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Degenerate Series Representations of the q-Deformed Algebra $\mathrm{so}'_q(r,s)$

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Abstract. The q-deformed algebra $\mathrm{so}'_q(r,s)$ is a real form of the q-deformed algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$, $n = r+s$, which differs from the quantum algebra $U_q(\mathrm{so}(n,\mathbb{C}))$ of Drinfeld and Jimbo. We study representations of the most degenerate series of the algebra $\mathrm{so}'_q(r,s)$. The formulas of action of operators of these representations upon the basis corresponding to restriction of representations onto the subalgebra $\mathrm{so}'_q(r)\times\mathrm{so}'_q(s)$ are given. Most of these representations are irreducible. Reducible representations appear under some conditions for the parameters determining the representations. All irreducible constituents which appear in reducible representations of the degenerate series are found. All $*$-representations of $\mathrm{so}'_q(r,s)$ are separated in the set of irreducible representations obtained in the paper.

Key words: q-deformed algebras; irreducible representations; reducible representations

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1 Introduction

In this paper we consider most degenerate series representations of the q-deformed algebra $\mathrm{so}'_q(r,s)$, which is a real form of the complex q-deformed algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ defined in [1]. The algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ differs from the quantum algebra $U_q(\mathrm{so}(n,\mathbb{C}))$ defined by Drinfeld [2] and Jimbo [3] (see also [4]). Drinfeld and Jimbo defined $U_q(\mathrm{so}(n,\mathbb{C}))$ by means of Cartan subalgebras and root subspaces of the Lie algebra $\mathrm{so}(n,\mathbb{C})$. However, the Lie algebra $\mathrm{so}(n,\mathbb{C})$ has a different structure based on the basis elements $I_{k,k-1} = E_{k,k-1} - E_{k-1,k}$ (where E_{is} is the matrix with elements $(E_{is})_{rt} = \delta_{ir}\delta_{st}$). The q-deformation of this structure leads to the algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$, determined in [1]. Later on it was shown that this q-deformation of $\mathrm{so}(n,\mathbb{C})$ is very useful in many directions of contemporary mathematics. Namely, representations of the algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ and of its real forms are closely related to the theory of q-orthogonal polynomials and q-special functions. Some ideas of such applications can be found in [5].

The algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ (especially its particular case $U'_q(\mathrm{so}(3,\mathbb{C}))$) is related to the algebra of observables in 2+1 quantum gravity on the Riemmanian surfaces (see, for example, [6]). A quantum analogue of the Riemannian symmetric space $SU(n)/SO(n)$ is constructed by means of the algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ [7]. It is clear that a construction of a quantum analogue of some pseudo-Riemannian symmetric spaces is connected with the q-deformed algebra $\mathrm{so}'_q(r,s)$.

A q-analogue of the theory of harmonic polynomials (q-harmonic polynomials on quantum vector space $\mathbb{R}^n_q$) is constructed by using the algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$. In particular, a q-analogue of separations of variables for the q-Laplace operator on $\mathbb{R}^n_q$ is given by means of this algebra and its subalgebras (see [8, 9]). The algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ also appears in the theory of links in the algebraic topology [10].

The representation theory of the q-deformed algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ differs from that for the Drinfeld–Jimbo algebra $U_q(\mathrm{so}(n,\mathbb{C}))$. One of these differences is related to the fact that the

Drinfeld–Jimbo algebra $U_q(\mathrm{so}(n,\mathbb{C}))$ admits the inclusion

$$U_q(\mathrm{so}(n,\mathbb{C})) \supset U_q(\mathrm{so}(n-2,\mathbb{C}))$$

and does not admit the inclusion

$$U_q(\mathrm{so}(n,\mathbb{C})) \supset U_q(\mathrm{so}(n-1,\mathbb{C})).$$

The algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ admits such an inclusion. This allows to construct Gel'fand–Tsetlin bases for finite dimensional representations of $U'_q(\mathrm{so}(n,\mathbb{C}))$ (see [1]).

As in the case of real forms of Drinfeld–Jimbo quantum algebras (see [11, 12, 13]) the real form $\mathrm{so}'_q(r,s)$ of $U'_q(\mathrm{so}(r+s,\mathbb{C}))$ is defined by means of introducing a $*$-operation into $U'_q(\mathrm{so}(r+s,\mathbb{C}))$. When $q\to 1$ then the q-deformed algebra $\mathrm{so}'_q(r,s)$ turns into the universal enveloping algebra $U(\mathrm{so}_{r,s})$ of the Lie algebra $\mathrm{so}_{r,s}$ which corresponds to the pseudo–orthogonal Lie group $SO_0(r,s)$.

Representations of the algebra $\mathrm{so}'_q(r,s)$, considered in this paper, are given by one continuous parameter. These representations are q-deformations of the representations of the classical Lie group $SO_0(r,s)$ considered in [14, 15] (see also [16, 17]). We derive several series of $*$-representations of the algebra $\mathrm{so}'_q(r,s)$. As in the case of the quantum algebra $U_q(\mathrm{su}_{1,1})$, the algebra $\mathrm{so}'_q(r,s)$ has the so-called strange series of $*$-representations, which is absent in the case of the Lie group $SO_0(r,s)$. When $q\to 1$ then matrix elements of operators of the strange series representations tend to the infinity and representations become senseless.

Everywhere below we consider that q is a positive number. We also suppose that $r>2$ and $s>2$. Representations of degenerate series of the algebra $\mathrm{so}'_q(r,1)$ were considered in [18]. Representations of the algebra $\mathrm{so}'_q(r,2)$ were studied in [19]. In fact, we generalize a part of the results of papers [18, 19].

