Generating Operator of *XXX* or Gaudin Transfer Matrices Has Quasi-Exponential Kernel[⋆]

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Abstract. Let M be the tensor product of finite-dimensional polynomial evaluation $Y(\mathfrak{gl}_N)$-modules. Consider the universal difference operator $\mathfrak{D} = \sum_{k=0}^{N}(-1)^k\mathfrak{T}_k(u)e^{-k\partial_u}$ whose coefficients $\mathfrak{T}_k(u) : M \to M$ are the XXX transfer matrices associated with M. We show that the difference equation $\mathfrak{D}f = 0$ for an M-valued function f has a basis of solutions consisting of quasi-exponentials. We prove the same for the universal differential operator $D = \sum_{k=0}^{N}(-1)^k\mathcal{S}_k(u)\partial_u^{N-k}$ whose coefficients $\mathcal{S}_k(u) : \mathcal{M} \to \mathcal{M}$ are the Gaudin transfer matrices associated with the tensor product $\mathcal{M}$ of finite-dimensional polynomial evaluation $\mathfrak{gl}_N[x]$-modules.

Key words: Gaudin model; *XXX* model; universal differential operator

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To the memory of Vadim Kuznetsov

1 Introduction

In a quantum integrable model one constructs a collection of one-parameter families of commuting linear operators (called transfer matrices) acting on a finite-dimensional vector space. In this paper we consider the vector-valued differential or difference linear operator whose coefficients are the transfer matrices. This operator is called the universal differential or difference operator. For the *XXX* and Gaudin type models we show that the kernel of the universal operator is generated by quasi-exponentials or sometimes just polynomials. This statement establishes a relationship between these quantum integrable models and that part of algebraic geometry, which studies the finite-dimensional spaces of quasi-exponentials or polynomials, in particular with Schubert calculus.

We plan to develop this relationship in subsequent papers. An example of an application of this relationship see in [5].

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Consider the complex Lie algebra $\mathfrak{gl}_N$ with standard generators e_{ab}, $a, b = 1, \dots, N$. Let $\boldsymbol{\Lambda} = (\Lambda^{(1)}, \dots, \Lambda^{(n)})$ be a collection of integral dominant $\mathfrak{gl}_N$-weights, where $\Lambda^{(i)} = (\Lambda_1^{(i)}, \dots, \Lambda_N^{(i)})$ for $i = 1, \dots, n$ and $\Lambda_N^{(i)} \in \mathbb{Z}_{\geqslant 0}$. Let $M = M_{\Lambda^{(1)}} \otimes \dots \otimes M_{\Lambda^{(n)}}$ be the tensor product of the corresponding finite-dimensional highest weight $\mathfrak{gl}_N$-modules. Let $K = (K_{ab})$ be an $N \times N$ matrix with complex entries. Let $z_1, \dots, z_n$ be distinct complex numbers. For $a, b = 1, \dots, N$, set

$$X_{ab}(u, \partial_u) = \delta_{ab}\, \partial_u - K_{ab} - \sum_{j=1}^{n} \frac{e_{ba}^{(j)}}{u - z_j},$$

where $\partial_u = d/du$. Following [13], introduce the differential operator

$$\mathcal{D}_{K,M}(u, \partial_u) = \sum_{\sigma \in S_N} (-1)^{\sigma}\, X_{1\,\sigma_1}(u, \partial_u)\, X_{2\,\sigma_2}(u, \partial_u) \cdots X_{N\sigma_N}(u, \partial_u).$$

The operator $\mathcal{D}_{K,M}(u, \partial_u)$ acts on M-valued functions in u. The operator is called *the universal differential operator* associated with M, K and $z_1, \dots, z_n$.

Introduce the coefficients $\mathcal{S}_{0,K,M}(u), \dots, \mathcal{S}_{N,K,M}(u)$:

$$\mathcal{D}_{K,M}(u, \partial_u) = \sum_{k=0}^{N} (-1)^k\, \mathcal{S}_{k,K,M}(u)\, \partial_u^{N-k}.$$

The coefficients are called *the transfer matrices of the Gaudin model* associated with M, K and $z_1, \dots, z_n$. The transfer matrices form a commutative family: $[\mathcal{S}_{k,K,M}(u), \mathcal{S}_{l,K,M}(v)] = 0$ for all k, l, u, v.

A quasi-exponential is a finite sum of functions of the form $e^{\lambda u} p(u)$, where $p(u)$ is a polynomial.

Theorem 1.1. *The kernel of the universal differential operator $\mathcal{D}_{K,M}(u, \partial_u)$ is generated by M-valued quasi-exponentials. Possible exponents λ appearing in the kernel are eigenvalues of the matrix K. In particular, if $K = 0$, then the kernel is generated by M-valued polynomials.*

For $K = 0$, the theorem was conjectured in [1] and proved in [5].

We describe possible degrees of polynomials $p(u)$ appearing in the kernel further in the paper.

The universal differential operator has singular points at $z_1, \dots, z_n$. We describe behavior of elements of the kernel at these points.

The tensor product M may be naturally regarded as the tensor product of polynomial evaluation modules over the current algebra $\mathfrak{gl}_N[x]$. Then the operators $X_{ab}(u)$, $\mathcal{D}_{K,M}(u, \partial_u)$ may be naturally defined in terms of the $\mathfrak{gl}_N[x]$-action on M.

Similarly, the tensor product M may be naturally regarded as the tensor product of polynomial evaluation modules over the Yangian $Y(\mathfrak{gl}_N)$. Then one may define a linear N-th order difference operator acting on M-valued functions in u. That operator is called the universal difference operator. Its coefficients commute and are called the transfer matrices of the associated XXX model.

We prove that the kernel of the universal difference operator is generated by quasi-exponentials. We describe the quasi-exponentials entering the kernel and their behavior at singular points of the universal difference operator.

The paper has the following structure. In Section 2 we make general remarks on quasi-exponentials.

In Section 3 we collect basic facts about the Yangian $Y(\mathfrak{gl}_N)$, the fundamental difference operator and the XXX transfer matrices. In Section 3.4 we formulate Theorem 3.8 which states that the kernel of the universal difference operator is generated by quasi-exponentials. Theorem 3.8 is our first main result for the XXX type models.

In Section 4 we prove a continuity principle for difference operators with quasi-exponential kernel. Under certain conditions we show that if a family of difference operators has a limiting difference operator, and if the kernel of each operator in the family is generated by quasi-exponentials, then the kernel of the limiting difference operator is generated by quasi-exponentials too.

Section 5 is devoted to the Bethe ansatz method for the XXX type models. Using the Bethe ansatz method we prove the special case of Theorem 3.8 in which M is the tensor product of vector representations of $\mathfrak{gl}_N$. Then the functoriality properties of the fundamental difference operator and our continuity principle allow us to deduce the general case of Theorem 3.8 from the special one.

In Section 6 we give a formula comparing the kernels of universal difference operators associated respectively with the tensor products $M_{\Lambda^{(1)}} \otimes \cdots \otimes M_{\Lambda^{(n)}}$ and $M_{\Lambda^{(1)}(a_1)} \otimes \cdots \otimes M_{\Lambda^{(n)}(a_n)}$, where $a_1, \dots, a_n$ are non-negative integers and $\Lambda^{(i)}(a_i) = \big(\Lambda_1^{(i)} + a_i, \dots, \Lambda_N^{(i)} + a_i\big)$.

In Section 7 we describe the quasi-exponentials entering the kernel of the universal difference operator and the behavior of functions of the kernel at singular points of the universal difference operator. Theorems 7.1, 7.2 and 7.3 form our second main result for the XXX type models.

In Sections 8–12 we develop an analogous theory for the universal differential operator of the Gaudin type models.

Section 8 contains basic facts about the current algebra $\mathfrak{gl}_N[x]$, the universal differential operator and the Gaudin transfer matrices. We formulate Theorem 8.4, which states that the kernel of the universal differential operator is generated by quasi-exponentials. This theorem is our first main result for the Gaudin type models.

Section 9 contains the continuity principle for the differential operators with quasi-exponential kernel.

Section 10 is devoted to the Bethe ansatz method for the Gaudin type model.

In Section 11 we compare kernels of the universal differential operators associated respectively with tensor products $M_{\Lambda^{(1)}} \otimes \cdots \otimes M_{\Lambda^{(n)}}$ and $M_{\Lambda^{(1)}(a_1)} \otimes \cdots \otimes M_{\Lambda^{(n)}(a_n)}$.

In Section 12 we describe the quasi-exponentials entering the kernel of the universal differential operator and the behavior of functions of the kernel at singular points of the universal differential operator. Theorems 12.1, 12.2 and 12.3 form our second main result for the Gaudin type models.

2 Spaces of quasi-exponentials

2.1 Quasi-exponentials

2.1.1. Define the operator τ acting on functions of u as $(\tau f)(u) = f(u+1)$. A function $f(u)$ will be called *one-periodic* if $f(u+1) = f(u)$. Meromorphic one-periodic functions form a field with respect to addition and multiplication.

2.1.2. Let Q be a nonzero complex number with fixed argument. Set $Q^u = e^{u \ln Q}$. We have $\tau Q^u = Q^u Q$.

Let $p \in \mathbb{C}[u]$ be a polynomial. The function $Q^u p$ will be called *a (scalar) elementary quasi-exponential in u.* A finite sum of elementary quasi-exponentials will be called *a (scalar) quasi-exponential.*

Let V be a complex vector space of finite dimension d. *A V-valued quasi-exponential* is a V-valued function of the form $\sum_a f_a(u) v_a$, where $f_a(u)$ are scalar quasi-exponentials, $v_a \in V$, and the sum is finite.

We say that a quasi-exponential $\sum_{ab} Q_a^u u^b v_{ab}$ is of degree less than k if $v_{ab} = 0$ for all $b \geqslant k$.

2.1.3. For given End (V)-valued rational functions $A_0(u), \dots, A_N(u)$ consider the difference operator

$$\mathfrak{D} = \sum_{k=0}^{N} A_k(u)\, \tau^{-k} \tag{2.1}$$

acting on V-valued functions in u.

We say that the kernel of $\mathfrak{D}$ *is generated by quasi-exponentials* if there exist Nd quasi-exponential functions with values in V such that each of these function belongs to the kernel of $\mathfrak{D}$ and these functions generate an Nd-dimensional vector space over the field of one-periodic meromorphic functions.

The following simple observation is useful.

Lemma 2.1. *Assume that a quasi-exponential $\sum_{ab} Q_a^u u^b v_{ab}$, with all numbers Q_a being different, lies in the kernel of $\mathfrak{D}$ defined in* (2.1)*. Then for every a, the quasi-exponential $Q_a^u \sum_b u^b v_{ab}$ lies in the kernel of $\mathfrak{D}$.*

The lemma follows from the fact that exponential functions with different exponents are linearly independent over the field of rational functions in u.

3 Generating operator of the XXX transfer matrices

3.1 Yangian $Y(\mathfrak{gl}_N)$

3.1.1. Let e_{ab}, $a, b = 1, \dots, N$, be the standard generators of the complex Lie algebra $\mathfrak{gl}_N$. We have $\mathfrak{gl}_N = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where

$$\mathfrak{n}^+ = \oplus_{a<b}\, \mathbb{C} \cdot e_{ab}, \qquad \mathfrak{h} = \oplus_{a=1}^{N}\, \mathbb{C} \cdot e_{aa}, \qquad \mathfrak{n}^- = \oplus_{a>b}\, \mathbb{C} \cdot e_{ab}.$$

For an integral dominant $\mathfrak{gl}_N$-weight $\Lambda \in \mathfrak{h}^*$, denote by M_Λ the irreducible finite dimensional $\mathfrak{gl}_N$-module with highest weight Λ.

For a $\mathfrak{gl}_N$-module M and a weight $\mu \in \mathfrak{h}^*$, denote by $M[\mu] \subset M$ the vector subspace of vectors of weight μ.

3.1.2. The Yangian $Y(\mathfrak{gl}_N)$ is the unital associative algebra with generators $T_{ab}^{\{s\}}$, $a, b = 1, \dots, N$ and $s = 1, 2, \dots$. Let

$$T_{ab} = \delta_{ab} + \sum_{s=1}^{\infty} T_{ab}^{\{s\}} u^{-s}, \qquad a, b = 1, \dots, N.$$

The defining relations in $Y(\mathfrak{gl}_N)$ have the form

$$(u - v)\, [T_{ab}(u), T_{cd}(v)] = T_{cb}(v) T_{ad}(u) - T_{cb}(u) T_{ad}(v),$$

for all a, b, c, d. The Yangian is a Hopf algebra with coproduct

$$\Delta \ :\ T_{ab}(u) \ \mapsto\ \sum_{c=1}^{N} T_{cb}(u) \otimes T_{ac}(u)$$

for all a, b.

3.1.3. We identify the elements of End $(\mathbb{C}^N)$ with $N \times N$-matrices. Let $E_{ab} \in$ End $(\mathbb{C}^N)$ denote the matrix with the only nonzero entry 1 at the intersection of the a-th row and b-th column.

Let $P = \sum_{a,b} E_{ab} \otimes E_{ba}$, $R(u) = u + P \in \mathrm{End}\,(\mathbb{C}^N \otimes \mathbb{C}^N)$ and $T(u) = \sum_{a,b} E_{ab} \otimes T_{ab}(u) \in \mathrm{End}\,(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)((u^{-1}))$. Then the defining relations for the Yangian can be written as the following equation of series in u^{-1} with coefficients in $\mathrm{End}\,(\mathbb{C}^N) \otimes \mathrm{End}\,(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)$,

$$R^{(12)}(u-v)\, T^{(13)}(u)\, T^{(23)}(v) = T^{(23)}(v)\, T^{(13)}(u)\, R^{(12)}(u-v).$$

3.1.4. A series $f(u)$ in u^{-1} is called monic if $f(u) = 1 + O(u^{-1})$.

For a monic series $f(u)$, there is an automorphism

$$\chi_f \;:\; Y(\mathfrak{gl}_N) \;\to\; Y(\mathfrak{gl}_N), \qquad T(u) \;\mapsto\; f(u)\, T(u).$$

The fixed point subalgebra in $Y(\mathfrak{gl}_N)$ with respect to all automorphisms χ_f is called the Yangian $Y(\mathfrak{sl}_N)$. Denote by $ZY(\mathfrak{gl}_N)$ the center of the Yangian $Y(\mathfrak{gl}_N)$.

Proposition 3.1. *The Yangian $Y(\mathfrak{gl}_N)$ is isomorphic to the tensor product $Y(\mathfrak{sl}_N) \otimes ZY(\mathfrak{gl}_N)$.*

See [4] for a proof.

3.1.5. Let V be an irreducible finite-dimensional $Y(\mathfrak{gl}_N)$-module. Then there exists a unique vector $v \in V$ such that

$$\begin{aligned} &T_{ab}(u)\, v = 0, && a > b, \\ &T_{aa}(u)\, v = c_a(u)\, v, && a = 1, \dots, N, \end{aligned}$$

for suitable monic series $c_a(u)$. Moreover,

$$\frac{c_a(u)}{c_{a+1}(u)} = \frac{P_a(u+a)}{P_a(u+a-1)}, \qquad a = 1, \dots, N-1, \tag{3.1}$$

for certain monic polynomials $P_a(u)$.

The polynomials $P_1, \dots, P_{N-1}$ are called *the Drinfeld polynomials* of the module V. The vector v is called *a highest weight vector* and the series $c_1(u), \dots, c_N(u)$ – *the Yangian highest weights* of the module V.

For any collection of monic polynomials $P_1, \dots, P_{N-1}$ there exists an irreducible finite-dimensional $Y(\mathfrak{gl}_N)$-module V such that the polynomials $P_1, \dots, P_{N-1}$ are the Drinfeld polynomials of V. The module V is uniquely determined up to twisting by an automorphism of the form χ_f.

The claim follows from Drinfeld's description of irreducible finite-dimensional $Y(\mathfrak{sl}_N)$-modules [2] and Proposition 3.1.

