Quadratic Algebra Approach to an Exactly Solvable Position-Dependent Mass Schrödinger Equation in Two Dimensions

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Abstract. An exactly solvable position-dependent mass Schrödinger equation in two dimensions, depicting a particle moving in a semi-infinite layer, is re-examined in the light of recent theories describing superintegrable two-dimensional systems with integrals of motion that are quadratic functions of the momenta. To get the energy spectrum a quadratic algebra approach is used together with a realization in terms of deformed parafermionic oscillator operators. In this process, the importance of supplementing algebraic considerations with a proper treatment of boundary conditions for selecting physical wavefunctions is stressed. Some new results for matrix elements are derived. This example emphasizes the interest of a quadratic algebra approach to position-dependent mass Schrödinger equations.

Key words: Schrödinger equation; position-dependent mass; quadratic algebra

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1 Introduction

Quantum mechanical systems with a position-dependent (effective) mass (PDM) have attracted a lot of attention and inspired intense research activities during recent years. They are indeed very useful in the study of many physical problems, such as electronic properties of semiconductors [1] and quantum dots [2], nuclei [3], quantum liquids [4], ^{3}He clusters [5], metal clusters [6], etc.

Looking for exact solutions of the Schrödinger equation with a PDM has become an interesting research topic because such solutions may provide a conceptual understanding of some physical phenomena, as well as a testing ground for some approximation schemes (for a list of references see, e.g., [7]). For such a purpose, use has been made of methods known in the constant-mass case and extended to a PDM context, such as point canonical transformations [8, 9, 10], Lie algebraic methods [11, 12, 13, 14], as well as supersymmetric quantum mechanical (SUSYQM) and shape-invariance techniques [15, 16].

Although mostly one-dimensional equations have been considered up to now, several works have recently paid attention to d-dimensional problems [7, 17, 18, 19, 20, 21, 22]. In [7] (henceforth referred to as I and whose equations will be quoted by their number preceded by I), we have analyzed d-dimensional PDM Schrödinger equations in the framework of first-order intertwining operators and shown that with a pair (H, H_1) of intertwined Hamiltonians we can associate another pair (R, R_1) of second-order partial differential operators related to the same intertwining operator and such that H (resp. H_1) commutes with R (resp. R_1). In the context of SUSYQM based on an sl(1/1) superalgebra, R and R_1 can be interpreted as SUSY partners, while H and H_1 are related to the Casimir operator of a larger gl(1/1) superalgebra.

In the same work, we have also applied our general theory to an explicit example, depicting a particle moving in a two-dimensional semi-infinite layer. This model may be of interest in the study of quantum wires with an abrupt termination in an environment that can be modelled by a dependence of the carrier effective mass on the position. It illustrates the influence of a uniformity breaking in a quantum channel on the production of bound states, as it was previously observed in the case of a quantum dot or a bend [23, 24].

From a theoretical viewpoint, our model has proved interesting too because it is solvable in two different ways: by separation of variables in the corresponding Schrödinger equation or employing SUSYQM and shape-invariance techniques. The former method relies upon the existence of an integral of motion L, while, as above-mentioned, the latter is based on the use of R. In other words, the three second-order partial differential operators H, L and R form a set of algebraically independent integrals of motion, which means that the system is superintegrable.

Let us recall that in classical mechanics [25], an integrable system on a d-dimensional manifold is a system which has d functionally independent (globally defined) integrals of motion in involution (including the Hamiltonian). Any system with more that d functionally independent integrals of motion is called superintegrable. It is maximally superintegrable if it admits the maximum number $2d-1$ of integrals of motion. The latter form a complete set so that any other integral of motion can be expressed in terms of them. In particular, the Poisson bracket of any two basic integrals, being again a constant of motion, can be written as a (in general) nonlinear function of them. Such results can be extended to quantum mechanics [26], so that for quantum counterparts of maximally superintegrable systems we get (in general) nonlinear associative algebras of algebraically independent observables, all of them commuting with H.

The simplest case corresponds to the class of two-dimensional superintegrable systems with integrals of motion that are linear and quadratic functions of the momenta. The study and classification of such systems, dating back to the 19th century and revived in the 1960ties [27, 28, 29], have recently been the subject of intense research activities and substantial progress has been made in this area (see [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48] and references quoted therein). In particular, it has been shown that their integrals of motion generate a quadratic Poisson algebra (in the classical case) or a quadratic associative algebra (in the quantum one) with a Casimir of sixth degree in the momenta and the general form of these algebras has been uncovered [36, 45, 46, 47, 48]. Algebras of this kind have many similarities to the quadratic Racah algebra QR(3) (a special case of the quadratic Askey–Wilson algebra QAW(3)) [31, 32]. They actually coincide with QR(3) whenever one of their parameters vanishes. The eigenvalues and eigenfunctions of the superintegrable system Hamiltonian can be found from the finite-dimensional irreducible representations of these algebras. The latter can be determined by a ladder-operator method [31, 32, 33, 34] or through a realization [35, 36] in terms of (generalized) deformed parafermionic operators [49], which are a finite-dimensional version of deformed oscillator operators [50].

Since our two-dimensional PDM model belongs to this class of superintegrable systems, it is interesting to analyze it in the light of such topical and innovative theories. This is one of the purposes of the present paper, which will therefore provide us with a third method for solving the PDM Schrödinger equation. In such a process, we will insist on the necessity of supplementing algebraic calculations with a proper treatment of the wavefunction boundary conditions imposed by the physics of the problem – a point that is not always highlighted enough.

Another purpose of this work is to stress the interest of a quadratic algebra approach to PDM Schrödinger equations. If the presence of such an algebra was already noted before in a one-dimensional example [51], this is indeed – as far as the author knows – the first case where an algebra of this kind is used as a tool for solving a physical problem in a PDM context.

