Symmetry, Integrability and Geometry: Methods and Applications SIGMA **3** (2007), 061, 50 pages

Completely Integrable Systems Associated with Classical Root Systems*

Toshio OSHIMA

Graduate School of Mathematical Sciences, University of Tokyo,
3-8-1, Komaba, Meguro-ku, Tokyo 153-8914, Japan
E-mail: *oshima@ms.u-tokyo.ac.jp*
URL: http://akagi.ms.u-tokyo.ac.jp/

Received December 14, 2006, in final form March 19, 2007; Published online April 25, 2007
Original article is available at http://www.emis.de/journals/SIGMA/2007/061/

Abstract. We study integrals of completely integrable quantum systems associated with classical root systems. We review integrals of the systems invariant under the corresponding Weyl group and as their limits we construct enough integrals of the non-invariant systems, which include systems whose complete integrability will be first established in this paper. We also present a conjecture claiming that the quantum systems with enough integrals given in this note coincide with the systems that have the integrals with constant principal symbols corresponding to the homogeneous generators of the B_n-invariants. We review conditions supporting the conjecture and give a new condition assuring it.

Key words: completely integrable systems; Calogero–Moser systems; Toda lattices with boundary conditions

2000 Mathematics Subject Classification: 81R12; 70H06

To the memory of Professor Vadim B. Kuznetsov

1 Introduction

A Schrödinger operator

$$P = -\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + R(x) \tag{1.1}$$

with the potential function $R(x)$ of n variables $x = (x_1, \dots, x_n)$ is called *completely integrable* if there exist n differential operators $P_1, \dots, P_n$ such that

$$\begin{aligned} &[P_i, P_j] = 0 \qquad (1 \le i < j \le n),\\ &P \in \mathbb{C}[P_1, \dots, P_n],\\ &P_1, \dots, P_n \quad \text{are algebraically independent.} \end{aligned} \tag{1.2}$$

In this paper, we explicitly construct the integrals $P_1, \dots, P_n$ for completely integrable potential functions $R(x)$ of the form

$$R(x) = \sum_{1\le i<j\le n}\big(u_{ij}^-(x_i - x_j) + u_{ij}^+(x_i + x_j)\big) + \sum_{k=1}^{n} v_k(x_k) \tag{1.3}$$

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue 'Integrable Systems and Related Topics'. The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html appearing in other papers. The Schrödinger operators with these commuting differential operators treated in this paper include Calogero–Moser–Sutherland systems (cf. [5, 22, 27, 28, 36]), Heckman–Opdam's hypergeometric systems (cf. [34] for type A_{n-1}, [11] in general), their extensions (cf. [1, 8, 14, 23, 24, 25, 30]) and finite Toda lattices corresponding to (extended) Dynkin diagrams for classical root systems (cf. [2, 9, 10, 17, 26, 33, 38]) and those with boundary conditions (cf. [8, 13, 18, 19, 20, 21, 23, 30]).

Put $\partial_j = \partial/\partial x_j$ for simplicity. We denote by $\sigma(Q)$ the principal symbol of a differential operator of Q. For example, $\sigma(P) = -(1/2)(\xi_1^2 + \cdots + \xi_n^2)$.

We note that [40] proves that the potential function is of the form (1.3) if

$$\sigma(P_k) = \sum_{1\le j_1<\cdots<j_k\le n} \xi_{j_1}^2 \cdots \xi_{j_k}^2 \qquad \text{for} \qquad k = 1, \ldots, n. \tag{1.4}$$

In this case we say that $R(x)$ is an integrable potential function of *type* B_n or of the *classical type*. Moreover when $R(x)$ is symmetric with respect to the coordinate $(x_1, \ldots, x_n)$ and invariant under the coordinate transformation $(x_1, x_2, \ldots, x_n) \mapsto (-x_1, x_2, \ldots, x_n)$, then $R(x)$ is determined by [32] for $n \ge 3$ and by [24] for $n = 2$ and P_k are calculated by [29].

Classifications of the integrable potential functions under certain conditions are given in [23, 24, 25, 30, 37, 40] etc. In Section 9 we review them and we present Conjecture which claims that the potential functions given in this note exhaust those of the completely integrable systems satisfying (1.4). We also give a new condition which assures Conjecture.

If $v_k = 0$ for $k = 1, \ldots, n$, we can expect $\sigma(P_k) = \sum\limits_{1\le j_1<\cdots<j_k\le n} \xi_{j_1}^2 \cdots \xi_{j_k}^2$ for $k = 1, \ldots, n-1$ and $\sigma(P_n) = \xi_1\xi_2\cdots\xi_n$ and we say the integrable potential function is of *type* D_n. If $v_k = 0$ and $u_{ij}^+ = 0$ for $k = 1, \ldots, n$ and $1 \le i < j \le n$, we can expect $P_1 = \partial_1 + \cdots + \partial_n$, $\sigma(P_k) = \sum\limits_{1\le j_1<\cdots<j_k\le n} \xi_{j_1} \cdots \xi_{j_k}$ for $k = 2, \ldots, n$ and we say that the integrable potential function is of *type* A_{n-1}. Note that the integrable potential function of type A_{n-1} or D_n is of type B_n.

The elliptic potential function of type A_{n-1} with

$$u_{ij}^-(t) = C\wp(t; 2\omega_1, 2\omega_2) + C', \qquad u_{ij}^+(t) = v_k(t) = 0 \qquad (C, C' \in \mathbb{C})$$

(cf. [28]) and that of type B_n with

$$u_{ij}^-(t) = v_{ij}^+(t) = A\wp(t; 2\omega_1, 2\omega_2),$$
$$v_k(t) = \sum_{j=0}^{3} C_j\wp(t + \omega_j; 2\omega_1, 2\omega_2) - \frac{C}{2}, \qquad (A, C_i, C \in \mathbb{C})$$

introduced by [12] are most fundamental and their integrability and the integrals of higher order are established by [25, 29, 32]. Here $\wp(t; 2\omega_1, 2\omega_2)$ is the Weierstrass elliptic function whose fundamental periods are $2\omega_1$ and $2\omega_2$ and

$$\omega_0 = 0, \qquad \omega_1 + \omega_2 + \omega_3 = 0.$$

Other potential functions are suitable limits of these elliptic potential functions, which is shown in [8, 13, 33] etc. We will study integrable systems by taking analytic continuations of the integrals given in [25] with respect to a suitable parameter, which is done for the invariant systems (of type A_{n-1}) by [32] and (of types B_n and D_n) by [29] and for the systems of type A_{n-1} by [33]. The main purpose of this note is to give the explicit expression of the operators $P_1, \ldots, P_n$ in (1.2) in this unified way. Namely we construct enough commuting integrals of the non-invariant systems from those of the invariant systems given by [24, 29, 32]. Such study of the systems of types A_{n-1}, B_2, B_n $(n \ge 3)$ and D_n are explained in Sections 3, 4, 5, 6, respectively.

Since the integrals of the system of type A_{n-1} are much simpler than those of type B_n, we review the above analytic continuation for the systems of type A_{n-1} in Section 3 preceding to the study for the systems of type B_n. There are many series of completely integrable systems of type B_2, which we review and classify in Section 4 with taking account of the above unified way.

We present 8 series of potential functions of type B_n in Section 5. There are 3 (elliptic, trigonometric or hyperbolic and rational) series of the invariant potentials of type B_n whose enough integrals are constructed by [25] and [29]. The complete integrability of the remaining 5 series of the potential functions is shown in Section 5, which is conjectured by [8] (4 series), partially proved by [18, 19, 21] (3 series) and announced by [30] (5 series). The complete integrability of two series among them seems to be first established in this note. Note that when $n \geq 3$, our systems which do not belong to these 8 series of type B_n are the Calogero–Moser systems with elliptic potentials and the finite Toda lattices of type $A^{(1)}_{n-1}$, whose complete integrability is known.

The main purpose of our previous study in [23, 24, 25, 30] is classification of the completely integrable systems associated with classical root systems. In this note we explicitly give integrals of all the systems classified in our previous study with reviewing known integrals.

Since our expression of P_k is natural, we can easily define their classical limits without any ambiguity and get completely integrable Hamiltonians of dynamical systems together with their enough integrals. This is clarified in Section 7.

In Section 8 we examine ordinary differential operators which are analogues of the Schrödinger operators studied in this note.

2 Notation and preliminary results

Let $\{e_1, \ldots, e_n\}$ be the natural orthonormal base of the Euclidean space $\mathbb{R}^n$ with the inner product

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j \qquad \text{for} \qquad x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbb{R}^n.$$

Here $e_j = (\delta_{1j}, \ldots, \delta_{nj}) \in \mathbb{R}^n$ with Kronecker's delta δ_{ij}.

Let $\alpha \in \mathbb{R}^n \setminus \{0\}$. The reflection w_α with respect to α is a linear transformation of $\mathbb{R}^n$ defined by $w_\alpha(x) = x - \frac{2\langle\alpha,x\rangle}{\langle\alpha,\alpha\rangle}\alpha$ for $x \in \mathbb{R}^n$. Furthermore we define a differential operator ∂_α by

$$(\partial_\alpha\varphi)(x) = \frac{d}{dt}\varphi(x + t\alpha)\bigg|_{t=0}$$

and then $\partial_j = \partial_{e_j}$.

The root system $\Sigma(B_n)$ of type B_n is realized in $\mathbb{R}^n$ by

$$\begin{aligned}
&\Sigma(A_{n-1})^+ = \{e_i - e_j;\ 1 \leq i < j \leq n\},\\
&\Sigma(D_n)^+ = \{e_i \pm e_j;\ 1 \leq i < j \leq n\},\\
&\Sigma(B_n)^+_S = \{e_k;\ 1 \leq k \leq n\},\\
&\Sigma(B_n)^+ = \Sigma(D_n)^+ \cup \Sigma(B_n)^+_S,\\
&\Sigma(F) = \{\alpha, -\alpha;\ \alpha \in \Sigma(F)^+\} \qquad \text{for} \quad F = A_{n-1}, \quad D_n \quad \text{or} \quad B_n.
\end{aligned}$$

The Weyl groups $W(B_n)$ of type B_n, $W(D_n)$ of type D_n and $W(A_{n-1})$ of type A_{n-1} are the groups generated by w_α for $\alpha \in \Sigma(B_n)$, $\Sigma(D_n)$ and $\Sigma(A_{n-1})$, respectively. The Weyl group

$W(A_{n-1})$ is naturally identified with the permutation group $\mathfrak{S}_n$ of the set $\{1,\dots,n\}$ with n elements. Let ϵ be the group homomorphism of $W(B_n)$ defined by

$$\epsilon(w)=\begin{cases}\ \ 1 & \text{if}\quad w\in W(D_n),\\ -1 & \text{if}\quad w\in W(B_n)\setminus W(D_n).\end{cases}$$

The potential function (1.3) is of the form

$$R(x)=\sum_{\alpha\in\Sigma(D_n)^+}u_\alpha(\langle\alpha,x\rangle)+\sum_{\alpha\in\Sigma(B_n)^+_S}v_\beta(\langle\beta,x\rangle)$$

with functions u_α and v_β of one variable. For simplicity we will denote

$$\begin{aligned}&u_\alpha(x)=u_{-\alpha}(x)=u_\alpha(\langle\alpha,x\rangle)\qquad\text{for}\quad\alpha\in\Sigma(D_n)^+,\\&v_\beta(x)=v_{-\beta}(x)=v_\beta(\langle\beta,x\rangle)\qquad\text{for}\quad\beta\in\Sigma(B_n)^+_S,\\&u^\pm_{ij}(x)=u_{e_i\pm e_j}(x),\qquad v_k(x)=v_{e_k}(x).\end{aligned}$$

Lemma 1. *For a bounded open subset U of $\mathbb{C}$, there exists an open neighborhood V of 0 in $\mathbb{C}$ such that the following statements hold.*

i) *The function $\lambda\sinh^{-1}\lambda z$ is holomorphically extended to $(z,\lambda)\in(U\setminus\{0\})\times V$ and the function is $1/z$ when $\lambda=0$.*

ii) *Suppose $\operatorname{Re}\lambda>0$. Then the functions*

$$e^{2\lambda t}\sinh^{-2}\lambda(z\pm t)\qquad\textit{and}\qquad e^{4\lambda t}\big(\sinh^{-2}\lambda(z\pm t)-\cosh^{-2}\lambda(z\pm t)\big)$$

are holomorphically extended to $(z,q)\in U\times V$ with $q=e^{-2\lambda t}$ and the functions are $4e^{\mp2\lambda z}$ and $16e^{\mp4\lambda z}$, respectively, when $q=0$.

Proof. The claims are clear from

$$\begin{aligned}&\lambda^{-1}\sinh\lambda z=z+\sum_{j=1}^\infty\frac{\lambda^{2j}z^{2j+1}}{(2j+1)!},\\&4e^{-2\lambda t}\sinh^2\lambda(z\pm t)=e^{\pm2\lambda z}(1-e^{-2\lambda t}e^{\mp2\lambda z})^2,\\&\sinh^{-2}\lambda z-\cosh^{-2}\lambda z=4\sinh^{-2}2\lambda z.\end{aligned}$$

■

The elliptic functions $\wp$ and ζ of Weierstrass type are defined by

$$\begin{aligned}&\wp(z)=\wp(z;2\omega_1,2\omega_2)=\frac{1}{z^2}+\sum_{\omega\neq0}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right),\\&\zeta(z)=\zeta(z;2\omega_1,2\omega_2)=\frac{1}{z}+\sum_{\omega\neq0}\left(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\right),\end{aligned}$$

where the sum ranges over all non-zero periods $2m_1\omega_1+2m_2\omega_2$ (m_1, $m_2\in\mathbb{Z}$) of $\wp$. The following are some elementary properties of these functions (cf. [41]).

$$\wp(z)=\wp(z+2\omega_1)=\wp(z+2\omega_2),\tag{2.1}$$
$$\zeta'(z)=-\wp(z),\tag{2.2}$$
$$\begin{aligned}&(\wp')^2=4\wp^3-g_2\wp-g_3=4(\wp-e_1)(\wp-e_2)(\wp-e_3),\\&e_\nu=\wp(\omega_\nu)\qquad\text{for}\qquad\nu=1,2,3,\qquad\omega_3=-\omega_1-\omega_2\qquad\text{and}\qquad\omega_0=0,\end{aligned}\tag{2.3}$$

$$\wp(2z) = \frac{1}{4}\sum_{\nu=0}^{4}\wp(z+\omega_\nu) = \frac{(12\wp(z)^2-g_2)^2}{16\wp'(z)^2} - 2\wp(z),$$

$$\wp(z;2\omega_2,2\omega_1) = \wp(z;2\omega_1,2\omega_2), \tag{2.4}$$

$$\wp(z+\omega_1;2\omega_1,2\omega_2) = e_1 + \frac{(e_1-e_2)(e_1-e_3)}{\wp(z;2\omega_1,2\omega_2)-e_1}, \tag{2.5}$$

$$\wp(z;\sqrt{-1}\lambda^{-1}\pi,\infty) = \lambda^2\sinh^{-2}\lambda z + \frac{1}{3}\lambda^2, \tag{2.6}$$

$$\wp(z;\infty,\infty) = z^{-2}, \tag{2.7}$$

$$\wp(z;\omega_1,2\omega_2) = \wp(z;2\omega_1,2\omega_2) + \wp(z+\omega_1;2\omega_1,2\omega_2) - e_1, \tag{2.8}$$

$$\begin{vmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ \wp(z_3) & \wp'(z_3) & 1 \end{vmatrix} = 0 \qquad \text{if} \qquad z_1+z_2+z_3=0, \tag{2.9}$$

$$\begin{aligned}
&\wp(z;2\omega_1,2\omega_2) = -\frac{\eta_1}{\omega_1} + \lambda^2\sinh^{-2}\lambda z + \sum_{n=1}^{\infty}\frac{8n\lambda^2 e^{-4n\lambda\omega_2}}{1-e^{-4n\lambda\omega_2}}\cosh 2n\lambda z,\\
&\eta_1 = \zeta(\omega_1;2\omega_1,2\omega_2) = \frac{\pi^2}{\omega_1}\left(\frac{1}{12} - 2\sum_{n=1}^{\infty}\frac{ne^{-4n\lambda\omega_2}}{1-e^{-4n\lambda\omega_2}}\right),\\
&\tau = \frac{\omega_2}{\omega_1}, \qquad q = e^{\pi i\tau} = e^{-2\lambda\omega_2} \qquad \text{and} \qquad \lambda = \frac{\pi}{2\sqrt{-1}\omega_1}.
\end{aligned} \tag{2.10}$$

Here the sums in (2.10) converge if

$$2\mathrm{Im}\,\frac{\omega_2}{\omega_1} > \frac{|z|}{|\omega_1|}.$$

Let $0 \le k < 2m$. Then (2.10) means

$$\begin{aligned}
&\wp\left(z+\frac{k}{m}\omega_2;2\omega_1,2\omega_2\right) = -\frac{\eta_1}{\omega_1} + 4\lambda^2\left(\frac{q^{k/m}e^{-2\lambda z}}{(1-e^{-2\lambda z}q^{k/m})^2} + \sum_{n=1}^{\infty}\frac{q^{n(2-k/m)}e^{2n\lambda z}}{1-q^{2n}}\right),\\
&-\frac{\eta_1}{\omega_1} = 4\lambda^2\left(\frac{1}{12} - 2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right).
\end{aligned}$$

Lemma 2. *Let k and m be integers satisfying $0<k<2m$. Put*

$$\begin{aligned}
&\wp_0(z;2\omega_1,2\omega_2) = \wp(z;2\omega_1,2\omega_2) + \frac{\eta_1}{\omega_1},\\
&\lambda = \frac{\pi}{2\sqrt{-1}\omega_1} \qquad \textit{and} \qquad t = q^{1/m} = e^{\pi i\omega_2/(m\omega_1)}.
\end{aligned}$$

Then for any bounded open set U in $\mathbb{C}\times\mathbb{C}$, there exists a neighborhood V of the origin of $\mathbb{C}$ such that the following statements hold.

i) *$\wp_0(z;2\omega_1,2\omega_2)-\lambda^2\sinh^{-2}\lambda z$ and $\wp_0(z+\omega_1;2\omega_1,2\omega_2)+\lambda^2\cosh^{-2}\lambda z$ are holomorphic functions of $(z,\lambda,q)\in U\times V$ and vanish when $q=0$.*

ii) *$\wp_0(z+(k/m)\omega_2;2\omega_1,2\omega_2)$ is holomorphic for $(z,\lambda,t)\in U\times V$ and has zeros of order $\min\{k,2m-k\}$ along the hyperplane defined by $t=0$ and satisfies*

$$\begin{aligned}
&t^{-k}\wp_0\left(z+\frac{k}{m}\omega_2;2\omega_1,2\omega_2\right)\bigg|_{t=0} = 4\lambda^2 e^{-2\lambda z} \qquad (0<k<m),\\
&t^{-k}\wp_0\left(z+\frac{k}{m}\omega_2;2\omega_1,2\omega_2\right)\bigg|_{t=0} = 8\lambda^2\cosh 2\lambda z \qquad (k=m),
\end{aligned}$$

$$t^{k-2m}\wp_0\Big(z+\frac{k}{m}\omega_2;2\omega_1,2\omega_2\Big)\Big|_{t=0}=4\lambda^2e^{2\lambda z}\qquad (m<k<2m).$$

For our later convenience we list up some limiting formula discussed above. Fix ω_1 with $\sqrt{-1}\omega_1>0$ and let $\omega_2\in\mathbb{R}$ with $\omega_2>0$. Then $\lambda=\pi/(2\sqrt{-1}\omega_1)>0$ and

$$\sinh^2\lambda(z+\omega_1)=-\cosh^2\lambda z,\qquad \cosh 2\lambda(z+\omega_1)=-\cosh 2\lambda z,\tag{2.11}$$

$$\lim_{\lambda\to 0}\lambda^2\sinh^{-2}\lambda z=\frac{1}{z^2},\tag{2.12}$$

$$\lim_{N\to\pm\infty}e^{2\lambda|N|}\sinh^{-2}\lambda(z+N)=4e^{\mp 2\lambda z},\tag{2.13}$$

$$\lim_{\omega_2\to+\infty}\wp_0(z;2\omega_1,2\omega_2)=\lambda^2\sinh^{-2}\lambda z,\tag{2.14}$$

$$\lim_{\omega_2\to+\infty}\wp_0(z+\omega_1;2\omega_1,2\omega_2)=-\lambda^2\cosh^{-2}\lambda z,\tag{2.15}$$

$$\lim_{\omega_2\to\infty}e^{2r\lambda\omega_2}\wp_0(z+r\omega_2;2\omega_1,2\omega_2)=4\lambda^2e^{-2\lambda z}\qquad\text{if}\qquad 0<r<1,\tag{2.16}$$

$$\lim_{\omega_2\to\infty}e^{2\lambda\omega_2}\wp_0(z+\omega_2;2\omega_1,2\omega_2)=8\lambda^2\cosh 2\lambda z,\tag{2.17}$$

$$\lim_{\omega_2\to\infty}e^{2(2-r)\lambda\omega_2}\wp_0(z+r\omega_2;2\omega_1,2\omega_2)=4\lambda^2e^{2\lambda z}\qquad\text{if}\qquad 1<r<2.\tag{2.18}$$

3 Type A_{n-1} ($n\geq 3$)

The completely integrable Schrödinger operator of type A_{n-1} is of the form

$$P=-\frac12\sum_{j=1}^n\frac{\partial^2}{\partial x_j^2}+\sum_{1\leq i<j\leq n}u_{ij}^-(x_i-x_j).$$

Denoting

$$u_{e_i-e_j}(x)=u_{e_j-e_i}(x)=u_{ij}^-(x_i-x_j),$$

we put

$$\begin{aligned}P_k&=\sum_{0\leq\nu\leq k/2}\frac{1}{2^\nu\nu!(k-2\nu)!(n-k)!}\sum_{w\in\mathfrak{S}_n}u_{w(e_1-e_2)}u_{w(e_3-e_4)}\cdots u_{w(e_{2\nu-1}-e_{2\nu})}\partial_{w(e_{2\nu+1})}\cdots\partial_{w(e_k)}\\&=\sum_{0\leq\nu\leq k/2}\sum u_{i_1i_2}^-\cdots u_{i_{2\nu-1}i_{2\nu}}^-\partial_{i_{2\nu+1}}\cdots\partial_{i_k}\end{aligned}\tag{3.1}$$

according to the integrals given in [25, 32]. We will examine the functions $u_{ij}^-(t)$ which satisfy

$$[P_i,P_j]=0\qquad\text{for}\qquad 1\leq i<j\leq n.\tag{3.2}$$

Here we note that

$$\begin{aligned}&P=P_2-\frac12P_1^2,\qquad P_1=\partial_1+\cdots+\partial_n,\\&P_2=\sum_{1\leq i<j\leq n}\partial_i\partial_j+\sum_{1\leq i<j\leq n}u_{ij}^-(x_i-x_j),\\&P_3=\sum_{1\leq i<j<k\leq n}\partial_i\partial_j\partial_k+\sum_{k=1}^n\sum_{\substack{1\leq i<j\leq n\\ i\neq k,\ j\neq k}}u_{ij}^-(x_i-x_j)\partial_k.\end{aligned}$$

$$
\begin{aligned}
P_4 &= \sum_{1\le i<j<k<\ell\le n} \partial_i\partial_j\partial_k\partial_\ell + \sum_{1\le k<\ell\le n} \sum_{\substack{1\le i<j\le n\\ i\ne k,\ell,\ j\ne k,\ell}} u_{ij}^-\partial_k\partial_\ell + \sum_{1\le i<j<k<\ell\le n} (u_{ij}^-u_{k\ell}^- + u_{ik}^-u_{j\ell}^- + u_{i\ell}^-u_{jk}^-)\\
&= \sum \partial_i\partial_j\partial_k\partial_\ell + \sum u_{ij}^-\partial_k\partial_\ell + \sum u_{ij}^-u_{k\ell}^-,\\
P_5 &= \sum \partial_i\partial_j\partial_k\partial_\ell\partial_m + \sum u_{ij}^-\partial_k\partial_\ell\partial_m + \sum u_{ij}^-u_{k\ell}^-\partial_m,\\
P_6 &= \sum \partial_i\partial_j\partial_k\partial_\ell\partial_m\partial_\nu + \sum u_{ij}^-\partial_k\partial_\ell\partial_m\partial_\nu + \sum u_{ij}^-u_{k\ell}^-\partial_m\partial_\nu + \sum u_{ij}^-u_{k\ell}^-u_{m\nu}^-,\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Since $W(A_{n-1})$ is naturally isomorphic to the permutation group $\mathfrak{S}_n$ of the set $\{1,\dots,n\}$, we will identify them. In [25, 32], the integrable potentials of type A_{n-1} which are invariant under the action of $\mathfrak{S}_n$ are determined and moreover (3.2) with (3.1) is proved. They are

$$
u_{e_i-e_j}(x) = u(\langle e_i-e_j, x\rangle) \tag{3.3}
$$

with an even function u and

(Ellip-A_{n-1}) *Elliptic potential of type* A_{n-1}:

$$
\begin{aligned}
&u(t) = C\wp_0(t;2\omega_1,2\omega_2),\\
&R_E(A_{n-1};x_1,\dots,x_n;C,2\omega_1,2\omega_2) = C\sum_{1\le i<j\le n}\wp_0(x_i-x_j;2\omega_1,2\omega_2),
\end{aligned}
$$

(Trig-A_{n-1}) *Trigonometric potential of type* A_{n-1}:

$$
u(t) = C\sinh^{-2}\lambda t, \qquad R_T(A_{n-1};x_1,\dots,x_n;C,\lambda) = C\sum_{1\le i<j\le n}\sinh^{-2}\lambda(x_i-x_j),
$$

(Rat-A_{n-1}) *Rational potential of type* A_{n-1}:

$$
u(t) = \frac{C}{t^2}, \qquad R_R(A_{n-1};x_1,\dots,x_n;C) = \sum_{1\le i<j\le n}\frac{C}{(x_i-x_j)^2}.
$$

We review how the integrability of (Ellip-A_{n-1}) implies the integrability of other systems. Since it follows from (2.14) that

$$
\begin{aligned}
&\lim_{\omega_2\to\infty} R_E\left(A_{n-1};x;\frac{C}{\lambda^2},2\omega_1,2\omega_2\right) = R_T(A_{n-1};x;C,\lambda),\\
&u(t) = \lim_{\omega_2\to\infty}\frac{C}{\lambda^2}\wp_0(t;2\omega_1,2\omega_2) = C\sinh^{-2}\lambda t,
\end{aligned}
$$

the integrability (3.2) for (Trig-A_{n-1}) follows from that for (Ellip-A_{n-1}) by the analytic continuation of $u_{e_i-e_j}(x)$ and P_k with respect to q (cf. (2.10), (3.1), (3.3) and Lemma 2 i)).