It is well-known that the algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ has finite dimensional irreducible representations of two types: representations of the classical type (at $q\to 1$ they tend to the corresponding representations of the Lie algebra $\mathrm{so}(n,\mathbb{C})$) and representations of the non-classical type (there exists no analogue of these representations in the case of $\mathrm{so}(n,\mathbb{C})$). The algebra $\mathrm{so}'_q(r,s)$ has no degenerate series representations of the non-classical type. The reason is that the "compact" algebras $\mathrm{so}'_q(n)$ have no degenerate irreducible representations of the non-classical type.

2 The q-deformed algebra $\mathrm{so}'_q(r,s)$

The algebra $\mathrm{so}'_q(r,s)$ is a real form of the q-deformed algebra $U'_q(\mathrm{so}(r+s,\mathbb{C}))$ which is separated by the $*$-operation. The algebra $U'_q(\mathrm{so}(r+s,\mathbb{C}))$ is defined in [1].

The classical universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$ is generated by the elements $I_{i,i-1}$, $i=2,3,\ldots,n$, that satisfy the relations

$$I_{i,i-1}I^2_{i+1,i}-2I_{i+1,i}I_{i,i-1}I_{i+1,i}+I^2_{i+1,i}I_{i,i-1}=-I_{i,i-1}, \tag{1}$$
$$I^2_{i,i-1}I_{i+1,i}-2I_{i,i-1}I_{i+1,i}I_{i,i-1}+I_{i+1,i}I^2_{i,i-1}=-I_{i+1,i}, \tag{2}$$
$$[I_{i,i-1},I_{j,j-1}]=0, \qquad |i-j|>1 \tag{3}$$

(they follow from the well-known commutation relations for the generators I_{ij} of the Lie algebra $\mathrm{so}(n,\mathbb{C})$). In the approach to the q-deformed orthogonal algebra of paper [1], a q-deformation of the associative algebra $U(\mathrm{so}(n,\mathbb{C}))$ is defined by deforming the relations (1)–(3). These q-deformed relations are of the form

$$I_{i,i-1}I^2_{i+1,i}-aI_{i+1,i}I_{i,i-1}I_{i+1,i}+I^2_{i+1,i}I_{i,i-1}=-I_{i,i-1}, \tag{4}$$
$$I^2_{i,i-1}I_{i+1,i}-aI_{i,i-1}I_{i+1,i}I_{i,i-1}+I_{i+1,i}I^2_{i,i-1}=-I_{i+1,i}, \tag{5}$$

$$[I_{i,i-1}, I_{j,j-1}] = 0, \quad |i-j| > 1, \tag{6}$$

where $a = q^{1/2}+q^{-1/2}$ and $[\cdot,\cdot]$ denotes the usual commutator. Obviously, in the limit $q\to 1$ formulas (4)–(6) give relations (1)–(3). Remark that relations (4) and (5) differ from the q-deformed Serre relations in the approach of Jimbo and Drinfeld to the quantum algebras $U_q(\mathrm{so}(n,\mathbb{C}))$ by appearance of nonzero right hand sides and by possibility of reduction

$$U'_q(\mathrm{so}(n,\mathbb{C})) \supset U'_q(\mathrm{so}(n-1,\mathbb{C})).$$

Below, by the algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ we mean the q-deformed algebra defined by formulas (4)–(6).

The "compact" real form $\mathrm{so}'_q(n)$ of the algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ is defined by the involution given as

$$I^*_{i,i-1} = -I_{i,i-1}, \qquad i = 2,3,\ldots,n. \tag{7}$$

The "noncompact" real form $\mathrm{so}'_q(r,n-r)$ of $U'_q(\mathrm{so}(n,\mathbb{C}))$ is determined by the involution

$$I^*_{i,i-1} = -I_{i,i-1}, \qquad i \neq r+1, \qquad I^*_{r+1,r} = I_{r+1,r}. \tag{8}$$

It would be more correct to use the notation $U'_q(\mathrm{so}_q(r,n-r))$ for $\mathrm{so}'_q(r,n-r)$. We use the last notation since it is simpler.

The q-deformed algebra $\mathrm{so}'_q(n)$ contains the subalgebra $\mathrm{so}'_q(n-1)$. This fact allows us to consider Gel'fand–Tsetlin bases of carrier spaces of representations of $\mathrm{so}'_q(n)$ [1]. The q-deformed algebra $\mathrm{so}'_q(n-r,r)$ contains the subalgebra $\mathrm{so}'_q(n-r)\times\mathrm{so}'_q(r)$.

3 Representations of $\mathrm{so}'_q(n)$

Irreducible representations of the algebra $\mathrm{so}'_q(r,s)$ are described by means of finite dimensional irreducible representations of the subalgebras $\mathrm{so}'_q(r)$ and $\mathrm{so}'_q(s)$. Therefore, we describe representations of $\mathrm{so}'_q(n)$ which will be used below.

Irreducible finite dimensional representations of the algebra $\mathrm{so}'_q(3)$ (belonging to the classical type) are given by integral or half-integral nonnegative number l. We denote these representations by T_l. The carrier space of the representation T_l has the orthonormal basis

$$|m\rangle, \qquad m = -l, -l+1, \ldots, l,$$

and the operators $T_l(I_{21})$ and $T_l(I_{32})$ act upon this basis as (see [18])

$$T_l(I_{21})|m\rangle = \mathrm{i}[m]_q|m\rangle, \quad \mathrm{i} = \sqrt{-1}, \tag{9}$$

$$\begin{aligned} T(I_{32})|m\rangle &= d(m)\left([l-m]_q[l+m+1]_q\right)^{1/2}|m+1\rangle \\ &\quad - d(m-1)\left([l+m]_q[l-m+1]_q\right)^{1/2}|m-1\rangle, \end{aligned} \tag{10}$$