This paper is organized as follows. In Section 2, the two-dimensional PDM model of I is briefly reviewed and some important comments on its mathematical structure are made in conjunction with the physics of the problem. In Section 3, a quadratic algebra associated with such a model is then introduced and its classical limit is obtained. The finite-dimensional irreducible representations of the algebra are determined in Section 4. Finally, Section 5 contains the conclusion.

2 Exactly solvable and superintegrable PDM model in a two-dimensional semi-infinite layer

In I we considered a particle moving in a two-dimensional semi-infinite layer of width π/q, parallel to the x-axis and with impenetrable barriers at the boundaries. The variables x, y vary in the domain

$$D: \qquad 0 < x < \infty, \qquad -\frac{\pi}{2q} < y < \frac{\pi}{2q},$$

and the wavefunctions have to satisfy the conditions

$$\psi(0, y) = 0, \qquad \psi\left(x, \pm\frac{\pi}{2q}\right) = 0. \tag{2.1}$$

The mass of the particle is $m(x) = m_0 M(x)$, where the dimensionless function $M(x)$ is given by

$$M(x) = \operatorname{sech}^2 qx. \tag{2.2}$$

In units wherein $\hbar = 2m_0 = 1$, the Hamiltonian of the model can be written as

$$H^{(k)} = -\partial_x \frac{1}{M(x)} \partial_x - \partial_y \frac{1}{M(x)} \partial_y + V_{\text{eff}}^{(k)}(x), \tag{2.3}$$

where we adopt the general form (I2.2) and

$$V_{\text{eff}}^{(k)}(x) = -q^2 \cosh^2 qx + q^2 k(k-1) \operatorname{csch}^2 qx \tag{2.4}$$

is an effective potential. This function includes some terms depending on the ambiguity parameters [52], which allow any ordering of the noncommutating momentum and PDM operators (see equation (I2.3)). In (2.4), the constant k is assumed positive and we have set an irrelevant additive constant v_0 to zero.

As shown in I, both the operators

$$L = -\partial_y^2$$

and

$$\begin{aligned} R^{(k)} &= \eta^{(k)\dagger} \eta^{(k)} \\ &= -\cosh^2 qx \sin^2 qy\, \partial_x^2 + 2 \sinh qx \cosh qx \sin qy \cos qy\, \partial_{xy}^2 - \sinh^2 qx \cos^2 qy\, \partial_y^2 \\ &\quad + q \sinh qx \cosh qx (1 - 4\sin^2 qy) \partial_x + q(1 + 4\sinh^2 qx) \sin qy \cos qy \partial_y \\ &\quad + q^2(\sinh^2 qx - \sin^2 qy - 3\sinh^2 qx \sin^2 qy) - q^2 k(1 + \operatorname{csch}^2 qx \sin^2 qy) \\ &\quad + q^2 k^2 \operatorname{csch}^2 qx \sin^2 qy, \end{aligned}$$

where

$$\begin{aligned} \eta^{(k)\dagger} &= -\cosh qx \sin qy\, \partial_x + \sinh qx \cos qy\, \partial_y - q \sinh qx \sin qy - qk \operatorname{csch} qx \sin qy, \\ \eta^{(k)} &= \cosh qx \sin qy\, \partial_x - \sinh qx \cos qy\, \partial_y + q \sinh qx \sin qy - qk \operatorname{csch} qx \sin qy, \end{aligned}$$

commute with $H^{(k)}$, although not with one another. Hence one may diagonalize either $H^{(k)}$ and L or $H^{(k)}$ and $R^{(k)}$ simultaneously. This leads to two alternative bases for the Hamiltonian eigenfunctions, corresponding to the eigenvalues

$$E_N^{(k)} = q^2(N+2)(N+2k+1), \qquad N = 0, 1, 2, \ldots, \tag{2.5}$$

with degeneracies

$$\deg(N) = \left[\frac{N}{2}\right] + 1, \tag{2.6}$$

where $[N/2]$ stands for the integer part of $N/2$.

The first basis is obtained by separating the variables x, y in the PDM Schrödinger equation and its members, associated with the eigenvalues $(l+1)^2q^2$ of L, read

$$\psi_{n,l}^{(k)}(x,y) = \phi_{n,l}^{(k)}(x)\chi_l(y), \qquad n, l = 0, 1, 2, \ldots, \tag{2.7}$$

with $N = 2n + l$,

$$\begin{aligned} \phi_{n,l}^{(k)}(x) &= \mathcal{N}_{n,l}^{(k)}(\tanh qx)^k(\operatorname{sech} qx)^{l+2}P_n^{(k-\frac{1}{2},l+1)}(1-2\tanh^2 qx), \\ \chi_l(y) &= \begin{cases} \sqrt{\dfrac{2q}{\pi}}\cos[(l+1)qy] & \text{for } l = 0, 2, 4, \ldots, \\ \sqrt{\dfrac{2q}{\pi}}\sin[(l+1)qy] & \text{for } l = 1, 3, 5, \ldots, \end{cases} \end{aligned} \tag{2.8}$$

and $\mathcal{N}_{n,l}^{(k)}$ a normalization constant given in equation (I3.18).

