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SU_2 Nonstandard Bases: Case of Mutually Unbiased Bases

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Abstract. This paper deals with bases in a finite-dimensional Hilbert space. Such a space can be realized as a subspace of the representation space of SU_2 corresponding to an irreducible representation of SU_2. The representation theory of SU_2 is reconsidered via the use of two truncated deformed oscillators. This leads to replacement of the familiar scheme $\{j^2, j_z\}$ by a scheme $\{j^2, v_{ra}\}$, where the two-parameter operator v_{ra} is defined in the universal enveloping algebra of the Lie algebra su_2. The eigenvectors of the commuting set of operators $\{j^2, v_{ra}\}$ are adapted to a tower of chains $SO_3 \supset C_{2j+1}$ $(2j \in \mathbb{N}^*)$, where C_{2j+1} is the cyclic group of order $2j + 1$. In the case where $2j + 1$ is prime, the corresponding eigenvectors generate a complete set of mutually unbiased bases. Some useful relations on generalized quadratic Gauss sums are exposed in three appendices.

Key words: symmetry adapted bases; truncated deformed oscillators; angular momentum; polar decomposition of su_2; finite quantum mechanics; cyclic systems; mutually unbiased bases; Gauss sums

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1 Introduction

Utilisation of linear combinations of simultaneous eigenstates $|jm\rangle$ of the square j^2 and the component j_z of a generalized angular momentum is widespread in physics. For instance, in molecular physics and condensed matter physics, we employ state vectors of the type

$$|ja\Gamma\gamma\rangle = \sum_{m=-j}^{j} |jm\rangle \langle jm|ja\Gamma\gamma\rangle , \tag{1.1}$$

where Γ stands for an irreducible representation of a subgroup G^* of the group SU_2, γ is a label for differentiating the various partners of Γ (in the case where $\dim\Gamma \geq 2$) and a denotes a multiplicity label necessary when Γ occurs several times in the irreducible representation (j) of SU_2. The group G^* is the spinor or double group of a (generally finite) subgroup G of SO_3, G being a point group of molecular or crystallographic interest. The vectors (1.1) are referred to as symmetry adapted vectors or symmetry adapted functions in the context of molecular orbital theory [1].

The state vector $|ja\Gamma\gamma\rangle$ is an eigenvector of j^2 and of the projection operator

$$P^{\Gamma}_{\gamma} = \frac{\dim\Gamma}{|G^*|} \sum_{R\in G^*} \overline{D^{\Gamma}(R)_{\gamma\gamma}} P_R, \tag{1.2}$$

where the bar indicates complex conjugation and $|G^*|$ is the order of G^*. Here, we use $D^{\Gamma}(R)_{\gamma\gamma'}$ to denote the $\gamma\gamma'$ matrix element of the matrix representation D^{Γ} associated with Γ. In addition, the operator P_R acts on $|ja\Gamma\gamma\rangle$ as

$$P_R\,|ja\Gamma\gamma\rangle = \sum_{\gamma'=1}^{\dim\Gamma} D^{\Gamma}(R)_{\gamma'\gamma}\,|ja\Gamma\gamma'\rangle\,. \tag{1.3}$$

Note that it is always possible to assume that the matrix representation D^{Γ} in (1.2) and (1.3) is unitary. Thus for fixed j, the vectors $|ja\Gamma\gamma\rangle$ are eigenvectors of the set $\{j^2, P_\gamma^\Gamma : \gamma = 1, 2, \ldots, \dim\Gamma\}$ of commuting operators.

In the situation where j is an integer, realizations of (1.1) on the sphere S^2 are known as G-harmonics and play an important role in chemical physics and quantum chemistry [2, 3]. More generally, state vectors (1.1) with j integer or half of an odd integer are of considerable interest in electronic spectroscopy of paramagnetic ions in finite symmetry [4, 5] and/or in rotational-vibrational spectroscopy of molecules [6, 7].

It is to be noted that in many cases the labels a and γ can be characterized (at least partially) by irreducible representations of a chain of groups containing G^* and having SU_2 as head group. In such cases, the state vectors (1.1) transform according to irreducible representations of the groups of the chain under consideration.

It is also possible to give more physical significance to the label a. For this purpose, let us consider an operator v defined in the enveloping algebra of SU_2 and invariant under the group G. The operators v and j^2 obviously commute. According to Wigner's theorem, the eigenvectors of j^2 and v are of type (1.1), where a stands for an eigenvalue of v. In that case, the label a may be replaced by an eigenvalue λ of v. (We assume that there is no state labeling problem, i.e., the triplets $\lambda\Gamma\gamma$ completely label the state vectors within the irreducible representation (j) of SU_2.) We are thus led to supersymmetry adapted vectors of the type

$$|j\lambda\Gamma\gamma\rangle = \sum_a |ja\Gamma\gamma\rangle\, U_{a\lambda},$$

where the unitary matrix U diagonalizes the matrix v set up on the set $\{|ja\Gamma\gamma\rangle : a \text{ ranging}\}$. Integrity bases to obtain v were given for different subgroups G of SO_3 [8, 9, 10].

As a résumé, there are several kinds of physically interesting bases for the irreducible representations of SU_2. The standard basis, associated with the commuting set $\{j^2, j_z\}$, corresponds to the canonical group-subgroup chain $\mathrm{SU}_2 \supset \mathrm{U}_1$. For this chain, we have $\Gamma := m$ and there is no need for the labels a and γ. Another kind of group-subgroup basis can be obtained by replacing U_1 by another subgroup G^* of the group SU_2. Among the various $\mathrm{SU}_2 \supset G^*$ symmetry adapted bases (SABs), we may distinguish (i) the weakly SABs $\{|ja\Gamma\gamma\rangle : a, \Gamma, \gamma \text{ ranging}\}$ for which the symmetry adapted vectors are eigenvectors of j^2 and of the projection operators of G^* and (ii) the strongly SABs $\{|ja\Gamma\gamma\rangle : \lambda, \Gamma, \gamma \text{ ranging}\}$ for which the supersymmetry adapted vectors are eigenvectors of j^2 and of an operator defined in the enveloping algebra of SU_2 and invariant under the group G. Both for strongly and weakly SABs, the restriction of SU_2 to G^* yields a decomposition of the irreducible representation (j) of SU_2 into a direct sum of irreducible representations Γ of G^*.

It is the object of the present work to study nonstandard bases of SU_2 with a tower of chains $\mathrm{SU}_2 \supset C^*_{2j+1}$ or $\mathrm{SO}_3 \supset C_{2j+1}$, with $2j \in \mathbb{N}^*$, where C_{2j+1} is the cyclic group of order $2j+1$. In other words, the chain of groups used here depends on the irreducible representation of SU_2 to be considered. We shall establish a connection, mentioned in [11], between the obtained bases and the so-called mutually unbiased bases (MUBs) used in quantum information.

The organisation of this paper is as follows. In Section 2, we construct the Lie algebra of SU_2 from two quon algebras A_1 and A_2 corresponding to the same deformation parameter q taken as a root of unity. Section 3 deals with an alternative to the $\{j^2, j_z\}$ scheme, viz., the $\{j^2, v_{ra}\}$ scheme, which corresponds, for fixed j, to a set of polar decompositions of SU_2 with

$a = 0, 1, \ldots, 2j$. Realizations of the operators v_{ra} in the enveloping algebra of SU$_2$ are given in Section 3. The link with MUBs is developed in Section 4. In a series of appendices, we give some useful relations satisfied by generalized quadratic Gauss sums.

2 A quon realization of the algebra su$_2$

2.1 Two quon algebras

Following the works in [12, 13, 14, 15, 16], we define two quon algebras $A_i = \{a_{i-}, a_{i+}, N_i\}$ with $i = 1$ and 2 by

$$a_{i-}a_{i+} - qa_{i+}a_{i-} = 1, \qquad [N_i, a_{i\pm}] = \pm a_{i\pm}, \qquad N_i^\dagger = N_i, \qquad (a_{i\pm})^k = 0,$$
$$\forall\, x_1 \in A_1,\ \forall\, x_2 \in A_2 :\ [x_1, x_2] = 0,$$

where

$$q = \exp\left(\frac{2\pi i}{k}\right), \quad k \in \mathbb{N} \setminus \{0, 1\}. \tag{2.1}$$

The generators $a_{i\pm}$ and N_i of A_i are linear operators. As in the classical case $q = 1$, we say that a_{i+} is a creation operator, a_{i-} an annihilation operator and N_i a number operator. Note that the case $k = 2$ corresponds to fermion operators and the case $k \to \infty$ to boson operators. In other words, each of the algebras A_i describes fermions for $q = -1$ and bosons for $q = 1$. The nilpotency conditions $(a_{i\pm})^k = 0$ can be understood as describing a generalized exclusion principle for particles of fractional spin $1/k$ (the Pauli exclusion principle corresponds to $k = 2$). Let us mention that algebras similar to A_1 and A_2 with $N_1 = N_2$ were introduced in [17, 18, 19] to define k-fermions which are, like anyons, objects interpolating between fermions (corresponding to $k = 2$) and bosons (corresponding to $k \to \infty$).

2.2 Representation of the quon algebras

We can find several Hilbertian representations of the algebras A_1 and A_2. Let $\mathcal{F}(i)$ be two truncated Fock–Hilbert spaces of dimension k corresponding to two truncated harmonic oscillators $(i = 1, 2)$. We endow each space $\mathcal{F}(i)$ with an orthonormalized basis $\{|n_i) : n_i = 0, 1, \ldots, k-1\}$. As a generalization of the representation given in [17, 18, 19], we have the following result.