where

$$d(m) = \left(\frac{[m]_q[m+1]_q}{[2m]_q[2m+2]_q}\right)^{1/2}$$

and $[b]_q$ is a q-number defined by the formula

$$[b]_q = \frac{q^{b/2}-q^{-b/2}}{q^{1/2}-q^{-1/2}}.$$

Let us describe finite dimensional irreducible representations of the algebra $\mathrm{so}'_q(n)$, $n > 3$, which are of class 1 with respect to the subalgebra $\mathrm{so}'_q(n-1)$ [9]. As in the classical case, these representations are given by highest weights $(m_n, 0, \ldots, 0)$, where m_n is a nonnegative integer. We denote these representations by T_{m_n}. Under restriction to the subalgebra $\mathrm{so}'_q(n-1)$, the representation T_{m_n} contains (with unit multiplicity) those and only those irreducible representations $T_{m_{n-1}}$ of this subalgebra for which we have

$$m_n \geq m_{n-1} \geq 0.$$

Exactly in the same way as in the case of the classical group $SO(n)$, we introduce the Gel'fand–Tsetlin basis of the carrier space of the representation T_{m_n} by using the successive reduction

$$\mathrm{so}'_q(n) \supset \mathrm{so}'_q(n-1) \supset \mathrm{so}'_q(n-2) \supset \cdots \supset \mathrm{so}'_q(3) \supset \mathrm{so}'_q(2).$$

We denote the basis elements of this space by

$$|m_n, m_{n-1}, m_{n-2}, \ldots, m_3, m_2\rangle,$$

where $m_n \geq m_{n-1} \geq m_{n-2} \geq \cdots \geq |m_2|$ and m_{n-i} determines the representation $T_{m_{n-i}}$ of $\mathrm{so}'_q(n-i)$. With respect to this basis the operator $T(I_{n,n-1})$ of the representation T_{m_n} of $\mathrm{so}'_q(n)$ is given by the formula

$$\begin{aligned} & T_{m_n}(I_{n,n-1})|m_n, m_{n-1}, \ldots, m_2\rangle \\ & \quad = ([m_n + m_{n-1} + n - 2]_q [m_n - m_{n-1}]_q)^{1/2} R(m_{n-1}) |m_n, m_{n-1}+1, \ldots, m_2\rangle \\ & \quad - ([m_n + m_{n-1} + n - 3]_q [m_n - m_{n-1} + 1]_q)^{1/2} R(m_{n-1}-1) |m_n, m_{n-1}-1, \ldots, m_2\rangle, \end{aligned} \tag{11}$$

where

$$R(m_{n-1}) = \left(\frac{[m_{n-1} + m_{n-2} + n - 3]_q [m_{n-1} - m_{n-2} + 1]_q}{[2m_{n-1} + n - 3]_q [2m_{n-1} + n - 1]_q} \right)^{1/2}.$$

The other operators $T(I_{i,i-1})$ are given by the same formulas with the corresponding change for m_n and m_{n-1} or at $i = 3$, 2 by the formulas for irreducible representations of the algebra $\mathrm{so}'_q(3)$, described above.

The representations T_{m_n} are characterized by the property that under restriction to the subalgebra $\mathrm{so}'_q(n-1)$ the restricted representations contain a trivial irreducible representation of $\mathrm{so}'_q(n-1)$ (that is, a representation with highest weight $(0, 0, \ldots, 0)$). This is why one says that the representation T_{m_n} is of class 1 with respect to $\mathrm{so}'_q(n-1)$. The irreducible representations T_{m_n} exhaust all irreducible representations of class 1 of the algebra $\mathrm{so}'_q(n)$.

4 Representations of the degenerate principal series

We shall consider infinite dimensional representations of the associative algebra $\mathrm{so}'_q(r, s)$. Moreover, we admit representations by unbounded operators. There exist non-equivalent definitions of representations of associative algebras by unbounded or bounded operators (see [20, 21]). In order to have a natural definition of a representation of $\mathrm{so}'_q(r, s)$ we take into account the following items:

(I) We shall deal also with $*$-representations (it is well-known that these representations are an analogue of unitary representations of Lie groups). Therefore, for each representation operator there should exist an adjoint operator. This means that a representation space have to be defined on a Hilbert space.

(II) Unbounded operators cannot be defined on the whole Hilbert space. However, existence of an adjoint operator A^* to an unbounded operator A means that the operator A must be defined on an everywhere dense subspace in the Hilbert space.

(III) In order to be able to consider products of representation operators, there must exist an everywhere dense subspace of the Hilbert space which enter to a domain of definition of each representation operator.

Therefore, we give the following definition of a representation of $\mathrm{so}'_q(r,s)$. A representation T of the associative algebra $\mathrm{so}'_q(r,s)$ is an algebraic homomorphism from $\mathrm{so}'_q(r,s)$ into an algebra of linear (bounded or unbounded) operators on a Hilbert space $\mathcal{H}$ for which the following conditions are fulfilled:

a) the restriction of T onto the "compact" subalgebra $\mathrm{so}'_q(r)\times\mathrm{so}'_q(s)$ decomposes into a direct sum of its finite dimensional irreducible representations (given by highest weights) with finite multiplicities;

(b) operators of a representation T are determined on an everywhere dense subspace $\mathcal{W}$ of $\mathcal{H}$, containing all subspaces which are carrier spaces of irreducible finite dimensional representations of $\mathrm{so}'_q(r)\times\mathrm{so}'_q(s)$ from the restriction of T.

In other words, our representations of $\mathrm{so}'_q(r,s)$ are Harish-Chandra modules of $\mathrm{so}'_q(r,s)$ with respect to $\mathrm{so}'_q(r)\times\mathrm{so}'_q(s)$.

There exist different non-equivalent definitions of irreducibility of representations of associative algebras by unbounded operators (see [20]). Since unbounded representation operators are not defined on all elements of the Hilbert space, then we cannot define irreducibility as in the finite dimensional case. A natural definition is the following one. A representation T of $\mathrm{so}'_q(r,s)$ on $\mathcal{H}$ is called irreducible if $\mathcal{H}$ has no non-trivial invariant subspaces such that its closure does not coincide with $\mathcal{H}$. If operators of a representation T obey the relations

$$T(I_{i,i-1})^* = -T(I_{i,i-1}), \qquad i\ne r+1, \qquad T(I_{r+1,r})^* = T(I_{r+1,r}) \tag{12}$$

(compare with formulas (8)) on a common domain $\mathcal{W}$ of definition, then T is called a $*$-representation.