The integrability for (Rat-A_{n-1}) is similarly follows in view of Lemma 1 with

$$
\begin{aligned}
&\lim_{\lambda\to 0} R_T(A_{n-1};x;\lambda^2C,\lambda) = R_R(A_{n-1};x;C),\\
&u(t) = \lim_{\lambda\to 0}\lambda^2C\sinh^{-2}\lambda t.
\end{aligned}
$$

This argument using the analytic continuation for the proof of (3.2) is given in [32].

As is shown [33], there are two other integrable potentials of type A_{n-1} to which this argument can be applied.

(Toda-$A_{n-1}^{(1)}$) *Toda potential of type* $A_{n-1}^{(1)}$:

$$R_L(A_{n-1}^{(1)};x;C,\lambda)=\sum_{i=1}^{n-1}Ce^{\lambda(x_i-x_{i+1})}+Ce^{\lambda(x_n-x_1)},$$

(Toda-A_{n-1}) *Toda potential of type* A_{n-1}:

$$R_L(A_{n-1};x;C,\lambda)=\sum_{i=1}^{n-1}Ce^{\lambda(x_i-x_{i+1})}.$$

The integrability (3.2) for these potentials similarly follows in view of Lemmas 1, 2 and

$$\begin{aligned}
&\lim_{\omega_2\to\infty}R_E\left(A_{n-1};x_1-\frac{2\omega_2}{n},\dots,x_k-\frac{2k\omega_2}{n},\dots,x_n-2\omega_2;\frac{e^{(4/n)\lambda\omega_2}}{4\lambda^2}C,2\omega_1,2\omega_2\right)\\
&\quad=R_L(A_{n-1}^{(1)};x;C,-2\lambda),\\
&u_{e_i-e_j}(x)=\lim_{\omega_2\to\infty}\frac{e^{(4/n)\lambda\omega_2}}{4\lambda^2}C\wp_0\left(x_i-x_j+\frac{2(j-i)\omega_2}{n};2\omega_1,2\omega_2\right)\\
&\quad=\begin{cases}Ce^{-2\lambda(x_i-x_{i+1})} & \text{if}\quad 1<j=i+1\le n,\\ Ce^{-2\lambda(x_n-x_1)} & \text{if}\quad i=1\quad\text{and}\quad j=n,\\ 0 & \text{if}\quad 1\le i<j\le n\quad\text{and}\quad j-i\ne 1,\ n-1\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{N\to\infty}R_T\left(A_{n-1};x_1-N,\dots,x_n-nN;\frac{e^{2\lambda N}}{4}C,\lambda\right)=R_L(A_{n-1};x;C,-2\lambda),\\
&u_{e_i-e_j}(x)=\lim_{N\to\infty}\frac{e^{2\lambda N}}{4}C\sinh^{-2}\lambda(x_i-x_j+(j-i)N)\\
&\quad=\begin{cases}Ce^{-2\lambda(x_i-x_{i+1})} & \text{if}\quad 1<j=i+1\le n,\\ 0 & \text{if}\quad 1\le i<j\le n\quad\text{and}\quad j\ne i+1,\end{cases}
\end{aligned}$$

respectively, if $\operatorname{Re}\lambda>0$. The restriction $\operatorname{Re}\lambda>0$ is removed also by the analytic continuation.

Thus the following theorem is obtained by the analytic continuation of the integrals (3.1) of (Ellip-A_{n-1}) whose commutativity (3.2) is assured by [32].

Theorem 1 (A_{n-1}, [10, 11, 28, 32, 33, 34], etc.). *The Schrödinger operators with the potential functions* (Ellip-A_{n-1}), (Trig-A_{n-1}), (Rat-A_{n-1}), (Toda-$A_{n-1}^{(1)}$) *and* (Toda-A_{n-1}) *are completely integrable and their integrals are given by* (3.1) *with* $u_{e_i-e_j}(x)$ *in the above.*

Remark 1. i) There are quite many papers studying these Schrödinger operators of type A_{n-1}. The proof of this theorem using analytic continuation is explained in [33].

ii) The complete integrability (3.2) for (Ellip-A_{n-1}) is first established by [32, Theorem 5.2], whose proof is as follows. The equations $[P_1,P_k]=[P_2,P_k]=0$ for $k=1,\dots,n$ are easily obtained by direct calculations with the formula (2.9) (cf. [32, Lemma 5.1]). Then the relation $[P_2,[P_i,P_j]]=0$ and periodicity and symmetry of P_k imply $[P_i,P_j]=0$ (cf. [32, Lemma 3.5]). Note that the proof of the integrability given in [28, § 5 Proposition 2 and Appendix E] is not correct as is clarified in [32, Remark 3.7] (cf. [33, § 4.2]).

Note that the complete integrability for (Trig-A_{n-1}) is shown in [34].

iii) If $\operatorname{Re}\lambda>0$, we also have

$$\lim_{N\to\infty}R_L\big(A_{n-1}^{(1)};x_k+kN;Ce^{-2\lambda N};-2\lambda\big)=R_L(A_{n-1};x;C,-2\lambda).$$

iv) Note that

$$R_L(A_{n-1}; x_k + N_k; C, \lambda) = \sum_{i=1}^{n-1} C e^{\lambda(N_i - N_{i+1})} \cdot e^{\lambda(x_i - x_{i+1})}.$$

Hence $e^{\lambda(x_1-x_2)} - e^{\lambda(x_2-x_3)}$ gives the potential function of a completely integrable system of type A_{n-1} with $n = 3$ but the potential function

$$\lim_{\lambda\to 0} \lambda^{-1}\bigl(e^{\lambda(x_1-x_2)} - e^{\lambda(x_2-x_3)}\bigr) = x_1 - 2x_2 + x_3$$

does not give such a system because it does not satisfy (9.3) in Remark 13.

Considering the limit of the parameters of the integrable potential function, we should take care of the limit of integrals.

v) Let $P_n(t)$ denote the differential operator P_n in (3.1) defined by replacing u_{ij}^- by $\tilde{u}_{ij}^- = u_{ij}^- + t$ with a constant $t \in \mathbb{C}$. Then

$$\begin{aligned} &P_n(t) = \sum_{0\le k\le n/2} \frac{(2k)!}{2^k k!} P_{n-2k} t^k \qquad \text{with} \qquad P_0 = 1, \\ &[P_n(s), P_n(t)] = 0 \qquad \text{for} \qquad s,\, t \in \mathbb{C}. \end{aligned} \tag{3.4}$$

In fact, the term $u_{12}^- u_{34}^- \cdots u_{2j-1,2j}^- \partial_{2j+1}\partial_{2j+2}\cdots\partial_{n-2k}$ appears only in the coefficient of t^k in the right hand side of (3.4) and it is contained in the terms

$$\tilde{u}_{i_{n-2k+1}i_{n-2k+2}}^- \cdots \tilde{u}_{i_{n-1}i_n}^- \tilde{u}_{12}^- \cdots \tilde{u}_{2j-1,2j}^- \partial_{2j+1}\cdots\partial_{n-2k}$$

of $P_n(t)$, where the number of the possibilities of these $\tilde{u}_{i_{n-2k+1}i_{n-2k+2}} \cdots \tilde{u}_{i_{n-1}i_n}^-$ is $(2k)!/(2^k k!)$ because $\{i_{n-2k+1}, i_{n-2k+2}, \ldots, i_n\} = \{n-2k+1, \ldots, n\}$.

vi) Since

$$P_{k-1} = (n-k+1)[P_k, x_1 + \cdots + x_n] \qquad \text{for} \qquad k = 2, \ldots . n,$$

$[P_k, P_2] = 0$ implies $[P_{k-1}, P_2] = 0$ by the Jacobi identity. Here we note that

$$[u_{12}^- u_{34}^- \cdots u_{2j-1,2j}^- \partial_{2j+1}\cdots\partial_{k-1}\partial_\nu, x_\nu] = u_{12}^- u_{34}^- \cdots u_{2j-1,2j}^- \partial_{2j+1}\cdots\partial_{k-1}$$

for $\nu = k, k+1, \ldots, n$.

vi) The potential functions of (Trig-A_{n-1}), (Rat-A_{n-1}) and (Toda-A_{n-1}) are specializations of more general integrable potential functions of type B_n (cf. Definition 5).

In the following diagram we show the relations among integrable potentials of type A_{n-1} by taking limits.

Hierarchy of Integrable Potentials of Type A_{n-1} ($n \ge 3$)

$$\begin{array}{ccccc} & & \text{Toda-}A_{n-1}^{(1)} & \to & \text{Toda-}A_{n-1} \\ & \nearrow & & \nearrow & \\ \text{Ellip-}A_{n-1} & \to & \text{Trig-}A_{n-1} & \to & \text{Rat-}A_{n-1} \end{array}$$

4 Type B_2

In this section we study the following commuting differential operators P and P_2.

$$
\begin{aligned}
&P = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + R(x,y),\\
&P_2 = \frac{\partial^4}{\partial x^2 \partial y^2} + S \qquad \text{with} \qquad \operatorname{ord} S < 4,\\
&[P, P_2] = 0.
\end{aligned} \tag{4.1}
$$

The Schrödinger operators P of type B_2 in this section are known to be completely integrable. They are listed in [23, 24, 30]. We review them and give the explicit expression of P_2.

First we review the arguments given in [23, 24]. Since P is self-adjoint, we may assume P_2 is also self-adjoint by replacing P_2 by its self-adjoint part if necessary. Here for

$$
A = \sum a_{ij}(x,y)\frac{\partial^{i+j}}{\partial x^i \partial y^j}
$$

we define

$$
{}^tA = \sum (-1)^{i+j}\frac{\partial^{i+j}}{\partial x^i \partial y^j} a_{ij}(x,y)
$$

and A is called self-adjoint if ${}^tA = A$.

Lemma 3 ([23]). *Suppose P and P_2 are self-adjoint operators satisfying* (4.1)*. Then*

$$
\begin{aligned}
&R(x,y) = u^+(x+y) + u^-(x-y) + v(x) + w(y),\\
&P_2 = \left(\frac{\partial^2}{\partial x \partial y} + u^-(x-y) - u^+(x+y)\right)^2 - 2w(y)\frac{\partial^2}{\partial x^2} - 2v(x)\frac{\partial^2}{\partial y^2} + 4v(x)w(y) + T(x,y)
\end{aligned} \tag{4.2}
$$

and the function $T(x,y)$ satisfies

$$
\begin{aligned}
&\frac{\partial T(x,y)}{\partial x} = 2(u^+(x+y) - u^-(x-y))\frac{\partial w(y)}{\partial y} + 4w(y)\frac{\partial}{\partial y}(u^+(x+y) - u^-(x-y)),\\
&\frac{\partial T(x,y)}{\partial y} = 2(u^+(x+y) - u^-(x-y))\frac{\partial v(x)}{\partial x} + 4v(x)\frac{\partial}{\partial x}(u^+(x+y) - u^-(x-y)).
\end{aligned} \tag{4.3}
$$

Conversely, if a function $T(x,y)$ satisfies (4.3) *for suitable functions $u^\pm(t)$, $v(t)$ and $w(t)$, then* (4.1) *is valid for $R(x,y)$ and P_2 defined by* (4.2)*.*

Remark 2. i) If $w(y) = 0$, then $T(x,y)$ does not depend on x.

ii) The self-adjointness of P_2 and the vanishing of the third order term of $[P, P_2]$ imply that P_2 should be of the form (4.2) with a suitable function $T(x,y)$. Then the vanishing of the first order term implies (4.3). The last claim in Lemma 3 is obtained by direct calculation.

Since $T(x,y)$ satisfying (4.3) is determined by $(u^-, u^+; v, w)$ up to the difference of constants, we will write $T(u^-, u^+; v, w)$ for the corresponding $T(x,y)$ which is an element of the space of meromorphic functions of (x,y) modulo constant functions and define $Q(u^-, u^+; v, w)$ by

$$
T(u^-, u^+; v, w) = 2(u^-(x-y) + u^+(x+y))(v(x) + w(y)) - 4Q(u^-, u^+; v, w). \tag{4.4}
$$

Note that $T(u^-, u^+; v, w)$ and $Q(u^-, u^+; v, w)$ are defined only if the function $T(x,y)$ satisfying (4.3) exists. The following lemma is a direct consequence of (4.3) and this definition of Q.

Lemma 4 ([23]). *Suppose $T(u^-,u^+;v,w)$ and $T(u_i^-,u_i^+;v_j,w_j)$ are defined. Then for any C, C_i, $C_j' \in \mathbb{C}$ the left hand sides of the following identities are also defined and*

$$T(u^-(t)+C,u^+(t)+C;v(t),w(t))) = T(u^-(t),u^+(t);v(t),w(t)),$$
$$Q(u^-(t),u^+(t);v(t)+C,w(t)+C)) = Q(u^-(t),u^+(t);v(t),w(t)),$$
$$Q(u^-(Ct),u^+(Ct);v(Ct),w(Ct)) = Q(u^-(t),u^+(t);v(t),w(t))|_{x\to Cx,\, y\to Cy},$$
$$Q\left(\sum_{i=1}^{2} A_i u_i^-, \sum_{i=1}^{2} A_i u_i^+; \sum_{j=1}^{2} C_j v_j, \sum_{j=1}^{2} C_j w_j\right) = \sum_{i=1}^{2}\sum_{j=1}^{2} A_i C_j Q(u_i^-,u_i^+;v_j,w_j).$$

Hence the left hand sides give pairs P and P_2 with $[P,P_2]=0$.

For simplicity we will use the notation

$$Q(u^-,u^+;v) = Q(u^-,u^+;v,v), \quad Q(u;v,w) = Q(u,u;v,w), \quad Q(u;v) = Q(u,u;v,v). \tag{4.5}$$

The same convention will be also used for $T(u^-,u^+;v,w)$. The integrable potentials of type B_2 in this note are classified into three kinds. The potentials of the first kind are the unified integrable potentials which are in the same form as those of type B_n with $n \geq 3$, which we call *normal integrable potentials* of type B_2.

The integrable potentials of type B_2 admit a special transformation called *dual* which does not exist in B_n with $n \geq 3$. Hence there are normal potentials and their dual in the invariant integrable potentials of type B_2. Because of this duality, there exist another kind of invariant integrable potentials of type B_2, which we call *special integrable potentials* of type B_2.

In this section we present $\big(R(x,y),T(x,y)\big)$ as suitable limits of elliptic functions as in the previous section since it helps to study the potentials of type B_n in Section 5. We reduce the complete integrability of the limits to that of a systems with elliptic potentials. But we can also check (4.3) by direct calculations (cf. Remark 5).

4.1 Normal case

In this subsection we study the integrable systems (4.1) with (4.2) which have natural extension to type B_n for $n \geq 3$ and have the form

$$u^-(t) = Au_0^-(t), \qquad u^+(t) = Au_0^+(t), \qquad v(t) = \sum_{j=0}^{3} C_j v_j(t), \qquad w(t) = \sum_{j=0}^{3} C_j w_j(t)$$

with any A, C_0, C_1, C_2, $C_3 \in \mathbb{C}$. These systems are expressed by the symbol

$$(\langle u_0^-\rangle, \langle u_0^+\rangle; \langle v_0,v_1,v_2,v_3\rangle, \langle w_0,w_1,w_2,w_3\rangle).$$

The most general system is the following (Ellip-B_2) defined by elliptic functions, which is called Inozemtsev model [12].

Theorem 2 (B_2: Normal case, [12, 24, 23, 30] etc.). *The operators P and P_2 defined by the following pairs of $R(x,y)$ and $T(x,y)$ satisfy* (4.1) *and* (4.2).

(Ellip-B_2): $\quad (\langle\wp(t)\rangle; \langle\wp(t),\wp(t+\omega_1),\wp(t+\omega_2),\wp(t+\omega_3)\rangle)$

$$v(x) = \sum_{j=0}^{3} C_j \wp(x+\omega_j), \qquad w(y) = \sum_{j=0}^{3} C_j \wp(y+\omega_j),$$

$$
\begin{aligned}
&u^-(x-y) = A\wp(x-y), \qquad u^+(x+y) = A\wp(x+y),\\
&R(x,y) = A(\wp(x-y)+\wp(x+y)) + \sum_{j=0}^{3} C_j(\wp(x+\omega_j)+\wp(y+\omega_j)),\\
&T(x,y) = 2A(\wp(x-y)+\wp(x+y))\left(\sum_{j=0}^{3} C_j\big(\wp(x+\omega_j)+\wp(y+\omega_j)\big)\right)\\
&\qquad - 4A\sum_{j=0}^{3} C_j\wp(x+\omega_j)\wp(y+\omega_j).
\end{aligned}
$$

(Trig-B_2): $(\langle \sinh^{-2}\lambda t\rangle; \langle \sinh^{-2}\lambda t, \sinh^{-2}2\lambda t, \sinh^2\lambda t, \sinh^2 2\lambda t\rangle)$

$$
\begin{aligned}
&v(x) = C_0\sinh^{-2}\lambda x + C_1\cosh^{-2}\lambda x + C_2\sinh^2\lambda x + \frac{1}{4}C_3\sinh^2 2\lambda x,\\
&w(y) = C_0\sinh^{-2}\lambda y + C_1\cosh^{-2}\lambda y + C_2\sinh^2\lambda y + \frac{1}{4}C_3\sinh^2 2\lambda y,\\
&u^-(x-y) = A\sinh^{-2}\lambda(x-y), \qquad u^+(x+y) = A\sinh^{-2}\lambda(x+y),\\
&R(x,y) = A\big(\sinh^{-2}\lambda(x-y)+\sinh^{-2}\lambda(x+y)\big) + C_0\big(\sinh^{-2}\lambda x + \sinh^{-2}\lambda y\big)\\
&\qquad + C_1\big(\cosh^{-2}\lambda x + \cosh^{-2}\lambda y\big) + C_2\big(\sinh^2\lambda x + \sinh^2\lambda y\big)\\
&\qquad + \frac{1}{4}C_3\big(\sinh^2 2\lambda x + \sinh^2 2\lambda y\big),\\
&T(x,y) = 2A(\sinh^{-2}\lambda(x-y)+\sinh^{-2}\lambda(x+y))\\
&\qquad \times\Big(C_0\big(\sinh^{-2}\lambda x + \sinh^{-2}\lambda y\big) + C_1(\cosh^{-2}\lambda x + \cosh^{-2}\lambda y)\\
&\qquad + C_2(\sinh^2\lambda x + \sinh^2\lambda y) + \frac{1}{4}C_3(\sinh^2 2\lambda x + \sinh^2 2\lambda y)\Big)\\
&\qquad - 4A\Big(C_0\sinh^{-2}\lambda x\cdot\sinh^{-2}\lambda y - C_1\cosh^{-2}\lambda x\cdot\cosh^{-2}\lambda y\\
&\qquad + C_3(\sinh^2\lambda x + \sinh^2\lambda y + 2\sinh^2\lambda x\cdot\sinh^2\lambda y)\Big).
\end{aligned}
$$

(Rat-B_2): $(\langle t^{-2}\rangle; \langle t^{-2}, t^2, t^4, t^6\rangle)$

$$
\begin{aligned}
&v(x) = C_0x^{-2} + C_1x^2 + C_2x^4 + C_3x^6, \qquad w(y) = C_0y^{-2} + C_1y^2 + C_2y^4 + C_3y^6,\\
&u^-(x-y) = \frac{A}{(x-y)^2}, \qquad u^+(x+y) = \frac{A}{(x+y)^2},\\
&R(x,y) = \frac{A}{(x-y)^2} + \frac{A}{(x+y)^2} + C_0(x^{-2}+y^{-2}) + C_1(x^2+y^2) + C_2(x^4+y^4) + C_3(x^6+y^6),\\
&T(x,y) = 2(u^-(x-y)+u^+(x+y))(v(x)+w(y)) - 4A\bigg(\frac{C_0}{x^2y^2} + C_2(x^2+y^2)\\
&\qquad + C_3(x^4+y^4+3x^2y^2)\bigg) = 8A\frac{2C_0 + 2C_1x^2y^2 + C_2x^2y^2(x^2+y^2) + 2C_3x^4y^4}{(x^2-y^2)^2}.
\end{aligned}
$$

(Toda-$D_2^{(1)}$-bry): $(\langle \cosh 2\lambda t\rangle; \langle \sinh^{-2}\lambda t, \sinh^{-2}2\lambda t\rangle, \langle \sinh^{-2}\lambda t, \sinh^{-2}2\lambda t\rangle)$

$$
\begin{aligned}
&v(x) = C_0\sinh^{-2}\lambda x + C_1\sinh^{-2}2\lambda x, \qquad w(y) = C_2\sinh^{-2}\lambda y + C_3\sinh^{-2}2\lambda y,\\
&u^-(x-y) = A\cosh 2\lambda(x-y), \qquad u^+(x+y) = A\cosh 2\lambda(x+y),\\
&R(x,y) = A\cosh 2\lambda(x-y) + A\cosh 2\lambda(x+y)\\
&\qquad + C_0\sinh^{-2}\lambda x + C_1\sinh^{-2}2\lambda x + C_2\sinh^{-2}\lambda y + C_3\sinh^{-2}2\lambda y,
\end{aligned}
$$

$$T(x,y) = 8A(C_0\cosh 2\lambda y + C_2\cosh 2\lambda x).$$

(Toda-$B_2^{(1)}$-bry): $(\langle e^{-2\lambda t}\rangle; \langle e^{2\lambda t}, e^{4\lambda t}\rangle, \langle \sinh^{-2}\lambda t, \sinh^{-2}2\lambda t\rangle)$

$$\begin{aligned}
&v(x) = C_0e^{2\lambda x} + C_1e^{4\lambda x}, \qquad w(y) = C_2\sinh^{-2}\lambda y + C_3\sinh^{-2}2\lambda y,\\
&u^-(x-y) = Ae^{-2\lambda(x-y)}, \qquad u^+(x+y) = Ae^{-2\lambda(x+y)},\\
&R(x,y) = Ae^{-2\lambda(x-y)} + Ae^{-2\lambda(x+y)} + C_0e^{2\lambda x} + C_1e^{4\lambda x} + C_2\sinh^{-2}\lambda y + C_3\sinh^{-2}2\lambda y,\\
&T(x,y) = 4A\big(C_0\cosh 2\lambda y + 2C_2e^{-2\lambda x}\big).
\end{aligned}$$

(Trig-$A_1^{(1)}$-bry): $(\langle \sinh^{-2}\lambda t\rangle, 0; \langle e^{-2\lambda t}, e^{-4\lambda t}, e^{2\lambda t}, e^{4\lambda t}\rangle)$

$$\begin{aligned}
&v(x) = C_0e^{-2\lambda x} + C_1e^{-4\lambda x} + C_2e^{2\lambda x} + C_3e^{4\lambda x},\\
&w(y) = C_0e^{-2\lambda y} + C_1e^{-4\lambda y} + C_2e^{2\lambda y} + C_3e^{4\lambda y},\\
&u^-(x-y) = A\sinh^{-2}\lambda(x-y), \qquad u^+(x+y) = 0,\\
&R(x,y) = A\sinh^{-2}\lambda(x-y) + C_0(e^{-2\lambda x} + e^{-2\lambda y}) + C_1(e^{-4\lambda x} + e^{-4\lambda y})\\
&\qquad\quad + C_2(e^{2\lambda x} + e^{2\lambda y}) + C_3(e^{4\lambda x} + e^{4\lambda y}),\\
&T(x,y) = 2A\sinh^{-2}\lambda(x-y)(C_0\big(e^{-2\lambda x} + e^{-2\lambda y}\big)\\
&\qquad\quad + 2C_1e^{-2\lambda(x+y)} + C_2\big(e^{2\lambda x} + e^{2\lambda y}\big) + 2C_3e^{2\lambda(x+y)}).
\end{aligned}$$

(Toda-$C_2^{(1)}$): $(\langle e^{-2\lambda t}\rangle, 0; \langle e^{2\lambda t}, e^{4\lambda t}\rangle, \langle e^{-2\lambda t}, e^{-4\lambda t}\rangle)$

$$\begin{aligned}
&v(x) = C_0e^{2\lambda x} + C_1^{4\lambda x}, \qquad w(y) = C_2e^{-2\lambda y} + C_3e^{-4\lambda y},\\
&u^-(x-y) = Ae^{-2\lambda(x-y)}, \qquad u^+(x+y) = 0,\\
&R(x,y) = Ae^{-2\lambda(x-y)} + C_0e^{2\lambda x} + C_1^{4\lambda x} + C_2e^{-2\lambda y} + C_3e^{-4\lambda y},\\
&T(x,y) = 2A\big(C_0e^{2\lambda y} + C_2e^{-2\lambda x}\big).
\end{aligned}$$

(Rat-A_1-bry): $(\langle t^{-2}\rangle, 0; \langle t, t^2, t^3, t^4\rangle)$

$$\begin{aligned}
&v(x) = C_0x + C_1x^2 + C_2x^3 + C_3x^4, \qquad w(y) = C_0y + C_1y^2 + C_2y^3 + C_3y^4,\\
&u^-(x-y) = \frac{A}{(x-y)^2}, \qquad u^+(x+y) = 0,\\
&R(x,y) = \frac{A}{(x-y)^2} + C_0(x+y) + C_1(x^2+y^2) + C_2(x^3+y^3) + C_3(x^4+y^4),\\
&T(x,y) = \frac{2A}{(x-y)^2}(C_0(x+y) + C_1(x^2+y^2) + C_2xy(x+y) + 2C_3x^2y^2).
\end{aligned}$$

Remark 3. For example, $(\langle t^{-2}\rangle, 0; \langle t, t^2, t^3, t^4\rangle)$ in the above (Rat-A_1-bry) means

$$u^-(t) = At^{-2}, \qquad u^+(t) = 0, \qquad v(t) = w(t) = C_0t + C_1t^2 + C_2t^3 + C_3t^4$$

with a convention similar to that in (4.5).

We will review the proof of the above theorem after certain remarks.

Remark 4. All the invariant integrable potentials of type B_2 together with P_2 are determined by [24, 25]. They are classified into three cases. In the normal case they are (Ellip-B_2), (Trig-B_2) and (Rat-B_2) which have the following unified expression of the invariant potentials given by [32, Lemma 7.3], where the periods $2\omega_1$ and $2\omega_2$ may be infinite (cf. (2.6) and (2.7))

$$R(x,y) = A\wp(x-y) + A\wp(x+y) + \frac{C_4\wp(x)^4 + C_3\wp(x)^3 + C_2\wp(x)^2 + C_1\wp(x) + C_0}{\wp'(x)^2}$$

$$+\frac{C_4\wp(y)^4+C_3\wp(y)^3+C_2\wp(y)^2+C_1\wp(y)+C_0}{\wp'(y)^2},$$
$$T(x,y)=4A(\wp(x)-\wp(y))^{-2}\biggl(C_4\wp(x)^2\wp(y)^2+\frac{C_3}{2}\wp(x)^2\wp(y)$$
$$+\frac{C_3}{2}\wp(x)\wp(y)^2+C_2\wp(x)\wp(y)+\frac{C_1}{2}\wp(x)+\frac{C_1}{2}\wp(y)+C_0\biggr).$$

This is the original form we found in the classification of the invariant integrable systems of type B_n (cf. [25]). Later we knew Inozemtzev model and in fact, when the periods are finite, (2.3) and (2.5) show that the above potential function corresponds to (Ellip-B_2).