Proposition 1. *The relations*

$$a_{1+}|n_1) = \left(\left[n_1 + s + \frac{1}{2}\right]_q\right)^a |n_1 + 1), \qquad a_{1+}|k-1) = 0, \tag{2.2}$$

$$a_{1-}|n_1) = \left(\left[n_1 + s - \frac{1}{2}\right]_q\right)^c |n_1 - 1), \qquad a_{1-}|0) = 0, \tag{2.3}$$

$$a_{2+}|n_2) = \left(\left[n_2 + s + \frac{1}{2}\right]_q\right)^b |n_2 + 1), \qquad a_{2+}|k-1) = 0, \tag{2.4}$$

$$a_{2-}|n_2) = \left(\left[n_2 + s - \frac{1}{2}\right]_q\right)^d |n_2 - 1), \qquad a_{2-}|0) = 0, \tag{2.5}$$

and

$$N_1|n_1) = n_1|n_1), \qquad N_2|n_2) = n_2|n_2)$$

define a family of representations of A_1 and A_2 depending on two independent parameters, say a and b, with $a + c = b + d = 1$.

In equations (2.2)–(2.5), we take $s = 1/2$. Furthermore, we use

$$[x]_q = \frac{1-q^x}{1-q}, \qquad x \in \mathbb{R}.$$

We shall also use the q-deformed factorial defined by

$$[n]_q! = [1]_q\,[2]_q \cdots [n]_q\,, \qquad n \in \mathbb{N}^*, \qquad [0]_q! = 1.$$

We continue with the representation of $A_1 \otimes A_2$ afforded by $a = 0$ and $b = 1$. The operators on $A_1 \otimes A_2$ act on the finite-dimensional Hilbert space $\mathcal{F}_k = \mathcal{F}(1) \otimes \mathcal{F}(2)$ of dimension k^2. The set $\{|n_1, n_2) = |n_1) \otimes |n_2) : n_1, n_2 = 0, 1, \ldots, k-1\}$ constitute an orthonormalized basis of $\mathcal{F}_k$. We denote $(\ |\ )$ the scalar product on $\mathcal{F}_k$ so that

$$(n_1', n_2'|n_1, n_2) = \delta_{n_1', n_1}\, \delta_{n_2', n_2}.$$

2.3 Two basic operators

Following [20, 21], we define the two linear operators

$$h = \sqrt{N_1\,(N_2+1)}, \qquad v_{ra} = s_1 s_2, \tag{2.6}$$

with

$$s_1 = q^{a(N_1+N_2)/2} a_{1+} + e^{i\phi_r/2} \frac{1}{[k-1]_q!} (a_{1-})^{k-1}, \tag{2.7}$$

$$s_2 = a_{2-} q^{-a(N_1-N_2)/2} + e^{i\phi_r/2} \frac{1}{[k-1]_q!} (a_{2+})^{k-1}, \tag{2.8}$$

where a and ϕ_r are two real parameters. The parameter ϕ_r is taken in the form

$$\phi_r = \pi(k-1)r, \qquad r \in \mathbb{R}.$$

It is immediate to show that the action of h and v_{ra} on $\mathcal{F}_k$ is given by

$$h|n_1, n_2) = \sqrt{n_1(n_2+1)}|n_1, n_2), \qquad n_i = 0, 1, 2, \ldots, k-1, \quad i = 1, 2 \tag{2.9}$$

and

$$\begin{aligned} v_{ra}|n_1, n_2) &= q^{an_2}|n_1+1, n_2-1), \qquad n_1 \neq k-1, \quad n_2 \neq 0, && (2.10)\\ v_{ra}|k-1, n_2) &= e^{i\phi_r/2} q^{-a(k-1-n_2)/2}|0, n_2-1), \qquad n_2 \neq 0, &&\\ v_{ra}|n_1, 0) &= e^{i\phi_r/2} q^{a(k+n_1)/2}|n_1+1, k-1), \qquad n_1 \neq k-1, &&\\ v_{ra}|k-1, 0) &= e^{i\phi_r}|0, k-1). && (2.11) \end{aligned}$$

The operators h and v_{ra} satisfy interesting properties. First, it is obvious that the operator h is Hermitian. Second, the operator v_{ra} is unitary. In addition, the action of v_{ra} on the space $\mathcal{F}_k$ is cyclic. More precisely, we can check that

$$(v_{ra})^k = e^{i\pi(k-1)(a+r)} I, \tag{2.12}$$

where I is the identity operator.

From the Schwinger work on angular momentum [22], we introduce

$$J = \frac{1}{2}\,(n_1+n_2)\,, \qquad M = \frac{1}{2}\,(n_1-n_2)\,.$$

Consequently, we can write

$$|n_1, n_2) = |J + M, J - M).$$

We shall use the notation

$$|J, M\rangle := |J + M, J - M)$$

for the vector $|J + M, J - M)$. For a fixed value of J, the label M can take $2J + 1$ values $M = J, J - 1, \dots, -J$.

For fixed k, the following value of J

$$j = \frac{1}{2}(k - 1)$$

is admissible. For a given value of $k \in \mathbb{N} \setminus \{0, 1\}$, the $k = 2j + 1$ vectors $|j, m\rangle$ belong to the vector space $\mathcal{F}_k$. Let $\varepsilon(j)$ be the subspace of $\mathcal{F}_k$, of dimension k, spanned by the k vectors $|j, m\rangle$. We can thus associate $\varepsilon(j)$ for $j = 1/2, 1, 3/2, \dots$ with the values $k = 2, 3, 4, \dots$, respectively. We shall denote as S the spherical basis

$$S = \{|j, m\rangle : m = j, j - 1, \dots, -j\} \tag{2.13}$$

of the space $\varepsilon(j)$. The rewriting of equations (2.9)–(2.11) in terms of the vectors $|j, m\rangle$ shows that $\varepsilon(j)$ is stable under h and v_{ra}. More precisely, we have the following result.

Proposition 2. *The action of the operators h and v_{ra} on the subspace $\varepsilon(j)$ of $\mathcal{F}_k$ can be described by*

$$h|j, m\rangle = \sqrt{(j + m)(j - m + 1)}|j, m\rangle$$

and

$$v_{ra}|j, m\rangle = (1 - \delta_{m,j})\, q^{(j-m)a}|j, m + 1\rangle + \delta_{m,j} e^{i2\pi jr}|j, -j\rangle,$$

which are a simple rewriting, in terms of the vectors $|j, m\rangle$, of equations (2.9), (2.10) *and* (2.11), *respectively.*

We can check that the operator h is Hermitian and the operator v_{ra} is unitary on the space $\varepsilon(j)$. Equation (2.12) can be rewritten as

$$(v_{ra})^{2j+1} = e^{i2\pi j(a+r)} I, \tag{2.14}$$

which reflects the cyclic character of v_{ra} on $\varepsilon(j)$.

2.4 The $\mathbf{su_2}$ algebra

We are now in a position to give a realization of the Lie algebra su_2 of the group SU_2 in terms of the generators of A_1 and A_2. Let us define the three operators

$$j_+ = h v_{ra}, \qquad j_- = v_{ra}^\dagger h, \qquad j_z = \frac{1}{2}(h^2 - v_{ra}^\dagger h^2 v_{ra}). \tag{2.15}$$

It is straightforward to check that the action on the vector $|j, m\rangle$ of the operators defined by equation (2.15) is given by

$$j_+|j, m\rangle = q^{+(j-m+s-1/2)a}\sqrt{(j - m)(j + m + 1)}|j, m + 1\rangle, \tag{2.16}$$

$$j_-|j, m\rangle = q^{-(j+m+s+1/2)a}\sqrt{(j + m)(j - m + 1)}|j, m - 1\rangle \tag{2.17}$$

and

$$j_z|j, m\rangle = m|j, m\rangle.$$

Consequently, we get the following result.

Proposition 3. *We have the commutation relations*

$$[j_z, j_+] = +j_+, \qquad [j_z, j_-] = -j_-, \qquad [j_+, j_-] = 2j_z, \tag{2.18}$$

which correspond to the Lie algebra su_2*.*

We observe that the latter result (2.18) does not depend on the parameters a and r. On the contrary, the action of $j_\pm$ on $|j,m\rangle$ depends on a but not on r; the familiar Condon and Shortley phase convention used in atomic spectroscopy amounts to take $a = 0$ in equations (2.16) and (2.17). The decomposition (2.15) of the Cartan operator j_z and the shift operators j_+ and j_- in terms of h and v_{ra} constitutes a two-parameter polar decomposition of su_2. Note that one-parameter polar decompositions were obtained (i) in [23] for SU_2 in a completely different context and (ii) in [24, 25, 26, 27] for SU_n in the context of deformations.

2.5 The W_∞ algebra

In the following, we shall restrict a to take the values $a = 0, 1, \ldots, 2j$. By defining the linear operator z through

$$z|j,m\rangle = q^{j-m}|j,m\rangle, \tag{2.19}$$

we can rewrite v_{ra} as

$$v_{ra} = v_{r0}z^a, \qquad a = 0, 1, \ldots, 2j. \tag{2.20}$$

The operators v_{ra} and z satisfy the q-commutation relation

$$v_{ra}z - qzv_{ra} = 0.$$

Let us now introduce

$$t_m = q^{-m_1m_2/2}(v_{ra})^{m_1}z^{m_2}, \qquad m = (m_1, m_2) \in \mathbb{N}^{*2}.$$

Then, it is easy to obtain the following result.

Proposition 4. *We have the commutator*

$$[t_m, t_n] = 2i\sin\left(\frac{\pi}{2j+1}m \wedge n\right)t_{m+n},$$

where

$$m = (m_1, m_2) \in \mathbb{N}^{*2}, \qquad n = (n_1, n_2) \in \mathbb{N}^{*2},$$
$$m \wedge n = m_1n_2 - m_2n_1, \qquad m + n = (m_1 + n_1, m_2 + n_2),$$

so that the linear operators t_m *span the infinite-dimensional Lie algebra* W_∞ *introduced in* [28]*.*

This result parallels the ones obtained, on one hand, from a study of k-fermions and of the Dirac quantum phase operator through a q-deformation of the harmonic oscillator [17, 18, 19] and, on the other hand, from an investigation of correlation measure for finite quantum systems [24, 25, 26, 27].