To define a representation T of $\mathrm{so}'_q(r,s)$ it is sufficient to give the operators $T(I_{i,i-1})$, $i=2,3,\dots,r+s$, satisfying relations (4)–(6) on a common domain of definition. Let us define representations of $\mathrm{so}'_q(r,s)$ belonging to the *degenerate principal series*. They are given by a complex number λ and a number $\epsilon\in\{0,1\}$. We denote the corresponding representation by $T_{\epsilon\lambda}$. The space $\mathcal{H}(T_{\epsilon\lambda})$ of the representation $T_{\epsilon\lambda}$ is an orthogonal sum of the subspaces $\mathcal{V}(m,\mathbf{0};m',\mathbf{0})$, which are the carrier spaces of the finite dimensional representations of $\mathrm{so}'_q(r)\times\mathrm{so}'_q(s)$ with highest weights $(m,\mathbf{0};m',\mathbf{0})$ such that

$$m+m'\equiv\epsilon \pmod 2.$$

Here $(m,\mathbf{0})$ and $(m',\mathbf{0})$, $m\ge 0$, $m'\ge 0$, are highest weights of irreducible representations of the subalgebras $\mathrm{so}'_q(r)$ and $\mathrm{so}'_q(r)$, respectively, and $\mathbf{0}$ denotes the set of zero coordinates (in the case of the subalgebra $\mathrm{so}'_q(3)$, $\mathbf{0}$ must be omitted). We assume that the basis vectors from (11) are orthonormal in $\mathcal{V}(m,\mathbf{0};m',\mathbf{0})$. Therefore, we have

$$\mathcal{H}(T_{\epsilon\lambda}) = \bigoplus_{m+m'\equiv\epsilon \pmod 2} \mathcal{V}(m,\mathbf{0};m',\mathbf{0}), \tag{13}$$

where we suppose that the sum means a closure of the corresponding linear span. A linear span of the subspaces $\mathcal{V}(m,\mathbf{0};m',\mathbf{0})$ determines an everywhere dense subspace on which all operators of the representation $T_{\epsilon\lambda}$ are defined. Recall that we suppose that $r>2$ and $s>2$.

We choose in the subspaces $\mathcal{V}(m,\mathbf{0};m',\mathbf{0})$ the orthonormal bases which are products of the bases corresponding to irreducible representations of the subalgebras $\mathrm{so}'_q(r)$ and $\mathrm{so}'_q(s)$ introduced in Section 3 (Gel'fand–Tsetlin bases). Elements of such bases are labeled by double Gel'fand–Tsetlin patterns which will be denoted as

$$|M\rangle = |m,k,j,\ldots;m',k',j',\ldots\rangle. \tag{14}$$

It is clear that entries of patterns (14) obey the following conditions:

$$m\ge k\ge j\ge\cdots, \qquad m'\ge k'\ge j'\ge\cdots \tag{15}$$

if $r>3$ and $s>3$. Thus, elements of the orthonormal basis of the carrier space of the representation $T_{\epsilon\lambda}$ are labeled by all patterns (14) satisfying betweenness conditions (15) and the equality $m+m'\equiv\epsilon \pmod 2$.

In order to define the representations $T_{\epsilon\lambda}$ we give explicit formulas for the operators $T_{\epsilon\lambda}(I_{i,i-1})$, $i=2,3,\ldots,r+s$. The operators $T_{\epsilon\lambda}(I_{i,i-1})$, $i=2,3,\ldots,r$, act upon basis elements (14) by the formulas of Section 3 as operators of the corresponding irreducible representations of the subalgebra $\mathrm{so}'_q(r)$. It is clear that these operators act only upon entries k, j, $\ldots$ of vectors (14) and do not change entries m, m', k', j', $\ldots$. The operators $T_{\epsilon\lambda}(I_{i,i-1})$, $i=r+2,r+3,\ldots,r+s$, act upon basis elements (14) by formulas of Section 3 as operators of corresponding irreducible representations of the subalgebra $\mathrm{so}'_q(s)$. The operator $T_{\epsilon\lambda}(I_{r+1,r})$ acts upon vectors (14) by the formula

$$\begin{aligned}
&T_{\epsilon\lambda}(I_{r+1,r})|m,k,j,\ldots;m',k',j',\ldots\rangle\\
&\quad= K_mL_{m'}[\lambda+m+m']_q|m+1,k,j,\ldots;m'+1,k',j',\ldots\rangle\\
&\quad- K_mL_{m'-1}[\lambda+m-m'-s+2]_q|m+1,k,j,\ldots;m'-1,k',j',\ldots\rangle\\
&\quad+ K_{m-1}L_{m'}[\lambda-m+m'-r+2]_q|m-1,k,j,\ldots;m'+1,k',j',\ldots\rangle\\
&\quad- K_{m-1}L_{m'-1}[\lambda-m-m'-r-s+4]_q|m-1,k,j,\ldots;m'-1,k',j',\ldots\rangle,
\end{aligned} \tag{16}$$

where

$$K_m=\left(\frac{[m-k+1]_q[m+k+r-2]_q}{[2m+r]_q[2m+r-2]_q}\right)^{1/2}, \tag{17}$$

$$L_{m'}=\left(\frac{[m'-k'+1]_q[m'+k'+s-2]_q}{[2m'+s]_q[2m'+s-2]_q}\right)^{1/2}. \tag{18}$$

So, this operator changes only the entries m and m' in vectors (14).

