From $\mathfrak{su}(2)$ Gaudin Models to Integrable Tops*

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Abstract. In the present paper we derive two well-known integrable cases of rigid body dynamics (the Lagrange top and the Clebsch system) performing an algebraic contraction on the two-body Lax matrices governing the (classical) $\mathfrak{su}(2)$ Gaudin models. The procedure preserves the linear r-matrix formulation of the ancestor models. We give the Lax representation of the resulting integrable systems in terms of $\mathfrak{su}(2)$ Lax matrices with rational and elliptic dependencies on the spectral parameter. We finally give some results about the many-body extensions of the constructed systems.

Key words: Gaudin models; spinning tops

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Dedicated to the memory of Vadim B. Kuznetsov (1963–2005)

1 Introduction

The Gaudin models were introduced in 1976 by M. Gaudin [5] and attracted considerable interest among theoretical and mathematical physicists, playing a distinguished role in the realm of integrable systems. Their peculiar properties, holding both at the classical and at the quantum level, are deeply connected with the long-range nature of the interaction described by its commuting Hamiltonians, which in fact yields a typical "mean field" dynamics.

Indeed the Gaudin models describe completely integrable classical and quantum long-range spin chains. The original Gaudin model was formulated as a quantum spin model related to the Lie algebra $\mathfrak{su}(2)$ [5]. Later it was realized that such models can be associated with any semi-simple complex Lie algebra $\mathfrak{g}$ [6, 10] and a solution of the corresponding classical Yang–Baxter equation [2, 23]. An important feature of Gaudin models is that they can be formulated in the framework of the r-matrix approach. In particular, they admit a linear r-matrix structure that characterizes both the classical and the quantum models, and holds whatever be the dependence (rational (XXX), trigonometric (XXZ), elliptic (XYZ)) on the spectral parameter. In this context, it is possible to see Gaudin models as appropriate "semiclassical" limits of the integrable Heisenberg magnets [26], which admit a quadratic r-matrix structure.

In the 80's, the rational Gaudin model was studied by Sklyanin [24] and Jurčo [10] from the point of view of the quantum inverse scattering method. Precisely, Sklyanin studied the $\mathfrak{su}(2)$ rational Gaudin models, diagonalizing the commuting Hamiltonians by means of separation of variables and stressing the connection between his procedure and the functional Bethe Ansatz.

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On the other hand, the algebraic structure encoded in the linear r-matrix algebra allowed Jurčo to use the algebraic Bethe Ansatz to simultaneously diagonalize the set of commuting Hamiltonians in all cases when $\mathfrak{g}$ is a semi-simple Lie algebra. We have to mention here also the the work of Reyman and Semenov-Tian-Shansky [22]. Classical Hamiltonian systems associated with Lax matrices of the Gaudin-type were studied by them in the context of a general group-theoretic approach.

Vadim Kuznetsov, to whom this work is dedicated, widely studied Gaudin models, especially from the point of view of their separability properties [11, 12, 13] and of their integrable discretizations through Bäcklund transformations [8, 14]. In [14] we collaborated with him showing that the Lagrange top can be obtained through an algebraic contraction procedure performed on the two-body $\mathfrak{su}(2)$ rational Gaudin model. Such a derivation of the Lagrange system preserves the linear r-matrix algebra of the ancestor model, and it has been used as a tool to construct an integrable discretization starting from a known one for the rational $\mathfrak{su}(2)$ Gaudin model [8].

The purpose of the present paper is twofold: on one hand we recall the procedure we used in [14] to obtain the Lagrange top from the two-body $\mathfrak{su}(2)$ rational Gaudin model; on the other hand we show how the same technique can be used to derive a special case of the Clebsch system (i.e. the motion of a free rigid body in an ideal incompressible fluid) starting from the elliptic $\mathfrak{su}(2)$ Gaudin model. In the last Section we show how to construct many-body extensions starting from the obtained Lax matrices governing the Lagrange top and the Clebsch system.

2 A short review of $\mathfrak{su}(2)$ Gaudin models

The aim of this Section is to give a terse survey of the main features of $\mathfrak{su}(2)$ Gaudin models. In particular we shall describe them in terms of their (linear) r-matrix formulation, providing their Lax matrices and r-matrices. For further details we remand at the references [5, 6, 8, 10, 11, 17, 20, 22, 24, 25, 26].

Let us choose the following basis of the linear space $\mathfrak{su}(2)$:

$$\sigma_1 \doteq \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_2 \doteq \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 \doteq \frac{1}{2}\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}.$$

We recall that the correspondence

$$\mathbb{R}^3 \ni \mathbf{a} \doteq (a^1, a^2, a^3) \;\longleftrightarrow\; \mathbf{a} \doteq \frac{1}{2}\begin{pmatrix} -\mathrm{i}\, a^3 & -\mathrm{i}\, a^1 - a^2 \\ -\mathrm{i}\, a^1 + a^2 & \mathrm{i}\, a^3 \end{pmatrix} \in \mathfrak{su}(2),$$

is an isomorphism between $(\mathfrak{su}(2), [\,\cdot,\cdot\,])$ and the Lie algebra $(\mathbb{R}^3, \times)$, where $\times$ stands for the vector product. This allows us to identify $\mathbb{R}^3$ vectors and $\mathfrak{su}(2)$ matrices. We supply $\mathfrak{su}(2)$ with the scalar product $\langle\,\cdot,\cdot\,\rangle$ induced from $\mathbb{R}^3$, namely $\langle\,\mathbf{a},\mathbf{b}\,\rangle = -2\,\mathrm{tr}\,(\mathbf{a}\,\mathbf{b}) = 2\,\mathrm{tr}\,(\mathbf{b}\,\mathbf{a}^\dagger)$, $\forall\,\mathbf{a},\mathbf{b} \in \mathfrak{su}(2)$. This scalar product allows us to identify the dual space $\mathfrak{su}^*(2)$ with $\mathfrak{su}(2)$, so that the coadjoint action of the algebra becomes the usual Lie bracket with minus.

The Lie–Poisson algebra of the N-body $\mathfrak{su}(2)$ Gaudin models is given by (minus) $\oplus^N \mathfrak{su}^*(2)$. We will denote by $\{y_i^\alpha\}_{\alpha=1}^3$, $1 \le i \le N$, the set of the (time-dependent) coordinate functions relative to the i-th copy of $\mathfrak{su}(2)$. Consequently, the Lie–Poisson brackets on $\oplus^N \mathfrak{su}^*(2)$ read

$$\left\{ y_i^\alpha, y_j^\beta \right\} = -\delta_{i,j}\, \epsilon_{\alpha\beta\gamma}\, y_i^\gamma, \tag{2.1}$$

with $1 \le i, j \le N$. Here $\epsilon_{\alpha\beta\gamma}$ is the skew-symmetric tensor with $\epsilon_{123} = 1$. The brackets (2.1) are degenerate: they possess the N Casimir functions

$$C_i \doteq \frac{1}{2}\langle\, \mathbf{y}_i, \mathbf{y}_i\,\rangle, \qquad 1 \le i \le N, \tag{2.2}$$

that provide a trivial dynamics.

