algebraic properties of the curvature operator to the underlying geometry of the manifold.

Key words: algebraic curvature tensor; anti-self-dual; conformal Jacobi operator; conformal Osserman manifold; Jacobi operator; Jacobi–Tsankov; Jacobi–Videv; mixed-Tsankov; Osserman manifold; Ricci operator; self-dual; skew-symmetric curvature operator; skew-Tsankov; skew-Videv; Walker manifold; Weyl conformal curvature operator

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This article is dedicated to the memory of N. Blažić (who passed away 10 October 2005) and to the memory of T. Branson (who passed away 11 March 2006). They were coauthors, friends, and talented mathematicians.

1 Introduction

In this article we shall survey just a few of the many recent developments in Differential Geometry which relate algebraic properties of various operators naturally associated with the curvature of a pseudo-Riemannian manifold to the underlying geometric properties of the manifolds involved.

We introduce the following notational conventions. Let $\mathcal{M} = (M, g)$ be a pseudo-Riemannian manifold of signature (p, q) and dimension $m = p + q$. We say that $\mathcal{M}$ is Riemannian if $p = 0$,

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i.e. if g is positive definite. We say that $\mathcal{M}$ is Lorentzian if $p = 1$. Let

$$S_P^{\pm}(\mathcal{M}) = \{\xi \in T_P M : g(\xi, \xi) = \pm 1\}$$

be the *pseudo-spheres* of unit spacelike $(+)$ and unit timelike $(-)$ vectors. Let ∇ be the Levi-Civita connection and let

$$\mathcal{R}(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}$$

be the associated *skew-symmetric curvature operator*. If $\{e_i\}$ is a local frame for the tangent bundle, we let $g_{ij} := g(e_i, e_j)$ and let g^{ij} be the inverse matrix. The *Jacobi operator* and the *Ricci operator* are the self-adjoint endomorphisms defined, respectively, by:

$$\mathcal{J}(x) : y \to \mathcal{R}(y, x)x \qquad \text{and} \qquad \rho : x \to \sum_{ij} g^{ij} \mathcal{R}(x, e_i) e_j. \tag{1.1}$$

One also defines the *curvature tensor* $R \in \otimes^4 T^* M$, the *scalar curvature* τ, the *Weyl conformal curvature operator* $\mathcal{W}$, and the *conformal Jacobi operator* $\mathcal{J}_W$, respectively, by:

$$\begin{aligned} & R(x, y, z, w) = g(\mathcal{R}(x, y)z, w), \\ & \tau := \mathrm{Tr}(\rho) = \textstyle\sum_{ijkl} g^{il} g^{jk} R(e_i, e_j, e_k, e_l), \\ & \mathcal{W}(x, y): \quad z \to \mathcal{R}(x, y)z - \{(m-1)(m-2)\}^{-1} \tau \{g(y, z)x - g(x, z)y\} \\ & \qquad\qquad + (m-2)^{-1} \{g(\rho y, z)x - g(\rho x, z)y + g(y, z)\rho x - g(x, z)\rho y\}, \\ & \mathcal{J}_W(x): \quad y \to \mathcal{W}(y, x)x. \end{aligned} \tag{1.2}$$

Motivated by the seminal paper of Osserman [23], one studies the spectral properties of the Jacobi operator $\mathcal{J}$ and of the conformal Jacobi operator $\mathcal{J}_W$ and makes the following definitions:

Definition 1. Let $\mathcal{M}$ be a pseudo-Riemannian manifold.

1. $\mathcal{M}$ is *pointwise Osserman* if $\mathcal{J}$ has constant eigenvalues on $S_P^+(\mathcal{M})$ and on $S_P^-(\mathcal{M})$ for every $P \in M$.
2. $\mathcal{M}$ is *pointwise conformally Osserman* if $\mathcal{J}_W$ has constant eigenvalues on $S_P^+(\mathcal{M})$ and on $S_P^-(\mathcal{M})$ for every $P \in M$.

We refer to [16] for a more complete discussion of Osserman geometry as that lies beyond the scope of our present endeavors.

Similarly, motivated by the seminal papers of Stanilov and Videv [26], of Tsankov [27], and of Videv [28] one studies the commutativity properties of these operators:

Definition 2. Let $\mathcal{M}$ be a pseudo-Riemannian manifold.

1. $\mathcal{M}$ is *Jacobi–Tsankov* if $\mathcal{J}(\xi_1)\mathcal{J}(\xi_2) = \mathcal{J}(\xi_2)\mathcal{J}(\xi_1)$ for all ξ_i.
2. $\mathcal{M}$ is *mixed-Tsankov* if $\mathcal{R}(\xi_1, \xi_2)\mathcal{J}(\xi_3) = \mathcal{J}(\xi_3)\mathcal{R}(\xi_1, \xi_2)$ for all ξ_i.
3. $\mathcal{M}$ is *skew-Tsankov* if $\mathcal{R}(\xi_1, \xi_2)\mathcal{R}(\xi_3, \xi_4) = \mathcal{R}(\xi_3, \xi_4)\mathcal{R}(\xi_1, \xi_2)$ for all ξ_i.
4. $\mathcal{M}$ is *Jacobi–Videv* if $\mathcal{J}(\xi)\rho = \rho\mathcal{J}(\xi)$ for all ξ.
5. $\mathcal{M}$ is *skew-Videv* if $\mathcal{R}(\xi_1, \xi_2)\rho = \rho\mathcal{R}(\xi_1, \xi_2)$ for all ξ_i. This has also been called *Ricci semi-symmetric* by some authors.

In this brief note, we survey some recent results concerning these concepts; we refer to [16, 17, 18] for a discussion of some previous results in this area.

Our first task is to pass to the algebraic setting.