To close this section, we note that the (Weyl–Pauli) operators z and v_{ra} can be used to generate the (Pauli) group $\mathcal{P}_{2j+1}$ introduced in [29] (see also [30]). The group $\mathcal{P}_{2j+1}$ is a finite subgroup of $\mathrm{GL}(2j+1, \mathbb{C})$ and consists of generalized Pauli matrices. It is spanned by two generators. In fact, the $(2j+1)^3$ elements of $\mathcal{P}_{2j+1}$ can be generated by v_{00} ($r = 0$, $a = 0$) and z for $2j+1$ odd and by v_{10} ($r = 1$, $a = 0$) and $e^{i\pi/(2j+1)}z$ for $2j+1$ even.

3 An alternative basis for the representation of SU_2

3.1 The $\{j^2, v_{ra}\}$ scheme

It is immediate to check that the Casimir operator j^2 of su_2 can be written as

$$j^2 = h^2 + j_z^2 - j_z = v_{ra}^{\dagger} h^2 v_{ra} + j_z^2 + j_z$$

or

$$j^2 = \frac{1}{4}(N_1 + N_2)(N_1 + N_2 + 2). \tag{3.1}$$

Thus, the operators j^2 and v_{ra} can be expressed in terms of the generators of A_1 and A_2, see equations (2.6)–(2.8), and (3.1). It is a simple matter of calculation to prove that j^2 commutes with v_{ra} for any value of a and r. Therefore, for fixed a and r, the commuting set $\{j^2, v_{ra}\}$ provides us with an alternative to the familiar commuting set $\{j^2, j_z\}$ of angular momentum theory.

3.2 Eigenvalues and eigenvectors

The eigenvalues and the common eigenvectors of the complete set of commuting operators $\{j^2, v_{ra}\}$ can be easily found by using standard techniques. This leads to the following result.

Proposition 5. *The spectra of the operators v_{ra} and j^2 are given by*

$$v_{ra}|j\alpha; ra\rangle = q^{j(a+r)-\alpha}|j\alpha; ra\rangle, \qquad j^2|j\alpha; ra\rangle = j(j+1)|j\alpha; ra\rangle, \tag{3.2}$$

where

$$|j\alpha; ra\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} q^{(j+m)(j-m+1)a/2 - jmr + (j+m)\alpha}|j, m\rangle, \qquad \alpha = 0, 1, \ldots, 2j \tag{3.3}$$

and q is given by (2.1) *with $k = 2j+1$. The spectrum of v_{ra} is nondegenerate. For fixed j, a, and r, the $2j+1$ eigenvectors $|j\alpha; ra\rangle$, with $\alpha = 0, 1, \ldots, 2j$, of the operator v_{ra} generate an orthonormalized basis $B_{ra} = \{|j\alpha; ra\rangle : \alpha = 0, 1, \ldots, 2j\}$ of the space $\epsilon(j)$. In addition, we have*

$$|\langle j, m|j\alpha; ra\rangle| = \frac{1}{\sqrt{2j+1}}, \qquad m = j, j-1, \ldots, -j, \quad \alpha = 0, 1, \ldots, 2j, \tag{3.4}$$

so that the bases B_{ra} and S are mutually unbiased.

Let us recall that two orthonormalized bases $\{|a\alpha\rangle : \alpha = 0, 1, \ldots, d-1\}$ and $\{|b\beta\rangle : \beta = 0, 1, \ldots, d-1\}$ of a d-dimensional Hilbert space over $\mathbb{C}$, with an inner product denoted as $\langle\,|\,\rangle$, are said to be mutually unbiased [31, 32, 33, 34, 35, 36, 37, 38, 39] if and only if

$$|\langle a\alpha|b\beta\rangle| = \delta_{a,b}\delta_{\alpha,\beta} + (1 - \delta_{a,b})\frac{1}{\sqrt{d}}. \tag{3.5}$$

The correspondence between equations (3.4) and (3.5) is as follows. In equation (3.4), $2j+1$ corresponds to d while the symbols jra, α, j, and m correspond to the symbols a, α, b, and β, respectively.

3.3 Representation of SU_2

The representation theory of SU_2 can be transcribed in the $\{j^2, v_{ra}\}$ scheme. For fixed a and r, the nonstandard basis B_{ra} turns out to be an alternative to the standard or spherical basis S. In the $\{j^2, v_{ra}\}$ scheme, the rotation matrix elements for the rotation R of SO_3 assumes the form

$$D^{(j)}(R)_{\alpha\alpha'} = \frac{1}{2j+1} \sum_{m=-j}^{j} \sum_{m'=-j}^{j} q^{-\rho(j,m,a,r,\alpha)+\rho(j,m',a,r,\alpha')} \, \mathcal{D}^{(j)}(R)_{mm'} \tag{3.6}$$

in terms of the standard matrix elements $\mathcal{D}^{(j)}(R)_{mm'}$ corresponding to the $\{j^2, j_z\}$ scheme. In equation (3.6), the function ρ is defined by

$$\rho(J, M, x, y, z) = \frac{1}{2}(J+M)(J-M+1)x - JMy + (J+M)z. \tag{3.7}$$

Then, the behavior of the vector $|j\alpha; ra\rangle$ under an arbitrary rotation R is given by

$$P_R|j\alpha; ra\rangle = \sum_{\alpha'=0}^{2j} |j\alpha'; ra\rangle \, D^{(j)}(R)_{\alpha'\alpha}, \tag{3.8}$$

where P_R stands for the operator associated with R. In the case where R is a rotation around the z-axis, equation (3.8) takes a simple form as shown in the following result.

Proposition 6. *If $R(\varphi)$ is a rotation of an angle*

$$\varphi = p\frac{2\pi}{2j+1}, \qquad p = 0, 1, 2, \ldots, 2j \tag{3.9}$$

around the z-axis, then we have

$$P_{R(\varphi)}\, |j\alpha; ra\rangle = q^{jp}|j\alpha'; ra\rangle, \quad \alpha' = \alpha - p \pmod{2j+1}, \tag{3.10}$$

so that the set $\{|j\alpha; ra\rangle : \alpha \text{ ranging}\}$ *is stable under* $P_{R(\varphi)}$.

Consequently, the set $\{|j\alpha; ra\rangle : \alpha = 0, 1, \ldots, 2j\}$ spans a reducible representation of dimension $2j+1$ of the cyclic subgroup C_{2j+1} of SO_3. It can be seen that this representation is nothing but the regular representation of C_{2j+1}. Thus, this representation contains each irreducible representation of C_{2j+1} once and only once. The nonstandard basis B_{ra} presents some characteristics of a group-subgroup type basis in the sense that B_{ra} carries a representation of a subgroup of SO_3. However, this representation is reducible (except for $j = 0$). Therefore, the label $\alpha; ar$ does not correspond to some irreducible representation of a subgroup of SU_2 or $\mathrm{SO}(3) \sim \mathrm{SU}_2/Z_2$ so that the basis B_{ra} also exhibits some characteristics of a nongroup-subgroup type basis.

3.4 Wigner–Racah algebra of SU_2

We are now ready to give the starting point for a study of the Wigner–Racah algebra of SU_2 in the $\{j^2, v_{ra}\}$ scheme. In such a scheme, the coupling or Clebsch–Gordan coefficients read

$$\begin{aligned}(j_1 j_2 \alpha_1 \alpha_2 | j_3 \alpha_3)_{ra} = {} & [(2j_1+1)(2j_2+1)(2j_3+1)]^{-\frac{1}{2}} \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} (j_1 j_2 m_1 m_2 | j_3 m_3) \\ & \times q_1^{-\rho(j_1,m_1,a,r,\alpha_1)} \, q_2^{-\rho(j_2,m_2,a,r,\alpha_2)} \, q_3^{\rho(j_3,m_3,a,r,\alpha_3)},\end{aligned} \tag{3.11}$$

where the function ρ is given by equation (3.7). In equation (3.11), $(j_1 j_2 m_1 m_2 | j_3 m_3)$ is a standard Clebsch–Gordan coefficient in the $\{j^2, j_z\}$ scheme and

$$q_\ell = \exp\left(i\frac{2\pi}{2j_\ell + 1}\right), \qquad \ell = 1, 2, 3.$$

The algebra of the new coupling coefficients (3.11) can be developed in a way similar to the one [40] known in the $\{j^2, j_z\}$ scheme (see [20] for the basic ideas). In particular, following the technique developed in [4, 5], the familiar 6-j and 9-j symbols of Wigner can be expressed in terms of the coupling coefficients defined by equation (3.11).

3.5 Realization of v_{ra}

The operator v_{ra} can be expressed in the enveloping algebra of SU_2. A possible way to find a realization of v_{ra} in terms of the generators $j_\pm$ and j_z of SU_2 is as follows.

