Exact Solutions for Equations of Bose–Fermi Mixtures in One-Dimensional Optical Lattice

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Abstract. We present two new families of stationary solutions for equations of Bose–Fermi mixtures with an elliptic function potential with modulus k. We also discuss particular cases when the quasiperiodic solutions become periodic ones. In the limit of a sinusoidal potential ($k \to 0$) our solutions model a quasi-one dimensional quantum degenerate Bose–Fermi mixture trapped in optical lattice. In the limit $k \to 1$ the solutions are expressed by hyperbolic function solutions (vector solitons). Thus we are able to obtain in an unified way quasi-periodic and periodic waves, and solitons. The precise conditions for existence of every class of solutions are derived. There are indications that such waves and localized objects may be observed in experiments with cold quantum degenerate gases.

Key words: Bose–Fermi mixtures; one dimensional optical lattice

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1 Introduction

Over the last decade, the field of cold degenerate gases has been one of the most active areas in physics. The discovery of Bose–Einstein Condensates (BEC) in 1995 (see e.g. [1, 2]) greatly stimulated research of ultracold dilute Boson-Fermion mixtures. This interest is driven by the desire to understand strongly interacting and strongly correlated systems, with applications in solid-state physics, nuclear physics, astrophysics, quantum computing, and nanotechnologies.

An important property of Bose–Fermi mixtures wherein the fermion component is dominant is that the mixture tends to exhibit essentially three-dimensional character even in a strongly elongated trap. During the last decade, great progress has been achieved in the experimental realization of Bose–Fermi mixtures [3, 4], in particular Bose–Fermi mixtures in one-dimensional lattices. Optical lattices provide a powerful tool to manipulate matter waves, in particular solitons. The Pauli exclusion principle results in the extension of the fermion cloud in the transverse direction over distances comparable to the longitudinal dimension of the excitations. It has been shown recently, however, that the quasi-one-dimensional situation can nevertheless be realized in a Bose–Fermi mixture due to strong localization of the bosonic component [5, 6]. With account of the effectiveness of the optical lattice in managing systems of cold atoms, their effect on the dynamics of Bose–Fermi mixtures is of obvious interest. Some of the aspects of this problem have already been explored within the framework of the mean-field approximation. In particular, the dynamics of the Bose–Fermi mixtures were explored from the point of view of designing quantum dots [8]. The localized states of Bose–Fermi mixtures with attractive (repulsive) Bose–Fermi interactions are viewed as a matter-wave realization of quantum dots and antidots. The case of Bose–Fermi mixtures in optical lattices is investigated in detail and the existence of gap solitons is shown. In particular, in [8] it is obtained that the gap solitons can trap a number of fermionic bound-state levels inside both for repulsive and attractive boson-boson interactions. The time-dependent dynamical mean-field-hydrodynamic model to study the formation of fermionic bright solitons in a trapped degenerate Fermi gas mixed with a Bose–Einstein condensate in a quasi-one-dimensional cigar-shaped geometry is proposed in [9]. Similar model is used to study mixing-demixing in a degenerate fermion-fermion mixture in [10]. Modulational instability, solitons and periodic waves in a model of quantum degenerate boson-fermion mixtures are obtained in [11].

Our aim is to derive two new classes of quasi-periodic exact solutions of the time dependent mean field equations of Bose–Fermi mixture in one-dimensional lattice. We also study some limiting cases of these solutions. The paper is organized as follows. In Section 2 we give the basic equations. Section 3 is devoted to derivation of the first class quasi-periodic solutions with non-trivial phases. A system of $N_f + 1$ equations, which reduce quasi-periodic solutions to periodic are derived. In Section 4 we present second class (type B) nontrivial phase solutions. In Section 5 we obtain 14 classes of elliptic solutions. Section 6 is devoted to two special limits, to hyperbolic and trigonometric functions. In Section 7 preliminary results about the linear stability of solutions are given. Section 8 summarizes the main conclusions of the paper.

2 Basic equations

At mean field approximation we consider the following $N_f + 1$ coupled equations [7, 8, 12, 11]

$$i\hbar\frac{\partial\Psi^b}{\partial t} + \frac{1}{2m_{\mathrm{B}}}\frac{\partial^2\Psi^b}{\partial x^2} - V\Psi^b - g_{\mathrm{BB}}|\Psi^b|^2\Psi^b - g_{\mathrm{BF}}\rho_f\Psi^b = 0, \tag{2.1}$$

$$i\hbar\frac{\partial\Psi^f_j}{\partial t} + \frac{1}{2m_{\mathrm{F}}}\frac{\partial^2\Psi^f_j}{\partial x^2} - V\Psi^f_j - g_{\mathrm{BF}}|\Psi^b|^2\Psi^f_j = 0, \tag{2.2}$$

where $\rho_f = \sum\limits_{i=1}^{N_f}|\Psi^f_i|^2$ and

$$g_{\mathrm{BB}} = \frac{2a_{\mathrm{BB}}}{a_{\mathrm{s}}}, \qquad g_{\mathrm{BF}} = \frac{2a_{\mathrm{BF}}}{a_{\mathrm{s}}\alpha}, \qquad \alpha = \frac{m_{\mathrm{B}}}{m_{\mathrm{F}}}, \qquad a_{\mathrm{s}} = \sqrt{\frac{\hbar}{m_{\mathrm{B}}\omega_\perp}},$$

a_{BB} and a_{BF} are the scattering lengths for s-wave collisions for boson-boson and boson-fermion interactions, respectively. In recent experiments [13, 14] the quantum degenerate mixtures of ^{40}K and ^{87}Rb are studied where $m_{\mathrm{B}} = 87m_{\mathrm{p}}$, $m_{\mathrm{B}} = 40m_{\mathrm{p}}$ and $\omega_\perp = 215$ Hz. Equations (2.1), (2.2) have been studied numerically in [7]. The formation of localized structures containing bosons and fermions has been reported in the particular case in which the interspecies scattering length a_{BF} is negative, which is the case of the ^{40}K-^{87}Rb mixture. An appropriate class of periodic potentials to model the quasi-1D confinement produced by a standing light wave is given by [15]

$$V = V_0\,\mathrm{sn}^2(\alpha x, k),$$

where $\mathrm{sn}\,(\alpha x, k)$ denotes the Jacobian elliptic sine function with elliptic modulus $0 \le k \le 1$.

Experimental realization of two-component Bose–Einstein condensates have stimulated considerable attention in general [16] and in particular in the quasi-1D regime [17, 18] when the Gross–Pitaevskii equations for two interacting Bose–Einstein condensates reduce to coupled nonlinear Schrödinger (CNLS) equations with an external potential. In specific cases the two component CNLS equations can be reduced to the Manakov system [19] with an external potential.

Important role in analyzing these effects was played by the elliptic and periodic solutions of the above-mentioned equations. Such solutions for the one-component nonlinear Schrödinger equation are well known, see [20] and the numerous references therein. Elliptic solutions for the CNLS and Manakov system were derived in [21, 22, 23].

In the presence of external elliptic potential explicit stationary solutions for NLS were derived in [15, 24, 25]. These results were generalized to the n-component CNLS in [18]. For 2-component CNLS explicit stationary solutions are derived in [26].

