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On the Applications of a New Technique to Solve Linear Differential Equations, with and without Source*

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Abstract. A general method for solving linear differential equations of arbitrary order, is used to arrive at new representations for the solutions of the known differential equations, both without and with a source term. A new quasi-solvable potential has also been constructed taking recourse to the above method.

Key words: Euler operator; monomials; quasi-exactly solvable models

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1 A new procedure for solving linear differential equations

Linear differential equations play a crucial role in various branches of science and mathematics. Second order differential equations routinely manifest in the study of quantum mechanics, in connection with Schrödinger equation. There are various techniques available to solve a given differential equation, e.g., power series method, Laplace transforms, etc. Not many general methods applicable to differential equations of arbitrary order exist in the literature (see [1] and references therein). We make use of a general method for solving linear differential equations of arbitrary order to construct new representations for the solutions of the known second order linear differential equations, both without and with a source term. This method has found applications in solving linear single and multi variable differential equations. The solutions of linear differential equation with a source and development of a new Quasi-Exactly Soluble (QES) system are the new results of this paper.

1.1 Case (i). Linear differential equation without a source term

After appropriate manipulation, any single variable linear differential equation can be brought to the following form

$$[F(D) + P(x, d/dx)]\, y(x) = 0, \tag{1}$$

where $D \equiv x\frac{d}{dx}$, $F(D) = \sum_n a_n D^n$ is a diagonal operator in the space of monomials spanned by x^n and a_n's are some parameters. Here $P(x, d/dx) = \sum_{i,j} c_{i,j} x^i (\frac{d}{dx})^j$, where $c_{i,j} = 0$ if $i = j$.

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Notice that, since $F(D)$ is a diagonal operator, $1/F(D)$ is also well defined in the space of monomials. The following ansatz

$$y(x) = C_\lambda \left\{ \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{F(D)} P(x, d/dx) \right]^m \right\} x^\lambda \equiv C_\lambda \hat{G}_\lambda x^\lambda \tag{2}$$

is a solution of the above equation, provided $F(D)x^\lambda = 0$ and the coefficient of x^λ in $P(x, \frac{d}{dx})^m x^\lambda$ should be zero [6]. In order to realise (2) as a power series we impose the requirement that $P(x, \frac{d}{dx})$ lowers (i.e. $c_{i,j} = 0$ for $i \geq j$) or raises the degree of monomials. Substituting equation (2), modulo C_λ, in equation (1)

$$\begin{aligned}
&(F(D) + P(x, d/dx)) \left\{ \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{F(D)} P(x, d/dx) \right]^m \right\} x^\lambda \\
&= F(D) \left[1 + \frac{1}{F(D)} P(x, d/dx) \right] \left\{ \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{F(D)} P(x, d/dx) \right]^m \right\} x^\lambda \\
&= F(D) \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{F(D)} P(x, d/dx) \right]^m x^\lambda \\
&+ F(D) \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{F(D)} P(x, d/dx) \right]^{m+1} x^\lambda \\
&= F(D)x^\lambda - F(D) \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{F(D)} P(x, d/dx) \right]^{m+1} x^\lambda \\
&+ F(D) \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{F(D)} P(x, d/dx) \right]^{m+1} x^\lambda = 0.
\end{aligned}$$

Equation (2) connects the solution of a given differential equation to the monomials. In order to show that, this rather straightforward procedure indeed yields non-trivial results, we explicitly work out a few examples. Consider the Hermite differential equation, which arises in the context of quantum harmonic oscillator,

$$\left[D - n - \frac{1}{2} \frac{d^2}{dx^2} \right] H_n(x) = 0.$$

Here $F(D) = D - n$ and $F(D)x^\lambda = 0$ yields $\lambda = n$. Hence

$$H_n(x) = C_n \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{D - n} (-1/2)(d^2/dx^2) \right]^m x^n.$$

Using $[D, (d^2/dx^2)] = -2(d^2/dx^2)$ it is easy to see that

$$\left[\frac{1}{(D - n)} (-1/2)(d^2/dx^2) \right]^m x^n = (-1/2)^m (d^2/dx^2)^m \prod_{l=1}^{m} \frac{1}{(-2l)} x^n$$

and

$$H_n(x) = C_n \sum_{m=0}^{\infty} (-1/4)^m \frac{1}{m!} (d^2/dx^2)^m x^n = C_n e^{-\frac{1}{4} \frac{d^2}{dx^2}} x^n,$$

this is a well-known result. Similar expression also holds for the Lagurre polynomial which matches with the one found in [5]. Below, we list new representations for the solutions of some frequently encountered differential equations in various branches of physics and mathematics [4]. Notice that, the cases of confluent hypergeometric, hypergeometric, Chebyshev type II and Jacobi solutions are given in [6], and reproduced here for the sake of completeness.