The first step is to develop v_{ra} on the basis of the Racah unit tensor $\mathbf{u}^{(k)}$ [41]. Let us recall that the components $u_p^{(k)}$ of $\mathbf{u}^{(k)}$, $p = k, k-1, \ldots, -k$, are defined by

$$\langle j, m | u_p^{(k)} | j, m' \rangle = (-1)^{j-m} \begin{pmatrix} j & k & j \\ -m & p & m' \end{pmatrix},$$

where the symbol $(\cdots)$ stands for a 3-jm Wigner symbol. The Hilbert–Schmidt scalar product of $u_p^{(k)}$ by $u_q^{(\ell)}$ satisfies

$$\mathrm{Tr}_{\epsilon(j)}\left((u_p^{(k)})^\dagger u_q^{(\ell)}\right) = \Delta(j, j, k)\, \delta_{k,\ell}\, \delta_{p,q}\, \frac{1}{2k+1}, \tag{3.12}$$

where $\Delta(j, j, k) = 1$ if j, j, and k satisfy the triangular inequality and is zero otherwise. Therefore, the coefficients $b_{kq}(ra)$ of the development

$$v_{ra} = \sum_{k=0}^{2j} \sum_{p=-k}^{k} b_{kp}(ra) u_p^{(k)} \tag{3.13}$$

can be easily calculated from equations (3.12) and (3.13). This yields

$$b_{kp}(ra) = (2k+1)\mathrm{Tr}_{\epsilon(j)}\left((u_p^{(k)})^\dagger v_{ra}\right). \tag{3.14}$$

By developing the rhs of (3.14), it is possible to obtain

$$\begin{aligned} b_{kp}(ra) = {} & \delta_{p,1}\, (2k+1) \sum_{m=-j}^{j-1} q^{(j-m)a} (-1)^{j-m-1} \begin{pmatrix} j & k & j \\ -m-1 & 1 & m \end{pmatrix} \\ & + \delta_{k,2j}\, \delta_{p,-2j}\, \sqrt{4j+1} e^{i2\pi jr}. \end{aligned} \tag{3.15}$$

The second step is to express $u_p^{(k)}$ in the enveloping algebra of SU_2. This can be achieved by using the formulas given in [42, 43]. Indeed, the operator $u_p^{(k)}$ acting on $\varepsilon(j)$ reads

$$\begin{aligned} u_p^{(k)} = {} & \left[\frac{(k-p)!}{(k+p)!(2j-k)!(2j+k+1)!}\right]^{1/2} (-1)^{k+p} j_+^p \left[(-1)^p \frac{(2j-p)!(k+p)!}{p!(k-p)!}\right. \\ & \left. + (1 - \delta_{k,p}) \sum_{z=p+1}^{k} (-1)^z \frac{(2j-z)!(k+z)!}{z!(k-z)!(z-p)!} \prod_{t=1}^{z-p} (j + j_z + p - z + t)\right] \end{aligned} \tag{3.16}$$

for $p \geq 0$. The formula for $p < 0$ may be derived from (3.16) with the help of the Hermitian conjugation property $u_{-p}^{(k)} = (-1)^p (u_p^{(k)})^\dagger$. Alternatively, $u_p^{(k)}$ with $p < 0$ may be obtained by changing p, j_+ and j_z into $-p$, $-j_-$ and $-j_z$ respectively, in the rhs of (3.16) and by multiplying the expression so-obtained by $(-1)^{k+p}$. This yields

$$u_p^{(k)} = \left[\frac{(k+p)!}{(k-p)!(2j-k)!(2j+k+1)!}\right]^{1/2} (-1)^p j_-^{-p} \left[(-1)^p \frac{(2j+p)!(k-p)!}{(-p)!(k+p)!}\right.$$
$$\left. + (1-\delta_{k,-p}) \sum_{z=-p+1}^{k} (-1)^z \frac{(2j-z)!(k+z)!}{z!(k-z)!(z+p)!} \prod_{t=1}^{z-p} (j - j_z - p - z + t)\right] \tag{3.17}$$

for $p \leq 0$.

As a conclusion we have the following result.

Proposition 7. *The development of v_{ra} in the enveloping algebra of SU_2 is given by* (3.13)–(3.17)*.*

By way of illustration, let us consider the case $j = 1/2$, 1, and 3/2 for $r = 0$ and $a = 0$.

Case $j = 1/2$

$$v_{00} = \sqrt{3}\left(u_{-1}^{(1)} - u_1^{(1)}\right) \Rightarrow v_{00} = j_+ + j_-.$$

Case $j = 1$

$$v_{00} = \sqrt{5}u_{-2}^{(2)} - \sqrt{6}u_1^{(1)} \Rightarrow v_{00} = \frac{1}{\sqrt{2}} j_+ + \frac{1}{2}(j_-)^2.$$

Case $j = 3/2$

$$v_{00} = -\left(1+\sqrt{3}\right)\sqrt{\frac{6}{5}}u_1^{(1)} + \sqrt{7}u_{-3}^{(3)} + \left(\sqrt{3}-2\right)\sqrt{\frac{7}{5}}u_1^{(3)}$$
$$\Rightarrow \quad v_{00} = \frac{1}{\sqrt{3}} j_+ + \left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right) j_+ \left(j_z + \frac{3}{2}\right)\left(j_z - \frac{1}{2}\right) + \frac{1}{6}(j_-)^3.$$

A program in MAPLE was run to get v_{0a} for higher values of j. The results were returned under the form

$$j_+ \times (\text{a polynomial in } j_z) + \frac{1}{(2j)!}(j_-)^{2j}.$$

For $a = 0$ as above, one can remark and prove that the polynomial in j_z is of degree $\leq 2\lceil j \rceil - 2$ and that its coefficients are elements of the field $\mathbb{Q}[\sqrt{2}, \sqrt{3}, \dots]$ which form a set of vectors over $\mathbb{Q}$ of rank $\leq \lceil j \rceil$. Since these two upper bounds are reached for most of the values of j for which v_{00} was computed, one expects in general $\lceil j \rceil - 2$ independent linear relations between the coefficients. These relations seemingly exhibit some regularities.

3.6 Connection between B_{ra} and B_{sb}

The operators v_{ra} and v_{sb} do not commute in general. For example, in the situation where $a = b = 0$, a necessary and sufficient condition to have $[v_{r0}, v_{s0}] = 0$ is

$$js = jr + t, \qquad t \in \mathbb{Z}.$$

Going back to the general case $a \neq b$, the operators v_{ra} and v_{sb} satisfy the property

$$\mathrm{Tr}_{\varepsilon(j)}\left(v_{ra}^\dagger v_{sb}\right) = \delta_{a,b}(2j+1) + e^{i(\phi_s - \phi_r)} - 1. \tag{3.18}$$

A simple development of the rhs of equation (3.18) and the use of equation (3.2) lead to

$$\sum_{\alpha=0}^{2j}\sum_{\beta=0}^{2j} q^{\alpha-\beta}|\langle j\alpha;ra|j\beta;sb\rangle|^2 = \delta_{a,b}q^{j(r-s)}(2j+1) + q^{j(a+r-b-s)}[e^{i(\phi_s-\phi_r)}-1]. \tag{3.19}$$

Let us now consider the overlap between two bases B_{ra} and B_{sb} corresponding to the schemes $\{j^2, v_{ra}\}$ and $\{j^2, v_{sb}\}$, respectively. We have

$$\langle j\alpha;ra|j'\beta;sb\rangle = \delta_{j,j'}\,\frac{1}{2j+1}\sum_{m=-j}^{j} q^{\rho(j,m,b-a,s-r,\beta-\alpha)}. \tag{3.20}$$

From equation (3.20), we see that the overlap $\langle j\alpha;ra|j\beta;sb\rangle$ depends solely on the difference $\alpha-\beta$ rather than on α and β separately. Hence, equation (3.19) can be reduced to

$$\sum_{\alpha=0}^{2j} q^{\alpha}\,|\langle j\alpha;ra|j0;sb\rangle|^2 = \delta_{a,b}q^{j(r-s)} + \frac{1}{2j+1}q^{j(a+r-b-s)}\left[e^{i(\phi_s-\phi_r)}-1\right],$$

a relation that also follows from repeated applications of equations (3.9) and (3.10) to the lhs of (3.19).

In the special case where $b=a$, we get

$$\langle j\alpha;ra|j'\beta;sa\rangle = \delta_{j,j'}\,\frac{1}{2j+1}\,q^{j(\beta-\alpha)}\,\frac{\sin\pi(jr-\alpha-js+\beta)}{\sin\frac{\pi}{2j+1}(jr-\alpha-js+\beta)}$$

for $jr-\alpha-js+\beta\not\equiv 0 \pmod{2j+1}$ and

$$\langle j\alpha;ra|j'\beta;sa\rangle = \delta_{j,j'}\,(-1)^{2jk}\,q^{j(\beta-\alpha)}$$

for $jr-\alpha-js+\beta=(2j+1)k$ with $k\in\mathbb{Z}$. It is clear that for $s=r$, we recover that the basis B_{ra} is orthonormalized since equation (3.20) gives

$$\langle j\alpha;ra|j\beta;ra\rangle = \delta_{\alpha,\beta}. \tag{3.21}$$

The case $b\neq a$ is much more involved. For $b\neq a$ and $r=s$, equation (3.20) is amenable in the form of a generalized quadratic Gauss sum $S(u,v,w)$. Such a sum is defined by

$$S(u,v,w) = \sum_{k=0}^{|w|-1} e^{i\pi(uk^2+vk)/w}, \tag{3.22}$$

where u, v, and w are integers such that $uw\neq 0$ and $uw+v$ is an even integer [44]. As a matter of fact, we have the following result.

Proposition 8. *For $b\neq a$, the overlap $\langle j\alpha;ra|j\beta;rb\rangle$ can be written as*

$$\langle j\alpha;ra|j\beta;rb\rangle = \frac{1}{w}S(u,v,w), \tag{3.23}$$

where

$$u=a-b, \qquad v=-(a-b)(2j+1)-2(\alpha-\beta), \qquad w=2j+1,$$

with $a-b=\pm1,\pm2,\ldots,\pm2j$ and $\alpha,\beta=0,1,\ldots,2j$. Furthermore, for $2j+1$ prime we have

$$|\langle j\alpha;ra|j\beta;rb\rangle| = \frac{1}{\sqrt{2j+1}}, \tag{3.24}$$

with $a-b=\pm1,\pm2,\ldots,\pm2j$ and $\alpha,\beta=0,1,\ldots,2j$.

The proof of Proposition 8 is given in Appendix A. Along this vein, relations between generalized quadratic Gauss sums and the absolute value of a particular Gaussian sum are presented in Appendices B and C, respectively.