The $\mathfrak{su}(2)$ rational, trigonometric and elliptic Gaudin models are governed respectively by the following Lax matrices defined on the loop algebra $\mathfrak{su}(2)[\lambda, \lambda^{-1}]$:

$$\mathcal{L}_{\mathcal{G}}^{r}(\lambda) \doteq \sigma_\alpha\, p^\alpha + \sum_{i=1}^{N} \frac{\sigma_\alpha\, y_i^\alpha}{\lambda - \lambda_i} = \mathbf{p} + \sum_{i=1}^{N} \frac{\mathbf{y}_i}{\lambda - \lambda_i}, \tag{2.3}$$

$$\mathcal{L}_{\mathcal{G}}^{t}(\lambda) \doteq \sum_{i=1}^{N} \frac{1}{\sin(\lambda - \lambda_i)} \left[\, \sigma_1\, y_i^1 + \sigma_2\, y_i^2 + \cos(\lambda - \lambda_i)\, \sigma_3\, y_i^3 \,\right], \tag{2.4}$$

$$\mathcal{L}_{\mathcal{G}}^{e}(\lambda) \doteq \sum_{i=1}^{N} \frac{1}{\mathrm{sn}(\lambda - \lambda_i)} \left[\, \mathrm{dn}(\lambda - \lambda_i)\, \sigma_1\, y_i^1 + \sigma_2\, y_i^2 + \mathrm{cn}(\lambda - \lambda_i)\, \sigma_3\, y_i^3 \,\right], \tag{2.5}$$

where the λ_i's, with $\lambda_i \neq \lambda_k$, $1 \leq i, k \leq N$, are complex parameters of the model. We remark that in equation (2.5) $\mathrm{cn}(\lambda), \mathrm{dn}(\lambda), \mathrm{sn}(\lambda)$ are the elliptic Jacobi functions of modulus k. In equation (2.3) $\mathbf{p}$ is a constant vector in $\mathbb{R}^3$. Its presence is necessary in the rational case in order to get a sufficient number of functionally independent integrals of motion.

It is well-known that the Lax matrices (2.3), (2.4) and (2.5) describe completely integrable systems on the Lie–Poisson manifold associated with $\oplus^N \mathfrak{su}^*(2)$. In particular they admit a linear r-matrix formulation, which ensures that all the spectral invariants of $\mathcal{L}_{\mathcal{G}}^{r}(\lambda)$, $\mathcal{L}_{\mathcal{G}}^{t}(\lambda)$, $\mathcal{L}_{\mathcal{G}}^{e}(\lambda)$ form a family of involutive functions. Let us give the following result.

Proposition 1. *The Lax matrices $\mathcal{L}_{\mathcal{G}}^{r}(\lambda)$, $\mathcal{L}_{\mathcal{G}}^{t}(\lambda)$, $\mathcal{L}_{\mathcal{G}}^{e}(\lambda)$ given in equations* (2.3), (2.4) *and* (2.5) *satisfy the linear r-matrix algebra*

$$\left\{ \mathcal{L}_{\mathcal{G}}^{r,t,e}(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes \mathcal{L}_{\mathcal{G}}^{r,t,e}(\mu) \right\} + \left[\, r_{r,t,e}(\lambda - \mu), \mathcal{L}_{\mathcal{G}}^{r,t,e}(\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{L}_{\mathcal{G}}^{r,t,e}(\mu) \,\right] = 0, \tag{2.6}$$

for all $\lambda, \mu \in \mathbb{C}$, with

$$r_{r,t,e}(\lambda) \doteq -f_{r,t,e}^{\alpha}(\lambda)\, \sigma_\alpha \otimes \sigma_\alpha, \tag{2.7}$$

and

$$f_r^\alpha(\lambda) \doteq \frac{1}{\lambda} \qquad \forall\, \alpha = 1, 2, 3,$$

$$(f_t^1(\lambda), f_t^2(\lambda), f_t^3(\lambda)) \doteq \left(\frac{1}{\sin(\lambda)}, \frac{1}{\sin(\lambda)}, \cot(\lambda) \right),$$

$$(f_e^1(\lambda), f_e^2(\lambda), f_e^3(\lambda)) \doteq \left(\frac{\mathrm{dn}(\lambda)}{\mathrm{sn}(\lambda)}, \frac{1}{\mathrm{sn}(\lambda)}, \frac{\mathrm{cn}(\lambda)}{\mathrm{sn}(\lambda)} \right).$$

In equation (2.6) $\mathbb{1}$ *denotes the 2×2 identity matrix and $\otimes$ stands for the tensor product in $\mathbb{C}^2 \otimes \mathbb{C}^2$.*

In the rational case the r-matrix is equivalent to $r_r(\lambda) = -\Pi/(2\,\lambda)$, where Π is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The complete set of integrals of the $\mathfrak{su}(2)$ rational, trigonometric and elliptic Gaudin models can be constructed computing the residues in $\lambda = \lambda_i$ of the characteristic curve $\det(\mathcal{L}_{\mathcal{G}}^{r,t,e}(\lambda) - \mu\,\mathbb{1}) = 0$ (or equivalently $\mu^2 = -(1/2)\,\mathrm{tr}[\,(\mathcal{L}_{\mathcal{G}}^{r,t,e}(\lambda))^2])$. The following results hold.

Proposition 2. *The hyperelliptic curve $\det(\mathcal{L}_{\mathcal{G}}^{r}(\lambda) - \mu\,\mathbb{1}) = 0$, $\lambda, \mu \in \mathbb{C}$, with $\mathcal{L}_{\mathcal{G}}^{r}(\lambda)$ given in equation* (2.3), *provides a set of $2N$ independent involutive integrals of motion given by*

$$H_i^r \doteq \langle\, \mathbf{p}, \mathbf{y}_i \,\rangle + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\langle\, \mathbf{y}_i, \mathbf{y}_j \,\rangle}{\lambda_i - \lambda_j}, \qquad \sum_{i=1}^{N} H_i^r = \sum_{i=1}^{N} \langle\, \mathbf{p}, \mathbf{y}_i \,\rangle, \tag{2.8}$$

$$C_i \doteq \frac{1}{2} \langle \mathbf{y}_i, \mathbf{y}_i \rangle.$$

The integrals $\{H_i^r\}_{i=1}^N$ *are first integrals of motion and the integrals* $\{C_i\}_{i=1}^N$ *are the Casimir functions given in equation* (2.2).

Proposition 3. *The curve* $\det(\mathcal{L}_{\mathcal{G}}^t(\lambda) - \mu \, \mathbb{1}) = 0$, $\lambda, \mu \in \mathbb{C}$, *with* $\mathcal{L}_{\mathcal{G}}^t(\lambda)$ *given in equation* (2.4), *provides a set of* $2N$ *independent involutive integrals of motion given by*

$$H_i^t \doteq \sum_{\substack{j=1 \\ j\neq i}}^N \frac{y_i^1 \, y_j^1 + y_i^2 \, y_j^2 + \cos(\lambda_i - \lambda_j) \, y_i^3 \, y_j^3}{\sin(\lambda_i - \lambda_j)}, \qquad \sum_{i=1}^N H_i^t = 0,$$

$$H_0^t \doteq \left(\sum_{i=1}^N y_i^3 \right)^2, \qquad C_i \doteq \frac{1}{2} \langle \mathbf{y}_i, \mathbf{y}_i \rangle.$$

The integrals $\{H_i^t\}_{i=0}^N$ *are first integrals of motion and the integrals* $\{C_i\}_{i=1}^N$ *are the Casimir functions given in equation* (2.2).

