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Paths and Tableaux Descriptions of Jacobi–Trudi Determinant Associated with Quantum Affine Algebra of Type C_n

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Abstract. We study the Jacobi–Trudi-type determinant which is conjectured to be the q-character of a certain, in many cases irreducible, finite-dimensional representation of the quantum affine algebra of type C_n. Like the D_n case studied by the authors recently, applying the Gessel–Viennot path method with an additional involution and a deformation of paths, we obtain an expression by a positive sum over a set of tuples of paths, which is naturally translated into the one over a set of tableaux on a skew diagram.

Key words: quantum group; q-character; lattice path; Young tableau

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1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let $\hat{\mathfrak{g}}$ be the corresponding untwisted affine Lie algebra. Let $U_q(\hat{\mathfrak{g}})$ be the quantum affine algebra, namely, the quantized universal enveloping algebra of $\hat{\mathfrak{g}}$ [4, 11]. In order to investigate the finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$ [5, 3], the *q-character* of $U_q(\hat{\mathfrak{g}})$ was introduced and studied in [7, 6]. So far, the explicit description of $\chi_q(V)$ is available only for a limited type of representations (e.g., the fundamental representations [6, 2]), and the description for general V is an open problem. There are also results by [21, 22, 9, 10], some of which will be mentioned below.

This is a continuation of our previous work [23, 24]. Firstly, let us briefly summarize some background. In [23], for $\mathfrak{g}$ of types A_n, B_n, C_n, and D_n, we conjecture that the q-characters of a certain family of, in many cases irreducible, finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$ are given by the determinant form $\chi_{\lambda/\mu,a}$, where λ/μ is a skew diagram and a is a complex parameter. We call $\chi_{\lambda/\mu,a}$ the Jacobi–Trudi determinant. For A_n and B_n, this is a reinterpretation of the conjecture for the spectra of the transfer matrices of the vertex models associated with the corresponding representations [1, 17]. When a diagram λ/μ is a rectangle of depth d with $d \le n$ for A_n, $d \le n-1$ for B_n and C_n, and $d \le n-2$ for D_n, the corresponding representation V is a so-called *Kirillov–Reshetikhin (KR) module* [13, 14]. In this case, the conjecture is known to be true from the facts: (i) The determinants $\chi_{\lambda/\mu,a}$ corresponding to the KR modules consist of a part of the unique solution of the set of polynomial relations called the *T-system* ([15, equation (2.17)] for A_n, [17, Theorem 5.1] for B_n, [16, Theorem 3.1] for C_n, [28, Theorem 3.1] for D_n). (ii) The q-characters of the KR modules also consists of the solution of the T-system ([22, Theorem 1.1] for A_n and D_n, [9, Theorem 3.4] for any X_n).

The characters of an irreducible representations of sl_n, or the Schur polynomials, are described by the semistandard tableaux on Young diagrams. As is well known, several objects of representation theory and combinatorics are related via the semistandard tableaux (e.g. [26]). Since $\chi_{\lambda/\mu,a}$ is expected to be the characters of representations of $U_q(\hat{\mathfrak{g}})$, it is natural to ask if there are such tableaux for $\chi_{\lambda/\mu,a}$. This is the question studied in [23, 24]. To do it, the paths method by [8] (see also [26]), which derives the semistandard tableaux purely combinatorially from the determinant expression of the Schur polynomials, seems suited to our situation.

Now let us briefly summarize the results in [23, 24]. For A_n and B_n [23], the application of the standard paths method by [8] immediately reproduces the known tableaux descriptions of $\chi_{\lambda/\mu,a}$ by [1, 17]. Here, by *tableaux description* we mean an expression of $\chi_{\lambda/\mu,a}$ by a positive sum over a certain set of tableaux on λ/μ. In contrast, for C_n [23], the situation is not as simple as the former cases, and we obtain a tableaux description of $\chi_{\lambda/\mu,a}$ only for some special cases (i.e., a skew diagram λ/μ of at most two columns or of at most three rows). In [24], we consider the same problem for D_n, where the situation is similar to C_n. By extending the idea of [23] to full generality, a tableaux description of $\chi_{\lambda/\mu,a}$ for a general skew diagram λ/μ is obtained.

In this paper, we come back to C_n again, and, using the method in [24], we now obtain a tableaux description of $\chi_{\lambda/\mu,a}$ for a general skew diagram λ/μ. Even though the formulation is quite parallel to D_n, the C_n case has slight technical complications in two aspects. First, the definitions of the duals of lower and upper paths in (3.3) and (3.4) are more delicate. Secondly, the translation of the extra rule into tableau language is less straightforward. In retrospect, these are the reasons why we could derive the tableaux description in a general case first for D_n. We expect that the method is also applicable to the twisted quantum affine algebras of classical type [27], and hopefully, even to the quantum affine algebras of exceptional type as well. We also expect that our tableaux description (and the corresponding one in the path picture) is useful to study the q-characters and the crystal bases [12, 25] of those representations. One of the advantage of our tableaux (or paths) is that it naturally resolves the multiplicity of the weight polynomials of q-characters so that the proposed algorithm by [6] to create the q-characters could be more naturally realized on the space of tableaux (or paths) for those representations having multiplicities.

The organization of the paper is as follows. In Section 2 we review the result of [23]. Namely, we define the Jacobi–Trudi determinant $\chi_{\lambda/\mu,a}$ for C_n; then, following the standard method by [8], we introduce lattice paths and express $\chi_{\lambda/\mu,a}$ as a sum over a certain set of tuples of paths. In Section 3, we apply the method of [24], with adequate modifications of the basic notions for C_n, and obtain an expression of $\chi_{\lambda/\mu,a}$ by a positive sum over a set of tuples of paths (Theorem 3.10). In Section 4, we translate the last expression into the tableaux description whose tableaux are determined by the horizontal, vertical, and extra rules (Theorems 4.3 and 4.10).

This paper is written as the companion to [24]. As we mentioned, once the basic objects are adequately set, the derivation of paths and tableaux descriptions is quite parallel to D_n. In particular, the proofs of the core propositions in Section 3 are the same almost word for word. For such propositions, instead of repeating the proofs, we only refer the corresponding propositions in [24] so that readers could focus on what are specific to the C_n case.

2 Jacobi–Trudi determinant and paths

In this section, we introduce the Jacobi–Trudi determinant $\chi_{\lambda/\mu,a}$ of type C_n and its associated paths. Most of the information is taken from [23]. See [23] for more information.

2.1 Jacobi–Trudi determinant of type C_n

A *partition* is a sequence of weakly decreasing non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ with finitely many non-zero terms $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$. The *length* $l(\lambda)$ of λ is the number of the non-zero integers. The *conjugate* is denoted by $\lambda' = (\lambda'_1, \lambda'_2, \dots)$. As usual, we identify a partition λ with a *Young diagram* $\lambda = \{(i,j) \in \mathbb{N}^2 \mid 1 \leq j \leq \lambda_i\}$, and also identify a pair of partitions such that $\lambda_i \geq \mu_i$ for any i, with a *skew diagram* $\lambda/\mu = \{(i,j) \in \mathbb{N}^2 \mid \mu_i + 1 \leq j \leq \lambda_i\}$.

Let

$$I = \{1, 2, \ldots, n, \overline{n}, \ldots, \overline{2}, \overline{1}\}. \tag{2.1}$$

Let $\mathcal{Z}$ be the commutative ring over $\mathbb{Z}$ generated by $z_{i,a}$'s, $i \in I$, $a \in \mathbb{C}$, with the following generating relations:

$$z_{i,a} z_{\overline{i},a-2n+2i-4} = z_{i-1,a} z_{\overline{i-1},a-2n+2i-4} \quad (i = 1, \ldots, n), \qquad z_{0,a} = z_{\overline{0},a} = 1. \tag{2.2}$$