It is to be noted that equation (3.24) can be proved equally well without using generalized quadratic Gauss sums. The following proof is an adaptation, in the framework of angular momentum, of the method developed in [45] (see also [46, 47, 48, 49]) in order to construct a complete set of MUBs in $\mathbb{C}^d$ with d prime.

Proof. We start from

$$v_{ra}z^n = v_{rb}, \qquad b = a + n, \qquad n \in \mathbb{Z},$$

which can be derived from equation (2.20). In view of Proposition 5, the action of the operator $v_{ra}z^n$ on the vector $|j\beta_0; rb\rangle$ leads to

$$v_{ra}z^n|j\beta_0; rb\rangle = q^{j(a+n+r)-\beta_0}|j\beta_0; rb\rangle. \tag{3.25}$$

Furthermore, equations (2.19) and (3.3) give

$$z^n|j\beta_0; rb\rangle = q^{2jn}|j\beta_1; rb\rangle, \qquad \beta_i = \beta_0 - in, \qquad i \in \mathbb{Z}, \quad n \in \mathbb{Z}. \tag{3.26}$$

Let us consider the scalar product $\langle j\alpha; ra|v_{ra}z^n|j\beta_0; rb\rangle$. This product can be calculated in two different ways owing to (3.25) and (3.26). We thus obtain

$$|\langle j\alpha; ra|v_{ra}|j\beta_1; rb\rangle| = |\langle j\alpha; ra|j\beta_0; rb\rangle|\,. \tag{3.27}$$

Since v_{ra} is unitary and satisfies (2.14), we can write

$$v_{ra} = \left(v_{ra}^{\dagger}\right)^{2j}(v_{ra})^{2j}\,v_{ra} = \left(v_{ra}^{\dagger}\right)^{2j}(v_{ra})^{2j+1} = e^{i2\pi(a+r)j}\left(v_{ra}^{\dagger}\right)^{2j}. \tag{3.28}$$

Finally, the introduction of (3.28) into (3.27) produces the master formula

$$|\langle j\alpha; ra|j\beta_1; rb\rangle| = |\langle j\alpha; ra|j\beta_0; rb\rangle|\,.$$

The number of different β_i modulo $2j+1$ that can be reached by repeated translations of β_0 is $(2j+1)/\gcd(2j+1, |n|)$. As a conclusion, equation (3.24) is true for $2j+1$ prime. ∎

4 Applications to cyclic systems and quantum information

We shall devote the rest of this paper to some applications involving state vectors of type (3.3). More specifically, we shall deal with cyclic systems, like ring shape molecules and 1/2-spin chains, and with MUBs of quantum information theory (quantum cryptography and quantum tomography). For the sake of comparison with some previous works, it is appropriate to leave the framework of angular momentum theory by transforming sums on m from $-j$ to j into sums on k from 0 to d with $d = 2j+1$. Furthermore, we shall limit ourselves in Section 4 to vanishing r-parameters. Consequently, we shall use the notation $|a\alpha\rangle := |j\alpha; 0a\rangle$ and $|k\rangle := |jm\rangle$ with $k = j + m$. Thus, equation (3.3) becomes

$$|a\alpha\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} q^{k(d-k)a/2+k\alpha}|k\rangle, \qquad a = 0, 1, \ldots, 2j, \quad \alpha = 0, 1, \ldots, 2j. \tag{4.1}$$

The notation $|a\alpha\rangle$ and $|k\rangle$ is especially adapted to the study of cyclic systems and MUBs.

4.1 Cyclic systems

Let us consider a ring shape molecule with N atoms (or aggregates) at the vertices of a regular polygon with N sides ($N = 6$ for the benzen molecule C_6H_6). The atoms are labelled by the integer n with $n = 0, 1, \ldots, N-1$. Hence, the cyclic character of the ring shape molecule makes it possible to identify the atom with the number n to the one with the number $n + kN$ where $k \in \mathbb{Z}$ (the location of an atom is defined modulo N). Let $|\varphi_n\rangle$ be the atomic state vector, or atomic orbital in quantum chemistry parlance, describing a π-electron located in the neighboring of site n. From symmetry considerations, the molecular state vector, or molecular orbital, for the molecule reads [50]

$$|\kappa_s\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{i2\pi ns/N} |\varphi_n\rangle,$$

with $s = 0, 1, \ldots, N-1$. As a result, the molecular orbital $|\kappa_s\rangle$ assumes the same form, up to a global phase factor, as the state $|a\alpha\rangle$ given by equation (4.1) with $a = 0$ and $\alpha = s$.

A similar result can be obtained for a one-dimensional chain of N 1/2-spins (numbered with $n = 0, 1, \ldots, N-1$) used as a modeling tool of a ferromagnetic substance. Here again, we have a cyclical symmetry since the spins numbered $n = N$ and $n = 0$ are considered to be identical. The spin waves can then be described by state vectors (see [50]) very similar to the ones given by equation (4.1) with again $a = 0$.

4.2 Mutually unbiased bases

The results in Section 3 can be applied to the derivation of MUBs. Proposition 8 provides us with a method for deriving MUBs that differs from the methods developed, used or discussed in [31]–[39] and in [51]–[78].

4.2.1 d arbitrary

Let V_a be the matrix of the operator v_{0a} in the computational basis

$$S = \{|k\rangle : k = d-1, d-2, \ldots, 0\}, \tag{4.2}$$

cf. equation (2.13). This matrix can be expressed in terms of the generators $E_{x,y}$ (with $x, y = 0, 1, \ldots, d-1$) of the unitary group U_d. The generator $E_{x,y}$ is defined by its matrix elements

$$(E_{x,y})_{ij} = \delta_{i,x}\, \delta_{j,y}, \qquad i, j = d-1, d-2, \ldots, 0.$$

Therefore, we have

$$V_a = E_{0,d-1} + \sum_{k=1}^{d-1} q^{(d-k)a} E_{k,k-1},$$

with $q = \exp(2\pi i/d)$, as far as we label the lines and columns of V_a according to the decreasing order $d-1, d-2, \ldots, 0$. The eigenvectors $\varphi(a\alpha)$ of matrix V_a are expressible in terms of the column vectors e_x (with $x = 0, 1, \ldots, d-1$) defined via

$$(e_x)_{i1} = \delta_{i,x}, \qquad i = d-1, d-2, \ldots, 0.$$

Indeed, we have

$$\varphi(a\alpha) = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} q^{k(d-k)a/2 + k\alpha} e_k.$$

This can be summed up by the following result.

Proposition 9. *We have the eigenvalue equation*

$$V_a \varphi(a\alpha) = q^{(d-1)a/2-\alpha} \varphi(a\alpha), \qquad \alpha = 0, 1, \ldots, d-1.$$

Furthermore, the generalized Hadamard matrix

$$H_a = \sum_{\alpha=0}^{d-1} \sum_{k=0}^{d-1} q^{k(d-k)a/2+k\alpha} E_{k,\alpha} \tag{4.3}$$

satisf ies

$$H_a^\dagger H_a = dI, \qquad H_a^\dagger V_a H_a = q^{(d-1)a/2} \, d \sum_{\alpha=0}^{d-1} q^{-\alpha} E_{\alpha,\alpha}$$

and thus reduces the endomorphism associated with V_a.

4.2.2 *d* prime

We are now in a position to establish contact with MUBs. We know that for a d-dimensional Hilbert space, with d prime ($d = p$) or a power of a prime ($d = p^e$), with p prime and e positive integer greater than 1, there exists a complete set of $d + 1$ pairwise MUBs [31, 32, 33, 34, 35, 36, 37, 38, 39, 51, 52, 53, 54, 55, 56, 57, 58, 59].

For $d = p = 2j + 1$ prime, the bases

$$B_{0a} = \{|a\alpha\rangle := |j\alpha; 0a\rangle : \alpha = 0, 1, \ldots, p-1\}, \qquad a = 0, 1, \ldots, p-1$$

satisfy (3.21) and (3.24). Consequently, they constitute an incomplete set of p MUBs. On the other hand, the bases S and B_{0a}, with fixed a, are mutually unbiased (see Proposition 5). Therefore, we have the following result.

Proposition 10. *For p prime, the spherical or computational basis S, given by* (2.13) *or* (4.2), *and the bases B_{0a} with $a = 0, 1, \ldots, p-1$, given by* (3.3) *or* (4.1), *constitute a complete set of $p + 1$ MUBs in $\mathbb{C}^p$. In matrix form, the basis vectors of each B_{0a} are given by the columns of the generalized Hadamard matrix* (4.3).

It is to be noted that each column of matrix H_a gives one of the p vectors of basis B_{0a}. For p prime, the $(p + 1)p$ vectors of the $p + 1$ MUBs are given by the columns of the p matrices H_a with $a = 0, 1, \ldots, p$ together with the columns of the p-dimensional identity matrix. We observe that Proposition 10 is valid for $p = 2$.

4.2.3 *d* not prime

Returning to the general case where d is arbitrary, it is clear that the bases B_{0a} with $a = 0, 1, \ldots, d-1$ do not constitute in general a set of d pairwise MUBs. However, some of them can be mutually unbiased. An easy way to test the unbiased character of the bases B_{0a} and B_{0b}, with $a \neq b$, is to calculate $H_a^\dagger H_b$; if the module of each matrix element of $H_a^\dagger H_b$ is equal to $\sqrt{d}$, then B_{0a} and B_{0b} are mutually unbiased.

The bases B_{0a} corresponding to $d = p = 2j + 1$, p prime, can serve to generate MUBs in the case $d = p^e = (2j + 1)^e$. For $d = p^e = (2j + 1)^e$, with p prime and e positive integer greater than 1, let us consider the p^e bases

$$B_{a_1 a_2 \cdots a_e} = B_{0a_1} \otimes B_{0a_2} \otimes \cdots \otimes B_{0a_e}, \qquad a_i \in \{0, 1, \ldots, p-1\}, \quad i = 1, 2, \ldots, e.$$

In this case, we face a degeneracy problem. Therefore, the basis vectors in $B_{a_1 a_2 \cdots a_e}$ must be reorganized (via linear combinations) in order to form, with the p^e-dimensional computational basis, a complete set of $p^e + 1$ MUBs in $\mathbb{C}^d$ (cf. the approaches of [45, 46, 47, 48, 49]).