3.1.6. Let V_1, V_2 be irreducible finite-dimensional $Y(\mathfrak{gl}_N)$-modules with respective highest weight vectors v_1, v_2. Then for the $Y(\mathfrak{gl}_N)$-module $V_1 \otimes V_2$, we have

$$\begin{aligned} &T_{ab}(u)\, v_1 \otimes v_2 = 0, && a > b, \\ &T_{aa}(u)\, v_1 \otimes v_2 = c_a^{(1)}(u)\, c_a^{(2)}(u)\, v_1 \otimes v_2, && a = 1, \dots, N. \end{aligned}$$

Let W be the irreducible subquotient of $V_1 \otimes V_2$ generated by the vector $v_1 \otimes v_2$. Then the Drinfeld polynomials of the module W equal the products of the respective Drinfeld polynomials of the modules V_1 and V_2.

3.1.7. A finite-dimensional irreducible $Y(\mathfrak{gl}_N)$-module V will be called *polynomial* if

$$c_N(u) = \frac{P_N(u+N)}{P_N(u+N-1)}. \tag{3.2}$$

for some monic polynomial $P_N(u)$.

For any collection of monic polynomials $P_1, \dots, P_N$ there exists a unique polynomial irreducible finite-dimensional $Y(\mathfrak{gl}_N)$-module V such that the polynomials $P_1, \dots, P_{N-1}$ are the Drinfeld polynomials of V and (3.2) holds.

3.1.8. There is a one-parameter family of automorphisms

$$\rho_z \ : \ Y(\mathfrak{gl}_N) \to Y(\mathfrak{gl}_N), \qquad T_{ab}(u) \mapsto T_{ab}(u-z),$$

where in the right hand side, $(u-z)^{-1}$ has to be expanded as a power series in u^{-1}.

The Yangian $Y(\mathfrak{gl}_N)$ contains the universal enveloping algebra $U(\mathfrak{gl}_N)$ as a Hopf subalgebra. The embedding is given by the formula $e_{ab} \mapsto T_{ba}^{\{1\}}$ for all a, b. We identify $U(\mathfrak{gl}_N)$ with its image.

The evaluation homomorphism $\epsilon : Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N)$ is defined by the rule: $T_{ab}^{\{1\}} \mapsto e_{ba}$ for all a, b and $T_{ab}^{\{s\}} \mapsto 0$ for all a, b and all $s > 1$.

For a $\mathfrak{gl}_N$-module V denote by $V(z)$ the $Y(\mathfrak{gl}_N)$-module induced from V by the homomorphism $\epsilon \cdot \rho_z$. The module $V(z)$ is called the evaluation module with the evaluation point z.

Let $\boldsymbol{\Lambda} = (\Lambda^{(1)}, \dots, \Lambda^{(n)})$ be a collection of integral dominant $\mathfrak{gl}_N$-weights, where $\Lambda^{(i)} = (\Lambda_1^{(i)}, \dots, \Lambda_N^{(i)})$ for $i = 1, \dots, n$. For generic complex numbers $z_1, \dots, z_n$, the tensor product of evaluation modules

$$M_{\boldsymbol{\Lambda}}(\boldsymbol{z}) = M_{\Lambda^{(1)}}(z_1) \otimes \dots \otimes M_{\Lambda^{(n)}}(z_n)$$

is an irreducible $Y(\mathfrak{gl}_N)$-module and the corresponding highest weight series $c_1(u), \dots, c_N(u)$ have the form

$$c_a(u) = \prod_{i=1}^{n} \frac{u - z_i + \Lambda_a^{(i)}}{u - z_i}. \tag{3.3}$$

The corresponding Drinfeld polynomials are

$$P_a(u) = \prod_{i=1}^{n} \prod_{s=\Lambda_{a+1}^{(i)}+1}^{\Lambda_a^{(i)}} (u - z_i + s - a)$$

for $a = 1, \dots, N-1$. The $Y(\mathfrak{gl}_N)$-module $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ is polynomial if $\Lambda_N^{(i)} \in \mathbb{Z}_{\geqslant 0}$ for all i. Then the polynomial $P_N(u)$ has the form

$$P_N(u) = \prod_{i=1}^{n} \prod_{s=1}^{\Lambda_N^{(i)}} (u - z_i + s - N).$$

3.1.9. Consider $\mathbb{C}^N$ as the $\mathfrak{gl}_N$-module with highest weight $(1, 0, \dots, 0)$.

For any complex numbers $z_1, \dots, z_n$, all irreducible subquotients of the $Y(\mathfrak{gl}_N)$-module $\mathbb{C}^N(z_1) \otimes \dots \otimes \mathbb{C}^N(z_n)$ are polynomial $Y(\mathfrak{gl}_N)$-modules. Moreover, for any polynomial irreducible finite-dimensional $Y(\mathfrak{gl}_N)$-module V, there exist complex numbers $z_1, \dots, z_n$ such that V is isomorphic to a subquotient of the $Y(\mathfrak{gl}_N)$-module $\mathbb{C}^N(z_1) \otimes \dots \otimes \mathbb{C}^N(z_n)$.

The numbers $z_1, \dots, z_n$ are determined by the formula

$$\prod_{i=1}^{n} (u - z_i) = \prod_{a=1}^{N} \prod_{s=0}^{a-1} P_a(u+s).$$

This formula follows from consideration of the action of the center of the Yangian $Y(\mathfrak{gl}_N)$ in the module $\mathbb{C}^N(z_1) \otimes \dots \otimes \mathbb{C}^N(z_n)$.

3.1.10. A finite-dimensional $Y(\mathfrak{gl}_N)$-module will be called *polynomial* if it is the direct sum of tensor products of polynomial irreducible finite-dimensional $Y(\mathfrak{gl}_N)$-modules.

If V is a polynomial finite-dimensional $Y(\mathfrak{gl}_N)$-module, then for any $a, b = 1, \dots, N$, the series $T_{ab}(u)|_V$ converges to an $\mathrm{End}\,(V)$-valued rational function in u.

3.1.11. Let $\pi : U(\mathfrak{gl}_N) \to \mathrm{End}\,(\mathbb{C}^N)$ be the representation homomorphism for the $\mathfrak{gl}_N$-module $\mathbb{C}^N$. Clearly, for any $x \in U(\mathfrak{gl}_N)$ we have

$$\big[\pi(x) \otimes 1 + 1 \otimes x,\, T(u)\big] = 0. \tag{3.4}$$

For a non-degenerate matrix $A \in \mathrm{End}\,(\mathbb{C}^N)$, define an automorphism ν_A of $Y(\mathfrak{gl}_N)$ by the formula

$$(\mathrm{id} \otimes \nu_A)\big(T(u)\big) = \sum_{ab} A^{-1} E_{ab} A \otimes T_{ab}(u).$$

Let V be a finite-dimensional Yangian module with the representation $\mu : Y(\mathfrak{gl}_N) \to \mathrm{End}\,(V)$ and $\tilde{\mu} : GL_N \to \mathrm{End}\,(V)$ the corresponding representation of the group GL_N. The automorphism ν_A induces a new Yangian module structure V^A on the same vector space with the representation $\mu_A = \mu \circ \nu_A$. Formula (3.4) yields that for any $x \in Y(\mathfrak{gl}_N)$,

$$\mu_A(x) = \tilde{\mu}(A)\, \mu(x) \big(\tilde{\mu}(A)\big)^{-1}, \tag{3.5}$$

that is, the $Y(\mathfrak{gl}_N)$-modules V and V^A are isomorphic. In particular, if V is a polynomial irreducible finite-dimensional $Y(\mathfrak{gl}_N)$-module, then V^A is a polynomial irreducible finite-dimensional $Y(\mathfrak{gl}_N)$-module too.

3.2 Universal difference operator

3.2.1. Let $Q = (Q_{ab})$ be an $N \times N$-matrix. Define

$$\mathcal{X}_{ab}(u, \tau) = \delta_{ab} - \sum_{c=1}^{N} Q_{ac}\, T_{cb}(u)\, \tau^{-1}, \qquad a, b = 1, \dots, N. \tag{3.6}$$

If V is a polynomial finite-dimensional $Y(\mathfrak{gl}_N)$-module, then $\mathcal{X}_{ab}(u, \tau)$ acts on V-valued functions in u,

$$f(u) \;\mapsto\; \delta_{ab} f(u) - \sum_{c=1}^{N} Q_{ac}\, T_{cb}(u) f(u-1).$$

Following [13], introduce the difference operator

$$\mathfrak{D}(u, \tau) = \sum_{\sigma \in S_N} (-1)^{\sigma}\, \mathcal{X}_{1\,\sigma_1}(u, \tau)\, \mathcal{X}_{2\,\sigma_2}(u, \tau) \cdots \mathcal{X}_{N\sigma_N}(u, \tau), \tag{3.7}$$

where the sum is over all permutations σ of $\{1, \dots, N\}$. The operator $\mathfrak{D}(u, \tau)$ will be called *the universal difference operator* associated with the matrix Q.

Lemma 3.2. *Let π be a map $\{1, \dots, N\} \to \{1, \dots, N\}$. If π is a permutation of $\{1, \dots, N\}$, then*

$$\sum_{\sigma \in S_N} (-1)^{\sigma}\, \mathcal{X}_{\pi_1\sigma_1}(u, \tau)\, \mathcal{X}_{\pi_2\sigma_2}(u, \tau) \cdots \mathcal{X}_{\pi_N\sigma_N}(u, \tau) = (-1)^{\pi}\, \mathfrak{D}(u, \tau).$$

If π is not bijective, then

$$\sum_{\sigma \in S_N} (-1)^{\sigma}\, \mathcal{X}_{\pi_1\sigma_1}(u, \tau)\, \mathcal{X}_{\pi_2\sigma_2}(u, \tau) \cdots \mathcal{X}_{\pi_N\sigma_N}(u, \tau) = 0.$$

The statement is Proposition 4.10 in [6].

3.2.2. Introduce the coefficients $\mathfrak{T}_0(u),\dots,\mathfrak{T}_N(u)$ of $\mathfrak{D}(u,\tau)$:

$$\mathfrak{D}(u,\tau)=\sum_{k=0}^{N}(-1)^k\,\mathfrak{T}_k(u)\,\tau^{-k}.$$

The coefficients $\mathfrak{T}_k(u)$ are called *the transfer matrices of the XXX type model* associated with Q. The main properties of the transfer matrices:

(i) the transfer matrices commute: $[\mathfrak{T}_k(u),\mathfrak{T}_l(v)]=0$ for all k, l, u, v,

(ii) if Q is a diagonal matrix, then the transfer matrices preserve the $\mathfrak{gl}_N$-weight: $[\mathfrak{T}_k(u),e_{aa}]=0$ for all k, a, u,

(iii) if Q is the identity matrix, then the transfer matrices commute with the subalgebra $U(\mathfrak{gl}_N)$: $[\mathfrak{T}_k(u),x]=0$ for all k,u and $x\in U(\mathfrak{gl}_N)$,

see [13, 6].

3.2.3. Evidently, $\mathfrak{T}_0(u)=1$. We also have $\mathfrak{T}_N(u)=\det Q\,\mathrm{qdet}\,T(u)$, where

$$\mathrm{qdet}\,T(u)=\sum_{\sigma\in S_N}(-1)^\sigma\,T_{1\,\sigma_1}(u)\,T_{2\,\sigma_2}(u-1)\cdots T_{N\sigma_N}(u-N+1),$$

and $\mathrm{qdet}\,T(u)=1+O(u^{-1})$.

Theorem 3.3. *The coefficients of the series* $\mathrm{qdet}\,T(u)$ *are free generators of the center of the Yangian* $Y(\mathfrak{gl}_N)$*.*

See [4] for a proof.

3.2.4. If V is a polynomial finite-dimensional $Y(\mathfrak{gl}_N)$-module, then the universal operator $\mathfrak{D}(u,\tau)$ induces a difference operator acting on V-valued functions in u. This operator will be called *the universal difference operator* associated with Q and V and denoted by $\mathfrak{D}_{Q,V}(u,\tau)$. The linear operators $\mathfrak{T}_k(u)|_V\in\mathrm{End}\,(V)$ will be called *the transfer matrices* associated with Q and V and denoted by $\mathfrak{T}_{k,Q,V}(u)$. They are rational functions in u.

Example 3.4. Let

$$V=M_{\boldsymbol{\Lambda}}(\boldsymbol{z})=M_{\Lambda^{(1)}}(z_1)\otimes\cdots\otimes M_{\Lambda^{(n)}}(z_n),$$

Consider the algebra $\bigl(U(\mathfrak{gl}_N)\bigr)^{\otimes n}$. For $a,b=1,\dots,N$ and $i=1,\dots,n$, define

$$e_{ab}^{(i)}=1^{\otimes(i-1)}\otimes e_{ab}\otimes 1^{\otimes(n-i)},\qquad L_{ab}^{(i)}(u,z)=\delta_{ab}+\frac{1}{u-z}\,e_{ba}^{(i)}.$$

The operator $T_{ab}(u)$ acts on $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ as

$$\sum_{c_1,\dots,c_{n-1}=1}^{N}L^{(n)}_{a\,c_{n-1}}(u-z_n)\,L^{(n-1)}_{c_{n-1}\,c_{n-2}}(u-z_{n-1})\cdots L^{(2)}_{c_2\,c_1}(u-z_2)\,L^{(1)}_{c_1 b}(u-z_1).$$

Lemma 3.5. *If* $\mathfrak{D}_Q(u,\tau)$ *is the universal difference operator associated with the matrix* Q *and* $\nu_A:Y(\mathfrak{gl}_N)\to Y(\mathfrak{gl}_N)$ *is the automorphism defined in Section* 3.1.11*, then*

$$\nu_A(\mathfrak{D}_Q(u,\tau))=\mathfrak{D}_{AQA^{-1}}(u,\tau)\tag{3.8}$$

is the universal difference operator associated with the matrix AQA^{-1}*.*

Proof. Consider matrices $Q = (Q_{ab})$, $T = (T_{ab}(u))$, and $\mathcal{X} = (\mathcal{X}_{ab}(u,\tau))$. Then formula (3.6), defining $\mathcal{X}_{ab}$, may be read as $\mathcal{X} = 1 - QT\tau^{-1}$. Formula (3.7) for the universal difference operator may be understood as the row determinant of the matrix $\mathcal{X}$, $\mathfrak{D}(u,\tau) = \det \mathcal{X}$. The definition of the automorphism ν_A reads as $(\mathrm{id} \otimes \nu_A)(T) = A^{-1}TA$. Then we have

$$\nu_A(\mathfrak{D}(u,\tau)) = \det\big(1 - QA^{-1}TA\tau^{-1}\big) = \det\big(A^{-1}\big(1 - AQA^{-1}T\tau^{-1}\big)A\big).$$

Now formula (3.8) will be proved if we were able to write that last determinant as the product:

$$\det\big(A^{-1}\big(1 - AQA^{-1}T\tau^{-1}\big)A\big) = \det\big(A^{-1}\big)\det\big(1 - AQA^{-1}T\tau^{-1}\big)\det A.$$

The last formula may be proved the same way as the standard formula $\det MN = \det M\,\det N$ in ordinary linear algebra, using two observations. The first is that the entries of A are numbers and commute with the entries of $\mathcal{X}$. The second observation is Lemma 3.2 describing the transformations of the row determinant of $\mathcal{X}$ with respect to row replacements. ■

3.2.5. Let V be a polynomial finite-dimensional Yangian module and $\tilde{\mu} : G_N \to GL(V)$ the associated GL_N-representation. Then formulae (3.5) and (3.8) yield

$$\mathfrak{D}_{AQA^{-1}}(u,\tau)|_V = \tilde{\mu}(A)\,\mathfrak{D}_Q(u,\tau)|_V\,\tilde{\mu}\big(A^{-1}\big).$$

3.3 More properties of transfer matrices

Let $Q = \mathrm{diag}\,(Q_1,\dots,Q_N) \in GL_N$. Let $\boldsymbol{\Lambda} = (\Lambda^{(1)},\dots,\Lambda^{(n)})$ be a collection of integral dominant $\mathfrak{gl}_N$-weights, where $\Lambda^{(i)} = (\Lambda^{(i)}_1,\dots,\Lambda^{(i)}_N)$ and $\Lambda^{(i)}_N = 0$ for $i = 1,\dots,n$.