When $\omega_1=\omega_2=\infty$, $\wp(t)=t^{-2}$ and $(\wp(x)-\wp(y))^{-2}=x^4y^4(x^2-y^2)^{-2}$ and

$$R(x,y)=\frac{A}{(x-y)^2}+\frac{A}{(x+y)^2}+\frac{1}{4}(C_4x^{-2}+C_3+C_2x^2+C_1x^4+C_0x^6)$$
$$+\frac{1}{4}(C_4y^{-2}+C_3+C_2y^2+C_1y^4+C_0y^6),$$
$$T(x,y)=2A(x^2-y^2)^{-2}(2C_4+C_3(x^2+y^2)+2C_2x^2y^2+C_1x^2y^2(x^2+y^2)+2C_0x^4y^4).$$

We review these invariant cases discussed in [24, 32]. Owing to the identity

$$2(u^--u^+)v'+4v((u^-)'-(u^+)')+\partial_y(2(u^-+u^+)(v+w)-4vw)$$
$$=2\begin{vmatrix} v & v' & 1\\ w & -w' & 1\\ u^- & -(u^-)' & 1\end{vmatrix}+2\begin{vmatrix} v & -v' & 1\\ w & -w' & 1\\ u^+ & (u^+)' & 1\end{vmatrix}$$

and (2.9), the right hand side of the above is zero and we have (4.3) when

$$u^-=C\wp(x-y)+C',\quad u^+=C\wp(x+y)+C',\quad v=C\wp(x)+C'\quad\text{and}\quad w=C\wp(y)+C'$$

with $T(x,y)=2(u^-+u^+)(v+w)-4vw$. Hence with $Q\bigl(\wp(t);\wp(t)\bigr)=\wp(x)\wp(y)$, the function $T(x,y)$ given by (4.4) satisfies (4.3) with the above $u^\pm$, v and w. Using the transformations $(x,y)\mapsto(x+\omega_j,y+\omega_j)$, we have

$$\wp(x+\omega_j)\wp(y+\omega_j)=Q(\wp(t+\omega_j);\wp(t+\omega_j))=Q(\wp(t);\wp(t+\omega_j))\tag{4.6}$$

for $j=0,1,2,3$ because the function $\wp(x\pm y)$ does not change under these transformations (cf. (2.1)). Thus we have Theorem 2 for (Ellip-B_2) in virtue of Lemma 3 and Lemma 4.

Here we note that $\wp$ may be replaced by $\wp_0$.

By the limit under $\omega_2\to\infty$, we have the following (Trig-B_2) from (Ellip-B_2). See the proof of [29, Proposition 6.1] for the precise argument.

(Trig-B_2):

$$Q(\sinh^{-2}\lambda t;\sinh^{-2}\lambda t)=\sinh^{-2}\lambda x\cdot\sinh^{-2}\lambda y,\tag{4.7}$$
$$Q(\sinh^{-2}\lambda t;\cosh^{-2}\lambda t)=-\cosh^{-2}\lambda x\cdot\cosh^{-2}\lambda y,\tag{4.8}$$
$$Q(\sinh^{-2}\lambda t;\sinh^{2}\lambda t)=0,\tag{4.9}$$
$$Q\biggl(\sinh^{-2}\lambda t;\frac{1}{4}\sinh^2 2\lambda t\biggr)=\sinh^2\lambda x+\sinh^2\lambda y+2\sinh^2\lambda x\cdot\sinh^2\lambda y.\tag{4.10}$$

The equations (4.7), (4.8) and (4.9) correspond to (2.14), (2.15) and (2.17), respectively. Moreover (2.8) should be noted and (4.10) corresponds to (2.17) with replacing (ω_1,λ) by $(\omega_1/2,2\lambda)$.

By the limit under $\lambda \to 0$, we have the following (Rat-B_2) from (Trig-B_2) as was shown in the proof of [29, Proposition 6.3]. Here we note (2.13) and

$$\begin{aligned}
&\cosh^{-2}\lambda t\cdot\sinh^2\lambda t = 1-\cosh^{-2}\lambda t,\\
&\cosh^{-2}\lambda t\cdot\sinh^4\lambda t = -1+\cosh^{-2}\lambda t+\sinh^2\lambda t,\\
&\cosh^{-2}\lambda t\cdot\sinh^6\lambda t = 1-\cosh^{-2}\lambda t-2\sinh^2\lambda t+\frac{1}{4}\sinh^2 2\lambda t,\\
&\lim_{\lambda\to 0}\lambda^{-2j}\cosh^{-2}\lambda t\cdot\sinh^{2j}\lambda t = t^{2j}\qquad\text{for}\qquad j=1,2,3,\\
&\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}=\frac{2(x^2+y^2)}{(x^2-y^2)^2}.
\end{aligned}$$

The result is as follows.

(Rat-B_2):

$$\begin{aligned}
Q(t^{-2};t^{-2}) &= x^{-2}y^{-2},\\
T(t^{-2};t^{-2}) &= 4(x^{-2}+y^{-2})((x-y)^{-2}+(x+y)^{-2})-4x^{-2}y^{-2}\\
&=\frac{4(x^2+y^2)^2-4(x^2-y^2)^2}{x^2y^2(x^2-y^2)^2}=\frac{16}{(x^2-y^2)^2},\\
Q(t^{-2};t^2) &= 0,\\
T(t^{-2};t^2) &= \frac{4(x^2+y^2)^2}{(x^2-y^2)^2}=\frac{16x^2y^2}{(x^2-y^2)^2}+4,\\
Q(t^{-2};t^4) &= x^2+y^2,\\
T(t^{-2},t^4) &= \frac{4(x^2+y^2)(x^4+y^4)}{(x^2-y^2)^2}-4(x^2+y^2)=\frac{8x^2y^2(x^2+y^2)}{(x^2-y^2)^2},\\
Q(t^{-2};t^6) &= x^4+y^4+3x^2y^2,\\
T(t^{-2},t^6) &= \frac{4(x^2+y^2)(x^6+y^6)}{(x^2-y^2)^2}-4(x^4+3x^2y^2+y^4)=\frac{16x^4y^4}{(x^2-y^2)^2}.
\end{aligned}$$

This expression of $T(x,y)$ for (Rat-B_2) is also given in Remark 4. Note that we ignore the difference of constants for Q and T.

Proof of Theorem 2. The three cases (Ellip-B_2), (Trig-B_2) and (Rat-B_2) have been explained. Note that if $u^{\pm}$, v, w and $T(u^-,u^+;v,w)$ (or $Q(u^-,u^+;v,w)$) are defined and they have an analytic parameter, Lemma 3 assures that their analytic continuations also define P and P_2 satisfying $[P,P_2]=0$.

(Toda-$D_2^{(1)}$-bry) $\leftarrow$ (Ellip-B_2): Replacing (x,y) by $(x+\omega_2,y)$, we have

$$\begin{aligned}
Q(\cosh 2\lambda t;0,\sinh^{-2}\lambda t) &= \lim_{\omega_2\to\infty} Q\biggl(\frac{e^{2\lambda\omega_2}}{8\lambda^2}\wp_0(t+\omega_2);\frac{1}{\lambda^2}\wp_0(t+\omega_2),\frac{1}{\lambda^2}\wp_0(t)\biggr)\\
&=\lim_{\omega_2\to\infty}\frac{e^{2\lambda\omega_2}}{8\lambda^4}\wp_0(x+\omega_2)\wp_0(y)=\cosh 2\lambda x\cdot\sinh^{-2}\lambda y,\\
Q(\cosh 2\lambda t;0,\cosh^{-2}\lambda t) &= \lim_{\omega_2\to\infty} Q\biggl(\frac{e^{2\lambda\omega_2}}{8\lambda^2}\wp_0(t+\omega_2);-\frac{1}{\lambda^2}\wp_0(t+\omega_1+\omega_2);-\frac{1}{\lambda_2}\wp_0(t+\omega_1)\biggr)\\
&=-\lim_{\omega_2\to\infty}\frac{e^{2\lambda\omega_2}}{8\lambda^4}\wp_0(x+\omega_1+\omega_2)\cdot\wp_0(y+\omega_1)\\
&=-\cosh 2\lambda x\cdot\cosh^{-2}\lambda y,\\
Q(\cosh 2\lambda t;\sinh^{-2}\lambda t,0) &= \sinh^{-2}\lambda x\cdot\cosh^2\lambda y,
\end{aligned}$$

$$Q(\cosh 2\lambda t;\cosh^{-2}\lambda t,0)=-\cosh^{-2}\lambda x\cdot\sinh^2\lambda y.$$

Hence

$$\begin{aligned}
T(\cosh 2\lambda t;0,\sinh^{-2}\lambda t)&=2\bigl(\cosh 2\lambda(x+y)+\cosh 2\lambda(x-y)\bigr)\cdot\sinh^{-2}\lambda y\\
&\qquad-4\cosh 2\lambda x\cdot\sinh^{-2}\lambda y=8\cosh 2\lambda x,\\
T(\cosh 2\lambda t;0,\cosh^{-2}\lambda t)&=2\bigl(\cosh 2\lambda(x+y)+\cosh 2\lambda(x-y)\bigr)\cdot\cosh^{-2}\lambda y\\
&\qquad+4\cosh 2\lambda x\cdot\cosh^{-2}\lambda y=8\cosh 2\lambda x,\\
T(\cosh 2\lambda t;0,\sinh^{-2}2\lambda t)&=T(\cosh 2\lambda t;\sinh^{-2}2\lambda t,0)=0,\\
T(\cosh 2\lambda t;\sinh^{-2}\lambda t,0)&=8\cosh 2\lambda y.
\end{aligned}$$

(Toda-$B_2^{(1)}$-bry) $\leftarrow$ (Toda-$D_2^{(1)}$-bry): Replacing (x,y) by $(x-N,y)$, we have

$$\begin{aligned}
T(e^{-2\lambda t};0,\sinh^{-2}\lambda t)&=\lim_{N\to\infty}T(2e^{-2\lambda N}\cosh 2\lambda(t-N);0,\sinh^{-2}\lambda t)\\
&=\lim_{N\to\infty}16e^{-2\lambda N}\cosh 2\lambda(x-N)=8e^{-2\lambda x},\\
T(e^{-2\lambda t};0,\sinh^{-2}2\lambda t)&=\lim_{N\to\infty}T(2e^{-2\lambda N}\cosh 2\lambda(t-N);0,\sinh^{-2}2\lambda t)=0,\\
T(e^{-2\lambda t};e^{2\lambda t},0)&=\lim_{N\to\infty}T\left(2e^{-2\lambda N}\cosh 2\lambda(t-N);\frac14 e^{2\lambda N}\sinh^{-2}\lambda(t-N),0\right)=4\cosh 2\lambda y,\\
T\bigl(e^{-2\lambda t};e^{4\lambda t},0\bigr)&=\lim_{N\to\infty}T\left(2e^{-2\lambda N}\cosh 2\lambda(t-N);\frac14 e^{4\lambda N}\sinh^{-2}2\lambda(t-N),0\right)=0.
\end{aligned}$$

(Trig-$C_2^{(1)}$) $\leftarrow$ (Trig-$B_2^{(1)}$-bry): Replacing (x,y) by $(x+N,y+N)$,

$$\begin{aligned}
T(e^{-2\lambda t},0;e^{2\lambda t},0)&=\lim_{N\to\infty}T(e^{-2\lambda t},e^{-2\lambda(t+2N)};e^{-2\lambda N}e^{2\lambda(t+N)},0)\\
&=\lim_{N\to\infty}e^{-2\lambda N}\cosh 2\lambda(y+N)=2e^{2\lambda y},\\
T(e^{-2\lambda t},0;e^{4\lambda t},0)&=\lim_{N\to\infty}T(e^{-2\lambda t},e^{-2\lambda(t+2N)};e^{-4\lambda N}e^{4\lambda(t+N)},0)=0.
\end{aligned}$$

By the transformation $(x,y,\lambda)\mapsto(y,x,-\lambda)$, we have

$$T(e^{-2\lambda t},0;0,e^{-2\lambda t})=2e^{-2\lambda x},\qquad T(e^{-2\lambda t},0;0,e^{-4\lambda t})=0.$$

(Trig-$A_1^{(1)}$-bry) $\leftarrow$ (Trig-B_2): Replacing (x,y) by $(x+N,y+N)$,

$$\begin{aligned}
Q\bigl(\sinh^{-2}\lambda t,0;e^{2\lambda t}\bigr)&=\lim_{N\to\infty}Q\left(\sinh^{-2}\lambda t,\sinh^{-2}\lambda(t+2N);\frac14 e^{2\lambda N}\sinh^{-2}\lambda(t+N)\right)\\
&=\lim_{N\to\infty}\frac14 e^{2\lambda N}\sinh^{-2}\lambda(x+N)\sinh^{-2}\lambda(y+N)=0,\\
T\bigl(\sinh^{-2}\lambda t,0;e^{2\lambda t}\bigr)&=2(e^{2\lambda x}+e^{2\lambda y})\sinh^{-2}\lambda(x-y),\\
Q(\sinh^{-2}\lambda t,0;e^{4\lambda t})&\lim_{N\to\infty}Q(\sinh^{-2}\lambda t,\sinh^{-2}\lambda(t+2N);4e^{-4\lambda N}\sinh^2 2\lambda(t+N))\\
&=16\lim_{N\to\infty}e^{-4\lambda N}(\sinh^2\lambda(x+N)+\sinh^2\lambda(y+N)\\
&\qquad+2\sinh^2\lambda(x+N)\sinh^2\lambda(y+N))=2e^{2\lambda(x+y)},\\
T(\sinh^{-2}\lambda t,0;e^{4\lambda t})&=2\sinh^{-2}\lambda(x-y)(e^{4\lambda x}+e^{4\lambda y})-8e^{2\lambda(x+y)}\\
&=2\sinh^{-2}\lambda(x-y)(e^{4\lambda x}+e^{4\lambda y}-e^{2\lambda(x+y)}(e^{\lambda(x-y)}-e^{-\lambda(x-y)})^2)\\
&=4e^{2\lambda(x+y)}\sinh^{-2}\lambda(x-y).
\end{aligned}$$

(Rat-A_1-bry) $\leftarrow$ (Trig-A_1-bry): Taking the limit $\lambda \to 0$,

$$
\begin{aligned}
Q(t^{-2},0;t) &= \lim_{\lambda\to 0} Q\left(\lambda^2 \sinh^{-2}\lambda t, 0; \frac{1}{2\lambda}(e^{2\lambda t}-1)\right) = 0,\\
T(t^{-2},0;t) &= \frac{2(x+y)}{(x-y)^2},\\
Q(t^{-2},0;t^2) &= \lim_{\lambda\to 0} Q\left(\lambda^2 \sinh^{-2}\lambda t, 0; \frac{1}{4\lambda^2}(e^{2\lambda t}+e^{-2\lambda t}-2)\right) = 0,\\
T(t^{-2},0;t^2) &= 2\frac{x^2+y^2}{(x-y)^2},\\
Q(t^{-2},0;t^3) &= \lim_{\lambda\to 0} Q\left(\lambda^2 \sinh^{-2}\lambda t, 0; \frac{1}{8\lambda^3}(e^{4\lambda t}-3e^{2\lambda t}-e^{-2\lambda t}+3)\right)\\
&= \lim_{\lambda\to 0}\frac{2}{8\lambda}(e^{2\lambda(x+y)}-1) = \frac{1}{2}(x+y),\\
T(t^{-2},0;t^3) &= 2\frac{x^3+y^3}{(x-y)^2} - 2(x+y) = 2\frac{xy(x+y)}{(x-y)^2},\\
Q(t^{-2},0;t^4) &= \lim_{\lambda\to 0} Q\left(\lambda^2 \sinh^{-2}\lambda t, 0; \frac{1}{16\lambda^4}(e^{4\lambda t}+e^{-4\lambda t}-4e^{2\lambda t}-4e^{-2\lambda t}+6)\right)\\
&= \lim_{\lambda\to 0}\frac{2}{16\lambda^2}(e^{2\lambda(x+y)}+e^{-2\lambda(x+y)}-2) = \frac{1}{2}(x+y)^2,\\
T(t^{-2},0;t^4) &= 2\frac{x^4+y^4}{(x-y)^2} - 2(x+y)^2 = \frac{4x^2y^2}{(x-y)^2}.
\end{aligned}
$$

Thus we have completed the proof of Theorem 2 by using Lemma 3 and Lemma 4. ■

Remark 5. Theorem 2 can be checked by direct calculations. For example, Remark 2 and the equations

$$
\begin{aligned}
&2(\varepsilon e^{-2\lambda(x+y)} - e^{-2\lambda(x-y)})(e^{2\lambda x})' + 4e^{2\lambda x}\frac{\partial}{\partial x}(\varepsilon e^{-2\lambda(x+y)} - e^{-2\lambda(x-y)})\\
&\qquad = 4\lambda(\varepsilon e^{-2\lambda y} - e^{2\lambda y}) - 8\lambda(\varepsilon e^{-2\lambda y} - e^{2\lambda y}) = \frac{\partial}{\partial y}(2(\varepsilon e^{-2\lambda y} + e^{2\lambda y})),\\
&2(\varepsilon e^{-2\lambda(x+y)} - e^{-2\lambda(x-y)})(e^{4\lambda x})' + 4e^{4\lambda x}\frac{\partial}{\partial x}(\varepsilon e^{-2\lambda(x+y)} - e^{-2\lambda(x-y)}) = 0
\end{aligned}
$$

with $\varepsilon = 1$ give $T(e^{-2\lambda t}; e^{2\lambda t}, 0)$ and $T(e^{-2\lambda t}; e^{4\lambda t}, 0)$ for (Trig-B_2-bry). Moreover the functions $T(e^{-2\lambda t}, 0; e^{2\lambda t}, 0)$ and $T(e^{-2\lambda t}, 0; e^{4\lambda t}, 0)$ for (Toda-$C_2^{(1)}$) also follow from these equations with $\varepsilon = 0$.

4.2 Special case

In this subsection we study the integrable systems (4.1) with (4.2) which are of the form

$$
\begin{aligned}
&R(x,y) = u^-(x-y) + u^+(x+y) + v(x) + w(y),\\
&u^-(t) = \sum_{j=0}^{1} A_j u_j^-(t), \qquad u^+(t) = \sum_{j=0}^{1} A_j u_j^+(t),\\
&v(t) = \sum_{j=0}^{1} C_j v_j(t), \qquad w(t) = \sum_{j=0}^{1} C_j w_j(t)
\end{aligned}
$$

with $A_0, A_1, C_0, C_1 \in \mathbb{C}$. The most general system (Ellip-B_2-S) in the following theorem is presented by [24] as an elliptic generalization of (Trig-B_2-S) found by [32].

Theorem 3 (B_2 : Special case, [23, 24, 30] etc.). *The operators P and P_2 defined by the following pairs of $R(x,y)$ and $T(x,y)$ satisfy* (4.1) *and* (4.2)*.*

(Ellip-B_2-S): $(\langle\wp(t;2\omega_1,2\omega_2),\wp(t;\omega_1,2\omega_2)\rangle;\langle\wp(t;\omega_1,2\omega_2),\wp(t;\omega_1,\omega_2)\rangle)$

$$\begin{aligned}
v(x)&=C_0\wp(x;\omega_1,2\omega_2)+C_1\wp(x;\omega_1,\omega_2),\qquad w(y)=C_0\wp(y;\omega_1,2\omega_2)+C_1\wp(y;\omega_1,\omega_2),\\
u^-(x-y)&=A_0\wp(x-y;2\omega_1,2\omega_2)+A_1\wp(x-y;\omega_1,2\omega_2),\\
u^+(x+y)&=A_0\wp(x+y;2\omega_1,2\omega_2)+A_1\wp(x+y;\omega_1,2\omega_2),\\
R(x,y)&=A_0\wp(x-y;2\omega_1,2\omega_2)+A_0\wp(x+y;2\omega_1,2\omega_2)\\
&\quad+A_1\wp(x-y;\omega_1,2\omega_2)+A_1\wp(x+y;\omega_1,2\omega_2)\\
&\quad+C_0\wp(x;\omega_1,2\omega_2)+C_0\wp(y;\omega_1,2\omega_2)+C_1\wp(x;\omega_1,\omega_2)+C_1\wp(y;\omega_1,\omega_2),\\
T(x,y)&=2(A_0\wp(x-y;2\omega_1,2\omega_2)+A_0\wp(x+y;2\omega_1,2\omega_2)\\
&\quad+A_1\wp(x-y;\omega_1,2\omega_2)+A_1\wp(x+y;\omega_1,2\omega_2))\\
&\quad\times(C_0\wp(x;\omega_1,2\omega_2)+C_0\wp(y;\omega_1,2\omega_2)+C_1\wp(x;\omega_1,\omega_2)+C_1\wp(y;\omega_1,\omega_2))\\
&\quad-4A_0C_0\sum_{j=0}^{1}\wp(x+\omega_j;2\omega_1,2\omega_2)\cdot\wp(y+\omega_j;2\omega_1,2\omega_2)\\
&\quad-4A_0C_1\sum_{j=0}^{3}\wp(x+\omega_j;2\omega_1,2\omega_2)\wp(y+\omega_j;2\omega_1,2\omega_2)\\
&\quad-4A_1C_0\wp(x;\omega_1,2\omega_2)\wp(y;\omega_1,2\omega_2)\\
&\quad-4A_1C_1\sum_{j=0}^{1}\wp(x+\omega_{2j};\omega_1,2\omega_2)\wp(y+\omega_{2j};\omega_1,2\omega_2).
\end{aligned}$$

(Trig-B_2-S): $(\langle\sinh^{-2}\lambda t,\sinh^{-2}2\lambda t\rangle;\langle\sinh^{-2}2\lambda t,\sinh^{2}2\lambda t\rangle)$

$$\begin{aligned}
v(x)&=C_0\sinh^{-2}2\lambda x+C_1\sinh^2 2\lambda x,\qquad w(y)=C_0\sinh^{-2}2\lambda y+C_1\sinh^2 2\lambda y,\\
u^-(x-y)&=A_0\sinh^{-2}\lambda(x-y)+A_1\sinh^{-2}2\lambda(x-y),\\
u^+(x+y)&=A_0\sinh^{-2}\lambda(x+y)+A_1\sinh^{-2}2\lambda(x+y),\\
R(x,y)&=A_0\sinh^{-2}\lambda(x+y)+A_0\sinh^{-2}\lambda(x-y)+A_1\sinh^{-2}2\lambda(x+y)\\
&\quad+A_1\sinh^{-2}2\lambda(x-y)+C_0\sinh^{-2}2\lambda x+C_0\sinh^{-2}2\lambda y\\
&\quad+C_1\sinh^2 2\lambda x+C_1\sinh^2 2\lambda y,\\
T(x,y)&=2(A_0\sinh^{-2}\lambda(x+y)+A_0\sinh^{-2}\lambda(x-y)+A_1\sinh^{-2}2\lambda(x+y)\\
&\quad+A_1\sinh^{-2}2\lambda(x-y))(C_0\sinh^{-2}2\lambda x+C_0\sinh^{-2}2\lambda y+C_1\sinh^2 2\lambda x\\
&\quad+C_1\sinh^2 2\lambda y)-A_0C_0\big(\sinh^{-2}\lambda x\cdot\sinh^{-2}\lambda y+\cosh^{-2}\lambda x\cdot\cosh^{-2}\lambda y\big)\\
&\quad-4A_0C_1(\sinh^2\lambda x+\sinh^2\lambda y+2\sinh^2\lambda x\cdot\sinh^2\lambda y)\\
&\quad-4A_1C_0\sinh^{-2}2\lambda x\cdot\sinh^{-2}2\lambda y.
\end{aligned}$$

(Rat-B_2-S): $(\langle t^{-2},t^2\rangle;\langle t^{-2},t^2\rangle)$

$$\begin{aligned}
v(x)&=C_0x^{-2}+C_1x^2,\qquad w(y)=C_0y^{-2}+C_1y^2,\\
u^-(x-y)&=\frac{A_0}{(x-y)^2}+A_1(x-y)^2,\qquad u^+(x+y)=\frac{A_0}{(x+y)^2}+A_1(x+y)^2,\\
R(x,y)&=\frac{A_0}{(x-y)^2}+\frac{A_0}{(x+y)^2}+A_1(x-y)^2+A_1(x+y)^2+\frac{C_0}{x^2}+\frac{C_0}{y^2}+C_1x^2+C_1y^2,\\
T(x,y)&=\frac{16A_0C_0+16A_0C_1x^2y^2}{(x^2-y^2)^2}+16A_1C_1x^2y^2.
\end{aligned}$$

(Toda-$D_2^{(1)}$-S-bry): $(\langle\cosh 2\lambda, \cosh 4\lambda t\rangle; \langle\sinh^{-2} 2\lambda t\rangle, \langle\sinh^{-2} 2\lambda t\rangle)$

$$
\begin{aligned}
&v(x) = C_0 \sinh^{-2} 2\lambda x, \qquad w(y) = C_1 \sinh^{-2} 2\lambda y,\\
&u^-(x-y) = A_0 \cosh 2\lambda(x-y) + A_1 \cosh 4\lambda(x-y),\\
&u^+(x+y) = A_0 \cosh 2\lambda(x+y) + A_1 \cosh 4\lambda(x+y),\\
&R(x,y) = A_0 \cosh 2\lambda(x-y) + A_0 \cosh 2\lambda(x+y) + A_1 \cosh 4\lambda(x-y)\\
&\qquad + A_1 \cosh 4\lambda(x+y) + C_0 \sinh^{-2} 2\lambda x + C_1 \sinh^{-2} 2\lambda y,\\
&T(x,y) = 8A_1(C_0 \cosh 4\lambda y + C_1 \cosh 4\lambda x).
\end{aligned}
$$