Definition 3. Let $\langle\cdot,\cdot\rangle$ be a non-degenerate bilinear form of signature (p,q) on a finite dimensional real vector space V. Let $R \in \otimes^4 V^*$ be a 4-tensor. We say that $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, R)$ is a *model* and that R is an *algebraic curvature tensor* if R satisfies the usual curvature identities for all x, y, z, and w:

$$\begin{aligned}&R(x,y,z,w) = -R(y,x,z,w) = R(z,w,x,y),\\&R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0.\end{aligned}$$

The associated *algebraic curvature operator* $\mathcal{R}$ is then defined by using the inner product to raise indices; this skew-symmetric operator is characterized by the identity:

$$\langle \mathcal{R}(x,y)z, w\rangle = R(x,y,z,w).$$

The Jacobi operator, the Ricci operator, the Weyl conformal curvature operator, and the conformal Jacobi operator are then defined as in equations (1.1) and (1.2). The concepts of Definitions 1 and 2 extend naturally to this setting.

If P is a point of a pseudo-Riemannian manifold $\mathcal{M}$, then the associated model is defined by

$$\mathfrak{M}(\mathcal{M}, P) := (T_P M, g_P, R_P).$$

We note that every model $\mathfrak{M}$ is geometrically realizable; this means that given $\mathfrak{M}$, there is $(\mathcal{M}, P)$ such that $\mathfrak{M}(\mathcal{M}, P)$ is isomorphic to $\mathfrak{M}$ – see, for example, the discussion in [17].

One has the following examples of algebraic curvature tensors.

Example 1.

1. If ψ is self-adjoint with respect to $\langle\cdot,\cdot\rangle$, one defines an algebraic curvature tensor

   $$R_\psi(x,y,z,w) = \langle\psi x, w\rangle\langle\psi y, z\rangle - \langle\psi x, z\rangle\langle\psi y, w\rangle.$$

   Taking $\psi = \mathrm{id}$ and rescaling yields the algebraic curvature tensor of constant sectional curvature c:

   $$R_c(x,y,z,w) = c\{\langle x, w\rangle\langle y, z\rangle - \langle x, z\rangle\langle y, w\rangle\}.$$

   One says that a model $\mathfrak{M}$ or a pseudo-Riemannian manifold $\mathcal{M}$ has *constant sectional curvature* c if $R = R_c$ for some constant c.

2. If ϕ is skew-adjoint with respect to $\langle\cdot,\cdot\rangle$, one defines an algebraic curvature tensor

   $$R_\phi(x,y,z,w) = \langle\phi y, z\rangle\langle\phi x, w\rangle - \langle\phi x, z\rangle\langle\phi y, w\rangle - 2\langle\phi x, y\rangle\langle\phi z, w\rangle.$$

Remark 1. The space of algebraic curvature tensors is spanned as a linear space by the tensors given in Example 1 (1) or in Example 1 (2) [13]; we also refer to [12].

Our first result is the equivalence of conditions (1) and (2) and of (4) and (5) in Definition 2; if $\mathfrak{M}$ is a model or if $\mathcal{M}$ is a pseudo-Riemannian manifold, then Jacobi–Tsankov and mixed-Tsankov are equivalent conditions. Similarly Jacobi–Videv and skew-Videv are equivalent conditions. This follows from the following result [20]:

Theorem 1. *Let $\mathfrak{M}$ be a model and let T be a self-adjoint linear transformation of V. Then the following assertions are equivalent:*

1. *$\mathcal{R}(x,y)T = T\mathcal{R}(x,y)$ for all x, $y \in V$.*

2. *$\mathcal{J}(x)T = T\mathcal{J}(x)$ for all $x \in V$.*
3. *$R(Tx, y, z, w) = R(x, Ty, z, w) = R(x, y, Tz, w) = R(x, y, z, Tw)$ for all x, y, z, w in V.*

Here is a brief outline of the remainder of this article. In Section 2, we study Jacobi–Tsankov models and manifolds. In Section 3, we study skew-Tsankov models and manifolds. In Section 4, we study Jacobi–Videv models and manifolds. In Section 5, we recall some general results concerning conformal Osserman geometry. In Section 6, we study these concepts in the context of Walker manifolds of signature (2,2).

2 Jacobi–Tsankov models and manifolds

We first turn to the Riemannian setting in the following result [9]:

Theorem 2. *If $\mathfrak{M}$ is a Jacobi–Tsankov Riemannian model, then $R = 0$.*

Proof. We can sketch the proof as follows. Since $\{\mathcal{J}(x)\}_{x\in V}$ form a family of commuting self-adjoint operators, we can simultaneously diagonalize these operators to decompose $V = \oplus_\lambda V_\lambda$ so $\mathcal{J}(x) = \lambda(x)\,\mathrm{id}$ on V_λ. If $x \in V$, decompose $x = \oplus x_\lambda$ for $x_\lambda \in V_\lambda$. Let

$$\mathcal{O} = \{x \in V : x_\lambda \neq 0 \quad \text{for all} \quad \lambda\};$$

this is an open dense subset of V. If $x \in \mathcal{O}$, since $\mathcal{J}(x)x = 0$, $\lambda(x) = 0$ for all λ. Since $\mathcal{O}$ is dense and $\lambda(\cdot)$ is continuous, $\lambda(x) = 0$ for all x so $\mathcal{J}(x) = 0$ for all x; the usual curvature symmetries now imply the full curvature tensor R vanishes. ■

Definition 4. One says that a model $\mathfrak{M}$ or a pseudo-Riemannian manifold $\mathcal{M}$ is *orthogonally Jacobi–Tsankov* if $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ for all vectors x and y with $x \perp y$.

One has the following classification result [9]; we also refer to a related result [27] if $\mathcal{M}$ is a hypersurface in $\mathbb{R}^{m+1}$.

Theorem 3.

