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Skew Divided Difference Operators and Schubert Polynomials*

Anatol N. KIRILLOV

Research Institute of Mathematical Sciences (RIMS), Sakyo-ku, Kyoto 606-8502, Japan
E-mail: *kirillov@kurims.kyoto-u.ac.jp*
URL: http://www.kurims.kyoto-u.ac.jp/~kirillov/

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Abstract. We study an action of the skew divided difference operators on the Schubert polynomials and give an explicit formula for structural constants for the Schubert polynomials in terms of certain weighted paths in the Bruhat order on the symmetric group. We also prove that, under certain assumptions, the skew divided difference operators transform the Schubert polynomials into polynomials with positive integer coefficients.

Key words: divided differences; nilCoxeter algebras; Schubert polynomials

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Dedicated to the memory of Vadim Kuznetsov

1 Introduction

In this paper we study the skew divided difference operators with applications to the "Littlewood–Richardson problem" in the Schubert calculus. By the Littlewood–Richardson problem in the Schubert calculus we mean the problem of finding a combinatorial rule for computing what one calls the structural constants for Schubert polynomials. These are the structural constants c_{uv}^w, $u, v, w \in S_n$, of the ring P_n/I_n, where P_n is the polynomial ring $\mathbf{Z}[x_1, \dots, x_n]$ and I_n is the ideal of P_n generated by the symmetric polynomials without constant terms, with respect to its $\mathbf{Z}$-free basis consisting of the classes of Schubert polynomials $\mathfrak{S}_w$, $w \in S_n$. Namely, the constants c_{uv}^w are defined via the decomposition of the product of two Schubert polynomials $\mathfrak{S}_u$ and $\mathfrak{S}_v$ modulo the ideal I_n:

$$\mathfrak{S}_v \mathfrak{S}_u \equiv \sum_{w \in S_n} c_{uv}^w \mathfrak{S}_w \ (\mathrm{mod}\ I_n). \tag{1.1}$$

Up to now such a rule is known in the case when u, v, w are the Grassmannian permutations (see, e.g., [11, p. 13] and [12, Chapter I, Section 9]) – this is the famous Littlewood–Richardson rule for Schur functions – and in some special cases, see e.g. [7, 8].

The skew divided difference operators were introduced by I. Macdonald in [11]. The simplest way to define the skew divided difference operators is based on the Leibniz rule for the divided difference operators ∂_w, $w \in S_n$, namely,

$$\partial_w(fg) = \sum_{w \succeq v} \left(\partial_{w/v} f\right) \partial_v g. \tag{1.2}$$

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The symbol $w \succeq v$ for $w, v \in S_n$, here and after, means that w dominates v with respect to the Bruhat order on the symmetric group S_n (see, e.g., [11, p. 6]). Formula (1.2) is reduced to the classical Leibnitz rule in the case when $w = (i, i+1)$ is a simple transposition:

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i g.$$

One of the main applications of the skew divided difference operators is an elementary and transparent algebraic proof of the Monk formula for Schubert polynomials (see [11, equation (4.15)]).

Our interest to the skew divided difference operators is based on their connection with the structural constants for Schubert polynomials. More precisely, if $w, v \in S_n$, and $w \succeq v$, then

$$\partial_{w/v}\left(\mathfrak{S}_u\right)|_{x=0} = c_{uv}^{w}. \tag{1.3}$$

The polynomial $\partial_{w/v}(\mathfrak{S}_u)$ is a homogeneous polynomial in $x_1, \ldots, x_n$ of degree $l(u)+l(v)-l(w)$ with integer coefficients. We make a conjecture that in fact

$$\partial_{w/v}(\mathfrak{S}_u) \in \mathbf{N}[x_1, \ldots, x_n], \tag{1.4}$$

i.e. the polynomial $\partial_{w/v}(\mathfrak{S}_u)$ has nonnegative integer coefficients. In the case $l(u)+l(v) = l(w)$, this conjecture follows from the geometric interpretation of the structural constants c_{uv}^{w} as the intersection numbers for Schubert cycles. For general $u, v, w \in S_n$ the conjecture is still open.

In Section 8 we prove conjecture (1.4) in the simplest nontrivial case (see Theorem 1) when w and v are connected by an edge in the Bruhat order on the symmetric group S_n. In other words, if $w = vt_{ij}$, where t_{ij} is the transposition that interchanges i and j, and $l(w) = l(vt_{ij})+1$. It is well-known [11, p. 30] that in this case the skew divided difference operator $\partial_{w/v}$ coincides with operator ∂_{ij}, i.e. $\partial_{w/v} = \partial_{ij}$. Our proof employs the generating function for Schubert polynomials ("Schubert expression" [5, 3, 4]) in the nilCoxeter algebra.

In Section 9 we consider another application of the skew divided difference operators, namely, we give an explicit (but still not combinatorial) formula for structural constants for Schubert polynomials in terms of weighted paths in the Bruhat order with weights taken from the nil-Coxeter algebra (see Theorem 2).

It is well known that there are several equivalent ways to define the skew Schur functions, see e.g., [11, 12]. Apart from the present paper, a few different definitions of skew Schubert polynomials have been proposed in [1, 10] and [2]. These definitions produce, in general, different polynomials.

2 Skew Schur functions

In this Section we review the definition and basic properties of the skew Schur functions. For more details and proofs, see [12, Chapter I, Section 5]. The main goal of this Section is to arise a problem of constructing skew Schubert polynomials with properties "similar" to the those for skew Schur functions (see properties (2.2)–(2.5) below).

Let $X_n = (x_1, \ldots, x_n)$ be a set of independent variables, and λ, μ be partitions, $\mu \subset \lambda$, $l(\lambda) \le n$.