For $\boldsymbol{m} = (m_1,\dots,m_N)$ denote by $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}] \subset M_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ the weight-subspace of $\mathfrak{gl}_N$-weight $\boldsymbol{m}$ and by $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}] \subset M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ the subspace of $\mathfrak{gl}_N$-singular vectors.

Note that the subspace $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ may be nonzero only if $\boldsymbol{m}$ is dominant integral, i.e. the integers $m_1,\dots,m_N$ have to satisfy the inequalities $m_1 \geqslant \dots \geqslant m_N$.

Consider the universal difference operator associated with Q and $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})$. Then the associated transfer matrices preserve $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ and we may consider the universal difference operator $\mathfrak{D}_{Q,\,M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u,\tau)$ acting on $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$-valued functions of u. We may write

$$\mathfrak{D}_{Q,\,M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u,\tau) = \sum_{k=0}^{N}(-1)^k\,\mathfrak{T}_{k,Q,\,M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u)\,\tau^{-k}.$$

As we know $\mathfrak{T}_{0,Q,M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u) = 1$, and we have

$$\mathfrak{T}_{k,Q,M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u) = \mathfrak{T}_{k0} + \mathfrak{T}_{k1}u^{-1} + \mathfrak{T}_{k2}u^{-2} + \cdots$$

for suitable $\mathfrak{T}_{ki} \in \mathrm{End}\,(M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}])$.

Theorem 3.6. *The followings statements hold.*

(i) The operators $\mathfrak{T}_{10},\mathfrak{T}_{20},\dots,\mathfrak{T}_{N0}$ and $\mathfrak{T}_{11},\mathfrak{T}_{21},\dots,\mathfrak{T}_{N1}$ are scalar operators. Moreover, the following relations hold:

$$x^N + \sum_{k=1}^{N}(-1)^k\,\mathfrak{T}_{k0}\,x^{N-k} = \prod_{i=1}^{N}(x - Q_i),$$

$$\sum_{k=1}^{N}(-1)^k\,\mathfrak{T}_{k1}\,x^{N-k} = -\prod_{i=1}^{N}(x - Q_i)\sum_{j=1}^{N}\frac{m_j\,Q_j}{x - Q_j}.$$

(ii) For $k = 1, \dots, N-1$,

$$\mathfrak{T}_{N,Q,M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u) = \left(\prod_{i=1}^{N} Q_i\right) \prod_{s=1}^{n} \prod_{i=1}^{N-1} \frac{u - z_s + \Lambda_i^{(s)} - i + 1}{u - z_s - i + 1},$$

$$\mathfrak{T}_{k,Q,M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u) = \tilde{\mathfrak{T}}_k(u) \prod_{i=1}^{k} \prod_{s=1}^{n} \frac{1}{u - z_s - i + 1},$$

where $\tilde{\mathfrak{T}}_k(u)$ is a polynomial in u of degree nk.

Proof. Part (i) follows from Proposition B.1 in [6]. Part (ii) follows from the definition of the universal difference operator and the fact that the coefficients of the series $\mathfrak{T}_N(u)$ belong to the center of $Y(\mathfrak{gl}_N)$. ■

3.3.1. Assume that Q is the identity matrix. Then the associated transfer matrices preserve $\operatorname{Sing} M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ and we may consider the universal difference operator $\mathfrak{D}_{Q=1,\operatorname{Sing} M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u,\tau)$ acting on $\operatorname{Sing} M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$-valued functions of u. We may write

$$\mathfrak{D}_{Q=1,\operatorname{Sing} M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u,\tau)\,\tau^N = \sum_{k=0}^{N} (-1)^k\, \mathfrak{S}_k(u)\, (\tau - 1)^{N-k} \tag{3.9}$$

for suitable coefficients $\mathfrak{S}_k(u)$.

Note that the operators $\mathfrak{S}_k(u)$ coincide with the action in $\operatorname{Sing} M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ of the modified transfer matrices $\mathcal{S}_k(u)$ from formula (10.4) of [6].

Theorem 3.7. *The following three statements hold.*

(i) We have $\mathfrak{S}_0(u) = 1$.

(ii) For $k = 1, \dots, N$, the coefficient $\mathfrak{S}_k(u)$ has the following Laurent expansion at $u = \infty$:

$$\mathfrak{S}_k(u) = \mathfrak{S}_{k,0} u^{-k} + \mathfrak{S}_{k,1} u^{-k-1} + \cdots,$$

where the operators $\mathfrak{S}_{1,0}, \dots, \mathfrak{S}_{N,0}$ are scalar operators.

(iii) For all d we have

$$\sum_{k=0}^{N} (-1)^k\, \mathfrak{S}_{k,0} \prod_{j=0}^{N-k-1} (d - j) = \prod_{s=1}^{N} (d - m_s - N + s).$$

Proof. Part (i) is evident.

Since Q is the identity matrix, formula (3.6) reads now as follows

$$\mathcal{X}_{ab}(u) = \delta_{ab} - T_{ab}(u)\,\tau^{-1} = \big(\delta_{ab}\,(\tau - 1) - T_{ab}^{\{1\}}\, u^{-1} - O\big(u^{-2}\big)\big)\,\tau^{-1}. \tag{3.10}$$

Then part (ii) is straightforward from formulas (3.9) and (3.7).

Let v be any vector in $\operatorname{Sing} M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ and d any number. To prove part (iii) we apply the difference operators in formula (3.9) to the function vu^d. The expansion at infinity of result of the application of the right side is

$$u^{d-N} \left(\sum_{k=0}^{N} (-1)^k \left[\prod_{j=0}^{N-k-1} (d - j) \right] \mathfrak{S}_{k,0}\, v + O\big(u^{-1}\big) \right).$$

So it remains to show that

$$\mathfrak{D}_{Q=1,\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u,\tau)\,\tau^N v\, u^d = u^{d-N}\left(\left[\prod_{s=1}^{N}(d-m_s-N+s)\right] v + O\big(u^{-1}\big)\right),$$

as u goes to infinity. Since $\tau^N u^d = u^d\big(1+O(u^{-1})\big)$, the last formula is equivalent to

$$\mathfrak{D}_{Q=1,\,\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u,\tau)\, v\, u^d = u^{d-N}\left(\left[\prod_{s=1}^{N}(d-m_s-N+s)\right] v + O\big(u^{-1}\big)\right). \tag{3.11}$$

To prove (3.11), observe that according to formula (3.10) we have

$$\mathcal{X}_{ab}(u)\, v u^d = u^{d-1}\left(\big(d\,\delta_{ab} - T^{\{1\}}_{ab}(u)\big)\, v + O\big(u^{-1}\big)\right).$$

Applying this remark to formula (3.7) we get

$$\begin{aligned}&\mathfrak{D}_{Q=1,\,\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u,\tau)\, v\, u^d\\ &\quad = u^{d-N}\left(\sum_{\sigma\in S_N}(-1)^\sigma\,\big((d-N+1)\,\delta_{1,\sigma_1} - T^{\{1\}}_{1\sigma_1}\big)\cdots\big(d\,\delta_{N,\sigma_N} - T^{\{1\}}_{N\sigma_N}\big)\, v + O\big(u^{-1}\big)\right).\end{aligned}$$

Each element $T^{\{1\}}_{ab}$ acts as $\sum_{i=1}^{n} e^{(i)}_{ba}$ in $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})$, which corresponds to the standard $\mathfrak{gl}_N$-action in the $\mathfrak{gl}_N$-module $M_{\boldsymbol{\Lambda}}$. Since v is a $\mathfrak{gl}_N$-singular vector of a $\mathfrak{gl}_N$-weight $(m_1,\dots,m_N)$, we have

$$\sum_{\sigma\in S_N}(-1)^\sigma\big((d-N+1)\,\delta_{1,\sigma_1} - T^{\{1\}}_{1\sigma_1}\big)\cdots\big(d\,\delta_{N,\sigma_N} - T^{\{1\}}_{N\sigma_N}\big)v = \left[\prod_{s=1}^{N}(d-m_s-N+s)\right] v.$$

Indeed only the identity permutation contributes nontrivially to the sum in the left side. This proves part (iii) of Theorem 3.7. ■

3.4 First main result

Theorem 3.8. *Let V be a polynomial finite-dimensional $Y(\mathfrak{gl}_N)$-module and $Q\in GL_N$. Consider the universal difference operator $\mathfrak{D}_{Q,V}(u,\tau)$ associated with Q and V. Then the kernel of $\mathfrak{D}_{Q,V}(u,\tau)$ is generated by quasi-exponentials.*

The theorem will be proved in Section 5.

4 Continuity principle for difference operators with quasi-exponential kernel

4.1 Independent quasi-exponentials

Let V be a complex vector space of dimension d. Let $p\in\mathbb{C}[u]$ be a monic polynomial of degree k. Consider the differential equation

$$p\left(\frac{d}{du}\right) f(u) = 0$$

for a V-valued function $f(u)$. Denote by W_p the complex vector space of its solutions. The map

$$\delta_p \;:\; W_p \;\to\; V^{\oplus k}, \quad w \;\mapsto\; \big(w(0), w'(0), \dots, w^{(k-1)}(0)\big),$$

assigning to a solution its initial condition at $u=0$, is an isomorphism.

Let $\lambda_1, \dots, \lambda_l$ be all distinct roots of the polynomial p of multiplicities $k_1, \dots, k_l$, respectively. Let $v_1, \dots, v_d$ be a basis of V. Then the quasi-exponentials

$$e^{\lambda_j u} u^a v_b, \qquad j = 1, \dots, l, \quad a = 0, \dots, k_j - 1, \quad b = 1, \dots, d, \tag{4.1}$$

form a basis in W_p.

Lemma 4.1. *Assume that $\lambda_a - \lambda_b \notin 2\pi i\,\mathbb{Z}$ for all $a \neq b$. Then the kd functions listed in* (4.1) *are linear independent over the field of one-periodic functions.*

4.2 Admissible difference operators

Let $A_0(u), \dots, A_N(u)$ be $\mathrm{End}\,(V)$-valued rational functions in u. Assume that each of these functions has limit as $u \to \infty$ and $A_0(u) = 1$ in $\mathrm{End}\,(V)$. For every k, let

$$A_k(u) = A^\infty_{k,0} + A^\infty_{k,1} u^{-1} + A^\infty_{k,2} u^{-2} + \cdots$$

be the Laurent expansion at infinity. Consider the algebraic equation

$$\det \big(A_{N,0} + x A_{N-1,0} + \cdots + x^{N-1} A_{1,0} + x^N \big) = 0 \tag{4.2}$$

with respect to variable x and the difference operator

$$\mathfrak{D} = \sum_{k=0}^{N} A_k(u)\,\tau^{-k}$$

acting on V-valued functions in u. Equation (4.2) will be called the characteristic equation for the difference operator $\mathfrak{D}$.

The operator $\mathfrak{D}$ will be called *admissible at infinity* if $\det A_{N,0} \neq 0$, or equivalently, if $x = 0$ is not a root of the characteristic equation.

Example 4.2. Let V be the tensor product of polynomial finite-dimensional irreducible $Y(\mathfrak{gl}_N)$-modules and $Q \in GL_N$. Let $\mathfrak{D}_{Q,V}(u,\tau)$ be the associated universal difference operator. Then $\mathfrak{D}_{Q,V}(u,\tau)$ is admissible at infinity, see Sections 3.2.3 and 3.3.

Lemma 4.3. *Assume that $\mathfrak{D}$ is admissible at infinity and a nonzero V-valued quasi-exponential $Q^u(u^d v_d + u^{d-1} v_{d-1} + \cdots + v_0)$ lies in the kernel of $\mathfrak{D}$. Then Q is a root of the characteristic equation* (4.2).

4.3 Continuity principle

Let $A_0(u,\epsilon), \dots, A_N(u,\epsilon)$ be $\mathrm{End}\,(V)$-valued rational functions in u analytically depending on $\epsilon \in [0,1)$. Assume that

- for every $\epsilon \in [0,1)$ the difference operator $\mathfrak{D}_\epsilon = \sum\limits_{k=0}^{N} A_k(u,\epsilon)\tau^{-k}$ is admissible at infinity,
- for every $\epsilon \in (0,1)$ the kernel of $\mathfrak{D}_\epsilon$ is generated by quasi-exponentials,
- there exists a natural number m such that for every $\epsilon \in (0,1)$ all quasi-exponentials generating the kernel of $\mathfrak{D}_\epsilon$ are of degree less than m.

Theorem 4.4. *Under these conditions the kernel of the difference operator $\mathfrak{D}_{\epsilon=0}$ is generated by quasi-exponentials.*

Proof. For every $\epsilon \in [0,1)$, the characteristic equation for $\mathfrak{D}_\epsilon$ has Nd roots counted with multiplicities. As ϵ tends to 0 the roots of the characteristic equation of $\mathfrak{D}_\epsilon$ tend to the roots of the characteristic equation of $\mathfrak{D}_{\epsilon=0}$. All these roots are nonzero numbers. For small positive ϵ the set of multiplicities of roots does not depend on ϵ.

The following lemma is evident.

Lemma 4.5. *There exist*

- *a number $\bar\epsilon$ with $0 < \bar\epsilon \leqslant 1$,*
- *for any ϵ, $0 < \epsilon < \bar\epsilon$, a way to order the roots of the characteristic equation of $\mathfrak{D}_\epsilon$ (we denote the ordered roots by $Q_1^\epsilon, \dots, Q_{Nd}^\epsilon$),*
- *a way to assign the logarithm q_j^ϵ to every root Q_j^ϵ*

such that for every j the number q_j^ϵ continuously depends on ϵ and $q_j^\epsilon = q_l^\epsilon$ whenever $Q_j^\epsilon = Q_l^\epsilon$.

Let m be the number described in Section 4.3. For every ϵ, $0 < \epsilon < \bar\epsilon$, we define $p_\epsilon \in \mathbb{C}[u]$ to be the monic polynomial of degree $k = mNd$, whose set of roots consists of m copies of each of the numbers $q_1^\epsilon, \dots, q_{Nd}^\epsilon$.

Let W_{p_ϵ} be the kd-dimensional vector space of quasi-exponentials assigned to the polynomial p_ϵ in Section 4.1. By assumptions of Theorem 4.1, for every ϵ, $0 < \epsilon < \bar\epsilon$, the space W_{p_ϵ} contains an Nd-dimensional subspace U_ϵ generating the kernel of $\mathfrak{D}_\epsilon$. This subspace determines a point in the Grassmannian $Gr(W_{p_\epsilon}, Nd)$ of Nd-dimensional subspaces of W_{p_ϵ}.