Let $\mathbb{Z}[[X]]$ be the formal power series ring over $\mathbb{Z}$ with variable X. Let $\mathcal{A}$ be the *non-commutative* ring generated by $\mathcal{Z}$ and $\mathbb{Z}[[X]]$ with relations

$$X z_{i,a} = z_{i,a-2} X, \qquad i \in I, a \in \mathbb{C}.$$

For any $a \in \mathbb{C}$, we define $E_a(z, X)$, $H_a(z, X) \in \mathcal{A}$ as

$$E_a(z, X) := \left\{ \overrightarrow{\prod_{1 \le k \le n}} (1 + z_{k,a} X) \right\} (1 - z_{n,a} X z_{\overline{n},a} X) \left\{ \overleftarrow{\prod_{1 \le k \le n}} (1 + z_{\overline{k},a} X) \right\}, \tag{2.3}$$

$$H_a(z, X) := \left\{ \overrightarrow{\prod_{1 \le k \le n}} (1 - z_{\overline{k},a} X)^{-1} \right\} (1 - z_{n,a} X z_{\overline{n},a} X)^{-1} \left\{ \overleftarrow{\prod_{1 \le k \le n}} (1 - z_{k,a} X)^{-1} \right\}, \tag{2.4}$$

where $\overrightarrow{\prod\limits_{1 \le k \le n}} A_k = A_1 \cdots A_n$ and $\overleftarrow{\prod\limits_{1 \le k \le n}} A_k = A_n \cdots A_1$. Then we have

$$H_a(z, X) E_a(z, -X) = E_a(z, -X) H_a(z, X) = 1. \tag{2.5}$$

For any $i \in \mathbb{Z}$ and $a \in \mathbb{C}$, we define $e_{i,a}$, $h_{i,a} \in \mathcal{Z}$ as

$$E_a(z, X) = \sum_{i=0}^{\infty} e_{i,a} X^i, \qquad H_a(z, X) = \sum_{i=0}^{\infty} h_{i,a} X^i,$$

with $e_{i,a} = h_{i,a} = 0$ for $i < 0$. Note that $e_{i,a} = 0$ if $i > 2n+2$ or $i = n+1$. It is worth mentioning that the equality

$$e_{2n+2-i,a} = -e_{i,a-2n+2i-2} \tag{2.6}$$

holds for any i (cf. [18, equation (2.14)]), though we do not use it in the rest of the paper. This follows from the following pseudo-antisymmetric property of $E_a(z, X)$,

$$E_a(z, X^{-1})|_{z_{i,a} \mapsto z_{\overline{i},a}, z_{\overline{i},a} \mapsto z_{i,a}} X^{2n+2} = -E_{a+2n}(z, X), \tag{2.7}$$

which is proved by successive applications of the relations (2.2).

Due to the relation (2.5), we have [20, equation (2.9)]

$$\det(h_{\lambda_i - \mu_j - i + j, a + 2(\lambda_i - i)})_{1 \le i,j \le l} = \det(e_{\lambda'_i - \mu'_j - i + j, a - 2(\mu'_j - j + 1)})_{1 \le i,j \le l'} \tag{2.8}$$

for any pair of partitions (λ, μ), where l and l' are any non-negative integers such that $l \ge l(\lambda), l(\mu)$ and $l' \ge l(\lambda'), l(\mu')$. For any skew diagram λ/μ, let $\chi_{\lambda/\mu,a}$ denote the determinant on the left- or right-hand side of (2.8). We call it the *Jacobi–Trudi determinant* associated with the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ of type C_n.

Let $d(\lambda/\mu) := \max\{\lambda'_i - \mu'_i\}$ be the *depth* of λ/μ. We conjecture that, if $d(\lambda/\mu) \le n$, the determinant $\chi_{\lambda/\mu,a}$ is the *q-character* for a certain finite-dimensional representation V of the quantum affine algebra of type C_n. If λ/μ is connected, we further expect that $\chi_{\lambda/\mu,a}$ is the q-character for the *irreducible* representation V whose highest weight monomial is

$$\prod_{j=1}^{l(\lambda')} Y_{\lambda'_j-\mu'_j,q^{a+2j-\lambda'_j-\mu'_j-1}}, \tag{2.9}$$

where $Y_{i,a}$ is the variables for the q-characters in [7].

The rest of the paper is devoted to providing the combinatorial description of the determinant $\chi_{\lambda/\mu,a}$.

2.2 Gessel–Viennot paths

Following [23], let us apply the Gessel–Viennot path method (see [26, Chapter 4.5] for good exposition) to the determinant $\chi_{\lambda/\mu,a}$ in (2.8) and the generating function $H_a(z,X)$ in (2.4). The function $H_a(z,X)$ is the generating function of the symmetric polynomials. So, one can employ the h-labeling of [26]. However, the middle factor produces the powers of the pair of variables $z_{n,a}z_{\overline{n},a-2}$, which is the source of the whole complexity [23].

Consider the lattice $\mathbb{Z}^2$. An *E-step* (resp. *N-step*) s is a step from a point u to a point v in the lattice of unit length in east (resp. north) direction. For any point $(x,y)\in\mathbb{R}^2$, we define the *height* by y and the *horizontal position* by x. Due to (2.4), we define a *path* p (of type C_n) as a sequence of consecutive steps $(s_1,s_2,\dots)$ which satisfies the following conditions:

(i) It starts from a point u at height $-n-1$ and ends at a point v at height $n+1$.

(ii) Each step s_i is an E- or N-step.

(iii) The number of E-steps at height 0 is even.

(iv) The E-steps do not occur at height $\pm(n+1)$,

We also write such a path p starting from u and ending at v as $u \xrightarrow{p} v$. See Fig. 1 for an example.

Remark 2.1. We slightly change the definition of paths from that of [23]. In [23], a path starts at height $-n$ and end at height n, satisfying (ii) and (iii). Here, we extend it by adding N-steps at its both ends. The reason of this change is to make the formulation in Section 3 as parallel as possible to the D_n case. With our new definition, a *boundary II-unit* in Definition 3.3 makes sense, and so does a *II-region* in Definition 3.4.

Let $\mathcal{P}$ be the set of all the paths. For any $p\in\mathcal{P}$, set

$$\begin{aligned} E(p) &:= \{s\in p \mid s \text{ is an E-step}\},\\ E_0(p) &:= \{s\in p \mid s \text{ is an E-step at height } 0\}\subset E(p). \end{aligned} \tag{2.10}$$

If $E_0(p)=\{s_j,s_{j+1},\dots,s_{j+2k-1}\}$, then let

$$E_0^1(p) := \{s_{j+1},s_{j+3},\dots,s_{j+2k-1}\}\subset E_0(p).$$

Fix $a\in\mathbb{C}$. The *h-labeling* (of type C_n) associated with a for a path $p\in\mathcal{P}$ is the pair of maps $L_a=(L_a^1,L_a^2)$ on $E(p)$ defined as follows: Suppose that a step $s\in E(p)$ starts at a point $w=(x,y)$. Then, we set

$$L_a^1(s)=\begin{cases} n+1+y, & \text{if } y<0,\\ \overline{n}, & \text{if } y=0 \text{ and } s\notin E_0^1(p),\\ n, & \text{if } y=0 \text{ and } s\in E_0^1(p),\\ \overline{n+1-y}, & \text{if } y>0, \end{cases} \tag{2.11}$$

$$L_a^2(s)=a+2x.$$

See Fig. 1.

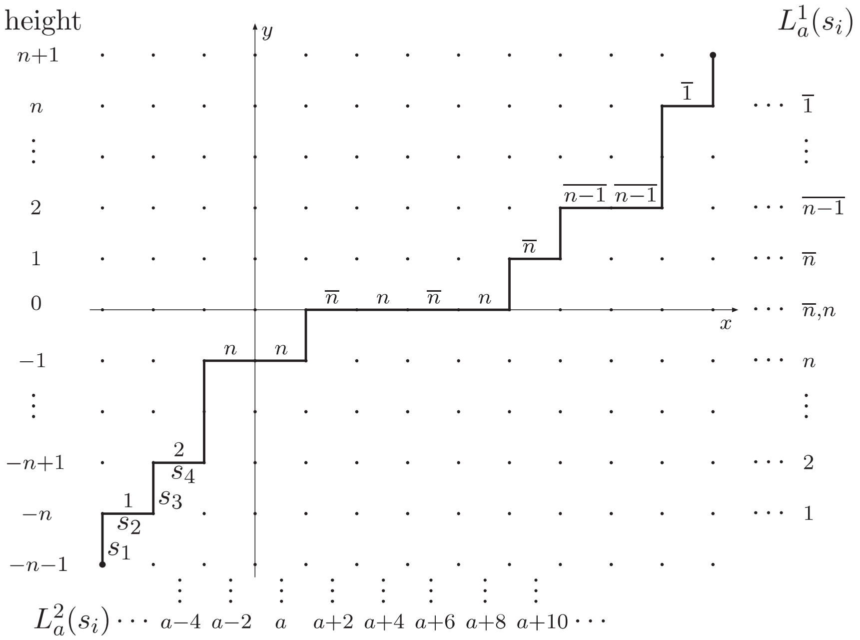


Figure 1. An example of a path of type C_n and its h-labeling.

Now we define the *weight* of $p \in \mathcal{P}$ as

$$z_a^p := \prod_{s \in E(p)} z_{L_a^1(s), L_a^2(s)} \in \mathcal{Z}.$$

By the definition of $E_a(z, X)$ in (2.3), for any $k \in \mathbb{Z}$, we have

$$h_{r,a+2k+2r-2}(z) = \sum_p z_a^p, \tag{2.12}$$

where the sum runs over all $p \in \mathcal{P}$ such that $(k, -n-1) \xrightarrow{p} (k+r, n+1)$.