Definition 1. The skew Schur function $s_{\lambda/\mu}(X_n)$ corresponding to the skew shape $\lambda - \mu$ is defined to be

$$s_{\lambda/\mu}(X_n) = \det\left(h_{\lambda_i-\mu_j-i+j}\right)_{1\le i,j\le n}, \tag{2.1}$$

where $h_k := h_k(X_n)$ is the complete homogeneous symmetric function of degree k in the variables $X_n = (x_1, \ldots, x_n)$.

Below we list the basic properties of skew Schur functions:

a) Combinatorial formula:

$$s_{\lambda/\mu}(X_n) = \sum_T x^{w(T)}, \tag{2.2}$$

where summation is taken over all semistandard tableaux T of the shape $\lambda - \mu$ with entries not exceeding n; here $w(T)$ is the weight of the tableau T (see, e.g., [12, p. 5]), and $x^{w(T)} := x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}$.

b) Connection with structural constants for Schur functions:

$$s_{\lambda/\mu} = \sum_{\nu,\ l(\nu)\leq n} c_{\mu\nu}^{\lambda} s_\nu, \tag{2.3}$$

where the coefficients $c_{\mu\nu}^{\lambda}$ (the structural constants, or the Littlewood–Richardson numbers) are defined through the decomposition

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^{\lambda} s_\lambda. \tag{2.4}$$

c) Littlewood–Richardson rule:

$$s_{\lambda/\mu} = \sum_{\nu, l(\nu)\leq n} |\mathrm{Tab}^0(\lambda-\mu,\nu)| s_\nu, \tag{2.5}$$

where $\mathrm{Tab}^0(\lambda-\mu,\nu)$ is the set of all semistandard tableaux T of shape $\lambda-\mu$ and weight ν such that the reading word $\mathrm{w}(T)$ of the tableaux T (see, e.g., [12, Chapter I, Section 9]) is a lattice word (ibid). Thus,

$$\mathrm{Mult}_{V_\lambda}(V_\nu \otimes V_\mu) = c_{\nu\mu}^{\lambda} = |\mathrm{Tab}^0(\lambda-\mu,\nu)|. \tag{2.6}$$

3 Divided difference operators

Definition 2. Let f be a function of x and y (and possibly other variables), the divided difference operators ∂_{xy} is defined to be

$$\partial_{xy} f = \frac{f(x,y) - f(y,x)}{x-y}. \tag{3.1}$$

The operator ∂_{xy} takes polynomials to polynomials and has degree -1. On a product fg, ∂_{xy} acts according to the Leibniz rule

$$\partial_{xy}(fg) = (\partial_{xy} f)g + (s_{xy} f)(\partial_{xy} g), \tag{3.2}$$

where s_{xy} interchanges x and y.

It is easy to check the following properties of divided difference operators ∂_{xy}:

a) $\partial_{xy} s_{xy} = -\partial_{xy}, \qquad s_{xy}\partial_{xy} = \partial_{xy},$

b) $\partial_{xy}^2 = 0,$

c) $\partial_{xy}\partial_{yz}\partial_{xy} = \partial_{yz}\partial_{xy}\partial_{yz},$

d) $\partial_{xy}\partial_{yz} = \partial_{xz}\partial_{xy} + \partial_{yz}\partial_{xz}.$

The next step is to define a family of divided difference operators ∂_i, $1 \leq i \leq n-1$, which act on the ring of polynomials in n variables.

Let $x_1, x_2, \ldots, x_n$ be independent variables, and let

$$P_n = \mathbf{Z}[x_1, \ldots, x_n].$$

For each i, $1 \le i \le n-1$, let

$$\partial_i = \partial_{x_i, x_{i+1}},$$

be the divided difference operator corresponding to the simple transposition $s_i = (i, i+1)$ which interchanges x_i and x_{i+1}.

Each ∂_i is a linear operator on P_n of degree -1. The divided difference operators ∂_i, $1 \le i \le n-1$, satisfy the following relations

$$\begin{array}{ll} i) & \partial_i^2 = 0, \quad \text{if} \quad 1 \le i \le n-1, \\ ii) & \partial_i \partial_j = \partial_j \partial_i, \quad \text{if} \quad 1 \le i, j \le n-1, \quad \text{and} \quad |i-j| > 1, \\ iii) & \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}, \quad \text{if} \quad 1 \le i \le n-2. \end{array}$$

Let $w \in S_n$ be a permutation; then w can be written as a product of simple transpositions $s_i = (i, i+1)$, $1 \le i \le n-1$, namely,

$$w = s_{i_1} \cdots s_{i_p}.$$

Such a representation (or the sequence $(i_1, \ldots, i_p)$) is called a reduced decomposition of w, if $p = l(w)$, where $l(w)$ is the length of w. For each $w \in S_n$, let $R(w)$ denote the set of all reduced decompositions of w, i.e. the set of all sequences $(i_1, \ldots, i_p)$ of length $p = l(w)$ such that $w = s_{i_1} \cdots s_{i_p}$.

For any sequence $\mathbf{a} = (a_1, \ldots, a_p)$ of positive integers, let us define $\partial_{\mathbf{a}} = \partial_{a_1} \cdots \partial_{a_p}$.

Proposition 1 ([11], Chapter II).

i) If a sequence $\mathbf{a} = (a_1, \ldots, a_p)$ is not reduced, i.e. not a reduced decomposition of any $w \in S_n$, then $\partial_{\mathbf{a}} = 0$.

ii) If $\mathbf{a}, \mathbf{b} \in R(w)$ then $\partial_{\mathbf{a}} = \partial_{\mathbf{b}}$.

From Proposition 1, *ii*) follows that one can define $\partial_w = \partial_{\mathbf{a}}$ unambiguously, where $\mathbf{a}$ is any reduced decomposition of w.

4 Schubert polynomials

In this section we recall the definition and basic properties of the Schubert polynomials introduced by A. Lascoux and M.-P. Schützenberger. Further details and proofs can be found in [11].