The second basis, resulting from the intertwining relation

$$\eta^{(k)}H^{(k)} = H_1^{(k)}\eta^{(k)}, \qquad H_1^{(k)} = H^{(k+1)} + 2q^2k,$$

and its Hermitian conjugate, can be built by successive applications of operators of type $\eta^{(k)\dagger}$,

$$\Psi_{N,N_0}^{(k)}(x,y) = \bar{\mathcal{N}}_{N,N_0}^{(k)}\eta^{(k)\dagger}\eta^{(k+1)\dagger}\cdots\eta^{(k+\nu-1)\dagger}\Psi_{N_0,N_0}^{(k+\nu)}(x,y), \tag{2.9}$$

on functions $\Psi_{N_0,N_0}^{(k+\nu)}(x,y)$, annihilated by $\eta^{(k+\nu)}$ and given in Eqs. (I3.28), (I3.32) and (I3.34). In (2.9), N_0 runs over 0, 2, 4,…, N or $N-1$, according to whether N is even or odd, while ν, defined by $\nu = N - N_0$, determines the $R^{(k)}$ eigenvalue

$$r_\nu^{(k)} = q^2\nu(\nu+2k), \qquad \nu = 0, 1, 2, \ldots. \tag{2.10}$$

Although an explicit expression of the normalization coefficient $\bar{\mathcal{N}}_{N,N_0}^{(k)}$ is easily obtained (see equation (I3.41)), this is not the case for $\Psi_{N,N_0}^{(k)}(x,y)$ (except for some low values of N and N_0), nor for the expansion of $\Psi_{N,N_0}^{(k)}(x,y)$ into the first basis eigenfunctions $\psi_{n,l}^{(k)}(x,y)$, which is given by rather awkward formulas (see equations (I3.46), (I3.51), (I3.55) and (I3.56)).

Before proceeding to a quadratic algebra approach to the problem in Section 3, it is worth making a few valuable observations, which were not included in I.

Mathematically speaking, the separable Schrödinger equation of our model admits four linearly independent solutions obtained by combining the two independent solutions of the second-order differential equation in x with those of the second-order differential equation in y. Among those four functions, only the combination $\psi_{n,l}^{(k)}(x,y)$, considered in (2.7), satisfies all the boundary conditions and is normalizable on D. It is indeed clear that the alternative solution to the differential equation in x is not normalizable, while that to the differential equation in y,

$$\bar{\chi}_l(y) \propto \begin{cases} \sin[(l+1)qy] & \text{for } l = 0, 2, 4, \ldots, \\ \cos[(l+1)qy] & \text{for } l = -1, 1, 3, 5, \ldots, \end{cases} \tag{2.11}$$

violates the second condition in equation (2.1). Hence the three remaining combinations provide unphysical functions.

Some mathematical considerations might also lead to another choice than L and $R^{(k)}$ for the basic integrals of motion complementing $H^{(k)}$. First of all, instead of L, one might select the operator $p_y = -\mathrm{i}\partial_y$, which obviously satisfies the condition $[H^{(k)}, p_y] = 0$. This would result in a linear and a quadratic (in the momenta) integrals of motion, generating a much simpler quadratic algebra than that to be considered in Section 3. It should be realized, however, that the eigenfunctions $e^{\mathrm{i}my}$ $(m \in \mathbb{Z})$ of p_y, being linear combinations of the physical and unphysical functions (2.8) and (2.11), are useless from a physical viewpoint. We are therefore forced to consider the second-order operator L instead of p_y.

Furthermore, it is straightforward to see that another pair of first-order differential operators

$$\bar{\eta}^{(k)\dagger} = -\cosh qx \cos qy\, \partial_x - \sinh qx \sin qy\, \partial_y - q \sinh qx \cos qy - qk \operatorname{csch} qx \cos qy, \tag{2.12}$$

$$\bar{\eta}^{(k)} = \cosh qx \cos qy\, \partial_x + \sinh qx \sin qy\, \partial_y + q \sinh qx \cos qy - qk \operatorname{csch} qx \cos qy, \tag{2.13}$$

intertwines with $H^{(k)}$ and $H_1^{(k)}$, i.e., satisfies the relation

$$\bar{\eta}^{(k)} H^{(k)} = H_1^{(k)} \bar{\eta}^{(k)}, \qquad H_1^{(k)} = H^{(k+1)} + 2q^2 k, \tag{2.14}$$

and its Hermitian conjugate. Such operators correspond to the choice $a = c = g = 0$, $b = d = 1$ in equation (I2.29).

As a consequence of (2.14), the operator

$$\begin{aligned}
\bar{R}^{(k)} &= \bar{\eta}^{(k)\dagger} \bar{\eta}^{(k)} \\
&= -\cosh^2 qx \cos^2 qy\, \partial_x^2 - 2 \sinh qx \cosh qx \sin qy \cos qy\, \partial_{xy}^2 - \sinh^2 qx \sin^2 qy\, \partial_y^2 \\
&\quad + q \sinh qx \cosh qx (1 - 4\cos^2 qy)\partial_x - q(1 + 4\sinh^2 qx) \sin qy \cos qy \partial_y \\
&\quad + q^2(\sinh^2 qx - \cos^2 qy - 3\sinh^2 qx \cos^2 qy) - q^2 k(1 + \operatorname{csch}^2 qx \cos^2 qy) \\
&\quad + q^2 k^2 \operatorname{csch}^2 qx \cos^2 qy,
\end{aligned}$$

commutes with $H^{(k)}$ and is therefore another integral of motion. It can of course be expressed in terms of $H^{(k)}$, L and $R^{(k)}$, as it can be checked that

$$H^{(k)} = L + R^{(k)} + \bar{R}^{(k)} + 2q^2 k.$$

However, we have now at our disposal three (dependent) integrals of motion L, $R^{(k)}$ and $\bar{R}^{(k)}$ in addition to $H^{(k)}$, so that we may ask the following question: what is the best choice for the basic integrals of motion from a physical viewpoint?