For any l-tuples of distinct points $u = (u_1, \dots, u_l)$ of height $-n-1$ and $v = (v_1, \dots, v_l)$ of height $n+1$, and any permutation $\sigma \in \mathfrak{S}_l$, let

$$\mathfrak{P}(\sigma; u, v) = \{p = (p_1, \dots, p_l) \mid p_i \in \mathcal{P},\ u_i \xrightarrow{p_i} v_{\sigma(i)} \text{ for } i = 1, \dots, l\},$$

and set

$$\mathfrak{P}(u, v) = \bigsqcup_{\sigma \in \mathfrak{S}_l} \mathfrak{P}(\sigma; u, v).$$

We define the *weight* z_a^p and the *sign* $(-1)^p$ of $p \in \mathfrak{P}(u, v)$ as

$$z_a^p := \prod_{i=1}^{l} z_a^{p_i}, \qquad (-1)^p := \operatorname{sgn} \sigma \quad \text{if } p \in \mathfrak{P}(\sigma; u, v). \tag{2.13}$$

For any skew diagram λ/μ, set $l = l(\lambda)$, and

$$\begin{aligned} u_\mu &= (u_1, \dots, u_l), \qquad & u_i &= (\mu_i + 1 - i, -n-1), \\ v_\lambda &= (v_1, \dots, v_l), \qquad & v_i &= (\lambda_i + 1 - i, n+1). \end{aligned}$$

Then, due to (2.12), the determinant (2.8) can be written as

$$\chi_{\lambda/\mu, a} = \sum_{p \in \mathfrak{P}(u_\mu, v_\lambda)} (-1)^p z_a^p. \tag{2.14}$$

Definition 2.2. We say that an intersecting pair (p_i, p_j) of paths is *specially intersecting* if it satisfies the following conditions:

1. The intersection of p_i and p_j occurs only at height 0.
2. $p_i(0) - p_j(0)$ is odd, where $p_i(0)$ is the horizontal position of the leftmost point on p_i at height 0.

Otherwise, we say that an intersecting pair (p_i, p_j) is *ordinarily intersecting*.

We can define a weight-preserving, sign-reversing involution on the set of all the tuples $p \in \mathfrak{P}(u_\mu, v_\lambda)$ which have some ordinarily intersecting pairs (p_i, p_j). Therefore, we have

Proposition 2.3 ([23, Proposition 5.3]). *For any skew diagram λ/μ,*

$$\chi_{\lambda/\mu,a} = \sum_{p \in P_1(\lambda/\mu)} (-1)^p z_a^p, \tag{2.15}$$

where $P_1(\lambda/\mu)$ is the set of all $p \in \mathfrak{P}(u_\mu, v_\lambda)$ which do not have any ordinarily intersecting pair of paths.

3 Paths description

In this section, we turn the expression (2.15) into an expression (3.11) by a positive sum over a set of tuples of paths ('paths description'), which will be naturally translated to the tableaux description Section 4.

3.1 Outline of formulation

As we mentioned in the introduction, the formulation here is quite parallel to the D_n case [24]. Before starting, let us briefly explain the idea behind the formulation.

The expression (2.15) still involves negative terms. Since $\chi_{\lambda/\mu,a}$ is supposed to be the q-character of a certain representation if $d(\lambda/\mu) \leq n$, these negative terms should be canceled by the positive terms of the same weight if $d(\lambda/\mu) \leq n$. (It turned out that the condition to have a positive sum can be slightly extended to (3.5).) This motivates us to construct a new type of weight-preserving, sign-reversing involution besides the standard one (the *first involution* in [24]) used for Proposition 2.3. A key is the relations (2.2) among our variables, which allow to deform a tuple of paths preserving its weight. This deformation is called the *expansion* and *folding* in [24]. Using them, one can construct the desired involution ι_2 (the *second involution* in [24]), and obtain an expression (3.10) by a positive sum.

However, it turns out that (3.10) is not our final answer yet, because a tuple of paths for (3.10) may be transposed (i.e., there is a pair of paths whose orders of the initial points and final points are transposed); in that case it cannot be translated it into a tableau on λ/μ à la Gessel–Viennot [8, 26]. To resolve this problem, we construct a weight-preserving map ϕ (the *folding map* in [24]), which transforms a tuple of paths of (3.10) into non-transposed one. Then, we finally obtain the desired expression (3.11) which is the counterpart of the tableaux description in Section 4.

Below we are going to define basic objects (lower/upper path, I/II-unit, even/odd I/II-region, etc.), all of which are introduced to construct the above mentioned maps ι_2 and ϕ. The differences between C_n and D_n, which originate from the ones of the generating functions and the relations of the variables, are absorbed into the definitions of these objects. With these objects, the construction of ι_2 and ϕ are done in a unified way for both C_n and D_n. For the reader who is only interested in the final result (3.11), only the definition of an *odd II-region* is crucial. See Fig. 3 to have idea of a II-region.

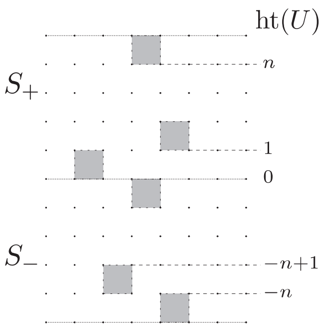


Figure 2. Examples of units.

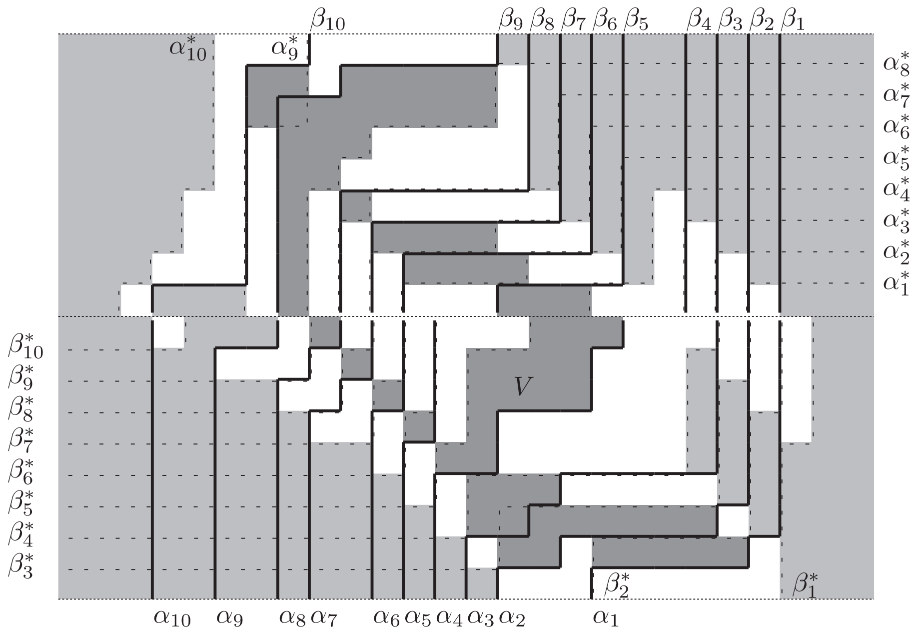


Figure 3. An example of $(\alpha;\beta)$ and its dual $(\alpha^*;\beta^*)$. The II-units of $(\alpha;\beta)$ are shaded, and, especially, the darkly shaded region represents a II-region V.

3.2 I- and II-regions

Here, we introduce some notions which are necessary to give the expression by a positive sum. See Figs. 2 and 3 for examples.

Let

$$S_+ := \mathbb{R} \times [0, n+1], \qquad S_- := \mathbb{R} \times [-n-1, 0]. \tag{3.1}$$

Definition 3.1. A *lower path* α (of type C_n) is a sequence of consecutive steps in S_- which starts at a point of height $-n-1$ and ends at a point of height 0 such that each step is an E- or N-step, and an E-step does not occur at height 0 and $-n-1$. Similarly, an *upper path* β (of type C_n) is a sequence of consecutive steps in S_+ which starts at a point of height 0 and ends at a point of height $n+1$ such that each step is an E- or N-step, and an E-step does not occur at height 0 and $n+1$.

For a lower path α and an upper path β, let $\alpha(r)$ and $\beta(r)$ be the horizontal positions of the leftmost points of α and the rightmost points β at height r, respectively.

Let

$$(\alpha;\beta) := (\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_l) \tag{3.2}$$

be a pair of an l-tuple α of lower paths and an l-tuple β of upper paths. We say that $(\alpha;\beta)$ is *nonintersecting* if (α_i,α_j) is not intersecting, and so is (β_i,β_j) for any $i\neq j$.