Let $\delta = \delta_n = (n-1, n-2, \ldots, 1, 0)$, so that

$$x^\delta = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

Definition 3 ([9]). For each permutation $w \in S_n$ the Schubert polynomial $\mathfrak{S}_w$ is defined to be

$$\mathfrak{S}_w = \partial_{w^{-1} w_0}(x^\delta), \tag{4.1}$$

where w_0 is the longest element of S_n.

Proposition 2 ([9, 11]).

i) For each permutation $w \in S_n$, $\mathfrak{S}_w$ is a polynomial in $x_1, \ldots, x_{n-1}$ of degree $l(w)$ with positive integer coefficients.

ii) Let $v, w \in S_n$. Then

$$\partial_v \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{wv^{-1}}, & if \quad l(wv^{-1}) = l(w) - l(v), \\ 0, & otherwise. \end{cases}$$

iii) The Schubert polynomials $\mathfrak{S}_w$, $w \in S_n$, form a $\mathbf{Z}$-linear basis in the space $\mathcal{F}_n$, where

$$\mathcal{F}_n = \left\{ f \in P_n \mid f = \sum_{\alpha \subset \delta} c_\alpha x^\alpha \right\}.$$

iv) The Schubert polynomials $\mathfrak{S}_w$, $w \in S_n$, form an orthogonal basis with respect to the pairing $\langle \ , \ \rangle_0$:

$$\langle \mathfrak{S}_w, \mathfrak{S}_u \rangle_0 = \begin{cases} 1, & if \quad u = w_0 w, \\ 0, & otherwise, \end{cases}$$

where by definition $\langle f, g \rangle_0 = \eta(\partial_{w_0}(fg)) := \partial_{w_0}(fg)|_{x=0}$, and $\eta(h) = h|_{x_1 = \cdots = x_n = 0}$ for any polynomial h in the variables $x_1, \ldots, x_n$.

v) (Stability) Let $m > n$ and let $i : S_n \hookrightarrow S_m$ be the natural embedding. Then

$$\mathfrak{S}_w = \mathfrak{S}_{i(w)}.$$

5 Skew divided difference operators

The skew divided difference operators $\partial_{w/v}$, $w, v \in S_n$, were introduced by I. Macdonald [11, Chapter II].

Let $w, v \in S_n$, and $w \succeq v$ with respect to the Bruhat order $\succeq$ on the symmetric group S_n. In other words, if $\mathbf{a} = (a_1, \ldots, a_p)$ is a reduced decomposition of w then there exists a subsequence $\mathbf{b} \subset \mathbf{a}$ such that $\mathbf{b}$ is a reduced decomposition of v (for more details, see, e.g., [11, equation (1.17)].

Definition 4 ([11]). Let $v, w \in S_n$, and $w \succeq v$ with respect to the Bruhat order, and $\mathbf{a} = (a_1, \ldots a_p) \in R(w)$. The skew divided difference operator $\partial_{w/v}$ is defined to be

$$\partial_{w/v} = v^{-1} \sum_{\mathbf{b} \subset \mathbf{a},\ \mathbf{b} \in R(v)} \phi(\mathbf{a}, \mathbf{b}), \tag{5.1}$$

where

$$\phi(\mathbf{a}, \mathbf{b}) = \prod_{i=1}^{p} \phi_i(\mathbf{a}, \mathbf{b}), \phi_i(\mathbf{a}, \mathbf{b}) = \begin{cases} s_{a_i}, & if \quad a_i \in \mathbf{b}, \\ \partial_{a_i}, & if \quad a_i \notin \mathbf{b}. \end{cases}$$

One can show (see, e.g., [11, p. 29]) that Definition 4 is independent of the reduced decomposition $\mathbf{a} \in R(w)$.

Below we list the basic properties of the skew divided difference operators $\partial_{w/v}$. For more details and proofs, see, e.g., [11]. The statement iv) of Proposition 3 below seems to be new.

Proposition 3.

i) Let $f, g \in P_n$, $w \in S_n$, then

$$\partial_w(fg) = \sum_{w \succeq v} (\partial_{w/v} f) \partial_v g. \tag{5.2}$$

More generally,

ii) Let $f, g \in P_n$, $u, w \in S_n$, and $w \succeq u$ with respect to the the Bruhat order. Then

$$\partial_{w/u}(fg) = \sum_{w \succeq v \succeq u} u^{-1} v (\partial_{w/v} f) \partial_{v/u} g \tag{5.3}$$

(generalized Leibnitz' rule).

iii) Let $w = vt$, where $l(w) = l(v) + 1$, and $t = t_{ij}$ is the transposition that interchanges i and j and fixes all other elements of $[1, n]$. Then

$$\partial_{w/v} = \partial_{ij}, \tag{5.4}$$

where $\partial_{ij} := \partial_{x_i x_j}$.

iv) Let w_0 be the longest element of S_n. Then

$$w_0 v \partial_{w_0/v} = \partial_{w_0 v}. \tag{5.5}$$

v) Let $u, v, w \in S_n$, $w \succeq u$, and $l(w) = l(u) + l(v)$. Then

$$\partial_{w/u} \mathfrak{S}_v = c_{uv}^w, \tag{5.6}$$

where c_{uv}^w are the structural constants for the Schubert polynomials $\mathfrak{S}_w$, $w \in S_n$; in other words,

$$\mathfrak{S}_u \mathfrak{S}_v \equiv \sum_{w \in S_n} c_{uv}^w \mathfrak{S}_w \pmod{I_n},$$

where I_n is the ideal generated by the elementary symmetric functions $e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)$.