(Toda-$B_2^{(1)}$-S-bry): $(\langle e^{-2\lambda t}, e^{-4\lambda t}\rangle; \langle e^{4\lambda t}\rangle, \langle\sinh^{-2} 2\lambda t\rangle)$

$$
\begin{aligned}
&v(x) = C_0 e^{4\lambda x}, \qquad w(y) = C_1 \sinh^{-2} 2\lambda y,\\
&u^-(x-y) = A_0 e^{-2\lambda(x-y)} + A_1 e^{-4\lambda(x-y)}, \qquad u^+(x+y) = A_0 e^{-2\lambda(x+y)} + A_1 e^{-4\lambda(x+y)},\\
&R(x,y) = A_0 e^{-2\lambda(x-y)} + A_0 e^{-2\lambda(x+y)} + A_1 e^{-4\lambda(x-y)} + A_1 e^{-4\lambda(x+y)} + C_0 e^{4\lambda x} + C_1 \sinh^{-2} 2\lambda y,\\
&T(x,y) = 4A_1\big(C_0 \cosh 4\lambda y + 2C_1 e^{-4\lambda x}\big).
\end{aligned}
$$

(Toda-$C_2^{(1)}$-S): $(\langle e^{-2\lambda t}, e^{-4\lambda t}\rangle, 0; \langle e^{4\lambda t}\rangle, \langle e^{-4\lambda t}\rangle)$

$$
\begin{aligned}
&v(x) = C_0 e^{4\lambda x}, \qquad w(y) = C_1 e^{-4\lambda y},\\
&u^-(x-y) = A_0 e^{-2\lambda(x-y)} + A_1 e^{-4\lambda(x-y)}, \qquad u^+(x+y) = 0,\\
&R(x,y) = A_0 e^{-2\lambda(x-y)} + A_1 e^{-4\lambda(x-y)} + C_0 e^{4\lambda x} + C_1 e^{-4\lambda y},\\
&T(x,y) = 2A_1\big(C_0 e^{4\lambda y} + C_1 e^{-4\lambda x}\big).
\end{aligned}
$$

(Trig-A_1-S-bry): $(\langle\sinh^{-2} \lambda t, \sinh^{-2} 2\lambda t\rangle, 0; \langle e^{-4\lambda t}, e^{4\lambda t}\rangle)$

$$
\begin{aligned}
&v(x) = C_0 e^{-4\lambda x} + C_1 e^{4\lambda x}, \qquad w(y) = C_0 e^{-4\lambda y} + C_1 e^{4\lambda y},\\
&u^-(x-y) = A_0 \sinh^{-2} \lambda(x-y) + A_1 \sinh^{-2} 2\lambda(x-y), \qquad u^+(x+y) = 0,\\
&R(x,y) = A_0 \sinh^{-2} \lambda(x-y) + A_1 \sinh^{-2} 2\lambda(x-y) + C_0 e^{-4\lambda x} + C_0 e^{-4\lambda y} + C_1 e^{4\lambda x} + C_1 e^{4\lambda y},\\
&T(x,y) = 2A_1 \sinh^{-2} 2\lambda(x-y)(C_0 e^{-4\lambda x} + C_0 e^{-4\lambda y} + C_1 e^{4\lambda x} + C_1 e^{4\lambda y})\\
&\qquad + 4A_0 \sinh^{-2} \lambda(x-y)(C_0 e^{-2\lambda(x+y)} + C_1 e^{2\lambda(x+y)}).
\end{aligned}
$$

(Rat^d-D_2-S-bry): $(\langle t^2, t^4\rangle; \langle t^{-2}\rangle, \langle t^{-2}\rangle)$

$$
\begin{aligned}
&v(x) = C_0 x^{-2}, \qquad w(y) = C_1 y^{-2},\\
&u^-(x-y) = A_0(x-y)^2 + A_1(x-y)^4, \qquad u^+(x+y) = A_0(x+y)^2 + A_1(x+y)^4,\\
&R(x,y) = 2A_0(x^2+y^2) + 2A_1(x^4 + 6x^2y^2 + y^4) + \frac{C_0}{x^2} + \frac{C_1}{y^2},\\
&T(x,y) = 32A_1(C_0 y^2 + C_1 x^2).
\end{aligned}
$$

Proof. (Ellip-B_2-S): We have the following from (4.6), Lemma 4 and (2.8).

$$
\begin{aligned}
&Q(\wp(t;\omega_1,2\omega_2); \wp(t;\omega_1,2\omega_2)) = \wp(x;\omega_1,2\omega_2)\wp(y;\omega_1,2\omega_2),\\
&Q(\wp(t;2\omega_1,2\omega_2); \wp(t;\omega_1,2\omega_2)) = Q(\wp(t;2\omega_1,2\omega_2); \wp(t;2\omega_1,2\omega_2)\\
&\qquad + \wp(t2A_1 \sinh^{-2} 2\lambda(x-y)(C_0 + \omega_1; 2\omega_1, 2\omega_2))\\
&\qquad = \wp(x;2\omega_1,2\omega_2)\wp(y;2\omega_1,2\omega_2) + \wp(x+\omega_1;2\omega_1,2\omega_2)\wp(y+\omega_1;2\omega_1,2\omega_2),\\
&Q(\wp(t;\omega_1,2\omega_2); \wp(t;\omega_1,\omega_2)) = \wp(x;\omega_1,2\omega_2)\wp(y;\omega_1,2\omega_2)
\end{aligned}
$$

$$+\wp(x+\omega_2;\omega_1,2\omega_2)\wp(y+\omega_2;\omega_1,2\omega_2),$$

$$Q(\wp(t;2\omega_1,2\omega_2);\wp(t;\omega_1,\omega_2))=\sum_{j=0}^{3}\wp(x+\omega_j;2\omega_1,2\omega_2)\wp(y+\omega_j;2\omega_1,2\omega_2).$$

(Rat-B_2-S) is given in [32, (7.13)] but it is easy to check (4.3) or prove the result as a limit of (Trig-B_2-S). (Rat^d-D_2-S-bry) follows from $T(t^2;t^{-2},0)=0$ and $T(t^4;t^{-2},0)=32y^2$ (cf. Remark 2 i)). Moreover (Trig-B_2-S), (Toda-$D_2^{(1)}$-S-bry), (Toda-$C_2^{(1)}$-S) and (Trig-A_1-S-bry) are obtained from the corresponding normal cases together with Lemma 4. For example, Q for (Trig-B_2-S) is given by (4.7), (4.9) and

$$\begin{aligned}Q(\sinh^{-2}\lambda t;\sinh^{-2}2\lambda t)&=Q\left(\sinh^{-2}\lambda t;\frac{1}{4}\sinh^{-2}\lambda t-\frac{1}{4}\cosh^{-2}\lambda t\right)\\&=\frac{1}{4}(\sinh^{-2}\lambda t\sinh^{-2}\lambda y+\cosh^{-2}\lambda x\cosh^{-2}\lambda y),\\Q(\sinh^{-2}2\lambda t;\sinh^{-2}2\lambda t)&=\sinh^{-2}2\lambda x\cdot\sinh^{-2}2\lambda y,\\Q(\sinh^{-2}2\lambda t;\sinh^{2}2\lambda t)&=0.\end{aligned}$$

Thus we get Theorem 3 from Theorem 2. ■

4.3 Duality

Definition 1 (Duality in B_2, [23]). Under the coordinate transformation

$$(x,y)\mapsto(X,Y)=\left(\frac{x+y}{\sqrt{2}},\frac{x-y}{\sqrt{2}}\right)$$

the pair (P,P^2-P_2) also satisfies (4.1), which we call the *duality* of the commuting differential operators of type B_2.

Denoting $\partial_x=\partial/\partial x$, $\partial_y=\partial/\partial y$ and put

$$L=P^2-P_2-\left(\frac{1}{2}\partial_x^2-\frac{1}{2}\partial_y^2+w-v\right)^2+u^-(\partial_x+\partial_y)^2+u^+(\partial_x-\partial_y)^2.$$

Then the order of L is at most 2 and the second order term of L is

$$\begin{aligned}&-(u^++u^-+v+w)(\partial_x^2+\partial_y^2)-2(u^--u^+)\partial_x\partial_y+2w\partial_x^2+2v\partial_y^2-(w-v)(\partial_x^2-\partial_y^2)\\&\quad+u^-(\partial_x+\partial_y)^2+u^+(\partial_x-\partial_y)^2=0.\end{aligned}$$

Since L is self-adjoint, L is of order at most 0 and the 0-th order term of L is

$$\begin{aligned}&-\frac{1}{2}(\partial_x^2+\partial_y^2)(u^++u^-+v+w)+(u^++u^-+v+w)^2-4vw-T-\partial_x\partial_y(u^--u^+)\\&\quad-\frac{1}{2}(\partial_x^2-\partial_y^2)(w-v)=(u^++u^-+v+w)^2-4vw-T\end{aligned}$$

and therefore we have the following proposition.

Proposition 1 ([23, 24]). i) *By the duality in Definition* 1 *the pair* $\big(R(x,y),T(x,y)\big)$ *changes into* $\big(\tilde R(x,y),\tilde T(x,y)\big)$ *with*

$$\tilde R(x,y)=v\left(\frac{x+y}{\sqrt{2}}\right)+w\left(\frac{x-y}{\sqrt{2}}\right)+u^+(\sqrt{2}x)+u^-(\sqrt{2}y),$$

$$\tilde{T}(x,y) = \tilde{R}(x,y)^2 - 4v\left(\frac{x+y}{\sqrt{2}}\right)w\left(\frac{x-y}{\sqrt{2}}\right) - T\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right).$$

ii) *Combining the duality with the scaling map $R(x,y) \mapsto c^{-2}R(cx,cy)$, the following pair $\big(R^d(x,y), T^d(x,y)\big)$ defines commuting differential operators if so does $\big(R(x,y), T(x,y)\big)$. This $R^d(x,y)$ is also called the dual of $R(x,y)$,*

$$R^d(x,y) = v(x+y) + w(x-y) + u^+(2x) + u^-(2y),$$
$$T^d(x,y) = R^d(x,y)^2 - 4v(x+y)w(x-y) - T(x+y, x-y).$$

Remark 6. i) We list up the systems of type B_2 given in Sections 4.1 and 4.2:

$(u^-(t), u^+(t); v(t), w(t))$	*Symbol*
$(\langle\wp(t)\rangle; \langle\wp(t), \wp(t+\omega_1), \wp(t+\omega_2), \wp(t+\omega_3)\rangle)$,	(Ellip-B_2)
$(\langle\wp(t;2\omega_1,2\omega_2), \wp(t;\omega_1,2\omega_2)\rangle; \langle\wp(t;\omega_1,2\omega_2), \wp(t;\omega_1,\omega_2)\rangle)$,	(Ellip-B_2-S)
$(\langle\sinh^{-2}\lambda t\rangle; \langle\sinh^{-2}\lambda t, \sinh^{-2}2\lambda t, \sinh^2\lambda t, \sinh^2 2\lambda t\rangle)$,	(Trig-B_2)
$(\langle\sinh^{-2}\lambda t, \sinh^{-2}2\lambda t\rangle; \langle\sinh^{-2}2\lambda t, \sinh^2 2\lambda t\rangle)$,	(Trig-B_2-S)
$(\langle t^{-2}\rangle; \langle t^{-2}, t^2,\ t^4,\ t^6\rangle)$,	(Rat-B_2)
$(\langle t^{-2}, t^2\rangle; \langle t^{-2}, t^2\rangle)$,	(Rat-B_2-S)
$(\langle\cosh 2\lambda t\rangle; \langle\sinh^{-2}\lambda t, \sinh^{-2}2\lambda t\rangle, \langle\sinh^{-2}\lambda t, \sinh^{-2}2\lambda t\rangle)$,	(Toda-$D_2^{(1)}$-bry)
$(\langle\cosh\lambda t, \cosh 2\lambda t\rangle; \langle\sinh^{-2}\lambda t\rangle, \langle\sinh^{-2}\lambda t\rangle)$,	(Toda-$D_2^{(1)}$-S-bry)
$(\langle e^{-2\lambda t}\rangle; \langle e^{2\lambda t},\ e^{4\lambda t}\rangle, \langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda t\rangle)$,	(Toda-$B_2^{(1)}$-bry)
$(\langle e^{-\lambda t}, e^{-2\lambda t}\rangle; \langle e^{2\lambda t}\rangle, \langle\sinh^{-2}\lambda t\rangle)$,	(Toda-$B_2^{(1)}$-S-bry)
$(\langle e^{-2\lambda t}\rangle, 0; \langle e^{2\lambda t}, e^{4\lambda t}\rangle, \langle e^{-2\lambda t},\ e^{-4\lambda t}\rangle)$,	(Toda-$C_2^{(1)}$)
$(\langle e^{-2\lambda t}, e^{-4\lambda t}\rangle, 0; \langle e^{4\lambda t}\rangle, \langle e^{-4\lambda t}\rangle)$,	(Toda-$C_2^{(1)}$-S)
$(\langle\sinh^{-2}\lambda t\rangle, 0; \langle e^{-2\lambda t}, e^{-4\lambda t}, e^{2\lambda t}, e^{4\lambda t}\rangle)$,	(Trig-A_1-bry)
$(\langle\sinh^{-2}\lambda t, \sinh^{-2}2\lambda t\rangle, 0; \langle e^{-4\lambda t},\ e^{4\lambda t}\rangle)$,	(Trig-A_1-S-bry)
$(\langle t^{-2}\rangle, 0; \langle t, t^2, t^3, t^4\rangle)$,	(Rat-A_1-bry)
$(\langle t^{-2}\rangle, \langle t^{-2}\rangle; \langle t^2, t^4\rangle)$.	(Rat-D_2-S-bry)

The first 6 cases above are classified by [24] as invariant systems. The systems such that at least two of $u^{\pm}$, v and w are not entire are classified by [23]. (Trig-$*$) and (Toda-$*$) in the above are classified by [30] as certain systems with periodic potentials.

We do not put $(\langle t^{-2}, t^2\rangle, 0; \langle t, t^2\rangle)$ in the list which corresponds to (Rat-A_1-S-bry) because its dual defines a direct sum of trivial operators (cf. Section 9).

ii) Since $1 - \sinh^{-2}t + 4\sinh^{-2}2t = \coth^2 t = t^2 - (2/3)t^4 + o(t^4)$ and $\sinh^2 t = t^2 + (2/3)t^4 + o(t^4)$, we have $\sinh^2 2\lambda x + \sinh^2 2\lambda y - 2\coth^2\lambda(x-y) - 2\coth^2\lambda(x+y) = 8\lambda^4(x^2+y^2)^2 + o(\lambda^4)$. Hence the potential function

$$\frac{A_0}{(x-y)^2} + \frac{A_0}{(x+y)^2} + \frac{C_0}{x^2} + \frac{C_0}{y^2} + C_1(x^2+y^2) + A_1(x^2+y^2)^2$$

is an analytic continuation of that of (Trig-B_2-S) but this is not a completely integrable potential function of type B_2

iii) The dual is indicated by superfix ^{d}. For example, the dual of (Ellip-B_2) is denoted by (Ellip^d-B_2) whose potential function is

$$R(x,y) = A\wp(2x) + A\wp(2y) + \sum_{j=0}^{3} C_j\big(\wp(x-y+\omega_j) + \wp(x+y+\omega_j)\big)$$

and the dual of (Toda-$C_2^{(1)}$) is

$$(\langle e^{-2\lambda t}, e^{-4\lambda t}\rangle, \langle e^{2\lambda t}, e^{4\lambda t}\rangle; 0, \langle e^{-4\lambda t}\rangle) \tag{Toda^d-$C_2^{(1)}$}$$

since the dual of $\bigl(u^-(t), u^+(t); v(t), w(t)\bigr)$ is $\bigl(w(t), v(t); u^+(2t), u^-(2t)\bigr)$. Similarly

$$(\langle \wp(t;\omega_1, 2\omega_2), \wp(t;\omega_1,\omega_2)\rangle; \langle \wp(2t; 2\omega_1, 2\omega_2), \wp(2t;\omega_1, 2\omega_2)\rangle). \tag{Ellip^d-B_2-S}$$

Since $4\wp(2t; 2\omega_1, 2\omega_2) = \wp(t;\omega_1,\omega_2)$ etc., (Ellip^d-B_2-S) coincides with (Ellip-B_2-S) by replacing (ω_1,ω_2) by $(2\omega_2,\omega_1)$.

Then we have the following diagrams and their duals, where the arrows with double lines represent specializations of parameters. For example, "Trig-$B_2 \overset{5:3}{\Rightarrow}$ Trig-BC_2-reg" means that 2 parameters (coupling constants C_2 and C_3) out of 5 in the potential function (Trig-B_2) are specialized to get the potential function (Trig-BC_2-reg) with 3 parameters. For the normal cases see Definition 5 and the diagrams in the last part of the next section (type B_n).

Hierarchy of Normal Integrable Potentials of type B_2

$$\begin{array}{ccccccc}
 & & \text{Trig-}BC_2\text{-reg} & \to & \text{Toda-}D_2\text{-bry} & & \\
 & & {\scriptstyle C_2=C_3=0}\Uparrow{\scriptstyle 5:3} & & {\scriptstyle C_0=C_1=0}\Uparrow{\scriptstyle 5:3} & & \\
 & & \text{Trig-}B_2 & \to & \text{Toda-}B_2^{(1)}\text{-bry} & \overset{5:3}{\underset{C_0=C_1=0}{\Rightarrow}} & \text{Toda-}B_2^{(1)} \\
 & \nearrow & & \nearrow & \downarrow & & \\
\text{Ellip-}B_2 & \to & \text{Toda-}D_2^{(1)}\text{-bry} & & \text{Toda-}C_2^{(1)} & \overset{5:3}{\underset{C_0=C_1=0}{\Rightarrow}} & \text{Toda-}BC_2 \\
 & \searrow & & & & & \\
 & & \text{Trig-}B_2 & \to & \text{Trig-}A_1\text{-bry} & \overset{5:3}{\underset{C_2=C_3=0}{\Rightarrow}} & \text{Trig-}A_1\text{-bry-reg} \\
 & & & \searrow & & \searrow & \\
 & & & & \text{Rat-}B_2 & \to & \text{Rat-}A_1\text{-bry}
\end{array}$$

Hierarchy of Special Integrable Potentials of type B_2

$$\begin{array}{ccccccc}
 & & \text{Trig}^{(d)}\text{-}B_2\text{-S-reg} & \to & \text{Toda}^{(d)}\text{-}D_2\text{-S-bry} & & \text{Rat}^{d}\text{-}D_2\text{-S-bry} \\
 & & {\scriptstyle C_1=0}\Uparrow{\scriptstyle 4:3} & & {\scriptstyle C_0=0}\Uparrow{\scriptstyle 4:3} & \nearrow & \\
 & & \text{Trig}^{(d)}\text{-}B_2\text{-S} & \to & \text{Toda}^{(d)}\text{-}B_2^{(1)}\text{-S-bry} & \overset{4:3}{\underset{C_1=0}{\Rightarrow}} & \text{Toda}^{(d)}\text{-}B_2^{(1)}\text{-S} \\
 & \nearrow & & \nearrow & \downarrow & & \\
\text{Ellip-}B_2\text{-S} & \to & \text{Toda}^{(d)}\text{-}D_2^{(1)}\text{-S-bry} & & \text{Toda}^{(d)}\text{-}C_2^{(1)}\text{-S} & \overset{4:3}{\underset{C_0=0}{\Rightarrow}} & \text{Toda}^{(d)}\text{-}B_2\text{-S} \\
 & \searrow & & & & & \\
 & & \text{Trig}^{(d)}\text{-}B_2\text{-S} & \to & \text{Trig}^{(d)}\text{-}A_1\text{-S-bry} & \overset{4:3}{\underset{C_1=0}{\Rightarrow}} & \text{Trig}^{(d)}\text{-}A_1\text{-S-bry-reg} \\
 & & & \searrow & & & \\
 & & & & \text{Rat-}B_2\text{-S} & &
\end{array}$$

Definition 2. We define some potential functions as specializations.

(Trig-B_2-S-reg) *Trigonometric special potential of type B_2 with regular boundary conditions* is (Trig-B_2-S) with $C_1 = 0$.

(Toda-D_2-S-bry) *Toda special potential of type D_2 with boundary conditions* is (Toda-$B_2^{(1)}$-S-bry) with $C_0 = 0$.

(Toda-$B_2^{(1)}$-S) *Toda special potential of type $B_2^{(1)}$* is (Toda-$B_2^{(1)}$-S-bry) with $C_1 = 0$.

(Toda-B_2-S) *Toda special potential of type* B_2 is (Toda-$C_2^{(1)}$-S-bry) with $C_0 = 0$.

(Trig-A_1-S-bry-reg) *Trigonometric special potential of type* A_1 *with regular boundary conditions* is (Trig-A_1-S-bry) with $C_1 = 0$.

Remark 7. We have some equivalences as follows:

(Ellip-B_2-S) = (Ellip^d-B_2-S),

(Rat-B_2-S) = (Rat^d-B_2-S),

(Trig-BC_2-reg) = (Trig^d-B_2-S-reg),

(Trig-A_1-bry-reg) = (Toda^d-D_2-S-bry),

(Toda-D_2-bry) = (Trig^d-A_1-S-bry-reg),

(Toda-$B_2^{(1)}$) = (Toda^d-$B_2^{(1)}$-S),

(Toda-BC_2) = (Toda^d-B_2-S).

5 Type B_n ($n \geq 3$)

In this section we construct integrals of the completely integrable systems of type B_n appearing in the following diagram. The diagram is also given in [8, Figure III.1], where (Toda-$B_n^{(1)}$-bry) is missing. The most general system (Ellip-B_n) is called Inozemtzev model (cf. [12]).

Hierarchy of Integrable Potentials with 5 parameters ($n \geq 2$)

$$\begin{array}{ccccccc}
 & & \text{Toda-}D_n^{(1)}\text{-bry} & \to & \text{Toda-}B_n^{(1)}\text{-bry} & \to & \text{Toda-}C_n^{(1)} \\
 & \nearrow & & \nearrow & & & \\
\text{Ellip-}B_n & \to & \text{Trig-}B_n & \to & \text{Rat-}B_n & & \\
 & & & \searrow & & \searrow & \\
 & & & & \text{Trig-}A_{n-1}\text{-bry} & \to & \text{Rat-}A_{n-1}\text{-bry}
\end{array}$$

5.1 Integrable potentials

Definition 3. The potential functions $R(x)$ of (1.1) are as follows:
Here A, C_0, C_1, C_2 and C_3 are any complex numbers.

(Ellip-B_n) *Elliptic potential of type* B_n:

$$\sum_{1\leq i<j\leq n} A(\wp(x_i - x_j; 2\omega_1, 2\omega_2) + \wp(x_i + x_j; 2\omega_1, 2\omega_2)) + \sum_{k=1}^{n}\sum_{j=0}^{3} C_j \wp(x_k + \omega_j; 2\omega_1, 2\omega_2),$$

(Trig-B_n) *Trigonometric potential of type* B_n:

$$\sum_{1\leq i<j\leq n} A(\sinh^{-2}\lambda(x_i - x_j) + \sinh^{-2}\lambda(x_i + x_j))$$
$$+ \sum_{k=1}^{n}\left(C_0 \sinh^{-2}\lambda x_k + C_1\cosh^{-2}\lambda x_k + C_2 \sinh^2\lambda x_k + \frac{C_3}{4}\sinh^2 2\lambda x_k\right),$$

(Rat-B_n) *Rational potential of type* B_n:

$$\sum_{1\leq i<j\leq n}\left(\frac{A}{(x_i - x_j)^2} + \frac{A}{(x_i + x_j)^2}\right) + \sum_{k=1}^{n}(C_0 x_k^{-2} + C_1 x_k^2 + C_2 x_k^4 + C_3 x_k^6),$$

(Trig-A_{n-1}-bry) *Trigonometric potential of type A_{n-1} with boundary conditions*:

$$\sum_{1\le i<j\le n} A\sinh^{-2}\lambda(x_i-x_j)+\sum_{k=1}^{n}(C_0e^{-2\lambda x_k}+C_1e^{-4\lambda x_k}+C_2e^{2\lambda x_k}+C_3e^{4\lambda x_k}),$$

(Toda-$B_n^{(1)}$-bry) *Toda potential of type $B_n^{(1)}$ with boundary conditions*:

$$\sum_{i=1}^{n-1}Ae^{-2\lambda(x_i-x_{i+1})}+Ae^{-2\lambda(x_{n-1}+x_n)}+C_0e^{2\lambda x_1}+C_1e^{4\lambda x_1}+C_2\sinh^{-2}\lambda x_n+C_3\sinh^{-2}2\lambda x_n,$$

(Toda-$C_n^{(1)}$) *Toda potential of type $C_n^{(1)}$*:

$$\sum_{i=1}^{n-1}Ae^{-2\lambda(x_i-x_{i+1})}+C_0e^{2\lambda x_1}+C_1e^{4\lambda x_1}+C_2e^{-2\lambda x_n}+C_3e^{-4\lambda x_n},$$

(Toda-$D_n^{(1)}$-bry) *Toda potential of type $D_n^{(1)}$ with boundary conditions*:

$$\sum_{i=1}^{n-1}Ae^{-2\lambda(x_i-x_{i+1})}+Ae^{-2\lambda(x_{n-1}+x_n)}+Ae^{2\lambda(x_1+x_2)}$$
$$+C_0\sinh^{-2}\lambda x_1+C_1\sinh^{-2}2\lambda x_1+C_2\sinh^{-2}\lambda x_n+C_3\sinh^{-2}2\lambda x_n,$$

(Rat-A_{n-1}-bry) *Rational potential of type A_{n-1} with boundary conditions*:

$$\sum_{1\le i<j\le n}\frac{A}{(x_i-x_j)^2}+\sum_{k=1}^{n}(C_0x_k+C_1x_k^2+C_2x_k^3+C_3x_k^4).$$

Remark 8. In these cases the Schrödinger operator P is of the form

$$P=-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+R(x),$$
$$R(x)=\sum_{1\le i<j\le n}(u_{e_i-e_j}(x)+u_{e_i+e_j}(x))+\sum_{j=0}^{3}C_jv^j(x),\qquad v^j(x)=\sum_{k=1}^{n}v^j_{e_k}(x).$$

Here

$$\partial_a u_\alpha(x)=\partial_b v^j_\beta(x)=0\qquad\text{if}\qquad a,b\in\mathbb{R}^n\qquad\text{satisfy}\qquad\langle a,\alpha\rangle=\langle b,\beta\rangle=0.$$

The complete integrability of the invariant systems (Ellip-B_n), (Trig-B_n) and (Rat-B_n) is established by [29]. We review their integrals and then we prove that the other five systems are also completely integrable by constructing enough integrals, which is announced by [30]. The complete integrability of (Trig-A_{n-1}-bry), (Toda-$C_n^{(1)}$), (Toda-$D_n^{(1)}$-bry) and (Rat-A_{n-1}-bry) is presented as an unknown problem by [8] and then that of (Toda-$B_n^{(1)}$-bry), (Toda-$C_n^{(1)}$) and (Toda-$D_n^{(1)}$-bry) are established by [18, 19, 21] using R-matrix method. The compete integrability of (Trig-A_{n-1}-bry) and (Rat-A_{n-1}-bry) seems to have not been proved.