1. *Let $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a Riemannian model. Then $\mathfrak{M}$ is orthogonally Jacobi–Tsankov if and only if one of the following conditions holds:*
   (a) *$R = cR_{\mathrm{id}}$ has constant sectional curvature c for some $c \in \mathbb{R}$.*
   (b) *$\dim(V)$ is even and $R = cR_\Theta$ is defined by Example 1 (2) where Θ is a Hermitian almost complex structure on $(V, \langle\cdot,\cdot\rangle)$ and where $c \in \mathbb{R}$.*
2. *Let $\mathcal{M}$ be a Riemannian manifold of dimension m.*
   (a) *If $m > 2$, then $\mathcal{M}$ is orthogonally Jacobi–Tsankov if and only if $\mathcal{M}$ has constant sectional curvature c.*
   (b) *If $m = 2$, then $\mathcal{M}$ is always orthogonally Jacobi–Tsankov.*

Definition 5. We say that a model $\mathfrak{M}$ or a pseudo-Riemannian manifold $\mathcal{M}$ is *conformally Jacobi–Tsankov* if $\mathcal{J}_W(x)\mathcal{J}_W(y) = \mathcal{J}_W(y)\mathcal{J}_W(x)$ for all x and y. We say that $\mathfrak{M}$ or $\mathcal{M}$ is *orthogonally conformally Jacobi–Tsankov* if $\mathcal{J}_W(x)\mathcal{J}_W(y) = \mathcal{J}_W(y)\mathcal{J}_W(x)$ for all vectors x and y with $x \perp y$.

Remark 2. These are conformal notions – if $\mathcal{M}$ is conformally equivalent to $\mathcal{M}_1$, then $\mathcal{M}$ is conformally Jacobi–Tsankov (resp. orthogonally conformally Jacobi–Tsankov) if and only if $\mathcal{M}_1$ is conformally Jacobi Tsankov (resp. orthogonally conformally Jacobi–Tsankov). We refer to [3] for further details.

We have the following useful result:

Theorem 4. *A Riemannian model $\mathfrak{M}$ is orthogonally conformally Jacobi–Tsankov if and only if $\mathcal{W} = 0$.*

Proof. Let $\mathcal{W}$ be the associated Weyl conformal curvature operator. Then $\mathcal{W}$ is an algebraic curvature tensor which is orthogonally-Jacobi Tsankov. Thus Theorem 3 yields either that $\mathcal{W} = c\mathcal{R}_{\mathrm{id}}$ or that $\mathcal{W} = c\mathcal{R}_{\Theta}$. Since the scalar curvature defined by the tensors $\mathcal{R}_{\mathrm{id}}$ and $\mathcal{R}_{\Theta}$ is non-zero, we may conclude $c = 0$. ■

There are non-trivial examples of Jacobi–Tsankov manifolds and models in the higher signature setting.

Definition 6. We say that a model $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, R)$ is *indecomposable* if there is no non-trivial orthogonal decomposition $V = V_1 \oplus V_2$ which induces a decomposition $R = R_1 \oplus R_2$.

We refer to [7] for the proof of the following result:

Theorem 5. *Let $\mathfrak{M}$ be a model.*

1. *If $\mathfrak{M}$ is Jacobi–Tsankov, then $\mathcal{J}(x)^2 = 0$ for all x in V.*
2. *If $\mathfrak{M}$ is Jacobi–Tsankov and Lorentzian, then $R = 0$.*
3. *Let $\mathfrak{M}$ be indecomposable with $\dim(\mathfrak{M}) < 14$. The following conditions are equivalent:*
   (a) *$V = U \oplus \bar{U}$ and $R = R_U \oplus 0$ where U and $\bar{U}$ are totally isotropic subspaces.*
   (b) *$\mathfrak{M}$ is Jacobi–Tsankov.*

   Either (3a) or (3b) implies that $\mathcal{R}(x,y)z \in \bar{U}$ and that $\mathcal{R}(x,y)\mathcal{R}(u,v)z = 0$ for all x, y, z, u, $v \in V$, that $\mathcal{J}(x)\mathcal{J}(y) = 0$ for all x, $y \in V$, and that $\mathfrak{M}$ is skew-Tsankov.

The condition $\mathcal{J}(x)^2 = 0$ does not imply that $\mathfrak{M}$ is Jacobi–Tsankov [7]:

Example 2. Let $\langle\cdot,\cdot\rangle$ be an inner product of signature $(4,4)$ on $\mathbb{R}^8$. Choose skew-symmetric endomorphisms $\{e_1, e_2, e_3, e_4\}$ so that

$$e_1^2 = e_2^2 = \mathrm{id}, \qquad e_3^2 = e_4^2 = -\,\mathrm{id}, \qquad \text{and} \qquad e_ie_j + e_je_i = 0 \quad \text{for} \quad i \neq j.$$

Note that this gives a suitable Clifford module structure to $\mathbb{R}^8$. Set

$$\phi_1 = e_1 + e_3 \qquad \text{and} \qquad \phi_2 = e_2 + e_4.$$

Adopt the notation of Example 1 (2) to define R_{ϕ_i}. Then

$$\mathfrak{M} := (\mathbb{R}^8, \langle\cdot,\cdot\rangle, R_{\phi_1} + R_{\phi_2})$$

is not Jacobi–Tsankov but satisfies $\mathcal{J}(x)^2 = 0$ for all x.

We have the following example [7] that shows that the structure of Theorem 5 (3a) is geometrically realizable:

Example 3. Let $(x_1, \ldots, x_p, y_1, \ldots, y_p)$ be the usual coordinates on $\mathbb{R}^{2p}$. Let $\mathcal{M} = (\mathbb{R}^{2p}, g)$ where $g(\partial_{x_i}, \partial_{y_j}) = \delta_{ij}$ and let $g(\partial_{x_i}, \partial_{x_j}) = g_{ij}(x)$. Then there exists a decomposition $T(\mathbb{R}^{2p}) = U \oplus \bar{U}$ where U and $\bar{U}$ are totally isotropic so that $\mathcal{R}(x,y)z \in \bar{U}$ and that $\mathcal{R}(x,y)\mathcal{R}(u,v)z = 0$ for all x, y, z, u, $v \in V$. Furthermore, for generic g, the model $\mathfrak{M}(\mathcal{M}, P)$ is indecomposable for all $P \in \mathbb{R}^{2p}$.