Up to this point, we have dealt with pairwise MUBs. We close this section with a few remarks concerning the number of bases which are unbiased with a given basis. In the proof of Proposition 8 given in Appendix A, one of the key arguments is that u or u' must be invertible modulo $2(2j+1)$, what is immediately checked since $2j+1$ is prime. This argument cannot be used when the dimension $d = 2j+1$ is a power of a prime, $d = p^e$ (p prime and e integer greater than 1). However, taking $p \neq 2$, let us consider the bases B_{ra} ($a = 0, 1, \ldots, d-1$) whose vectors are given by (3.3), with $j = (d-1)/2$. We remark that the number of bases B_{ra} (a ranging) that are unbiased with one of them is at least $\varphi(p^e) = p^e - p^{e-1}$, a remark that is also valid for arbitrary dimension. If $d = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, with $p_i \neq 2$ for $i = 1, 2, \ldots, n$, then the number of bases B_{ra} (a ranging) that are unbiased with one of them is at least

$$\varphi(d) = \prod_{i=1}^{n} p_i^{e_i} - p_i^{e_i - 1}.$$

These considerations can be expressed in a geometrical way in the case of a prime power dimension $d = p^e$, with $p \neq 2$. Any integer a between 0 and $p^e - 1$ can be written in the form

$$a = a_0 + a_1 p + \cdots + a_{e-1} p^{e-1},$$

with $0 \leq a_i \leq p-1$ for $i = 0, 1, \ldots, e-1$. Thus, any basis B_{ra} corresponds to the point of coordinates $(a_0, a_1, \ldots, a_{e-1})$ in an affine space of dimension e over the Galois field $\mathbb{Z}/p\mathbb{Z}$. Moreover, two bases B_{ra} and B_{rb} are mutually unbiased if and only if $a_0 - b_0 \neq 0$, what excludes a hyperplane of the affine space. Whenever d is a product of prime powers, all of the primes being different from 2, a generalization is straightforward by the use of the Chinese remainder theorem.

5 Concluding remarks

The originality of the present paper rests upon the development of the representation of the nondeformed group SU_2 by means of truncated deformed oscillators and of its application to cyclic systems and MUBs. The main results of this work may be summarized as follows.

From two deformed oscillator algebras, with a single deformation parameter taken as a root of unity, we generated a new polar decomposition of the nondeformed group SU_2. The latter decomposition is characterized by a two-parameter operator v_{ra} and differs from previous decompositions. Our decomposition proved to be especially adapted for generating the infinite-dimensional Lie algebra W_∞. The set $\{(v_{01})^n : n = 0, 1, \ldots, 2j\}$ acting on a subspace associated with the irreducible representation (j) of SU_2 spans the cyclic group C_{2j+1}. For fixed j, each operator v_{0a} ($a = 0, 1, \ldots, 2j$) corresponds to an irreducible representation of C_{2j+1}. This yields a tower of group-subgroup chains $\mathrm{SO}_3 \supset C_{2j+1}$. The operator v_{ra} is a pseudo-invariant under C_{2j+1} and commutes with the Casimir j^2 of SU_2. For an arbitrary irreducible representation (j) of SU_2, we developed v_{ra} in the enveloping algebra of SU_2. Such a development might be useful for a study of the Wigner–Racah agebra of SU_2 in a $\mathrm{SU}_2 \supset C^*_{2j+1}$ basis. The eigenstates $|j\alpha; ra\rangle$ of the set $\{j^2, v_{ra}\}$ can be seen as generalized discrete Fourier transforms (in the sense of [60, 61, 62, 63, 64]) of the eigenstates of the standard set $\{j^2, j_z\}$. This led us to strongly SABs $B_{ra} = \{|j\alpha; ra\rangle : \alpha = 0, 1, \ldots, 2j\}$ which are unbiased with the spherical basis $S = \{|jm\rangle : m = j, j-1, \ldots, -j\}$.

The statevectors $|j\alpha; ra\rangle$ are quite convenient to study cyclic systems (as for example ring-shape molecules) and MUBs. In particular, in the case where $2j+1$ is prime, we generated from the statevectors $|j\alpha; ra\rangle$ a complete set of MUBs for the finite-dimensional Hilbert space $\mathbb{C}^{2j+1}$. The obtained MUBs are given by a single formula which is easily codable on a computer. As a by-product, the close-form expression for the vectors $|j\alpha; ra\rangle$ generates generalized Hadamard matrices in arbitrary dimension. The new approach to MUBs developed in this work lies on the use of (i) deformations introduced in fractional supersymmetric quantum mechanics [17, 18, 19], (ii) angular momentum theory, and (iii) generalized quadratic Gauss sums (for which we gave useful formulas in the appendices). In this respect, it differs from the approaches developed in previous studies through the use of Galois fields and Galois rings, discrete Wigner functions, mutually orthogonal Latin squares, graph theory, and finite geometries (e.g., see [65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77] and references cited therein for former works). Our approach to MUBs in the framework of angular momentum should be particularly appropriate for dealing with entanglement of spin states.

To close this article, let us say a few more words about point (iii) just above. We carried on with the idea of building MUBs using additive characters and Gauss sums [33, 34, 35, 36, 37, 38, 51, 54, 55, 56, 57, 58, 59]. However, our own formulas come out with an extra 1/2 factor in the argument of the exponential and we needed to study a special kind of Gauss sums to prove mutual unbiasedness in prime dimension. These Gauss sums exhibit two features of previous works [33, 34, 35, 36, 37, 38, 65, 66], namely a second degree polynomial as an argument of the exponential and the doubling of the dimension in the denominator. Such combinations or comparisons between different methods have been hardly explored up to now, except in [72].

When one tries to figure out MUBs in a space the dimension of which is a nontrivial power of a prime, $d = p^e$, $e > 1$, one usually refers to the additive characters of the Galois field $\mathbb{F}_d$. Therefore, the method is specific to the case of prime power dimensions. In the end of Section 4.2, the same Gauss sums as above enabled us to get some partial results about unbiasedness for arbitrary dimensions. We first replaced Galois fields with the simplest Galois rings $\mathbb{Z}/p^e\mathbb{Z}$ and then generalized the so-obtained replacement through the use of the Chinese remainder theorem. Unfortunately, numerical tests show that all bases obtained in that way are not mutually unbiased, even in prime power dimension, though other regularities appear. In order to study unbiasedness in arbitrary dimension, it would be important to know whether or not there exist a structure which supports both an equivalent of the trace operator for Galois fields and an equivalent of the Chinese remainder theorem that could be combined efficiently.

Finally, it is also to be noted that in our geometrical interpretation, bases are associated to points rather than to striations of the space as in [33, 34, 35, 36, 37, 38] and [78].

A Proof of Proposition 8

The proof of (3.23) is straightforward. Let us focus on the proof of (3.24).

Proof. It is sufficient to combine (3.20) for $j = j'$ and $s = r$ with (3.7) and (3.22). This yields

$$(2j+1)\langle j\alpha; ra|j\beta; rb\rangle = S(u, v, 2j+1) = \sum_{k=0}^{2j} q^{(uk^2+vk)/2},$$

where $u = a - b$, $v = 2(\beta - \alpha) + (2j+1)(b-a)$, and $q = e^{2\pi i/(2j+1)}$.

For $j = 1/2$, the generalized quadratic Gauss sum $S(u, v, 2)$ can be easily calculated and we then check that (3.24) is satisfied for $2j + 1 = 2$.

We continue with $2j + 1$ equal to an odd prime number. In $S(u, v, 2j + 1)$, the integer u is such that $-2j \le u \le 2j$ and, for $2j + 1$ prime with $j \ne 1/2$, the integer v has the same parity as u. We shall thus consider in turn u even and u odd.

In the case u even, $\xi = u/2$ and $\eta = v/2$ are two integers. Then, we have

$$S(u, v, 2j+1) = \sum_{k=0}^{2j} q^{\xi k^2 + \eta k},$$

where the exponent of q may be taken modulo $2j+1$. A translation of the index k gives

$$S(u, v, 2j+1) = \sum_{k=0}^{2j} q^{\xi (k+t)^2 + \eta (k+t)}.$$

By choosing t such that $2\xi t + \eta = 0 \pmod{2j+1}$, we get

$$|S(u, v, 2j+1)| = \left| \sum_{k=0}^{2j} q^{\xi k^2} \right|. \tag{A.1}$$

The value of the rhs of (A.1) is $\sqrt{2j+1}$ [44]. Therefore, (3.24) is proved for $2j+1$ odd prime and u even.