Definition 4 ([25, 29]). Let $u_\alpha(x)$ and $T_I^o(v^j)$ are functions given for $\alpha\in\Sigma(D_n)$ and subsets $I=\{i_1,\dots,i_k\}\subset\{1,\dots,n\}$ such that

$$u_\alpha(x)=u_{-\alpha}(x)\qquad\text{and}\qquad\partial_y u_\alpha=0\qquad\text{for}\qquad y\in\mathbb{R}^n\qquad\text{with}\qquad\langle\alpha,y\rangle=0.$$

Define a differential operator

$$P_n(C)=\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{w\in\mathfrak{S}_n}\big(q_{\{w(1),\dots,w(k)\}}(C)\cdot\Delta^2_{\{w(k+1),\dots,w(n)\}}\big)$$

by

$$\Delta_{\{i_1,\dots,i_k\}}=\sum_{0\le j\le[\frac{k}{2}]}\frac{1}{2^kj!(k-2j)!}\times\sum_{w\in W(B_k)}\varepsilon(w)w(u_{e_{i_1}-e_{i_2}}(x)u_{e_{i_3}-e_{i_4}}(x)\cdots u_{e_{i_{2j-1}}-e_{i_{2j}}}(x)\cdot\partial_{i_{2j+1}}\partial_{i_{2j+2}}\cdots\partial_{i_k}), \tag{5.1}$$

$$q_{\{i_1,\dots,i_k\}}(C)=\sum_{\nu=1}^{k}\sum_{I_1\amalg\cdots\amalg I_\nu=\{i_1,\dots,i_k\}}T_{I_1}\cdots T_{I_\nu}, \tag{5.2}$$

$$T_I=(-1)^{\#I-1}\left(CS^o_I-\sum_{j=0}^{3}C_jT^o_I(v^j)\right), \tag{5.3}$$

where

$$S^o_{\{i_1,i_2,\dots,i_k\}}=\frac{1}{2}\sum_{w\in W(B_k)}w(u_{e_{i_1}-e_{i_2}}(x)u_{e_{i_2}-e_{i_3}}(x)\cdots u_{e_{i_{k-1}}-e_{i_k}}(x)),$$

$$S^o_\varnothing=0,\ S^o_{\{k\}}=1,\qquad S^o_{\{i,j\}}=2u_{e_i-e_j}(x)+2u_{e_i+e_j}(x),$$

$$T^o_\varnothing(v^j)=0,\qquad T^o_{\{k\}}(v^j)=2v^j_{e_k}(x)\qquad\text{for}\qquad 1\le k\le n,$$

$$q_\varnothing=1,\qquad q_{\{i\}}=T_{\{i\}},\qquad q_{\{i_1i_2\}}=T_{\{i_1\}}T_{\{i_2\}}+T_{\{i_1,i_2\}},\qquad\dots$$

In the above, we identify $W(B_k)$ with the reflection group generated by $w_{e_{i_k}}$ and $w_{e_{i_\nu}-e_{i_{\nu+1}}}$ ($\nu=1,\dots,k-1$). The sum in (5.2) runs over all the partitions of the set I and the order of the subsets $I_1,\dots,I_\nu$ is ignored.

Replacing ∂_j by ξ_j for $j=1,\dots,n$ in the definition of $\Delta_{\{i_1,\dots,i_k\}}$ and $P_n(C)$, we define functions $\bar\Delta_{\{i_1,\dots,i_k\}}$ and $\bar P_n(C)$ of (x,ξ), respectively.

We will define $u_\alpha(x)$ and $T^o_I(v^j)$ so that

$$[P_n(C),P_n(C')]=0\qquad\text{for}\qquad C,\,C'\in\mathbb{C}. \tag{5.4}$$

Then putting

$$q^o_I=q_I\big|_{C=0},$$

$$P_n=P_n(0)=\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{w\in\mathfrak{S}_n}(q^o_{\{w(1),\dots,w(k)\}}\Delta^2_{\{w(k+1),\dots,w(n)\}}), \tag{5.5}$$

$$P_{n-j}=\sum_{i=j}^{n}\sum_{k=i}^{n}\frac{(-1)^{i-j}}{i!(k-i)!(n-k)!}\sum_{w\in\mathfrak{S}_n}\sum_{I_1\amalg\cdots\amalg I_j=w(\{1,\dots,i\})}S^o_{I_1}\cdots S^o_{I_j}q^o_{w(\{i+1,\dots,k\})}\Delta^2_{w(\{k+1,\dots,n\})}, \tag{5.6}$$

we have $P_n(C)=\sum_{j=0}^{n}C^jP_{n-j}$ and (1.4) and then

$$[P_i,P_j]=0\qquad\text{for}\qquad 1\le i<j\le n,$$

$$-\frac{P_1}{2}=-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+\sum_{1\le i<j\le n}(u_{e_i-e_j}(x)+u_{e_i+e_j}(x))+\sum_{j=0}^{3}C_jv^j(x). \tag{5.7}$$

Remark 9. i) When $n = 2$, $T(x, y)$ in the last section corresponds to T_{12}, namely

$$T(x_1, x_2) = T_{12}\big|_{C=0}.$$

ii) Put

$$U(x) = \sum_{1\le i<j\le n} (u^-_{ij}(x_i - x_j) + u^+_{ij}(x_i + x_j)) \qquad \text{and} \qquad V(x) = \sum_{k=1}^{n} v_k(x_k)$$

in (1.3) and let $T_I(U; V)$ be the corresponding T_I given by (5.3). Then [29, Remark 4.3] says

$$T_I(c_0U; c_1V + c_2W) = c_0^{\#I-1}c_1T_I(U; V) + c_0^{\#I-1}c_2T_I(U; W) \qquad \text{for} \qquad c_i \in \mathbb{C}. \tag{5.8}$$

iii) The definition (5.3) may be replaced by

$$T_I = (-1)^{\#I-1}\left(C\sum_{\nu=1}^{\#I}\sum_{I_1\amalg\cdots\amalg I_\nu=I}(-2\lambda)^{\nu-1}(\nu-1)!\cdot S^o_{I_1}\cdots S^o_{I_\nu} - \sum_{j=0}^{3} C_jT^o_I(v^j)\right) \tag{5.9}$$

because λ can be any complex number in [29, Lemma 5.2 ii)] when $v = C$. Note that we fixed $\lambda = 1$ in [29]. Combining (5.9) and (5.8), we may put

$$T_I = (-1)^{\#I-1}\left(CS^o_I + c'\sum_{\nu\ge2}c^{\nu-1}\sum_{I_1\amalg\cdots\amalg I_\nu=I}S^o_{I_1}\cdots S^o_{I_\nu} - \sum_{j=0}^{3}C_jT^o_I(v^j)\right)$$

for any c, $c' \in \mathbb{C}$ and hence

$$T_I = (-1)^{\#I-1}\left(CS^o_I + \sum_{\nu\ge2}c_\nu\sum_{I_1\amalg\cdots\amalg I_\nu=I}S^o_{I_1}\cdots S^o_{I_\nu} - \sum_{j=0}^{3}C_jT^o_I(v^j)\right)$$

for any $c_2, c_3, \ldots \in \mathbb{C}$.

Theorem 4 (Ellip-B_n, [25], [29, Theorem 7.2]). *Put*

$$\begin{aligned} &u_{e_i\pm e_j}(x) = A\wp_0(x_i \pm x_j; 2\omega_1, 2\omega_2) && \textit{for} \quad 1 \le i < j \le n,\\ &v^j_{e_k}(x) = \wp_0(x_k + \omega_j; 2\omega_1, 2\omega_2) && \textit{for} \quad 1 \le k \le n \quad \textit{and} \quad 0 \le j \le 3 \end{aligned}$$

and

$$T^o_I(v^j) = \sum_{\nu=1}^{\#I}\sum_{I_1\amalg\cdots\amalg I_\nu=I}(-A)^{\nu-1}(\nu-1)!\cdot S_{I_1}(v^j)\cdots S_{I_\nu}(v^j), \tag{5.10}$$

$$S_{\{i_1,\ldots,i_k\}}(v^j) = \sum_{w\in W(B_k)} v^j_{w(e_{i_1})}(x)u_{w(e_{i_1}-e_{i_2})}(x)u_{w(e_{i_2}-e_{i_3})}(x)\cdots u_{w(e_{i_{k-1}}-e_{i_k})}(x). \tag{5.11}$$

Then (5.4) *holds.*

Example 1. Put $v^j_k = v^j_{e_k}$, $\tilde{v}_k = \sum_{j=0}^{3} C_jv^j_k$, and $w^\pm_{ij} = u^-_{ij} \pm u^+_{ij}$. Then

$$\begin{aligned} &\Delta_{\{1\}} = \partial_1,\\ &\Delta_{\{1,2\}} = \partial_1\partial_2 + u^-_{12} - u^+_{12} = \partial_1\partial_2 + w^-_{12},\\ &\Delta_{\{1,2,3\}} = \partial_1\partial_2\partial_3 + w^-_{12}\partial_3 + w^-_{23}\partial_1 + w^-_{13}\partial_2, \end{aligned}$$

$$\Delta_{\{1,2,3,4\}} = \partial_1\partial_2\partial_3\partial_4 + w_{34}^-\partial_1\partial_2 + w_{24}^-\partial_1\partial_3 + w_{23}^-\partial_1\partial_4 + w_{14}^-\partial_2\partial_3 + w_{13}^-\partial_2\partial_4$$
$$\quad + w_{12}^-\partial_3\partial_4 + w_{12}^-w_{34}^- + w_{13}^-w_{24}^- + w_{14}^-w_{23}^-,$$
$$S_1(v^j) = 2v_1^j,$$
$$S_{\{1,2\}}(v^j) = 2v_1^j u_{12}^- + 2v_1^j u_{12}^+ + 2v_2^j u_{12}^- + 2v_2^j u_{12}^+ = 2(v_1^j + v_2^j)(u_{12}^- + u_{12}^+),$$
$$S_{\{1,2,3\}}(v^j) = 2v_1^j u_{12}^- u_{23}^- + 2v_1^j u_{12}^- u_{23}^+ + 2v_1^j u_{12}^+ u_{23}^- + 2v_1^j u_{12}^+ u_{23}^+ + \cdots$$
$$= 2(v_1^j + v_3^j)w_{12}^+ w_{23}^+ + 2(v_1^j + v_2^j)w_{23}^+ w_{13}^+ + 2(v_2^j + v_3^j)w_{12}^+ w_{13}^+,$$
$$T_{\{1\}} = CS_{\{1\}} - \sum_{j=0}^{3} C_j T_{\{1\}}^o(v^j) = C - 2\tilde{v}_1,$$
$$T_{\{1,2\}} = -CS_{\{1,2\}}^o + \sum_{j=0}^{3} C_j T_{\{1,2\}}^o(v^j),$$
$$q_{\{1\}} = T_{\{1\}},$$
$$q_{\{1,2\}} = T_{\{1\}}T_{\{2\}} + T_{\{1,2\}},$$
$$q_{\{1,2,3\}} = T_{\{1\}}T_{\{2\}}T_{\{3\}} + T_{\{1,2\}}T_{\{3\}} + T_{\{1,3\}}T_{\{2\}} + T_{\{1\}}T_{\{2,3\}} + T_{\{1,2,3\}}.$$

If $T^o(v^j)$ and $S_{i_1,\dots,i_k}(v^j)$ are given by (5.10) and (5.11), then

$$T_{\{1\}}^o(v^j) = 2v_1^j,$$
$$T_{\{1,2\}}^o(v^j) = S_{\{1,2\}}(v^j) - AT_{\{1\}}^o T_{\{2\}}^o = 2(v_1^j + v_2^j)w_{12}^+ - 4Av_1^j v_2^j,$$
$$T_{\{1,2,3\}}^o(v^j) = S_{\{1,2,3\}}(v^j) - 2A(v_1^j S_{\{2,3\}}(v^j) + v_2^j S_{\{1,3\}}(v^j) + v_3^j S_{\{1,2\}}(v^j)) + 16A^2 v_1^j v_2^j v_3^j.$$

In particular, if $n = 2$, then

$$P_2(C) = \Delta_{\{1,2\}}^2 + q_{\{1\}}\Delta_{\{2\}}^2 + q_{\{2\}}\Delta_{\{1\}}^2 + q_{\{1,2\}} = \left(\partial_1\partial_2 + u_{12}^- - u_{12}^+\right)^2$$
$$+ T_{\{1\}}\partial_2^2 + T_{\{2\}}\partial_1^2 + T_{\{1\}}T_{\{2\}} - CS_{\{1,2\}}^o + \sum_{j=0}^{3} C_j T_{\{1,2\}}^o(v^j)$$
$$= \left(\partial_1\partial_2 + u_{12}^- - u_{12}^+\right)^2 + (C - 2\tilde{v}_1)\partial_2^2 + (C - 2\tilde{v}_2)\partial_1^2 + (C - 2\tilde{v}_1)(C - 2\tilde{v}_2)$$
$$- 2C(u_{12}^- + u_{12}^+) + 2(\tilde{v}_1 + \tilde{v}_2)(u_{12}^- + u_{12}^+) - 4A\sum_{j=0}^{3} C_j v_1^j v_2^j = C^2 - 2P\cdot C + P_2$$

with

$$P = -\frac{1}{2}(\partial_1^2 + \partial_2^2) + \tilde{v}_1 + \tilde{v}_2 + u_{12}^- + u_{12}^+,$$
$$P_2 = \left(\partial_1\partial_2 + u_{12}^- - u_{12}^+\right)^2 - 2\tilde{v}_1\partial_2^2 - 2\tilde{v}_2\partial_1^2 + 4\tilde{v}_1\tilde{v}_2 + 2(\tilde{v}_1 + \tilde{v}_2)(u_{12}^- + u_{12}^+) - 4A\sum_{j=0}^{3} C_j v_1^j v_2^j,$$

which should be compared with (4.2), (4.4) and (4.6).

In general

$$P_1 = \sum_{k=1}^{n}(\Delta_{\{k\}}^2 + q_{\{k\}}^o) - \sum_{1\le i<j\le n} S_{\{i,j\}}^o = \sum_{k=1}^{n}(\partial_k^2 - 2\tilde{v}_k^j) - 2\sum_{1\le i<j\le n} w_{ij}^+,$$
$$P_2 = \sum_{1\le i<j\le n} \Delta_{\{i,j\}}^2 + \sum_{\substack{1\le i\le n\\ 1\le j\le n,\, i\ne j}} \sum_{j=1}^{n} q_{\{i\}}^o \Delta_{\{j\}}^2 + \sum_{1\le i<j\le n} q_{\{i,j\}}^o$$

$$- \sum_{\substack{1\le i<j\le n\\ 1\le k\le n,\, k\ne i,\, j}} S^o_{\{i,j\}}(\Delta^2_{\{k\}} + q^o_{\{k\}}) + \sum_{\substack{1\le i<j\le n\\ i<k<\ell\le n\\ j\ne k,\,\ell}} S^o_{\{i,j\}} S^o_{\{k,\ell\}} + \sum_{1\le i<j<k\le n} S^o_{\{i,j,k\}}$$

$$= \sum (\partial_i\partial_j + w^-_{ij})^2 - \sum 2\tilde v_i \partial_j^2 + \sum 4\tilde v_i \tilde v_j + \sum C_j T^o_{\{k,\ell\}}(v^j)$$
$$- 2\sum w^+_{ij}(\partial_k^2 - 2\tilde v_k) + 4\sum w^+_{ij} w^+_{k\ell} + 2\sum w^+_{ij} w^+_{jk}. \tag{5.12}$$

Here if (5.10) and (5.11) are valid, then

$$T^o_{\{k,\ell\}}(v^j) = 2(v^j_k + v^j_\ell) w^+_{k\ell} - 4A v^j_k v^j_\ell. \tag{5.13}$$

The commuting operator P_3 of the 6-th order is

$$P_3 = \sum \Delta^2_{\{i,j,k\}} + \sum q^o_{\{k\}}\Delta^2_{\{i,j\}} + \sum q^o_{\{i,j,k\}} - \sum S^o_{\{k,\ell\}}\Delta^2_{\{i,j\}}$$
$$- \sum S^o_{\{k,\ell\}} q^o_{\{i\}} \Delta^2_{\{j\}} - \sum S^o_{\{k,\ell\}} q^o_{\{i,j\}} + \sum S^o_{\{i,j,k,\ell,m\}} + \sum S^o_{\{i,j,k,\ell\}} S^o_{\{\mu,\nu\}}$$
$$+ \sum S^o_{\{i,j,k\}} S^o_{\{\ell,\mu,\nu\}} + \sum S^o_{\{i,j,k\}} S^o_{\{\ell,m\}} S^o_{\{\mu,\nu\}} + \sum S^o_{\{i_1,i_2\}} S^o_{\{j_1,j_2\}} S^o_{\{k_1,k_2\}} S^o_{\{\ell_1,\ell_2\}}.$$

In Theorem 4 the Schrödinger operator is

$$P = -\frac{1}{2}\sum_{k=0}^{n} \frac{\partial^2}{\partial x_k^2} + A \sum_{1\le i<j\le n} (\wp(x_i - x_j) + \wp(x_i + x_j)) + \sum_{j=0}^{3} C_j \sum_{k=1}^{n} \wp(x_k + \omega_j)$$

and the operator P_2 satisfying $[P, P_2] = 0$ is given by (5.12) and (5.13) with

$$\tilde v_k = \sum_{\nu=0}^{3} C_\nu \wp(x_k + \omega_\nu), \qquad v^j_k = \wp(x_k + \omega_j), \qquad w^\pm_{ij} = A(\wp(x_i - x_j) \pm \wp(x_i + x_j)).$$

5.2 Analytic continuation of integrals

Theorem 5 (Toda-$D^{(1)}_n$-bry). *For*

$$u_{e_i - e_j}(x) = \begin{cases} Ae^{-2\lambda(x_i - x_{i+1})} & (j = i+1), \\ 0 & (|j-i| > 1), \end{cases}$$
$$u_{e_i + e_j}(x) = \begin{cases} Ae^{2\lambda(x_1 + x_2)} & (i+j = 3), \\ Ae^{-2\lambda(x_{n-1} + x_n)} & (i+j = 2n-1), \\ 0 & (i+j \notin \{3, 2n-1\}), \end{cases} \tag{5.14}$$
$$v^0_k(x) = \delta_{1k}\sinh^{-2}\lambda x_1, \qquad v^1_k(x) = \delta_{1k}\sinh^{-2} 2\lambda x_1,$$
$$v^2_k(x) = \delta_{nk}\sinh^{-2}\lambda x_n, \qquad v^3_k(x) = \delta_{nk}\sinh^{-2} 2\lambda x_n,$$

we have commuting integrals P_j by (5.5), (5.1), (5.2), (5.3) *and*

$$S^o_{\{k\}} = 1 \quad \textit{for} \quad 1 \le k \le n,$$
$$S^o_I = 0 \quad \textit{if} \quad I \ne \{k, k+1, \ldots, \ell\} \quad \textit{for} \quad 1 \le k < \ell \le n,$$
$$S^o_{\{k,k+1,\ldots,\ell\}} = 2A^{\ell-k+1}(e^{-2\lambda(x_k - x_\ell)} + \delta_{1k} e^{2\lambda(x_1 + x_\ell)} + \delta_{\ell n} e^{-2\lambda(x_k + x_n)}),$$
$$T^0_{\{k\}}(v^j) = 2v^j_k(x) \quad \textit{for} \quad 0 \le j \le 3, \quad k = 1, \ldots, n,$$
$$T^0_I(v^0) = 0 \quad \textit{if} \quad I \ne \{1, \ldots, k\} \quad \textit{for} \quad k = 1, \ldots, n,$$
$$T^0_I(v^2) = 0 \quad \textit{if} \quad I \ne \{k, \ldots, n\} \quad \textit{for} \quad k = 1, \ldots, n,$$

$$T^0_I(v^1) = T^0_I(v^3) = 0 \qquad \textit{if} \qquad \#I > 1,$$
$$T^0_{\{1,\dots,k\}}(v^0) = 8A^{k-1}(e^{2\lambda x_k} + \delta_{kn}e^{-2\lambda x_n}) \qquad \textit{for} \qquad k \geq 2,$$
$$T^0_{\{n-k+1,\dots,n\}}(v^2) = 8A^{k-1}(e^{-2\lambda x_{n-k+1}} + \delta_{kn}e^{2\lambda x_1}) \qquad \textit{for} \qquad k \geq 2.$$

Proof. Put

$$\tilde{x} = \left(x_1 - \frac{1-1}{n-1}\omega_2, \dots, x_k - \frac{k-1}{n-1}\omega_2, \dots, x_n - \frac{n-1}{n-1}\omega_2\right),$$
$$\tilde{u}_{e_i \mp e_j}(\tilde{x}) = A\frac{e^{2\lambda\omega_2/(n-1)}}{4\lambda^2}\wp_0\left(x_i - \frac{i-1}{n-1}\omega_2 \mp \left(x_j - \frac{j-1}{n-1}\omega_2\right); 2\omega_1, 2\omega_2\right),$$
$$\tilde{v}^j_k(\tilde{x}) = \frac{(-1)^j}{\lambda^2}\wp_0\left(x_k - \frac{k-1}{n-1}\omega_2 + \omega_j; 2\omega_1, 2\omega_2\right) \qquad \text{for} \qquad 0 \leq j \leq 3, \qquad 1 \leq k \leq n.$$

When $\omega_2 \to \infty$, $\tilde{u}_{e_i \mp e_j}(\tilde{x})$ and $\tilde{v}^\ell_k$ $(\ell = 0, 1, 2, 3)$ converge to $u_{e_i \mp e_j}(x)$ in (5.14) and

$$v^0_k(x) = \delta_{1k}\sinh^{-2}\lambda x_1, \quad v^1_k(x) = \delta_{1k}\cosh^{-2}\lambda x_1,$$
$$v^2_k(x) = \delta_{nk}\sinh^{-2}\lambda x_n, \quad v^3_k(x) = \delta_{nk}\cosh^{-2}\lambda x_n,$$

respectively. Under the notation in Theorem 4, let $\tilde{S}_I(\tilde{v}^\ell)$ and $\tilde{T}^o_I(\tilde{v}^\ell)$ be the functions defined in the same way as $S_I(v^\ell)$ and $T^o_I(v^\ell)$, respectively, where $(u_{e_i \mp e_j}(x), v^\ell_k(x))$ are replaced by $(\tilde{u}_{e_i \mp e_j}(x), \tilde{v}^\ell_k(\tilde{x}))$. Then by taking the limits for $\omega_2 \to \infty$, $\tilde{T}^o_I(\tilde{v}^\ell)$ converge to the following $\bar{T}^o_I(v^\ell)$

$$\bar{T}^0_I(v^0) = \bar{T}^0_I(v^1) = 0 \qquad \text{if} \qquad I \neq \{1, \dots, k\} \qquad \text{for} \qquad k = 1, \dots, n,$$
$$\bar{T}^0_I(v^2) = \bar{T}^0_I(v^3) = 0 \qquad \text{if} \qquad I \neq \{k, \dots, n\} \qquad \text{for} \qquad k = 1, \dots, n.$$

If $k \geq 2$, then

$$\begin{aligned}
\bar{T}^0_{\{1,\dots,k\}}(v^0) &= \lim_{\omega_2\to\infty}\sum_{\nu=1}^{\#I}\sum_{I_1\amalg\cdots\amalg I_\nu = I}\left(-A\frac{e^{2\lambda\omega_2/(n-1)}}{4\lambda^2}\right)^{\nu-1}(\nu-1)!\tilde{S}_{I_1}(\tilde{v}^0)\cdots\tilde{S}_{I_\nu}(\tilde{v}^0)\\
&= \lim_{\omega_2\to\infty}\tilde{S}_{\{1,\dots,k\}}(\tilde{v}^0) - \lim_{\omega_2\to\infty}\tilde{S}_{\{1\}}(\tilde{v}^0)A\frac{e^{2\lambda\omega_2/(n-1)}}{4\lambda^2}\tilde{S}_{\{2,\dots,k\}}(\tilde{v}^0)\\
&= 2A^{k-1}\sinh^{-2}\lambda x_1(e^{-2\lambda(x_1-x_k)} + e^{2\lambda(x_1+x_k)} + \delta_{kn}e^{2\lambda(x_1-x_n)} + \delta_{kn}e^{-2\lambda(x_1+x_n)})\\
&\quad - 2\sinh^{-2}\lambda x_1 \cdot 2A^{k-1}(e^{2\lambda x_k} + \delta_{kn}e^{-2\lambda x_n}) = 8A^{k-1}(e^{2\lambda x_k} + \delta_{kn}e^{-2\lambda x_n}),\\
\bar{T}^0_{\{1,\dots,k\}}(v^1) &= 2A^{k-1}\cosh^{-2}\lambda x_1(e^{-2\lambda(x_1-x_k)} + e^{2\lambda(x_1+x_k)} + \delta_{kn}e^{2\lambda(x_1-x_n)} + \delta_{kn}e^{-2\lambda(x_1+x_n)})\\
&\quad + 4A^{k-1}\cosh^{-2}\lambda x_1(e^{2\lambda x_k} + \delta_{kn}e^{-2\lambda x_n}) = 8A^{k-1}(e^{2\lambda x_k} + \delta_{kn}e^{-2\lambda x_n}),\\
\bar{T}^0_{\{n-k+1,\dots,n\}}(v^2) &= 8A^{k-1}(e^{-2\lambda x_{n-k+1}} + \delta_{kn}e^{2\lambda x_1}),\\
\bar{T}^0_{\{n-k+1,\dots,n\}}(v^3) &= 8A^{k-1}(e^{-2\lambda x_{n-k+1}} + \delta_{kn}e^{2\lambda x_1}).
\end{aligned}$$

Replacing v^1 and v^3 by $(1/4)(v^0 - v^1)$ and $(1/4)(v^2 - v^3)$, respectively, we have the theorem by the analytic continuation given in Lemma 2. ∎

As is proved by [29], suitable limits of the functions in Theorem 4 give the following theorem.