To prove that these operators really give a representation of the algebra $\mathrm{so}'_q(r,s)$ we have to verify that the relations

$$\begin{aligned}
&T_{\epsilon\lambda}(I_{r+1,r})^2T_{\epsilon\lambda}(I_{r,r-1})-\big(q^{\frac12}+q^{-\frac12}\big)T_{\epsilon\lambda}(I_{r+1,r})T_{\epsilon\lambda}(I_{r,r-1})T_{\epsilon\lambda}(I_{r+1,r})\\
&\quad+T_{\epsilon\lambda}(I_{r,r-1})T_{\epsilon\lambda}(I_{r+1,r})^2=-T_{\epsilon\lambda}(I_{r,r-1}),
\end{aligned} \tag{19}$$

$$\begin{aligned}
&T_{\epsilon\lambda}(I_{r+1,r})T_{\epsilon\lambda}(I_{r,r-1})^2-\big(q^{\frac12}+q^{-\frac12}\big)T_{\epsilon\lambda}(I_{r,r-1})T_{\epsilon\lambda}(I_{r+1,r})T_{\epsilon\lambda}(I_{r,r-1})\\
&\quad+T_{\epsilon\lambda}(I_{r,r-1})^2T_{\epsilon\lambda}(I_{r+1,r})=-T_{\epsilon\lambda}(I_{r+1,r}),
\end{aligned} \tag{20}$$

as well as relations (19) and (20), in which $T_{\epsilon\lambda}(I_{r,r-1})$ is replaced by $T_{\epsilon\lambda}(I_{r+2,r+1})$, and the relations

$$[T_{\epsilon\lambda}(I_{i,i-1}),T_{\epsilon\lambda}(I_{j,j-1})]=0, \qquad |i-j|>1, \tag{21}$$

where $[\cdot,\cdot]$ is the usual commutator, are fulfilled.

Fulfilment of these relations can be shown by a direct calculation. Namely, we act by both their parts upon vector (14), then collect coefficients at the same resulting basis vectors and show that the relations obtained are correct. We do not give here these direct calculations.

Irreducibility of representations $T_{\epsilon\lambda}$ is studied by means of the following proposition.

Proposition 1. *The representation $T_{\epsilon\lambda}$ is irreducible if each of the numbers $[\lambda+m+m']_q$, $[\lambda+m-m'-s+2]_q$, $[\lambda-m+m'-r+2]_q$, $[\lambda-m-m'-r-s+4]_q$ in* (16) *vanishes only if the vector on the right hand side of* (16) *with the coefficient, containing this number, does not belong to the Hilbert space $\mathcal{H}(T_{\epsilon\lambda})$.*

This proposition is proved in the same way as the corresponding proposition in the classical case (see Proposition 8.5 in Section 8.5 of [16]). Thus, the main role under studying irreducibility of the representations $T_{\epsilon\lambda}$ is played by the coefficients at vectors in (16) numerated in Proposition 1.

5 Irreducibility

To study the representations $T_{\epsilon\lambda}$ we take into account properties of the function

$$w(z) = [z]_q = \frac{q^{z/2}-q^{-z/2}}{q^{1/2}-q^{-1/2}} = \frac{e^{hz/2}-e^{-hz/2}}{e^{h/2}-e^{-h/2}}.$$

where $q=e^h$. Namely, we have

$$w(z+4\pi k\mathrm{i}/h) = w(z), \qquad w(z+2\pi k\mathrm{i}/h) = -w(z), \qquad k \text{ are odd integers.}$$

These relations mean that the following proposition holds:

Proposition 2. *For arbitrary λ the pairs of representations $T_{\epsilon\lambda}$ and $T_{\epsilon,\lambda+4\pi k\mathrm{i}/h}$ are coinciding and the pairs of representations $T_{\epsilon\lambda}$ and $T_{\epsilon,\lambda+2\pi k\mathrm{i}/h}$ are equivalent.*

Therefore, we may consider only representations $T_{\epsilon\lambda}$ with $0 \le \operatorname{Im}\lambda < 2\pi/h$.

Most of the representations $T_{\epsilon\lambda}$ are irreducible. Nevertheless, some of them are reducible. As in the classical case (see [14, 15, 16, 17]), reducibility appears because of vanishing of some of the coefficients $[\lambda+m+m']_q$, $[\lambda+m-m'-s+2]_q$, $[\lambda-m+m'-r+2]_q$, $[\lambda-m-m'-r-s+4]_q$ in formula (16). Using this fact we derive (in the same way as in the classical case; see [16]) the following theorem:

Theorem 1. *If r and s are both even, then the representation $T_{\epsilon\lambda}$ is irreducible if and only if λ is not an integer such that $\lambda \equiv \epsilon \pmod 2$. If one of the numbers r and s is even and the other one is odd, then the representation $T_{\epsilon\lambda}$ is irreducible if and only if λ is not an integer. If r and s are both odd, then the representation $T_{\epsilon\lambda}$ is irreducible if and only if λ is not an integer or if λ is an integer such that*

$$\lambda \equiv \epsilon \pmod 2, \qquad 0 < \lambda < \tfrac{1}{2}(r+s)-2.$$

Irreducible representations $T_{\epsilon\lambda}$ admit additional equivalence relations.

Proposition 3. *The pairs of irreducible representations $T_{\epsilon\lambda}$ and $T_{\epsilon,r+s-\lambda-2}$ are equivalent.*

This equivalence is proved by constructing an explicit form of intertwining operators. These operators are diagonal in the basis (14) and can be evaluated exactly in the same way as in the classical case.