3 Stationary solutions with non-trivial phases

We restrict our attention to stationary solutions of these CNLS

$$\Psi^b(x,t) = q_0(x)\exp\left(-i\frac{\omega_0}{\hbar}t + i\Theta_0(x) + i\kappa_0\right), \tag{3.1}$$

$$\Psi_j^f(x,t) = q_j(x)\exp\left(-i\frac{\omega_j}{\hbar}t + i\Theta_j(x) + i\kappa_{0,j}\right), \tag{3.2}$$

where $j = 1,\ldots,N_f$, κ_0, $\kappa_{0,j}$, are constant phases, q_j and Θ_0, $\Theta_j(x)$ are real-valued functions connected by the relation

$$\Theta_0(x) = \mathcal{C}_0\int_0^x \frac{dx'}{q_0^2(x')}, \qquad \Theta_j(x) = \mathcal{C}_j\int_0^x \frac{dx'}{q_j^2(x')}, \tag{3.3}$$

$\mathcal{C}_0, \mathcal{C}_j$, $j = 1,\ldots,N_f$ being constants of integration. Substituting the ansatz (3.1), (3.2) in equations (2.1) and separating the real and imaginary part we get

$$\begin{aligned}
&\frac{1}{2m_{\mathrm{B}}}q_0^3 q_{0xx} - g_{\mathrm{BB}}q_0^6 - Vq_0^4 - g_{\mathrm{BF}}\left(\sum_{i=1}^{N_f} q_i^2\right)q_0^4 + \omega_0 q_0^4 = \frac{1}{2m_{\mathrm{B}}}\mathcal{C}_0^2, \\
&\frac{1}{2m_{\mathrm{F}}}q_j^3 q_{jxx} - g_{\mathrm{BF}}q_0^2 q_j^4 - Vq_j^4 + \omega_j q_j^4 = \frac{1}{2m_{\mathrm{F}}}\mathcal{C}_j^2.
\end{aligned} \tag{3.4}$$

We seek solutions for q_0^2 and q_j^2, $j = 1,\ldots,N_f$ as a quadratic function of $\mathrm{sn}\,(\alpha x,k)$:

$$q_0^2 = A_0\mathrm{sn}^2(\alpha x,k) + B_0, \qquad q_j^2 = A_j\mathrm{sn}^2(\alpha x,k) + B_j. \tag{3.5}$$

Inserting (3.5) in (3.4) and equating the coefficients of equal powers of $\mathrm{sn}\,(\alpha x,k)$ results in the following relations among the solution parameters ω_j, $\mathcal{C}_j$, A_j and B_j and the characteristic of the optical lattice V_0, α and k:

$$A_0 = \frac{\alpha^2k^2 - m_FV_0}{m_{\mathrm{F}}g_{\mathrm{BF}}}, \qquad \sum_{j=1}^{N_f} A_j = \frac{\alpha^2k^2}{g_{\mathrm{BF}}}\left(\frac{1}{m_{\mathrm{B}}} - \frac{g_{\mathrm{BB}}}{m_{\mathrm{F}}g_{\mathrm{BF}}}\right) - \frac{V_0}{g_{\mathrm{BF}}}\left(1 - \frac{g_{\mathrm{BB}}}{g_{\mathrm{BF}}}\right), \tag{3.6}$$

$$\begin{aligned}
\omega_0 &= \frac{\alpha^2(k^2+1)}{2m_{\mathrm{B}}} + g_{\mathrm{BB}}B_0 + g_{\mathrm{BF}}\sum_{i=1}^{N_f} B_i + \frac{\alpha^2k^2}{2m_{\mathrm{B}}}\frac{B_0}{A_0}, \\
\omega_j &= \frac{\alpha^2(k^2+1)}{2m_{\mathrm{F}}} + g_{\mathrm{BF}}B_0 + \frac{\alpha^2k^2}{2m_{\mathrm{F}}}\frac{B_j}{A_j},
\end{aligned} \tag{3.7}$$

$$\mathcal{C}_0^2 = \frac{\alpha^2B_0}{A_0}(A_0+B_0)(A_0+B_0k^2), \qquad \mathcal{C}_j^2 = \frac{\alpha^2B_j}{A_j}(A_j+B_j)(A_j+B_jk^2), \tag{3.8}$$

where $j = 1,\ldots,N_f$. Next for convenience we introduce

$$B_0 = -\beta_0A_0, \qquad B_j = -\beta_jA_j, \qquad j = 1,\ldots,N_f,$$

Table 1. $W = g_{\rm BF} m_{\rm F} W_{\rm B}/(m_{\rm B} W_{\rm F})$.

1	$\beta_0 \le 0$	$\beta_j \le 0$	$A_0 \ge 0$	$A_j \ge 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \lessgtr W$	$V_0 \lessgtr \alpha^2 k^2/m_{\rm F}$
2	$\beta_0 \le 0$	$1 \le \beta_j \le 1/k^2$	$A_0 \ge 0$	$A_j \le 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \gtrless W$	$V_0 \lessgtr \alpha^2 k^2/m_{\rm F}$
3	$1 \le \beta_0 \le 1/k^2$	$\beta_j \le 0$	$A_0 \le 0$	$A_j \ge 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \gtrless W$	$V_0 \gtrless \alpha^2 k^2/m_{\rm F}$
4	$1 \le \beta_0 \le 1/k^2$	$1 \le \beta_j \le 1/k^2$	$A_0 \le 0$	$A_j \le 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \lessgtr W$	$V_0 \gtrless \alpha^2 k^2/m_{\rm F}$

then

$$\mathcal{C}_0^2 = \alpha^2 A_0^2 \beta_0(\beta_0 - 1)(1 - \beta_0 k^2), \qquad \mathcal{C}_j^2 = \alpha^2 A_j^2 \beta_j(\beta_j - 1)(1 - \beta_j k^2), \qquad j = 1, \dots, N_f.$$

In order for our results (3.5) to be consistent with the parametrization (3.1)–(3.3) we must ensure that both $q_0(x)$ and $\Theta_0(x)$ are real-valued, and also $q_j(x)$ and $\Theta_j(x)$ are real-valued; this means that $C_0^2 \ge 0$ and $q_0^2(x) \ge 0$ and also $C_j^2 \ge 0$ and $q_j^2(x) \ge 0$ (see Table 1, $W_{\rm B} = (\alpha^2 k^2 - m_{\rm B} V_0)$, $W_{\rm F} = (\alpha^2 k^2 - m_{\rm F} V_0)$). An elementary analysis shows that with $l = 0, \dots, N_f$ one of the following conditions must hold

$$\text{a)} \quad A_l \ge 0, \quad \beta_l \le 0, \qquad \text{b)} \quad A_l \le 0, \quad 1 \le \beta_l \le \frac{1}{k^2}.$$

Although our main interest is to analyze periodic solutions, note that the solutions Ψ^b, Ψ_j^f in (2.1), (2.2) *are not* always periodic in x. Indeed, let us first calculate explicitly $\Theta_0(x)$ and $\Theta_j(x)$ by using the well known formula, see e.g. [27]:

$$\int_0^x \frac{du}{\wp(\alpha u) - \wp(\alpha v)} = \frac{1}{\wp'(\alpha v)} \left[2x\zeta(\alpha v) + \frac{1}{\alpha} \ln \frac{\sigma(\alpha u - \alpha v)}{\sigma(\alpha u + \alpha v)} \right],$$

where $\wp$, ζ, σ are standard Weierstrass functions.