The restriction in Theorem 5 that $\dim(V) < 14$ is essential. We have the following [7]:

Example 4. Let $\{\alpha_i, \alpha_i^*, \beta_{i,1}, \beta_{i,2}, \beta_{4,1}, \beta_{4,2}\}_{1\le i\le 3}$ be a basis for $\mathbb{R}^{14}$. Define $\mathcal{M}_{6,8}$ by:

$$\begin{aligned}
&\langle\alpha_i, \alpha_i^*\rangle = \langle\beta_{i,1}, \beta_{i,2}\rangle, \quad 1 \le i \le 3; \qquad \langle\beta_{4,1}, \beta_{4,1}\rangle = \langle\beta_{4,2}, \beta_{4,2}\rangle = -\tfrac{1}{2}; \qquad \langle\beta_{4,1}, \beta_{4,2}\rangle = \tfrac{1}{4};\\
&R_{\alpha_2,\alpha_1,\alpha_1,\beta_{2,1}} = R_{\alpha_3,\alpha_1,\alpha_1,\beta_{3,1}} = R_{\alpha_3,\alpha_2,\alpha_2,\beta_{3,2}} = 1,\\
&R_{\alpha_1,\alpha_2,\alpha_2,\beta_{1,2}} = R_{\alpha_1,\alpha_3,\alpha_3,\beta_{1,1}} = R_{\alpha_2,\alpha_3,\alpha_3,\beta_{2,2}} = 1,\\
&R_{\alpha_1,\alpha_2,\alpha_3,\beta_{4,1}} = R_{\alpha_1,\alpha_3,\alpha_2,\beta_{4,1}} = R_{\alpha_2,\alpha_3,\alpha_1,\beta_{4,2}} = R_{\alpha_2,\alpha_1,\alpha_3,\beta_{4,2}} = -\tfrac{1}{2}.
\end{aligned}$$

Then $\mathfrak{M}_{6,8}$ has signature $(6, 8)$, $\mathfrak{M}_{6,8}$ is Jacobi–Tsankov, $\mathfrak{M}_{6,8}$ is not skew-Tsankov, and there exist x and y so that $\mathcal{J}(x)\mathcal{J}(y) \neq 0$.

Furthermore, this example is geometrically realizable [10]:

Example 5. Take coordinates $\{x_i, x_i^*, y_{i,1}, y_{i,2}, y_{4,1}, y_{4,2}\}_{i=1,2,3}$ for $\mathbb{R}^{14}$. Let $a_{i,j} \in \mathbb{R}$ and let $\mathcal{M}_{6,8} := (\mathbb{R}^{14}, g)$ where:

$$\begin{aligned}
&g(\partial_{x_i}, \partial_{x_i^*}) = g(\partial_{y_{i,1}}, \partial_{y_{i,2}}) = 1, \qquad g(\partial_{y_{4,1}}, \partial_{y_{4,1}}) = g(\partial_{y_{4,2}}, \partial_{y_{4,2}}) = -\tfrac{1}{2},\\
&g(\partial_{y_{4,1}}, \partial_{y_{4,2}}) = \tfrac{1}{4}, \qquad g(\partial_{x_1}, \partial_{x_1}) = -2a_{2,1}x_2y_{2,1} - 2a_{3,1}x_3y_{3,1},\\
&g(\partial_{x_2}, \partial_{x_2}) = -2a_{3,2}x_3y_{3,2} - 2a_{1,2}x_1y_{1,2}, \qquad g(\partial_{x_3}, \partial_{x_3}) = -2a_{1,1}x_1y_{1,1} - 2a_{2,2}x_2y_{2,2},\\
&g(\partial_{x_1}, \partial_{x_2}) = 2(1 - a_{2,1})x_1y_{2,1} + 2(1 - a_{1,2})x_2y_{1,2},\\
&g(\partial_{x_2}, \partial_{x_3}) = x_1y_{4,1} + 2(1 - a_{3,2})x_2y_{3,2} + 2(1 - a_{2,2})x_3y_{2,2},\\
&g(\partial_{x_1}, \partial_{x_3}) = x_2y_{4,2} + 2(1 - a_{3,1})x_1y_{3,1} + 2(1 - a_{1,1})x_3y_{1,1}.
\end{aligned}$$

Then $\mathcal{M}$ has the model $\mathfrak{M}_{6,8}$ and $\mathcal{M}$ is locally symmetric if and only if

$$\begin{aligned}
&a_{1,1} + a_{2,2} + a_{3,1}a_{3,2} = 2, \qquad 3a_{2,1} + 3a_{3,1} + 3a_{1,2}a_{1,1} = 4,\\
&3a_{1,2} + 3a_{3,2} + 3a_{2,1}a_{2,2} = 4.
\end{aligned}$$

We note that the relations of Example 5 have non-trivial solutions. One may take, for example, $a_{1,1} = a_{2,2} = 1$, $a_{1,2} = a_{2,1} = \frac{2}{3}$, and $a_{3,1} = a_{3,2} = 0$.

3 Skew-Tsankov models and manifolds

Riemannian skew-Tsankov models are completely classified [8]:

Theorem 6. *Let $\mathfrak{M}$ be a Riemannian skew-Tsankov model. Then there is an orthogonal direct sum decomposition $V = V_1 \oplus \cdots \oplus V_k \oplus U$ where $\dim(V_k) = 2$ and where $R = R_1 \oplus \cdots \oplus R_k \oplus 0$.*

Proof. One has that $\{\mathcal{R}(\xi, \eta)\}_{\xi,\eta\in V}$ is a collection of commuting skew-adjoint endomorphisms. As the inner product is definite, there exists an orthogonal decomposition of V so that each endomorphism $\mathcal{R}(\xi, \eta)$ decomposes as a direct sum of 2×2 blocks

$$\begin{pmatrix} 0 & a(\xi,\eta) \\ -a(\xi,\eta) & 0 \end{pmatrix}.$$

The desired result then follows from the curvature symmetries. ■

The situation in the geometric context is less clear. We refer to [8] for the following 3-dimensional and 4-dimensional examples which generalize previous examples found in [27]. We say that $\mathcal{M}$ is an irreducible Riemannian manifold if there is no local product decomposition.

Example 6.