Sometimes it is useful to have the representations $T_{\epsilon\lambda}$ in somewhat different basis, namely, in the basis

$$|m,k,j,\ldots;\ m',k',j',\ldots\rangle'$$

which is related to the basis (14) by the formulas

$$\begin{aligned}
&|m_0+\epsilon+i,k,j,\ldots;m_0-i,k',j',\ldots\rangle\\
&\quad=\frac{\prod\limits_{t=1}^{m_0}[-\lambda+\epsilon+r+s+2t-4]_q^{1/2}}{\prod\limits_{t=1}^{m_0}[\lambda+\epsilon+2t-2]_q^{1/2}}\frac{\prod\limits_{t=1}^{i}[-\lambda+\epsilon+r+2t-2]_q^{1/2}}{\prod\limits_{t=1}^{i}[\lambda+\epsilon-s+2t]_q^{1/2}}\\
&\quad\times|m_0+\epsilon+i,k,j,\ldots;m_0-i,k',j',\ldots\rangle',\\
&|m_0+\epsilon-i,k,j,\ldots;m_0+i,k',j',\ldots\rangle\\
&\quad=\frac{\prod\limits_{t=1}^{m_0}[-\lambda+\epsilon+r+s+2t-4]_q^{1/2}}{\prod\limits_{t=1}^{m_0}[\lambda+\epsilon+2t-2]_q^{1/2}}\frac{\prod\limits_{t=1}^{i}[\lambda+\epsilon-s-2t+2]_q^{1/2}}{\prod\limits_{t=1}^{i}[-\lambda+\epsilon+r-2t]_q^{1/2}}\\
&\quad\times|m_0+\epsilon-i,k,j,\ldots;m_0+i,k',j',\ldots\rangle'.
\end{aligned}$$

In the new basis the operators $T_{\epsilon\lambda}(I_{i,i-1})$, $i\neq r+1$, have the same form as in the basis (14), and the operator $T_{\epsilon\lambda}(I_{r+1,r})$ is of the form

$$\begin{aligned}
&T_{\epsilon\lambda}(I_{r+1,r})|m,k,j,\ldots;m',k',j',\ldots\rangle'\\
&\quad=K_mL_{m'}\{[\lambda+m+m']_q[-\lambda+m+m'+r+s-2]_q\}^{1/2}\\
&\quad\times|m+1,k,j,\ldots;m'+1,k',j',\ldots\rangle'\\
&\quad-K_mL_{m'-1}\{[\lambda+m-m'-s+2]_q[-\lambda+m-m'+r]_q\}^{1/2}\\
&\quad\times|m+1,k,j,\ldots;m'-1,k',j',\ldots\rangle'\\
&\quad+K_{m-1}L_{m'}\{[\lambda-m+m'-r+2]_q[-\lambda-m+m'+s]_q\}^{1/2}\\
&\quad\times|m-1,k,j,\ldots;m'+1,k',j',\ldots\rangle'\\
&\quad-K_{m-1}L_{m'-1}\{[\lambda-m-m'-r-s+4]_q[-\lambda-m-m'+2]_q\}^{1/2}\\
&\quad\times|m-1,k,j,\ldots;m'-1,k',j',\ldots\rangle'.
\end{aligned}\tag{22}$$

Formulas (16) and (22) are used to select $*$-representations in the set of all irreducible representations $T_{\epsilon\lambda}$ of the algebra $\mathrm{so}'_q(r,s)$. This selection is fulfilled in the same way as in the case of the q-deformed algebras $\mathrm{so}'_q(2,1)$ and $\mathrm{so}'_q(3,1)$ in [18], that is, by a direct check that the relations (8) are satisfied. This selection leads to the following theorem.

Theorem 2. *All irreducible representations $T_{\epsilon\lambda}$ of $\mathrm{so}'_q(r,s)$ with $\lambda=-\overline{\lambda}+r+s-2$ are $*$-representations (the principal degenerate series of $*$-representations). All irreducible representations $T_{\epsilon\lambda}$ with $\mathrm{Im}\,\lambda=\pi/h$ are $*$-representations (the strange series). If r and s are both even or both odd, then all irreducible representations $T_{0\lambda}$, $\frac12(r+s)-1<\lambda<\frac12(r+s)$, for even $\frac12(r+s)$ and all irreducible representations $T_{1\lambda}$, $\frac12(r+s)-1<\lambda<\frac12(r+s)$, for odd $\frac12(r+s)$ are $*$-representations (the supplementary series). If among the integers r and s one is odd and another is even, then all irreducible representations $T_{\epsilon\lambda}$, $\frac12(r+s)-1<\lambda<\frac12(r+s-1)$ are $*$-representations (supplementary series).*

This theorem describes all $*$-representations in the set of irreducible representations $T_{\epsilon\lambda}$. However, there are equivalent representations in the formulation of Theorem 2. All possible equivalences are given by equivalence relations described above or by their combinations (products).

6 Reducible representations $T_{\epsilon\lambda}$

Let us study a structure of reducible representations $T_{\epsilon\lambda}$ of the algebra $\mathrm{so}'_q(r,s)$. Vanishing of some coefficients in formula (16) or (22) leads to appearing of invariant subspaces in the carrier space of the representation $T_{\epsilon\lambda}$. Analysis of reducibility and finding of all irreducible constituents in $T_{\epsilon\lambda}$ are done in the same way as in the classical case [15, 16, 17]. For this reason, we shall formulate the results of such analysis without detailed proof.

Let us also note that, as in the classical case, reducible representations $T_{\epsilon\lambda}$ and $T_{\epsilon,-\lambda+r+s-2}$ contain the same (equivalent) irreducible constituents. This is easily seen from formula (22). Studying the representations $T_{\epsilon\lambda}$, we have to distinguish the cases of odd and even r and s since in different cases a structure of reducible representations $T_{\epsilon\lambda}$ is different. Below, we investigate all reducible representations $T_{\epsilon\lambda}$ (which are excluded in Theorem 1).