In the case a) we replace v by iv_0 and v by iv_j, set $\mathrm{sn}^2(i\alpha v_0; k) = \beta_0 < 0$, $\mathrm{sn}^2(i\alpha v_j; k) = \beta_j < 0$ and

$$e_1 = \frac{1}{3}(2 - k^2), \qquad e_2 = \frac{1}{3}(2k^2 - 1), \qquad e_3 = -\frac{1}{3}(1 + k^2),$$

and rewrite the l.h.s in terms of Jacobi elliptic functions:

$$\int_0^x \frac{du\, \mathrm{sn}^2(i\alpha v; k) \mathrm{sn}^2(\alpha u; k)}{\mathrm{sn}^2(i\alpha v; k) - \mathrm{sn}^2(\alpha u; k)} = -\beta_0 x - \beta_0^2 \int_0^x \frac{du}{\mathrm{sn}^2(\alpha u, k) - \beta_0},$$

and for $j = 1, \dots, N_f$ we have

$$\int_0^x \frac{du\, \mathrm{sn}^2(i\alpha v; k) \mathrm{sn}^2(\alpha u; k)}{\mathrm{sn}^2(i\alpha v; k) - \mathrm{sn}^2(\alpha u; k)} = -\beta_j x - \beta_j^2 \int_0^x \frac{du}{\mathrm{sn}^2(\alpha u, k) - \beta_j}.$$

Skipping the details we find the explicit form of

$$\Theta_0(x) = C_0 \int_0^x \frac{du}{A_0(\mathrm{sn}^2(\alpha u; k) - \beta_0} = -\tau_0 x + \frac{i}{2} \ln \frac{\sigma(\alpha x + i\alpha v_0)}{\sigma(\alpha x - i\alpha v_0)},$$
$$\tau_0 = i\alpha\zeta(i\alpha v_0) + \frac{\alpha}{\beta_0} \sqrt{-\beta_0(1 - \beta_0)(1 - k^2\beta_0)}.$$

and for $\Theta_j(x)$, $j = 1, \dots, N_f$ we have

$$\Theta_j(x) = C_j \int_0^x \frac{du}{A_j(\mathrm{sn}^2(\alpha u; k) - \beta_j)} = -\tau_j x + \frac{i}{2} \ln \frac{\sigma(\alpha x + i\alpha v_j)}{\sigma(\alpha x - i\alpha v_j)}, \tag{3.9}$$

$$\tau_j = i\alpha\zeta(i\alpha v_j) + \frac{\alpha}{\beta_j}\sqrt{-\beta_j(1-\beta_j)(1-k^2\beta_j)}.$$

These formulae provide an explicit expression for the solutions Ψ^b, Ψ_j^f with nontrivial phases; note that for real values of v_0 $\Theta_0(x)$, v_j $\Theta_j(x)$ are also real. Now we can find the conditions under which $Q_j(x,t)$ are periodic. Indeed, from (3.9) we can calculate the quantities T_0, T_j satisfying:

$$\Theta_0(x+T_0) - \Theta_0(x) = 2\pi p_0, \qquad \Theta_j(x+T_j) - \Theta_j(x) = 2\pi p_j, \qquad j = 1,\ldots,N_f.$$

Then Ψ^b, Ψ_j^f will be periodic in x with periods $T_0 = 2m_0\omega/\alpha$, $T_j = 2m_j\omega/\alpha$ if there exist pairs of integers m_0, p_0, and m_j, p_j, such that:

$$\frac{m_0}{p_0} = -\pi\left[\alpha v_0\zeta(\omega) + \omega\tau_0/\alpha\right]^{-1}, \qquad \frac{m_j}{p_j} = -\pi\left[\alpha v_j\zeta(\omega) + \omega\tau_j/\alpha\right]^{-1}, \qquad j = 1,\ldots,N_f.$$

where ω (and ω') are the half-periods of the Weierstrass functions.

4 Type B nontrivial phase solutions

For the first time solutions of this type were derived in [15, 24, 25] for the case of nonlinear Schrödinger equation and in [18] for the n-component CNLSE. For Bose–Fermi mixtures solutions of this type are possible

- when we have two lattices $V_{\rm B}$ and $V_{\rm F}$,
- when $m_{\rm B} = m_{\rm F}$.

We seek the solutions in one of the following forms:

$$q_0^2 = A_0\mathrm{sn}\,(\alpha x,k) + B_0, \qquad q_j^2 = A_j\mathrm{sn}\,(\alpha x,k) + B_j, \tag{4.1}$$
$$q_0^2 = A_0\mathrm{cn}\,(\alpha x,k) + B_0, \qquad q_1^2 = A_j\mathrm{cn}\,(\alpha x,k) + B_j, \tag{4.2}$$
$$q_0^2 = A_0\mathrm{dn}\,(\alpha x,k) + B_0, \qquad q_1^2 = A_j\mathrm{dn}\,(\alpha x,k) + B_j, \qquad j = 1,\ldots,N_f. \tag{4.3}$$

In the first case (4.1) we have

$$V_{\rm B} = \frac{3\alpha^2k^2}{8m_{\rm B}}, \qquad V_{\rm F} = \frac{3\alpha^2k^2}{8m_{\rm F}}$$
$$A_0 = -\frac{\alpha^2k^2}{4m_{\rm F}g_{\rm BF}}\frac{B_j}{A_j}, \qquad \frac{B_1}{A_1} = \cdots = \frac{B_{N_f}}{A_{N_F}}, \qquad \sum_j A_j = -\frac{\alpha^2k^2}{4m_{\rm B}g_{\rm BF}}\frac{B_0}{A_0} - \frac{A_0 g_{\rm BB}}{g_{\rm BF}},$$
$$\omega_0 = \frac{\alpha^2(k^2+1)}{8m_{\rm B}} + g_{\rm BB}B_0 + g_{\rm BF}B_1 - \frac{\alpha^2k^2}{8m_{\rm B}}\frac{B_0^2}{A_0^2},$$
$$\omega_j = \frac{\alpha^2(k^2+1)}{8m_{\rm F}} + g_{\rm BF}B_0 - \frac{\alpha^2k^2}{8m_{\rm F}}\frac{B_j^2}{A_j^2},$$
$$\mathcal{C}_0^2 = \frac{\alpha^2}{4A_0^2}(B_0^2 - A_0^2)(A_0^2 - B_0^2k^2), \qquad \mathcal{C}_j^2 = \frac{\alpha^2}{4A_j^2}(B_j^2 - A_j^2)(A_j^2 - B_j^2k^2).$$

We remark that due to relations $\frac{B_1}{A_1} = \cdots = \frac{B_{N_f}}{A_{N_F}}$ we have that all q_j of the fermion fields are proportional to q_1.

5 Examples of elliptic solutions

Using the general solution equations (3.6)–(3.8) we have the following special cases: (these solutions are possible only when we have some restrictions on $g_{\rm BB}$, $g_{\rm BF}$, and V_0 see the Table 1)

Example 1. Suppose that $B_0 = B_j = 0$. Therefore we have

$$q_0(x) = \sqrt{A_0}\,\mathrm{sn}\,(\alpha x, k), \qquad q_j = \sqrt{A_j}\,\mathrm{sn}\,(\alpha x, k), \tag{5.1}$$

$$A_0 = \frac{\alpha^2 k^2 - m_{\rm F} V_0}{m_{\rm F} g_{\rm BF}}, \qquad \sum_j A_j = \frac{\alpha^2 k^2}{g_{\rm BF}}\left(\frac{1}{m_{\rm B}} - \frac{g_{\rm BB}}{m_F g_{\rm FB}}\right) - \frac{V_0}{g_{\rm BF}}\left(1 - \frac{g_{\rm BB}}{g_{\rm BF}}\right). \tag{5.2}$$

For the frequencies ω_0 and ω_j we have

$$\omega_0 = \frac{\alpha^2(1+k^2)}{2m_{\rm B}}, \qquad \omega_j = \frac{\alpha^2(1+k^2)}{2m_{\rm F}}.$$

as well as $\mathcal{C}_0 = \mathcal{C}_j = 0$.