The map $\rho_{p_\epsilon} : W_{p_\epsilon} \to V^{\oplus k}$ identifies the Grassmannian $\mathrm{Gr}(W_{p_\epsilon}, Nd)$ with the Grassmannian $\mathrm{Gr}(V^{\oplus k}, Nd)$. The points $\rho_{p_\epsilon}(U_\epsilon)$ all lie in the compact manifold $\mathrm{Gr}(V^{\oplus k}, Nd)$ and the set of all such points has an accumulation point $\tilde U \in \mathrm{Gr}(V^{\oplus k}, Nd)$ as ϵ tends to zero. Then $\rho^{-1}_{p_{\epsilon=0}}(\tilde U) \subset \mathrm{Gr}(W_{p_{\epsilon=0}}, Nd)$ is an Nd-dimensional subspace of V-valued quasi-exponentials. Using Lemma 4.1, we conclude that the space $\rho^{-1}_{p_{\epsilon=0}}(\tilde U)$ generates the kernel of $\mathfrak{D}_{\epsilon=0}$. ∎

5 Bethe ansatz

5.1 Preliminaries

Consider $\mathbb{C}^N$ as the $\mathfrak{gl}_N$-module with highest weight $(1,0,\dots,0)$. For complex numbers $z_1, \dots,$ gz_n, denote $\boldsymbol{z} = (z_1, \dots, z_n)$, and

$$M(\boldsymbol{z}) = \mathbb{C}^N(z_1) \otimes \cdots \otimes \mathbb{C}^N(z_n),$$

which is a polynomial $Y(\mathfrak{gl}_N)$-module. Let

$$M(\boldsymbol{z}) = \oplus_{m_1,\dots,m_N} M(\boldsymbol{z})[m_1, \dots, m_N]$$

be its $\mathfrak{gl}_N$-weight decomposition with respect to the Cartan subalgebra of diagonal matrices. The weight subspace $M(\boldsymbol{z})[m_1, \dots, m_N]$ is nonzero if and only if $m_1, \dots, m_N$ are non-negative integers.

Assume that $Q = \operatorname{diag}(Q_1, \dots, Q_N)$ is a diagonal non-degenerate $N \times N$-matrix with distinct diagonal entries, and consider the universal difference operator

$$\mathfrak{D}_{Q,\,M(\boldsymbol{z})}(u,\tau) = \sum_{k=0}^{N} (-1)^k\, \mathfrak{T}_{k,Q,M(\boldsymbol{z})}(u)\, \tau^{-k}.$$

associated with $M(\boldsymbol{z})$ and Q. Acting on $M(\boldsymbol{z})$-valued functions the operator $\mathfrak{D}_{Q,M(\boldsymbol{z})}(u,\tau)$ preserves the weight decomposition.

In this section we shall study the kernel of this difference operator, restricted to $M(\boldsymbol{z})[m_1, \dots,$ $m_N]$-valued functions.

5.2 Bethe ansatz equations associated with a weight subspace

Consider a nonzero weight subspace $M(\boldsymbol{z})[m_1,\dots,m_N]$. Introduce $\boldsymbol{l}=(l_1,\dots,l_{N-1})$ with $l_j=m_{j+1}+\dots+m_N$. We have $n\geqslant l_1\geqslant\dots\geqslant l_{N-1}\geqslant 0$. Set $l_0=l_N=0$ and $l=l_1+\dots+l_{N-1}$. We shall consider functions of l variables

$$\boldsymbol{t}=\big(t_1^{(1)},\dots,t_{l_1}^{(1)},t_1^{(2)},\dots,t_{l_2}^{(2)},\dots,t_1^{(N-1)},\dots,t_{l_{N-1}}^{(N-1)}\big).$$

The following system of l algebraic equations with respect to l variables $\boldsymbol{t}$ is called *the Bethe ansatz equations* associated with $M(\boldsymbol{z})[m_1,\dots,m_N]$ and Q,

$$\begin{aligned}
&Q_1\prod_{s=1}^{n}\big(t_j^{(1)}-z_s+1\big)\prod_{\substack{j'=1\\ j'\neq j}}^{l_1}\big(t_j^{(1)}-t_{j'}^{(1)}-1\big)\prod_{j'=1}^{l_2}\big(t_j^{(1)}-t_{j'}^{(2)}\big)\\
&\qquad=Q_2\prod_{s=1}^{n}\big(t_j^{(1)}-z_s\big)\prod_{\substack{j'=1\\ j'\neq j}}^{l_1}\big(t_j^{(1)}-t_{j'}^{(1)}+1\big)\prod_{j'=1}^{l_2}\big(t_j^{(1)}-t_{j'}^{(2)}-1\big),\\
&Q_a\prod_{j'=1}^{l_{a-1}}\big(t_j^{(a)}-t_{j'}^{(a-1)}+1\big)\prod_{\substack{j'=1\\ j'\neq j}}^{l_a}\big(t_j^{(a)}-t_{j'}^{(a)}-1\big)\prod_{j'=1}^{l_{a+1}}\big(t_j^{(a)}-t_{j'}^{(a+1)}\big)\\
&\qquad=Q_{a+1}\prod_{j'=1}^{l_{a-1}}\big(t_j^{(a)}-t_{j'}^{(a-1)}\big)\prod_{\substack{j'=1\\ j'\neq j}}^{l_a}\big(t_j^{(a)}-t_{j'}^{(a)}+1\big)\prod_{j'=1}^{l_{a+1}}\big(t_j^{(a)}-t_{j'}^{(a+1)}-1\big),\\
&Q_{N-1}\prod_{j'=1}^{l_{N-2}}\big(t_j^{(a)}-t_{j'}^{(a-1)}+1\big)\prod_{\substack{j'=1\\ j'\neq j}}^{l_{N-1}}\big(t_j^{(a)}-t_{j'}^{(a)}-1\big)\\
&\qquad=Q_N\prod_{j'=1}^{l_{N-2}}\big(t_j^{(a)}-t_{j'}^{(a-1)}\big)\prod_{\substack{j'=1\\ j'\neq j}}^{l_{N-1}}\big(t_j^{(a)}-t_{j'}^{(a)}+1\big).
\end{aligned}\tag{5.1}$$

Here the equations of the first group are labeled by $j=1,\dots,l_1$, the equations of the second group are labeled by $a=2,\dots,N-2$, $j=1,\dots,l_a$, the equations of the third group are labeled by $j=1,\dots,l_{N-1}$.

A solution $\tilde{\boldsymbol{t}}$ of system (5.1) will be called *off-diagonal* if $\tilde{t}_j^{(a)}\neq\tilde{t}_{j'}^{(a)}$ for any $a=1,\dots,N-1$, $1\leqslant j\leqslant j'\leqslant l_a$, and $\tilde{t}_j^{(a)}\neq\tilde{t}_{j'}^{(a+1)}$ for any $a=1,\dots,N-2$, $j=1,\dots,l_a$, $j'=1,\dots,l_{a+1}$.

5.3 Weight function and Bethe ansatz theorem

Denote by $\omega(\boldsymbol{t},\boldsymbol{z})$ *the universal weight function* associated with the weight subspace $M(\boldsymbol{z})[m_1,\dots,m_N]$. The universal weight function is defined in formula (6.2) in [6], see explicit formula (5.2) below. At this moment, it is enough for us to know that this function is an $M(\boldsymbol{z})[m_1,\dots,m_N]$-valued polynomial in $\boldsymbol{t}$, $\boldsymbol{z}$.

If $\tilde{\boldsymbol{t}}$ is an off-diagonal solution of the Bethe ansatz equations, then the vector $\omega(\tilde{\boldsymbol{t}},\boldsymbol{z})\in M(\boldsymbol{z})[m_1,\dots,m_N]$ is called *the Bethe vector* associated with $\tilde{\boldsymbol{t}}$.

Theorem 5.1. *Let Q be a diagonal matrix and $\tilde{\boldsymbol{t}}$ an off-diagonal solution of the Bethe ansatz equations* (5.1)*. Assume that the Bethe vector $\omega(\tilde{\boldsymbol{t}},\boldsymbol{z})$ is nonzero. Then the Bethe vector is an eigenvector of all transfer-matrices $\mathfrak{T}_{k,Q,M(\boldsymbol{z})}(u)$, $k=0,\dots,N$.*

The statement follows from Theorem 6.1 in [6]. For $k=1$, the result is established in [3].

The eigenvalues of the Bethe vector are given by the following construction. Set

$$\chi^1(u,\boldsymbol{t},\boldsymbol{z}) = Q_1 \prod_{s=1}^{n} \frac{u-z_s+1}{u-z_s} \prod_{j=1}^{l_1} \frac{u-t_j^{(1)}-1}{u-t_j^{(1)}},$$

$$\chi^a(u,\boldsymbol{t},\boldsymbol{z}) = Q_a \prod_{j=1}^{l_{a-1}} \frac{u-t_j^{(a-1)}-1}{u-t_j^{(1)}+1} \prod_{j=1}^{l_a} \frac{u-t_j^{(1)}-1}{u-t_j^{(1)}},$$

for $a=2,\dots,N$. Define the functions $\lambda_k(u,\boldsymbol{t},\boldsymbol{z})$ by the formula

$$\bigl(1-\chi^1(u,\boldsymbol{t},\boldsymbol{z})\,\tau^{-1}\bigr)\cdots\bigl(1-\chi^N(u,\boldsymbol{t},\boldsymbol{z})\,\tau^{-1}\bigr) = \sum_{k=0}^{N} (-1)^k\,\lambda_k(u,\boldsymbol{t},\boldsymbol{z})\,\tau^{-k}.$$

Then

$$\mathfrak{T}_{k,Q,M(\boldsymbol{z})}(u)\,\omega(\tilde{\boldsymbol{t}},\boldsymbol{z}) = \lambda_k(u,\tilde{\boldsymbol{t}},\boldsymbol{z})\,\omega(\tilde{\boldsymbol{t}},\boldsymbol{z})$$

for $k=0,\dots,N$, see Theorem 6.1 in [6].

5.4 Difference operator associated with an off-diagonal solution

Let $\tilde{\boldsymbol{t}}$ be an off-diagonal solution of the Bethe ansatz equations. The scalar difference operator

$$\mathfrak{D}_{\tilde{\boldsymbol{t}}}(u,\tau) = \sum_{k=0}^{N} (-1)^k\,\lambda_k(u,\tilde{\boldsymbol{t}},\boldsymbol{z})\,\tau^{-k}$$

will be called the associated *fundamental difference operator*.

Theorem 5.2. *The kernel of $\mathfrak{D}_{\tilde{\boldsymbol{t}}}(u,\tau)$ is generated by quasi-exponentials of degree bounded from above by a function in n and N.*

This is Proposition 7.6 in [9], which is a generalization of Proposition 4.8 in [8].

5.5 Completeness of the Bethe ansatz

Theorem 5.3. *Let $z_1,\dots,z_n$ and $Q=\operatorname{diag}(Q_1,\dots,Q_N)$ be generic. Then the Bethe vectors form a basis in $M(\boldsymbol{z})[m_1,\dots,m_N]$.*

Theorem 5.3 will be proved in Section 5.6.

Corollary 5.4. *Theorems* 5.3 *and* 4.4 *imply Theorem* 3.8.

Proof. Theorems 5.2 and 5.3 imply that the statement of Theorem 3.8 holds if the tensor product $M(\boldsymbol{z})$ is considered for generic $\boldsymbol{z}$ and generic diagonal Q. Then according to the remark in Section 3.2.5, the statement of Theorem 3.8 holds if the tensor product $M(\boldsymbol{z})$ is considered for generic $\boldsymbol{z}$ and generic (not necessarily diagonal) Q. Then the remark in Section 3.1.9 and Theorem 4.4 imply that the statement of Theorem 3.8 holds for the tensor product of any polynomial finite-dimensional irreducible $Y(\mathfrak{gl}_N)$-modules and any $Q\in GL_N$. Hence the statement of Theorem 3.8 holds for direct sums of tensor products of polynomial finite-dimensional irreducible $Y(\mathfrak{gl}_N)$-modules and any $Q\in GL_N$. ■

5.6 Proof of Theorem 5.3

5.6.1. For a nonzero weight subspace $M(\boldsymbol{z})[m_1,\dots,m_N]\subset M(\boldsymbol{z})$ denote by $d[m_1,\dots,m_N]$ its dimension. Let $n\geqslant l_1\geqslant\dots\geqslant l_{N-1}\geqslant 0$ be the numbers defined in Section 5.2.

A vector $\boldsymbol{a}=(a_1,\dots,a_n)$ with coordinates a_i from the set $\{0,2,3,\dots,N\}$ will be called *admissible* if for any $j=1,\dots,N-1$ we have $l_j = \#\{\,a_i\mid i=1,\dots,n,\text{ and } a_i>j\,\}$ In other words, $\boldsymbol{a}$ is admissible if $m_j = \#\{\,a_i\mid i=1,\dots,n,\text{ and } a_i=j\}$.

If $\boldsymbol{a}=(a_1,\dots,a_n)$ is admissible, then for $j=1,\dots,N-1$, there exists a unique increasing map $\rho_{\boldsymbol{a},i}:\{1,\dots,l_i\}\to\{1,\dots,n\}$ such that $\#\rho^{-1}_{\boldsymbol{a},i}(j)=1$ if $a_j>i$ and $\#\rho^{-1}_{\boldsymbol{a},i}(j)=0$ if $a_j\leqslant i$.

We order admissible vectors lexicographically: we say that $\boldsymbol{a}>\boldsymbol{a}'$ if $a_N=a'_N$, $a_{N-1}=a'_{N-1}$, $\dots$, $a_i=a'_i$, $a_{i-1}>a_{i-1}$ for some i.

5.6.2. Let $v=(1,0,\dots,0)\in\mathbb{C}^N$ be the highest weight vector. Consider the set of vectors $e_{\boldsymbol{a}}\boldsymbol{v}=e_{a_1,1}v\otimes\dots\otimes e_{a_n,1}v\in M(\boldsymbol{z})$ labeled by admissible indices $\boldsymbol{a}$. Here $e_{a_i,1}v$ denotes v if $a_i=0$. Then this set of vectors is a basis of the weight subspace $M(\boldsymbol{z})[m_1,\dots,m_N]$. In particular, the total number of admissible indices equals $d[m_1,\dots,m_N]$.

5.6.3. For a function $f(u_1,\dots,u_k)$ set

$$\operatorname{Sym}_{u_1,\dots,u_k} f(u_1,\dots,u_k)=\sum_{\sigma\in S_k} f(u_{\sigma_1},\dots,u_{\sigma_k}),$$

where the sum is over all permutations σ of $\{1,\dots,k\}$.

Lemma 5.5. *The universal weight function $\omega(\boldsymbol{t},\boldsymbol{z})$ is given by the rule*

$$\omega(\boldsymbol{t},\boldsymbol{z})=\prod_{s=1}^{n}\prod_{j=1}^{l_1}\big(t^{(1)}_j-z_s\big)\prod_{b=2}^{N-1}\prod_{i=1}^{l_{b-1}}\prod_{j=1}^{l_b}\big(t^{(b)}_j-t^{(b-1)}_i\big) \tag{5.2}$$
$$\times\sum_{\boldsymbol{a}}\operatorname{Sym}_{t^{(1)}_1,\dots,t^{(1)}_{l_1}}\cdots\operatorname{Sym}_{t^{(N-1)}_1,\dots,t^{(N-1)}_{l_{N-1}}}\Bigg[\prod_{\substack{s=1\\ a_s>1}}^{n}\Bigg(\frac{1}{t^{(1)}_{\rho^{-1}_{\boldsymbol{a},1}(s)}-z_s}\prod_{r=1}^{s-1}\frac{t^{(1)}_{\rho^{-1}_{\boldsymbol{a},1}(s)}-z_r+1}{t^{(1)}_{\rho^{-1}_{\boldsymbol{a},1}(s)}-z_r}\Bigg)$$
$$\times\prod_{b=2}^{N-1}\prod_{\substack{s=1\\ a_s>b}}^{n}\Bigg(\frac{1}{t^{(b)}_{\rho^{-1}_{\boldsymbol{a},b}(s)}-t^{(b-1)}_{\rho^{-1}_{\boldsymbol{a},b-1}(s)}}\prod_{\substack{r=1\\ a_r\geqslant b}}^{s-1}\frac{t^{(b)}_{\rho^{-1}_{\boldsymbol{a},b}(s)}-t^{(b-1)}_{\rho^{-1}_{\boldsymbol{a},b-1}(r)}+1}{t^{(b)}_{\rho^{-1}_{\boldsymbol{a},b}(s)}-t^{(b-1)}_{\rho^{-1}_{\boldsymbol{a},b-1}(r)}}\Bigg)\Bigg]e_{\boldsymbol{a}}\boldsymbol{v},$$

where the sum is over all admissible $\boldsymbol{a}$.