Proof. We refer the reader to [11, p. 30] for proofs of statements i)–iii).

iv) To prove the identity (5.5), we will use the formula (5.2) and the following result due to I. Macdonald [11, equation (5.7)]):

$$\partial_{w_0}(fg) = \sum_{w \in S_n} \epsilon(w) \partial_w(w_0 f) \partial_{w w_0}(g), \tag{5.7}$$

where for each permutation $w \in S_n$, $\epsilon(w) = (-1)^{l(w)}$ is the sign (signature) of w.

Using the generalized Leibnitz formula (5.2), we can write the LHS (5.7) as follows:

$$\partial_{w_0}(fg) = \sum_{w_0 \succeq v} v(\partial_{w_0/v} f) \partial_v(g). \tag{5.8}$$

Comparing the RHS of (5.7) and that of (5.8), we see that

$$v(\partial_{w_0/v} f) = \epsilon(v w_0) \partial_{v w_0}(w_0 f).$$

To finish the proof of equality (5.5), it remains to apply the following formula [11, equation (2.12)]:

$$\partial_{w_0 w w_0} = \epsilon(w) w_0 \partial_w w_0.$$

v) We consider formula (5.6) as a starting point for applications of the skew divided difference operators to the problem of finding a combinatorial formula for the structural constants c_{uv}^w ("Littlewood–Richardson problem" for Schubert polynomials, see Section 2). Having in mind some applications of (5.6) (see Sections 8 and 9), we reproduce below the proof of (5.6) given by I. Macdonald [11, p. 112]. It follows from Proposition 2 *ii*) and Proposition 3 *i*) that

$$c_{uv}^w = \partial_w(\mathfrak{S}_u\mathfrak{S}_v) = \sum_{w \succeq v_1} v_1 \left(\partial_{w/v_1}\mathfrak{S}_u\right) \partial_{v_1}\mathfrak{S}_v.$$

In the latter sum the only nonzero term appears when $v_1 = v$. Hence,

$$c_{uv}^w = \partial_w(\mathfrak{S}_u\mathfrak{S}_v) = v\partial_{w/v}(\mathfrak{S}_u) = \partial_{w/v}(\mathfrak{S}_u),$$

since $\deg \partial_{w/v}(\mathfrak{S}_u) = 0$. ■

It is well-known (and follows, for example, from Proposition 2, *i*) and *ii*)) that for each $v, w \in S_n$

$$\partial_v\left(\mathfrak{S}_w\right) \in \mathbf{N}[x_1, \dots, x_{n-1}].$$

More generally, we make the following conjecture.

Conjecture 1. *For any $u, v, w \in S_n$,*

$$\partial_{w/u}\mathfrak{S}_v \in \mathbf{N}[X_n],$$

i.e. $\partial_{w,u}(\mathfrak{S}_v)$ is a polynomial in $x_1, \dots, x_n$ with nonnegative integer coefficients.

Example 1. Take $w = s_2s_1s_3s_2s_1 \in S_4$, $v = s_2s_1 \in S_4$, and $\mathbf{a} = (2,1,3,2,1) \in R(w)$. There are three possibilities to choose $\mathbf{b}$ such that $\mathbf{b} \subset \mathbf{a}$, $\mathbf{b} \in R(v)$, namely, $\mathbf{b} = (2,1,\cdot,\cdot,\cdot)$, $\mathbf{b} = (2,\cdot,\cdot,\cdot,1)$ and $\mathbf{b} = (\cdot,\cdot,\cdot,2,1)$. Hence,

$$\begin{aligned}\partial_{w/v} &= s_1s_2s_2s_1\partial_3\partial_2\partial_1 + s_1s_2s_2\partial_1\partial_3\partial_2s_1 + s_1s_2\partial_2\partial_1\partial_3s_2s_1\\ &= \partial_3\partial_2\partial_1 - \partial_1\partial_3\partial_{13} - \partial_{13}\partial_2\partial_{14}.\end{aligned}$$

Using this expression for the divided difference operator $\partial_{w/v}$, one can find

a) $\partial_{w/v}(x_1^3x_2^2) = x_1^2 + x_1x_4 + x_4^2 \equiv x_2x_3 \pmod{I_4}$.

Thus,

$$\partial_{w/v}(x_1^3x_2^2) \equiv \mathfrak{S}_{23} - \mathfrak{S}_{13} + \mathfrak{S}_{21} \pmod{I_4}.$$

Here $\mathfrak{S}_{23}$ means $\mathfrak{S}_{s_2s_3}$, not $\mathfrak{S}_{(2,4)}$. Similar remarks apply to other similar symbols here and after.

We used the following formulae for Schubert polynomials:

$$\mathfrak{S}_{23} = x_1x_2 + x_1x_3 + x_2x_3, \qquad \mathfrak{S}_{13} = x_1^2 + x_1x_2 + x_1x_3, \qquad \mathfrak{S}_{12} = x_1^2.$$

b) $\partial_{w/v}(x_1^3x_2^2x_3) \equiv x_2^2x_3 \pmod{I_4}$,

and

$$\partial_{w/v}(x_1^3x_2^2x_3) \equiv \mathfrak{S}_{121} + \mathfrak{S}_{232} - \mathfrak{S}_{123} - \mathfrak{S}_{213} - \mathfrak{S}_{312} \pmod{I_4}.$$

c) $\partial_{13}(x_1^3x_2x_3) = x_1x_2x_3(x_1 + x_3) \equiv -x_1x_2^2x_3 \pmod{I_4}$.

Let us note that $x_1^3x_2x_3 = \mathfrak{S}_{12321}$.

These examples show that in general

- the "intersection" numbers $\langle \partial_{w/v}(\mathfrak{S}_u), \mathfrak{S}_\tau \rangle_0$ may have negative values;
- coefficients c_α in the decomposition $\partial_{w/v}(\mathfrak{S}_u) \equiv \sum_{\alpha \subset \delta_n} c_\alpha x^\alpha \pmod{I_n}$ may take negative values.