Proposition 4. *The curve* $\det(\mathcal{L}_{\mathcal{G}}^e(\lambda) - \mu \, \mathbb{1}) = 0$, $\lambda, \mu \in \mathbb{C}$, *with* $\mathcal{L}_{\mathcal{G}}^e(\lambda)$ *given in equation* (2.5), *provides a set of* $2N$ *independent involutive integrals of motion given by*

$$H_i^e \doteq \sum_{\substack{j=1 \\ j\neq i}}^N \frac{\mathrm{dn}(\lambda_i - \lambda_j) \, y_i^1 \, y_j^1 + y_i^2 \, y_j^2 + \mathrm{cn}(\lambda_i - \lambda_j) \, y_i^3 \, y_j^3}{\mathrm{sn}(\lambda_i - \lambda_j)}, \qquad \sum_{i=1}^N H_i^e = 0,$$

$$H_0^e \doteq \sum_{\substack{i,j=1 \\ i\neq j}}^N \left[y_i^1 \, y_j^1 \, g_1(\lambda_i - \lambda_j) + y_i^2 \, y_j^2 \, g_2(\lambda_i - \lambda_j) + y_i^3 \, y_j^3 \, g_3(\lambda_i - \lambda_j) \right],$$

with

$$g_1(\lambda) \doteq \frac{\theta_{11}' \, \theta_{10}'(\lambda)}{\theta_{10} \, \theta_{11}(\lambda)}, \qquad g_2(\lambda) \doteq \frac{\theta_{11}' \, \theta_{00}'(\lambda)}{\theta_{00} \, \theta_{11}(\lambda)}, \qquad g_3(\lambda) \doteq \frac{\theta_{11}' \, \theta_{01}'(\lambda)}{\theta_{01} \, \theta_{11}(\lambda)},$$

and

$$C_i \doteq \frac{1}{2} \langle \mathbf{y}_i, \mathbf{y}_i \rangle.$$

Here $\theta_{\alpha\beta}(\lambda)$, $\alpha, \beta = 0, 1$, *is the theta function*[1], *and* $\theta_{\alpha\beta} \doteq \theta_{\alpha\beta}(0)$, $\theta_{\alpha\beta}' \doteq (d/d\lambda)_{\lambda=0} \theta_{\alpha\beta}(\lambda)$. *The integrals* $\{H_i^e\}_{i=0}^N$ *are first integrals of motion and the integrals* $\{C_i\}_{i=1}^N$ *are the Casimir functions given in equation* (2.2).

In the rational case it is possible to select a simple and remarkable Hamiltonian. It is given by the following linear combination of the integrals of motion $\{H_i^r\}_{i=1}^N$ given in equation (2.8):

$$\sum_{i=1}^N \eta_i \, H_i^r = \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^N \frac{\eta_i - \eta_j}{\lambda_i - \lambda_j} \langle \mathbf{y}_i, \mathbf{y}_j \rangle + \sum_{i=1}^N \eta_i \langle \mathbf{p}, \mathbf{y}_i \rangle, \tag{2.9}$$

[1]We are using the notation adopted in [26]:

$$\theta_{\alpha\beta}(\lambda) \doteq \theta_{\alpha\beta}(\lambda, \tau) = \sum_{n\in\mathbb{Z}} \exp\left[\pi \, \mathrm{i} \left(n + \frac{\alpha}{2} \right)^2 \tau + 2 \pi \, \mathrm{i} \left(n + \frac{\alpha}{2} \right) \left(n + \frac{\beta}{2} \right) \right],$$

$\alpha, \beta = 0, 1$, where τ is a complex number in the upper half plane.

where the η_i's with $\eta_i \neq \eta_k$, $1 \leq i, k \leq N$, are arbitrary complex numbers. An interesting specialization of the Hamiltonian (2.9) is obtained considering $\eta_i = \lambda_i$, $1 \leq i \leq N$:

$$\mathcal{H}^r_{\mathcal{G}} \doteq \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \langle \mathbf{y}_i, \mathbf{y}_j \rangle + \sum_{i=1}^{N} \lambda_i \langle \mathbf{p}, \mathbf{y}_i \rangle. \tag{2.10}$$

Proposition 5. *The equations of motion w.r.t. the Hamiltonian* (2.10) *are given by*

$$\dot{\mathbf{y}}_i = \left[\lambda_i \, \mathbf{p} + \sum_{j=1}^{N} \mathbf{y}_j \, , \, \mathbf{y}_i \right], \qquad 1 \leq i \leq N, \tag{2.11}$$

where $\dot{\mathbf{y}}_i \doteq d\mathbf{y}_i/dt$. *Equations* (2.11) *admit the following Lax representation:*

$$\dot{\mathcal{L}}^r_{\mathcal{G}}(\lambda) = \left[\mathcal{L}^r_{\mathcal{G}}(\lambda), \mathcal{M}^{(r,-)}_{\mathcal{G}}(\lambda) \right] = -\left[\mathcal{L}^r_{\mathcal{G}}(\lambda), \mathcal{M}^{(r,+)}_{\mathcal{G}}(\lambda) \right],$$

with the matrix $\mathcal{L}^r_{\mathcal{G}}(\lambda)$ *given in equation* (2.3) *and*

$$\mathcal{M}^{(r,-)}_{\mathcal{G}}(\lambda) \doteq \sum_{i=1}^{N} \frac{\lambda_i \, \mathbf{y}_i}{\lambda - \lambda_i}, \qquad \mathcal{M}^{(r,+)}_{\mathcal{G}}(\lambda) \doteq \lambda \, \mathbf{p} + \sum_{i=1}^{N} \mathbf{y}_i. \tag{2.12}$$

Proof. A direct computation. ■

3 Contraction of $\mathfrak{su}(2)$ Gaudin models: the two-body case

In the present section we fix $N = 2$, namely we consider two-body $\mathfrak{su}(2)$ Gaudin models.