Theorem 6 (Trig-B_n, [29, Proposition 6.1]). *For complex numbers λ, C, $C_0, \dots, C_3$ and A with $\lambda \neq 0$, we have (5.7) by putting*

$$u_{e_i \pm e_j}(x) = A\sinh^{-2}\lambda(x_i \pm x_k),$$

$$v^0_{e_k}(x) = \sinh^{-2}\lambda x_k, \qquad v^1_{e_k}(x) = \cosh^{-2}\lambda x_k,$$
$$v^2_{e_k}(x) = \sinh^{2}\lambda x_k, \qquad v^3_{e_k}(x) = \frac{1}{4}\sinh^{2} 2\lambda x_k$$

and

$$\begin{aligned}T_I &= (-1)^{\#I-1}(CS^o_I - C_0T^o_I(v^0) - C_1T^o_I(v^1) - C_2S_I(v^2) - C_3S_I(v^3)\\ &\quad + 2C_3\sum_{I_1\amalg I_2=I}(S_{I_1}(v^2)S_{I_2}(v^2) + S_{I_1}(v^2)S^o_{I_2} + S^o_{I_1}S_{I_2}(v^2))),\\ T^o_I(v^0) &= \sum_{\nu=1}^{\#I}\sum_{I_1\amalg\cdots\amalg I_\nu=I}(-A)^{\nu-1}(\nu-1)!S_{I_1}(v^0)\cdots S_{I_\nu}(v^0),\\ T^o_I(v^1) &= \sum_{\nu=1}^{\#I}\sum_{I_1\amalg\cdots\amalg I_\nu=I}A^{\nu-1}(\nu-1)!S_{I_1}(v^1)\cdots S_{I_\nu}(v^1).\end{aligned}$$

Theorem 7 (Trig-A_{n-1}-bry). *For*

$$u_{e_i-e_j}(x) = A\sinh^{-2}\lambda(x_i-x_j), \qquad u_{e_i+e_j}(x) = 0,$$
$$v^0_{e_k}(x) = e^{-2\lambda x_k}, \qquad v^1_{e_k}(x) = e^{-4\lambda x_k}, \qquad v^2_{e_k}(x) = e^{2\lambda x_k}, \qquad v^3_{e_k}(x) = e^{4\lambda x_k}$$

we have (5.7) *by putting*

$$T_I = (-1)^{\#I-1}\left(CS^o_I - \sum_{j=0}^{3}C_jS_I(v^j) + 2\sum_{I_1\amalg I_2=I}(C_1S_{I_1}(v^0)S_{I_2}(v^0) + C_3S_{I_1}(v^2)S_{I_2}(v^2))\right).$$

Proof. Putting

$$\begin{aligned}&\tilde u_{e_i\pm e_j} = A\sinh^{-2}\lambda\big((x_i+N)\pm(x_j+N)\big), \qquad \tilde v^0_k = \frac{1}{4}e^{2\lambda N}\sinh^{-2}\lambda(x_k+N),\\ &\tilde v^1_k = \frac{1}{4}e^{4\lambda N}\sinh^{-2}2\lambda(x_k+N) = \frac{1}{16}e^{4\lambda N}\big(\sinh^{-2}2\lambda(x_k+N) - \cosh^{-2}2\lambda(x_k+N)\big),\\ &\tilde v^2_k = 4e^{-2\lambda N}\sinh^2\lambda(x_k+N), \qquad \tilde v^3_k = 4e^{-4\lambda N}\sinh^2 2\lambda(x_k+N),\\ &\tilde x = (x_1+N, x_2+N, \ldots, x_n+N)\end{aligned}$$

under the notation in Theorem 6, we have

$$\begin{aligned}&(\bar u_{e_i-e_j}, \bar u_{e_i+e_j}, \bar v^0_k, \bar v^1_k, \bar v^2_k, \bar v^3_k) := \lim_{N\to\infty}(\tilde u_{e_i-e_j}, \tilde u_{e_i+e_j}, \tilde v^0_k, \tilde v^1_k, \tilde v^2_k, \tilde v^3_k)\\ &\qquad = (A\sinh^{-2}\lambda(x_i-x_k), 0, e^{-2\lambda x_k}, e^{-4\lambda x_k}e^{2\lambda x_k}, e^{4\lambda x_k}),\\ &\lim_{N\to\infty}\frac{1}{4}e^{2\lambda N}T^o_I(v^0)(\tilde x) = \bar S_I(\bar v^0),\\ &\lim_{N\to\infty}\frac{1}{16}e^{4\lambda N}\big(T^o_I(v^0)(\tilde x) - T^o_I(v^1)(\tilde x)\big) = \bar S_I(\bar v^1) - 2\sum_{I_1\amalg I_2=I}\bar S_{I_1}(\bar v^0)\bar S_{I_2}(\bar v^0),\\ &\lim_{N\to\infty}4e^{-2\lambda N}T^o_I(v^2)(\tilde x) = \bar S_I(\bar v^2),\\ &\lim_{N\to\infty}4e^{-2\lambda N}T^o_I(v^3)(\tilde x) = \bar S_I(\bar v^3) - 2\sum_{I_1\amalg I_2=I}\bar S_{I_1}(\bar v^2)\bar S_{I_2}(\bar v^2).\end{aligned}$$

Here $\bar S_I(\bar v^\ell)$ are defined by (5.11) with $u_{e_i\pm e_j}$ and $v^\ell_{e_k}$ replaced by $\bar u_{e_i\pm e_j}$ and $\bar v^\ell_{e_k}$, respectively. Then the theorem is clear. ■

Theorem 8 (Toda-$B_n^{(1)}$-bry). *For the potential function defined by*

$$\begin{aligned}
&u_{e_i-e_j}(x) = \begin{cases} Ae^{-2\lambda(x_i-x_{i+1})} & \textit{if} \quad j = i+1, \\ 0 & \textit{if} \quad 1 \le i < i+1 < j \le n, \end{cases} \\
&u_{e_i+e_j}(x) = \begin{cases} Ae^{-2\lambda(x_{n-1}+x_n)} & \textit{if} \quad i = n-1, \quad j = n, \\ 0 & \textit{if} \quad 1 \le i < j \le n \quad \textit{and} \quad i \ne n-1, \end{cases} \\
&v_k^0(x) = \delta_{k1}e^{2\lambda x_1}, \qquad v_k^1(x) = \delta_{k1}e^{4\lambda x_1}, \\
&v_k^2(x) = \delta_{kn}\sinh^{-2}\lambda x_n, \qquad v_k^3(x) = \delta_{kn}\sinh^{-2}2\lambda x_n,
\end{aligned} \tag{5.15}$$

we have (5.7) *with*

$$\begin{aligned}
&S^o_{\{k\}} = 1 \quad \textit{for} \quad 1 \le k \le n, \\
&S^o_I = 0 \quad \textit{if} \quad I \ne \{k, k+1, \dots, \ell\} \quad \textit{for} \quad 1 \le k < \ell \le n, \\
&S^o_{\{k,k+1,\dots,\ell\}} = 2A^{\ell-k+1}(e^{-2\lambda(x_k-x_\ell)} + \delta_{\ell n}e^{-2\lambda(x_k+x_n)}), \\
&T^0_{\{k\}}(v^j) = 2v_k^j(x) \quad \textit{for} \quad 0 \le j \le 3, \quad k = 1, \dots, n, \\
&T^0_I(v^0) = 0 \quad \textit{if} \quad I \ne \{1, \dots, k\} \quad \textit{for} \quad k = 1, \dots, n, \\
&T^0_I(v^2) = 0 \quad \textit{if} \quad I \ne \{k, \dots, n\} \quad \textit{for} \quad k = 1, \dots, n, \\
&T^0_I(v^1) = T^0_I(v^3) = 0 \quad \textit{if} \quad \#I > 1, \\
&T^0_{\{1,\dots,k\}}(v^0) = 2A^{k-1}(e^{2\lambda x_k} + \delta_{kn}e^{-2\lambda x_n}) \quad \textit{for} \quad k \ge 2, \\
&T^0_{\{n-k+1,\dots,n\}}(v^2) = 8A^{k-1}e^{-2\lambda x_{n-k+1}} \quad \textit{for} \quad k \ge 2.
\end{aligned}$$

Proof. Suppose $\operatorname{Re}\lambda > 0$. In (Toda-$D_n^{(1)}$-bry) put

$$\begin{aligned}
&\tilde{x} = (x_1 - (n-1)N, \dots, x_k - (n-k)N, \dots, x_n - (n-n)N), \\
&\tilde{u}_{e_i-e_j} = \begin{cases} Ae^{-2\lambda N}e^{-2\lambda(x_i-(n-i)N-x_{i+1}+(n-i-1)N)} & (j = i+1), \\ 0 & (|j-i| > 1), \end{cases} \\
&\tilde{u}_{e_i+e_j} = \begin{cases} Ae^{-2\lambda N}e^{2\lambda(x_1-(n-1)N+x_2-(n-2)N)} & (i+j = 3), \\ Ae^{-2\lambda N}e^{-2\lambda(x_{n-1}-N+x_n)} & (i+j = 2n-1), \\ 0 & (i+j \notin \{3, 2n-1\}), \end{cases} \\
&\tilde{v}_k^0 = \delta_{1k}\frac{e^{2\lambda(n-1)N}}{4}\sinh^{-2}\lambda(x_1 - (n-1)N), \\
&\tilde{v}_k^1 = \delta_{1k}\frac{e^{4\lambda(n-1)N}}{4}\sinh^{-2}2\lambda(x_1 - (n-1)N), \\
&\tilde{v}_k^2 = \delta_{nk}\sinh^{-2}\lambda x_n, \qquad \tilde{v}_k^3 = \delta_{nk}\sinh^{-2}2\lambda x_n
\end{aligned}$$

and we have (5.15) by the limit $N \to \infty$. Moreover for $k \ge 2$, it follows from Theorem 5 and (5.8) that

$$\begin{aligned}
\tilde{T}_{\{1,\dots,k\}}(\tilde{v}^1) &= \tilde{T}_{\{1,\dots,k\}}(\tilde{v}^3) = 0, \\
\tilde{T}_{\{1,\dots,k\}}(\tilde{v}^0) &= \frac{1}{4}e^{2\lambda(n-1)N}(Ae^{-2\lambda N})^{k-1}(8e^{2\lambda(x_k-(n-k)N)} + 8\delta_{kn}e^{-2\lambda x_n}) \\
&= 2A^{k-1}(e^{2\lambda x_k} + \delta_{kn}e^{-2\lambda x_n}), \\
\tilde{T}_{\{n-k+1,\dots,n\}}(\tilde{v}^2) &= (Ae^{-2\lambda N})^{k-1}(8e^{-2\lambda(x_{n-k+1}-(k-1)N)} + 8\delta_{1k}e^{2\lambda(x_1-(n-1)N)}) \\
&= 8A^{k-1}e^{-2\lambda x_{n-k+1}},
\end{aligned}$$

which implies the theorem. ■

Theorem 9 (Toda-$C_n^{(1)}$). *For the potential function defined by*

$$
\begin{aligned}
&u_{e_i-e_j}(x) = \begin{cases} Ae^{-2\lambda(x_i-x_{i+1})} & \text{if} \quad j=i+1, \\ 0 & \text{if} \quad 1\le i<i+1<j\le n, \end{cases} \\
&u_{e_i+e_j}(x) = 0 \quad \text{for} \quad 1\le i<j\le n, \\
&v_k^0(x) = \delta_{k1}e^{2\lambda x_1}, \qquad v_k^1(x) = \delta_{k1}e^{4\lambda x_1}, \\
&v_k^2(x) = \delta_{kn}e^{-2\lambda x_n}, \qquad v_k^3(x) = \delta_{kn}e^{-4\lambda x_n},
\end{aligned}
$$

we have (5.7) *with*

$$
\begin{aligned}
&S^o_{\{k\}} = 1 \quad \text{for} \quad 1\le k\le n, \\
&S^o_I = 0 \quad \text{if} \quad I\ne\{k,k+1,\dots,\ell\} \quad \text{for} \quad 1\le k<\ell\le n, \\
&S^o_{\{k,k+1,\dots,\ell\}} = 2A^{\ell-k+1}e^{-2\lambda(x_k-x_\ell)}, \\
&T^0_{\{k\}}(v^j) = 2v_k^j(x) \quad \text{for} \quad 0\le j\le 3, \quad k=1,\dots,n, \\
&T^0_I(v^0) = 0 \quad \text{if} \quad I\ne\{1,\dots,k\} \quad \text{for} \quad k=1,\dots,n, \\
&T^0_I(v^2) = 0 \quad \text{if} \quad I\ne\{k,\dots,n\} \quad \text{for} \quad k=1,\dots,n, \\
&T^0_I(v^1) = T^0_I(v^3) = 0 \quad \text{if} \quad \#I>1, \\
&T^0_{\{1,\dots,k\}}(v^0) = 2A^{k-1}e^{2\lambda x_k} \quad \text{for} \quad k\ge 2, \\
&T^0_{\{n-k+1,\dots,n\}}(v^2) = 2A^{k-1}e^{-2\lambda x_{n-k+1}} \quad \text{for} \quad k\ge 2.
\end{aligned}
$$

Proof. Substituting x_k by x_k+R for $k=1,\dots,n$ and multiplying v_k^0, v_k^1, v_k^2 and v_k^3 by $e^{-2\lambda R}$, $e^{-4\lambda R}$, $(1/4)e^{2\lambda R}$ and $(1/4)e^{4\lambda R}$, respectively, we have the claim from Theorem 8. ■

Theorem 10 (Rat-A_{n-1}-bry). *We have* (5.7) *if*

$$
\begin{aligned}
&u_{e_i-e_j}(x) = \frac{A}{(x_i-x_j)^2}, \qquad u_{e_i+e_j}(x) = 0, \qquad v_k^j(x) = x_k^{j+1}, \\
&T_I = (-1)^{\#I-1}\Bigg(CS^o_I - \sum_{j=0}^{3} C_jS_I(v^j) + \sum_{I_1\amalg I_2=I} C_1\big(S_{I_1}(v^0)S^o_{I_2} + S^o_{I_1}S_{I_1}(v^0)\big) \\
&\qquad + \sum_{I_1\amalg I_2=I} C_3\big(S_{I_1}(v^1)S^o_{I_2} + S_{I_1}(v^0)S_{I_2}(v^0) + S^o_{I_1}S_{I_2}(v^1)\big)\Bigg).
\end{aligned}
$$

Proof. Put

$$
\begin{aligned}
&\tilde u_{e_i-e_j} = \lambda^2\sinh^{-2}\lambda(x_i-x_j), \qquad \tilde u_{e_i+e_j} = 0, \\
&\tilde v_k^0 = \frac{1}{2\lambda}(e^{2\lambda x_k}-1), \qquad \tilde v_k^1 = \frac{1}{4\lambda^2}(e^{2\lambda x_k}+e^{-2\lambda x_k}-2), \\
&\tilde v_k^2 = \frac{1}{8\lambda^3}(e^{4\lambda x_k}-3e^{2\lambda x_k}-e^{-2\lambda x_k}+3), \quad \tilde v_k^3 = \frac{1}{16\lambda^4}(e^{4\lambda x_k}+e^{-4\lambda x_k}-4e^{2\lambda x_k}-4e^{-2\lambda x_k}+6).
\end{aligned}
$$

Then taking $\lambda\to 0$ we have the required potential function.

Owing to (Trig-A_{n-1}-bry) and Remark 9, we have

$$
\begin{aligned}
&\lim_{\lambda\to 0}\tilde S_I\Big(\sum \tilde v_k^j\Big) = \bar S_I\Big(\sum x_k^{j+1}\Big), \\
&\lim_{\lambda\to 0}\lambda^2\frac{1}{8\lambda^3}\Big(\tilde S_{I_1}\Big(\sum e^{2\lambda x_k}\Big)\tilde S_{I_2}\Big(\sum e^{2\lambda x_k}\Big) - 4\tilde S^o_{I_1}\tilde S^o_{I_2}\Big)
\end{aligned}
$$

$$= \frac{1}{2}\Big(\bar{S}_{I_1}\Big(\sum x_k\Big)\bar{S}^o_{I_2} + \bar{S}^o_{I_1}\bar{S}_{I_2}\Big(\sum x_k\Big)\Big),$$

$$\lim_{\lambda\to 0}\lambda^2\frac{1}{16\lambda^4}\Big(\tilde{S}_{I_1}\Big(\sum e^{2\lambda x_k}\Big)\tilde{S}_{I_2}\Big(\sum e^{2\lambda x_k}\Big) + \tilde{S}_{I_1}\Big(\sum e^{-2\lambda x_k}\Big)\tilde{S}_{I_2}\Big(\sum e^{-2\lambda x_k}\Big) - 8\tilde{S}^o_{I_1}\tilde{S}^o_{I_2}\Big)$$
$$= \frac{1}{2}\Big(\bar{S}_{I_1}\Big(\sum x_k^2\Big)\bar{S}^o_{I_2} + \bar{S}_{I_1}\Big(\sum x_k\Big)\bar{S}_{I_2}\Big(\sum x_k\Big) + \bar{S}^o_{I_1}\bar{S}_{I_2}\Big(\sum x_k\Big)\Big).$$

and thus the theorem. ■

As is proved by [29], suitable limits of the functions in Theorem 6 give the following theorem.

Theorem 11 (Rat-B_n, [29, Proposition 6.3]). *Put*

$$u_{e_i-e_j}(x) = \frac{A}{(x_i-x_j)^2}, \qquad u_{e_i+e_j}(x) = \frac{A}{(x_i+x_j)^2}$$
$$v^0_k(x) = x_k^{-2}, \qquad v^1_k(x) = x_k^2, \qquad v^2_k(x) = x_k^4, \qquad v^3_k(x) = x_k^6.$$

Then (5.7) *holds with*

$$T_I = (-1)^{\#I-1}\Bigg(CS^o_I - C_0T^o_I(v^0) - C_1S_I(v^1) - C_2S_I(v^2)$$
$$+ 2C_2\sum_{I_1\amalg I_2=I}(S_{I_1}(v^1)S^o_{I_2} + S^o_{I_1}S_{I_2}(v^1)) - 2C_3S^o_I(v^3)$$
$$+ C_3\sum_{I_1\amalg I_2=I}(S_{I_1}(v^1)S_{I_2}(v^1) + 2S_{I_1}(v^2)S^o_{I_2} + 2S^o_{I_1}S_{I_2}(v^2))$$
$$- 24C_3\sum_{I_1\amalg I_2\amalg I_3=I}(S_{I_1}(v^1)S^o_{I_2}S^o_{I_3} + S^o_{I_1}S_{I_2}(v^1)S^o_{I_3} + S^o_{I_1}S^o_{I_2}S_{I_3}(v^1))\Bigg),$$
$$T^o_I(v^0) = \sum_{\nu=1}^{\#I}\sum_{I_1\amalg\cdots\amalg I_\nu=I}(-A)^{\nu-1}(\nu-1)!\cdot S_{I_1}(v^0)\cdots S_{I_\nu}(v^0).$$

Definition 5. We define some potential functions as specializations of potential functions in Definition 3.

(Trig-A_{n-1}-bry-reg) *Trigonometric potential of type A_{n-1} with regular boundary conditions* is (Trig-A_{n-1}-bry) with $C_2 = C_3 = 0$.

(Trig-A_{n-1}) *Trigonometric potential of type A_{n-1}* is (Trig-A_{n-1}-bry) with $C_0 = C_1 = C_2 = C_3 = 0$.

(Trig-BC_n-reg) *Trigonometric potential of type BC_n with regular boundary conditions* is (Trig-B_n) with $C_2 = C_3 = 0$.

(Toda-D_n-bry) *Toda potential of type D_n with boundary conditions* is (Toda-$B^{(1)}_n$-bry) with $C_0 = C_1 = 0$.

(Toda-$B^{(1)}_n$) *Toda potential of type $B^{(1)}_n$* is (Toda-$B^{(1)}_n$-bry) with $C_2 = C_3 = 0$.

(Toda-$D^{(1)}_n$) *Toda potential of type $D^{(1)}_n$* is (Toda-$D^{(1)}_n$-bry) with $C_0 = C_1 = C_2 = C_3 = 0$.

(Toda-A_{n-1}) *Toda potential of type A_{n-1}* is (Toda-$C^{(1)}_n$) with $C_0 = C_1 = C_2 = C_3 = 0$.

(Toda-BC_n) *Toda potential of type B_n* is (Toda-$C^{(1)}_n$) with $C_0 = C_1 = 0$.

(Ellip-D_n) *Elliptic potential of type D_n* is (Ellip-B_n) with $C_0 = C_1 = C_2 = C_3 = 0$.

(Trig-D_n) *Trigonometric potential of type D_n* is (Trig-B_n) with $C_0 = C_1 = C_2 = C_3 = 0$.

(Rat-D_n) *Rational potential of type* D_n is (Rat-B_n) with $C_0 = C_1 = C_2 = C_3 = 0$.

(Toda-D_n) *Toda potential of type* D_n is (Toda-$B_n^{(1)}$-bry) with $C_0 = C_1 = C_2 = C_3 = 0$.

(Rat-B_n-2) *Rational potential of type* B_n-2 is (Rat-B_n) with $C_2 = C_3 = 0$.

(Rat-A_{n-1}-bry2) *Rational potential of type* A_{n-1} *with* 2*-boundary conditions* is (Rat-A_{n-1}-bry) with $C_2 = C_3 = 0$. In this case, we may assume $C_0 = 0$ or $C_1 = 0$ by the transformation $x_k \mapsto x_k + c$ $(k = 1, \dots, n)$ with a suitable $c \in \mathbb{C}$.

Then we have the following diagrams for $n \geq 3$. Note that we don't write all the arrows in the diagrams (ex. (Toda-D_n-bry) $\to$ (Toda-BC_n)) and the meaning of the symbol $\overset{5:3}{\Rightarrow}$ is same as in the diagram for type B_2. Namely, 5 parameters (coupling constant) in the potential are reduced to 3 parameters by a certain restriction.

Hierarchy of Elliptic-Trigonometric-Rational Integrable Potentials

$$
\begin{array}{ccccccc}
 & & \text{Rat-}B_n\text{-2} & & \text{Ellip-}D_n & & \\
 & & \Uparrow_{5:3} & & \downarrow & & \\
 & & \text{Rat-}B_n & \nwarrow & \text{Trig-}D_n & \to & \text{Rat-}D_n \\
 & & \uparrow & & \Uparrow_{3:1} & & \\
\text{Ellip-}B_n & \to & \text{Trig-}B_n & \overset{5:3}{\Rightarrow} & \text{Trig-}BC_n\text{-reg} & & \text{Ellip-}A_{n-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Trig-}A_{n-1}\text{-bry} & \overset{5:3}{\Rightarrow} & \text{Trig-}A_{n-1}\text{-bry-reg} & \overset{3:1}{\Rightarrow} & \text{Trig-}A_{n-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Rat-}A_{n-1}\text{-bry} & \overset{5:3}{\Rightarrow} & \text{Rat-}A_{n-1}\text{-bry2} & \overset{3:1}{\Rightarrow} & \text{Rat-}A_{n-1}
\end{array}
$$

Hierarchy of Toda Integrable Potentials

$$
\begin{array}{ccccccc}
 & & \text{Trig-}BC_n\text{-reg} & \to & \text{Toda-}D_n\text{-bry} & \overset{3:1}{\Rightarrow} & \text{Toda-}D_n \\
 & & \Uparrow_{5:3} & & \Uparrow_{5:3} & & \Uparrow_{3:1} \\
 & & \text{Trig-}B_n & \to & \text{Toda-}B_n^{(1)}\text{-bry} & \overset{5:3}{\Rightarrow} & \text{Toda-}B_n^{(1)} \\
 & \nearrow & & \nearrow & & \searrow & \\
\text{Ellip-}B_n & \to & \text{Toda-}D_n^{(1)}\text{-bry} & \overset{5:1}{\Rightarrow} & \text{Toda-}D_n^{(1)} & & \text{Toda-}C_n^{(1)} \\
 & \Searrow^{5:1} & & \nearrow & & & \Downarrow_{5:3} \\
 & & \text{Ellip-}D_n & & \text{Trig-}A_{n-1} & & \text{Toda-}BC_n \\
 & & & \nearrow & & \searrow & \Downarrow_{3:1} \\
 & & \text{Ellip-}A_{n-1} & \to & \text{Toda-}A_{n-1}^{(1)} & \to & \text{Toda-}A_{n-1}
\end{array}
$$

6 Type D_n $(n \geq 3)$

Theorem 12 (Type D_n). *The Schrödinger operators* (Ellip-D_n)*,* (Trig-D_n)*,* (Rat-D_n)*,* (Toda-$D_n^{(1)}$) *and* (Toda-D_n) *are in the commutative algebra of differential operators generated by* $P_1, P_2, \dots, P_{n-1}$ *and* $\Delta_{\{1,\dots,n\}}$ *which are the corresponding operators for* (Ellip-B_n)*,* (Trig-B_n)*,* (Rat-B_n)*,* (Toda-$D_n^{(1)}$-bry)*,* (Toda-D_n-bry) *with* $C_0 = C_1 = C_2 = C_3 = 0$*, respectively.*

Proof. This theorem is proved by [29] in the cases (Ellip-D_n), (Trig-D_n), (Rat-D_n). Other two cases have been defined by suitable analytic continuation and therefore the claim is clear. ∎

Remark 10. In the above theorem we have $P_n = \Delta^2_{\{1,\dots,n\}}$ because $q^o_I = 0$ if $I \neq \varnothing$. Then $[P_j, P_n] = 0$ implies $[P_j, \Delta_{\{1,\dots,n\}}] = 0$.

Hierarchy of Integrable Potentials of Type D_n ($n \geq 3$)

$$\begin{array}{ccccc} & & \text{Toda-}D_n^{(1)} & \to & \text{Toda-}D_n \\ & \nearrow & & \nearrow & \\ \text{Ellip-}D_n & \to & \text{Trig-}D_n & \to & \text{Rat-}D_n \end{array}$$

7 Classical limits

For functions $f(\xi, x)$ and $g(\xi, x)$ of $(\xi, x) = (\xi_1, \dots, \xi_n, x_1, \dots, x_n)$, we define their Poisson bracket by

$$\{f, g\} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial \xi_k} \frac{\partial g}{\partial x_k} - \frac{\partial g}{\partial \xi_k} \frac{\partial f}{\partial x_k} \right).$$

Theorem 13. *Put*

$$\bar{P}(\xi, x) = -\frac{1}{2} \sum_{k=1}^{n} \xi_k^2 + R(x).$$

Then for the integrable potential function $R(x)$ given in this note, the functions $\bar{P}_k(\xi, x)$ and $\bar{\Delta}_{\{1,\dots,n\}}(\xi, x)$ of (ξ, x) defined by replacing ∂_ν by ξ_ν $(\nu = 1, \dots, n)$ in the definitions of P_k and $\Delta_{\{1,\dots,n\}}$ in Sections 3, 4 *and* 5 *satisfy*

$$\big\{\bar{P}_i(\xi, x), \bar{P}_j(\xi, x)\big\} = \big\{\bar{P}(\xi, x), \bar{P}_k(\xi, x)\big\} = 0 \qquad \textit{for} \quad 1 \leq i < j \leq n \quad \textit{and} \quad 1 \leq k \leq n.$$

Hence $\bar{P}(\xi, x)$ are Hamiltonians of completely integrable dynamical systems.