From now on, let λ/μ be a skew diagram, and we set $l=l(\lambda)$. Let

$$\mathcal{H}(\lambda/\mu) := \left\{ (\alpha;\beta)=(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_l) \,\middle|\, \begin{array}{l} (\alpha;\beta)\text{ is nonintersecting,} \\ \alpha_i(-n-1)=\mu_i+1-i, \\ \beta_i(n+1)=\lambda_i+1-i \end{array} \right\}.$$

For any $(\alpha;\beta)\in\mathcal{H}(\lambda/\mu)$, we define $\alpha_i^*(r)$, $\beta_i^*(-r)$, $(i=1,\dots,l;\ r=0,\dots,n)$ as

$$\alpha_i^*(r) = \begin{cases} \alpha_{i-r}(-r)-1-r & i-r\geq 1, \\ +\infty & i-r\leq 0, \end{cases} \tag{3.3}$$

$$\beta_i^*(-r) = \begin{cases} \beta_{i+r}(r)+1+r & i+r\leq l, \\ -\infty & i+r\geq l+1. \end{cases} \tag{3.4}$$

Since $(\alpha;\beta)$ is nonintersecting, we have $\alpha_i^*(r)\leq\alpha_i^*(r+1)$ and $\beta_i^*(-r)\leq\beta_i^*(-r+1)$. Therefore, one can naturally identifies the data $\{\alpha_i^*(r)\}_{r=0}^n$ with an upper path α_i^* (an 'infinite' upper path if it contains $+\infty$). The lower path β_i^* is obtained similarly. It is easy to check that $(\alpha^*;\beta^*)$ is nonintersecting. We call $(\alpha^*;\beta^*)$ the *dual* of (α,β). As we mention in the introduction, the definition of $(\alpha^*;\beta^*)$ is more delicate than the one for D_n, where $\alpha_i^*(r):=\alpha_i(-r)-1$ and $\beta_i^*(-r):=\beta_i(r)+1$. It is so defined that the involution ι_2 in Proposition 3.7 is weight-preserving under the relation (2.2).

For any skew diagram λ/μ, we call the following condition the *positivity condition*:

$$d(\lambda/\mu)\leq n+1, \tag{3.5}$$

or equivalently, $\lambda_{i+n+1}\leq\mu_i$ for any i. We call this the 'positivity condition', because (3.5) guarantees that $\chi_{\lambda/\mu,a}$ is a positive (nonnegative, strictly to say) sum (see Theorem 3.8). By the definition, we have

Lemma 3.2 (cf. [24, Lemma 4.2]). *Let λ/μ be a skew diagram satisfying the positivity condition* (3.5)*, and let $(\alpha;\beta)\in\mathcal{H}(\lambda/\mu)$. Then,*

$$\beta_{i+1}(n+1)\leq\alpha_i^*(n+1), \qquad \beta_{i+1}^*(-n-1)\leq\alpha_i(-n-1). \tag{3.6}$$

A *unit* $U\subset S_\pm$ is a unit square with its vertices on the lattice. The *height* $\mathrm{ht}(U)$ of $U\subset S_+$ (resp. $U\subset S_-$) is given by the height of the lower edge (resp. the upper edge) of U.

Definition 3.3. Let $(\alpha;\beta)\in\mathcal{H}(\lambda/\mu)$. For any unit $U\subset S_\pm$, let $\pm r=\mathrm{ht}(U)$ and let a and $a'=a+1$ be the horizontal positions of the left and the right edges of U. Set $\beta_{l+1}(r)=\beta_{l+1}^*(-r)=-\infty$ and $\alpha_0(-r)=\alpha_0^*(r)=+\infty$. Then

1. U is called a I-*unit* of $(\alpha;\beta)$ if there exists some i $(0\leq i\leq l)$ such that

$$\begin{aligned} &\alpha_i^*(r)\leq a<a'\leq\beta_{i+1}(r), && \text{if } U\subset S_+, \\ &\alpha_i(-r)\leq a<a'\leq\beta_{i+1}^*(-r), && \text{if } U\subset S_-. \end{aligned} \tag{3.7}$$

2. U is called a II-*unit* of $(\alpha;\beta)$ if there exists some i $(0\leq i\leq l)$ such that

$$\begin{aligned} &\beta_{i+1}(r)\leq a<a'\leq\alpha_i^*(r), && \text{if } U\subset S_+, \\ &\beta_{i+1}^*(-r)\leq a<a'\leq\alpha_i(-r), && \text{if } U\subset S_-. \end{aligned} \tag{3.8}$$

Furthermore, a II-unit U of $(\alpha;\beta)$ is called a *boundary* II-unit if one of the following holds:

(i) $U \subset S_+$, and (3.8) holds for $i \le r$, $i = l$, or $r = n$.

(ii) $U \subset S_-$, and (3.8) holds for $i = 0$, $i \ge l - r$, or $r = n$.

For a unit $U \subset S_+$ with vertices (x, y), $(x+1, y)$, $(x, y+1)$, $(x+1, y+1)$, the *dual* $U^* \subset S_-$ is a unit with vertices $(x+1+y, -y)$, $(x+2+y, -y)$, $(x+1+y, -y-1)$, $(x+2+y, -y-1)$. Conversely, we define $(U^*)^* = U$. Let U and U' be units. If the upper-left or the lower-right vertex of U is also a vertex of U', then we say that U and U' are *adjacent* and write $U \diamond U'$.

So far, the definitions are specific for type C_n. From now till the end of Section 3, all the statements are literally the same as type D_n [24].

Fix $(\alpha;\beta) \in \mathcal{H}(\lambda/\mu)$. Let $\mathcal{U}_{\mathrm{I}}$ be the set of all I-units of $(\alpha;\beta)$, and let $\tilde{\mathcal{U}}_{\mathrm{I}} := \bigcup_{U \in \mathcal{U}_{\mathrm{I}}} U$, where the union is taken for U as a subset of $S_+ \sqcup S_-$. Let $\sim$ be the equivalence relation in $\mathcal{U}_{\mathrm{I}}$ generated by the relation $\diamond$, and $[U]$ be its equivalence class of $U \in \mathcal{U}_{\mathrm{I}}$. We call $\bigcup_{U' \in [U]} U'$ a *connected component* of $\tilde{\mathcal{U}}_{\mathrm{I}}$. For II-units, $\mathcal{U}_{\mathrm{II}}$, $\tilde{\mathcal{U}}_{\mathrm{II}}$ and its connected component are defined similarly.

Now we introduce the main concept in the section.

Definition 3.4. Let λ/μ be a skew diagram satisfying the positivity condition (3.5), and let $(\alpha;\beta) \in \mathcal{H}(\lambda/\mu)$.

1. A connected component V of $\tilde{\mathcal{U}}_{\mathrm{I}}$ is called a I-*region* of $(\alpha;\beta)$ if it contains at least one I-unit of height 0.
2. A connected component V of $\tilde{\mathcal{U}}_{\mathrm{II}}$ is called a II-*region* of $(\alpha;\beta)$ if it satisfies the following conditions:

   (i) V contains at least one II-unit of height 0.

   (ii) V does not contain any boundary II-unit.

Proposition 3.5 (cf. [24, Proposition 4.6]). *If V is a* I- *or* II-*region, then $V^* = V$, where for a union of units $V = \bigcup U_i$, we define $V^* = \bigcup U_i^*$.*

3.3 Second involution

Following the D_n case [24], let us derive an expression of $\chi_{\lambda/\mu,a}$ by a positive sum from (2.15).

Since the proofs for D_n are applicable almost word for word to all the statements below, we omit them. For interested readers, we provide some technical information in Appendix A.

From now on, we assume that λ/μ satisfies the positivity condition (3.5). For each $p \in P_1(\lambda/\mu)$, one can uniquely associate $(\alpha;\beta) \in \mathcal{H}(\lambda/\mu)$ by removing all the E-steps from p. We denote by $\pi(p)$ the element $(\alpha;\beta)$ corresponding to the path p. A I- or II-region of $(\alpha;\beta) = \pi(p)$ is also called a I- *or* II-*region of* p. If $h := \alpha_i(0) - \beta_{i+1}(0)$ is a non-positive number (resp. a positive number), then we call a pair (α_i, β_{i+1}) an *overlap* (resp. a *hole*). Furthermore, if h is an even number (resp. an odd number), then we say that (α_i, β_{i+1}) is *even* (resp. *odd*). For any I-region V (resp. II-region V) of $p \in P_1(\lambda/\mu)$ with $(\alpha;\beta) = \pi(p)$, we set

$$n(V) := \#\left\{ i \;\middle|\; \begin{array}{l} (\alpha_i, \beta_{i+1}) \text{ is an even overlap (resp. an even hole)} \\ \text{which intersects with } V \text{ at height } 0 \end{array} \right\}. \tag{3.9}$$

For example, $n(V) = 2$ for V in Fig. 3.

Definition 3.6. We say that a I- or II-region V is *even* (resp. *odd*) if $n(V)$ is even (resp. odd).

Let $P_{\mathrm{odd}}(\lambda/\mu)$ be the set of all $p \in P_1(\lambda/\mu)$ which have at least one odd I- or II-region of p.

Proposition 3.7 (cf. [24, Proposition 4.12]). *There exists a weight-preserving, sign-reversing involution* $\iota_2 : P_{\mathrm{odd}}(\lambda/\mu) \to P_{\mathrm{odd}}(\lambda/\mu)$.