It is well-known that the Inönü–Wigner contraction of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, i.e. a Lie algebra isomorphic to $\mathfrak{so}(4)$, gives the real Euclidean algebra $\mathfrak{e}(3)$ [9]. Let us define the isomorphism $\phi_\varepsilon : \mathfrak{su}^*(2) \oplus \mathfrak{su}^*(2) \to \mathfrak{su}^*(2) \oplus \mathfrak{su}^*(2)$ by the map

$$\phi_\varepsilon : (\mathbf{y}_1, \mathbf{y}_2) \longmapsto (\mathbf{m}, \mathbf{a}) \doteq (\mathbf{y}_1 + \mathbf{y}_2, \varepsilon \, (\nu_1 \, \mathbf{y}_1 + \nu_2 \, \mathbf{y}_2)), \tag{3.1}$$

where $\nu_1, \nu_2 \in \mathbb{C}$, $\nu_1 \neq \nu_2$ and $0 < \varepsilon \leq 1$ plays the role of a contraction parameter. In the limit $\varepsilon \to 0$ the Lie–Poisson brackets on $\mathfrak{su}^*(2) \oplus \mathfrak{su}^*(2)$ are mapped by ϕ_ε into the Lie–Poisson brackets on $\mathfrak{e}^*(3) \cong \mathfrak{su}^*(2) \oplus_s \mathbb{R}^3$:

$$\left\{ m^\alpha, m^\beta \right\} = -\epsilon_{\alpha\beta\gamma} \, m^\gamma, \qquad \left\{ m^\alpha, a^\beta \right\} = -\epsilon_{\alpha\beta\gamma} \, a^\gamma, \qquad \left\{ a^\alpha, a^\beta \right\} = 0. \tag{3.2}$$

Obviously, the map ϕ_ε is not an isomorphism after the contraction limit $\varepsilon \to 0$. The Lie–Poisson brackets (3.2) are degenerate: they possess the two Casimir functions

$$K_1 \doteq \langle \mathbf{m}, \mathbf{a} \rangle, \qquad K_2 \doteq \frac{1}{2} \langle \mathbf{a}, \mathbf{a} \rangle. \tag{3.3}$$

A direct calculation shows that if $H(\mathbf{y}_1, \mathbf{y}_2)$ and $G(\mathbf{y}_1, \mathbf{y}_2)$ are two involutive functions w.r.t. the Lie–Poisson brackets on $\mathfrak{su}^*(2) \oplus \mathfrak{su}^*(2)$ then, in the contraction limit $\varepsilon \to 0$, the functions $\phi_\varepsilon(H(\mathbf{y}_1, \mathbf{y}_2))$ and $\phi_\varepsilon(G(\mathbf{y}_1, \mathbf{y}_2))$ are in involution w.r.t. the Lie–Poisson brackets on $\mathfrak{e}^*(3)$.

Our aim is now to apply the contraction map ϕ_ε defined in equation (3.1) to the Lax matrices of the two-body $\mathfrak{su}(2)$ Gaudin models, i.e. the matrices in equations (2.3), (2.4) and (2.5) with $N = 2$. To do this a second ingredient is needed: as shown in [14, 17, 18, 20] we have to consider the *pole coalescence* $\lambda_i = \varepsilon \, \nu_i$, $i = 1, 2$. This fusion procedure can be considered as the analytical counterpart of the algebraic contraction given by the map in equation (3.1).

A straightforward computation leads to the following statement [14, 20].

Proposition 6. *In the limit $\varepsilon \to 0$, the isomorphism* (3.1) *maps the Lax matrices* (2.3)*,* (2.4) *and* (2.5) *with $\lambda_i = \varepsilon\,\nu_i$, $i = 1, 2$, respectively into the Lax matrices*

$$\mathcal{L}^r(\lambda) \doteq \mathbf{p} + \frac{\mathbf{m}}{\lambda} + \frac{\mathbf{a}}{\lambda^2}, \tag{3.4}$$

$$\begin{aligned} \mathcal{L}^t(\lambda) \doteq {} & \frac{1}{\sin(\lambda)} \left[\sigma_1\, m^1 + \sigma_2\, m^2 + \cos(\lambda)\, \sigma_3\, m^3 \right] \\ & + \frac{1}{\sin^2(\lambda)} \left[\cos(\lambda)\, (\sigma_1\, a^1 + \sigma_2\, a^2) + \sigma_3\, a^3 \right], \end{aligned} \tag{3.5}$$

$$\begin{aligned} \mathcal{L}^e(\lambda) \doteq {} & \frac{1}{\mathrm{sn}(\lambda)} \left[\mathrm{dn}(\lambda)\, \sigma_1\, m^1 + \sigma_2\, m^2 + \mathrm{cn}(\lambda)\, \sigma_3\, m^3 \right] \\ & + \frac{1}{\mathrm{sn}^2(\lambda)} \left[\mathrm{cn}(\lambda)\, \sigma_1\, a^1 + \mathrm{cn}(\lambda)\, \mathrm{dn}(\lambda)\, \sigma_2\, a^2 + \mathrm{dn}(\lambda)\, \sigma_3\, a^3 \right]. \end{aligned} \tag{3.6}$$

The Lax matrices given in equations (3.4), (3.5) and (3.6) describe completely integrable systems on the Lie–Poisson manifold associated with $\mathfrak{e}^*(3)$. The remarkable feature of the above procedure is that the contracted models inherit the linear r-matrix algebra (2.6) of the ancestor system. The following proposition holds [14, 17, 18].

Proposition 7. *The Lax matrices $\mathcal{L}^r(\lambda)$, $\mathcal{L}^t(\lambda)$, $\mathcal{L}^e(\lambda)$ given in equations* (3.4)*,* (3.5) *and* (3.6) *satisfy the linear r-matrix algebra*

$$\left\{ \mathcal{L}^{r,t,e}(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes \mathcal{L}^{r,t,e}(\mu) \right\} + \left[r_{r,t,e}(\lambda - \mu), \mathcal{L}^{r,t,e}(\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{L}^{r,t,e}(\mu) \right] = 0, \tag{3.7}$$

for all $\lambda, \mu \in \mathbb{C}$, with $r_{r,t,e}(\lambda)$ given in equation (2.7)*.*

3.1 A Lagrange top arising from the rational $\mathfrak{su}(2)$ Gaudin model

Recall that the (3-dimensional) Lagrange case of the rigid body motion around a fixed point in a homogeneous field is characterized by the following data: the inertia tensor is given by $\mathrm{diag}(1, 1, \alpha)$, $\alpha \in \mathbb{R}$, which means that the body is rotationally symmetric with respect to the third coordinate axis, and the fixed point lies on the symmetry axis [1, 4, 14, 22].

As noticed in [14] the Lagrange top can be obtained from the two-body rational $\mathfrak{su}(2)$ Gaudin model performing the contraction procedure previously described.

Let us recall the main features of the dynamics of the Lagrange top (in the rest frame). The equations of motion are given by:

$$\dot{\mathbf{m}} = [\,\mathbf{p}, \mathbf{a}\,], \qquad \dot{\mathbf{a}} = [\,\mathbf{m}, \mathbf{a}\,], \tag{3.8}$$

where $\mathbf{m} \in \mathbb{R}^3$ is the vector of kinetic momentum of the body, $\mathbf{a} \in \mathbb{R}^3$ is the vector pointing from the fixed point to the center of mass of the body and $\mathbf{p} \doteq (0, 0, p)$ is the constant vector along the external field. An external observer is mainly interested in the motion of the symmetry axis of the top on the surface $\langle\, \mathbf{a}, \mathbf{a}\,\rangle = \text{constant}$. For an actual integration of this flow in terms of elliptic functions see [7].