6 Analog of skew divided differences in the Bracket algebra

In this Section for each $v, w \in S_n$ we construct the element $[w/v]$ in the Bracket algebra $\mathcal{E}_n^0$ which is an analog of the skew divided difference operators $\partial_{w/v}$. The Bracket algebra $\mathcal{E}_n^0$ was introduced in [4]. By definition, the Bracket algebra $\mathcal{E}_n^0$ (of type A_{n-1}) is the quadratic algebra (say, over $\mathbf{Z}$) with generators $[ij]$, $1 \le i < j \le n$, which satisfy the following relations

$$\begin{array}{ll} (i) & [ij]^2 = 0, \quad \text{for } \ i<j; \\ (ii) & [ij][jk] = [jk][ik] + [ik][ij], \quad [jk][ij] = [ik][jk] + [ij][ik], \quad \text{for } \ i<j<k; \\ (iii) & [ij][kl] = [kl][ij] \quad \text{whenever} \quad \{i,j\} \cap \{k,l\} = \varnothing, \quad i<j \ \text{ and } \ k<l. \end{array}$$

For further details, see [4, 6].

Note that $[ij] \to \partial_{ij}$, $1 \le i < j \le n$, defines a representation of the algebra $\mathcal{E}_n^0$ in P_n.

Now, let $v, w \in S_n$, and $w \succeq v$ with respect to the Bruhat order on S_n. Let $\mathbf{a} \in R(w)$ be a reduced decomposition of w. We define the element $[w/v]$ in the Bracket algebra $\mathcal{E}_n^0$ to be

$$[w/v] = v^{-1} \sum_{\mathbf{b} \subset \mathbf{a}, \ \mathbf{b} \in R(v)} \phi(\mathbf{a}, b),$$

where

$$\phi(\mathbf{a}, \mathbf{b}) = \prod_i \phi_i(\mathbf{a}, \mathbf{b}), \qquad \text{and} \qquad \phi_i = \begin{cases} s_{a_i}, & a_i \in \mathbf{b}, \\ [a_i \quad a_i+1], & a_i \notin \mathbf{b}. \end{cases}$$

Note that the right-hand side of the definition of $[w/v]$ can be interpreted inside the crossed product of $\mathcal{E}_n^0$ by S_n (which is also called a skew group algebra in this case) with respect to the action of S_n on $\mathcal{E}_n^0$ defined by

$$w \cdot [w/v] = [w(i)w(j)] \qquad (\text{which means} \quad -[w(j)w(i)] \ \text{ if } \ w(i) > w(j)),$$

eventually giving an element of $\mathcal{E}_n^0$.

Remark 1. Let $w, v \in S_n$, and $w \succeq v$. One can show that the element $[w/v] \in \mathcal{E}_n^0$ is independent of the reduced decomposition $\mathbf{a} \in R(w)$.

Conjecture 2. *The element $[w/v] \in \mathcal{E}_n^0$ can be written as a linear combination of monomials in the generators $[ij]$, $i < j$, with nonnegative integer coefficients.*

Example 2. Take $w = s_2s_1s_3s_2s_1 \in S_4$, $v = s_2s_1 \in S_4$. Then

$$\begin{aligned} [w/v] &= [34][23][12] - [12][34][13] - [13][23][14] \\ &= [34][12][13] + [34][13][23] - [12][34][13] - [13][23][14] \\ &= [13][14][23] + [14][34][23] - [13][23][14] = [14][34][23]. \end{aligned}$$

7 Skew Schubert polynomials

Definition 5. Let $v, w \in S_n$, and $w \succeq v$ with respect to the Bruhat order. The skew Schubert polynomial $\mathfrak{S}_{w/v}$ is defined to be

$$\mathfrak{S}_{w/v} = \partial_{v^{-1}w_0/w^{-1}w_0}(x^{\delta_n}). \tag{7.1}$$

Example 3. a) Let $w = s_1s_2s_3s_1 \in S_4$, and $v = s_1 \in S_4$. Then $v^{-1}w_0 = s_2s_1s_3s_2s_1$, $w^{-1}w_0 = s_2s_1$, and

$$\begin{aligned}\mathfrak{S}_{w/v} &= \partial_{21321/21}(x_1^3x_2^2x_3) = (x_1^2 + x_1x_4 + x_4^2)x_2 \\ &\equiv \mathfrak{S}_{121} + \mathfrak{S}_{232} - \mathfrak{S}_{124} - \mathfrak{S}_{213} - \mathfrak{S}_{132} \pmod{I_4}.\end{aligned}$$

b) Take $w = s_3s_2 \in S_4$ and $v = s_3 \in S_4$. Then $v^{-1}w_0 = s_1s_2s_1s_3s_2$, $w^{-1}w_0 = s_1s_2s_3s_2$, and $\partial_{v^{-1}w_0/w^{-1}w_0} = \partial_{13}$. Thus

$$\mathfrak{S}_{w/v} = x_1^2x_2x_3(x_2 + x_3) \equiv -x_1^3x_2x_3 \pmod{I_4}.$$

It is clear that if $w, v \in S_n$, and $w \succeq v$, then $\mathfrak{S}_{w/v}$ is a homogeneous polynomial of degree $\binom{n}{2} - l(w) + l(v)$ with integer coefficients. It would be a corollary of Conjecture 1 that skew the Schubert polynomials have in fact positive integer coefficients.

Proposition 4.
i) Let $v \in S_n$, and $w_0 \in S_n$ be the longest element. Then

$$\mathfrak{S}_{w_0/v} = \mathfrak{S}_v. \tag{7.2}$$

ii) Let $w \in S_n$, then $\mathfrak{S}_{w/1} = w_0ww_0\mathfrak{S}_{ww_0}$.

Proof of (7.2) follows from (5.5) and (7.1).