It follows from Proposition 3.7 that the contributions of $P_{\mathrm{odd}}(\lambda/\mu)$ to the sum (2.15) cancel each other. Let $P_2(\lambda/\mu) := P_1(\lambda/\mu)\backslash P_{\mathrm{odd}}(\lambda/\mu)$, i.e., the set of all $p \in P_1(\lambda/\mu)$ which satisfy the following conditions:

(i) p does not have any ordinarily intersecting pair (p_i, p_j).

(ii) p does not have any odd I- or II-region.

For any $p \in P_2(\lambda/\mu)$, $(-1)^p = 1$ holds. Thus, the sum (2.15) reduces to a positive sum, and we have an expression by a positive sum,

Theorem 3.8 (cf. [24, Theorem 4.13]). *For any skew diagram* λ/μ *satisfying the positivity condition* (3.5), *we have*

$$\chi_{\lambda/\mu,a} = \sum_{p \in P_2(\lambda/\mu)} z_a^p. \tag{3.10}$$

3.4 Paths description

Since a tuple of paths $p \in \mathfrak{P}(\sigma; u_\mu, v_\lambda)$ is naturally translated into a tableau of shape λ/μ if and only if $\sigma = \mathrm{id}$, we introduce another set of paths as follows. Let $P(\lambda/\mu)$ be the set of all $p \in \mathfrak{P}(\mathrm{id}; u_\mu, v_\lambda)$ such that

(i) p does not have any ordinarily intersecting *adjacent* pair (p_i, p_{i+1}).

(ii) p does not have any odd II-region.

Here, an odd II-region of $p \in P(\lambda/\mu)$ is defined in the same way as that of $p \in P_1(\lambda/\mu)$. The following fact is not so trivial.

Proposition 3.9 (cf. [24, Proposition 5.1]). *There exists a weight-preserving bijection*

$$\phi : P_2(\lambda/\mu) \to P(\lambda/\mu).$$

The map ϕ is called the *folding map* in [24]. From Theorem 3.8 and Proposition 3.9, we immediately have

Theorem 3.10 (Paths description, cf. [24, Theorem 5.2]). *For any skew diagram* λ/μ *satisfying the positivity condition* (3.5), *we have*

$$\chi_{\lambda/\mu,a} = \sum_{p \in P(\lambda/\mu)} z_a^p. \tag{3.11}$$

4 Tableaux description

4.1 Tableaux description

Define a total order in I in (2.1) by

$$1 \prec 2 \prec \cdots \prec n \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

A *tableau* T of shape λ/μ is the skew diagram λ/μ with each box filled by one entry of I. For a tableau T and $a \in \mathbb{C}$, we define the *weight* of T as

$$z_a^T = \prod_{(i,j)\in\lambda/\mu} z_{T(i,j),a+2(j-i)},$$

where $T(i,j)$ is the entry of T at (i,j).

Definition 4.1. A tableau T (of shape λ/μ) is called an *HV-tableau* if it satisfies the following conditions:

(**H**) horizontal rule. Each $(i,j) \in \lambda/\mu$ satisfies both of the following conditions:

(i) $T(i,j) \preceq T(i,j+1)$ or $(T(i,j),T(i,j+1)) = (\overline{n},n)$.
(ii) $(T(i,j-1),T(i,j),T(i,j+1)) \neq (\overline{n},\overline{n},n),(\overline{n},n,n)$.

(**V**) vertical rule. Each $(i,j) \in \lambda/\mu$ satisfies one of the following conditions:

(i) $T(i,j) \prec T(i+1,j)$.
(ii) $T(i,j) = T(i+1,j) = n$, $(i+1,j-1) \in \lambda/\mu$, $T(i+1,j-1) = \overline{n}$.
(iii) $T(i,j) = T(i+1,j) = \overline{n}$, $(i,j+1) \in \lambda/\mu$, $T(i,j+1) = n$.

The rule (**H**) appears in [19] for a skew diagram of one row. We write the set of all HV-tableaux of shape λ/μ by $\mathrm{Tab}_{\mathrm{HV}}(\lambda/\mu)$.

Let $P_{\mathrm{HV}}(\lambda/\mu)$ be the set of all $p \in \mathfrak{P}(\mathrm{id}; u_\mu, v_\lambda)$ which do not have any ordinarily intersecting adjacent pair (p_i, p_{i+1}). With any $p \in P_{\mathrm{HV}}(\lambda/\mu)$, we associate a tableau T of shape λ/μ as follows: For any $j = 1, \dots, l$, let $E(p_j) = \{s_{i_1}, s_{i_2}, \dots, s_{i_m}\}$ $(i_1 < i_2 < \cdots < i_m)$ be the set defined as in (2.10), and set

$$T(j, \mu_j + k) = L_a^1(s_{i_k}), \qquad k = 1, \dots, m,$$

where L_a^1 is the first component of the h-labeling (2.11). It is easy to see that T satisfies the horizontal rule (**H**) because of the definition of the h-labeling, and satisfies the vertical rule (**V**) because p does not have any ordinarily intersecting adjacent pair. Therefore, if we set $\mathcal{T}_{\mathrm{h}} : p \mapsto T$, we have

Proposition 4.2 (cf. [24, Proposition 5.5]). *The map*

$$\mathcal{T}_{\mathrm{h}} : P_{\mathrm{HV}}(\lambda/\mu) \to \mathrm{Tab}_{\mathrm{HV}}(\lambda/\mu)$$

is a weight-preserving bijection.

Note that $P(\lambda/\mu) \subset P_{\mathrm{HV}}(\lambda/\mu)$. Let $\mathrm{Tab}(\lambda/\mu) := \mathcal{T}_{\mathrm{h}}(P(\lambda/\mu))$. In other words, $\mathrm{Tab}(\lambda/\mu)$ is the set of all the tableaux T which satisfy (**H**), (**V**), and the following *extra rule*:

(**E**) The corresponding $p = \mathcal{T}_{\mathrm{h}}^{-1}(T)$ does not have any odd II-region.

By Theorem 3.10 and Proposition 4.2, we obtain a tableaux description of $\chi_{\lambda/\mu,a}$, which is the main result of the paper.

Theorem 4.3 (Tableaux description, cf. [24, Theorem 5.6]). *For any skew diagram λ/μ satisfying the positivity condition* (3.5)*, we have*

$$\chi_{\lambda/\mu,a} = \sum_{T \in \mathrm{Tab}(\lambda/\mu)} z_a^T.$$

Example 4.4. Let $n = 4$. Consider the following two HV-tableaux which differ in only one letter:

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 4 & 4 \\ \hline \overline{4} & \overline{4} & \overline{3} \\ \hline \overline{3} & \overline{2} & \overline{2} \\ \hline \end{array}\,, \qquad T' = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & 4 & 4 \\ \hline \overline{4} & \overline{4} & \overline{3} \\ \hline \overline{3} & \overline{2} & \overline{2} \\ \hline \end{array}\,. \tag{4.1}$$

By Fig. 4, we see that $T \notin \mathrm{Tab}(\lambda/\mu)$ because $\mathcal{T}_{\mathrm{h}}^{-1}(T)$ has an odd II-region, while $T' \in \mathrm{Tab}(\lambda/\mu)$ because $\mathcal{T}_{\mathrm{h}}^{-1}(T')$ does not so.

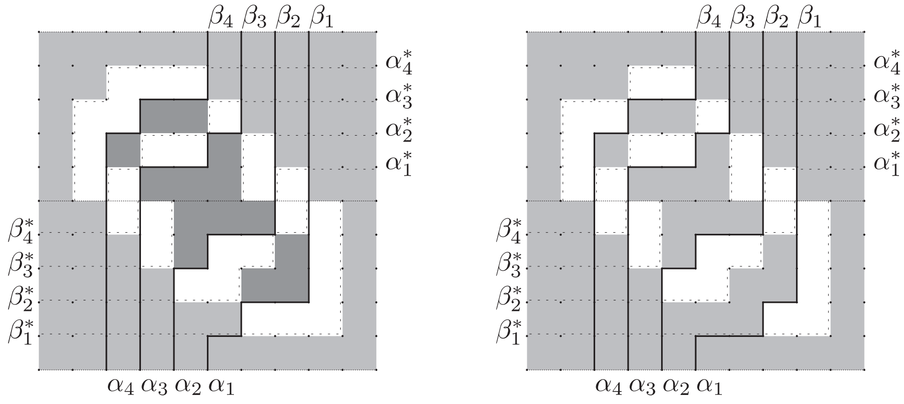


Figure 4. The tuples of paths corresponding to T and T' in (4.1). The II-units of $(\alpha;\beta)$ are shaded, and, especially, the darkly shaded region represents a II-region.

4.2 Transformation into e-picture

The paths we have used so far are so-called 'h-paths' in the Gessel–Viennot method. To translate the extra rule (**E**) into tableau language in a more explicit way, it is convenient to transform the definition of II-regions by h-paths into the one by 'e-paths'. This procedure is not necessary for D_n, where e-paths are employed from beginning [24].