Example 2. Let $B_0 = -A_0$ and $B_j = -A_j$ hold true. Then we have

$$q_0(x) = \sqrt{-A_0}\,\mathrm{cn}\,(\alpha x, k), \qquad q_j(x) = \sqrt{-A_j}\,\mathrm{cn}\,(\alpha x, k). \tag{5.3}$$

The coefficients A_0 and A_j have the same form as (5.2). The frequencies ω_0 and ω_j now look as follows

$$\omega_0 = \frac{\alpha^2(1-2k^2)}{2m_{\rm B}} + V_0, \qquad \omega_j = \frac{\alpha^2(1-2k^2)}{2m_{\rm F}} + V_0.$$

The constants $\mathcal{C}_0$ and $\mathcal{C}_j$ are equal to zero again.

Example 3. $B_0 = -A_0/k^2$ and $B_j = -A_j/k^2$. In this case we obtain

$$\begin{aligned} q_0(x) &= \frac{\sqrt{-A_0}}{k}\mathrm{dn}\,(\alpha x, k), \qquad q_j(x) = \frac{\sqrt{-A_j}}{k}\mathrm{dn}\,(\alpha x, k),\\ \omega_0 &= \frac{\alpha^2(k^2-2)}{2m_{\rm B}} + \frac{V_0}{k^2}, \qquad \omega_j = \frac{\alpha^2(k^2-2)}{2m_{\rm F}} + \frac{V_0}{k^2}. \end{aligned} \tag{5.4}$$

As before $\mathcal{C}_0 = \mathcal{C}_j = 0$.

Example 4. $B_0 = 0$ and $B_j = -A_j$. The result reads

$$\begin{aligned} q_0(x) &= \sqrt{A_0}\,\mathrm{sn}\,(\alpha x, k), \qquad q_j(x) = \sqrt{-A_j}\,\mathrm{cn}\,(\alpha x, k),\\ \omega_0 &= \frac{\alpha^2(1-k^2)}{2m_{\rm B}} + V_0 + A_0 g_{\rm BB}, \qquad \omega_j = \frac{\alpha^2}{2m_{\rm F}}. \end{aligned} \tag{5.5}$$

By analogy with the previous examples the constants A_0, A_j, $\mathcal{C}_0$ and $\mathcal{C}_j$ are given by formulae (5.2) and $\mathcal{C}_0$, $\mathcal{C}_j$ are all zero.

Example 5. $B_0 = 0$ and $B_j = -A_j/k^2$. Thus one gets

$$\begin{aligned} q_0(x) &= \sqrt{A_0}\,\mathrm{sn}\,(\alpha x, k), \qquad q_j(x) = \frac{\sqrt{-A_j}}{k}\mathrm{dn}\,(\alpha x, k),\\ \omega_0 &= \frac{\alpha^2(k^2-1)}{2m_{\rm B}} + \frac{V_0}{k^2} + \frac{A_0 g_{\rm BB}}{k^2}, \qquad \omega_j = \frac{\alpha^2 k^2}{2m_{\rm F}}. \end{aligned} \tag{5.6}$$

Table 2. Trivial phase solutions in the generic case. We use the quantity $W = g_{\mathrm{BF}} m_{\mathrm{F}} W_{\mathrm{B}}/(m_{\mathrm{B}} W_{\mathrm{F}})$.

1	$q_0 = \sqrt{A_0}\mathrm{sn}\,(\alpha x, k)$ $q_j = \sqrt{A_j}\mathrm{sn}\,(\alpha x, k)$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \lessgtr W$	$V_0 \lessgtr \alpha^2 k^2/m_{\mathrm{F}}$
2	$q_0 = \sqrt{-A_0}\mathrm{cn}\,(\alpha x, k)$ $q_j = \sqrt{-A_j}\mathrm{cn}\,(\alpha x, k)$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \lessgtr W$	$V_0 \gtrless \alpha^2 k^2/m_{\mathrm{F}}$
3	$q_0 = \sqrt{-A_0}\mathrm{dn}\,(\alpha x, k)/k$ $q_j = \sqrt{-A_j}\mathrm{dn}\,(\alpha x, k)/k$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \lessgtr W$	$V_0 \gtrless \alpha^2 k^2/m_{\mathrm{F}}$
4	$q_0 = \sqrt{A_0}\mathrm{sn}\,(\alpha x, k)$ $q_j = \sqrt{-A_j}\mathrm{cn}\,(\alpha x, k)$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \gtrless W$	$V_0 \lessgtr \alpha^2 k^2/m_{\mathrm{F}}$
5	$q_0 = \sqrt{A_0}\mathrm{sn}\,(\alpha x, k)$ $q_j = \sqrt{-A_j}\mathrm{dn}\,(\alpha x, k)/k$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \gtrless W$	$V_0 \lessgtr \alpha^2 k^2/m_{\mathrm{F}}$
6	$q_0 = \sqrt{-A_0}\mathrm{cn}\,(\alpha x, k)$ $q_j = \sqrt{A_j}\mathrm{sn}\,(\alpha x, k)$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \gtrless W$	$V_0 \gtrless \alpha^2 k^2/m_{\mathrm{F}}$
7	$q_0 = \sqrt{-A_0}\mathrm{cn}\,(\alpha x, k)$ $q_j = \sqrt{-A_j}\mathrm{dn}\,(\alpha x, k)/k$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \lessgtr W$	$V_0 \gtrless \alpha^2 k^2/m_{\mathrm{F}}$
8	$q_0 = \sqrt{-A_0}\mathrm{dn}\,(\alpha x, k)/k$ $q_j = \sqrt{A_j}\mathrm{sn}\,(\alpha x, k)$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \gtrless W$	$V_0 \gtrless \alpha^2 k^2/m_{\mathrm{F}}$
9	$q_0 = \sqrt{-A_0}\mathrm{dn}\,(\alpha x, k)/k$ $q_j = \sqrt{-A_j}\mathrm{cn}\,(\alpha x, k)$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \lessgtr W$	$V_0 \gtrless \alpha^2 k^2/m_{\mathrm{F}}$

Example 6. Let $B_0 = -A_0$ and $B_j = 0$. Hence we have

$$q_0(x) = \sqrt{-A_0}\mathrm{cn}\,(\alpha x, k), \qquad q_j(x) = \sqrt{A_j}\mathrm{sn}\,(\alpha x, k),$$
$$\omega_0 = \frac{\alpha^2}{2m_{\mathrm{B}}} - g_{\mathrm{BB}} A_0, \qquad \omega_j = \frac{\alpha^2(1-k^2)}{2m_{\mathrm{F}}} + V_0.$$

Example 7. Let $B_0 = -A_0$ and $B_j = -A_j/k^2$. We obtain

$$q_0(x) = \sqrt{-A_0}\mathrm{cn}\,(\alpha x, k), \qquad q_j(x) = \frac{\sqrt{-A_j}}{k}\mathrm{dn}\,(\alpha x, k),$$
$$\omega_0 = \frac{V_0}{k^2} - \frac{\alpha^2}{2m_{\mathrm{B}}} + \frac{1-k^2}{k^2} A_0 g_{\mathrm{BB}}, \qquad \omega_j = V_0 - \frac{\alpha^2 k^2}{2m_{\mathrm{F}}}.$$

Example 8. Suppose $B_0 = -A_0/k^2$ and $B_j = 0$. Then

$$q_0(x) = \frac{\sqrt{-A_0}}{k}\mathrm{dn}\,(\alpha x, k), \qquad q_j(x) = \sqrt{A_j}\mathrm{sn}\,(\alpha x, k),$$
$$\omega_0 = \frac{\alpha^2 k^2}{2m_{\mathrm{B}}} - \frac{g_{\mathrm{BB}} A_0}{k^2}, \qquad \omega_j = \frac{\alpha^2(k^2-1)}{2m_{\mathrm{F}}} + \frac{V_0}{k^2}.$$

Example 9. Let $B_0 = -A_0/k^2$ and $B_j = -A_j$. Thus

$$q_0(x) = \frac{\sqrt{-A_0}}{k}\mathrm{dn}\,(\alpha x, k), \qquad q_j(x) = \sqrt{-A_j}\mathrm{cn}\,(\alpha x, k),$$
$$\omega_0 = V_0 - \frac{\alpha^2 k^2}{2m_{\mathrm{B}}} + \frac{k^2-1}{k^2} g_{\mathrm{BB}} A_0, \qquad \omega_j = \frac{V_0}{k^2} - \frac{\alpha^2}{2m_{\mathrm{F}}}.$$

All these cases when $V_0 = 0$ and $j = 2$ are derived for the first time in [11].