This problem is easily settled by noting that the zero modes of $\bar{\eta}^{(k)}$,

$$\bar{\omega}_s^{(k)}(x, y) = (\tanh qx)^k (\operatorname{sech} qx)^{s+1} (\sin qy)^s,$$

violate the second condition in equation (2.1) for any real value of s and therefore lead to unphysical functions. This contrasts with what happens for the zero modes $\omega_s^{(k)}(x, y)$ of $\eta^{(k)}$, given in (I3.28), which are physical functions for $s > 0$ and can therefore be used to build the functions $\Psi_{N,N_0}^{(k)}(x, y)$ considered in (2.9), as it was shown in (I3.32). We conclude that the physics of the model imposes the choice of L and $R^{(k)}$ as basic integrals of motion.

3 Quadratic associative algebra and its classical limit

It has been shown [36, 47] that for any two-dimensional quantum superintegrable system with integrals of motion A, B, which are second-order differential operators, one can construct a quadratic associative algebra generated by A, B, and their commutator C. This operator is not independent of A, B, but since it is a third-order differential operator, it cannot be written as a polynomial function of them. The general form of the quadratic algebra commutation relations is

$$[A,B] = C, \tag{3.1}$$
$$[A,C] = \alpha A^2 + \gamma\{A,B\} + \delta A + \epsilon B + \zeta, \tag{3.2}$$
$$[B,C] = aA^2 - \gamma B^2 - \alpha\{A,B\} + dA - \delta B + z. \tag{3.3}$$

Here $\{A,B\} \equiv AB + BA$,

$$\delta = \delta(H) = \delta_0 + \delta_1 H, \qquad \epsilon = \epsilon(H) = \epsilon_0 + \epsilon_1 H, \qquad \zeta = \zeta(H) = \zeta_0 + \zeta_1 H + \zeta_2 H^2,$$
$$d = d(H) = d_0 + d_1 H, \qquad z = z(H) = z_0 + z_1 H + z_2 H^2,$$

and α, γ, a, δ_i, ϵ_i, ζ_i, d_i, z_i are some constants. Note that it is the Jacobi identity $[A,[B,C]] = [B,[A,C]]$ that imposes some relations between coefficients in (3.2) and (3.3).

Such a quadratic algebra closes at level 6 [47] or, in other words, it has a Casimir operator which is a sixth-order differential operator [36],

$$\begin{aligned} K &= C^2 + \tfrac{2}{3}aA^3 - \tfrac{1}{3}\alpha\{A,A,B\} - \tfrac{1}{3}\gamma\{A,B,B\} + \left(\tfrac{2}{3}\alpha^2 + d + \tfrac{2}{3}a\gamma\right)A^2 \\ &\quad + \left(\tfrac{1}{3}\alpha\gamma - \delta\right)\{A,B\} + \left(\tfrac{2}{3}\gamma^2 - \epsilon\right)B^2 + \left(\tfrac{2}{3}\alpha\delta + \tfrac{1}{3}a\epsilon + \tfrac{1}{3}d\gamma + 2z\right)A \\ &\quad + \left(-\tfrac{1}{3}\alpha\epsilon + \tfrac{2}{3}\gamma\delta - 2\zeta\right)B + \tfrac{1}{3}\gamma z - \tfrac{1}{3}\alpha\zeta \\ &= k_0 + k_1 H + k_2 H^2 + k_3 H^3, \end{aligned} \tag{3.4}$$

where k_i are some constants and $\{A,B,C\} \equiv ABC + ACB + BAC + BCA + CAB + CBA$.

For our two-dimensional PDM model, described by the Hamiltonian defined in equations (2.2)–(2.4), we shall take

$$A = R, \qquad B = L, \tag{3.5}$$

where, for simplicity's sake, we dropped the superscript (k) because no confusion can arise outside the SUSYQM context.

To determine their commutation relations, it is worth noting first that their building blocks, the first-order differential operators ∂_y, $\eta^\dagger$ and η, generate another quadratic algebra together with the other set of intertwining operators $\bar{\eta}^\dagger$, $\bar{\eta}$, given in (2.12) and (2.13). Their commutation relations are indeed easily obtained as

$$[\partial_y, \eta] = q\bar{\eta}, \qquad [\partial_y, \bar{\eta}] = -q\eta, \qquad [\eta, \bar{\eta}] = q\partial_y, \tag{3.6}$$
$$[\eta, \eta^\dagger] = 2q^2k(1+\xi^2), \qquad [\bar{\eta}, \bar{\eta}^\dagger] = 2q^2k(1+\bar{\xi}^2), \qquad [\eta, \bar{\eta}^\dagger] = -q\partial_y + 2q^2k\xi\bar{\xi}, \tag{3.7}$$

and their Hermitian conjugates. In (3.7), we have defined

$$\xi = -(2qk)^{-1}(\eta + \eta^\dagger) = \operatorname{csch} qx \sin qy, \qquad \bar{\xi} = -(2qk)^{-1}(\bar{\eta} + \bar{\eta}^\dagger) = \operatorname{csch} qx \cos qy.$$

Interestingly, ∂_y, η and $\bar{\eta}$ (as well as ∂_y, $\eta^\dagger$ and $\bar{\eta}^\dagger$) close an sl(2) subalgebra.

From these results, it is now straightforward to show that the operator C in (3.1) is given by

$$C = q\{\partial_y, \eta^\dagger\bar{\eta} + \bar{\eta}^\dagger\eta\}$$

and that the coefficients in (3.2) and (3.3) are

$$\begin{aligned}&\alpha=\gamma=8q^2, \qquad \delta=8q^2[q^2(2k-1)-H], \qquad \epsilon=16q^4(k-1)(k+1),\\ &\zeta=8q^4(k-1)(2q^2k-H), \qquad a=0, \qquad d=16q^4, \qquad z=8q^4(2q^2k-H).\end{aligned} \tag{3.8}$$

On inserting the latter in (3.4), we obtain for the value of the Casimir operator

$$K=-4q^4[2q^2(7k-6)-3H](2q^2k-H).$$

It is worth noting that since $a=0$ in (3.3), we actually have here an example of quadratic Racah algebra QR(3) [31].