Definition 4.5. A *lower e-path* γ (of type C_n) is a sequence of consecutive steps in S_- which starts at a point of height $-n-1$ and ends at a point of height 0 such that each step is a W(est)- or N-step, and a W-step occurs at most once at each height and does not occur at height 0, $-n-1$. Similarly, an *upper e-path* δ (of type C_n) is a sequence of consecutive steps in S_+ which starts at a point of height 0 and ends at a point of height $n+1$ such that each step is a W- or N-step, and a W-step occurs at most once at each height and does not occur at height 0, $n+1$.

For a lower e-path γ and an upper e-path δ, let $\gamma(r)$ (resp. $\delta(r)$) be the horizontal position of the rightmost point of γ (resp. the leftmost point of δ) at height r.

Fix a given $p \in P_{\mathrm{HV}}(\lambda/\mu)$, and let $T = \mathcal{T}_{\mathrm{h}}(p)$, and $l' = \lambda_1$. With p we associate a pair of an l'-tuple γ of lower e-paths and an l'-tuple δ of upper e-paths,

$$(\gamma;\delta) := (\gamma_1,\dots,\gamma_{l'};\delta_1,\dots,\delta_{l'}), \tag{4.2}$$

where, for each i, we set $\gamma_i(-n-1) = i-\mu'_i$, $\delta_i(n+1) = i-\lambda'_i$, and we identify the E-steps of p at nonzero heights corresponding to the letters in the ith column of T with the W-steps of γ_i and δ_i (ignoring the direction of W- and E-steps). We write $(\gamma;\delta) = \pi'(p)$. The *dual* γ_i^* of γ_i is the upper e-path with $\gamma_i^*(r) = \gamma_i(-r)-1-r$ $(r=0,\dots,n)$. Similarly, The *dual* δ_i^* of δ_i is the lower e-path with $\delta_i^*(-r) = \delta_i(r)+1+r$ $(r=0,\dots,n)$. See Fig. 5 for examples of notions in this subsection.

Definition 4.6. Let $(\gamma;\delta)$ be as above. For any unit $U \subset S_\pm$, let $\pm r = \mathrm{ht}(U)$ and let a and $a' = a+1$ be the horizontal positions of the left and the right edges of U. Then, U is called a II$'$-*unit* of $(\gamma;\delta)$ if there exists some i $(1 \le i \le l')$ such that

$$\begin{aligned} &\gamma_i^*(r) \le a < a' \le \delta_i(r), && \text{if } U \subset S_+,\\ &\gamma_i(-r) \le a < a' \le \delta_i^*(-r), && \text{if } U \subset S_-. \end{aligned} \tag{4.3}$$

Furthermore, a II$'$-unit U of $(\gamma;\delta)$ is called a *boundary* II$'$-unit if (4.3) holds for $r=n$.

A connected component of the union $\tilde{\mathcal{U}}_{\mathrm{II}'}$ of all II$'$-units of $(\gamma;\delta)$ is defined similarly as before.

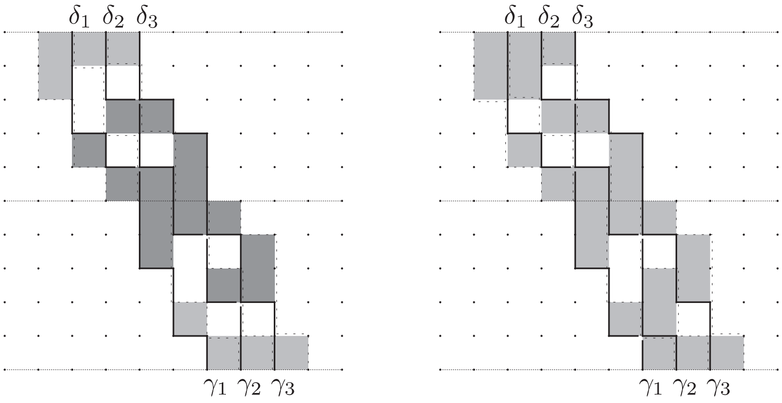


Figure 5. $(\gamma;\delta)$ corresponding to T and T' in (4.1). The II$'$-units of $(\gamma;\delta)$ are shaded, and, especially, the darkly shaded region represents a II$'$-region.

Definition 4.7. A connected component V of $\tilde{\mathcal{U}}_{\mathrm{II}'}$ is called a II$'$-*region* of $(\gamma;\delta)$ if it satisfies the following conditions:

(i) V contains at least one II$'$-unit of height 0.

(ii) V does not contain any boundary II$'$-unit.

In Figs. 4 and 5, we can see that the II-region of $(\alpha;\beta)$ and the II$'$-region of $(\gamma;\delta)$ coincide. In fact,

Proposition 4.8. *Let $(\alpha;\beta)=\pi(p)$ and $(\gamma;\delta)=\pi'(p)$. Then, V is a* II*-region of $(\alpha;\beta)$ if and only if V is a* II$'$*-region of $(\gamma;\delta)$.*

Proof. It is not difficult to show the following facts:

(1) A II$'$-unit of $(\gamma;\delta)$ is a II- unit of $(\alpha;\beta)$.

(2) Let U be a II-unit of $(\alpha;\beta)$ satisfying one of the following conditions:

(i) $U\subset S_+$, and (3.8) holds for some i with $r<i<l$.

(ii) $U\subset S_-$, and (3.8) holds for some i with $0<i<l-r$.

Then, U is a II$'$-unit of $(\gamma;\beta)$.

(3) Let U be a II-unit of $(\alpha;\beta)$ satisfying one of the following conditions:

(i) $U\subset S_+$, and (3.8) holds for some i with $i\le r$ or $i=l$.

(ii) $U\subset S_-$, and (3.8) holds for some i with $i=0$ or $i\ge l-r$.

Let U' be a unit which belongs to a II$'$-region of $(\gamma;\delta)$. Then, U and U' are not adjacent to each other.

Now, the if part of the proposition follows from (1), (2), and (3), while the only if part follows from (1) and (2). ■

4.3 Extra rule in terms of tableau

With Proposition 4.8, it is now straightforward to translate the extra rule (**E**) into tableau language as type D_n [24]. We only give the result.

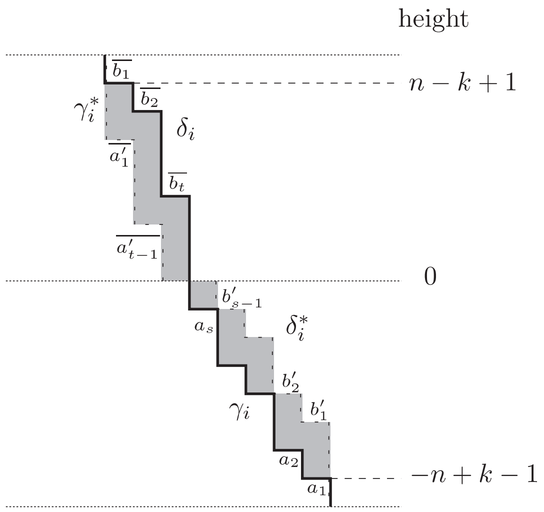


Figure 6. An example of a pair of lower and upper e-paths (γ_i, δ_i) whose part corresponds to a LU-configuration of type 1 as in (4.5).

Fix an HV-tableau T of shape λ/μ. For any $a_1, \dots, a_m \in I$, let $C(a_1, \dots, a_m)$ be a configuration in T as follows:

$$\begin{array}{|c|}\hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_m \\ \hline \end{array} \tag{4.4}$$

We call it an *L-configuration* if it satisfies

(i) $1 \preceq a_1 \prec \cdots \prec a_m \preceq n$.

(ii) If $a_m = n$, $\mathrm{Pos}(a_m) = (i, j)$, and $(i, j-1) \in \lambda/\mu$, then $T(i, j-1) \neq \overline{n}$.

Here and below, $\mathrm{Pos}(a_m)$, for example, means the position of a_m in T. We call it a *U-configuration* if it satisfies

(i) $\overline{n} \preceq a_1 \prec \cdots \prec a_m \preceq \overline{1}$.

(ii) If $a_1 = \overline{n}$, $\mathrm{Pos}(a_1) = (i, j)$, and $(i, j+1) \in \lambda/\mu$, then $T(i, j+1) \neq n$.

An L-configuration is identified with a part of lower paths for T under the map $\mathcal{T}_{\mathrm{h}}$, while a U-configuration is so with a part of upper paths.