In the case u odd, let us introduce the canonical additive character of $\mathbb{Z}/(2(2j+1))\mathbb{Z}$

$$\psi : (\mathbb{Z}/(2(2j+1))\mathbb{Z}, +) \longrightarrow (\mathbb{C}, \times) : x \longmapsto q^{y/2},$$

with $y \in \mathbb{Z}$ a representative of x modulo $2(2j+1)$. Consequently, we have

$$S(u, v, 2j+1) = \sum_{k=0}^{2j} \psi(uk^2 + vk),$$

where the argument of ψ stands for a residue modulo $2(2j+1)$. In order to apply the translation trick and to get rid of the linear term, as in the even case, k has to range over a complete set of residues modulo $2(2j+1)$. For this purpose, we may for instance consider the extra sum

$$\sum_{\ell=2j+1}^{2(2j+1)-1} \psi(u\ell^2 + v\ell) = \sum_{k=0}^{2j} \psi(uk^2 + 2(2j+1)uk + u(2j+1)^2 + vk + v(2j+1)). \tag{A.2}$$

The second term of the argument of ψ in the rhs of (A.2) vanishes under ψ. Moreover,

$$u(2j+1)^2 + v(2j+1) = 2(2j+1)uj + (u+v)(2j+1) \equiv 0 \pmod{2(2j+1)} \tag{A.3}$$

since $u+v$ is even. Hence, the extra sum is equal to $S(u, v, 2j+1)$ so that

$$S(u, v, 2j+1) = \frac{1}{2} \sum_{k=0}^{2(2j+1)-1} \psi(uk^2 + vk).$$

Now let us carry out the translation

$$u(k+t)^2 + v(k+t) = uk^2 + (2ut + v)k + ut^2 + vt. \tag{A.4}$$

Since u is odd and between $-2j$ and $2j$, it is invertible modulo $2(2j+1)$. Choosing $t \equiv u^{-1} \pmod{2(2j+1)}$, we see that

$$|S(u, v, 2j+1)| = |S(u, v+2, 2j+1)|, \tag{A.5}$$

where an increase of v by 2 amounts for an increase of $\beta-\alpha$ by 1. Therefore, the modules in the lhs of (3.24) do not depend on $\beta-\alpha$. To show that they are independent of $a-b$, we need only to remember that the overlaps $\langle j\alpha;ra|j\beta;rb\rangle$ are coefficients connecting two orthonormalized bases. Consequently

$$\sum_{\alpha=0}^{2j} |\langle j\alpha;ra|j\beta;rb\rangle|^2 = 1$$

and

$$\forall\, \alpha = 0,1,\ldots,2j: \ (2j+1)\,|\langle j\alpha;ra|j\beta;rb\rangle|^2 = 1,$$

so that (3.24) is proved for $2j+1$ prime and u odd. ■

At this point it is interesting to emphasize that the method we have developed to handle the odd case works in the even case too. Suppose $u = 2^n u'$, with u' not divisible by 2. In the translation relation (A.4), the term $2ut$ should be replaced by $2^{n+1}u't$, where u' is invertible modulo $2(2j+1)$. Thus $v+2$ in (A.5) is replaced by $v+2^{n+1}$ and an increase of v by 2^{n+1} amounts for an increase of $\beta-\alpha$ by 2^n. Since 2^n is coprime with $2j+1$, all values of $\beta-\alpha$ will be swept over modulo $2j+1$ and the result follows.

B Relations between generalized quadratic Gauss sums

As a by-product of this work, it is worthwhile to mention that the method of translation among a complete set of residues, recurrent in the present paper, can be used to derive relations between generalized quadratic Gauss sums. The Gauss sum $S(u,v,w)$, with u, v, and w integers such that $w \neq 0$ and $uw+v$ even, see (3.22), can be rewritten as

$$S(u,v,w) = q^{(ut^2+vt)/2} \sum_{k=-t}^{|w|-1-t} q^{[uk^2+(v+2ut)k]/2}, \qquad t \in \mathbb{Z}, \tag{B.1}$$

with q to be formally replaced by $e^{2\pi i/w}$. Moreover, as a more general version of (A.3), we have

$$uw^2 + vw = (uw+v)w \equiv 0 \pmod{2w},$$

which shows that, in spite of the factor $1/2$ in the exponent of q, a translation by w of any of the indices k does not modify the sum in (B.1). Hence, (B.1) leads to

$$S(u,v,w) = q^{(ut^2+vt)/2} S(u, v+2ut, w). \tag{B.2}$$

For t ranging and fixed u, v, and w, the number of different values of $v+2ut$ modulo $2(2j+1)$ is $|w|/\gcd(u,w)$; the corresponding Gauss sums are equal up to a phase factor. We now give two applications of formula (B.2).

First, for u and n integers and w odd integer, we obtain

$$S(u, 2n-uw, w) = q^{-(w-1)(w+1)u/8+(w-1)n/2} S(u, 2n-u, w) \tag{B.3}$$

as a particular case of (B.2). In fact, one can show that for w odd, (B.3) is equivalent to the general formula (B.2).

As a second application of (B.2), note that if there exists t such that $ut+v \equiv 0$ modulo w, then (B.2) yields

$$S(u,v,w) = \pm S(u,-v,w). \tag{B.4}$$

The minus sign may occur solely when u, v, and w are even. To see when it effectively occurs, let us consider the equation in t

$$ut + v \equiv w \pmod{2w} \tag{B.5}$$

and let v_2 be the 2-valuation of integers. If $v_2(u) \leq v_2(w)$ and (B.5) has an odd solution, or if $v_2(u) \geq v_2(w) + 1$ and (B.5) has a solution, then there is a minus sign. Otherwise there is a plus sign. For fixed u and w, the number of values of v for which (B.4) appears with a minus sign is $|w|/\gcd(u, w)$. Now, by using again the translation method, we can show that

$$S(u, v, w) = \sum_{k=-|w|+1}^{0} q^{(uk^2+vk)/2} = S(u, -v, w), \tag{B.6}$$

a result that also follows by applying twice the reciprocity theorem [44] for generalized quadratic Gauss sums whenever $u \neq 0$. A comparison of (B.4) and (B.6) shows that certain Gauss sums $S(u, v, w)$ vanish. One may check this numerically for ($u = 2$; $v = 2, 6, 10, 14$; $w = 8$) and ($u = 4$; $v = 2, 6, 10$; $w = 6$).

C On a Gaussian sum

As a corollary of Proposition 8, the sum rule

$$\left| \sum_{k=0}^{d-1} e^{i\pi[k(d-k)\lambda + 2k\mu]/d} \right| = \sqrt{d} \tag{C.1}$$

holds for d prime, $|\lambda| = 1, 2, \ldots, d-1$ and $|\mu| = 0, 1, \ldots, d-1$. The proof of (C.1) follows from the introduction of $k = j + m$ and $d = 2j + 1$ in (3.20), (3.7), and (3.24). Equation (C.1) can be derived also by adapting the results of Exercises 23 (p. 47) and 12 (p. 44) of [44] (a result kindly communicated to the authors by B.C. Berndt and R.J. Evans).

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References

[1] Melvin M.A., Simplification in finding symmetry-adapted eigenfunctions, *Rev. Modern Phys.* **28** (1956), 18–44.

[2] Altmann S.L., Cracknell A.P., Lattice harmonics I. Cubic groups, *Rev. Modern Phys.* **37** (1965), 19–32.

[3] Altmann S.L., Bradley C.J., Lattice harmonics II. Hexagonal close-packed lattice, *Rev. Modern Phys.* **37** (1965), 33–45.

[4] Kibler M., Ionic and paramagnetic energy levels: algebra. I, *J. Mol. Spectrosc.* **26** (1968), 111–130.

[5] Kibler M., Energy levels of paramagnetic ions: algebra. II, *Int. J. Quantum Chem.* **3** (1969), 795–822.

[6] Michelot F., Moret-Bailly J., Expressions algébriques approchées de symboles de couplage et de fonctions de base adaptés à la symétrie cubique, *J. Phys. (Paris)* **36** (1975), 451—460.

[7] Champion J.P., Pierre G., Michelot F., Moret-Bailly J., Composantes cubiques normales des tenseurs sphériques, *Can. J. Phys.* **55** (1977), 512–520.

[8] Patera J., Winternitz P., A new basis for the representations of the rotation group. Lamé and Heun polynomials, *J. Math. Phys.* **14** (1973), 1130–1139.

[9] Patera J., Winternitz P., On bases for irreducible representations of O(3) suitable for systems with an arbitrary finite symmetry group, *J. Chem. Phys.* **65** (1976), 2725–2731.

[10] Michel L., Invariants polynomiaux des groupes de symétrie moléculaire et cristallographique, in Group Theoretical Methods in Physics, Editors R.T. Sharp and B. Kolman, Academic Press, New York, 1977, 75–91.

[11] Kibler M.R., Planat M., A SU(2) recipe for mutually unbiased bases, *Internat. J. Modern Phys. B* **20** (2006), 1802–1807, quant-ph/0601092.

[12] Arik M., Coon D.D., Hilbert spaces of analytic functions and generalized coherent states, *J. Math. Phys.* **17** (1976), 524–527.

[13] Biedenharn L.C., The quantum group $\mathrm{SU}_q(2)$ and a q-analogue of the boson operators, *J. Phys. A: Math. Gen.* **22** (1989), L873–L878.

[14] Sun C.-P., Fu H.-C., The q-deformed boson realisation of the quantum group $\mathrm{SU}(n)_q$ and its representations, *J. Phys. A: Math. Gen.* **22** (1989), L983–L986.

[15] Macfarlane A.J., On q-analogues of the quantum harmonic oscillator and the quantum group $\mathrm{SU}(2)_q$, *J. Phys. A: Math. Gen.* **22** (1989), 4581–4588.

[16] Daoud M., Kibler M., Fractional supersymmetric quantum mechanics as a set of replicas of ordinary supersymmetric quantum mechanics, *Phys. Lett. A* **321** (2004), 147–151, math-ph/0312019.

[17] Daoud M., Hassouni Y., Kibler M., The k-fermions as objects interpolating between fermions and bosons, in Symmetries in Science X, Editors B. Gruber and M. Ramek, Plenum Press, New York, 1998, 63–67, quant-ph/9710016.

[18] Daoud M., Hassouni Y., Kibler M., Generalized supercoherent states, *Phys. Atom. Nuclei* **61** (1998), 1821–1824, quant-ph/9804046.

[19] Daoud M., Kibler M.R., Fractional supersymmetry and hierarchy of shape invariant potentials, *J. Math. Phys.* **47** (2006), 122108, 11 pages, quant-ph/0609017.

[20] Kibler M., Daoud M., Variations on a theme of quons: I. A non standard basis for the Wigner–Racah algebra of the group SU(2), *Recent Res. Devel. Quantum Chem.* **2** (2001), 91–99.