1. Let $M = (0, \infty) \times N$ where N is a Riemann surface with scalar curvature $\tau_N \not\equiv 1$. Give M the warped product metric $ds^2 = dt^2 + t^2 ds_N^2$. Then $\mathcal{M} := (M, g_M)$ is an irreducible skew-Tsankov manifold with $\tau_M = t^{-2}(\tau_N - 1)$.
2. Let (x_1, x_2, x_3, x_4) be the usual coordinates on $\mathbb{R}^4$. Let $\mathcal{M}_\beta = (\mathbb{R}^4, g)$ where $ds^2 = x_3^2 dx_1^2 + (x_3 + \beta x_4)^2 dx_2^2 + dx_3^2 + dx_4^2$. Then $\mathcal{M}_\beta$ is an irreducible skew-Tsankov manifold with $\tau_\beta = -2x_3^{-1}(x_3 + \beta x_4)^{-1}$. M_β is not isometric to $M_{\bar\beta}$ for $0 < \beta < \bar\beta$.

In the higher signature setting, we note that Example 3 provides examples of neutral signature pseudo-Riemannian manifolds with $\mathcal{R}(x, y)\mathcal{R}(z, w) = 0$ for all x, y, z, w. There are, however, less trivial examples.

Definition 7. We say $\mathcal{M}$ is 3-skew nilpotent if

1. There exist ξ_i with $\mathcal{R}(\xi_1, \xi_2)\mathcal{R}(\xi_3, \xi_4) \neq 0$ and
2. For all ξ_i, one has $\mathcal{R}(\xi_1, \xi_2)\mathcal{R}(\xi_3, \xi_4)\mathcal{R}(\xi_5, \xi_6) = 0$.

We refer to [14] for the proof of:

Example 7. Let $(x, u_1, \ldots, u_{m-2}, y)$ be coordinates on $\mathbb{R}^m$. Let $f = f(\vec{u})$ be smooth. Let Ξ be a non-degenerate bilinear form on $\mathbb{R}^{m-2}$. Consider $\mathcal{M} := (\mathbb{R}^m, g)$ where the non-zero components of g are given by:

$$g(\partial_x, \partial_x) = -2f(\vec{u}), \qquad g(\partial_x, \partial_y) = 1, \qquad g(\partial_{u_a}, \partial_{u_b}) = \Xi_{ab}.$$

Then $\mathcal{M}$ is skew-Tsankov and 3-skew nilpotent; it need not be Jacobi–Tsankov.

4 Jacobi–Videv models and manifolds

One says $\mathcal{M}$ is Einstein if ρ is a scalar multiple of the identity. More generally:

Definition 8. One says $\mathcal{M}$ is *pseudo-Einstein* if ρ either has a single real eigenvalue λ or has exactly two eigenvalues which are complex conjugates μ and $\bar\mu$.

It is immediate that pseudo-Einstein implies Einstein in the Riemannian setting as ρ is diagonalizable if the metric is positive definite.

We refer to [19] for the proof of the following result; see also [22] for related work in the 4-dimensional context.

Theorem 7. *Let $\mathfrak{M}$ be an indecomposable model which is Jacobi–Videv. Then $\mathfrak{M}$ is pseudo-Einstein.*

Proof. Let $m := \dim(V)$. Let $\lambda \in \mathbb{C}$ have non-negative real part. Set

$$V_\lambda := \{v \in V : (T - \lambda)^m (T - \bar\lambda)^m v = 0\}.$$

We then have the Jordan decomposition $V = \oplus_\lambda V_\lambda$ as an orthogonal direct sum of generalized eigenspaces of ρ. Since $\mathcal{J}(x)$ preserves this decomposition, it follows that $\mathcal{J} = \oplus_\lambda \mathcal{J}_\lambda$. The curvature symmetries then imply that $\mathcal{R} = \oplus_\lambda \mathcal{R}_\lambda$. Since $\mathfrak{M}$ is assumed indecomposable, there is only one $V_\lambda \neq \{0\}$ and thus $\mathfrak{M}$ is pseudo-Einstein. ■

This shows, in the Riemannian setting, that an indecomposable model is Jacobi–Videv if and only if it is Einstein. The condition that $\mathfrak{M}$ is pseudo-Einstein does not, however, imply that $\mathfrak{M}$ is Jacobi–Videv in the higher signature setting as the following [20] shows:

Example 8. Let $\{x_1, x_2, x_3, x_4\}$ be coordinates on $\mathbb{R}^4$. Let $\mathcal{M} = (\mathbb{R}^4, g)$ where

$$g(\partial_{x_1}, \partial_{x_4}) = g(\partial_{x_2}, \partial_{x_2}) = g(\partial_{x_3}, \partial_{x_3}) = 1 \qquad \text{and} \qquad g(\partial_{x_1}, \partial_{x_3}) = e^{x_2}.$$

Then $\mathcal{M}$ is a homogeneous Lorentz manifold and $\mathcal{M}$ is pseudo-Einstein with $\mathrm{Rank}(\rho) = 2$, $\mathrm{Rank}(\rho^2) = 1$, and $\mathrm{Rank}(\rho^3) = 0$. Thus $\mathcal{M}$ is pseudo-Einstein. However $\mathcal{M}$ is not Jacobi–Videv.

We also have [20]

Example 9. Let $\{x_1, x_2, x_3, x_4\}$ be coordinates on $\mathbb{R}^4$. Let $\mathcal{M} = (\mathbb{R}^4, g)$ where

$$\begin{aligned} &g(\partial_{x_1}, \partial_{x_3}) = g(\partial_{x_2}, \partial_{x_4}) = 1, \qquad g(\partial_{x_3}, \partial_{x_3}) = -g(\partial_{x_4}, \partial_{x_4}) = s x_1 x_2, \\ &g(\partial_{x_3}, \partial_{x_4}) = \tfrac{s}{2}(x_2^2 - x_1^2). \end{aligned}$$

Then $\mathcal{M}$ is locally symmetric of signature $(2, 2)$, $\mathcal{M}$ is Jacobi–Videv, $\mathcal{M}$ is skew-Tsankov, and $\mathcal{M}$ is conformal Osserman. $\mathcal{M}$ is neither Jacobi–Tsankov nor Osserman. $\mathcal{M}$ is pseudo-Einstein with $\rho^2 = -s^2 \,\mathrm{id}$.