5.1 Mixed trivial phase solution

Example 10. When $B_0 = 0$, $B_1 = 0$, $B_2 = -A_2$, $B_j = -A_j/k^2$, $j = 3, \dots, N_f$ the solutions obtain the form

$$q_0 = \sqrt{A_0}\mathrm{sn}\,(\alpha x, k), \qquad q_1 = \sqrt{A_1}\mathrm{sn}\,(\alpha x, k),$$
$$q_2 = \sqrt{-A_2}\mathrm{cn}\,(\alpha x, k), \qquad q_j = \sqrt{-A_j}\mathrm{dn}\,(\alpha x, k)/k, \qquad j = 3, \dots, N_f.$$

Using equations (3.6)–(3.8) we have

$$A_0 = \frac{\alpha^2 k^2 - V_0 m_F}{m_F g_{\mathrm{BF}}}, \qquad \sum_{j=1}^{N_f} A_j = \alpha^2 k^2 \left(\frac{1}{m_B g_{\mathrm{BF}}} - \frac{g_{\mathrm{BB}}}{m_F g_{\mathrm{BF}}^2} \right) - V_0 \left(\frac{1}{g_{\mathrm{BF}}} - \frac{g_{\mathrm{BB}}}{g_{\mathrm{BF}}^2} \right),$$
$$\omega_0 = \frac{\alpha^2(k^2-1)}{2m_B} + \frac{g_{\mathrm{BF}}}{k^2}\left(A_1 + (1-k^2)A_2\right) + \frac{g_{\mathrm{BB}}A_0}{k^2} + \frac{V_0}{k^2},$$
$$\omega_1 = \frac{\alpha^2(1+k^2)}{2m_F}, \qquad \omega_2 = \frac{1}{2m_F}\alpha^2, \qquad \omega_j = \frac{\alpha^2 k^2}{2m_F}, \qquad j = 3, \dots, N_F.$$

Example 11. Let $B_0 = B_1 = 0$ and $B_j = -A_j$ where $j = 2, \dots, N_f$. Therefore the solutions read

$$q_0(x) = \sqrt{A_0}\mathrm{sn}\,(\alpha x, k), \qquad q_1(x) = \sqrt{A_1}\mathrm{sn}\,(\alpha x, k), \qquad q_j(x) = \sqrt{-A_j}\mathrm{cn}\,(\alpha x, k).$$

Then we obtain for frequencies the following results

$$\omega_0 = \frac{\alpha^2(1-k^2)}{2m_{\mathrm{B}}} + V_0 + g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}A_1, \qquad \omega_1 = \frac{\alpha^2(1+k^2)}{2m_{\mathrm{F}}}, \qquad \omega_j = \frac{\alpha^2}{2m_{\mathrm{F}}}.$$

Example 12. Suppose $B_0 = -A_0$, $B_1 = 0$, $B_2 = -A_2$ and $B_j = -A_j/k^2$ where $j = 3, \dots, N_f$. The solutions have the form

$$q_0(x) = \sqrt{-A_0}\mathrm{cn}\,(\alpha x, k), \qquad q_1(x) = \sqrt{A_1}\mathrm{sn}\,(\alpha x, k),$$
$$q_2(x) = \sqrt{-A_2}\mathrm{cn}\,(\alpha x, k), \qquad q_j(x) = \sqrt{-A_j}\mathrm{dn}\,(\alpha x, k)/k.$$

The frequencies are

$$\omega_0 = \frac{V_0}{k^2} - \frac{\alpha^2}{2m_{\mathrm{B}}} + \frac{1-k^2}{k^2}(g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}A_2) + \frac{g_{\mathrm{BF}}}{k^2}A_1, \qquad \omega_1 = V_0 + \frac{\alpha^2(1-k^2)}{2m_{\mathrm{F}}},$$
$$\omega_2 = V_0 + \frac{\alpha^2(1-2k^2)}{2m_{\mathrm{F}}}, \qquad \omega_j = V_0 - \frac{\alpha^2 k^2}{2m_{\mathrm{F}}}.$$

Example 13. Let $B_0 = -A_0$, $B_1 = -A_1$ and $B_j = -A_j/k^2$ for $j = 2, \dots, N_f$. Then

$$q_0(x) = \sqrt{-A_0}\mathrm{cn}\,(\alpha x, k), \qquad q_1(x) = \sqrt{-A_1}\mathrm{cn}\,(\alpha x, k), \qquad q_j(x) = \sqrt{-A_j}\mathrm{dn}\,(\alpha x, k)/k,$$
$$\omega_0 = \frac{V_0}{k^2} - \frac{\alpha^2}{2m_{\mathrm{B}}} + \frac{1-k^2}{k^2}\left(g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}A_1\right), \qquad \omega_1 = V_0 + \frac{\alpha^2(1-2k^2)}{2m_{\mathrm{F}}},$$
$$\omega_j = V_0 - \frac{\alpha^2 k^2}{2m_{\mathrm{F}}}.$$

Example 14. Let $B_0 = -A_0/k^2$, $B_1 = -A_1$ and $B_j = -A_j/k^2$ for $j = 2, \dots, N_f$. Hence

$$q_0(x) = \sqrt{-A_0}\mathrm{dn}\,(\alpha x, k)/k, \quad q_1(x) = \sqrt{-A_1}\mathrm{cn}\,(\alpha x, k), \qquad q_j(x) = \sqrt{-A_j}\mathrm{dn}\,(\alpha x, k)/k,$$
$$\omega_0 = \frac{\alpha^2(k^2-2)}{2m_{\mathrm{B}}} + \frac{V_0}{k^2} + \frac{1-k^2}{k^2}(g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}A_1),$$
$$\omega_1 = \frac{V_0}{k^2} - \frac{\alpha^2}{2m_{\mathrm{F}}}, \qquad \omega_j = \frac{V_0}{k^2} + \frac{\alpha^2(k^2-2)}{2m_{\mathrm{F}}}.$$

Certainly these examples do not exhaust all possible combinations of solutions and it is easy to extend this list.