Moreover if the potential function $R(x)$ is of type D_n, then

$$\big\{\bar{\Delta}_{\{1,\dots,n\}}(\xi, x), \bar{P}_k(\xi, x)\big\} = \big\{\bar{\Delta}_{\{1,\dots,n\}}(\xi, x), \bar{P}(\xi, x)\big\} = 0 \qquad \textit{for} \qquad 1 \leq k \leq n.$$

Proof. If $R(x)$ is a potential function of (Ellip-A_{n-1}), (Ellip-B_n) or (Ellip-D_n), the claim is proved in [29, 32]. Since the claim keeps valid under suitable holomorphic continuations with respect to the parameters which are given in the former sections, we have the theorem. ■

Remark 11. Since our operators P_k are expressed by operators $P_k^\nu = \sum_i p^\nu_{k,i}(x) q^\nu_{k,i}(\partial)$ such that the polynomials $q^\nu_{k,i}(\partial)$ satisfy $[p^\nu_{k,i}(x), q^\nu_{k,i}(\partial)] = 0$, there is no ambiguity in the definition of the classical limits by replacing ∂_ν by ξ_ν. In another word, if we have given the above integrals $\bar{P}_j(x, \xi)$ of the classical limit, we have a natural unique quantization of them.

8 Analogue for one variable

Putting $n = 1$ for the Schrödinger operator P of type A_n in Section 3 or of type B_n in Section 5, we examine the ordinary differential equation $Pu = Cu$ with $C \in \mathbb{C}$ (cf. [41, § 10.6]). We will write the operators $Q = P - C$.

(Ellip-B_1) The Heun equation (cf. [32, § 8], [41, pp. 576])

$$-\frac{1}{2} \frac{d^2}{dt^2} + \sum_{j=0}^{3} C_j \wp(t + \omega_j) - C.$$

(Ellip-A_1) The Lamé equation

$$-\frac{1}{2}\frac{d^2}{dt^2} + A\wp(t) - C.$$

(Trig-BC_1-reg) The Gauss hypergeometric equation

$$-\frac{1}{2}\frac{d^2}{dt^2} + \frac{C_0}{\sinh^2 \lambda t} + \frac{C_1}{\sinh^2 2\lambda t} - C.$$

(Trig-A_1) The Legendre equation

$$-\frac{1}{2}\frac{d^2}{dt^2} + \frac{C_0}{\sinh^2 \lambda t} - C.$$

(Trig-B_1) with $C_0 = C_1 = C_3 = 0$. The (Modified) Mathieu equation

$$-\frac{1}{2}\frac{d^2}{dt^2} + C_2 \cosh 2\lambda t - C.$$

(Rat-B_1-2) Equation of the paraboloid of revolution

$$-\frac{1}{2}\frac{d^2}{dt^2} + \frac{C_0}{t^2} + C_1 t^2 - C.$$

This is the Weber equation if $C_0 = 0$. With $s = t^2$ and using the unknown function $t^{\frac{1}{2}}u$, the above equation is reduced to the Whittaker equation:

$$-\frac{1}{2}\frac{d^2}{ds^2} + \frac{C_0'}{s^2} + \frac{C_1'}{s} - C'.$$

(Rat-A_0-bry2) with $C_2 = C_3 = 0$:

$$-\frac{1}{2}\frac{d^2}{dt^2} + C_0 t + C_1 t^2 - C.$$

If $C_1 \neq 0$, this is transformed into the Weber equation under the coordinate $s = t + C_2/(2C_1)$. If $C_1 = 0$, this is the Stokes equation which is reduced to the Bessel equation. In particular, the Airy equation corresponds to $C = C_1 = 0$.
(Toda-BC_1)

$$-\frac{1}{2}\frac{d^2}{dt^2} + C_0 e^{-2t} + C_1 e^{-4t} - C,$$

which is transformed into (Rat-B_1-2) by putting $s = e^{-t}$. In particular
(Toda-A_1)

$$-\frac{1}{2}\frac{d^2}{dt^2} + C_0 e^{-2t} - C$$

is reduced to the Bessel equation.
(Rat-A_1) the Bessel equation

$$-\frac{1}{2}\frac{d^2}{dt^2} + \frac{C_0}{t^2} - C.$$

In fact, the equation $-u''/2 + C_0u/t^2 = Cu$ is equivalent to

$$\left(\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{2C_0 + 1/4}{t^2} + 2C\right)t^{-1/2}u = 0$$

since

$$t^{-\frac{1}{2}} \circ \frac{d^2}{dt^2} \circ t^{\frac{1}{2}} = \frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{1}{4t^2}.$$

Hence if $C \neq 0$, the function $v = t^{-1/2}u$ satisfies the following Bessel equation with $s = \sqrt{-2C}t$:

$$\frac{d^2v}{ds^2} + \frac{1}{s}\frac{dv}{ds} - \left(1 - \frac{C_0 + 1/8}{Cs^2}\right)v = 0.$$

Hierarchy of ordinary differential equations

$$\begin{array}{ccccccccc}
\text{Heun} & \overset{5:2}{\Rightarrow} & \text{Lamé} & \to & \text{Legendre} & & & & \\
 & \searrow & & & \Uparrow_{3:2} & \searrow & & & \\
\text{Mathieu} & \overset{2:5}{\Leftarrow} & \text{Trig-}B_1 & \overset{5:3}{\Rightarrow} & \text{Gauss} & & \text{Bessel (Stokes)} & \overset{2:1}{\to} & \text{Airy} \\
 & & \downarrow_{5:4} & \searrow & & \searrow & \Uparrow_{3:2} & & \uparrow_{2:1} \\
 & & \text{Rat-}A_0\text{-bry} & & \text{Rat-}B_1 & \overset{5:3}{\Rightarrow} & \text{Whittaker} & \overset{3:2}{\to} & \text{Weber}
\end{array}$$

9 A classification

We present a conjecture which characterizes the systems listed in this note.

Let P be the Schrödinger operator with the expression (1.1) and consider the condition

$$\text{there exist } P_1, \dots, P_n \text{ such that } \begin{cases} P \in \mathbb{C}[P_1, \dots, P_n], \\ [P_i, P_j] = 0 \qquad (1 \le i < j \le n), \\ \sigma(P_k) = \sum\limits_{1 \le j_1 < \cdots < j_k \le n} \xi_{j_1}^2 \cdots \xi_{j_k}^2 \qquad (1 \le k \le n). \end{cases} \tag{9.1}$$

Note that all the completely integrable systems given in Sections 3, 4 or 5 satisfy this condition.

Conjecture. *Suppose P satisfies* (9.1)*. Under a suitable affine transformation of the coordinate $x \in \mathbb{C}^n$ which keeps the algebra $\mathbb{C}\Big[\sum\limits_{k=1}^n \partial_k^2, \sum\limits_{k=1}^n \partial_k^4, \dots, \sum\limits_{k=1}^n \partial_k^{2n}\Big]$ invariant, P is transformed into an integrable Schrödinger operator studied in Sections* 3, 4 *or* 5, *(namely $u_{ij}^{\pm}$, v_k and w_k in* (1.3) *are suitable analytic continuations of the corresponding functions of the invariant elliptic systems) or in general a direct sum of such operators and/or trivial operators*

$$(A_1) \qquad \frac{d^2}{dx^2} + v(x)$$

with arbitrary functions $v(x)$ of one variable.

Here the direct sum of the two operators $P_j(x, \partial_x) = \sum\limits_{\alpha \in \{0,1,\dots\}^{n_j}} a_\alpha(x)\partial_x^\alpha$ of $x \in \mathbb{C}^{n_j}$ for $j = 1, 2$ means the operator $P_1(x, \partial_x) + P_2(y, \partial_y)$ of $(x, y) \in \mathbb{C}^{n_1+n_2}$.

We review known conditions assuring this conjecture and give another condition (cf. Theorem 19 and Remark 17). We also review related results on the classification of completely integrable quantum systems associated with classical root systems.

Remark 12. The condition

$$\text{there exists } P_2 \text{ such that } [P,P_2]=0 \text{ and } \sigma(P_2)=\sum_{1\le i<j\le n}\xi_i^2\xi_j^2 \tag{9.2}$$

may be sufficient to assure the claim of the conjecture.

Remark 13 (Type A_2). If $n=2$ and if there exists P_3 satisfying

$$\sigma(P_3)=\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1 \quad \text{and} \quad [P,P_3]=[\partial_1+\partial_2+\partial_3,P]=[\partial_1+\partial_2+\partial_3,P_3]=0,$$

then Conjecture is true.

In fact this case is reduced to solving the equation

$$\begin{vmatrix} u(x) & u'(x) & 1\\ v(y) & v'(y) & 1\\ w(z) & w'(z) & 1\end{vmatrix}=0 \qquad \text{for} \qquad x+y+z=0 \tag{9.3}$$

for three unknown functions $u(t)$, $v(t)$ and $w(t)$, which is solved by [3, 4]. Here $u(t)=u_{e_1-e_2}(t)$, $v(t)=u_{e_2-e_3}(t)$ and $w(t)=u_{e_1-e_3}(-t)$.

9.1 Pairwise interactions and meromorphy

Theorem 14 ([40]). *The potential function $R(x)$ of P satisfying* (9.1) *is of the form*

$$R(x)=\sum_{\alpha\in\Sigma(B_n)^+}u_\alpha(\langle\alpha,x\rangle) \tag{9.4}$$

with meromorphic functions $u_\alpha(t)$ of one variable.

Remark 14. i) The condition (9.2) assures

$$R(x)=\sum_{\alpha\in\Sigma(B_n)^+}u_\alpha(\langle\alpha,x\rangle)+\sum_{1\le i<j<k\le n}C_{ijk}x_ix_jx_k$$

with $C_{ijk}\in\mathbb{C}$ and thus the above theorem is proved in the invariant case (cf. Section 9.2) by [32] or in the case of Type B_2 by [23] or in the case of Type A_{n-1}. This theorem is proved in [40] by using $[P,P_2]=[P,P_3]=0$.

ii) Suppose $n=2$ and the operators

$$P=-\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)+R(x_1,x_2),$$
$$T=\sum_{i=0}^m c_j\frac{\partial^m}{\partial x_1^i\partial x_2^{m-i}}+\sum_{i+j\le m-2}T_{i,j}(x_1,x_2)\frac{\partial^{i+j}}{\partial x_1^i\partial x_2^j}$$

satisfy $[P,T]=0$ and $\sigma_m(T)\notin\mathbb{C}[\sigma(P)]$. Here $c_j\in\mathbb{C}$. Then [31, Theorem 8.1] shows that there exist functions $u_{\nu,i}(t)$ of one variable such that

$$R(x_1,x_2)=\sum_{\nu=1}^L\sum_{i=0}^{m_\nu-1}(b_\nu x_1+a_\nu x_2)^i u_{\nu,i}(a_\nu x_1-b_\nu x_2)$$

by putting

$$\left(\xi\frac{\partial}{\partial\tau}-\tau\frac{\partial}{\partial\xi}\right)\sum_{i=0}^m c_i\xi^{m-i}\tau^i=\prod_{\nu=1}^L(a_\nu\xi-b_\nu\tau)^{m_\nu}.$$

Here $(a_\nu,b_\nu)\in\mathbb{C}^2\setminus\{(0,0)\}$ and $a_\nu b_\mu\ne a_\mu b_\nu$ if $\mu\ne\nu$.

Definition 6. By the expression (9.4), put

$$S = \{\alpha \in \Sigma(B_n)^+ \, ; \, u'_\alpha \neq 0\}$$

and let $W(S)$ be the Weyl group generated by $\{w_\alpha \, ; \, \alpha \in S\}$ and moreover put $\bar{S} = W(S)S$.

Theorem 15 ([23] for Type B_2, [40] in general). *If the root system $\bar{S}$ has no irreducible component of rank one, then* (9.2) *assures that any function $u_\alpha(t)$ extends to a meromorphic function on $\mathbb{C}$.*

Remark 15 ([32, (6.4)–(6.5)], [40, § 3]). The condition (9.2) is equivalent to

$$S^{ij} = S^{ji} \qquad (1 \le i < j \le n) \tag{9.5}$$

with

$$\begin{aligned} S^{ij} = {} & \left(\partial_i^2 v_i(x_i) + \sum_{\nu \in I(i,j)} \partial_i^2 \big(u^+_{i\nu}(x_i + x_\nu) + u^-_{i\nu}(x_i - x_\nu)\big)\right)(u^+_{ij}(x_i + x_j) - u^-_{ij}(x_i - x_j)) \\ & + 3\left(\partial_i v_i(x_i) + \sum_{\nu \in I(i,j)} \partial_i \big(u^+_{i\nu}(x_i + x_\nu) + u^-_{i\nu}(x_i - x_\nu)\big)\right)(\partial_i u^+_{ij}(x_i + x_j) - \partial_i u^-_{ij}(x_i - x_j)) \\ & + 2\left(v_i(x_i) + \sum_{\nu \in I(i,j)} (u^+_{i\nu}(x_i + x_\nu) + u^-_{i\nu}(x_i - x_\nu))\right)(\partial_i^2 u^+_{ij}(x_i + x_j) - \partial_i^2 u^-_{ij}(x_i - x_j)) \\ & + \sum_{\nu \in I(i,j)} (\partial_i^2 u^+_{i\nu}(x_i + x_\nu) - \partial_i^2 u^-_{i\nu}(x_i - x_\nu))(u^+_{j\nu}(x_j + x_\nu) - u^-_{j\nu}(x_j - x_\nu)). \end{aligned}$$

Here $I(i,j) = \{1, 2, \ldots, n\} \setminus \{i, j\}$.

Lemma 5. *Suppose P satisfies* (9.2) *and* (9.4)*. Let S_0 be a subset of $\bar{S}$ such that*

$$S_0 \subset \sum_{i=1}^m \mathbb{R}e_i \qquad \text{and} \qquad \bar{S} \setminus S_0 \subset \sum_{i=m+1}^n \mathbb{R}e_i$$

with a suitable m. Then the Schrödinger operator

$$P' = -\frac{1}{2}\sum_{i=1}^m \partial_i^2 + \sum_{\alpha \in S_0 \cap S} u(\langle \alpha, x \rangle)$$

on $\mathbb{R}^m$ admits a differential operator P'_2 on $\mathbb{R}^m$ satisfying $[P', P'_2] = 0$ and $\sigma(P'_2) = \sum\limits_{1 \le i < j \le n}^{m} \xi_i^2 \xi_j^2$, that is, the condition (9.2) *with replacing P by P'.*

Proof. This lemma clearly follows from the equivalent condition (9.5) given in Remark 15. ∎

9.2 Invariant case

Theorem 16 ([24, 25, 29, 32]). *Assume that P in* (1.1) *is invariant under the Weyl group $W = W(A_{n-1})$, $W(B_n)$ or $W(D_n)$ with $n \ge 3$, or $W = W(B_2)$. If we have* (1.2) *with*

$$P_1 = \partial_1 + \partial_2 + \cdots + \partial_n \qquad \text{if} \qquad W = W(A_{n-1}),$$

$$\sigma(P_k) = \begin{cases} \displaystyle\sum_{1\le j_1<j_2<\cdots<j_k\le n} \xi_{j_1}\xi_{j_2}\cdots\xi_{j_k} & \text{if} \quad W = W(A_{n-1}) \quad \text{and} \quad 1 \le k \le n, \\ \displaystyle\sum_{1\le j_1<j_2<\cdots<j_k\le n} \xi_{j_1}^2\xi_{j_2}^2\cdots\xi_{j_k}^2 & \text{if} \quad W = W(B_n) \quad \text{and} \quad 1 \le k \le n, \\ \displaystyle\sum_{1\le j_1<j_2<\cdots<j_k\le n} \xi_{j_1}^2\xi_{j_2}^2\cdots\xi_{j_k}^2 & \text{if} \quad W = W(D_n) \quad \text{and} \quad 1 \le k < n, \end{cases}$$

$$\sigma(P_n) = \xi_1\xi_2\cdots\xi_n \qquad \text{if} \qquad W = W(D_n),$$

Conjecture is true.

Remark 16. The condition

$$[P, P_1] = [P, P_3] = 0 \qquad \text{if} \qquad W = W(A_{n-1}),$$
$$[P, P_2] = 0 \qquad \text{if} \qquad W = W(B_n) \qquad \text{or} \qquad W = W(D_n)$$

together with (9.4) is sufficient for the proof of this theorem.

9.3 Enough singularities

Put $\Xi = \{\alpha \in \Sigma(B_n)^+ \,;\, u_\alpha(t) \text{ is not entire}\}$.

Theorem 17. i) ([23]) *Suppose $n = 2$ and let $\bar{S}$ be of type B_2. If $\#\Xi \ge 2$, then Conjecture is true.*

ii) ([40]) *If $\bar{S}$ is of type A_{n-1} or of type B_n and moreover the reflections w_α for $\alpha \in \Xi$ generate $W(A_{n-1})$ or $W(B_n)$, respectively, then Conjecture is true.*

This theorem follows from the following key Lemma.

Lemma 6 ([23, 37, 40]). *Suppose* (9.1) *and moreover that there exist α and β in S such that $\alpha \ne \beta$, $\langle\alpha,\beta\rangle \ne 0$ and $u_\alpha(t)$ has a singularity at $t = t_0$. Then $u_\alpha(t - t_0)$ is an even function with a pole of order two at the origin and*

$$u_{w_\alpha(\beta)}\left(t - 2t_0\frac{\langle\alpha,\gamma\rangle}{\langle\alpha,\alpha\rangle}\right) = u_\beta(t) \qquad \text{if} \qquad w_\alpha(\beta) \in \Sigma(B_n)^+,$$
$$u_{-w_\alpha(\beta)}\left(-t + 2t_0\frac{\langle\alpha,\gamma\rangle}{\langle\alpha,\alpha\rangle}\right) = u_\beta(t) \qquad \text{if} \qquad -w_\alpha(\beta) \in \Sigma(B_n)^+. \tag{9.6}$$

Corollary 1. *Suppose the assumption in Lemma* 6.

i) *If $u_\alpha(t)$ has another singularity at $t_1 \ne t_0$, then*

$$u_\gamma\left(t + 2(t_1 - t_0)\frac{\langle\alpha,\gamma\rangle}{\langle\alpha,\alpha\rangle}\right) = u_\gamma(t) \qquad \text{for} \qquad \gamma \in S. \tag{9.7}$$

ii) *Assume that u_α has poles at 0, t_0 and t_1 such that t_0 and t_1 are linearly independent over $\mathbb{R}$. Then $u_\beta(t)$ is a doubly periodic function and therefore $u_\beta(t)$ has poles and hence $u_\alpha(t)$ is also a doubly periodic function. We may moreover assume that u_β has a pole at 0 by a parallel transformation of the variable x.*

Case I: *Suppose $\alpha = e_i - e_j$, $\beta = e_j - e_k$ with $1 \le i < j < k \le n$.*

$$u_{e_i-e_j}(t) = u_{e_j-e_k}(t) = u_{e_i-e_k}(t) = C\wp(t; 2\omega_1, 2\omega_2) + C'$$

with suitable $C, C' \in \mathbb{C}$, which corresponds to (Ellip-A_2).

Case II: *Suppose $\alpha = e_i - e_j$ and $\beta = e_j$ with $1 \le i < j \le n$.*

Then $\big(u_{e_i-e_j}(t), u_{e_i+e_j}(x), u_{e_i}(t), u_{e_j}(t)\big)$ is (Ellip-B_2), (Ellip-B_2-S) *or* (Ellip^d-B_2).

iii) If $\bar{S}$ is of type A_{n-1} or B_n or D_n and one of $u_\alpha(t)$ is a doubly periodic function with poles, then P transforms into (Ellip-A_{n-1}) or (Ellip-B_n) or (Ellip-D_n) under a suitable parallel transformation on $\mathbb{C}^n$.

Proof of Corollary 1. i) is a direct consequence of Lemma 6. iii) follows from ii). We have only to show ii).

Case I: It follows from (9.6) that $u_\alpha(t) = u_\beta(t) = u_{e_i-e_k}(t)$ and they are even functions. Let

$$\Gamma_{2\omega_1,2\omega_2} = \{2m_1\omega_1 + 2m_2\omega_2\,;\, m_1, m_2 \in \mathbb{Z}\}$$

be the set of poles of u_α. Then (9.7) implies $u_\beta(t+2\omega_1) = u_\beta(t+2\omega_2) = u_\beta(t)$. Since $2\omega_1$ and $2\omega_2$ are periods of $\wp(t)$ and there exists only one double pole in the fundamental domain defined by these periods, we have the claim.

Case II: It follows from (9.6) that $u_{e_i-e_j}(t) = u_{e_i+e_j}(t)$ and $u_{e_i}(t) = u_{e_k}(t)$ and they are even functions. Let $\Gamma_{2\omega_1,2\omega_2}$ be the poles of $u_{e_i-e_j}(t)$. Then (9.7) means $u_{e_i}(t+2\omega_1) = u_{e_i}(t+2\omega_2) = u_{e_i}(t)$. Considering the poles of $u_{e_i-e_k}(t)$ with (9.7), we have four possibilities of poles of u_{e_i}:

$$\begin{aligned}
&\text{(Case II-0): } \Gamma_{2\omega_1,2\omega_2},\\
&\text{(Case II-1): } \Gamma_{2\omega_1,2\omega_2} \bigcup(\omega_1 + \Gamma_{2\omega_1,2\omega_2}),\\
&\text{(Case II-2): } \Gamma_{2\omega_1,2\omega_2} \bigcup(\omega_2 + \Gamma_{2\omega_1,2\omega_2}),\\
&\text{(Case II-3): } \Gamma_{2\omega_1,2\omega_2} \bigcup(\omega_1 + \Gamma_{2\omega_1,2\omega_2}) \bigcup(\omega_2 + \Gamma_{2\omega_1,2\omega_2}).
\end{aligned}$$

Here we note that (Case II-1) changes into (Case II-2) if we exchange ω_1 and ω_2. Then we have

$$\begin{aligned}
&\text{(Case II-0): } u_{e_i-e_j}(t+4\omega_1) = u_{e_i-e_j}(t+4\omega_2) = u(t),\\
&\text{(Case II-2): } u_{e_i-e_j}(t+4\omega_1) = u_{e_i-e_j}(t+2\omega_2) = u(t),\\
&\text{(Case II-3): } u_{e_i-e_j}(t+2\omega_1) = u_{e_i-e_j}(t+2\omega_2) = u(t).
\end{aligned}$$

Thus (Case II-0), (Case II-2) and (Case II-3) are reduced to (Ellip^d-B_2), (Ellip-B_2-S) and (Ellip-B_2), respectively. ■

Let $\mathcal{H}$ be a finite set of mutually non-parallel vectors in $\mathbb{R}^n$ and suppose

$$P = -\frac{1}{2}\sum_{j=1}^{n} \partial_j^2 + R(x), \qquad R(x) = \sum_{\alpha\in\mathcal{H}} C_\alpha \frac{\langle\alpha,\alpha\rangle}{\langle\alpha,x\rangle^2} + \tilde{R}(x).$$

Here C_α are nonzero complex numbers and $\tilde{R}(x)$ is real analytic at the origin. We assume that $\mathcal{H}$ is *irreducible*, namely,

$$\mathbb{R}^n = \sum_{\alpha\in\mathcal{H}} \mathbb{R}\alpha,$$

$$\varnothing \neq \forall\,\mathcal{H}' \subsetneq \mathcal{H} \Rightarrow \exists\,\alpha \in \mathcal{H}' \qquad \text{and} \qquad \exists\,\beta \in \mathcal{H}\setminus\mathcal{H}' \qquad \text{with} \qquad \langle\alpha,\beta\rangle \neq 0.$$

Definition 7. The potential function $R(x)$ of a Schrödinger operator is *reducible* if $R(x)$ and $\mathbb{R}^n$ is decomposed as $R(x) = R_1(x) + R_2(x)$ and $\mathbb{R}^n = V_1 \oplus V_2$ such that

$$0 \subsetneq V_1 \subsetneq \mathbb{R}^n, \quad V_2 = V_1^\perp, \quad \partial_{v_2} R_1(x) = \partial_{v_1} R_2(x) = 0 \quad \text{for} \quad \forall\, v_2 \in V_2 \quad \text{and} \quad \forall\, v_1 \in V_1.$$

If $R(x)$ is not reducible, $R(x)$ is called to be irreducible.

Theorem 18 ([37]). *Suppose $n \geq 2$ and there exists a differential operator Q with $[P,Q] = 0$ whose principal symbol does not depend on x and is not a polynomial of $\sum_{i=0}^{n} \xi_i^2$. Put $W = \{w_\alpha; \alpha \in \mathcal{H}\}$. If*

$$2C_\alpha \neq k(k+1) \qquad \textit{for} \qquad k \in \mathbb{Z} \qquad \textit{and} \qquad \alpha \in \mathcal{H}, \tag{9.8}$$

then W is a finite reflection group and $\sigma(Q)$ is W-invariant.

9.4 Periodic potentials

The following theorem is a generalization of the result in [30].

Theorem 19. *Assume $R(x)$ is of the form* (9.4) *with meromorphic functions $u_\alpha(t)$ on $\mathbb{C}$ and*

$$R\biggl(x + \frac{2\pi\sqrt{-1}\alpha}{\langle\alpha,\alpha\rangle}\biggr) = R(x) \qquad \textit{for} \qquad \alpha \in \Sigma(B_n) \tag{9.9}$$

and moreover assume that

> *the root system $\bar{S}$ does not contain an irreducible component of type B_2 or even if $\bar{S}$ contains an irreducible component $\bar{S}_2 = \{\pm e_i \pm e_j, \pm e_i, \pm e_j\}$ of type B_2, the origin $s = 0$ is not an isolated essential singularity of $u_\alpha(\log s)$ for $\alpha \in \bar{S}_2 \bigcap \Sigma(D_n)^+$.*

Then Conjecture is true.

Remark 17. i) The integrable systems classified in this note which satisfy the assumption of Theorem 19 under a suitable coordinate system are (Ellip-$*$) and (Trig-$*$) and (Toda-$*$), which are the systems given in this note whose potential functions are not rational.

ii) The assumption (9.9) implies that $u_\alpha(\log s)$ is a meromorphic function on $\mathbb{C} \setminus \{0\}$ for any $\alpha \in \Sigma(B_n)^+$. It means that the corresponding Schrödinger operator is naturally defined on the Cartan subgroup of $Sp(n,\mathbb{C})$ with a meromorphic potential function.