Example 10. Setting

$$\begin{aligned} &g(\partial_{x_1}, \partial_{x_3}) = g(\partial_{x_2}, \partial_{x_4}) = 1, \qquad g(\partial_{x_3}, \partial_{x_3}) = -g(\partial_{x_4}, \partial_{x_4}) = \tfrac{s}{2}(x_2^2 - x_1^2), \\ &g(\partial_{x_3}, \partial_{x_4}) = -s x_1 x_2 \end{aligned}$$

yields a local symmetric space of signature $(2, 2)$ which is Einstein. This manifold is Jacobi–Videv and skew-Tsankov. It is neither Jacobi–Tsankov, Osserman, nor conformal Osserman.

We can give a general ansatz which constructs such examples in the algebraic setting; we do not know if these examples are geometrically realizable in general:

Example 11. Let $\mathfrak{M} = (V, (\cdot, \cdot), R)$ be a model. We complexify and let

$$U := V \otimes_{\mathbb{R}} \mathbb{C}.$$

We extend $(\cdot, \cdot)$ and R to be complex multi-linear. Let $\{e_i\}$ be an orthonormal basis for V. Let $\{e_i^+ := e_i, e_i^- := \sqrt{-1} e_i\}$ be a basis for the underlying real vector space $U := V \oplus \sqrt{-1} V$. Let $\Re$ and $\Im$ denote the real and imaginary parts of a complex number, respectively. It is then immediate that

$$\langle \cdot, \cdot \rangle := \Re\{(\cdot, \cdot)\} \qquad \text{and} \qquad S(\cdot, \cdot, \cdot, \cdot) = \Im\{R(\cdot, \cdot, \cdot, \cdot)\}$$

define a model $\mathfrak{N} := (U, \langle \cdot, \cdot \rangle, S)$. One has that the non-zero components of $\langle \cdot, \cdot \rangle$ are $\langle e_i^+, e_i^+ \rangle = 1$ and $\langle e_i^-, e_i^- \rangle = -1$. Thus the metric has neutral signature. Furthermore, the non-zero components of S are given by:

$$\begin{aligned} &S(e_i^-, e_j^+, e_k^+, e_l^+) = S(e_i^+, e_j^-, e_k^+, e_l^+) = S(e_i^+, e_j^+, e_k^-, e_l^+) \\ &\qquad = S(e_i^+, e_j^+, e_k^+, e_l^-) = R(e_i, e_j, e_k, e_l), \\ &S(e_i^+, e_j^-, e_k^-, e_l^-) = S(e_i^-, e_j^+, e_k^-, e_l^-) = S(e_i^-, e_j^-, e_k^+, e_l^-) \\ &\qquad = S(e_i^-, e_j^-, e_k^-, e_l^+) = -R(e_i, e_j, e_k, e_l). \end{aligned}$$

We refer to [20] for the proof of the following result:

Theorem 8. *Adopt the notation of Example* 11. *If $\mathfrak{M}$ is a Riemannian Einstein model with $\rho_{\mathfrak{M}} = s \,\mathrm{id}$, then $\mathfrak{N}$ is a Jacobi–Videv pseudo-Einstein neutral signature model with $\rho_{\mathfrak{N}}^2 = -4s^2 \,\mathrm{id}$.*

Definition 9. Let $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model. Let $\{v_1, \dots, v_k\}$ be an orthonormal basis for a non-degenerate k-plane $\pi \subset V$. Let $\varepsilon_i := \langle v_i, v_i\rangle$ be ± 1. One defines the *higher order Jacobi operator* by setting:

$$\mathcal{J}(\pi) := \sum_{i=1}^{k} \varepsilon_i \mathcal{J}(v_i).$$

The operator $\mathcal{J}(\pi)$ is independent of the particular orthonormal basis chosen; we refer to [21, 24, 25] for a further discussion of this operator. If $\pi = V$, then $\mathcal{J}(\pi) = \rho$. If $\pi = \mathrm{Span}(x)$ where x is a unit spacelike vector, then $\mathcal{J}(\pi) = \mathcal{J}(x)$. Thus $\mathcal{J}(\pi)$ can be thought of as interpolating between the Jacobi operator and the Ricci operator.

Definition 10. Let $\mathfrak{M}$ be a model of signature (p, q). We say that (r, s) is *admissible* if and only if

$$0 \le r \le p, \qquad 0 \le s \le q, \qquad \text{and} \qquad 1 \le r + s \le m - 1.$$

Equivalently, (r, s) is *admissible* if and only if the Grassmannian of linear subspaces of signature (r, s) has positive dimension.

One has the following useful characterization [19]:

Theorem 9. *The following properties are equivalent for* $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, R)$*:*

1. $\mathfrak{M}$ *is Jacobi–Videv, i.e.* $\mathcal{J}(x)\rho = \rho\mathcal{J}(x)$ *for all* $x \in V$*.*
2. *There exists* (r, s) *admissible so* $\mathcal{J}(\pi)\mathcal{J}(\pi^\perp) = \mathcal{J}(\pi^\perp)\mathcal{J}(\pi)$ *for every non-degenerate subspace* π *of signature* (r, s)*.*
3. *There exists* (r, s) *admissible so* $\mathcal{J}(\pi)\rho = \rho\mathcal{J}(\pi)$ *for every non degenerate subspace* π *of signature* (r, s)*.*
4. $\mathcal{J}(\pi)\mathcal{J}(\pi^\perp) = \mathcal{J}(\pi^\perp)\mathcal{J}(\pi)$ *for every non-degenerate linear subspace* π*.*
5. $\mathcal{J}(\pi)\rho = \rho\mathcal{J}(\pi)$ *for every non-degenerate linear subspace* $\pi \subset V$*.*

5 Conformal Osserman geometry

We refer to [1, 3] for the proof of the following result:

Theorem 10. *Let* $\mathcal{M}$ *be a conformally Osserman pseudo-Riemannian manifold of dimension* m*.*