Legendre polynomial

$$P_n(x) = C_n e^{-\{1/(2[D+n+1])\}(d^2/dx^2)} x^n.$$

Associated Legendre polynomial

$$P_n^m(x) = C_n(1-x^2)^{m/2} e^{-\{1/(2[D+n+m+1])\}(d^2/dx^2)} x^{n-m}.$$

Bessel function

$$J_{\pm\nu}(x) = C_{\pm\nu} e^{-\{1/(2[D\pm\nu])\}x^2} x^{\pm\nu}.$$

Generalized Bessel function

$$u_\pm(x) = C_\pm e^{-\{\beta\gamma^2/(2[D+\alpha\pm\beta\nu])\}x^{2\beta}} x^{\beta\nu-\alpha\pm\beta\nu}.$$

Gegenbauer polynomial

$$C_n^\lambda(x) = C_n e^{-\{1/(2[D+n+2\lambda])\}(d^2/dx^2)} x^n.$$

Hypergeometric function

$$y_\pm(\alpha,\beta;\gamma;x) = C_\pm e^{-\{(1/(D+\lambda_\pm)\}\hat{A}} x^{-\lambda_\mp},$$

where $\lambda_\pm$ is either α or β and $\hat{A} \equiv x\frac{d^2}{dx^2} + \gamma\frac{d}{dx}$. All the above series solutions have descending powers of x. In order to get the series in the ascending powers, one has to replace x by $\frac{1}{x}$ in the original differential equation and generate the solutions via equation (2). However, the number of solutions will remain the same. One can also generate the series solutions by multiplying the original differential equations with x^2, and then, rewriting $x^2\frac{d^2}{dx^2} = D(D-1) = F(D)$.

The solution for the following equation with periodic potential

$$\frac{d^2y}{dx^2} + a\cos(x)y = 0 \tag{3}$$

can be found after multiplying equation (3) by x^2 and rewriting $x^2\frac{d^2}{dx^2}$ as $(D-1)D$ to be

$$y(x) = \sum_{m,\{n_i\}=0}^{\infty} \frac{(-a)^m}{m!}\left\{\prod_{i=1}^{m}\frac{(-1)^{n_i}}{(2n_i)!}\right\} \times \left\{\prod_{r=1}^{m}\frac{\left(2\left[m+\lambda/2-r+\sum\limits_{i=1}^{m+1-r} n_i\right]\right)!}{\left(2\left[m+\lambda/2+1-r+\sum\limits_{i=1}^{m+1-r} n_i\right]\right)!}\right\} x^{2\left(m+\sum\limits_{i=1}^{m} n_i+\lambda/2\right)},$$

where $\lambda = 0$ or 1. In the same manner, one can write down the solutions for the Mathieu's equation as well.

Chebyshev polynomials $T_n(x)$ and $U_n(x)$

$$T_n(x) = C_n e^{-\left\{\frac{1}{2}\frac{1}{(D+n)}\frac{d^2}{dx^2}\right\}} x^n$$

and

$$U_n(x) = C'_n e^{-\left\{\frac{1}{2}\frac{1}{(D+n+2)}\frac{d^2}{dx^2}\right\}} x^n,$$

where C_n and C'_n are appropriate normalization constants.

Jacobi polynomial

$$J_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{\infty}\left[\frac{1}{(D-n)(D+\alpha+\beta+1)}\left(\frac{d^2}{dx^2}+(\beta-\alpha)\frac{d}{dx}\right)\right]^m x^n,$$

Schläfli, Whittaker and for that matter, any solution of a second order linear differential equation without a source term can be solved in a manner identical to the above cases.

1.2 Case (ii). Linear differential equation with a source term

Consider,

$$(F(D) + P(x, d/dx))\, y(x) = Q(x).$$

Now the solution can be found in a straightforward way as

$$F(D)\left(1 + \frac{1}{F(D)}P(x, d/dx)\right) y(x) = Q(x),$$

and

$$y(x) = \sum_{m=0}^{\infty}(-1)^m\left[\frac{1}{F(D)}P(x, d/dx)\right]^m \frac{1}{F(D)}Q(x).$$

We list below a few examples for the above case.

Neumann's polynomial

$$O_n(x) = \left\{\sum_{r=0}^{\infty}(-1)^r\left[\frac{1}{[(D+1)^2-n^2]}x^2\right]^r\left(\frac{1}{[(D+1)^2-n^2]}\right)\right\} \times \left(x\cos^2(n\pi/2) + n\sin^2(n\pi/2)\right).$$

Lommel function

$$w(x) = \sum_{m=0}^{\infty}(-1)^m\left[\frac{1}{(D^2-\nu^2)}x^2\right]^m \frac{1}{(D^2-\nu^2)}x^{\mu+1}.$$

The cases of Struve, Anger and Weber functions are identical to the above ones.

It is a priori not transparent that all the solutions of a given differential equation can be obtained through the present approach. If F is a polynomial of the same degree as the order of the differential equation, and has distinct roots, the linearly independent solutions can be obtained through this approach. As has been pointed out in an earlier paper [8] a given single variable differential equation can be cast in the desired form, as required by the present approach, in more than one way through multiplication by powers of x. In a number of cases, these lead to different solutions. However, the case of degenerate roots, as also the case of inhomogeneous equations needs separate consideration, which we hope to investigate in future.