6 Vector soliton solutions

6.1 Vector bright-bright soliton solutions

When $k \to 1$, $\mathrm{sn}\,(\alpha x, 1) = \tanh(\alpha x)$ and $B_0 = -A_0$, $B_j = -A_j$ we obtain that the solutions read

$$q_0 = \sqrt{-A_0}\frac{1}{\cosh(\alpha x)}, \qquad q_j = \sqrt{-A_j}\frac{1}{\cosh(\alpha x)},$$

where $A_0 \le 0$ as well as $A_j \le 0$. Using equations (3.6)–(3.8) we have

$$\begin{aligned}
&A_0 = \frac{\alpha^2 - V_0 m_{\mathrm{F}}}{m_{\mathrm{F}} g_{\mathrm{BF}}}, \qquad V = V_0 \tanh^2(\alpha x),\\
&\sum_{j=1}^{N_f} A_j = \frac{\alpha^2}{g_{\mathrm{BF}}}\left(\frac{1}{m_{\mathrm{B}}} - \frac{g_{\mathrm{BB}}}{m_{\mathrm{F}} g_{\mathrm{BF}}}\right) - \frac{V_0}{g_{\mathrm{BF}}}\left(1 - \frac{g_{\mathrm{BB}}}{g_{\mathrm{BF}}}\right),\\
&\omega_0 = V_0 - \frac{1}{2m_B}\alpha^2, \qquad \omega_j = V_0 - \frac{1}{2m_F}\alpha^2.
\end{aligned}$$

As a consequence of the restrictions on A_0 and A_j one can get the following unequalities

$$\begin{aligned}
&g_{\mathrm{BF}} > 0, \qquad V_0 \ge \frac{\alpha^2}{m_{\mathrm{F}}}, \qquad g_{\mathrm{BB}} \le \frac{(\alpha^2 - m_{\mathrm{B}} V_0) m_{\mathrm{F}}}{(\alpha^2 - m_{\mathrm{F}} V_0) m_{\mathrm{B}}} g_{\mathrm{BF}},\\
&g_{\mathrm{BF}} < 0, \qquad V_0 \le \frac{\alpha^2}{m_{\mathrm{F}}}, \qquad g_{\mathrm{BB}} \ge \frac{(\alpha^2 - m_{\mathrm{B}} V_0) m_{\mathrm{F}}}{(\alpha^2 - m_{\mathrm{F}} V_0) m_{\mathrm{B}}} g_{\mathrm{BF}}.
\end{aligned}$$

Vector bright soliton solution when $V_0 = 0$ is derived for the first time in [11].

6.2 Vector dark-dark soliton solutions

When $k \to 1$ and $B_0 = B_j = 0$ are satisfied the solutions read

$$q_0(x) = \sqrt{A_0}\tanh(\alpha x), \qquad q_j(x) = \sqrt{A_j}\tanh(\alpha x).$$

The natural restrictions $A_0 \ge 0$ and $A_j \ge 0$ lead to

$$\begin{aligned}
&g_{\mathrm{BF}} > 0, \qquad g_{\mathrm{BB}} \le \frac{(\alpha^2 - m_{\mathrm{B}} V_0) m_{\mathrm{F}}}{(\alpha^2 - m_{\mathrm{F}} V_0) m_{\mathrm{B}}} g_{\mathrm{BF}}, \qquad V_0 \le \alpha^2/m_{\mathrm{F}},\\
&g_{\mathrm{BF}} < 0, \qquad g_{\mathrm{BB}} \ge \frac{(\alpha^2 - m_{\mathrm{B}} V_0) m_{\mathrm{F}}}{(\alpha^2 - m_{\mathrm{F}} V_0) m_{\mathrm{B}}} g_{\mathrm{BF}}, \qquad V_0 \ge \alpha^2/m_{\mathrm{F}},\\
&A_0 = \frac{\alpha^2 - m_{\mathrm{F}} V_0}{m_{\mathrm{F}} g_{\mathrm{BF}}}, \qquad \sum_j A_j = \frac{\alpha^2}{g_{\mathrm{BF}}}\left(\frac{1}{m_{\mathrm{B}}} - \frac{g_{\mathrm{BB}}}{m_F g_{\mathrm{FB}}}\right) - \frac{V_0}{g_{\mathrm{BF}}}\left(1 - \frac{g_{\mathrm{BB}}}{g_{\mathrm{BF}}}\right).
\end{aligned} \tag{6.1}$$

For the frequencies ω_0 and ω_j and the constants $\mathcal{C}_0$ and $\mathcal{C}_j$ we have

$$\omega_0 = \frac{\alpha^2}{m_{\mathrm{B}}}, \qquad \omega_j = \frac{\alpha^2}{m_{\mathrm{F}}}, \qquad \mathcal{C}_0 = \mathcal{C}_j = 0. \tag{6.2}$$

6.3 Vector bright-dark soliton solutions

When $k \to 1$, $B_0 = -A_0$ and $B_j = 0$, we have

$$q_0(x) = \frac{\sqrt{-A_0}}{\cosh(\alpha x)}, \qquad q_j(x) = \sqrt{A_j}\tanh(\alpha x),$$
$$\omega_0 = \frac{\alpha^2}{2m_{\rm B}} - g_{\rm BB}A_0, \qquad \omega_j = V_0, \qquad \mathcal{C}_0 = \mathcal{C}_j = 0.$$

The parameters A_0 and A_j are given by (6.1). In this case we have the following restrictions

$$g_{\rm BF} > 0, \qquad g_{\rm BB} \geq \frac{(\alpha^2 - m_{\rm B}V_0)m_{\rm F}}{(\alpha^2 - m_{\rm F}V_0)m_{\rm B}} g_{\rm BF}, \qquad V_0 \geq \alpha^2/m_{\rm F},$$
$$g_{\rm BF} < 0, \qquad g_{\rm BB} \leq \frac{(\alpha^2 - m_{\rm B}V_0)m_{\rm F}}{(\alpha^2 - m_{\rm F}V_0)m_{\rm B}} g_{\rm BF}, \qquad V_0 \leq \alpha^2/m_{\rm F}.$$

6.4 Vector dark-bright soliton solutions

When $k \to 1$ and provided that $B_0 = 0$ and $B_j = -A_j$ the result is

$$q_0(x) = \sqrt{A_0}\tanh(\alpha x), \qquad q_j(x) = \frac{\sqrt{-A_j}}{\cosh(\alpha x)}, \qquad \omega_0 = V_0 + A_0 g_{\rm BB}, \qquad \omega_j = \frac{\alpha^2}{2m_{\rm F}}.$$

By analogy with the previous examples the constants A_0, A_j, $\mathcal{C}_0$ and $\mathcal{C}_j$ are given by formulae (6.1) and (6.2) respectively. The restrictions now are

$$g_{\rm BF} > 0, \qquad g_{\rm BB} \geq \frac{(\alpha^2 - m_{\rm B}V_0)m_{\rm F}}{(\alpha^2 - m_{\rm F}V_0)m_{\rm B}} g_{\rm BF}, \qquad V_0 \leq \alpha^2/m_{\rm F},$$
$$g_{\rm BF} < 0, \qquad g_{\rm BB} \leq \frac{(\alpha^2 - m_{\rm B}V_0)m_{\rm F}}{(\alpha^2 - m_{\rm F}V_0)m_{\rm B}} g_{\rm BF}, \qquad V_0 \geq \alpha^2/m_{\rm F}.$$

6.5 Vector dark-dark-bright soliton solutions

Let $B_0 = B_1 = 0$ and $B_j = -A_j$ where $j = 2, \ldots, N_f$. Therefore the solutions read

$$q_0(x) = \sqrt{A_0}\tanh(\alpha x), \qquad q_1(x) = \sqrt{A_1}\tanh(\alpha x), \qquad q_j(x) = \sqrt{-A_j}\,\mathrm{sech}(\alpha x).$$

Then we obtain for frequencies the following results

$$\omega_0 = V_0 + g_{\rm BB}A_0 + g_{\rm BF}A_1, \qquad \omega_1 = \frac{\alpha^2}{m_{\rm F}}, \qquad \omega_j = \frac{\alpha^2}{2m_{\rm F}}.$$

These examples are by no means exhaustive.