[21] Daoud M., Kibler M., Fractional supersymmetric quantum mechanics, *Phys. Part. Nuclei (Suppl. 1)* **33** (2002), S43–S51.

[22] Schwinger J., On angular momentum, in Quantum Theory of Angular Momentum, Editors L.C. Biedenharn and H. van Dam, Academic Press, New York, 1965, 229–279.

[23] Lévy-Leblond J.-M., Azimuthal quantization of angular momentum, *Rev. Mex. Fís.* **22** (1973), 15–23.

[24] Chaichian M., Ellinas D., On the polar decomposition of the quantum SU(2) algebra, *J. Phys. A: Math. Gen.* **23** (1990), L291–L296.

[25] Ellinas D., Phase operators via group contraction, *J. Math. Phys.* **32** (1991), 135–141.

[26] Ellinas D., Quantum phase angles and su(∞), *J. Mod. Opt.* **38** (1991), 2393–2399.

[27] Ellinas D., Floratos E.G., Prime decomposition and correlation measure of finite quantum systems, *J. Phys. A: Math. Gen.* **32** (1999), L63–L69, quant-ph/9806007.

[28] Fairlie D.B., Fletcher P., Zachos C.K., Infinite-dimensional algebras and a trigonometric basis for the classical Lie algebras, *J. Math. Phys.* **31** (1990), 1088–1094.

[29] Patera J., Zassenhaus H., The Pauli matrices in n dimensions and finest gradings of simple Lie algebras of type A_{n-1}, *J. Math. Phys.* **29** (1988), 665–673.

[30] Weyl H., The theory of groups and quantum mechanics, Dover Publications, New York, 1950.

[31] Delsarte P., Goethals J.M., Seidel J.J., Bounds for systems of lines and Jacobi polynomials, *Philips Res. Repts.* **30** (1975), 91–105.

[32] Ivanović I.D., Geometrical description of quantum state determination, *J. Phys. A: Math. Gen.* **14** (1981), 3241–3245.

[33] Wootters W.K., Quantum mechanics without probability amplitudes, *Found. Phys.* **16** (1986), 391–405.

[34] Wootters W.K., A Wigner-function formulation of finite-state quantum mechanics, *Ann. Phys. (N.Y.)* **176** (1987), 1–21.

[35] Wootters W.K., Fields B.D., Optimal state-determination by mutually unbiased measurements, *Ann. Phys. (N.Y.)* **191** (1989), 363–381.

[36] Gibbons K.S., Hoffman M.J., Wootters W.K., Discrete phase space based on finite fields, *Phys. Rev. A* **70** (2004), 062101, 23 pages, quant-ph/0401155.

[37] Wootters W.K., Picturing qubits in phase space, *IBM J. Res. Dev.* **48** (2004), 99, 12 pages, quant-ph/0306135.

[38] Wootters W.K., Quantum measurements and finite geometry, *Found. Phys.* **36** (2006), 112–126, quant-ph/0406032.

[39] Calderbank A.R., Cameron P.J., Kantor W.M., Seidel J.J., Zopf 4-Kerdock codes orthogonal spreads and extremal euclidean line-sets, *Proc. London Math. Soc.* **75** (1997), 436–480.

[40] Edmonds A.R., Angular momentum in quantum mechanics, Princeton University Press, Princeton, 1960.

[41] Racah G., Theory of complex spectra. II, *Phys. Rev.* **62** (1942), 438–462.

[42] Grenet G., Kibler M., On the operator equivalents, *Phys. Lett. A* **68** (1978), 147–150.

[43] Kibler M., Grenet G., On the SU_2 unit tensor, *J. Math. Phys.* **21** (1980), 422–439.

[44] Berndt B.C., Evans R.J., Williams K.S., Gauss and Jacobi sums, Wiley, New York, 1998.

[45] Bandyopadhyay S., Boykin P.O., Roychowdhury V., Vatan F., A new proof for the existence of mutually unbiased bases, *Algorithmica* **34** (2002), 512–528, quant-ph/0103162.

[46] Lawrence J., Brukner Č., Zeilinger A., Mutually unbiased binary observable sets on N qubits, *Phys. Rev. A* **65** (2002), 032320, 5 pages, quant-ph/0104012.

[47] Klimov A.B., Sánchez-Soto L.L., de Guise H., Multicomplementary operators via finite Fourier transform, *J. Phys. A: Math. Gen.* **38** (2005), 2747–2760, quant-ph/0410155.

[48] Klimov A.B., Sánchez-Soto L.L., de Guise H., A complementary-based approach to phase in finite-dimensional quantum systems, *J. Opt. B: Quantum Semiclass. Opt.* **7** (2005), 283–287, quant-ph/0410135.

[49] Romero J.L., Björk G., Klimov A.B., Sánchez-Soto L.L., Structure of the sets of mutually unbiased bases for N qubits, *Phys. Rev. A* **72** (2005), 062310, 8 pages, quant-ph/0508129.

[50] Le Bellac M., Physique quantique, EDP Sciences/CNRS Éditions, Paris, 2003.

[51] Chaturvedi S., Aspects of mutually unbiased bases in odd-prime-power dimensions, *Phys. Rev. A* **65** (2002), 044301, 3 pages, quant-ph/0109003.

[52] Pittenger A.O., Rubin M.H., Mutually unbiased bases, generalized spin matrices and separability, *Linear Algebr. Appl.* **390** (2004), 255–278, quant-ph/0308142.

[53] Pittenger A.O., Rubin M.H., Wigner functions and separability for finite systems, *J. Phys. A: Math. Gen.* **38** (2005), 6005–6036, quant-ph/0501104.

[54] Saniga M., Planat M., Rosu H., Mutually unbiased bases and finite projective planes, *J. Opt. B: Quantum Semiclass. Opt.* **6** (2004), L19–L20, math-ph/0403057.

[55] Saniga M., Planat M., Sets of mutually unbiased bases as arcs in finite projectives planes, *Chaos Solitons Fractals* **26** (2005), 1267–1270, quant-ph/0409184.

[56] Saniga M., Planat M., Hjelmslev geometry of mutually unbiased bases, *J. Phys. A: Math. Gen.* **39** (2006), 435–440, math-ph/0506057.

[57] Planat M., Rosu H., Mutually unbiased phase states, phase uncertainties, and Gauss sums, *Eur. Phys. J. D* **36** (2005), 133–139.

[58] Planat M., Saniga M., Galois algebras of squeezed quantum phase states, *J. Opt. B: Quantum Semiclass. Opt.* **7** (2005), S484–S489.

[59] Saniga M., Planat M., Finite geometries in quantum theory: from Galois (fields) to Hjelmslev (rings), *Internat. J. Modern Phys. B* **20** (2006), 1885–1892.

[60] Vourdas A., SU(2) and SU(1,1) phase states, *Phys. Rev. A* **41** (1990), 1653–1661.

[61] Vourdas A., The angle-angular momentum quantum phase space, *J. Phys. A: Math. Gen.* **29** (1996), 4275–4288.

[62] Vourdas A., Quantum systems with finite Hilbert space, *Rep. Prog. Phys.* **67** (2004), 267–320.

[63] Vourdas A., Galois quantum systems, *J. Phys. A: Math. Gen.* **38** (2005), 8453–8471.

[64] Vourdas A., Galois quantum systems, irreducible polynomials and Riemann surfaces, *J. Math. Phys.* **47** (2006), 092104, 15 pages.

[65] Klappenecker A., Rötteler M., Constructions of mutually unbiased bases, *Lect. Notes Comput. Sci.* **2948** (2004), 137–144, quant-ph/0309120.

[66] Klappenecker A., Rötteler M., Shparlinski I.E., Winterhof A., On approximately symmetric informationally complete positive operator-valued measures and related systems of quantum states, *J. Math. Phys.* **46** (2005), 082104, 17 pages, quant-ph/0503239.

[67] Grassl M., On SIC-POVMs and MUBs in dimension 6, in Proceedings ERATO Conference on Quantum Information Science (EQIS 2004), Editor J. Gruska, Tokyo, 2005, 60–61, quant-ph/0406175.

[68] Durt T., If 1=2+3, then 1=2.3: Bell states, finite groups, and mutually unbiased bases, a unifying approach, *J. Phys. A: Math. Gen.* **38** (2005), 5267–5283, quant-ph/0401046.

[69] Colin S., Corbett J., Durt T., Gross D., About SIC POVMs and discrete Wigner distributions, *J. Opt. B: Quantum Semiclass. Opt.* **7** (2005), S778–S785.

[70] Durt T., About the mean king's problem and discrete Wigner distributions, *Internat. J. Modern Phys. B* **20** (2006), 1742–1760.

[71] Wocjan P., Beth T., New construction of mutually unbiased bases in square dimensions, *Quantum Inf. Comput.* **5** (2005), 93–101, quant-ph/0407081.

[72] Archer C., There is no generalization of known formulas for mutually unbiased bases, *J. Math. Phys.* **46** (2005), 022106, 11 pages, quant-ph/0312204.

[73] Bengtsson I., MUBs, polytopes, and finite geometries, quant-ph/0406174.

[74] Bengtsson I., Ericsson Å., Mutually unbiased bases and the complementary polytope, *Open Syst. Inf. Dyn.* **12** (2005), 107–120, quant-ph/0410120.

[75] Bengtsson I., Bruzda W., Ericsson Å., Larsson J.-Å., Tadej W., Życkowski K., Mubs and Hadamards of order six, quant-ph/0610161.

[76] Bengtsson I., Three ways to look at mutually unbiased bases, quant-ph/0610216.

[77] Boykin P.O., Sitharam M., Tiep P.H., Wocjan P., Mutually unbiased bases and orthogonal decompositions of Lie algebras, quant-ph/0506089.

[78] Hayashi A., Horibe M., Hashimoto T., Mean king's problem with mutually unbiased bases and orthogonal Latin squares, *Phys. Rev. A* **71** (2005), 052331, 4 pages, quant-ph/0502092.