The lemma follows from formula (3.3), formula (6.2) in [6], and Corollaries 3.5, 3.7 in [14].

5.6.4. Let $z_1,\dots,z_n$ be real numbers such that $z_{i+1}-z_i>N$. Assume that $Q=\operatorname{diag}(1,q,\dots,q^{N-1})$, where q is a nonzero parameter. Then the Bethe ansatz equations (5.1) take the form

$$\prod_{s=1}^{n}\big(t^{(1)}_j-z_s+1\big)\prod_{\substack{j'=1\\ j'\neq j}}^{l_1}\big(t^{(1)}_j-t^{(1)}_{j'}-1\big)\prod_{j'=1}^{l_2}\big(t^{(1)}_j-t^{(2)}_{j'}\big) \tag{5.3}$$
$$=q\prod_{s=1}^{n}\big(t^{(1)}_j-z_s\big)\prod_{\substack{j'=1\\ j'\neq j}}^{l_1}\big(t^{(1)}_j-t^{(1)}_{j'}+1\big)\prod_{j'=1}^{l_2}\big(t^{(1)}_j-t^{(2)}_{j'}-1\big),$$

$$\prod_{j'=1}^{l_{a-1}} \big(t_j^{(a)} - t_{j'}^{(a-1)} + 1\big) \prod_{\substack{j'=1\\ j'\neq j}}^{l_a} \big(t_j^{(a)} - t_{j'}^{(a)} - 1\big) \prod_{j'=1}^{l_{a+1}} \big(t_j^{(a)} - t_{j'}^{(a+1)}\big)$$
$$= q \prod_{j'=1}^{l_{a-1}} \big(t_j^{(a)} - t_{j'}^{(a-1)}\big) \prod_{\substack{j'=1\\ j'\neq j}}^{l_a} \big(t_j^{(a)} - t_{j'}^{(a)} + 1\big) \prod_{j'=1}^{l_{a+1}} \big(t_j^{(a)} - t_{j'}^{(a+1)} - 1\big),$$
$$\prod_{j'=1}^{l_{N-2}} \big(t_j^{(a)} - t_{j'}^{(a-1)} + 1\big) \prod_{\substack{j'=1\\ j'\neq j}}^{l_{N-1}} \big(t_j^{(a)} - t_{j'}^{(a)} - 1\big) = q \prod_{j'=1}^{l_{N-2}} \big(t_j^{(a)} - t_{j'}^{(a-1)}\big) \prod_{\substack{j'=1\\ j'\neq j}}^{l_{N-1}} \big(t_j^{(a)} - t_{j'}^{(a)} + 1\big).$$

For $q = 0$ the right hand sides of equations (5.3) equal zero and to solve the equations for $q = 0$ one needs to find common zeros of the left hand sides.

Let $\boldsymbol{a} = (a_1, \dots, a_N)$ be an admissible index. Define $\tilde{\boldsymbol{t}}(\boldsymbol{a}, 0) = (\tilde{t}_j^{(i)}(\boldsymbol{a}, 0))$ to be the point in $\mathbb{C}^l$ with coordinates $\tilde{t}_j^{(i)}(\boldsymbol{a}, 0) = z_{\rho_{\boldsymbol{a},i}(j)} - i$ for all i, j. Then $\tilde{\boldsymbol{t}}(\boldsymbol{a}, 0)$ is an off-diagonal solution of system (5.3) for $q = 0$. That solution has multiplicity one.

Hence for every small nonzero q, there exists a unique point $\tilde{\boldsymbol{t}}(\boldsymbol{a}, q) = (\tilde{t}_j^{(i)}(\boldsymbol{a}, q))$, such that

- $\tilde{\boldsymbol{t}}(\boldsymbol{a}, q)$ is an off-diagonal solution of system (5.3) with the same q,
- $\tilde{\boldsymbol{t}}(\boldsymbol{a}, q)$ holomorphically depends on q and tends to $\tilde{\boldsymbol{t}}(\boldsymbol{a}, 0)$ as q tends to zero.

Therefore, for all i, j, we have

$$\tilde{t}_j^{(i)}(\boldsymbol{a}, q) = z_{\rho_{\boldsymbol{a},i}(j)} - i + O(q). \tag{5.4}$$

Lemma 5.6. *As q tends to zero, the Bethe vector $\omega(\tilde{\boldsymbol{t}}(\boldsymbol{a}, q), \boldsymbol{z})$ has the following asymptotics:*

$$\omega(\tilde{\boldsymbol{t}}(\boldsymbol{a}, q), \boldsymbol{z}) = C_{\boldsymbol{a}}\, e_{\boldsymbol{a}} v + O(q) + \cdots,$$

where $C_{\boldsymbol{a}}$ is a nonzero number and the dots denote a linear combination of basis vectors $e_{\boldsymbol{a}'} v$ with indices $\boldsymbol{a}'$ lexicographically greater than $\boldsymbol{a}$.

The lemma follows from formulae (5.4) and (5.2).

Lemma 5.6 implies Theorem 5.3.

6 Comparison theorem

6.1 The automorphism χ_f and the universal difference operator

Let $f(u)$ be a rational function in u whose Laurent expansion at $u = \infty$ has the form $f(u) = 1 + O(u^{-1})$. Then the map $\chi_f : T(u) \mapsto f(u)\, T(u)$ defines an automorphism of the Yangian $Y(\mathfrak{gl}_N)$, see Section 3.1.4.

For a Yangian module V, denote by V^f the representation of the Yangian on the same vector space given by the rule

$$X|_{V^f} = (\chi_f(X))|_V$$

for any $X \in Y(\mathfrak{gl}_N)$.

Lemma 6.1. *Let V be an irreducible finite-dimensional $Y(\mathfrak{gl}_N)$-module with highest weight series $c_1(u), \dots, c_N(u)$. Then the Yangian module V^f is irreducible with highest weight series $f(u)c_1(u), \dots, f(u)c_N(u)$.*

Lemma 6.2. *Let V be a finite-dimensional polynomial $Y(\mathfrak{gl}_N)$-module. Then the corresponding transfer matrices satisfy the relation*

$$\mathfrak{T}_a(u)|_{V^f} = f(u)\cdots f(u-a+1)\,\mathfrak{T}_a(u)|_V$$

for $a = 1, \dots, N$.

6.2 Comparison of kernels

Let V be a finite-dimensional polynomial $Y(\mathfrak{gl}_N)$-module. Consider the universal difference operators $\mathfrak{D}_{Q,V}(u,\tau)$ and $\mathfrak{D}_{Q,V^f}(u,\tau)$. Let $C(u)$ be a function satisfying the equation

$$C(u) = f(u)\,C(u-1).$$

Then a V-valued function $g(u)$ belongs to the kernel of $\mathfrak{D}_{Q,V}(u,\tau)$ if and only if the function $C(u)g(u)$ belongs to the kernel of $\mathfrak{D}_{Q,V^f}(u,\tau)$.

6.2.1. Let $\Lambda = (\Lambda_1, \dots, \Lambda_N)$ be a dominant integral $\mathfrak{gl}_N$-weight and a a complex number. Denote $\Lambda(a) = (\Lambda_1 + a, \dots, \Lambda_N + a)$. Let $V = M_\Lambda(z)$ be an evaluation module and

$$f(u) = \frac{u-z}{u-z-a}.$$

Then $V^f = M_{\Lambda(a)}(z+a)$, see Section 3.1.8.

Let $\boldsymbol{\Lambda} = (\Lambda^{(1)}, \dots, \Lambda^{(n)})$ be a collection of integral dominant $\mathfrak{gl}_N$-weights, where $\Lambda^{(i)} = (\Lambda^{(i)}_1, \dots, \Lambda^{(i)}_N)$ for $i = 1, \dots, n$. Let $\boldsymbol{a} = (a_1, \dots, a_n)$ be a collection on complex numbers. Introduce the new collection $\boldsymbol{\Lambda}(\boldsymbol{a}) = (\Lambda^{(1)}(a_1), \dots, \Lambda^{(n)}(a_n))$.

For complex numbers $z_1, \dots, z_n$, consider the tensor products of evaluation modules

$$M_{\boldsymbol{\Lambda}}(\boldsymbol{z}) = M_{\Lambda^{(1)}}(z_1) \otimes \cdots \otimes M_{\Lambda^{(n)}}(z_n),$$
$$M_{\boldsymbol{\Lambda}(\boldsymbol{a})}(\boldsymbol{z}+\boldsymbol{a}) = M_{\Lambda^{(1)}(a_1)}(z_1 + a_1) \otimes \cdots \otimes M_{\Lambda^{(n)}(a_n)}(z_n + a_n).$$

Let $V = M_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ and

$$f(u) = \prod_{s=1}^{n} \frac{u - z_s}{u - z_s - a_s}.$$

Then $V^f = M_{\boldsymbol{\Lambda}(\boldsymbol{a})}(\boldsymbol{z}+\boldsymbol{a})$.

Theorem 6.3. *An $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})$-valued function $g(u)$ belongs to the kernel of $\mathfrak{D}_{Q,M_{\boldsymbol{\Lambda}}(\boldsymbol{z})}(u,\tau)$ if and only if the function $C(u)g(u)$ belongs to the kernel of $\mathfrak{D}_{Q,M_{\boldsymbol{\Lambda}(\boldsymbol{a})}(\boldsymbol{z}+\boldsymbol{a})}(u,\tau)$, where*

$$C(u) = \prod_{s=1}^{n} \frac{\Gamma(u - z_s + 1)}{\Gamma(u - z_s - a_s + 1)}.$$

6.2.2. Assume that $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ is a polynomial Yangian module and $\boldsymbol{a} = (a_1, \dots, a_N)$ are non-negative integers. Then $C(u)$ is a polynomial and $M_{\boldsymbol{\Lambda}(\boldsymbol{a})}(\boldsymbol{z}+\boldsymbol{a})$ is a polynomial Yangian module. In this case Theorem 6.3 allows us to compare the quasi-exponential kernels of the difference operators $\mathfrak{D}_{Q,M_{\boldsymbol{\Lambda}}(\boldsymbol{z})}(u,\tau)$ and $\mathfrak{D}_{Q,M_{\boldsymbol{\Lambda}(\boldsymbol{a})}(\boldsymbol{z}+\boldsymbol{a})}(u,\tau)$.

7 The kernel in the tensor product of evaluation modules

7.1 The second main result

Let $Q = \operatorname{diag}(Q_1, \dots, Q_N) \in GL_N$. Let $\boldsymbol{\Lambda} = (\Lambda^{(1)}, \dots, \Lambda^{(n)})$ be a collection of integral dominant $\mathfrak{gl}_N$-weights, where $\Lambda^{(i)} = (\Lambda^{(i)}_1, \dots, \Lambda^{(i)}_N)$ and $\Lambda^{(i)}_N = 0$ for $i = 1, \dots, n$.

Choose a weight subspace $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[m_1, \dots, m_N] \subset M_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ and consider the universal difference operator $\mathfrak{D}_{Q, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}$ associated with Q and $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[m_1, \dots, m_N]$.

Theorem 7.1. *Assume that the nonzero numbers $Q_1, \dots, Q_N$ are distinct and the argument of each of them is chosen. Then for any $i = 1, \dots, N$ and any nonzero vector $v_0 \in M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[m_1, \dots, m_N]$, there exists an $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[m_1, \dots, m_N]$-valued quasi-exponential*

$$Q_i^u \big(v_0\, u^{m_i} + v_1\, u^{m_i - 1} + \dots + v_{m_i} \big) \tag{7.1}$$

which lies in the kernel of $\mathfrak{D}_{Q, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}$. Moreover, all such quasi-exponentials generate the kernel of $\mathfrak{D}_{Q, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}$.

Proof. On one hand, by Theorem 3.8, the kernel of $\mathfrak{D}_{Q, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}$ is generated by quasi-exponentials. On the other hand, by Lemma 4.3 and part (i) of Theorem 3.6, any quasi-exponential lying in the kernel of $\mathfrak{D}_{Q, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}$ must be of the form (7.1), where v_0 is a nonzero vector. Since the kernel is of dimension $N \cdot \dim M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$, there exists a quasi-exponential of the form (7.1) with an arbitrary nonzero v_0. ■

7.1.1. For $i = 1, \dots, n$, denote $S_i = \{ z_i - 1,\, z_i - \Lambda^{(i)}_{N-1} - 2, \dots, z_i - \Lambda^{(i)}_1 - N \}$.

Theorem 7.2. *Assume that index i is such that $z_i - z_j \notin \mathbb{Z}$ for any $j \neq i$. Let $f(u)$ be an $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$-valued quasi-exponential lying in the kernel of $\mathfrak{D}_{Q, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}$. Then*

(i) $f(u)$ is uniquely determined by its values

$$v_N = f(z_i - 1),\ v_{N-1} = f\big(z_i - \Lambda^{(i)}_{N-1} - 2\big),\ \dots,\ v_1 = f\big(z_i - \Lambda^{(i)}_1 - N\big) \in M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$$

at the points of S_i.

(ii) If $v_N = v_{N-1} = \dots = v_j = 0$ for some $j > 1$, then $f(z_i - k) = 0$ for $k = 1, 2, \dots, \Lambda^{(i)}_{j-1} + N - j + 1$.

(iii) For any vectors $v_N, v_{N-1}, \dots, v_1 \in M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$, there exists a quasi-exponential $f(u)$ which lies in the kernel of $\mathfrak{D}_{Q, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}$ and takes these values at S_i.

Proof. Consider the polynomial difference operator

$$\begin{aligned}
\tilde{\mathfrak{D}} &= \prod_{s=1}^{n} \prod_{j=1}^{N-1} (u - z_s - j + 1)\, \mathfrak{D}_{Q, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]} \\
&= \prod_{s=1}^{n} \prod_{j=1}^{N-1} (u - z_s - j + 1) + \prod_{s=1}^{n} \prod_{j=2}^{N-1} (u - z_s - j + 1)\, \tilde{\mathfrak{T}}_1(u)\, \tau^{-1} + \cdots \\
&\quad + \prod_{s=1}^{n} (u - z_s - N + 2)\, \tilde{\mathfrak{T}}_{N-2}(u)\, \tau^{-N+2} + \tilde{\mathfrak{T}}_{N-1}(u)\, \tau^{-N+1} \\
&\quad + (-1)^N \prod_{i=1}^{N} Q_i \prod_{s=1}^{n} \prod_{i=1}^{N-1} \big(u - z_s + \Lambda^{(s)}_i - i + 1\big) \tau^{-N},
\end{aligned}$$

see notation in part (i) of Theorem 3.6. We have $(\tilde{\mathfrak{D}} f)(u) = 0$.

Condition $(\tilde{\mathfrak{D}}f)(z_i+N-2)=0$ gives the equation $f(z_i-2)=\mathrm{const}\, f(z_i-1)$ and thus the value $f(z_i-1)$ determines the value $f(z_i-2)$. Condition $(\tilde{\mathfrak{D}}f)(z_i+N-3)=0$ gives the equation $f(z_i-3)=\mathrm{const}\, f(z_i-2)$ and hence the value $f(z_i-2)$ determines the values $f(z_i-3)$. We may continue on this reasoning up to equation $f(z_i-\Lambda^{(i)}_{N-1}-1)=\mathrm{const}\, f(z_i-\Lambda^{(i)}_{N-1})$ which shows that the value $f(z_i-\Lambda^{(i)}_{N-1})$ determines the value $f(z_i-\Lambda^{(i)}_{N-1}-1)$.

Condition $(\tilde{\mathfrak{D}}f)(z_i+N-\Lambda^{(i)}_{N-1}-2)=0$ does not determine the value $f(z_i-\Lambda^{(i)}_{N-1}-2)$, since $f(z_i-\Lambda^{(i)}_{N-1}-2)$ enters that condition with coefficient 0. But condition $(\tilde{\mathfrak{D}}f)(z_i+N-\Lambda^{(i)}_{N-1}-3)=0$ gives the equation $f(z_i-\Lambda^{(i)}_{N-1}-3)=\mathrm{const}\, f(z_i-\Lambda^{(i)}_{N-1}-2)$ which shows that the value $f(z_i-\Lambda^{(i)}_{N-1}-3)$ is determined by the value $f(z_i-\Lambda^{(i)}_{N-1}-2)$.