6.6 Nontrivial phase, trigonometric limit

In this section we consider a trap potential of the form $V_{\rm trap} = V_0\cos(2\alpha x)$, as a model for an optical lattice. Our potential V is similar and differs only with additive constant. When $k \to 0$, $\mathrm{sn}\,(\alpha x, 0) = \sin(\alpha x)$

$$q_0^2 = A_0\sin^2(\alpha x) + B_0, \qquad q_j^2 = A_j\sin^2(\alpha x) + B_j, \tag{6.3}$$
$$V = V_0\sin^2(\alpha x) = \frac{1}{2}(V_0 - V_0\cos(2\alpha x)), \tag{6.4}$$

Table 3. $W = g_{\mathrm{BF}} m_{\mathrm{F}} W_{\mathrm{B}}/(m_{\mathrm{B}} W_{\mathrm{F}})$.

1	$\beta_0 \le 0$	$\beta_j \le 0$	$A_0 \ge 0$	$A_j \ge 0$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \lessgtr g_{\mathrm{BF}}$	$V_0 \lessgtr 0$
2	$\beta_0 \le 0$	$\beta_j \ge 1$	$A_0 \ge 0$	$A_j \le 0$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \gtrless g_{\mathrm{BF}}$	$V_0 \lessgtr 0$
3	$\beta_0 \ge 1$	$\beta_j \le 0$	$A_0 \le 0$	$A_j \ge 0$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \gtrless g_{\mathrm{BF}}$	$V_0 \gtrless 0$
4	$\beta_0 \ge 1$	$\beta_j \ge 1$	$A_0 \le 0$	$A_j \le 0$	$g_{\mathrm{BF}} \gtrless 0$	$g_{\mathrm{BB}} \lessgtr g_{\mathrm{BF}}$	$V_0 \gtrless 0$

Using equations (3.6)–(3.8) again we obtain the following result when (see Table 3)

$$A_0 = -\frac{V_0}{g_{\mathrm{BF}}}, \qquad \sum_{j=1}^{N_f} A_j = -\frac{V_0}{g_{\mathrm{BF}}}\left(1 - \frac{g_{\mathrm{BB}}}{g_{\mathrm{BF}}}\right),$$

$$\omega_0 = \frac{1}{2m_{\mathrm{B}}}\alpha^2 + B_0 g_{\mathrm{BB}} + g_{\mathrm{BF}}\sum_{i=1}^{N_f} B_i, \qquad \omega_j = \frac{1}{2m_{\mathrm{F}}}\alpha^2 + g_{\mathrm{BF}} B_0,$$

$$\mathcal{C}_0^2 = \alpha^2 B_0(A_0 + B_0), \qquad \mathcal{C}_j^2 = \alpha^2 B_j(A_j + B_j),$$

where

$$\Theta_0(x) = \arctan\left(\sqrt{\frac{A_0 + B_0}{B_0}}\tan(\alpha x)\right), \quad \Theta_j(x) = \arctan\left(\sqrt{\frac{A_j + B_j}{B_j}}\tan(\alpha x)\right).$$

This solution is the most important from the physical point of view [8].

7 Linear stability, preliminary results

To analyze linear stability of our initial system of equations we seek solutions in the form

$$\psi_0(x,t) = (q_0(x) + \varepsilon\phi_0(x,t))\exp\left(-\frac{i\omega_0}{\hbar}t + i\Theta_0(x) + i\kappa_0\right),$$

$$\psi_j(x,t) = (q_1(x) + \varepsilon\phi_j(x,t))\exp\left(-\frac{i\omega_j}{\hbar}t + i\Theta_1(x) + i\kappa_1\right).$$

and obtain the following linearized equations

$$\hbar\begin{pmatrix}\mathbf{\Phi}_0\\ \mathbf{\Phi}_1\\ \vdots\\ \mathbf{\Phi}_{N_f}\end{pmatrix}_{,t} = \begin{pmatrix}\mathbf{\Lambda}_0 & \mathbf{U_1} & \mathbf{U_2} & \dots & \mathbf{U_{N_f}}\\ \mathbf{V_1} & \mathbf{\Lambda}_1 & 0 & \dots & 0\\ \mathbf{V_2} & 0 & \mathbf{\Lambda}_2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{V_{N_f}} & 0 & 0 & \dots & \mathbf{\Lambda}_{N_f}\end{pmatrix}\begin{pmatrix}\mathbf{\Phi}_0\\ \mathbf{\Phi}_1\\ \vdots\\ \mathbf{\Phi}_{N_f}\end{pmatrix}, \qquad \mathbf{\Phi}_0 = \begin{pmatrix}\phi_0^{\mathrm{R}}\\ \phi_0^{\mathrm{I}},\end{pmatrix}, \qquad \mathbf{\Phi}_j = \begin{pmatrix}\phi_j^{\mathrm{R}}\\ \phi_j^{\mathrm{I}}\end{pmatrix},$$

where

$$\mathbf{\Lambda}_0 = \begin{pmatrix}S_0 & L_{0,-}\\ L_{0,+} & S_0\end{pmatrix}, \quad \mathbf{U_j} = \begin{pmatrix}0 & 0\\ U_{0,j} & 0\end{pmatrix}, \quad \mathbf{\Lambda}_j = \begin{pmatrix}S_j & L_{j,-}\\ L_{j,+} & S_j\end{pmatrix}, \quad \mathbf{V_j} = \begin{pmatrix}0 & 0\\ U_{1,j} & 0\end{pmatrix},$$

$$S_0 = -\frac{\mathcal{C}_0}{m_{\mathrm{B}} q_0}\partial_x\left(\frac{1}{q_0}\right), \qquad L_{0,-} = -\frac{1}{2m_{\mathrm{B}}}\left(\partial_{xx}^2 - \frac{\mathcal{C}_0^2}{q_0^4}\right) + V + g_{\mathrm{BB}} q_0^2 + g_{\mathrm{BF}} q_1^2 - \omega_0,$$

$$U_{0,j} = -2g_{\mathrm{BF}} q_0^2, \qquad L_{0,+} = \frac{1}{2m_{\mathrm{B}}}\left(\partial_{xx}^2 - \frac{\mathcal{C}_0^2}{q_0^4}\right) - V - 3g_{\mathrm{BB}} q_0^2 - g_{\mathrm{BF}} q_1^2 + \omega_0,$$

$$S_j = -\frac{\mathcal{C}_j}{m_{\mathrm{F}} q_j}\partial_x\left(\frac{1}{q_j}\right), \qquad L_{j,-} = -\frac{1}{2m_{\mathrm{F}}}\left(\partial_{xx}^2 - \frac{\mathcal{C}_j^2}{q_0^4}\right) + V + g_{\mathrm{BF}} q_0^2 - \omega_j,$$

$$U_{1,j} = -2g_{\rm BF}q_0q_j, \qquad L_{j,+} = \frac{1}{2m_{\rm F}}\left(\partial^2_{xx} - \frac{\mathcal{C}_j^2}{q_0^4}\right) - V - g_{\rm BF}q_0^2 + \omega_j, \qquad j = 1,\dots,N_f.$$

The analysis of the latter matrix system is a difficult problem and only numerical simulations are possible. Recently a great progress was achieved for analysis of linear stability of periodic solutions of type (3.1), (3.2) (see e.g. [15, 24, 25, 18, 26] and references therein). Nevertheless the stability analysis is known only for solutions of type (5.1)–(5.6) and solutions with nontrivial phase of type (6.3) and (6.4). Linear analysis of soliton solutions is well developed, but it is out scope of the present paper.