Let (L, U) be a pair of an L-configuration $L = C(a_1, \dots, a_s)$ and a U-configuration $U = C(\overline{b_t}, \dots, \overline{b_1})$ in the same column. We call it an *LU-configuration* of T if it satisfies one of the following two conditions:

Condition 1. *LU-configuration of type 1.* (L, U) has the form

$$\left.\begin{array}{|c|}\hline a_1 \\ \hline \vdots \\ \hline a_s \\ \hline \overline{b_t} \\ \hline \vdots \\ \hline \overline{b_1} \\ \hline \end{array}\right\} n{-}k{+}2 \tag{4.5}$$

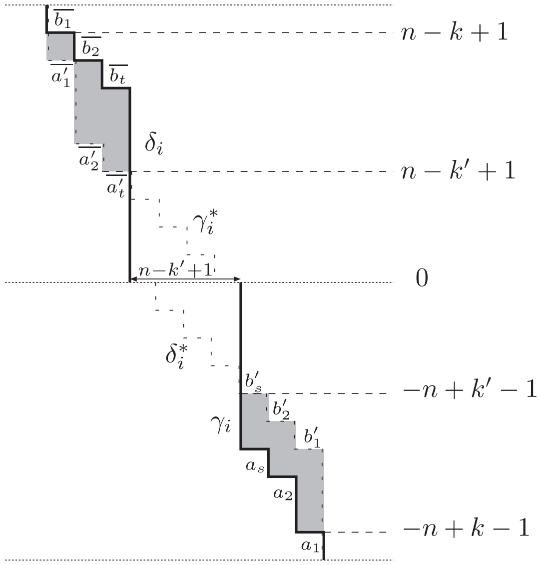


Figure 7. An example of a pair of lower and upper e-paths (γ_i, δ_i) whose part corresponds to a LU-configuration of type 2 as in (4.9).

for some k with $1 \le k \le n$, $n-k+2 = s+t$, and

$$a_1 = k, \qquad \overline{b_1} = \overline{k}, \tag{4.6}$$

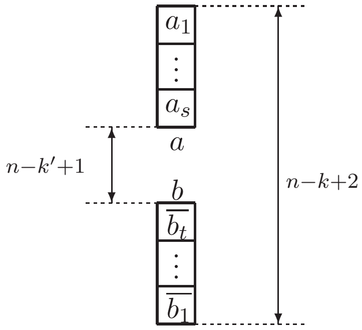
$$a_{i+1} \preceq b'_i, \quad (1 \le i \le s-1), \qquad \overline{b_{i+1}} \succeq \overline{a'_i}, \quad (1 \le i \le t-1), \tag{4.7}$$

where $a'_1 \prec \cdots \prec a'_t$ and $b'_1 \prec \cdots \prec b'_s$ are defined as

$$\begin{aligned} &\{a_1, \dots, a_s\} \sqcup \{a'_1, \dots, a'_{t-1}\} = \{k, k+1, \dots, n\}, \quad a'_t = \overline{n}, \\ &\{b_1, \dots, b_t\} \sqcup \{b'_1, \dots, b'_{s-1}\} = \{k, k+1, \dots, n\}, \quad b'_s = \overline{n}. \end{aligned} \tag{4.8}$$

See Fig. 6 for the corresponding part in the paths.

Condition 2. *LU-configuration of type 2.* (L, U) has the form



(4.9)

for some k and k' with $1 \le k < k' \le n$, $n-k+2 = n-k'+1+s+t$, and

$$a_1 = k, \qquad \overline{b_1} = \overline{k}, \qquad a'_t = k', \qquad \overline{b'_s} = \overline{k'}, \tag{4.10}$$

$$a \succ k' \qquad \text{or} \qquad \begin{bmatrix} \text{if } a = n, \mathrm{Pos}(a) = (i,j), \text{ then} \\ (i, j-1) \in \lambda/\mu, T(i, j-1) = \overline{n} \end{bmatrix}, \tag{4.11}$$

$$b \prec \overline{k'} \qquad \text{or} \qquad \begin{bmatrix} \text{if } b = \overline{n}, \mathrm{Pos}(b) = (i,j), \text{ then} \\ (i, j+1) \in \lambda/\mu, T(i, j+1) = n \end{bmatrix}, \tag{4.12}$$

$$a_{i+1} \preceq b'_i, \quad (1 \le i \le s-1), \qquad \overline{b_{i+1}} \succeq \overline{a'_i}, \quad (1 \le i \le t-1), \tag{4.13}$$

where $a'_1 \prec \cdots \prec a'_t$ and $b'_1 \prec \cdots \prec b'_s$ are defined as

$$\begin{aligned} &\{a_1, \dots, a_s\} \sqcup \{a'_1, \dots, a'_t\} = \{k, k+1, \dots, k'\}, \\ &\{b_1, \dots, b_t\} \sqcup \{b'_1, \dots, b'_s\} = \{k, k+1, \dots, k'\}. \end{aligned} \tag{4.14}$$

See Fig. 7 for the corresponding parts in the paths.

We say that an L-configuration $L = C(a_1, \dots, a_m)$ in the jth column of T is *boundary* if $\mathrm{Pos}(a_1) = (\mu'_j + 1, j)$, i.e., if a_1 is at the top of the jth column, and m is the largest number such that $L \cap L' = \varnothing$ for any LU-configuration (L', U'). Similarly, a U-configuration $U = C(a_1, \dots, a_m)$ in the jth column of T is *boundary* if $\mathrm{Pos}(a_m) = (\lambda'_j, j)$, i.e., if a_m is at the bottom of the jth column, and m is the largest number such that $U \cap U' = \varnothing$ for any LU-configuration (L', U').

Let $(L, U) = (C(a_1, \dots, a_s), C(\overline{b_t}, \dots, \overline{b_1}))$ be an LU-configuration, and set $a'_1 \prec \cdots \prec a'_t$ and $b'_1 \prec \cdots \prec b'_s$ as in (4.8) (resp. as in (4.14)) if (L, U) is of type 1 (resp. of type 2). We say that an L-configuration L' is *right-adjacent* to (L, U) if L' is in the right-next column to L; furthermore, there exists some pair of an entry e of L' and an entry a_i of L such that e is right-next to a_i and $e \prec b'_i$. Similarly, we say that a U-configuration U' is *left-adjacent* to (L, U) if U' is in the left-next column to U; furthermore, there exists some pair of an entry e of U' and an entry $\overline{b_i}$ of U such that e is left-next to $\overline{b_i}$ and $e \succ \overline{a'_i}$, where $\overline{\overline{n}} = n$. Then, we say that an LU-configuration (L', U') is *adjacent* to (L, U) if one of the following is satisfied, and write it by $(L, U) \diamond (L', U')$:

(i) L' is right-adjacent to (L, U).

(ii) L is right-adjacent to (L', U').

(iii) U' is left-adjacent to (L, U).

(iv) U is left-adjacent to (L', U').

For any tableau T, let $\mathcal{LU}(T)$ be the set of all LU-configurations of T. Then, the adjacent relation $\diamond$ of the LU-configurations generates an equivalence relation $\sim$ in $\mathcal{LU}(T)$.

Definition 4.9. For any $(L, U) \in \mathcal{LU}(T)$, let $[(L, U)] \subset \mathcal{LU}(T)$ be the equivalence class of (L, U) with respect to $\sim$, and let $R = R(L, U) := \bigcup_{(L', U') \in [(L, U)]} (L', U')$ be the corresponding configuration in T. We call R a II-*region* of T, if the following is satisfied:

(i) No boundary L-configuration L is right-adjacent to L' for any LU-configuration (L', U') in R.

(ii) No boundary U-configuration U is left-adjacent to U' for any LU-configuration (L', U') in R.

Moreover, we say that R is *odd* if the number of the type 1 LU-configurations in R is odd.

Then, an odd II-region of $T = \mathcal{T}_{\mathrm{h}}(p)$ corresponds to an odd II-region of p, and therefore, Theorem 4.3 is rewritten as follows:

Theorem 4.10 (Tableaux description, cf. [24, Theorem 5.6]). *For any skew diagram λ/μ satisfying the positivity condition* (3.5)*, we have*

$$\chi_{\lambda/\mu, a} = \sum_{T \in \mathrm{Tab}(\lambda/\mu)} z_a^T,$$

where $\mathrm{Tab}(\lambda/\mu)$ *is the set of all the tableaux of shape λ/μ which satisfy the* (**H**)*,* (**V**)*, and the following extra rule* ($\mathbf{E}'$)*:*

($\mathbf{E}'$) *T does not have any odd* II*-region.*

Example 4.11. Consider the tableaux T and T' in Example 4.4. The configuration

$$\begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 2 & 2 \\ \hline 3 & 4 & 4 \\ \hline \overline{4} & \overline{4} & \overline{3} \\ \hline \overline{3} & \overline{2} & \overline{2} \\ \hline \end{array} \subset T \tag{4.15}$$

is an odd II-region of T, while the configuration (the boxed part in the following)

$$\begin{array}{|c|c|c|} \cline{3-3} \multicolumn{2}{c|}{} & 2 \\ \hline 3 & 4 & 4 \\ \hline \overline{4} & \overline{4} & \overline{3} \\ \hline \overline{3} & \overline{2} & \overline{2} \\ \hline \end{array} \subset T' \tag{4.16}$$

is not a II-region of T' because the boundary U-configuration $C(\overline{2})$ in the second column is left-adjacent to the LU-configuration $(C(2,4), C(\overline{3},\overline{2}))$ in the third column.