1. *If* $\mathcal{M}$ *is Riemannian and if* m *is odd, then* $\mathcal{M}$ *is locally conformally flat.*
2. *If* $\mathcal{M}$ *is Riemannian, if* $m \equiv 2 \mod 4$*, if* $m \ge 10$*, and if* $\mathcal{W}(P) \ne 0$*, then there is an open neighborhood of* P *in* M *which is conformally equivalent to an open subset of either complex projective space with the Fubini–Study metric or the negative curvature dual.*
3. *If* $\mathcal{M}$ *is Lorentzian, then* $\mathcal{M}$ *is locally conformally flat.*

We also recall the following result [2, 5]:

Theorem 11. *Let* $\mathcal{M}$ *be a 4-dimensional model of arbitrary signature.*

1. $\mathfrak{M}$ *is conformally Osserman if and only if* $\mathcal{M}$ *is either self-dual or anti-self-dual.*
2. *If* $\mathfrak{M}$ *is Riemannian, then* $\mathfrak{M}$ *is conformally Osserman if and only if there exists a quaternion structure* $\{I, J, K\}$ *on* V *and constants* λ_I*,* λ_J*,* λ_K *with* $\lambda_I + \lambda_J + \lambda_K = 0$ *so that* $R = \lambda_I R_I + \lambda_J R_J + \lambda_K R_K$ *where* R_I*,* R_J*, and* R_K *are given by Example* 1 (2).

6 Walker geometry

One says $\mathcal{M}$ is a *Walker manifold* of signature $(2,2)$ if it admits a parallel totally isotropic 2-plane field; this implies [29, 30] that locally $\mathcal{M}$ is isometric to a metric on $\mathbb{R}^4$ with non-zero components

$$\begin{aligned} &g(\partial_{x_1},\partial_{x_3}) = g(\partial_{x_2},\partial_{x_4}) = 1, && g(\partial_{x_3},\partial_{x_4}) = g_{34},\\ &g(\partial_{x_3},\partial_{x_3}) = g_{33}, && g(\partial_{x_4},\partial_{x_4}) = g_{44}. \end{aligned}$$

The geometry of Walker manifolds with $g_{34} = 0$ has been studied in [11]. We impose a different condition by setting $g_{33} = g_{44} = 0$ so the non-zero components of the metric are given by:

$$g(\partial_{x_1},\partial_{x_3}) = g(\partial_{x_2},\partial_{x_4}) = 1 \qquad \text{and} \qquad g(\partial_{x_3},\partial_{x_4}) = g_{34}. \tag{6.1}$$

By Theorem 11, $\mathcal{M}$ is conformally Osserman if and only if $\mathcal{M}$ is either self-dual or anti-self-dual. One has [5] that:

Theorem 12. *Let $\mathcal{M} = (\mathbb{R}^4, g)$ where g is given by equation* (6.1)*.*

1. *$\mathcal{M}$ is self-dual if and only if $g_{34} = x_1 p(x_3,x_4) + x_2 q(x_3,x_4) + s(x_3,x_4)$.*
2. *$\mathcal{M}$ is anti-self-dual if and only if $g_{34} = x_1 p(x_3,x_4) + x_2 q(x_3,x_4) + s(x_3,x_4) + \xi(x_1,x_4) + \eta(x_2,x_3)$ with $p_{/3} = q_{/4}$ and $g_{34}p_{/3} - x_1 p_{/34} - x_2 p_{/33} - s_{/34} = 0$.*

We refer to [4] for the following results:

Theorem 13. *Let $\mathcal{M} = (\mathbb{R}^4, g)$ where g is given by equation* (6.1)*.*

1. *The following conditions are equivalent:*
   - (a) *$\mathcal{M}$ is Osserman.*
   - (b) *$\mathcal{M}$ is Einstein.*
   - (c) *$\rho = 0$.*
   - (d) *$g_{34} = x_1 p(x_3,x_4) + x_2 q(x_3,x_4) + s(x_3,x_4)$ where $p = -2a_4(a_0 + a_3x_3 + a_4x_4)^{-1}$, and $q = -2a_3(a_0 + a_3x_3 + a_4x_4)^{-1}$ for $(a_0,a_3,a_4) \neq (0,0,0)$.*
   - (e) *$\mathcal{J}(x)^2 = 0$ for all x.*
   - (f) *$\mathcal{M}$ is Jacobi–Tsankov.*
2. *The following conditions are equivalent:*
   - (a) *$\mathcal{M}$ is Jacobi–Videv.*
   - (b) *$\mathcal{M}$ is skew-Tsankov.*
   - (c) *$g_{34} = x_1 p(x_3,x_4) + x_2 q(x_3,x_4) + s(x_3,x_4)$ where $p_{/3} = q_{/4}$.*

A feature of these examples is that the warping functions are affine functions of x_1 and x_2. We return to the general setting of Walker signature $(2,2)$ geometry. Let ∇ be a torsion free connection on a 2-dimensional manifold N. Let (x_3,x_4) be local coordinates on N. We expand

$$\nabla_{\partial_{x_i}}\partial_{x_j} = \sum_k \Gamma_{ij}{}^k \partial_{x_k} \qquad \text{for} \quad i,j,k = 3,4$$

to define the Christoffel symbols of ∇. Let $\omega = x_1 dx_3 + x_2 dx_4 \in T^*N$; the pair (x_1,x_2) gives the dual fiber coordinates. Let $\xi = \xi_{ij}(x_3,x_4) \in C^\infty(S^2(T^*N))$ be an auxiliary symmetric bilinear form.