Before proceeding to a study of its finite-dimensional irreducible representations in Section 4, it is interesting to consider its classical limit. For such a purpose, since we have adopted units wherein $\hbar=2m_0=1$, we have first to make a change of variables and of parameters restoring a dependence on $\hbar$ (but keeping $2m_0=1$ for simplicity's sake) before letting $\hbar$ go to zero.

An appropriate transformation is

$$X=\hbar x, \qquad Y=\hbar y, \qquad P_X=-\mathrm{i}\hbar\partial_X, \qquad P_Y=-\mathrm{i}\hbar\partial_Y, \qquad Q=\frac{q}{\hbar}, \qquad K=\hbar k.$$

On performing it on the Hamiltonian given in equations (2.2)–(2.4), we obtain

$$H=-\hbar^2(\partial_X\cosh^2 QX\partial_X+\partial_Y\cosh^2 QX\partial_Y)-\hbar^2Q^2\cosh^2 QX+Q^2K(K-\hbar)\operatorname{csch}^2 QX,$$

yielding the classical Hamiltonian

$$H_{\mathrm{c}}=\lim_{\hbar\to 0}H=\cosh^2 QX(P_X^2+P_Y^2)+Q^2K^2\operatorname{csch}^2 QX.$$

A similar procedure applied to the intertwining operators leads to

$$\begin{aligned}\eta_{\mathrm{c}}&=\lim_{\hbar\to 0}\eta=\mathrm{i}\cosh QX\sin QY P_X-\mathrm{i}\sinh QX\cos QY P_Y-QK\operatorname{csch} QX\sin QY,\\ \bar{\eta}_{\mathrm{c}}&=\lim_{\hbar\to 0}\bar{\eta}=\mathrm{i}\cosh QX\cos QY P_X+\mathrm{i}\sinh QX\sin QY P_Y-QK\operatorname{csch} QX\cos QY,\end{aligned}$$

together with $\eta_{\mathrm{c}}^*=\lim_{\hbar\to 0}\eta^\dagger$ and $\bar{\eta}_{\mathrm{c}}^*=\lim_{\hbar\to 0}\bar{\eta}^\dagger$, while the operators quadratic in the momenta give rise to the functions

$$\begin{aligned}L_{\mathrm{c}}&=\lim_{\hbar\to 0}L=P_Y^2,\\ R_{\mathrm{c}}&=\lim_{\hbar\to 0}R=\cosh^2 QX\sin^2 QY P_X^2-2\sinh QX\cosh QX\sin QY\cos QY P_XP_Y\\ &\qquad+\sinh^2 QX\cos^2 QY P_Y^2+Q^2K^2\operatorname{csch}^2 QX\sin^2 QY,\\ \bar{R}_{\mathrm{c}}&=\lim_{\hbar\to 0}\bar{R}=\cosh^2 QX\cos^2 QY P_X^2+2\sinh QX\cosh QX\sin QY\cos QY P_XP_Y\\ &\qquad+\sinh^2 QX\sin^2 QY P_Y^2+Q^2K^2\operatorname{csch}^2 QX\cos^2 QY,\end{aligned}$$

satisfying the relation

$$H_{\mathrm{c}}=L_{\mathrm{c}}+R_{\mathrm{c}}+\bar{R}_{\mathrm{c}}.$$

The quadratic associative algebra (3.1)–(3.4) is now changed into a quadratic Poisson algebra, whose defining relations can be determined either by taking the limit $\lim_{\hbar\to 0}(\mathrm{i}\hbar)^{-1}[O,O']=\{O_{\mathrm{c}},O'_{\mathrm{c}}\}_{\mathrm{P}}$ or by direct calculation of the Poisson brackets $\{O_{\mathrm{c}},O'_{\mathrm{c}}\}_{\mathrm{P}}$:

$$\{A_{\mathrm{c}},B_{\mathrm{c}}\}_{\mathrm{P}}=C_{\mathrm{c}},$$

$$\{A_{\rm c}, C_{\rm c}\}_{\rm P} = \alpha_{\rm c} A_{\rm c}^2 + 2\gamma_{\rm c} A_{\rm c} B_{\rm c} + \delta_{\rm c} A_{\rm c} + \epsilon_{\rm c} B_{\rm c} + \zeta_{\rm c},$$
$$\{B_{\rm c}, C_{\rm c}\}_{\rm P} = a_{\rm c} A_{\rm c}^2 - \gamma_{\rm c} B_{\rm c}^2 - 2\alpha_{\rm c} A_{\rm c} B_{\rm c} + d_{\rm c} A_{\rm c} - \delta_{\rm c} B_{\rm c} + z_{\rm c}.$$

Here

$$C_{\rm c} = \lim_{\hbar\to 0} \frac{C}{{\rm i}\hbar} = 2QP_Y(\eta_{\rm c}^* \bar{\eta}_{\rm c} + \bar{\eta}_{\rm c}^* \eta_{\rm c})$$

and

$$\alpha_{\rm c} = \gamma_{\rm c} = -8Q^2, \qquad \delta_{\rm c} = 8Q^2 H_{\rm c}, \qquad \epsilon_{\rm c} = -16Q^4K^2, \qquad \zeta_{\rm c} = a_{\rm c} = d_{\rm c} = z_{\rm c} = 0.$$