Lemma 7. *Assume $n = 2$,* (9.2), (9.9), *$\bar{S}$ is of type B_2 and moreover $u_\alpha(\log s)$ are holomorphic for $\alpha \in \Sigma(B_2)^+$ and $0 < |s| \ll 1$. If the origin is at most a pole of $u_\beta(\log s)$ for $\beta \in \Sigma(D_2)^+$, the origin is also at most a pole of $u_\alpha(\log s)$ for $\alpha \in \Sigma(B_2)^+$.*

Proof. Use the notation as in (4.2). Put

$$u^-(\log s) = U_0^- + \sum_{\nu=r}^{\infty} \nu U_\nu^- s^\nu, \qquad u^+(\log s) = U_0^+ + \sum_{\nu=m}^{\infty} \nu U_\nu^+ s^\nu,$$
$$v(\log s) = V_0 + \sum_{\nu=-\infty}^{\infty} \nu V_\nu s^\nu, \qquad w(\log s) = W_0 + \sum_{\nu=-\infty}^{\infty} \nu W_\nu s^\nu.$$

with U_ν^-, U_ν^+, V_ν, $W_\nu \in \mathbb{C}$, $rm \neq 0$ and $(U_r^-, U_m^+) \neq 0$. Then as is shown in [30] the condition for the existence of $T(x,y)$ in (4.3) is equivalent to

$$pq(2p-q)(p-q)(V_{2p-q}U_{q-p}^+ + V_q U_{p-q}^- + W_{q-2p}U_p^+ - W_q U_p^-) = 0 \qquad \text{for} \quad p,q \in \mathbb{Z}. \tag{9.10}$$

Hence if $p < r$ and $p < m$,

$$p(p-k)(p+k)k(V_{p+k}U_{-k}^+ + V_{p-k}U_k^-) = 0 \qquad \text{for} \qquad k \in \mathbb{Z}.$$

Case $U_r^- \neq 0$: Put $k = r$. Suppose q is negative with a sufficiently large absolute value. Then $V_q = (-U_{-r}^+/U_r^-)V_{q+2r}$, which implies $V_q = 0$ since $\sum_{\nu=-\infty}^{\infty} \nu V_\nu s^\nu$ converges for $0 < |s| \ll 1$.

Suppose q is negative with a sufficiently large absolute value compared to p. Then by the relation $W_{q-2p}U_p^+ - W_q U_p^- = 0$ we similarly conclude $W_q = 0$.

Case $U_m^+ \neq 0$: Putting $k = -m$, we have the same conclusion as above in the same way. ■

Proof of Theorem 19. Lemma 5 assures that we may assume $\bar{S}$ is an irreducible root system. We may moreover assume that the rank of $\bar{S}$ is greater than one.

Suppose that there exists $\gamma \in S$ such that the origin is neither a removable singularity nor an isolated singularity of $u_\gamma(\log s)$. Then $u_\gamma(t)$ is a doubly periodic function with poles. Owing to Corollary 1, $\sum\limits_{\alpha\in S_0} u_\alpha(\langle\alpha, x\rangle)$ is reduced to the potential function of (Ellip-A_{n-1}) or (Ellip-B_n) or (Ellip-D_n).

Thus we may assume that the origin is a removable singularity or an isolated singularity of $u_\alpha(\log s)$ for any $\alpha \in S$.

Let $\alpha, \beta \in S \bigcap \Sigma(D_n)$ with $\alpha \neq \beta$ and $\langle\alpha, \beta\rangle \neq 0$. Put $\gamma = w_\alpha\beta$ or $\gamma = -w_\alpha\beta$ so that $\gamma \in \Sigma(D_n)^+$. Then [30] shows that $u(t) = u_\alpha(t)$, $v(t) = u_\beta(t)$ and $w(t) = u_\gamma(-t)$ satisfy (9.3). Then Remark 13 says that the origin is at most a pole of $u(\log s)$, $v(\log s)$ and $w(\log s)$.

Let $\alpha \in S \bigcap \Sigma(D_n)$ and $\beta \in S \setminus \Sigma(D_n)$ with $\langle\alpha, \beta\rangle \neq 0$. Let W be the reflection group generated by w_α and w_β and put $S^o = W\{\alpha, \beta\} \bigcap \Sigma(B_n)$. Then [30] also shows that

$$R(x) = \sum_{\gamma\in S^o} u_\gamma(\langle\gamma, x\rangle)$$

defines an integrable potential function of type B_2. Hence Lemma 7 assures that the origin is at most a pole of $u_\alpha(\log s)$ for $\alpha \in S^o$.

Since S is irreducible, the origin is at most a pole of $u_\alpha(\log s)$ for $\alpha \in S$. Then Theorem 19 follows from [30]. ■

9.5 Uniqueness

We give some remarks on the operator which commutes with the Schrödinger operator P.

Remark 18 ([32, Lemma 3.1 ii)]). If differential operators Q and Q' satisfy $[Q, Q'] = 0$, $\sigma(Q') = \sum\limits_{j=1}^{n} \xi_j^N$ and $\mathrm{ord}(Q) \leq N - 2$, then Q has a constant principal symbol, that is, $\sigma(Q)$ does not depend on x.

Hence if there exist differential operators $Q_1, \ldots, Q_n$ with constant principal symbols such that $\sigma(Q_1), \ldots, \sigma(Q_n)$ are algebraically independent and moreover they satisfy $[Q_i, Q_j] = 0$ for $1 \leq i < j \leq n$, then any operator Q satisfying $[Q, Q_j] = 0$ for $j = 1, \ldots, n$ has a constant principal symbol. In particular, if a differential operator Q satisfies $[Q, P_k] = 0$ for P_k in (1.2) and (1.4) with $k = 1, \ldots, n$, then $\sigma(Q)$ does not depend on x.

Remark 19. Assume that a differential operator Q commutes with a Schrödinger operator P and moreover assume that there exist linearly independent vectors $c_j \in \mathbb{C}^n$ for $j = 1, \ldots, n$ such that the operator is invariant under the parallel transformations $x \mapsto x + c_j$ for $j = 1, \ldots, n$. Then $\sigma(Q)$ does not depend on x (cf. [32, Lemma 3.1 i)]).

Furthermore assume that P is of type (Ellip-F) or (Trig-F) or (Rat-F) with $F = A_{n-1}$ or B_n or D_n. If the condition (9.8) holds or Q is $W(F)$-invariant, it follows from Theorem 18 or [29, Proposition 3.6] that Q is in the ring $\mathbb{C}[P_1, \ldots, P_n]$ generated by the $W(F)$-invariant commuting differential operators. If the condition (8.11) is not valid, $\sigma(Q)$ is not necessarily $W(F)$-invariant (cf. [7, 35, 39]).

Remark 20 ([32, Theorem 3.2]). Let P be the Schrödinger operator in Theorem 16. Under the notation in Theorem 16 suppose P_k are W-invariant for $1 \leq k \leq n$. Then the ring $\mathbb{C}[P_1, \ldots, P_n]$ is uniquely determined by P and Q, where $Q = P_3$ if $W = W(A_{n-1})$ and $Q = P_2$ if $W = W(B_n)$ or $W(D_n)$.

Remark 21. If $P_c = -(1/2)\sum_{j=1}^{n}\partial_j^2 + cR(x)$ is a Schrödinger operator with a coupling constant $c \in \mathbb{C}$ such that P_c admits a non-trivial commuting differential operator Q_c of order four for any $c \in \mathbb{C}$, then the operator P_c may be a system stated in Conjecture under a suitable coordinate system.

The following example satisfies neither this condition nor the condition (9.8). It does not admit commuting differential operators (1.2) satisfying (1.4) if $m \neq 0, -1$.

Example 2. It is shown in [6, 35] that the Schrödinger operator

$$P = -\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + \sum_{1\le i<j<n}\frac{m(m+1)}{(x_i-x_j)^2} + \sum_{i=1}^{n-1}\frac{m+1}{(x_i-\sqrt{m}x_n)^2}$$

is completely integrable for any m and *algebraically integrable* if m is an integer.

The following example shows that the Schrödinger operator P does not necessarily determine the commuting system $\mathbb{C}[P_1, \dots, P_n]$.

Example 3. Let α, β, γ and λ be complex numbers. Put $(A_0, A_1, C_0, C_1) = (\alpha, \gamma/2-\lambda/2, \beta, \lambda)$ for (Rat-B_2-S) in Theorem 3 (cf. [32, Remark 3.7]). Then the Schrödinger operator

$$P_{\alpha,\beta,\gamma} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + (x^2+y^2)\left(\frac{2\alpha}{(x^2-y^2)^2} + \frac{\beta}{x^2y^2} + \gamma\right)$$

commutes with

$$\begin{aligned} Q_{\alpha,\beta,\gamma,\lambda} = {} & \left(\frac{\partial^2}{\partial x\partial y} + \frac{4\alpha xy}{(x^2-y^2)^2} - 2(\gamma-\lambda)xy\right)^2 - 2\left(\frac{\beta}{y^2} + \lambda y^2\right)\frac{\partial^2}{\partial x^2} - 2\left(\frac{\beta}{x^2} + \lambda x^2\right)\frac{\partial^2}{\partial y^2} \\ & + 4\left(\frac{\beta}{x^2} + \lambda x^2\right)\left(\frac{\beta}{y^2} + \lambda y^2\right) + \frac{16\alpha\lambda x^2y^2 + 16\alpha\beta}{(x^2-y^2)^2} + 8\lambda(\gamma-\lambda)x^2y^2 \end{aligned}$$

for any $\lambda \in \mathbb{C}$. Note that $[Q_{\alpha,\beta,\gamma,\lambda}, Q_{\alpha,\beta,\gamma,\lambda'}] \neq 0$ if $\lambda \neq \lambda'$ and these operators are $W(B_2)$-invariant. The half of the coefficient of the term λ of $Q_{\alpha,\beta,\gamma,\lambda}$ considered as a polynomial function of λ is

$$S_{\alpha,\beta,\gamma} = -\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)^2 + 2\alpha\left(\frac{xy}{(x-y)^2} - \frac{xy}{(x+y)^2}\right) + 2\beta\left(\frac{y^2}{x^2} + \frac{x^2}{y^2}\right) + 4\gamma x^2y^2.$$

In particular, $P = -(1/2)(\partial_x^2 + \partial_y^2) + \gamma(x^2+y^2)$ commutes with $\partial_x\partial_y - 2\gamma xy$ and $x\partial_y - y\partial_x$.

Note that if $R(x)$ is a polynomial function on $\mathbb{C}^n$, the condition $\left[-(1/2)\sum_{j=1}^{n}\partial_j^2 + R(x), Q\right] = 0$ for a differential operator Q implies that the coefficients of Q are polynomial functions (cf. [32, Lemma 3.4]).

9.6 Regular singularities

Definition 8 ([16]). Put $\vartheta_k = t_k\partial/\partial t_k$ and $Y_k = \{t = (t_1, \dots, t_n) \in \mathbb{C}^n\,;\, t_k = 0\}$. Then a differential operator Q of the variable t is said to have *regular singularities* along the set of walls $\{Y_1, \dots, Y_n\}$ if

$$Q = q(\vartheta_1, \dots, \vartheta_n) + \sum_{k=1}^{n} t_kQ_k(t, \vartheta).$$

Here q is a polynomial of n variables and Q_k are differential operators with the form

$$Q_k(t,\vartheta) = \sum a_\alpha(t)\vartheta_1^{\alpha_1}\cdots\vartheta_n^{\alpha_n}$$

and $a_\alpha(t)$ are analytic at $t=0$. In this case we define

$$\sigma_*(Q) = q(\xi_1,\dots,\xi_n)$$

and $\sigma_*(Q)$ is called the *indicial polynomial* of Q.

Theorem 20. *Let $R(t)$ be a holomorphic function defined on a neighborhood of the origin of $\mathbb{C}^n$. Let Q_1 and Q_2 be differential operators of t which have regular singularities along the set of walls $\{Y_1,\dots,Y_n\}$. Suppose $\sigma_*(Q_1)=\sigma_*(Q_2)$ and $[Q_1,P]=[Q_2,P]=0$ with the Schrödinger operator*

$$P = -\frac{1}{2}\left(\vartheta_n^2 + \sum_{j=1}^{n-1}\bigl(\vartheta_{j+1}-\vartheta_j\bigr)^2\right) + R(t).$$

Then $Q_1=Q_2$.

Proof. Put $t_j = e^{-(x_j-x_{j+1})}$ for $j=1,\dots,n-1$ and $t_n = e^{-x_n}$. Then $\partial_j = \vartheta_{j+1}-\vartheta_j$ for $j=1,\dots,n-1$ and $\partial_n = -\vartheta_n$. Under the coordinate system $x=(x_1,\dots,x_n)$ Remark 19 says that Q_1-Q_2 has a constant principal symbol, which implies $Q_1=Q_2$ because $\sigma_*(Q_1-Q_2)=0$. ■

A more general result than this theorem is given in [31]. The following corollary is a direct consequence of this theorem.

Corollary 2. *Put $t_j = e^{-\lambda(x_j-x_{j+1})}$ for $j=1,\dots,n-1$ and $t_n=e^{-\lambda x_n}$. Suppose P is the Schrödinger operator of type* (Trig-A_{n-1}), (Trig-A_{n-1}-bry-reg), (Trig-BC_n-reg), (Trig-D_n), (Toda-A_{n-1}), (Toda-BC_n) *or* (Toda-D_n).

i) *P and P_k for $k=1,\dots,n$ have regular singularities along the set of walls $\{Y_1,\dots,Y_n\}$.*

ii) *Let Q be a differential operator which has regular singularities along the set of walls $\{Y_1,\dots,Y_n\}$ and satisfies $[Q,P]=0$. If $\sigma_*(Q)=\sigma_*(\tilde{Q})$ for an operator $\tilde{Q}\in\mathbb{C}[P_1,\dots,P_n]$, then $Q=\tilde{Q}$.*

Remark 22. i) This corollary assures that certain radial parts of invariant differential operators on a symmetric space correspond to our completely integrable systems with regular singularities and the map σ_* corresponds to the Harish-Chandra isomorphism (cf. [31]).

ii) The system (Trig-BC_n-reg) is Heckman–Opdam's hypergeometric system [11] of type BC_n. Since (Trig-BC_n-reg) is a generalization of Gauss hypergeometric system related to the root system $\Sigma(B_n)$, the systems in the following diagram are considered to be generalizations of Gauss hypergeometric system and its limits (cf. Section 8). They form a class whose eigenfunctions should be easier to be analyzed than those of other systems in this note.

Hierarchy starting from (Trig-BC_n-reg)

$$\begin{array}{ccccc}
 & & & & \text{Toda-}A_{n-1} \\
 & & & \nearrow & \\
\text{Rat-}D_n & & \text{Trig-}A_{n-1} & \rightarrow & \text{Rat-}A_{n-1} \\
\Uparrow 3{:}1 & & \Uparrow 3{:}1 & & \Uparrow 3{:}1 \\
\text{Rat-}B_n\text{-}2 & & \text{Trig-}A_{n-1}\text{-bry-reg} & \rightarrow & \text{Rat-}A_{n-1}\text{-bry2} \\
\uparrow & \nearrow & & & \\
\text{Trig-}BC_n\text{-reg} & \rightarrow & \text{Toda-}D_n\text{-bry} & \rightarrow & \text{Toda-}BC_n \\
\Downarrow 3{:}1 & & \Downarrow 3{:}1 & & \Downarrow 3{:}1 \\
\text{Trig-}D_n & \rightarrow & \text{Toda-}D_n & \rightarrow & \text{Toda-}A_{n-1} \\
\downarrow & \searrow & & & \\
\text{Trig-}A_{n-1} & & \text{Rat-}D_n & &
\end{array}$$

9.7 Other forms

If a Schrödinger operator P is in the commutative algebra $\mathbb{D} = \mathbb{C}[P_1, \ldots, P_n]$, then the differential operator $\tilde{P} := \psi(x)^{-1} P \circ \psi(x)$ with a function $\psi(x)$ is in the commutative algebra $\tilde{\mathbb{D}} = \mathbb{C}[\psi(x)^{-1} P_1 \circ \psi(x), \ldots, \psi(x)^{-1} P_n \circ \psi(x)]$ of differential operators. Then

$$\tilde{P} = -\frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j} + \tilde{R}(x), \tag{9.11}$$

$$\frac{\partial \psi(x)}{\partial x_j} = a_j(x) \qquad \text{for} \qquad j = 1, \ldots, n. \tag{9.12}$$

Conversely, if a function $\psi(x)$ satisfies (9.12) for a differential operator $\tilde{P}$ of the form (9.11), then $P = \psi(x) \tilde{P} \circ \psi(x)^{-1}$ is of the form (1.1), which we have studied in this note.

If $\psi(x)$ is a function satisfying

$$\frac{1}{2\psi(x)} \sum_{j=1}^{n} \frac{\partial^2 \psi}{\partial x_j^2}(x) = R(x),$$

then

$$\tilde{P} = \psi(x)^{-1} \left(-\frac{1}{2} \sum_{j=1}^{n} \partial_j^2 + R(x) \right) \circ \psi(x) = -\frac{1}{2} \sum_{j=1}^{n} \partial_j^2 - \psi(x)^{-1} \sum_{j=1}^{n} \frac{\partial \psi}{\partial x_j}(x) \partial_j.$$

Note that

$$e^{-\phi(x)} \frac{\partial e^{\phi(x)}}{\partial x_j} = \frac{\partial \phi(x)}{\partial x_j}, \qquad e^{-\phi(x)} \sum_{j=1}^{n} \frac{\partial^2 e^{\phi(x)}}{\partial x_j^2} = \sum_{j=1}^{n} \frac{\partial^2 \phi(x)}{\partial x_j^2} + \sum_{j=1}^{n} \left(\frac{\partial \phi(x)}{\partial x_j} \right)^2.$$

Putting

$$\phi(x) = m \sum_{1 \le i < j \le n} \log \sinh \lambda (x_i - x_j),$$

we have

$$\begin{aligned}
\frac{\partial \phi(x)}{\partial x_k} &= \lambda m \sum_{1 \le i \le n,\, i \ne k} \coth \lambda (x_k - x_i), \\
\sum_{j=1}^{n} \frac{\partial^2 \phi(x)}{\partial x_j^2} + \sum_{k=1}^{n} \Big(\frac{\partial \phi(x)}{\partial x_k} \Big)^2 &= -2\lambda^2 m \sum_{1 \le i < j \le n} \sinh^{-2} \lambda (x_i - x_j) \\
&\quad + 2\lambda^2 m^2 \sum_{1 \le i < j \le n} \coth^2 \lambda (x_i - x_j) + \lambda^2 m^2 \frac{n(n-1)(n-2)}{3} \\
&= 2\lambda^2 m(m-1) \sum_{1 \le i < j \le n} \sinh^{-2} \lambda (x_i - x_j) + \lambda^2 m^2 \frac{n(n^2-1)}{3}
\end{aligned}$$

since

$$\coth \alpha \cdot \coth \beta + \coth \beta \cdot \coth \gamma + \coth \gamma \cdot \coth \alpha = -1 \qquad \text{if} \qquad \alpha + \beta + \gamma = 0.$$

Hence

$$\tilde{P} = -\frac{1}{2} \sum_{j=1}^{n} \partial_j^2 - m \sum_{1 \le i < j \le n} \lambda \coth \lambda (x_i - x_j)(\partial_i - \partial_j),$$

$$\psi(x) = \prod_{1\le i<j\le n} \lambda^m \sinh^m \lambda(x_i - x_j),$$

$$\psi(x)\circ\tilde{P}\circ\psi^{-1}(x) = -\frac{1}{2}\sum_{j=1}^{n}\partial_j^2 + \sum_{1\le i<j\le n}\frac{m(m-1)\lambda^2}{\sinh^2\lambda(x_i-x_j)} + \frac{m^2 n(n^2-1)\lambda^2}{6} \tag{9.13}$$

and $\tilde{P}$ is transformed into the Schrödinger operator of type (Trig-A_{n-1}).

Now we put

$$\phi(x) = m_0\sum_{1\le i<j\le n}(\log\sinh\lambda(x_i-x_j) + \log\sinh\lambda(x_i+x_j)) + m_1\sum_{1\le k\le n}\log\sinh\lambda x_k$$
$$+ m_2\sum_{1\le k\le n}\log\sinh 2\lambda x_k$$

and we have

$$\frac{\partial\phi(x)}{\partial x_k} = \lambda m_0\sum_{1\le i\le n,\, i\ne k}(\coth\lambda(x_k+x_i)+\coth\lambda(x_k-x_i)) + \lambda m_1\coth\lambda x_k + 2\lambda m_2\coth 2\lambda x_k,$$

$$\coth\lambda x_k\coth 2\lambda x_k = 1 + \frac{1}{2}\sinh^{-2}\lambda x_k,$$

$$\sum_{\{i,j,k\}=I}(2\coth\lambda(x_k+x_i)\coth\lambda(x_k-x_i) + 2\coth\lambda(x_k+x_i)\coth\lambda(x_k-x_j)$$
$$+ \coth\lambda(x_k+x_i)\coth\lambda(x_k+x_j) + \coth\lambda(x_k-x_i)\coth\lambda(x_k-x_j))$$
$$= \sum_{\{i,j,k\}=I}(\coth\lambda(x_k+x_i)\coth\lambda(x_k+x_j) + \coth\lambda(x_i-x_j)\coth\lambda(x_i+x_k)$$
$$+ \coth\lambda(x_j-x_i)\coth\lambda(x_j+x_k)) + \sum_{\{i,j,k\}=I}(\coth\lambda(x_k-x_i)\coth\lambda(x_k-x_j)) = 8$$

$$\text{for}\quad I\subset\{1,\dots,n\}\quad\text{with}\quad \#I=3,$$

$$\coth\lambda(x_k+x_i)+\coth\lambda(x_k-x_i) = \frac{\sinh 2\lambda x_k}{\sinh\lambda(x_k+x_i)\sinh\lambda(x_k-x_i)},$$

$$\frac{\cosh 2\lambda x_k - \cosh 2\lambda x_i}{\sinh\lambda(x_k+x_i)\sinh\lambda(x_k-x_i)} = \frac{2\cosh^2\lambda x_k - 2\cosh^2\lambda x_i}{\sinh\lambda(x_k+x_i)\sinh\lambda(x_k-x_i)} = 2,$$

$$\sum_{k=1}^{n}\left(\frac{\partial\phi(x)}{\partial x_k}\right)^2 = 2\lambda^2 m_0^2\sum_{1\le i<j\le n}(\coth^2\lambda(x_i-x_j)+\coth^2\lambda(x_i+x_j))$$
$$+ \lambda^2 m_1^2\sum_{k=1}^{n}\coth^2\lambda x_k + 4\lambda^2 m_2^2\sum_{k=1}^{n}\coth^2 2\lambda x_k + 2\lambda^2 m_1 m_2\sum_{k=1}^{n}\sinh^{-2}\lambda x_k$$
$$+ \frac{4\lambda^2 m_0^2 n(n-1)(n-2)}{3} + 2\lambda^2 m_0(m_1+2m_2)n(n-1) + 4\lambda^2 m_1 m_2 n,$$

$$\sum_{j=1}^{n}\frac{\partial^2\phi(x)}{\partial x_j^2} + \sum_{k=1}^{n}\left(\frac{\partial\phi(x)}{\partial x_k}\right)^2 = 2\lambda^2 m_0(m_0-1)\sum_{1\le i<j\le n}(\sinh^{-2}\lambda(x_i-x_j)+\sinh^{-2}\lambda(x_i+x_j))$$
$$+ \lambda^2 m_1(m_1+2m_2-1)\sum_{k=1}^{n}\sinh^{-2}\lambda x_k + 4\lambda^2 m_2(m_2-1)\sum_{k=1}^{n}\sinh^{-2} 2\lambda x_k$$
$$+ \lambda^2\left(\left(\frac{2}{3}m_0(2n-1) + 2m_1 + 4m_2\right)m_0(n-1) + (m_1+2m_2)^2\right)n.$$

Hence

$$\tilde{P} = -\frac{1}{2}\sum_{j=1}^{n}\partial_j^2 - \sum_{k=1}^{n}\lambda\biggl(\sum_{1\le i<j\le n} m_0(\coth\lambda(x_i-x_k)+\coth\lambda(x_i+x_k))$$
$$+ m_1\coth\lambda x_k + 2m_2\coth 2\lambda x_k\biggr)\partial_k,$$
$$\psi(x) = \prod_{1\le i<j\le n}(\sinh^{m_0}\lambda(x_i-x_j)\sinh^{m_0}\lambda(x_i+x_j))\prod_{k=1}^{n}\sinh^{m_1}\lambda x_k\prod_{k=1}^{n}\sinh^{m_2}2\lambda x_k \quad (9.14)$$

and $\tilde{P}$ is transformed into the Schrödinger operator of type (Trig-BC_n-reg):

$$\psi(x)\circ\tilde{P}\circ\psi^{-1}(x) = -\frac{1}{2}\sum_{j=1}^{n}\partial_j^2 + m_0(m_0-1)\sum_{1\le i<j\le n}\biggl(\frac{\lambda^2}{\sinh^2\lambda(x_i-x_j)}+\frac{\lambda^2}{\sinh^2\lambda(x_i+x_j)}\biggr)$$
$$+\sum_{k=1}^{n}\frac{m_1(m_1+2m_2-1)\lambda^2}{2\sinh^2\lambda x_k}+\sum_{k=1}^{n}\frac{2m_2(m_2-1)\lambda^2}{\sinh^2 2\lambda x_k}$$
$$+\lambda^2\biggl(\frac{m_0^2}{3}(2n-1)(n-1)+m_0(m_1+2m_2)(n-1)+\frac{(m_1+2m_2)^2}{2}\biggr)n.$$

Remark 23. As is shown in [15, Theorem 5.24 in Ch. II], the operator (9.13) or (9.14) gives the radial part of the differential equation satisfied by the zonal spherical function of a Riemannian symmetric space G/K of the non-compact type which corresponds to the Laplace–Beltrami operator on G/K. Here G is a real connected semisimple Lie group with a finite center, K is a maximal compact subgroup of G and the numbers $2m$, $2m_0$, $2m_1$ and $2m_2$ correspond to the multiplicities of the roots of the restricted root system for G.

Similarly the following operator $\tilde{P}$ is used to characterize the K-fixed Whittaker vector v on $G = GL(n,\mathbb{R})$

$$\tilde{P} = -\frac{1}{2}\sum_{j=1}^{n}\partial_j^2 + \sum_{j=1}^{n}\biggl(\frac{n+1}{4}-\frac{j}{2}\biggr)\partial_j + C\sum_{j=1}^{n-1}e^{2(x_j-x_{j+1})}, \qquad \psi(x) = e^{\sum_{j=1}^{n}(j/2-n+1/4)x_j},$$
$$\psi(x)\circ\tilde{P}\circ\psi^{-1}(x) = -\frac{1}{2}\sum_{j=1}^{n}\partial_j^2 + C\sum_{j=1}^{n-1}e^{2(x_j-x_{j+1})}+\frac{n(n^2-1)}{48}.$$

Namely v is a simultaneous eigenfunction of the invariant differential operators on G/K and satisfies $v(nx) = \chi(n)v(x)$ with $n\in N$ and $x\in G/K$. Here $G = KAN$ is an Iwasawa decomposition of G and χ is a nonsingular character of the nilpotent Lie group N. Then $v|_A$ is a simultaneous eigenfunction of the commuting algebra of differential operators determined by $\tilde{P}$.

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