It is an interesting task to find the Monk formula for skew Schubert polynomials, in other words, to describe the decomposition of the product $(x_1+\cdots+x_r)\mathfrak{S}_{w/v}$, $w, v \in S_n$, $1 \le r \le n-1$, in terms of Schubert polynomials.

8 Proof of Conjecture 1 for divided difference operators ∂_{ij}

First of all we recall the definition of the nilCoxeter algebra NC_n and the construction of the Schubert expression $\mathfrak{S}^{(n+1)} \in \mathbf{N}[x_1,\ldots,x_n][NC_n]$, where $\mathbf{N}[x_1,\ldots,x_n][NC_n]$ denotes the set of all non-negative integral linear combinations of the elements of the form $x_1^{m_1}\cdots x_n^{m_n} \otimes e_w$, $m_1,\ldots,m_n \in \mathbf{N}$, $w \in S_n$, in $\mathbf{Z}[x_1,\ldots,x_n] \otimes_{\mathbf{Z}} NC_n$. Similar remarks apply to similar notation below.

The study of action of divided difference operators ∂_{ij}, $1 \le i < j \le n$, on the Schubert expression $\mathfrak{S}^{(n+1)}$ is the main step of our proof of Conjecture 1 for the skew divided difference operators corresponding to the edges in the Bruhat order on the symmetric group S_{n+1}. In exposition we follow to [5, 3, 4].

Definition 6. The nilCoxeter algebra NC_n is the algebra (say, over $\mathbf{Z}$) with generators e_i, $1 \le i \le n$, which satisfy the following relations

(*i*) $e_i^2 = 0$, for $1 \le i \le n$,

(*ii*) $e_ie_j = e_je_i$, for $1 \le i,j \le n$, $|i-j| > 1$,

(*iii*) $e_ie_je_i = e_je_ie_j$, for $1 \le i,j \le n$, $|i-j| = 1$.

For each $w \in S_{n+1}$ let us define $e_w \in NC_n$ to be $e_w = e_{a_1}\cdots e_{a_p}$, where $(a_1,\ldots,a_p)$ is any reduced decomposition of w. The elements e_w, $w \in S_{n+1}$, are well-defined and form a $\mathbf{Z}$-basis in the nilCoxeter algebra NC_n.

Now we are going to define the Schubert expression $\mathfrak{S}^{(n+1)}$ which is a noncommutative generating function for the Schubert polynomials. Namely,

$$\mathfrak{S}^{(n+1)} = \sum_{w \in S_{n+1}} \mathfrak{S}_w e_w \in \mathbf{N}[x_1, \ldots, x_n][NC_n].$$

The basic property of the Schubert expression $\mathfrak{S}^{(n+1)}$ is that it admits the following factorization [5]:

$$\mathfrak{S}^{(n+1)} = A_1(x_1) \cdots A_n(x_n), \tag{8.1}$$

where $A_i(x) = \prod_{j=n}^{i} (1 + xe_j) = (1 + xe_n)(1 + xe_{n-1}) \cdots (1 + xe_i)$.

Now we are ready to formulate and prove the main result of this Section, namely, the following positivity theorem:

Theorem 1. *Let $1 \le i < j \le n+1$, $w \in S_{n+1}$. Then*

$$\partial_{ij}\mathfrak{S}_w \in \mathbf{N}[x_1, \ldots, x_{n+1}].$$

Proof. Our starting point is the Lemma below which is a generalization of the Statement 4.19 from Macdonald's book [11]. Before to state the Lemma, we need to introduce a few notation.

Let $X = (x_1, \ldots, x_n)$ be the set of variables and $\mu = (\mu_1, \ldots, \mu_p)$ be a composition of size n. We assume that $\mu_j \neq 0$, $1 \le j \le p$, and put by definition $\mu_0 = 0$.

Denote by X_j the set of variables $(x_{\mu_1+\cdots+\mu_{j-1}+1}, \ldots, x_{\mu_1+\cdots+\mu_j})$, $1 \le j \le p$.

Let $w \in S^{(n)}$ be a permutation such that the code of w has length $\le n$. The Schubert polynomial $\mathfrak{S}_w(X)$ can be uniquely expressed in the form

$$\mathfrak{S}_w(X) = \sum d^w_{u_1,\cdots,u_p} \prod_{j=1}^{p} \mathfrak{S}_{u_j}(X_j),$$

summed over permutations $u_1 \in S^{(\mu_1)}, \ldots, u_p \in S^{(\mu_p)}$, see [11, Chapter IV].

Lemma 1. *The coefficients $d^w_{u_1,\ldots,u_p}$ defined above, are non-negative integers.*

The proof of Lemma proceed by induction on $l(u_p)$, and follows very close to that given in [11]. We omit details.

It follows from the Lemma above that it is enough to prove Theorem 1 only for the transposition $(i,j) = (1,n)$. Thus, we are going to prove that $\partial_{1n}\mathfrak{S}_w \in \mathbf{N}[x_1, \ldots, x_n]$. For this goal, let us consider the Schubert expression $\mathfrak{S}^{(n+1)} = A_1(x_1)A_2(x_2) \cdots A_n(x_n)$, see (8.1). We are going to prove that

$$\partial_{1n}\mathfrak{S}^{(n+1)} = \sum_{w \in S_{n+1}} \alpha_w(x) e_w,$$

where $\alpha_w(x) \in \mathbf{N}[x_1, \ldots, x_n]$ for all $w \in S_{n+1}$. Using the Leibniz rule (3.2), we can write

$$\begin{aligned} \partial_{1n}\mathfrak{S}^{(n+1)} &= \partial_{1n}(A_1(x_1)A_2(x_2) \cdots A_n(x_n)) \\ &= \partial_{1n}(A_1(x_1))A_2(x_2) \cdots A_n(x_n) + A_1(x_n)A_2(x_2) \cdots A_{n-1}(x_{n-1})\partial_{1n}(A_n(x_n)). \end{aligned}$$