Such a Poisson algebra has a vanishing Casimir:

$$K_{\rm c} = \lim_{\hbar\to 0} K = 0.$$

4 Finite-dimensional irreducible representations of the quadratic associative algebra

The quadratic algebra (3.1)–(3.4) can be realized in terms of (generalized) deformed oscillator operators $\mathcal{N}$, $b^\dagger$, b, satisfying the relations [50]

$$[\mathcal{N}, b^\dagger] = b^\dagger, \qquad [\mathcal{N}, b] = -b, \qquad b^\dagger b = \Phi(\mathcal{N}), \qquad bb^\dagger = \Phi(\mathcal{N}+1),$$

where the structure function $\Phi(x)$ is a 'well-behaved' real function such that

$$\Phi(0) = 0, \qquad \Phi(x) > 0 \quad \text{for} \quad x > 0. \tag{4.1}$$

This deformed oscillator algebra has a Fock-type representation, whose basis states $|m\rangle$, $m = 0, 1, 2, \ldots,$[1] fulfil the relations

$$\begin{aligned}
&\mathcal{N}|m\rangle = m|m\rangle, \\
&b^\dagger|m\rangle = \sqrt{\Phi(m+1)}\,|m+1\rangle, \qquad m = 0, 1, 2, \ldots, \\
&b|0\rangle = 0, \\
&b|m\rangle = \sqrt{\Phi(m)}\,|m-1\rangle, \qquad m = 1, 2, \ldots.
\end{aligned} \tag{4.2}$$

We shall be more specifically interested here in a subclass of deformed oscillator operators, which have a $(p+1)$-dimensional Fock space, spanned by $|p, m\rangle \equiv |m\rangle$, $m = 0, 1, \ldots, p$, due to the following property

$$\Phi(p+1) = 0 \tag{4.3}$$

of the structure function, implying that

$$(b^\dagger)^{p+1} = b^{p+1} = 0.$$

These are so-called (generalized) deformed parafermionic oscillator operators of order p [49]. The general form of their structure function is given by

$$\Phi(x) = x(p+1-x)(a_0 + a_1 x + a_2 x^2 + \cdots + a_{p-1} x^{p-1}),$$

[1] We adopt here the unusual notation $|m\rangle$ in order to avoid confusion between the number of deformed bosons and the quantum number n introduced in (2.7).

where $a_0, a_1, \dots, a_{p-1}$ may be any real constants such that the second condition in (4.1) is satisfied for $x = 1, 2, \dots, p$.

A realization of the quadratic algebra (3.1)–(3.4) in terms of deformed oscillator operators $\mathcal{N}$, $b^\dagger$, b reads [36]

$$A = A(\mathcal{N}), \tag{4.4}$$
$$B = \sigma(\mathcal{N}) + b^\dagger \rho(\mathcal{N}) + \rho(\mathcal{N}) b, \tag{4.5}$$

where $A(\mathcal{N})$, $\sigma(\mathcal{N})$ and $\rho(\mathcal{N})$ are some functions of $\mathcal{N}$, which, in the $\gamma \neq 0$ case, are given by

$$A(\mathcal{N}) = \frac{\gamma}{2}\left[(\mathcal{N}+u)^2 - \frac{1}{4} - \frac{\epsilon}{\gamma^2}\right], \tag{4.6}$$
$$\sigma(\mathcal{N}) = -\frac{\alpha}{4}\left[(\mathcal{N}+u)^2 - \frac{1}{4}\right] + \frac{\alpha\epsilon - \gamma\delta}{2\gamma^2} - \frac{\alpha\epsilon^2 - 2\gamma\delta\epsilon + 4\gamma^2\zeta}{4\gamma^4}\frac{1}{(\mathcal{N}+u)^2 - \frac{1}{4}}, \tag{4.7}$$
$$\rho^2(\mathcal{N}) = \frac{1}{3\cdot 2^{12}\gamma^8(\mathcal{N}+u)(\mathcal{N}+u+1)[2(\mathcal{N}+u)+1]^2}, \tag{4.8}$$

with the structure function

$$\begin{aligned}
\Phi(x) = {} & -3072\gamma^6 K[2(\mathcal{N}+u)-1]^2 \\
& - 48\gamma^6(\alpha^2\epsilon - \alpha\gamma\delta + a\gamma\epsilon - d\gamma^2)[2(\mathcal{N}+u)-3][2(\mathcal{N}+u)-1]^4[2(\mathcal{N}+u)+1] \\
& + \gamma^8(3\alpha^2 + 4a\gamma)[2(\mathcal{N}+u)-3]^2[2(\mathcal{N}+u)-1]^4[2(\mathcal{N}+u)+1]^2 \\
& + 768(\alpha\epsilon^2 - 2\gamma\delta\epsilon + 4\gamma^2\zeta)^2 \\
& + 32\gamma^4(3\alpha^2\epsilon^2 - 6\alpha\gamma\delta\epsilon + 2a\gamma\epsilon^2 + 2\gamma^2\delta^2 - 4d\gamma^2\epsilon + 8\gamma^3 z + 4\alpha\gamma^2\zeta) \\
& \times [2(\mathcal{N}+u)-1]^2[12(\mathcal{N}+u)^2 - 12(\mathcal{N}+u) - 1] \\
& - 256\gamma^2(3\alpha^2\epsilon^3 - 9\alpha\gamma\delta\epsilon^2 + a\gamma\epsilon^3 + 6\gamma^2\delta^2\epsilon - 3d\gamma^2\epsilon^2 + 2\gamma^4\delta^2 + 2d\gamma^4\epsilon + 12\gamma^3\epsilon z \\
& - 4\gamma^5 z + 12\alpha\gamma^2\epsilon\zeta - 12\gamma^3\delta\zeta + 4\alpha\gamma^4\zeta)[2(\mathcal{N}+u)-1]^2.
\end{aligned} \tag{4.9}$$

These functions depend upon two (so far undetermined) constants, u and the eigenvalue of the Casimir operator K (which we denote by the same symbol).