Definition 11. The *deformed Riemannian extension* is the Walker metric on T^*N defined by setting [15]

$$\begin{aligned}
&g(\partial_{x_1},\partial_{x_3}) = g(\partial_{x_2},\partial_{x_4}) = 1,\\
&g(\partial_{x_3},\partial_{x_3}) = -2x_1{\Gamma_{33}}^3(x_3,x_4) - 2x_2{\Gamma_{33}}^4(x_3,x_4) + \xi_{33}(x_3,x_4),\\
&g(\partial_{x_3},\partial_{x_4}) = -2x_1{\Gamma_{34}}^3(x_3,x_4) - 2x_2{\Gamma_{34}}^4(x_3,x_4) + \xi_{34}(x_3,x_4),\\
&g(\partial_{x_4},\partial_{x_4}) = -2x_1{\Gamma_{44}}^3(x_3,x_4) - 2x_2{\Gamma_{44}}^4(x_3,x_4) + \xi_{44}(x_4,x_4).
\end{aligned}$$

Definition 12. Let $\rho_N(x,y) := \operatorname{Tr}(z \to \mathcal{R}_\nabla(z,x)y)$ be the affine Ricci tensor. We may decompose this 2-tensor into symmetric and anti-symmetric parts by defining:

$$\begin{aligned}
&\rho_N^s(x,y) := \tfrac{1}{2}(\rho_N(x,y) + \rho_N(y,x)) \qquad \text{and},\\
&\rho_N^a(x,y) := \tfrac{1}{2}(\rho_N(x,y) - \rho_N(y,x)).
\end{aligned}$$

The Jacobi operator is defined by setting $\mathcal{J}_\nabla(x) : y \to \mathcal{R}_\nabla(y,x)x$. We say that $\mathcal{N} := (N,\nabla)$ is *affine Osserman* if $\mathcal{J}_\nabla(x)$ is nilpotent or, equivalently, if $\operatorname{Spec}\{\mathcal{J}_\nabla(x)\} = \{0\}$ for all x.

We refer to [4] for the proof of the following result:

Theorem 14.

1. *$\mathcal{M}$ is skew-Tsankov if and only if $\rho_N^a = 0$.*
2. *$\mathcal{M}$ is Osserman if and only if $\mathcal{N}$ is affine Osserman if and only if $\rho_N^s = 0$.*
3. *$\rho_N^a = 0$ or $\rho_N^s = 0$ if and only if $\mathcal{M}$ is Jacobi–Videv.*
4. *$\rho_N = 0$ if and only if $\mathcal{M}$ is Jacobi–Tsankov.*

Remark 3. This shows the notions Jacobi–Videv, and Jacobi–Tsankov, and skew-Tsankov are inequivalent notions.

If $\mathcal{M}$ is conformally Osserman, let m_λ be the minimal polynomial of $\mathcal{J}_W$ and let Spec_W be the spectrum of $\mathcal{J}_W$. One has [5]:

Theorem 15. *Let $\mathcal{M} = (\mathbb{R}^4, g)$ be the Walker manifold with non-zero metric components:*

$$g(\partial_{x_1},\partial_{x_3}) = g(\partial_{x_2},\partial_{x_4}) = 1, \qquad \text{and} \qquad g(\partial_{x_3},\partial_{x_4}) = g_{34}.$$

The following choices of g_{34} make $\mathcal{M}$ conformal Osserman with:

1. *The Jordan normal form does not change from point to point:*
   (a) *If $g_{34} = x_1^2 - x_2^2$, then $m_\lambda = \lambda(\lambda^2 - \frac{1}{4})$ and $\operatorname{Spec}_W = \{0,0,\pm\frac{1}{2}\}$.*
   (b) *If $g_{34} = x_1^2 + x_2^2$, then $m_\lambda = \lambda(\lambda^2 + \frac{1}{4})$ and $\operatorname{Spec}_W = \{0,0,\pm\frac{\sqrt{-1}}{2}\}$.*
   (c) *If $g_{34} = x_1x_4 + x_3x_4$, then $m_\lambda = \lambda^2$ and $\operatorname{Spec}_W = \{0\}$.*
   (d) *If $g_{34} = x_1^2$, then $m_\lambda = \lambda^3$ and $\operatorname{Spec}_W = \{0\}$.*
2. *$\operatorname{Spec}_W = \{0\}$ but the Jordan normal form changes from point to point.*
   (a) *If $g_{34} = x_2x_4^2 + x_3^2x_4$, then $m_\lambda = \lambda^3$ if $x_4 \neq 0$, $m_\lambda = \lambda^2$ if $x_4 = 0$ and $x_3 \neq 0$, and $m_\lambda = \lambda$ if $x_3 = x_4 = 0$.*
   (b) *If $g_{34} = x_2x_4^2 + x_3x_4$, then $m_\lambda = \lambda^3$ if $x_4 \neq 0$, and $m_\lambda = \lambda^2$ if $x_4 = 0$.*
   (c) *If $g_{34} = x_1x_3^2$, then $m_\lambda = \lambda^3$ if $x_3 \neq 0$, and $m_\lambda = \lambda$ if $x_3 = 0$.*
   (d) *If $g_{34} = x_1x_3 + x_2x_4$, then $m_\lambda = \lambda^2$ if $x_1x_3 + x_2x_4 \neq 0$, and $m_\lambda = \lambda$ if $x_1x_3 + x_2x_4 = 0$.*

3. *The eigenvalues can change from point to point:*

   (a) *If* $g_{34} = x_1^4 + x_1^2 - x_2^4 - x_2^2$, *then* $\mathrm{Spec}_W = \big\{0, 0, \pm\frac{1}{2}\sqrt{(6x_1^2+1)(6x_2^2+1)}\big\}$.

   (b) *If* $g_{34} = x_1^4 + x_1^2 + x_2^4 + x_2^2$, *then* $\mathrm{Spec}_W = \big\{0, 0, \pm\frac{1}{2}\sqrt{-(6x_1^2+1)(6x_2^2+1)}\big\}$.

   (c) *If* $g_{34} = x_1^3 - x_2^3$, *then* $\mathrm{Spec}_W = \{0, 0, \pm\frac{3}{2}\sqrt{x_1 x_2}\}$.

We conclude our discussion with the following result [6]:

Theorem 16. *Of the manifolds given above in Theorem* 15, *only the manifold with* $g_{34} = x_1^2$ *is curvature homogeneous and only the manifold with* $g_{34} = x_1x_4 + x_3x_4$ *is geodesically complete.*

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