6.1 The case of even r and s

Let λ be an even integer in $T_{0\lambda}$ and an odd integer in $T_{1\lambda}$. If $\lambda \le 0$ then in the carrier space $\mathcal{H}(T_{\epsilon\lambda})$ of the representation $T_{\epsilon\lambda}$ there exist invariant subspaces

$$\mathcal{H}^F_\lambda = \bigoplus_{m+m'\le -\lambda} \mathcal{V}(m,\mathbf{0};m',\mathbf{0}), \qquad \mathcal{H}^0_\lambda = \bigoplus_{\lambda-r+2\le m-m'\le -\lambda+s-2} \mathcal{V}(m,\mathbf{0};m',\mathbf{0}),$$

where $\mathcal{V}(m,\mathbf{0};m',\mathbf{0})$, is the subspace of $\mathcal{H}(T_{\epsilon\lambda})$, on which the irreducible representation of the subalgebra $\mathrm{so}'_q(r)\times\mathrm{so}'_q(s)$ with highest weight $(m,\mathbf{0};m',\mathbf{0})$ is realized. On the subspace $\mathcal{H}^F_\lambda$ the finite dimensional irreducible representation of the algebra $\mathrm{so}'_q(r,s)$ with highest weight $(-\lambda,\mathbf{0})$ is realized. An irreducible representation of $\mathrm{so}'_q(r,s)$ is realized on the quotient space $\mathcal{H}^0_\lambda/\mathcal{H}^F_\lambda$. We denote it by T^0_λ. A direct sum of two irreducible representations of $\mathrm{so}'_q(r,s)$ is realized on the quotient space $\mathcal{H}(T_{\epsilon\lambda})/\mathcal{H}^0_\lambda$. One of them acts on the direct sum of the subspaces $\mathcal{V}(m,\mathbf{0};m',\mathbf{0})$ for which $m-m' > -\lambda-s-2$ (we denote it by T^-_λ). The second one acts on the direct sum of the subspaces $\mathcal{V}(m,\mathbf{0};m',\mathbf{0})$ for which $m'-m > -\lambda+r-2$ (we denote it by T^+_λ). Figures showing distribution of subspaces $\mathcal{V}(m,\mathbf{0};m',\mathbf{0})$ between the subspaces $\mathcal{H}^F_\lambda$, $\mathcal{H}^0_\lambda$, $\mathcal{H}^+_\lambda$, $\mathcal{H}^-_\lambda$ (and also for other cases, considered below) are the same as in the classical case and can be found in [15], Chapter 8.

Let now λ be even in the representation $T_{0\lambda}$ and odd in the representation $T_{1\lambda}$, and let $0<\lambda\le \frac12(r+s)-2$. Then on the space $\mathcal{H}(T_{\epsilon\lambda})$ there exists only one invariant subspace $\mathcal{H}^0_\lambda$, which is the orthogonal sum of the subspaces $\mathcal{V}(m,\mathbf{0}\,;m',\mathbf{0})$ for which

$$m-m'\le -\lambda+s-2, \qquad m'-m\le -\lambda+r-2.$$

The representation of $\mathrm{so}'_q(r,s)$ on this subspace is irreducible (we denote it by T^0_λ). The direct sum of two irreducible representations of $\mathrm{so}'_q(r,s)$ is realized on the quotient space $\mathcal{H}(T_{\epsilon\lambda})/\mathcal{H}^0_\lambda$. For one of them we have $m-m' > -\lambda+s-2$ (we denote this irreducible representation by T^-_λ), and for another one $m'-m > -\lambda+r-2$ (we denote it by T^+_λ). For $\lambda=\frac12(r+s)-2$ the range of values of m and m' lies on one line in the coordinate space (m,m'). Physicists call such subrepresentations *ladder representations*.

For $\lambda=\frac12(r+s)-1$, the representation $T_{0\lambda}$ of $\mathrm{so}'_q(r,s)$, if this number λ is even, and the representation $T_{1\lambda}$, if this number is odd, is a direct sum of two irreducible representations T^-_λ and T^+_λ: for the first one we have $m'-m\le -\lambda+r-2$ and for the second one $m-m'\le -\lambda+s-2$.

Since the reducible representations $T_{\epsilon\lambda}$ and $T_{\epsilon,-\lambda+r+s-2}$ contain the same irreducible constituents, a structure of other reducible representations in this case is determined by that of the representations considered above.

In the same way as in the classical case, it is easy to verify that the following irreducible representations, considered here, are $*$-representations:

(a) all the representations T_λ^+ and T_λ^- (the discrete series);

(b) the representation $T^0_{(r+s-4)/2}$.

Proposition 4. *Irreducible representations $T_{\epsilon\lambda}$ and the irreducible representations T_λ^+, T_λ^-, T_λ^0 of this subsection exhaust all infinite dimensional irreducible representations of the algebra $\mathrm{so}'_q(r,s)$ with even r and s which consist under restriction to $\mathrm{so}'_q(r)\times\mathrm{so}'_q(s)$ of irreducible representations of this subalgebra only with highest weights of the form $(m,0,\dots,0)(m',0,\dots,0)$. Moreover, these representations are pairwise non-equivalent.*

This proposition is proved in the same way as in the case of the group $SO_0(r,s)$ (see Chapter 8 in [16]).

6.2 The case of even r and odd s

Let λ be a non-positive integer of the same evenness as $m+m'$ does. Then in the space $\mathcal{H}(T_{\epsilon\lambda})$ of the reducible representation $T_{\epsilon\lambda}$ there exist two invariant subspaces

$$\mathcal{H}_\lambda^F=\bigoplus_{m+m'\le-\lambda}\mathcal{V}(m,\mathbf{0};m',\mathbf{0}),\qquad \mathcal{H}_\lambda^0=\bigoplus_{m'-m\le-\lambda+r-2}\mathcal{V}(m,\mathbf{0};m',\mathbf{0}).$$

The irreducible finite dimensional representation T_λ^F of $\mathrm{so}_q(r,s)$ with the highest weight $(-\lambda,\mathbf{0})$ is realized in the first subspace. In the quotient spaces $\mathcal{H}_\lambda^0/\mathcal{H}_\lambda^F$ and $\mathcal{H}(T_{\epsilon\lambda})/\mathcal{H}_\lambda^0$ the irreducible representations of $\mathrm{so}'_q(r,s)$ are realized which will be denoted by T_λ^1 and T_λ^+ respectively. So, in this case the representation $T_{\epsilon\lambda}$ consists of three irreducible constituents.