First of all,

$$\partial_{1n}A_n(x_n) = \frac{1 + x_n e_n - 1 - x_1 e_n}{x_1 - x_n} = -e_n.$$

The next observation is

$$\partial_{1n} A_1(x_1) = \frac{A_1(x_n) - 1}{x_n} + f(x_1, x_n),$$

where $f(x_1, x_n) \in \mathbf{N}[x_1, x_n][NC_n]$. Indeed, if $A_1(x) = \sum\limits_{k=0}^{n} c_k x^k$, where $c_k \in NC_n$, $c_0 = 1$, then

$$\partial_{1n} A_1(x_1) = \sum_{k=1}^{n} c_k \frac{x_1^k - x_n^k}{x_1 - x_n} = \sum_{k=1}^{n} c_k x_n^{k-1} + f(x_1, x_n),$$

and $f(x_1, x_n) \in \mathbf{N}[x_1, x_n][NC_n]$, as it was claimed. Hence,

$$\begin{aligned} x_n \partial_{1n} \mathfrak{S}^{(n+1)} &= (A_1(x_n) - 1) A_2(x_2) \cdots A_{n-1}(x_{n-1})(1 + x_n e_n) \\ &\quad - A_1(x_n) A_2(x_2) \cdots A_{n-1}(x_{n-1}) e_n x_n + x_n F(x_1, \ldots, x_n) \\ &= A_1(x_n) A_2(x_2) \cdots A_{n-1}(x_{n-1}) - A_2(x_2) A_3(x_3) \cdots A_n(x_n) \\ &\quad + x_n F(x_1, \ldots, x_n), \end{aligned}$$

where $F(x_1, \ldots, x_n) \in \mathbf{N}[x_1, \ldots, x_n][NC_n]$. Thus, it is enough to prove that the difference

$$A_1(x_n) A_2(x_2) \cdots A_{n-1}(x_{n-1}) - A_2(x_2) A_3(x_3) \cdots A_n(x_n)$$

belongs to the set $\mathbf{N}[x_1, \ldots, x_n][NC_n]$. We will use the following result (see [5, 3]):

$$A_i(x) A_i(y) = A_i(y) A_i(x), \qquad 1 \le i \le n.$$

Thus, using a simple observation that $A_i(x) = A_{i+1}(x)(1 + x e_i)$, we have

$$\begin{aligned} & A_1(x_n) A_2(x_2) \cdots A_{n-1}(x_{n-1}) = A_2(x_n)(1 + x_n e_1) A_2(x_2) \cdots A_{n-1}(x_{n-1}) \\ &\quad = A_2(x_n) A_2(x_2) \cdots A_{n-1}(x_{n-1}) + x_n A_2(x_n) e_1 A_2(x_2) \cdots A_{n-1}(x_{n-1}) \\ &\quad = A_2(x_2) A_2(x_n) A_3(x_3) \cdots A_{n-1}(x_{n-1}) + x_n A_2(x_n) e_1 A_2(x_2) \cdots A_{n-1}(x_{n-1}) \\ &\quad = A_2(x_2) A_3(x_n)(1 + x_n e_2) A_3(x_3) \cdots A_{n-1}(x_{n-1}) + x_n A_2(x_n) e_1 A_2(x_2) \cdots A_{n-1}(x_{n-1}) \\ &\quad = \cdots = A_2(x_2) A_3(x_3) \cdots A_n(x_n) + x_n \sum_{i=1}^{n-1} \prod_{j=2}^{i} A_j(x_j) A_{i+1}(x_n) e_i \prod_{j=i+1}^{n-1} A_j(x_j). \end{aligned}$$

Let us denote the sum over i in (5.3) by $G(x_1, \ldots, x_n)$. It is clear that

$$G(x_1, \ldots x_n) \in \mathbf{N}[x_1, \ldots x_n][NC_n].$$

Thus the difference

$$A_1(x_n) A_2(x_2) \cdots A_{n-1}(x_{n-1}) - A_2(x_2) A_3(x_3) \cdots A_n(x_n) = G(x_1, \ldots, x_n)$$

also belongs to the set $\mathbf{N}[x_1, \ldots, x_n][NC_n]$. ■

9 Generating function for the Schubert polynomials structural constants c_{uv}^{w}

Let $w, v \in S_n$, $l(w) - l(v) \le 1$, and $w \succeq v$ with respect to the Bruhat order. For $1 \le i \le n$ and $1 \le s \le n-1$ we define the element $e_i^{(s)}(w/v)$ of the nilCoxeter algebra NC_n using the following rule

$$e_i^{(s)}(w/v) = \begin{cases} 0, & \text{if } \ w = v t_{(a,b)}, \ \text{ and simultaneously } \ a \ne s \ \text{ and } \ b \ne s, \\ e_{n-i}, & \text{if } \ w = v t_{(s,b)}, \ \text{ and } \ s < b, \\ -e_{n-i}, & \text{if } \ w = v t_{(b,s)}, \ \text{ and } \ b < s, \\ 1, & \text{if } \ w = v. \end{cases}$$

Theorem 2. *Let $u, w \in S_n$. Then*

$$\sum_{v \in S_n} c_{uv}^{w} e_v = \sum_{\{v_i^{(s)}\}_{s=1}^{n-1}} \prod_{s=1}^{n-1} \prod_{i=1}^{n-s} e_i^{(s)} \left(v_{i-1}^{(s)} / v_i^{(s)} \right), \tag{9.1}$$

summed over all sequences $\mathbf{v} = (v_i^{(s)})$ of permutations

$$\begin{aligned} w &= v_0^{(1)} \succeq v_1^{(1)} \succeq \cdots \succeq v_{n-1}^{(1)} = v_0^{(2)} \succeq v_1^{(2)} \succeq \cdots \succeq v_{n-2}^{(2)} \\ &= v_0^{(3)} \cdots = v_0^{(n-2)} \succeq v_1^{(n-2)} \succeq v_2^{(n-2)} = v_0^{(n-1)} \succeq v_1^{(n-1)} = u \end{aligned}$$

with restrictions

$$l(v_{i-1}^{(s)}) - l(v_i^{(s)}) \leq 1 \qquad \text{for all } i \text{ and } s.$$