A remarkable feature of the equations of motion (3.8) is that they do not depend explicitly on the anisotropy parameter α of the inertia tensor [4]. Moreover they are Hamiltonian equations with respect to the Lie–Poisson brackets of $\mathfrak{e}^*(3)$, see equation (3.2). The Hamiltonian function that generates the equations of motion (3.8) is given by

$$I_1^r \doteq \frac{1}{2} \langle\, \mathbf{m}, \mathbf{m}\,\rangle + \langle\, \mathbf{p}, \mathbf{a}\,\rangle, \tag{3.9}$$

and the complete integrability of the model is ensured by the second integral of motion $I_2^r \doteq \langle\, \mathbf{p}, \mathbf{m}\,\rangle$. These involutive Hamiltonians can be obtained by computing the spectral invariants of the Lax matrix given in equation (3.4). The remaining two spectral invariants are given by the Casimir functions of the Lie–Poisson brackets of $\mathfrak{e}^*(3)$, see equation (3.3).

Proposition 8. *The Hamiltonian flow* (3.8) *generated by the Hamiltonian* (3.9) *admits the following Lax representation:*

$$\dot{\mathcal{L}}^r(\lambda) = \left[\,\mathcal{L}^r(\lambda), \mathcal{M}^{(r,-)}(\lambda)\,\right] = -\left[\,\mathcal{L}^r(\lambda), \mathcal{M}^{(r,+)}(\lambda)\,\right],$$

with the matrix $\mathcal{L}^r(\lambda)$ *given in equation* (3.4) *and*

$$\mathcal{M}^{(r,-)}(\lambda) \doteq \frac{\mathbf{a}}{\lambda}, \qquad \mathcal{M}^{(r,+)}(\lambda) \doteq \lambda\,\mathbf{p} + \mathbf{m}. \tag{3.10}$$

Proof. A direct verification. ∎

Remark 1. Using the contraction map (3.1) one can obtain equations (3.8) directly from equations (2.11) (with $N = 2$):

$$\begin{aligned}
\dot{\mathbf{m}} &= \dot{\mathbf{y}}_1 + \dot{\mathbf{y}}_2 = [\,\mathbf{p}, \varepsilon\,(\nu_1\,\mathbf{y}_1 + \nu_2\,\mathbf{y}_2)\,] = [\,\mathbf{p}, \mathbf{a}\,],\\
\dot{\mathbf{a}} &= \varepsilon(\nu_1\,\dot{\mathbf{y}}_1 + \nu_2\,\dot{\mathbf{y}}_2) = [\,\mathbf{y}_1 + \mathbf{y}_1, \varepsilon\,(\nu_1\,\mathbf{y}_1 + \nu_2\,\mathbf{y}_2)\,] + O(\varepsilon^2) \xrightarrow{\varepsilon\to 0} [\,\mathbf{m}, \mathbf{a}\,].
\end{aligned}$$

Performing the same procedure on the Hamiltonian $\mathcal{H}^r_{\mathcal{G}} \doteq \lambda_1\,H^r_1 + \lambda_2\,H^r_2$ given in equation (2.10) (with $N = 2$) and on the linear integral $H^r_1 + H^r_2 = \langle\,\mathbf{p}, \mathbf{y}_1 + \mathbf{y}_2\,\rangle$ we recover the integrals of motion of the Lagrange top. We have:

$$\mathcal{H}^r_{\mathcal{G}} = \frac{1}{2}\langle\,\mathbf{y}_1 + \mathbf{y}_2, \mathbf{y}_1 + \mathbf{y}_2\,\rangle - C_1 - C_2 + \langle\,\mathbf{p}, \varepsilon\,\nu_1\,\mathbf{y}_1 + \varepsilon\,\nu_1\,\mathbf{y}_2\,\rangle,$$

being $C_1 \doteq \langle\,\mathbf{y}_1, \mathbf{y}_1\,\rangle/2$, $C_2 \doteq \langle\,\mathbf{y}_2, \mathbf{y}_2\,\rangle/2$ just Casimir functions. Hence,

$$\mathcal{H}^r_{\mathcal{G}} \xrightarrow{\varepsilon\to 0} \frac{1}{2}\langle\,\mathbf{m}, \mathbf{m}\,\rangle + \langle\,\mathbf{p}, \mathbf{a}\,\rangle = I^r_1.$$

Finally, $H^r_1 + H^r_2 = \langle\,\mathbf{p}, \mathbf{y}_1 + \mathbf{y}_2\,\rangle = \langle\,\mathbf{p}, \mathbf{m}\,\rangle = I^r_2$. The same procedure allows one to recover the auxiliary matrices $\mathcal{M}^{(r,\pm)}(\lambda)$ given in equation (3.10) from the matrices $\mathcal{M}^{(r,\pm)}_{\mathcal{G}}(\lambda)$ given in equation (2.12).

3.2 A Clebsch system arising from the elliptic $\mathfrak{su}(2)$ Gaudin model

Let us now consider the Lax matrix given in equation (3.6) obtained performing the contraction procedure on the Lax matrix of the $\mathfrak{su}(2)$ elliptic Gaudin model with $N = 2$.

A direct computation shows that the spectral invariants of $\mathcal{L}^e(\lambda)$ are given by the following quadratic functions:

$$I^e_1 \doteq \frac{1}{2}\langle\,\mathbf{m}, \mathbf{m}\,\rangle - \frac{1}{2}\langle\,\mathbf{a}, B_1\,\mathbf{a}\,\rangle, \tag{3.11}$$

$$I^e_2 \doteq \frac{1}{2}\langle\,\mathbf{m}, A\,\mathbf{m}\,\rangle - \frac{1}{2}\langle\,\mathbf{a}, B_2\,\mathbf{a}\,\rangle, \tag{3.12}$$

$$K_1 \doteq \langle\,\mathbf{m}, \mathbf{a}\,\rangle, \qquad K_2 \doteq \frac{1}{2}\langle\,\mathbf{a}, \mathbf{a}\,\rangle,$$

where

$$B_1 \doteq \operatorname{diag}(0, k^2, k^2 - 1), \qquad B_2 \doteq \operatorname{diag}(0, 0, k^2 - 1), \qquad A \doteq \operatorname{diag}(1 - k^2, 1, 0).$$

Obviously, the choice $k = 0$ in the integrals (3.11) and (3.12) provides the spectral invariants of the trigonometric Lax matrix $\mathcal{L}^t(\lambda)$ given in equation (3.5). Thus the system described by $\mathcal{L}^t(\lambda)$ is a subcase of the one described by $\mathcal{L}^e(\lambda)$. The quadratic functions (3.11) and (3.12) are in involution w.r.t. the Lie–Poisson brackets on $\mathfrak{e}^*(3)$ thanks to the r-matrix formulation in equation (3.7).