If $0<\lambda<\frac12(r+s)-2$ and, besides, λ is an integer of the same evenness as $m+m'$ does, then in $\mathcal{H}(T_{\epsilon\lambda})$ there exists only one invariant subspace $\mathcal{H}_\lambda^0$ which is a direct sum of the subspaces $\mathcal{V}(m,\mathbf{0};m',\mathbf{0})$ for which $m'-m\le-\lambda+r-2$. The irreducible representations of the algebra $\mathrm{so}_q(r,s)$ are realized on $\mathcal{H}_\lambda^0$ and $\mathcal{H}(T_{\epsilon\lambda})/\mathcal{H}_\lambda^0$. We denote them by T_λ^1 and T_λ^+ respectively.

If $\lambda<\frac12(r+s)-2$ and, besides, λ is an integer such that $\lambda\equiv(m+m'+1)\pmod 2$, then only one invariant subspace exists in $\mathcal{H}(T_{\epsilon\lambda})$. This subspace is

$$\mathcal{H}_\lambda^0=\bigoplus_{m-m'\le-\lambda+s-2}\mathcal{V}(m,\mathbf{0};m',\mathbf{0}).$$

The irreducible representations of the algebra $\mathrm{so}'_q(r,s)$ are realized on $\mathcal{H}_\lambda^0$ and $\mathcal{H}(T_{\epsilon\lambda})/\mathcal{H}_\lambda^0$. We denote them by T_λ^2 and T_λ^-, respectively.

Since the reducible representations $T_{\epsilon\lambda}$ and $T_{\epsilon,-\lambda+r+s-2}$ contain the same irreducible constituents, a structure of other reducible representations in this case is determined by that of the representations considered above.

In the set of irreducible representations, considered here, only the representations T_λ^+ and T_λ^+ are $*$-representations.

The case of odd r and even s is considered absolutely in the same way, interchanging the roles of r and s as well as of m and m'. For this reason, we omit consideration of this case.

Proposition 5. *Irreducible representations $T_{\epsilon\lambda}$ and the irreducible representations T_λ^+, T_λ^-, T_λ^1, T_λ^2 of this subsection exhaust all infinite dimensional irreducible representations of the algebra $\mathrm{so}'_q(r,s)$ with even r and odd s which consist under restriction to $\mathrm{so}'_q(r)\times\mathrm{so}'_q(s)$ of irreducible representations of this subalgebra only with highest weights of the form $(m,0,\dots,0)(m',0,\dots,0)$. Moreover, these representations are pairwise non-equivalent.*

6.3 The case of odd r and s

If λ is a non-positive integer such that $\lambda \equiv (m+m') \pmod 2$, then in the space $\mathcal{H}(T_{\epsilon\lambda})$ of the representation $T_{\epsilon\lambda}$ there exists only one invariant subspace $\mathcal{H}^F_\lambda$ containing all those subspaces $\mathcal{V}(m,\mathbf{0};m',\mathbf{0})$ for which $m+m' \le -\lambda$. On this invariant subspace the irreducible finite dimensional representation of $\mathrm{so}'_q(r,s)$ with the highest weight $(-\lambda,\mathbf{0})$ is realized. On the quotient space $\mathcal{H}(T_{\epsilon\lambda})/\mathcal{H}^F_\lambda$ the irreducible representation of $\mathrm{so}'_q(r,s)$ is realized which is denoted by T^3_λ.

If $\lambda \le \frac{1}{2}(r+s)-2$ and, besides, λ is an integer such that $\lambda \equiv (m+m'+1) \pmod 2$, then in the space $\mathcal{H}(T_{\epsilon\lambda})$ there exists only one invariant subspace $\mathcal{H}^0_\lambda$ containing all the subspaces $\mathcal{V}(m,\mathbf{0};m',\mathbf{0})$ for which

$$m'-m \le -\lambda+r-2, \qquad m-m' \le -\lambda+s-2.$$

An irreducible representation of $\mathrm{so}'_q(r,s)$ is realized on $\mathcal{H}^0_\lambda$ (we denote it by T^0_λ). On the quotient space $\mathcal{H}(T_{\epsilon\lambda})/\mathcal{H}^0_\lambda$ a direct sum of two irreducible representations of $\mathrm{so}'_q(r,s)$ acts. For one of them we have $m'-m \le -\lambda+r-2$, and for the other $m-m' \le -\lambda+s-2$. We denote these irreducible representations by T^-_λ and T^+_λ, respectively.

If $\lambda = \frac{1}{2}(r+s)-1$, then the representation $T_{0\lambda}$ for odd $\frac{1}{2}(r+s)-1$ and the representation $T_{1\lambda}$ for even $\frac{1}{2}(r+s)-1$ decompose into a direct sum of two irreducible representations of $\mathrm{so}'_q(r,s)$ (we denote them by T^-_λ and T^+_λ). For the first representation we have $m'-m \le -\lambda+r-2$ and for the second one $m-m' \le -\lambda+s-2$.

Since the reducible representations $T_{\epsilon\lambda}$ and $T_{\epsilon,-\lambda+r+s-2}$ contain the same irreducible constituents, a structure of other reducible representations in this case is determined by that of the representations considered above.

In the set of irreducible representations, considered in this subsection, only the following ones are $*$-representations:

(a) all the representations T^+_λ and T^-_λ (the discrete series);

(b) the representation $T^0_{(r+s-4)/2}$.

Proposition 6. *Irreducible representations $T_{\epsilon\lambda}$ and the irreducible representations T^+_λ, T^-_λ, T^0_λ, T^3_λ of this subsection exhaust all infinite dimensional irreducible representations of the algebra $\mathrm{so}'_q(r,s)$ with odd r and s which consist under restriction to $\mathrm{so}'_q(r)\times\mathrm{so}'_q(s)$ of irreducible representations of this subalgebra only with highest weights of the form $(m,0,\ldots,0)(m',0,\ldots,0)$. Moreover, these representations are pairwise non-equivalent.*

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