Finally we discuss three special cases:

Case I. Let $B_0 = B_j = 0$ then for $j = 1,\dots,N_f$ and $q_0 = \sqrt{A_0}\,\mathrm{sn}\,(\alpha x, k)$, $q_j = \sqrt{A_j}\,\mathrm{sn}\,(\alpha x, k)$ we have the following linearized equations:

$$\hbar\phi^{\rm R}_{0,t} = -\frac{1}{2m_{\rm B}}\partial^2_{xx}\phi^{\rm I}_0 + \left(V_0 + g_{\rm BB}A_0 + g_{\rm BF}\sum_j A_j\right)\mathrm{sn}^2(\alpha x, k)\phi^{\rm I}_0 - \omega_0\phi^{\rm I}_0,$$

$$\hbar\phi^{\rm I}_{0,t} = \frac{1}{2m_{\rm B}}\partial^2_{xx}\phi^{\rm R}_0 - \left(V_0 + 3g_{\rm BB}A_0 + g_{\rm BF}\sum_j A_j\right)\mathrm{sn}^2(\alpha x, k)\phi^{\rm R}_0$$
$$+ \omega_0\phi^{\rm R}_0 - 2g_{\rm BF}A_0\mathrm{sn}^2(\alpha x, k)\sum_j\phi^{\rm R}_j,$$

$$\hbar\phi^{\rm R}_{j,t} = -\frac{1}{2m_{\rm F}}\partial^2_{xx}\phi^{\rm I}_j + (V_0 + g_{\rm BF}A_0)\,\mathrm{sn}^2(\alpha x, k)\phi^{\rm I}_j - \omega_j\phi^{\rm I}_j,$$

$$\hbar\phi^{\rm I}_{j,t} = \frac{1}{2m_{\rm F}}\partial^2_{xx}\phi^{\rm R}_j - (V_0 + g_{\rm BF}A_0)\,\mathrm{sn}^2(\alpha x, k)\phi^{\rm R}_j + \omega_j\phi^{\rm R} - 2g_{\rm BF}\sqrt{A_0A_j}\,\mathrm{sn}^2(\alpha x, k)\phi^{\rm R}_0.$$

Case II. Let $B_0 = -A_0$, $B_j = -A_j$ then for $q_0 = \sqrt{-A_0}\,\mathrm{cn}\,(\alpha x, k)$, $q_j = \sqrt{-A_j}\,\mathrm{cn}\,(\alpha x, k)$ we obtain the following linearized equations:

$$\hbar\phi^{\rm R}_{0,t} = -\frac{1}{2m_{\rm B}}\partial^2_{xx}\phi^{\rm I}_0 + \left(V_0 + g_{\rm BB}A_0 + g_{\rm BF}\sum_j A_j\right)\mathrm{sn}^2(\alpha x, k)\phi^{\rm I}_0$$
$$- \left(g_{\rm BB}A_0 + g_{\rm BF}\sum_j A_j + \omega_0\right)\phi^{\rm I}_0,$$

$$\hbar\phi^{\rm I}_{0,t} = \frac{1}{2m_{\rm B}}\partial^2_{xx}\phi^{\rm R}_0 + \left(3g_{\rm BB}A_0 + g_{\rm BF}\sum_j A_j + \omega_0\right)\phi^{\rm R}_0$$
$$- \left(V_0 + 3g_{\rm BB}A_0 + g_{\rm BF}\sum_j A_j\right)\mathrm{sn}^2(\alpha x, k)\phi^{\rm R}_0 + 2g_{\rm BF}A_0\big(1 - \mathrm{sn}^2(\alpha x, k)\big)\sum_j\phi^{\rm R}_j,$$

$$\hbar\phi^{\rm R}_{j,t} = -\frac{1}{2m_{\rm F}}\partial^2_{xx}\phi^{\rm I}_j + (V_0 + g_{\rm BF}A_0)\mathrm{sn}^2(\alpha x, k)\phi^{\rm I}_j - (g_{\rm BF}A_0 + \omega_j)\phi^{\rm I}_j,$$

$$\hbar\phi^{\rm I}_{j,t} = \frac{1}{2m_{\rm F}}\partial^2_{xx}\phi^{\rm R}_j - (V_0 + g_{\rm BF}A_0)\mathrm{sn}^2(\alpha x, k)\phi^{\rm R}_j + (g_{\rm BF}A_0 + \omega_j)\phi^{\rm R}_j$$
$$- 2g_{\rm BF}\sqrt{A_0A_j}\big(1 - \mathrm{sn}^2(\alpha x, k)\big)\phi^{\rm R}_0, \qquad j = 1,\dots,N_f.$$

Case III. Let $B_0 = -A_0/k^2$, $B_j = -A_j/k^2$ therefore the solutions are

$$q_0 = \sqrt{-A_0}\,\mathrm{dn}\,(\alpha x, k)/k, \qquad q_j = \sqrt{-A_j}\,\mathrm{dn}\,(\alpha x, k)/k,$$

and we obtain the following linearized equations

$$
\begin{aligned}
\hbar\phi^{\mathrm{R}}_{0,t} &= -\frac{1}{2m_{\mathrm{B}}}\partial^2_{xx}\phi^{\mathrm{I}}_0 + \left(V_0 + g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j\right)\mathrm{sn}^2(\alpha x,k)\phi^{\mathrm{I}}_0 \\
&\quad - \left(g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j + k^2\omega_0\right)\frac{\phi^{\mathrm{I}}_0}{k^2}, \\
\hbar\phi^{\mathrm{I}}_{0,t} &= \frac{1}{2m_{\mathrm{B}}}\partial^2_{xx}\phi^{\mathrm{R}}_0 + \left(3g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j + k^2\omega_0\right)\frac{\phi^{\mathrm{R}}_0}{k^2}, \\
&\quad - \left(V_0 + 3g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j\right)\mathrm{sn}^2(\alpha x,k)\phi^{\mathrm{R}}_0 + \frac{2g_{\mathrm{BF}}A_0(1-k^2\mathrm{sn}^2(\alpha,k))}{k^2}\sum_j \phi^{\mathrm{R}}_j, \\
\hbar\phi^{\mathrm{R}}_{j,t} &= -\frac{1}{2m_{\mathrm{F}}}\partial^2_{xx}\phi^{\mathrm{I}}_j + (V_0 + g_{\mathrm{BF}}A_0)\,\mathrm{sn}^2(\alpha x,k)\phi^{\mathrm{I}}_j - \frac{g_{\mathrm{BF}}A_0 + k^2\omega_j}{k^2}\phi^{\mathrm{I}}_j, \\
\hbar\phi^{\mathrm{I}}_{j,t} &= \frac{1}{2m_{\mathrm{F}}}\partial^2_{xx}\phi^{\mathrm{R}}_j - (V_0 + g_{\mathrm{BF}}A_0)\,\mathrm{sn}^2(\alpha x,k)\phi^{\mathrm{R}}_j + \frac{g_{\mathrm{BF}}A_0 + k^2\omega_j}{k^2}\phi^{\mathrm{R}}_j \\
&\quad - \frac{2g_{\mathrm{BF}}\sqrt{A_0A_j}\big(1-k^2\mathrm{sn}^2(\alpha,k)\big)\phi^{\mathrm{R}}_0}{k^2}, \qquad j = 1,\ldots,N_f.
\end{aligned}
$$

These cases are by no means exhaustive.

8 Conclusions

In conclusion, we have considered the mean field model for boson-fermion mixtures in optical lattice. Classes of quasi-periodic, periodic, elliptic solutions, and solitons have been analyzed in detail. These solutions can be used as initial states which can generate localized matter waves (solitons) through the modulational instability mechanism. This important problem is under consideration.

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