Let us now recall the main features of the (3-dimensional) Clebsch case of the free rigid body motion (in an ideal fluid) [22, 27]. This problem is traditionally described by a Hamiltonian system on $\mathfrak{e}^*(3)$ with the Hamiltonian function

$$H \doteq \frac{1}{2}\langle\, \mathbf{m}, \mathcal{A}\, \mathbf{m}\,\rangle - \frac{1}{2}\langle\, \mathbf{a}, \mathcal{B}\, \mathbf{a}\,\rangle, \tag{3.13}$$

where $(\mathbf{m}, \mathbf{a}) \in \mathfrak{e}^*(3)$ and the matrices $\mathcal{A} \doteq \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$ and $\mathcal{B} \doteq \operatorname{diag}(\beta_1, \beta_2, \beta_3)$ are such that the following relation holds:

$$\frac{\beta_1 - \beta_2}{\alpha_3} + \frac{\beta_2 - \beta_3}{\alpha_1} + \frac{\beta_3 - \beta_1}{\alpha_2} = 0,$$

namely

$$\alpha_1 = \frac{\beta_2 - \beta_3}{\gamma_2 - \gamma_3}, \qquad \alpha_2 = \frac{\beta_3 - \beta_1}{\gamma_3 - \gamma_1}, \qquad \alpha_3 = \frac{\beta_1 - \beta_2}{\gamma_1 - \gamma_2}, \tag{3.14}$$

for some matrix $\mathcal{C} \doteq \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3)$.

Taking into account equations (3.11)–(3.12) and (3.14) we see that $\mathcal{C} = \operatorname{diag}(0, k^2, k^2-1) = B_1$ for the Hamiltonian (3.11) and $\mathcal{C} = \operatorname{diag}(1-k^2, 1, 0) = A$ for the Hamiltonian (3.12). Thus $\mathcal{L}^e(\lambda)$ can be considered as the Lax matrix of a special case of the Clebsch system described by the Hamilton function (3.13).

We now derive Lax representations for the Hamiltonian flows corresponding to the Hamilton functions (3.11)–(3.12). They can be written in terms of $\mathfrak{su}(2)$ matrices with an elliptic dependence on the spectral parameter.

The equations of motion w.r.t. the integrals I_1^e and I_2^e read respectively

$$\dot{\mathbf{m}} = [\,\mathbf{a}, B_1\, \mathbf{a}\,], \qquad \dot{\mathbf{a}} = [\,\mathbf{m}, \mathbf{a}\,], \tag{3.15}$$

and

$$\dot{\mathbf{m}} = [\,A\, \mathbf{m}, \mathbf{m}\,] + [\,\mathbf{a}, B_2\, \mathbf{a}\,], \qquad \dot{\mathbf{a}} = [\,A\, \mathbf{m}, \mathbf{a}\,]. \tag{3.16}$$

A straightforward computation leads to the following result.

Proposition 9. *The Hamiltonian flow* (3.15) *generated by the Hamiltonian* (3.11) *admits the Lax representation:*

$$\dot{\mathcal{L}}^e(\lambda) = [\,\mathcal{L}^e(\lambda), \mathcal{M}_1^e(\lambda)\,],$$

with the matrix $\mathcal{L}^e(\lambda)$ *given in equation* (3.6) *and*

$$\mathcal{M}_1^e(\lambda) \doteq \frac{1}{\operatorname{sn}(\lambda)}\,[\,\operatorname{dn}(\lambda)\, \sigma_1\, a_1 + \sigma_2\, a_2 + \operatorname{cn}(\lambda)\, \sigma_3\, a_3\,].$$

The Hamiltonian flow (3.16) *generated by the Hamiltonian* (3.12) *admits the Lax representation:*

$$\dot{\mathcal{L}}^e(\lambda) = [\,\mathcal{L}^e(\lambda), \mathcal{M}_2^e(\lambda)\,],$$

with the matrix $\mathcal{L}^e(\lambda)$ *given in equation* (3.6) *and*

$$\begin{aligned}\mathcal{M}_2^e(\lambda) \doteq{}& \frac{1}{\operatorname{sn}^2(\lambda)}\,[\,\operatorname{cn}(\lambda)\, \sigma_1\, m_1 + \operatorname{cn}(\lambda)\operatorname{dn}(\lambda)\, \sigma_2\, m_2 + \operatorname{dn}(\lambda)\, \sigma_3\, m_3\,] \\ &+ \frac{1}{\operatorname{sn}^3(\lambda)}\,\{\,\operatorname{dn}(\lambda)\, \sigma_1\, a_1 + \operatorname{dn}^2(\lambda)\;\sigma_2\, a_2 + \operatorname{cn}(\lambda)\,[\operatorname{dn}^2(\lambda) + \operatorname{sn}^2(\lambda)]\, \sigma_3\, a_3\,\}.\end{aligned}$$

Remark 2. We note that the "traditional" Lax representations for the Hamiltonian flows (3.15)–(3.16) are given in terms of Lax matrices depending rationally on the spectral parameter [22, 27]. However a Lax representation with elliptic dependence on the spectral parameter for the Clebsch system is already known [3]. Hence the novelty of our results consists just in establishing of the connection between $\mathfrak{su}(2)$ elliptic Gaudin models and the Clebsch system.

4 Integrable chains of interacting tops

As shown in [17, 19, 20] one can construct integrable many-body systems starting with the one-body Lax matrices given in equations (3.4), (3.5) and (3.6). Such systems describe completely integrable (long-range) chains of interacting tops on the Lie–Poisson manifold associated with $\oplus^M \mathfrak{e}^*(3)$, being M the number of tops appearing in the chain. Moreover they admit the same linear r-matrix formulation given in equation (2.6) [17, 20].

Let us denote with $(\mathbf{m}_i, \mathbf{a}_i) \doteq (m_i^1, m_i^2, m_i^3, a_i^1, a_i^2, a_i^3) \in \mathfrak{e}^*(3)$ the pair of $\mathbb{R}^3$ vectors associated with the i-th top of the chain. Thus the Lie–Poisson brackets on $\oplus^M \mathfrak{e}^*(3)$ read

$$\{m_i^\alpha, m_j^\beta\} = -\delta_{i,j}\, \epsilon_{\alpha\beta\gamma}\, m_i^\gamma, \qquad \{m_i^\alpha, a_j^\beta\} = -\delta_{i,j}\, \epsilon_{\alpha\beta\gamma}\, a_i^\gamma, \qquad \{a_i^\alpha, a_j^\beta\} = 0,$$

with $1 \le i, j \le M$. The above brackets are degenerate: they possess the following $2M$ Casimir functions:

$$C_i^{(1)} \doteq \langle\, \mathbf{m}_i, \mathbf{a}_i \,\rangle, \qquad C_i^{(2)} \doteq \frac{1}{2} \langle\, \mathbf{a}_i, \mathbf{a}_i \,\rangle, \qquad 1 \le i \le M. \tag{4.1}$$

According to equations (3.4), (3.5) and (3.6) we can consider the following Lax matrices defined on $\mathfrak{su}(2)[\,\lambda, \lambda^{-1}]$:

$$\mathcal{L}^r_M(\lambda) \doteq \mathbf{p} + \sum_{i=1}^M \mathcal{L}^r_i(\lambda - \mu_i), \tag{4.2}$$

$$\mathcal{L}^t_M(\lambda) \doteq \sum_{i=1}^M \mathcal{L}^t_i(\lambda - \mu_i), \tag{4.3}$$

$$\mathcal{L}^e_M(\lambda) \doteq \sum_{i=1}^M \mathcal{L}^e_i(\lambda - \mu_i), \tag{4.4}$$

where the μ_i's with $\mu_i \ne \mu_k$, $1 \le i, k \le M$, are complex parameters of the models. The Lax matrix $\mathcal{L}^r_M(\lambda)$ describes a system of M interacting Lagrange tops, called *Lagrange chain* in [17], while the matrices $\mathcal{L}^t_M(\lambda), \mathcal{L}^e_M(\lambda)$ govern the dynamics of M interacting Clebsch systems. The latter models can be called *Clebsch chains.*

The following proposition holds [17, 20].