Now we may continue on this reasoning up to the equation $f(z_i-\Lambda^{(i)}_{N-2}-2)=\mathrm{const}\, f(z_i-\Lambda^{(i)}_{N-2}-1)$ which determines $f(z_i-\Lambda^{(i)}_{N-2}-2)$ if $f(z_i-\Lambda^{(i)}_{N-2}-1)$ is known. Condition $(\tilde{\mathfrak{D}}f)(z_i+N-\Lambda^{(i)}_{N-2}-3)=0$ does not determine the value $f(z_i-\Lambda^{(i)}_{N-2}-3)$, but condition $(\tilde{\mathfrak{D}}f)(z_i+N-\Lambda^{(i)}_{N-2}-4)=0$ gives the equation $f(z_i-\Lambda^{(i)}_{N-2}-4)=\mathrm{const}\, f(z_i-\Lambda^{(i)}_{N-2}-3)$ and so on. Repeating this reasoning we prove parts (i) and (ii) of the theorem.

The same reasoning shows that if $f(u)=0$ for $u\in S_i$, then the quasi-exponential $f(u)$ identically equals zero. Since the kernel of $\tilde{\mathfrak{D}}$ is generated by quasi-exponentials, we obtain part (iii) of the theorem. ■

7.1.2. Assume that Q is the identity matrix. Consider the subspace $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]\subset M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ of $\mathfrak{gl}_N$-singular vectors and the associated universal difference operator $\mathfrak{D}_{Q=1,\,\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u,\tau)$.

Theorem 7.3. *For any $i=1,\dots,N$ and any nonzero vector $v_0\in \mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[m_1,\dots,m_N]$, there exists a $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[m_1,\dots,m_N]$-valued polynomial*

$$v_0\, u^{m_i+N-i}+v_1\, u^{m_i+N-i-1}+\dots+v_{m_i+N-i} \tag{7.2}$$

which lies in the kernel of $\mathfrak{D}_{Q=1,\,\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}$. Moreover, all such polynomials generate the kernel of $\mathfrak{D}_{Q=1,\,\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}$.

Proof. On one hand, by Theorem 3.8, the kernel of $\mathfrak{D}_{Q=1,\,\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}$ is generated by quasi-exponentials. On the other hand, by part (i) of Theorem 3.6, any quasi-exponential lying in the kernel of $\mathfrak{D}_{Q=1,\,\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}$ must a polynomial. By Theorem 3.7 such a polynomial has to be of the form indicated in (7.2), where v_0 is a nonzero vector. Since the kernel is of dimension $N\cdot\dim\, \mathrm{Sing}\, M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$, there exists a polynomial of the form (7.2) with an arbitrary nonzero v_0. ■

7.2 Kernel of the fundamental difference operator associated with an eigenvector

Assume that $v\in M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ is an eigenvector of all transfer matrices,

$$\mathfrak{T}_{Q,\,M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u)\, v=\lambda_{k,v}(u)\, v, \qquad k=0,\dots,N.$$

Then the scalar difference operator

$$\mathcal{D}_v(u,\tau_u)=\sum_{k=0}^{N}(-1)^k\,\lambda_{k,v}(u)\,\tau_u^{-k}$$

will be called *the fundamental difference operator* associated with the eigenvector v.

Theorems 7.1, 7.2 and 7.3 give us information on the kernel of the fundamental difference operator.

Corollary 7.4.

(i) Assume that $Q \in GL_N$ is diagonal with distinct diagonal entries. Then the kernel of $\mathfrak{D}_v(u, \tau_u)$ is generated by quasi-exponentials $Q_1^u p_1(u), \dots, Q_N^u p_N(u)$, where for every i the polynomial $p_i(u) \in \mathbb{C}[u]$ is of degree m_i.

(ii) Assume that Q is the identity matrix and the eigenvector v belongs to Sing $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$. *Then the kernel of $\mathfrak{D}_v(u, \tau_u)$ is generated by suitable polynomials $p_1(u), \dots, p_N(u)$ of degree $m_1 + N - 1, \dots, m_N$, respectively.*

Corollary 7.5. *Assume the index i is such that $z_i - z_j \notin \mathbb{Z}$ for any $j \neq i$. Let $f(u)$ be a quasi-exponential lying in the kernel of $\mathfrak{D}_v(u, \tau_u)$. Then*

(i) $f(u)$ is uniquely determined by its values

$$v_N = f(z_i - 1), \quad v_{N-1} = f\big(z_i - \Lambda_{N-1}^{(i)} - 2\big), \quad \dots, \quad v_1 = f\big(z_i - \Lambda_1^{(i)} - N\big)$$

at the points of S_i.

(ii) If $v_N = v_{N-1} = \dots = v_j = 0$ for some $j > 1$, then $f(z_i - k) = 0$ for $k = 1, 2, \dots, \Lambda_{j-1}^{(i)} + N - j + 1$.

(iii) For any numbers $v_N, v_{N-1}, \dots, v_1$ there exists a quasi-exponential $f(u)$ which lies in the kernel of $\mathfrak{D}_v(u, \tau_u)$ and takes these values at S_i.

If v is a Bethe eigenvector, then these two corollaries were proved in [8] and [9].

8 Generating operator of the Gaudin transfer matrices

8.1 Current algebra $\mathfrak{gl}_N[x]$

8.1.1. Let $\mathfrak{gl}_N[x]$ be the Lie algebra of polynomials with coefficients in $\mathfrak{gl}_N$ with point-wise commutator. The elements $e_{ab}^{\{s\}} = e_{ab} x^s$ with $a, b = 1, \dots, N$, $s = 0, 1, \dots$, span $\mathfrak{gl}_N[x]$. We have $[e_{ab}^{\{r\}}, e_{cd}^{\{s\}}] = \delta_{bc} e_{ad}^{\{r+s\}} - \delta_{ad} e_{cb}^{\{r+s\}}$. We shall identify $\mathfrak{gl}_N$ with the subalgebra of $\mathfrak{gl}_N[x]$ of constant polynomials.

For $a, b = 1, \dots, N$, we set

$$L_{ab}(u) = \sum_{s=0}^{\infty} e_{ba}^{\{s\}} u^{-s-1}$$

and $L(u) = \sum_{a,b} E_{ab} \otimes L_{ab}(u) \in \operatorname{End}(\mathbb{C}^N) \otimes \mathfrak{gl}_N[x]((u^{-1}))$.

Let V be a $\mathfrak{gl}_N$-module. For $z \in \mathbb{C}$ denote by $M(z)$ the corresponding evaluation $\mathfrak{gl}_N[x]$-module on the same vector space, where we define $e_{ab}^{\{s\}}|_V = z^s e_{ab}$ for all s, a, b. Then the series $L_{ab}(u)|_V$ converges to the $\operatorname{End}(V)$-valued function $(u - z)^{-1} e_{ba}$, which is a rational function in u.

Let $\Lambda = (\Lambda_1, \dots, \Lambda_N)$ be an integral dominant $\mathfrak{gl}_N$-weight, M_Λ the corresponding irreducible highest weight $\mathfrak{gl}_N$-module, $M_\Lambda(z)$ the associated evaluation $\mathfrak{gl}_N[x]$-module. Then the module $M_\Lambda(z)$ is called *polynomial* if Λ_N is a non-negative integer.

8.1.2. Let $\boldsymbol{\Lambda} = (\Lambda^{(1)}, \dots, \Lambda^{(n)})$ be a collection of integral dominant $\mathfrak{gl}_N$-weights, where $\Lambda^{(i)} = (\Lambda_1^{(i)}, \dots, \Lambda_N^{(i)})$ for $i = 1, \dots, n$. For $z_1, \dots, z_n \in \mathbb{C}$, we consider the tensor product of $\mathfrak{gl}_N[x]$-evaluation modules:

$$\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z}) = M_{\Lambda^{(1)}}(z_1) \otimes \dots \otimes M_{\Lambda^{(n)}}(z_n).$$

For any a, b, the series $L_{ab}(u)$ acts on $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ by the formula

$$L_{ab}(u)|_{\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})} = \sum_{j=1}^{n} \frac{e_{ba}^{(j)}}{u - z_j}.$$

If $z_1, \dots, z_N$ are distinct, then $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ is an irreducible $\mathfrak{gl}_N[x]$-module. Let

$$M_{\Lambda^{(1)}} \otimes \cdots \otimes M_{\Lambda^{(n)}} = \oplus_{\Lambda} M_{\Lambda}$$

be the decomposition of the tensor product of $\mathfrak{gl}_N$-modules into the direct sum of irreducible $\mathfrak{gl}_N$-modules. Then for any $z \in \mathbb{C}$,

$$M_{\Lambda^{(1)}}(z) \otimes \cdots \otimes M_{\Lambda^{(n)}}(z) = \oplus_{\Lambda} M_{\Lambda}(z)$$

is the decomposition of the tensor product of evaluation $\mathfrak{gl}_N[x]$-modules into the direct sum of irreducible $\mathfrak{gl}_N[x]$-modules.

8.1.3. Consider $\mathbb{C}^N$ as the $\mathfrak{gl}_N$-module with highest weight $(1, 0, \dots, 0)$. Let $\Lambda = (\Lambda_1, \dots, \Lambda_N)$ be an integral dominant $\mathfrak{gl}_N$-weight with $\Lambda_N \in \mathbb{Z}_{\geqslant 0}$. Then there exists $k \in \mathbb{Z}_{\geqslant 0}$ such that $(\mathbb{C}^N)^{\otimes k}$ contains M_Λ as a $\mathfrak{gl}_N$-submodule.

The previous remarks show that for any $z_1, \dots, z_n \in \mathbb{C}$, all irreducible submodules of the $\mathfrak{gl}_N[x]$-module $\mathbb{C}^N(z_1) \otimes \cdots \otimes \mathbb{C}^N(z_n)$ are tensor products of polynomial evaluation $\mathfrak{gl}_N[x]$-modules. Moreover, if V is a $\mathfrak{gl}_N[x]$-module which is the tensor product of polynomial evaluation $\mathfrak{gl}_N[x]$-modules, then there exist $z_1, \dots, z_n$ such that V is isomorphic to a submodule of the $\mathfrak{gl}_N[x]$-module $\mathbb{C}^N(z_1) \otimes \cdots \otimes \mathbb{C}^N(z_n)$.

8.1.4. Let $\pi : U(\mathfrak{gl}_N) \to \mathrm{End}\,(\mathbb{C}^N)$ be the representation homomorphism for the $\mathfrak{gl}_N$-module $\mathbb{C}^N$. Clearly, for any $x \in U(\mathfrak{gl}_N)$ we have

$$\bigl[\pi(x) \otimes 1 + 1 \otimes x,\, L(u)\bigr] = 0. \tag{8.1}$$

For a non-degenerate matrix $A \in \mathrm{End}\,(\mathbb{C}^N)$, define an automorphism ν_A of $\mathfrak{gl}_N[x]$ by the formula

$$(\mathrm{id} \otimes \nu_A)\bigl(L(u)\bigr) = \sum_{ab} A^{-1} E_{ab} A \otimes L_{ab}(u).$$

Let V be a finite-dimensional $\mathfrak{gl}_N[x]$-module with the representation $\mu : \mathfrak{gl}_N[x] \to \mathrm{End}\,(V)$ and $\tilde{\mu} : GL_N \to \mathrm{End}\,(V)$ the corresponding representation of the group GL_N. The automorphism ν_A induces a new $\mathfrak{gl}_N[x]$-module structure V^A on the same vector space with the representation $\mu_A = \mu \circ \nu_A$. Formula (8.1) yields that for any $x \in \mathfrak{gl}_N[x]$,

$$\mu_A(x) = \tilde{\mu}(A)\, \mu(x) \bigl(\tilde{\mu}(A)\bigr)^{-1},$$

that is, the $\mathfrak{gl}_N[x]$-modules V and V^A are isomorphic. In particular, if V is the tensor product of polynomial evaluation $\mathfrak{gl}_N[x]$-modules, then V^A is the tensor product of polynomial evaluation $\mathfrak{gl}_N[x]$-modules too.

8.2 Fundamental differential operator

Let $K = (K_{ab})$ be an $N \times N$ matrix with complex entries. For $a, b = 1, \dots, N$, define the differential operator

$$X_{ab}(u, \partial_u) = \delta_{ab}\, \partial_u - K_{ab} - L_{ab}(u),$$

where $\partial_u = d/du$.

Following [13], introduce the differential operator

$$\mathcal{D}(u,\partial_u) = \sum_{\sigma\in S_N} (-1)^\sigma\, X_{1\,\sigma_1}(u,\partial_u)\, X_{2\,\sigma_2}(u,\partial_u)\cdots X_{N\sigma_N}(u,\partial_u),$$

where the sum is over all permutations σ of $\{1,\dots,N\}$. The operator $\mathcal{D}(u,\partial_u)$ will be called *the universal differential operator* associated with the matrix K.

Lemma 8.1. *Let π be a map $\{1,\dots,N\}\to\{1,\dots,N\}$. If π is a permutation of $\{1,\dots,N\}$, then*

$$\sum_{\sigma\in S_N} (-1)^\sigma\, X_{\pi_1\sigma_1}(u,\tau)\, X_{\pi_2\sigma_2}(u,\tau)\cdots X_{\pi_N\sigma_N}(u,\tau) = (-1)^\pi\, \mathcal{D}(u,\partial_u).$$

If π is not bijective, then

$$\sum_{\sigma\in S_N} (-1)^\sigma\, X_{\pi_1\sigma_1}(u,\tau)\, X_{\pi_2\sigma_2}(u,\tau)\cdots X_{\pi_N\sigma_N}(u,\tau) = 0.$$

The statement is Proposition 8.1 in [6].

8.2.1. Introduce the coefficients $\mathcal{S}_0(u),\dots,\mathcal{S}_N(u)$ of $\mathcal{D}(u,\partial_u)$:

$$\mathcal{D}(u,\partial_u) = \sum_{k=0}^{N} (-1)^k\, \mathcal{S}_k(u)\, \partial_u^{N-k},$$

in particular, $\mathcal{S}_0(u)=1$. The coefficients $\mathcal{S}_k(u)$ are called *the transfer matrices of the Gaudin type model* associated with K.

The main properties of the transfer matrices:

(i) the transfer matrices commute: $[\mathcal{S}_k(u),\mathcal{S}_l(v)]=0$ for all k, l, u, v,

(ii) if K is a diagonal matrix, then the transfer matrices preserve the $\mathfrak{gl}_N$-weight: $[\mathcal{S}_k(u),e_{aa}]=0$ for all k, a, u,

(iii) if K is the zero matrix, then the transfer matrices commute with the subalgebra $U(\mathfrak{gl}_N)$: $[\mathcal{S}_k(u),x]=0$ for all k,u and $x\in U(\mathfrak{gl}_N)$,

see [13, 6].

8.2.2. If V is the tensor product of evaluation finite-dimensional $\mathfrak{gl}_N[x]$-modules, then the universal operator $\mathcal{D}(u,\partial_u)$ induces a differential operator acting on V-valued functions in u. This operator will be called *the universal differential operator* associated with K and V and denoted by $\mathcal{D}_{K,V}(u,\partial_u)$. The linear operators $\mathcal{S}_k(u)|_V\in \mathrm{End}\,(V)$ will be called *the transfer matrices* associated with K and V and denoted by $\mathcal{S}_{k,K,V}(u)$. They are rational functions in u.