Such a realization is convenient to determine the representations of the quadratic algebra in a basis wherein the generator A is diagonal together with K (or, equivalently, H) because the former is already diagonal with eigenvalues $A(m)$. The $(p+1)$-dimensional representations, associated with $(p+1)$-fold degenerate energy levels, correspond to the restriction to deformed parafermionic operators of order p [36]. The first condition in (4.1) can then be used with equation (4.3) to compute u and K (or E) in terms of p and of the Hamiltonian parameters. A choice is then made between the various solutions that emerge from the calculations by imposing the second restriction in (4.1) for $x = 1, 2, \dots, p$.

In the present case, for the set of parameters (3.8), the complicated structure function (4.9) drastically simplifies to yield the factorized expression

$$\begin{aligned}
\Phi(x) = {} & 3\cdot 2^{30}q^{20}(2x+2u+k-1)(2x+2u+k-2)(2x+2u-k)(2x+2u-k-1) \\
& \times \left(2x+2u-\tfrac{1}{2}+\Delta\right)\left(2x+2u-\tfrac{3}{2}+\Delta\right)\left(2x+2u-\tfrac{1}{2}-\Delta\right)\left(2x+2u-\tfrac{3}{2}-\Delta\right),
\end{aligned}$$

where

$$\Delta = \sqrt{\left(k-\frac{1}{2}\right)^2 + \frac{E}{q^2}}.$$

Furthermore, the eigenvalues of the operator A become

$$A(m) = q^2(2m + 2u - k)(2m + 2u + k).$$

Since $A = R$ is a positive-definite operator, only values of u such that $A(m) \geq 0$ for $m = 0$, $1, \ldots, p$ should be retained.

On taking this into account, the first condition in (4.1) can be satisfied by choosing either $u = k/2$ or $u = (k+1)/2$, yielding

$$A(m) = 4q^2 m(m + k) \tag{4.10}$$

or

$$A(m) = 4q^2 \left(m + \tfrac{1}{2}\right) \left(m + k + \tfrac{1}{2}\right), \tag{4.11}$$

respectively. For $u = k/2$, equation (4.3), together with the second condition in (4.1), can be fulfilled in two different ways corresponding to $\Delta = 2p + k + 1 \pm \frac{1}{2}$ or

$$E = q^2 \left(2p + \tfrac{3}{2} \pm \tfrac{1}{2}\right) \left(2p + 2k + \tfrac{1}{2} \pm \tfrac{1}{2}\right). \tag{4.12}$$

The resulting structure function reads

$$\begin{aligned} \Phi(x) = {} & 3 \cdot 2^{38} q^{20} x(p + 1 - x) \left(x - \tfrac{1}{2}\right) \left(p + 1 \pm \tfrac{1}{2} - x\right) \left(x + k - \tfrac{1}{2}\right) (x + k - 1) \\ & \times \left(x + p + k + \tfrac{1}{4} \pm \tfrac{1}{4}\right) \left(x + p + k - \tfrac{1}{4} \pm \tfrac{1}{4}\right). \end{aligned} \tag{4.13}$$

Similarly, for $u = (k+1)/2$, we obtain

$$E = q^2 \left(2p + \tfrac{5}{2} \pm \tfrac{1}{2}\right) \left(2p + 2k + \tfrac{3}{2} \pm \tfrac{1}{2}\right) \tag{4.14}$$

and

$$\begin{aligned} \Phi(x) = {} & 3 \cdot 2^{38} q^{20} x(p + 1 - x) \left(x + \tfrac{1}{2}\right) \left(p + 1 \pm \tfrac{1}{2} - x\right) (x + k) \left(x + k - \tfrac{1}{2}\right) \\ & \times \left(x + p + k + \tfrac{5}{4} \pm \tfrac{1}{4}\right) \left(x + p + k + \tfrac{3}{4} \pm \tfrac{1}{4}\right). \end{aligned} \tag{4.15}$$

Our quadratic algebra approach has therefore provided us with a purely algebraic derivation of the eigenvalues of H and R in a basis wherein they are simultaneously diagonal. It now remains to see to which eigenvalues we can associate physical wavefunctions, i.e., normalizable functions satisfying equation (2.1). This will imply a correspondence between $|p, m\rangle$ and the functions $\Psi_{N,N-\nu}(x, y)$, defined in (2.9).

On comparing $A(m)$ to the known (physical) eigenvalues r_ν of R, given in (2.10), we note that the first choice (4.10) for $A(m)$ corresponds to even $\nu = 2m$ (hence to even N), while the second choice (4.11) is associated with odd $\nu = 2m + 1$ (hence with odd N). Appropriate values of p leading to the level degeneracies (2.6) are therefore $p = N/2$ and $p = (N-1)/2$, respectively. With this identification, both equations (4.12) and (4.14) yield the same result

$$E = q^2 \left(N + \tfrac{3}{2} \pm \tfrac{1}{2}\right) \left(N + 2k + \tfrac{1}{2} \pm \tfrac{1}{2}\right). \tag{4.16}$$

Comparison with (2.5) shows that only the upper sign choice in (4.16) leads to physical wavefunctions $\Psi_{N,N-\nu}(x, y)$.