Proposition 10. *The Lax matrices $\mathcal{L}^r_M(\lambda)$, $\mathcal{L}^t_M(\lambda)$, $\mathcal{L}^e_M(\lambda)$ given in equations* (4.2), (4.3) *and* (4.4) *satisfy the linear r-matrix algebra*

$$\left\{\mathcal{L}^{r,t,e}_M(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes \mathcal{L}^{r,t,e}_M(\mu)\right\} + \left[\, r_{r,t,e}(\lambda - \mu), \mathcal{L}^{r,t,e}_M(\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{L}^{r,t,e}_M(\mu) \,\right] = 0,$$

for all $\lambda, \mu \in \mathbb{C}$, with $r_{r,t,e}(\lambda)$ given in equation (2.7)*.*

We now construct the spectral invariants of the Lagrange chain and of the Clebsch chain with $k = 0$.

4.1 The Lagrange chain

The complete set of integrals of the model can be obtained in the usual way. In fact, a straightforward computation leads to the following statement.

Proposition 11. *The hyperelliptic curve* $\det(\mathcal{L}^r_M(\lambda) - \mu\,\mathbb{1}) = 0$, $\lambda, \mu \in \mathbb{C}$, *with* $\mathcal{L}^r_M(\lambda)$ *given in equation* (4.2) *reads*

$$-\mu^2 = \frac{1}{4}\langle\, \mathbf{p}, \mathbf{p}\,\rangle + \frac{1}{2}\sum_{i=1}^{M}\left[\frac{R_i^r}{\lambda-\mu_i} + \frac{S_i^r}{(\lambda-\mu_i)^2} + \frac{C_i^{(1)}}{(\lambda-\mu_i)^3} + \frac{C_i^{(2)}}{(\lambda-\mu_i)^4}\right],$$

where

$$R_i^r \doteq \langle\, \mathbf{p}, \mathbf{m}_i\,\rangle + \sum_{\substack{j=1\\ j\neq i}}^{M}\left[\frac{\langle\, \mathbf{m}_i, \mathbf{m}_j\,\rangle}{\mu_i-\mu_j} + \frac{\langle\, \mathbf{m}_i, \mathbf{a}_j\,\rangle - \langle\, \mathbf{m}_j, \mathbf{a}_i\,\rangle}{(\mu_i-\mu_j)^2} - 2\,\frac{\langle\, \mathbf{a}_i, \mathbf{a}_j\,\rangle}{(\mu_i-\mu_j)^3}\right],$$

$$S_i^r \doteq \langle\, \mathbf{p}, \mathbf{a}_i\,\rangle + \frac{1}{2}\langle\, \mathbf{m}_i, \mathbf{m}_i\,\rangle + \sum_{\substack{j=1\\ j\neq i}}^{M}\left[\frac{\langle\, \mathbf{a}_i, \mathbf{m}_j\,\rangle}{\mu_i-\mu_j} + \frac{\langle\, \mathbf{a}_i, \mathbf{a}_j\,\rangle}{(\mu_i-\mu_j)^2}\right].$$

The $2M$ *independent integrals* $\{R_i^r\}_{i=1}^M$ *and* $\{S_i^r\}_{i=1}^M$ *are involutive first integrals of motion and the integrals* $\{C_i^{(1)}\}_{i=1}^M$ *and* $\{C_i^{(2)}\}_{i=1}^M$ *are the Casimir functions given in equation* (4.1).

Notice that, as in the $\mathfrak{su}(2)$ rational Gaudin model, there is a linear integral given by $\sum_{i=1}^{M} R_i^r = \sum_{i=1}^{M}\langle\, \mathbf{p}, \mathbf{m}_i\,\rangle$. A natural choice for a physical Hamiltonian describing the dynamics of the model can be constructed considering a linear combination of the Hamiltonians $\{R_i\}_{i=1}^M$ and $\{S_i\}_{i=1}^M$ similar to the one considered for the rational Gaudin model, see equation (2.9):

$$\mathcal{H}^r_M \doteq \sum_{i=1}^{M}(\mu_i\, R_i^r + S_i^r) = \sum_{i=1}^{M}\langle\, \mathbf{p}, \mu_i\, \mathbf{m}_i + \mathbf{a}_i\,\rangle + \frac{1}{2}\sum_{i,j=1}^{M}\langle\, \mathbf{m}_i, \mathbf{m}_j\,\rangle. \tag{4.5}$$

If $M = 1$ the Hamiltonian (4.5) gives the sum of the two integrals of motion of the Lagrange top. Our aim is now to find the Hamiltonian flow generated by $\mathcal{H}^r_M$ and its Lax representation.

Proposition 12. *The equations of motion w.r.t. the Hamiltonian* (4.5) *are given by*

$$\dot{\mathbf{m}}_i = [\mathbf{p}, \mathbf{a}_i\,] + \left[\mu_i\, \mathbf{p} + \sum_{j=1}^{M}\mathbf{m}_j, \mathbf{m}_i\right],$$

$$\dot{\mathbf{a}}_i = \left[\mu_i\, \mathbf{p} + \sum_{j=1}^{M}\mathbf{m}_j, \mathbf{a}_i\right], \tag{4.6}$$

with $1 \le i \le M$. *Equations* (4.6) *admit the following Lax representation:*

$$\dot{\mathcal{L}}^r_M(\lambda) = \left[\,\mathcal{L}^r_M(\lambda), \mathcal{M}^{(r,-)}_M(\lambda)\right] = -\left[\,\mathcal{L}^r_M(\lambda), \mathcal{M}^{(r,+)}_M(\lambda)\right],$$

with the matrix $\mathcal{L}^r_M(\lambda)$ *given in equation* (4.2) *and*

$$\mathcal{M}^{(r,-)}_M(\lambda) \doteq \sum_{i=1}^{M}\frac{1}{\lambda-\mu_i}\left[\mu_i\, \mathbf{m}_i + \frac{\lambda\, \mathbf{a}_i}{\lambda-\mu_i}\right], \qquad \mathcal{M}^{(r,+)}_M(\lambda) \doteq \lambda\, \mathbf{p} + \sum_{i=1}^{M}\mathbf{m}_i.$$

Proof. A direct computation. ■

4.2 The Clebsch chain: the case $k = 0$

The complete set of integrals of motion of the Clebsch chain, with $k = 0$, is given in the following statement.