8.2.3. If $\mathcal{D}_K(u,\tau)$ is the universal differential operator associated with the matrix K and $\nu_A:\mathfrak{gl}_N[x]\to\mathfrak{gl}_N[x]$ is the automorphism defined in Section 8.1.4. Then Lemma 8.1 implies that

$$\nu_A(\mathcal{D}_K(u,\partial_u)) = \mathcal{D}_{AKA^{-1}}(u,\partial_u)$$

is the universal differential operator associated with the matrix AKA^{-1}, cf. Lemma 3.5.

Let V be the tensor product of finite-dimensional evaluation $\mathfrak{gl}_N[x]$-modules, and $\tilde\mu: G_N\to GL(V)$ the associated GL_N-representation. Then

$$\mathcal{D}_{AKA^{-1}}(u,\partial_u)|_V = \tilde\mu(A)\,\mathcal{D}_K(u,\partial_u)|_V\,\tilde\mu\big(A^{-1}\big).$$

8.3 More properties of the Gaudin type transfer matrices

Let $K = \operatorname{diag}(K_1, \dots, K_N)$. Let $\boldsymbol{\Lambda} = (\Lambda^{(1)}, \dots, \Lambda^{(n)})$ be a collection of integral dominant $\mathfrak{gl}_N$-weights, where $\Lambda^{(i)} = (\Lambda^{(i)}_1, \dots, \Lambda^{(i)}_N)$ and $\Lambda^{(i)}_N = 0$ for $i = 1, \dots, n$.

For $\boldsymbol{m} = (m_1, \dots, m_N)$ denote by $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}] \subset \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ the weight subspace of the $\mathfrak{gl}_N$-weight $\boldsymbol{m}$ and by $\operatorname{Sing} \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}] \subset \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ the subspace of $\mathfrak{gl}_N$-singular vectors.

Consider the universal differential operator

$$\mathcal{D}_{K,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})}(u, \partial_u) = \sum_{k=0}^{N} (-1)^k \, \mathcal{S}_{k,K,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})}(u) \, \partial_u^{N-k}$$

associated with K and $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})$. We have $\mathcal{S}_{0,K,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u) = 1$ and

$$\mathcal{S}_{k,K,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u) = \mathcal{S}_{k0} + \mathcal{S}_{k1} u^{-1} + \mathcal{S}_{k2} u^{-2} + \cdots$$

for suitable $\mathcal{S}_{ki} \in \operatorname{End}(\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}])$.

Theorem 8.2. *The following statements hold.*

(i) The operators $\mathcal{S}_{10}, \mathcal{S}_{20}, \dots, \mathcal{S}_{N0}$ and $\mathcal{S}_{11}, \mathcal{S}_{21}, \dots, \mathcal{S}_{N1}$ are scalar operators. Moreover, the following relations hold:

$$x^N + \sum_{k=1}^{N} (-1)^k \, \mathcal{S}_{k0} \, x^{N-k} = \prod_{i=1}^{N} (x - K_i),$$
$$\sum_{k=1}^{N} (-1)^k \, \mathcal{S}_{k1} \, x^{N-k} = -\prod_{i=1}^{N} (x - K_i) \sum_{j=1}^{N} \frac{m_j}{x - K_j}.$$

(ii) For $k = 1, \dots, N-1$, we have

$$\mathcal{S}_{k,Q,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})}(u) = \tilde{\mathcal{S}}_k(u) \prod_{s=1}^{n} \frac{1}{(u - z_s)^k}, \tag{8.2}$$

where $\tilde{\mathcal{S}}_k(u)$ is a polynomial in u of degree nk. Moreover, the operators

$$\bar{\mathcal{S}}_{k,r} = \tilde{\mathcal{S}}_k(z_r) \prod_{s=1, s\neq r}^{n} \frac{1}{(z_r - z_s)^k}$$

are scalar and

$$\sum_{k=0}^{N} (-1)^k \, \bar{\mathcal{S}}_{k,s} \prod_{j=0}^{N-r-1} (d - j) = \prod_{i=1}^{N} \big(d - \Lambda^{(r)}_i - N + i\big). \tag{8.3}$$

Proof. Part (i) follows from Proposition B.1 in [6]. The existence of presentation (8.2) follows from the definition of the universal differential operator. To prove equation (8.3) it is enough to notice that the leading singular term of the universal differential operator at $u = z_r$ is equal to the leading singular term of the universal differential operator associated with one evaluation module $M_{\Lambda^{(r)}}(z_r)$, which in its turn expresses via the quantum determinant. ■

8.3.1. Assume that K is the zero matrix. Then the associated transfer matrices preserve $\operatorname{Sing} \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ and we may consider the universal differential operator

$$\mathcal{D}_{K=0,\,\operatorname{Sing}\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u,\partial_u) = \sum_{k=0}^{N} (-1)^k\, \mathcal{S}_k(u)\, \partial_u^{N-k}$$

acting on $\operatorname{Sing} \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$-valued functions of u.

Theorem 8.3. *For $k = 1,\dots,N$, the coefficients $\mathcal{S}_k(u)$ have the following Laurent expansion at $u = \infty$:*

$$\mathcal{S}_k(u) = \mathcal{S}_{k,0}u^{-k} + \mathcal{S}_{k,1}u^{-k-1} + \cdots,$$

where the operators $\mathcal{S}_{1,0},\dots,\mathcal{S}_{N,0}$ are scalar operators. Moreover,

$$\sum_{k=0}^{N}(-1)^k\mathcal{S}_{k,0}\prod_{j=0}^{N-k-1}(d-j) = \prod_{s=1}^{N}(d-m_s-N+s).$$

The proof of Theorem 8.3 is similar to the proof of Theorem 3.7.

8.4 First main result in the Gaudin case

Theorem 8.4. *Let K be an $N\times N$-matrix and $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ the tensor product of polynomial $\mathfrak{gl}_N[x]$-modules. Consider the universal differential operator $\mathcal{D}_{K,\,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})}(u,\partial_u)$ associated with K and $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})$. Then the kernel of $\mathcal{D}_{K,\,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})}(u,\partial_u)$ is generated by quasi-exponentials.*

A statement of this type was conjectured in [1]. Theorem 8.4 will be proved in Section 10.

9 Continuity principle for differential operators with quasi-exponential kernel

9.1 Quasi-exponentials

Let V be a complex vector space of dimension d. Let $A_0(u),\dots,A_N(u)$ be $\operatorname{End}(V)$-valued rational functions in u. Assume that each of these functions has limit as $u\to\infty$ and $A_0(u)=1$ in $\operatorname{End}(V)$. Then the differential operator

$$\mathcal{D} = \sum_{k=0}^{N} A_k(u)\,\partial_u^{N-k},$$

acting on V-valued functions in u, will be called admissible at infinity.

For every k, let $A_k(u) = A^{\infty}_{k,0} + A^{\infty}_{k,1}u^{-1} + A^{\infty}_{k,2}u^{-2} + \cdots$ be the Laurent expansion at infinity. Consider the algebraic equation

$$\det\big(x^N + x^{N-1}A_{1,0} + \cdots + xA_{N-1,0} + A_{N,0}\big) = 0 \tag{9.1}$$

with respect to variable x.

Lemma 9.1. *If a nonzero V-valued quasi-exponential $e^{\lambda u}(u^d v_d + u^{d-1}v_{d-1} + \cdots + v_0)$ lies in the kernel of an admissible at infinity differential operator $\mathcal{D}$, then λ is a root of equation* (9.1).

9.2 Continuity principle

Let $A_0(u,\epsilon),\dots,A_N(u,\epsilon)$ be $\mathrm{End}\,(V)$-valued rational functions in u analytically depending on $\epsilon\in[0,1)$. Assume that

- for every $\epsilon\in[0,1)$ the difference operator $\mathcal{D}_\epsilon=\sum_{k=0}^{N}A_k(u,\epsilon)\,\partial_u^{N-k}$ is admissible at infinity,
- for every $\epsilon\in(0,1)$ the kernel of $\mathcal{D}_\epsilon$ is generated by quasi-exponentials,
- there exists a natural number m such that for every $\epsilon\in(0,1)$ all quasi-exponentials generating the kernel of $\mathcal{D}_\epsilon$ are of degree less than m.

Theorem 9.2. *Under these conditions the kernel of the differential operator $\mathcal{D}_{\epsilon=0}$ is generated by quasi-exponentials.*

The proof is similar to the proof of Theorem 4.4.

10 Bethe ansatz in the Gaudin case

10.1 Preliminaries

Consider $\mathbb{C}^N$ as the $\mathfrak{gl}_N$-module with highest weight $(1,0,\dots,0)$. For complex numbers $z_1,\dots,z_n$, denote

$$\mathcal{M}(\boldsymbol{z})=\mathbb{C}^N(z_1)\otimes\cdots\otimes\mathbb{C}^N(z_n),$$

which is the tensor product of polynomial $\mathfrak{gl}_N[x]$-modules. Let

$$\mathcal{M}(\boldsymbol{z})=\oplus_{m_1\geqslant\cdots\geqslant m_N\geqslant 0}\,\mathcal{M}(\boldsymbol{z})[\boldsymbol{m}]$$

be its $\mathfrak{gl}_N$-weight decomposition with respect to the Cartan subalgebra of diagonal matrices. Here $\boldsymbol{m}=(m_1,\dots,m_N)$.

Assume that $K=\mathrm{diag}\,(K_1,\dots,K_N)$ is a diagonal $N\times N$-matrix with distinct coordinates and consider the universal differential operator

$$\mathcal{D}_{K,\mathcal{M}(\boldsymbol{z})}(u,\partial_u)=\sum_{k=0}^{N}(-1)^k\,\mathcal{S}_{k,K,\mathcal{M}(\boldsymbol{z})}(u)\,\partial_u^{N-k}.$$

associated with $\mathcal{M}(\boldsymbol{z})$ and K. Acting on $\mathcal{M}(\boldsymbol{z})$-valued functions, the operator $\mathcal{D}_{K,\mathcal{M}(\boldsymbol{z})}(u,\partial_u)$ preserves the weight decomposition.

In this section we shall study the kernel of this operator restricted to $\mathcal{M}(\boldsymbol{z})[\boldsymbol{m}]$-valued functions.

10.2 Bethe ansatz equations associated with a weight subspace

Consider a nonzero weight subspace $\mathcal{M}(\boldsymbol{z})[\boldsymbol{m}]$. Introduce $\boldsymbol{l}=(l_1,\dots,l_{N-1})$ with $l_j=m_{j+1}+\cdots+m_N$. We have $n\geqslant l_1\geqslant\cdots\geqslant l_{N-1}\geqslant 0$. Set $l_0=l_N=0$ and $l=l_1+\cdots+l_{N-1}$. In what follows we shall consider functions of l variables

$$\boldsymbol{t}=\bigl(t_1^{(1)},\dots,t_{l_1}^{(1)},t_1^{(2)},\dots,t_{l_2}^{(2)},\dots,t_1^{(N-1)},\dots,t_{l_{N-1}}^{(N-1)}\bigr).$$

The following system of l algebraic equations with respect to l variables $\boldsymbol{t}$ is called *the Bethe ansatz equations* associated with $\mathcal{M}(\boldsymbol{z})[\boldsymbol{m}]$ and matrix K,

$$\sum_{s=1}^{n}\frac{1}{t_j^{(1)}-z_s}+\sum_{j'=1}^{l_2}\frac{1}{t_j^{(1)}-t_{j'}^{(2)}}-\sum_{\substack{j'=1\\ j'\neq j}}^{l_1}\frac{2}{t_j^{(1)}-t_{j'}^{(1)}}=K_2-K_1, \tag{10.1}$$

$$\sum_{j'=1}^{l_{a+1}} \frac{1}{t_j^{(a)} - t_{j'}^{(a+1)}} + \sum_{j'=1}^{l_{a-1}} \frac{1}{t_j^{(a)} - t_{j'}^{(a-1)}} - \sum_{\substack{j'=1 \\ j'\neq j}}^{l_a} \frac{2}{t_j^{(a)} - t_{j'}^{(a)}} = K_{a+1} - K_a,$$

$$\sum_{j'=1}^{l_{N-2}} \frac{1}{t_j^{(N-1)} - t_{j'}^{(N-2)}} - \sum_{\substack{j'=1 \\ j'\neq j}}^{l_{N-1}} \frac{2}{t_j^{(N-1)} - t_{j'}^{(N-1)}} = K_N - K_{N-1}.$$

Here the equations of the first group are labeled by $j = 1, \dots, l_1$, the equations of the second group are labeled by $a = 2, \dots, N-2$, $j = 1, \dots, l_a$, the equations of the third group are labeled by $j = 1, \dots, l_{N-1}$.

10.3 Weight function and Bethe ansatz theorem

We denote by $\omega(\boldsymbol{t}, \boldsymbol{z})$ *the universal weight function of the Gaudin type* associated with the weight subspace $\mathcal{M}(\boldsymbol{z})[\boldsymbol{m}]$. The universal weight function of the Gaudin type is defined in [12]. A convenient formula for $\omega(\boldsymbol{t}, \boldsymbol{z})$ is given in Appendix in [11] and Theorems 6.3 and 6.5 in [11].

If $\tilde{\boldsymbol{t}}$ is a solution of the Bethe ansatz equations, then the vector $\omega(\tilde{\boldsymbol{t}}, \boldsymbol{z}) \in \mathcal{M}(\boldsymbol{z})[\boldsymbol{m}]$ is called *the Bethe vector* associated with $\tilde{\boldsymbol{t}}$.

Theorem 10.1. *Let K be a diagonal matrix and $\tilde{\boldsymbol{t}}$ a solution of the Bethe ansatz equations* (10.1)*. Assume that the Bethe vector $\omega(\tilde{\boldsymbol{t}}, \boldsymbol{z})$ is nonzero. Then the Bethe vector is an eigenvector of all transfer-matrices $\mathcal{S}_{k,K,\mathcal{M}(\boldsymbol{z})}(u)$, $k = 0, \dots, N$.*

The statement follows from Theorem 9.2 in [6]. For $k = 1$, the result is established in [10].

The eigenvalues of the Bethe vector are given by the following construction. Set

$$\chi^1(u, \boldsymbol{t}, \boldsymbol{z}) = K_1 + \sum_{s=1}^{n} \frac{1}{u - z_s} - \sum_{i=1}^{l_1} \frac{1}{u - t_i^{(1)}},$$

$$\chi^a(u, \boldsymbol{t}, \boldsymbol{z}) = K_a + \sum_{i=1}^{l_{a-1}} \frac{1}{u - t_i^{(a-1)}} - \sum_{i=1}^{l_a} \frac{1}{u - t_i^{(a)}},$$

for $a = 2, \dots, N$. Define the functions $\lambda_k(u, \boldsymbol{t}, \boldsymbol{z})$ by the formula

$$\big(\partial_u - \chi^1(u, \boldsymbol{t}, \boldsymbol{z})\big) \cdots \big(\partial_u - \chi^N(u, \boldsymbol{t}, \boldsymbol{z})\big) = \sum_{k=0}^{N} (-1)^k \lambda_k(u, \boldsymbol{t}, \boldsymbol{z})\, \partial_u^{N-k}.$$

Then for $k = 0, \dots, N$,

$$\mathcal{S}_{k,Q,M(\boldsymbol{z})}(u)\, \omega(\tilde{\boldsymbol{t}}, \boldsymbol{z}) = \lambda_k(u, \tilde{\boldsymbol{t}}, \boldsymbol{z})\, \omega(\tilde{\boldsymbol{t}}, \boldsymbol{z}),$$

see Theorem 9.2 in [6].

10.4 Differential operator associated with a solution of the Bethe ansatz equations

Let $\tilde{\boldsymbol{t}}$ be a solution of system (10.1). The scalar difference operator

$$\mathcal{D}_{\tilde{\boldsymbol{t}}}(u, \tau) = \sum_{k=0}^{N} (-1)^k \lambda_k(u, \tilde{\boldsymbol{t}}, \boldsymbol{z})\, \partial_u^{N-k}$$

will be called the associated *fundamental differential operator.*

Theorem 10.2. *The kernel of $\mathcal{D}_{\tilde{\boldsymbol{t}}}(u, \tau)$ is generated by quasi-exponentials of degree bounded from above by a function in n and N.*

This is Proposition 6.4 in [9], which is a generalization of Lemma 5.6 in [7].