Example 4.12. Let λ/μ be a skew diagram of at most three rows. The following is an example of an odd II-region of $T \in \mathrm{Tab}_{\mathrm{HV}}(\lambda/\mu)$:

$$\begin{array}{c|c|c|c|c} \cline{3-4} \multicolumn{2}{c|}{} & n-1 & n-1 & a \\ \cline{2-4} & n & n & \overline{n} & \\ \cline{2-4} b & \overline{n} & \overline{n-1} & \overline{n-1} & \\ \cline{2-4} \end{array} , \tag{4.17}$$

where $a \succeq \overline{n}$ and $b \preceq n$ if they exist. The complete list of all the possible odd II-regions in T corresponds to the rules (**E-2R**) and (**E-3R**) of Theorem 5.7 in [23].

Example 4.13. Let λ/μ be a skew diagram of two columns satisfying the positivity condition (3.5). We note that if $T \in \mathrm{Tab}_{\mathrm{HV}}(\lambda/\mu)$ contains a type 1 LU-configuration, say, in the first column, then $\gamma_2(0) = \delta_2(0)$. Then, it is easy to prove that the extra rule ($\mathbf{E}'$) is equivalent to the following condition:

(**E-2C**) T does not have any odd type 1 LU-configuration as

$$\begin{array}{c|c|c} \cline{2-2} & a_1 & d_1 \\ \cline{2-2} & \vdots & \vdots \\ \cline{2-2} & a_s & d_s \\ \cline{2-2} c_t & \overline{b_t} & \\ \cline{2-2} \vdots & \vdots & \\ \cline{2-2} c_1 & \overline{b_1} & \\ \cline{2-2} \end{array} ,$$

where

(i) $(L, U) = (C(a_1, \ldots, a_s), C(\overline{b_t}, \ldots, \overline{b_1}))$ is a type 1 LU-configuration.

(ii) Let a'_i be the one in (4.8). For each $k = 1, \ldots, t$, if $\mathrm{Pos}(\overline{b_k}) = (i,j)$ and $(i, j-1) \in \lambda/\mu$, then $c_k := T(i, j-1) \preceq \overline{a'_k}$.

(iii) Let b'_i be the one in (4.8). For each $k = 1, \ldots, s$, if $\mathrm{Pos}(a_k) = (i,j)$ and $(i, j+1) \in \lambda/\mu$, then $d_k := T(i, j+1) \succeq b'_k$.

This proves Conjecture 5.9 in [23].

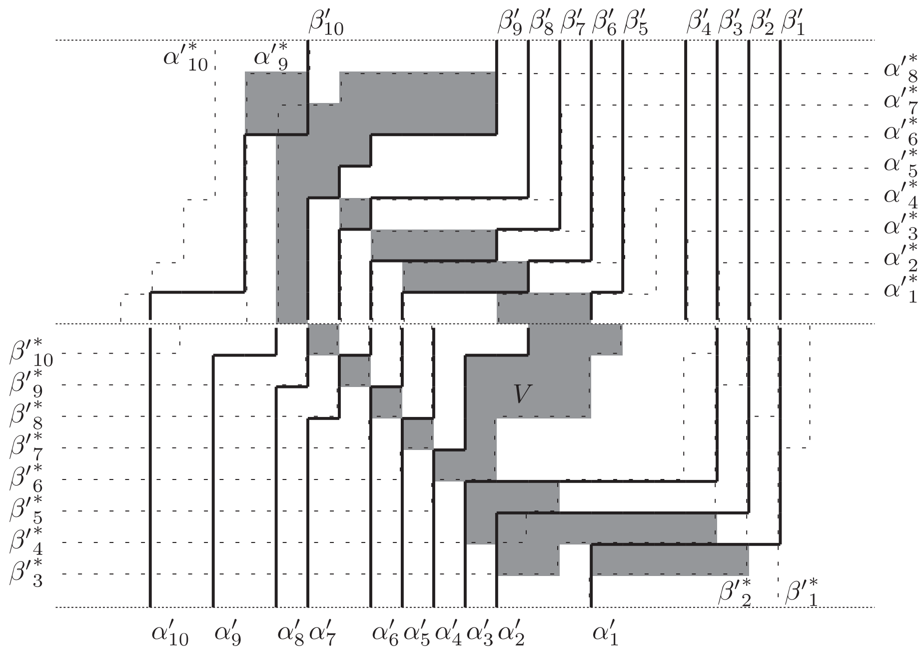


Figure 8. $(\alpha';\beta')$ is the folding of $(\alpha;\beta)$ in Figure 3 with respect to V. Conversely, $(\alpha;\beta)$ is the expansion of $(\alpha';\beta')$ with respect to V.

A Remarks on Section 3.3

We give some technical remarks on Section 3.3.

A.1 Expansion and folding

The involution ι_2 in Proposition 3.7 is defined by using the deformations of paths called *expansion* and *folding*. They are generalizations of the 'resolution of a (3.10) pair of paths' in [23] and its inverse. They are defined in exactly the same way as type D_n [24].

For any $(\alpha;\beta) \in \mathcal{H}(\lambda/\mu)$, let V be any I- or II-region of $(\alpha;\beta)$. Let α'_i be the lower path obtained from α_i by replacing the part $\alpha_i \cap V$ with $\beta^*_{i+1} \cap V$, and let β'_i be the upper path obtained from β_i by replacing the part $\beta_i \cap V$ with $\alpha^*_{i-1} \cap V$. Set $\varepsilon_V(\alpha;\beta) := (\alpha'_1,\dots,\alpha'_l;\beta'_1,\dots,\beta'_l)$. We have

Proposition A.1. *Let λ/μ be a skew diagram satisfying the positivity condition* (3.5)*. Then, for any $(\alpha;\beta) \in \mathcal{H}(\lambda/\mu)$, we have*

1. *For any* I- *or* II-*region V of $(\alpha;\beta)$, $\varepsilon_V(\alpha;\beta) \in \mathcal{H}(\lambda/\mu)$.*
2. *For any* I-*region V of $(\alpha;\beta)$, V is a* II-*region of $\varepsilon_V(\alpha;\beta)$.*
3. *For any* II-*region V of $(\alpha;\beta)$, V is a* I-*region of $\varepsilon_V(\alpha;\beta)$.*

We call the correspondence $(\alpha;\beta) \mapsto \varepsilon_V(\alpha;\beta)$ the *expansion* (resp. the *folding*) with respect to V, if V is a I-region (resp. a II-region) of $(\alpha;\beta)$. See Fig. 8 for an example.

A.2 I_k- and II_k-units

To construct the folding map ϕ in Proposition 3.9, which is a key to derive the tableaux description, we generalize the expansion and the folding to the k-*expansion* and the k-*folding* [24]. The original corresponds to $k = 1$. Like the $k = 1$ case, it starts from the definitions of I_k- and II_k-units, which are slightly modified for type C_n.

Definition A.2. Let $(\alpha;\beta) \in \mathcal{H}(\lambda/\mu)$. For any unit $U \subset S_\pm$, let $\pm r = \mathrm{ht}(U)$ and let a and $a' = a+1$ be the horizontal position of the left and right edges of U. Then, for any $k = 1,\dots,l-1$,

1. U is called a I_k-*unit* of $(\alpha;\beta)$ if there exists some i $(0 \le i \le l)$ such that

$$\begin{aligned} &\alpha_i^*(r) \le a < a' \le \beta_{i+k}(r), \quad \text{if } U \subset S_+, \\ &\alpha_i(-r) \le a < a' \le \beta_{i+k}^*(-r), \quad \text{if } U \subset S_-. \end{aligned} \tag{A.1}$$

2. U is called a II_k-*unit* of $(\alpha;\beta)$ if there exists some i $(0 \le i \le l)$ such that

$$\begin{aligned} &\beta_{i+k}(r) \le a < a' \le \alpha_i^*(r), \quad \text{if } U \subset S_+, \\ &\beta_{i+k}^*(-r) \le a < a' \le \alpha_i(-r), \quad \text{if } U \subset S_-. \end{aligned} \tag{A.2}$$

Here, we set $\beta_i(r) = \beta_i^*(-r) = -\infty$ and $\alpha_i(-r) = \alpha_i^*(r) = +\infty$ for any r and for any $i \ne 1,\dots,l$. Furthermore, a II-unit U of $(\alpha;\beta)$ is called a *boundary* II-unit if one of the following holds:

(i) $U \subset S_+$, and (A.2) holds for $i \le r$, $i \ge l+1-k$, or $r = n$.

(ii) $U \subset S_-$, and (A.2) holds for $i = 0$, $i \ge l+1-k-r$, or $r = n$.

Then, the rest of the definitions and the proof of Proposition 3.9 are exactly the same as type D_n.

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