Restricting ourselves to such a choice, we can now rewrite all the results obtained in this section in terms of N and ν instead of p and m. In particular, the two expressions (4.13) and (4.15) for the structure function can be recast in a single form $\Phi(m) \to \Phi_\nu$, where

$$\begin{aligned} \Phi_\nu = {} & 3 \cdot 2^{30} q^{20} \nu(\nu - 1)(\nu + 2k - 1)(\nu + 2k - 2)(N + \nu + 2k)(N + \nu + 2k + 1) \\ & \times (N - \nu + 2)(N - \nu + 3). \end{aligned} \tag{4.17}$$

More importantly, our quadratic algebra analysis provides us with an entirely new result, namely the matrix elements of the integral of motion L in the basis wherein H and R are simultaneously diagonal. On using indeed the correspondence $|p,m\rangle \to \Psi_{N,N-\nu}$, as well as the results in equations (4.2), (4.5), (4.7), (4.8) and (4.17), we obtain

$$L\Psi_{N,N-\nu} = \sigma_\nu \Psi_{N,N-\nu} + \tau_\nu \Psi_{N,N-\nu+2} + \tau_{\nu+2}\Psi_{N,N-\nu-2}, \tag{4.18}$$

where we have reset $\sigma(m) \to \sigma_\nu$, $\rho(m) \to \rho_\nu$ and defined $\tau_\nu = s_\nu \rho_{\nu-2}\sqrt{\Phi_\nu}$. The explicit form of the coefficients on the right-hand side of (4.18) is given by

$$\begin{aligned}\sigma_\nu = {} & \frac{q^2}{2(\nu+k-1)(\nu+k+1)}\{-(\nu+k-1)^2(\nu+k+1)^2 \\ & + [N^2+(2k+3)N+2k^2+2k+1](\nu+k-1)(\nu+k+1) \\ & - k(k-1)(N+k+1)(N+k+2)\},\end{aligned} \tag{4.19}$$

$$\begin{aligned}\tau_\nu^2 = {} & \frac{q^4}{16(\nu+k-2)(\nu+k-1)^2(\nu+k)}\nu(\nu-1)(\nu+2k-1)(\nu+2k-2) \\ & \times (N-\nu+2)(N-\nu+3)(N+\nu+2k)(N+\nu+2k+1).\end{aligned} \tag{4.20}$$

Note that τ_ν is determined up to some phase factor s_ν depending on the convention adopted for the relative phases of $\Psi_{N,N-\nu}$ and $\Psi_{N,N-\nu+2}$.

For $N = 4$, for instance, ν runs over 0, 2, 4, so that equations (4.18)–(4.20) become

$$\begin{aligned}L\Psi_{4,0} &= \frac{q^2}{k+3}\left[(13k+21)\Psi_{4,0} + 3s_4\sqrt{\frac{2(k+1)(2k+3)(2k+9)}{k+2}}\Psi_{4,2}\right], \\ L\Psi_{4,2} &= q^2\left[\frac{3s_4}{k+3}\sqrt{\frac{2(k+1)(2k+3)(2k+9)}{k+2}}\Psi_{4,0} + \frac{17k^2+76k+39}{(k+1)(k+3)}\Psi_{4,2}\right. \\ &\quad \left. + \frac{s_2}{k+1}\sqrt{\frac{10(k+3)(2k+1)(2k+7)}{k+2}}\Psi_{4,4}\right], \\ L\Psi_{4,4} &= \frac{q^2}{k+1}\left[s_2\sqrt{\frac{10(k+3)(2k+1)(2k+7)}{k+2}}\Psi_{4,2} + 5(k+3)\Psi_{4,4}\right].\end{aligned}$$

As a check, these results can be compared with those derived from the action of L on the expansions of $\Psi_{4,0}$, $\Psi_{4,2}$ and $\Psi_{4,4}$ in terms of the first basis eigenfunctions $\psi_{0,4}$, $\psi_{1,2}$ and $\psi_{2,0}$ (see, e.g., equations (I3.61) and (I3.49) for $\Psi_{4,0}$ and $\Psi_{4,4}$, respectively). This leads to the phase factors $s_2 = s_4 = -1$.

To conclude, it is worth mentioning that had we made the opposite choice in equation (3.5), i.e., $A = L$ and $B = R$, we would not have been able to use the deformed parafermionic realization (4.4), (4.5) to determine the energy spectrum. The counterpart of the parafermionic vacuum state would indeed have been a function annihilated by L and therefore constructed from the unphysical function $\bar{\chi}_{-1}(y)$ of equation (2.11).

5 Conclusion

In this paper, we have revisited the exactly solvable PDM model in a two-dimensional semi-infinite layer introduced in I. Here we have taken advantage of its superintegrability with two integrals of motion L and R that are quadratic in the momenta to propose a third method of solution in the line of some recent analyses of such problems.

We have first determined the explicit form of the quadratic associative algebra generated by L, R and their commutator. We have shown that it is a quadratic Racah algebra QR(3) and that its Casimir operator K is a second-degree polynomial in H. We have also obtained the quadratic Poisson algebra arising in the classical limit.

We have then studied the finite-dimensional irreducible representations of our algebra in a basis wherein K (or H) and R are diagonal. For such a purpose, we have used a simple procedure, proposed in [36], consisting in mapping this basis onto deformed parafermionic oscillator states of order p. Among the results so obtained for the energy spectrum, we have selected those with which physical wavefunctions can be associated. This has illustrated once again the well-known fact that in quantum mechanics the physics is determined not only by algebraic properties of operators, but also by the boundary conditions imposed on wavefunctions. Our analysis has provided us with an interesting new result, not obtainable in general form in the SUSYQM approach of I, namely the matrix elements of L in the basis wherein H and R are simultaneously diagonal.

As final points, it is worth observing that the approaches followed here are not the only ones available. First, one could have used a gauge transformation to relate equation (2.3) to a well-known superintegrable system in a Darboux space ([38, 48] and references quoted therein). Second, the irreducible representations of QR(3) could have been constructed by the ladder-operator method employed in [31, 32, 33, 34]. This would have allowed us to express the transformation matrix elements between the bases $\psi^{(k)}_{n,l}$ and $\Psi^{(k)}_{N,N_0}$ (denoted by $Z^{(k)}_{N_0;n,l}$ in I) in terms of Racah–Wilson polynomials.

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