10.5 Completeness of the Bethe ansatz

Theorem 10.3. *Let $z_1, \dots, z_n$ and $K = \operatorname{diag}(K_1, \dots, K_N)$ be generic. Then the Bethe vectors form a basis in $\mathcal{M}(\boldsymbol{z})[\boldsymbol{m}]$.*

Theorem 10.3 will be proved in Section 10.6.

Corollary 10.4. *Theorems* 10.3 *and* 9.2 *imply Theorem* 8.4.

Proof. Theorems 10.2 and 10.3 imply that the statement of Theorem 8.4 holds if the tensor product $\mathcal{M}(\boldsymbol{z})$ is considered for generic $\boldsymbol{z}$ and generic diagonal K. Then according to the remark in Section 8.2.3, the statement of Theorem 3.8 holds if the tensor product $\mathcal{M}(\boldsymbol{z})$ is considered for generic $\boldsymbol{z}$ and generic (not necessarily diagonal) K. Then the remark in Section 8.1.3 and Theorem 9.2 imply that the statement of Theorem 8.4 holds for the tensor product of any polynomial finite-dimensional $\mathfrak{gl}_N[x]$-modules and any K. ■

10.6 Proof of Theorem 10.3

10.6.1. For a nonzero weight subspace $\mathcal{M}(\boldsymbol{z})[\boldsymbol{m}] \subset \mathcal{M}(\boldsymbol{z})$ denote by $d[\boldsymbol{m}]$ its dimension. Let $n \geqslant l_1 \geqslant \cdots \geqslant l_{N-1} \geqslant 0$ be the numbers defined in Section 10.2.

A vector $\boldsymbol{a} = (a_1, \dots, a_n)$ with coordinates a_i from the set $\{0, 2, 3, \dots, N\}$ will be called *admissible* if for any $j = 1, \dots, N-1$ we have $l_j = \#\{\, a_i \mid i = 1, \dots, n, \text{ and } a_i > j\,\}$. In other words, $\boldsymbol{a}$ is admissible if $m_j = \#\{\, a_i \mid i = 1, \dots, n, \text{ and } a_i = j\}$.

If $\boldsymbol{a} = (a_1, \dots, a_n)$ is admissible, then for $j = 1, \dots, N-1$, there exists a unique increasing map $\rho_{\boldsymbol{a},i} : \{1, \dots, l_i\} \to \{1, \dots, n\}$ such that $\#\rho_{\boldsymbol{a},i}^{-1}(j) = 1$ if $a_j > i$ and $\#\rho_{\boldsymbol{a},i}^{-1}(j) = 0$ if $a_j \leqslant i$.

Let $v = (1, 0, \dots, 0) \in \mathbb{C}^N$ be the highest weight vector. The set of vectors $e_{\boldsymbol{a}}\boldsymbol{v} = e_{a_1,1}v \otimes \cdots \otimes e_{a_n,1}v \in \mathcal{M}(\boldsymbol{z})$, labeled by admissible indices $\boldsymbol{a}$, form a basis of $\mathcal{M}(\boldsymbol{z})[\boldsymbol{m}]$.

10.6.2. Let $z_1, \dots, z_n$ be distinct numbers. Assume that $K = \operatorname{diag}(1/q, 2/q, \dots, N/q)$, where q is a small nonzero parameter. Then the Bethe ansatz equations (10.1) take the form

$$
\begin{gathered}
\sum_{s=1}^{n} \frac{1}{t_j^{(1)} - z_s} + \sum_{j'=1}^{l_2} \frac{1}{t_j^{(1)} - t_{j'}^{(2)}} - \sum_{\substack{j'=1 \\ j' \neq j}}^{l_1} \frac{2}{t_j^{(1)} - t_{j'}^{(1)}} = \frac{1}{q}, \\
\sum_{j'=1}^{l_{a+1}} \frac{1}{t_j^{(a)} - t_{j'}^{(a+1)}} + \sum_{j'=1}^{l_{a-1}} \frac{1}{t_j^{(a)} - t_{j'}^{(a-1)}} - \sum_{\substack{j'=1 \\ j' \neq j}}^{l_a} \frac{2}{t_j^{(a)} - t_{j'}^{(a)}} = \frac{1}{q}, \\
\sum_{j'=1}^{l_{N-2}} \frac{1}{t_j^{(N-1)} - t_{j'}^{(N-2)}} - \sum_{\substack{j'=1 \\ j' \neq j}}^{l_{N-1}} \frac{2}{t_j^{(N-1)} - t_{j'}^{(N-1)}} = \frac{1}{q}.
\end{gathered}
\tag{10.2}
$$

Lemma 10.5. *For any admissible index $\boldsymbol{a} = (a_1, \dots, a_N)$ and small nonzero q, there exists a solution $\tilde{\boldsymbol{t}}(\boldsymbol{a}, q) = (\tilde{t}_j^{(i)}(\boldsymbol{a}, q))$ of system* (10.2) *such that*

$$
\tilde{t}_j^{(i)}(\boldsymbol{a}, q) = z_{\rho_{\boldsymbol{a},i}(j)} + \sum_{k=1}^{i} \frac{q}{a_{\rho_{\boldsymbol{a},i}(j)} - k} + O\big(q^2\big) \tag{10.3}
$$

for every i, j.

Lemma 10.6. *As q tends to zero, the Bethe vector $\omega(\tilde{\boldsymbol{t}}(\boldsymbol{a}, q), \boldsymbol{z})$ has the following asymptotics:*

$$
\omega(\tilde{\boldsymbol{t}}(\boldsymbol{a}, q), \boldsymbol{z}) = C_{\boldsymbol{a}}\, q^{-l} e_{\boldsymbol{a}} v + O\big(q^{-l+1}\big),
$$

where $C_{\boldsymbol{a}}$ is a nonzero number.

The lemma follows from formula (10.3) and the formula for the universal weight function in [11].

Lemma 10.6 implies Theorem 10.3.

11 Comparison theorem in the Gaudin case

11.1 The automorphism $\tilde{\chi}_f$ and the universal differential operator

For a series $f(u) = \sum_{s=0}^{\infty} f_s u^{-s-1}$ with coefficients in $\mathbb{C}$ we have an automorphism

$$\tilde{\chi}_f \;:\; \mathfrak{gl}_N[x] \;\to\; \mathfrak{gl}_N[x], \qquad e_{ab}^{\{s\}} \mapsto e_{ab}^{\{s\}} + f_s\,\delta_{ab}.$$

For a $\mathfrak{gl}_N[x]$-module V, denote by V^f the representation of $\mathfrak{gl}_N[x]$ on the same vector space given by the rule

$$X|_{V^f} = (\tilde{\chi}_f(X))|_V$$

for any $X \in \mathfrak{gl}_N[x]$.

Lemma 11.1. *Let $\Lambda = (\Lambda_1, \dots, \Lambda_N)$ be a dominant integral $\mathfrak{gl}_N$-weight and a a complex number. Denote $\Lambda(a) = (\Lambda_1 + a, \dots, \Lambda_N + a)$. Let $V = M_\Lambda(z)$ be an evaluation module and $f(u) = \frac{a}{u-z}$. Then $V^f = M_{\Lambda(a)}(z)$.*

11.1.1. Let $\boldsymbol{\Lambda} = (\Lambda^{(1)}, \dots, \Lambda^{(n)})$ be a collection of integral dominant $\mathfrak{gl}_N$-weights, where $\Lambda^{(i)} = (\Lambda_1^{(i)}, \dots, \Lambda_N^{(i)})$ for $i = 1, \dots, n$. Let $\boldsymbol{a} = (a_1, \dots, a_n)$ be a collection on complex numbers. Introduce the new collection $\boldsymbol{\Lambda}(\boldsymbol{a}) = (\Lambda^{(1)}(a_1), \dots, \Lambda^{(n)}(a_n))$.

For complex numbers $z_1, \dots, z_n$, consider the tensor products of evaluation modules

$$\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z}) = M_{\Lambda^{(1)}}(z_1) \otimes \cdots \otimes M_{\Lambda^{(n)}}(z_n),$$
$$\mathcal{M}_{\boldsymbol{\Lambda}(\boldsymbol{a})}(\boldsymbol{z}) = M_{\Lambda^{(1)}(a_1)}(z_1) \otimes \cdots \otimes M_{\Lambda^{(n)}(a_n)}(z_n).$$

Theorem 11.2. *For any matrix K, an $M_{\boldsymbol{\Lambda}}(\boldsymbol{z})$-valued function $g(u)$ belongs to the kernel of the differential operator $\mathcal{D}_{K,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})}(u, \partial_u)$ if and only if the function $C(u)g(u)$ belongs to the kernel of the differential operator $\mathcal{D}_{K,\mathcal{M}_{\boldsymbol{\Lambda}(\boldsymbol{a})}(\boldsymbol{z})}(u, \partial_u)$, where*

$$C(u) = \prod_{i=1}^{n} (u - z_i)^{a_i}.$$

11.1.2. Assume that $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ is the tensor product of polynomial $\mathfrak{gl}_N[x]$-modules and $\boldsymbol{a} = (a_1, \dots, a_N)$ are non-negative integers. Then $C(u)$ is a polynomial and $\mathcal{M}_{\boldsymbol{\Lambda}(\boldsymbol{a})}(\boldsymbol{z})$ is the tensor product of polynomial $\mathfrak{gl}_N[x]$-modules. In this case Theorem 11.2 allows us to compare the quasi-exponential kernels of the differential operators $\mathcal{D}_{K,\,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})}(u, \partial_u)$ and $\mathcal{D}_{K,\,\mathcal{M}_{\boldsymbol{\Lambda}(\boldsymbol{a})}(\boldsymbol{z})}(u, \partial_u)$.

12 The kernel in the tensor product of evaluation modules in the Gaudin case

12.1 The second main result in the Gaudin case

Let $K = \operatorname{diag}(K_1, \dots, K_N)$ be a diagonal matrix. Let $\boldsymbol{\Lambda} = (\Lambda^{(1)}, \dots, \Lambda^{(n)})$ be a collection of integral dominant $\mathfrak{gl}_N$-weights, where $\Lambda^{(i)} = (\Lambda_1^{(i)}, \dots, \Lambda_N^{(i)})$ and $\Lambda_N^{(i)} = 0$ for $i = 1, \dots, n$.

Choose a weight subspace $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}] \subset M_{\boldsymbol{\Lambda}}(\boldsymbol{z})$ and consider the universal differential operator $\mathcal{D}_{K,\,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u, \partial_u)$ associated with K and $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$.

Theorem 12.1. *Assume that the numbers $K_1, \dots, K_N$ are distinct. Then for any $i = 1, \dots, N$ and any nonzero vector $v_0 \in \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$, there exists an $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$-valued quasi-exponential*

$$e^{K_i u}(v_0 u^{m_i} + v_1 u^{m_i - 1} + \dots + v_{m_i}) \tag{12.1}$$

which lies in the kernel of $\mathcal{D}_{K,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u, \partial_u)$. Moreover, the $\mathbb{C}$-span of all such quasi-exponentials is the kernel of $\mathcal{D}_{K,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u, \partial_u)$.

Proof. On one hand, by Theorem 8.4, the kernel of $\mathcal{D}_{K,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u, \partial_u)$ is generated by quasi-exponentials. On the other hand, by Lemma 9.1 and part (i) of Theorem 8.2, any quasi-exponential lying in the kernel of $\mathcal{D}_{K,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u, \partial_u)$ must be of the form (12.1), where v_0 is a nonzero vector. Since the kernel is of dimension $N \cdot \dim \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$, there exists a quasi-exponential of the form (12.1) with an arbitrary nonzero v_0. ∎

Theorem 12.2. *For $i = 1, \dots, n$, $j = 1, \dots, N$, and any nonzero vector $v \in \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$, there exists an $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$-valued solution $f(u)$ of the differential equation $\mathcal{D}_{K,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u, \partial_u) f(u) = 0$ such that*

$$f(u) = v\,(u - z_i)^{\Lambda_j^{(i)} + N - j} + O\big((u - z_i)^{\Lambda_j^{(i)} + N - j + 1}\big).$$

The proof easily follows from part (ii) of Theorem 8.2.

12.1.1. Assume that K is the zero matrix. Consider the subspace $\operatorname{Sing} \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}] \subset \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ of $\mathfrak{gl}_N$-singular vectors and the associated universal differential operator $\mathcal{D}_{K=0,\operatorname{Sing}\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u,\partial_u)$.

Theorem 12.3. *For any $i = 1, \dots, N$ and any nonzero vector $v_0 \in \operatorname{Sing} \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[m_1, \dots, m_N]$, there exists a $\operatorname{Sing} \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$-valued polynomial*

$$v_0\, u^{m_i + N - i} + v_1\, u^{m_i + N - i - 1} + \dots + v_{m_i + N - i} \tag{12.2}$$

which lies in the kernel of $\mathcal{D}_{K=0,\operatorname{Sing}\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u, \partial_u)$. Moreover, the $\mathbb{C}$-span of all such polynomials is the kernel of $\mathfrak{D}_{K=0,\operatorname{Sing}\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u, \partial_u)$.

Proof. On one hand, by Theorem 8.4, the kernel of $\mathcal{D}_{K=0,\operatorname{Sing}\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u, \partial_u)$ is generated by quasi-exponentials. On the other hand, by part (i) of Theorem 8.2, any quasi-exponential lying in the kernel of $\mathcal{D}_{K=0,\operatorname{Sing}\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u, \partial_u)$ must a polynomial. By Theorem 8.3 such a polynomial has to be of the form indicated in (12.2), where v_0 is a nonzero vector. Since the kernel is of dimension $N \cdot \dim \operatorname{Sing} \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$, there exists a polynomial of the form (12.2) with an arbitrary nonzero v_0. ∎

12.2 Kernel of the fundamental differential operator associated with an eigenvector

Assume that $v \in \mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$ is an eigenvector of all transfer matrices,

$$\mathcal{S}_{K,\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]}(u)\, v = \lambda_{k,v}(u)\, v, \qquad k = 0, \dots, N.$$

Then the scalar differential operator

$$\mathcal{D}_v(u, \partial_u) = \sum_{k=0}^{N} (-1)^k\, \lambda_{k,v}(u)\, \partial_u^{N-k}$$

will be called *the fundamental differential operator* associated with the eigenvector v.

Theorems 12.1, 12.2 and 12.3 give us information on the kernel of the fundamental differential operator.

Corollary 12.4.

(i) Assume that K is diagonal with distinct diagonal entries. Then the kernel of $\mathcal{D}_v(u, \partial_u)$ is generated by quasi-exponentials $e^{K_1 u} p_1(u), \dots, e^{K_N u} p_N(u)$, where for every i the polynomial $p_i(u) \in \mathbb{C}[u]$ is of degree m_i.

(ii) Assume that K is the zero matrix and the eigenvector v belongs to Sing $\mathcal{M}_{\boldsymbol{\Lambda}}(\boldsymbol{z})[\boldsymbol{m}]$. *Then the kernel of $\mathcal{D}_v(u, \partial_u)$ is generated by suitable polynomials $p_1(u), \dots, p_N(u)$ of degree $m_1 + N - 1, \dots, m_N$, respectively.*

Corollary 12.5. *For $i = 1, \dots, n$, $j = 1, \dots, N$, there exists a solution $f(u)$ of the differential equation $\mathcal{D}_v(u, \partial_u)\, f(u) = 0$ such that*

$$f(u) = (u - z_i)^{\Lambda_j^{(i)} + N - j} + O\big((u - z_i)^{\Lambda_j^{(i)} + N - j + 1}\big).$$

If v is a Bethe eigenvector, then these two corollaries are proved in [7] and [9].

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