In the product in the RHS (9.1) *the factors are multiplied left-to-right, according to the increase of s.*

Proof. We start with rewriting of the LHS (9.1), namely showing that

$$\sum_{v \in S_n} c_{uv}^{w} e_v = \eta\big(\partial_{w/u} \mathfrak{S}^{(n)}\big),$$

where $\mathfrak{S}^{(n)}$ denotes the Schubert expression. Indeed,

$$\eta\big(\partial_{w/v} \mathfrak{S}^{(n)}\big) = \sum_{v \in S_n} \eta\left(\partial_{w/v}(\mathfrak{S}_v)\right) e_v = \sum_{v \in S_n} c_{uv}^{w} e_v.$$

The next step is to compute $\eta\left(\partial_{w/v}\mathfrak{S}\right)$ using the following lemma, which is obtained by repetitive use of the generalized Leibniz rule (5.3).

Lemma 2. *Let $w, u \in S_n$, and $f_1, \ldots, f_N \in P_n$. Then*

$$u\partial_{w/u}(f_1 \cdots f_N) = \sum_{w = v_0 \succeq v_1 \succeq \cdots \succeq v_{N-1} \succeq v_N = u} \prod_{i=1}^{N} v_i \left(\partial_{v_{i-1}/v_i}(f)\right).$$

We apply Lemma 2 to the Schubert expression

$$\mathfrak{S}^{(n)} = A_1(x_1) \cdots A_{n-1}(x_{n-1}) = \prod_{i=1}^{n-1} \prod_{k=n-1}^{i} (1 + x_i e_k). \tag{9.2}$$

On the rightmost side of (9.2), the factors are multiplied left-to-right according to the increase of i. As a result, we obtain

$$\eta\big(\partial_{w/u}\mathfrak{S}^{(n)}\big) = \sum_{\{v_0^{(s)} \succeq v_1^{(s)} \succeq \cdots \succeq v_{n-s}^{(s)}\}_{s=1}^{n-1}} \prod_{s=1}^{n-1} \prod_{i=1}^{n-s} \eta\big(\partial_{v_{i-1}^{(s)}/v_i^{(s)}}(1 + x_s e_{n-i})\big),$$

summed over all sequences of permutations $\{v_0^{(s)} \succeq v_1^{(s)} \succeq \cdots \succeq v_{n-s}^{(s)}\}_{s=1}^{n-1}$ such that $v_0^{(1)} = w$, $v_0^{(s+1)} = v_{n-s}^{(s)}$, $1 \leq s \leq n-2$, $v_1^{(n-1)} = u$.

Note that we omitted the action of the symmetric group elements since we apply η.

It is clear that we can assume $l(v_{i-1}^{(s)}) - l(v_i^{(s)}) \leq 1$ for all i, s, and under these conditions, we have

$$\eta\big(\partial_{v_{i-1}^{(s)}/v_i^{(s)}}(1 + x_s e_{n-i})\big) = e_i^{(s)}\big(v_{i-1}^{(s)}/v_i^{(s)}\big). \qquad \blacksquare$$

10 Open problems

Below we formulate a few problems related to the content of this paper.

1. Main problem. Let $w, v \in S_n$ and $w \succeq v$ with respect to the Bruhat order on the symmetric group S_n. Prove that polynomials $\partial_{w/v}(\mathfrak{S}_u)$ have nonnegative coefficients for each $u \in S_n$.

2. The generalized "Littlewood–Richardson problem" for Schubert polynomials. Let $u, v, w \in S_n$ and

$$\partial_{w/v}(\mathfrak{S}_u) = \sum_{\alpha} c_\alpha x^\alpha.$$

Find a combinatorial description of the coefficients $c_\alpha := c_\alpha(u, v, w)$.

Remark 2. If $l(w) = l(u) + l(v)$ and $w \succeq v$, then $\partial_{w/v}(\mathfrak{S}_u) = c^w_{uv}$, see [11, p. 112], or the present paper, Proposition 3, v).

3. Skew key polynomials. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a composition, $\lambda(\alpha)$ be a unique partition in the orbit $S_n \cdot \alpha$, and $w(\alpha) \in S_n$ be the shortest permutation such that $w(\alpha) \cdot \alpha = \lambda(\alpha)$. Let $v \in S_n$ be such that $w(\alpha) \succeq v$ with respect to the Bruhat order. Using in Definition 4 the isobaric divided difference operators $\pi_i := \partial_i x_i$, $1 \le i \le n-1$ (see, e.g., [11, p. 28]) instead of operators ∂_i one can define for each pair $w \succeq v$ the skew isobaric divided difference operator $\pi_{w/v} : P_n \to P_n$, where $P_n = \mathbf{Z}[x_1, \ldots, x_n]$.

We define the skew key polynomial $k_{\alpha/v}$ to be

$$k_{\alpha/v} = \pi_{w(\alpha)/v}\big(x^{\lambda(\alpha)}\big),$$

where $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ for any composition $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$. It is natural to ask whether or not the skew key polynomials have nonnegative coefficients?

4. Find a geometrical interpretation of the skew divided difference operators, the polynomials $\partial_{w/v}(\mathfrak{S}_u)$, and the skew key polynomials.

5. Does there exist a stable analog of the skew Schubert polynomials?

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