Proposition 13. *The curve* $\det(\mathcal{L}_M^t(\lambda) - \mu\,\mathbb{1}) = 0$, $\lambda, \mu \in \mathbb{C}$, *with* $\mathcal{L}_M^t(\lambda)$ *given in equation* (4.3) *reads*

$$
\begin{aligned}
-\mu^2 = H_0^t + \frac{1}{2}\sum_{i=1}^{M} \big[& R_i^t \cot(\lambda - \mu_i) + S_i^t \cot^2(\lambda - \mu_i) \\
& + C_i^{(1)} \cot^3(\lambda - \mu_i) + C_i^{(2)} \cot^4(\lambda - \mu_i) \big],
\end{aligned}
$$

where

$$
\begin{aligned}
H_0^t \doteq\; & \frac{1}{2}\sum_{i=1}^{M} \left[\, (m_i^1)^2 + (m_i^2)^2 \,\right] - \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{M} m_i^3\, m_j^3 + \frac{1}{2}\left(\sum_{i=1}^{M} a_i^3\right)^2 \\
& + \sum_{\substack{i,j=1\\ i\neq j}}^{M} \frac{1}{\sin(\mu_i - \mu_j)} \left[\, a_i^1\, m_j^1 + a_i^2\, m_j^2 + a_i^3\, m_j^3\, \cos(\mu_i - \mu_j) \,\right] \\
& + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{M} \frac{\cot(\mu_i - \mu_j)}{\sin(\mu_i - \mu_j)} \left[\, a_i^1\, a_j^1 + a_i^2\, a_j^2 + a_i^3\, a_j^3\, \cos(\mu_i - \mu_j) \,\right], \\
R_i^t \doteq\; & C_i^{(1)} + \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{M} \left(m_i^3\, a_j^3 - m_j^3\, a_i^3 \right) \\
& + \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{1}{\sin(\mu_i - \mu_j)} \left[\, m_i^1\, m_j^1 + m_i^2\, m_j^2 + m_i^3\, m_j^3\, \cos(\mu_i - \mu_j) \,\right] \\
& + \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\cot(\mu_i - \mu_j)}{\sin(\mu_i - \mu_j)} \big[\, m_i^1\, a_j^1 + m_i^2\, a_j^2 + m_i^3\, a_j^3\, \cos(\mu_i - \mu_j) \\
& - m_j^1\, a_i^1 - m_j^2\, a_i^2 - m_j^3\, a_i^3\, \cos(\mu_i - \mu_j) \,\big] \\
& - 2\sum_{\substack{j=1\\ j\neq i}}^{M} \frac{1}{\sin^3(\mu_i - \mu_j)} \left[\, a_i^1\, a_j^1 + a_i^2\, a_j^2 + a_i^3\, a_j^3\, \cos(\mu_i - \mu_j) \,\right], \\
S_i^t \doteq\; & C_i^{(2)} + \frac{1}{2}\left[\, (m_i^1)^2 + (m_i^2)^2 + (m_i^3)^2 \,\right] + \frac{1}{2}(a_i^3)^2 + \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{M} a_i^3\, a_j^3 \\
& + \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{1}{\sin(\mu_i - \mu_j)} \left[\, a_i^1\, m_j^1 + a_i^2\, m_j^2 + a_i^3\, m_j^3\, \cos(\mu_i - \mu_j) \,\right] \\
& + \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\cot(\mu_i - \mu_j)}{\sin(\mu_i - \mu_j)} \left[\, a_i^1\, a_j^1 + a_i^2\, a_j^2 + a_i^3\, a_j^3\, \cos(\mu_i - \mu_j) \,\right].
\end{aligned}
$$

The integrals H_0^t, $\{R_i^t\}_{i=1}^M$, $\{S_i^t\}_{i=1}^M$ *are involutive first integrals of motion (only* $2M$ *of them are independent). The integrals* $\{C_i^{(1)}\}_{i=1}^M$ *and* $\{C_i^{(2)}\}_{i=1}^M$ *are the Casimir functions given in equation* (4.1)*.*

5 Concluding remarks and open problems

In the present paper we have proposed an algebraic technique which enabled us to derive two (3-dimensional) integrable cases of rigid body dynamics (the Lagrange top and the Clebsch system) from two-body $\mathfrak{su}(2)$ Gaudin models. We remark that the explicit construction of the Lagrange top starting from the $\mathfrak{su}(2)$ rational two-body Gaudin system has been presented for the first time in [14]. To the best of our knowledge the derivation of the Clebsch system defined by the involutive Hamiltonians (3.11)–(3.12) starting from the $\mathfrak{su}(2)$ elliptic two-body Gaudin system is new, although the novelty is essentially in establishing of the connection between these two integrable systems.

Let us stress that the construction outlined here is just a top of an iceberg. In [17, 18, 19, 20] we presented a general and systematic reduction, based on generalized Inönü–Wigner contractions, of classical Gaudin models associated with a simple Lie algebra $\mathfrak{g}$. Suitable algebraic and pole coalescence procedures performed on the N-pole Gaudin Lax matrices, enabled us to construct one-body and many-body hierarchies of integrable models sharing the same (linear) r-matrix structure of the ancestor models. This technique can be applied to any simple Lie algebra $\mathfrak{g}$ and whatever be the dependence (rational, trigonometric, elliptic) on the spectral parameter. Fixing $\mathfrak{g} = \mathfrak{su}(2)$, we constructed the so called $\mathfrak{su}(2)$ *hierarchies* [18, 20]. In particular the Lagrange top corresponds to the first element ($N = 2$) of the $\mathfrak{su}(2)$ *rational hierarchy*, and the Clebsch system is the first element of the $\mathfrak{su}(2)$ *elliptic hierarchy*.

We studied also the problem of discretizing the Hamiltonian flows of the $\mathfrak{su}(2)$ rational Gaudin model. One of the authors (O.R.), together with Vadim Kuznetsov and Andy Hone, constructed in [8] one-point (complex) and two-point (real) Bäcklund transformations (BTs) for this model. Later on, in [14], again in collaboration with Vadim, we studied the problem of discretizing the dynamics of the Lagrange top using the BTs approach [15, 16].

In [20, 21], using a different approach, we have obtained a new integrable discretization for the Hamiltonian flow given in equation (2.11). It is expressed in terms of an explicit Poisson map and a suitable contraction performed on it enables us to construct discrete-time versions of the whole $\mathfrak{su}(2)$ *rational hierarchy*. Our results include, as a special case ($N = 2$), the discrete-time version of the Lagrange top found by Yu.B. Suris and A.I. Bobenko in [4]. Moreover, the same procedure enabled us to find an integrable discretization of the Hamiltonian flow (4.6), describing a discrete-time version of the Lagrange chain.

A natural extension of our discretizations could be the construction of a suitable approach for models with a trigonometric or elliptic dependence on the spectral parameter instead of a rational one. To the best of our knowledge there are no results in this direction in literature. We remark here that integrable discretizations for the flows (3.15)–(3.16) have been found by Yu.B. Suris, see [27, 28, 29], by using rational Lax matrices.

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