2 A new quasi-exactly solvable model

The above procedure is also applicable to differential equations having higher number of singularities e.g., Heun equation and its generalization [7]. The same can be used to generate quasi-solvable models. It is worth mentioning a quasi-solvable model based on Heun's equation has been studied; it has been shown that, this system lacks the $SL(2,R)$ of many well-known QES model [3, 9]. We now proceed to generalize this system for obtaining a new QES potential. The differential equation under consideration is given by:

$$f'' + \left\{ \frac{1}{2x} + \frac{1+2s}{x-1} + \frac{1}{2(x+\epsilon^2)} \right\} f' + \frac{(\alpha\beta x - q - \frac{\Omega}{x})}{x(x-1)(x+\epsilon^2)} f = 0. \tag{4}$$

When $\Omega = 0$ the above reduces to Heun equation. Under the following change of variable

$$x = \frac{\sinh^2 \frac{\rho y}{2}}{1 + \frac{1}{\epsilon^2} + \sinh^2 \frac{\rho y}{2}},$$

and with $\Psi = (1-x)^s f(x)$ the above equation can be cast in the form of Schrödinger eigenvalue problem. Here $s = (1 - E/\rho^2)^{1/2}$, where E is energy of the system. The constants α, β and q in the equation (4) are related to s in the following form:

$$\alpha = -\frac{5}{2} - s, \qquad \beta = \frac{3}{2} - s \qquad \text{and} \qquad q = (1-s^2)(1+\epsilon^2) - \frac{1}{2}s\epsilon^2 - \frac{1}{4}(1-2\epsilon^2).$$

The corresponding potential is given by

$$V(y) = \rho^2 \left\{ \frac{8\sinh^4 \frac{\rho y}{2} - 4(\frac{5}{\epsilon^2} - 1)\sinh^2 \frac{\rho y}{2} + 2(\frac{1}{\epsilon^4} - \frac{1}{\epsilon^2} - 2)}{8(1 + \frac{1}{\epsilon^2} + \sinh^2 \frac{\rho y}{2})^2} + \frac{\Omega}{\epsilon^2 \sinh^2 \frac{\rho y}{2}} \right\}.$$

For the purpose of finding solutions we cast the generalized Heun equation (4) in the form, $[F(D) + P]y(x) = 0$. Multiplying by x^2, we get

$$\left[-4x^2\epsilon^2 \frac{d^2}{dx^2} - 2\epsilon^2 x \frac{d}{dx} - 4\Omega \right] f(x) + \left[4x^3(\epsilon^2 - 1)\frac{d^2}{dx^2} + 2(3\epsilon^2 - 2 + 4s\epsilon^2)x^2 \frac{d}{dx} \right.$$
$$\left. - 4qx + 4x^4 \frac{d^2}{dx^2} + 8(1+s)x^3 \frac{d}{dx} + 4\alpha\beta x^2 \right] f(x) = 0.$$

In the above equation $F(D) = -4\epsilon^2 D^2 + 2\epsilon^2 D - 4\Omega$ and the condition $F(D)x^\xi = 0$ yields

$$\xi_\pm = \frac{1}{4}\left(1 \pm \sqrt{1 - 16\Omega/\epsilon^2}\right).$$

For polynomial solutions either of the roots of the equation must be an integer. Taking $\xi_- = m$ we obtain the following relation for the allowed values of m

$$2m^2 - m + 2\Omega/\epsilon^2 = 0.$$

Let us consider a case in which $\Omega = -\epsilon^2/2$. Taking care of normalization condition and positivity of energy, the allowed value of m is 1. This yields a solution of the form:

$$f(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

If the polynomial terminates at x^{n-1} then the coefficients a_n and a_{n+1} should be zero. Using this constraint, we get the following condition for n

$$s^2 + (2n-1) + n^2 - n - \frac{15}{4} = 0,$$

giving the allowed values of n as 2, 3. For $n = 2$, $s = 1/2$, $E = \frac{3}{4}\rho^2$ and $\epsilon^2 = 1$ we get the wave function

$$\psi(y) = N \left(\frac{2}{2 + \sinh^2(\rho y/2)} \right)^{1/2} \left(\frac{\sinh^2(\rho y/2)}{2 + \sinh^2(\rho y/2)} \right),$$

where N is the normalization constant. For $n = 3$, $s = -\frac{1}{2}$, which makes the wave function un-normalizable. This procedure may find applicability to unravel the symmetry properties of Heun equation and its generalization [2]. We have earlier studied the symmetry properties of confluent Hypergeometric and Hypergeometric equations using the present approach [8]. The fact that, the solutions are connected with monomials makes the search for the symmetry rather straightforward.

In the case when $\Omega \neq 0$, in equation (4), the additional term in Heun equation manifests in the potential in the Schrödinger equation, as a term having $\frac{1}{x^2}$ type singularity at the origin. This is very interesting since it is of the Calogero–Sutherland type, a system much investigated in the literature. Like the Calogero–Sutherland case, a Jastrow type factor in the wave function appears because of this singular interaction. At a formal level, this interaction modifies the singularity at the origin without adding any new singularity. We note that $x = 0$ is still a regular singular point, since the limit $x^2 \frac{\Omega}{x^2}$ as $x \to 0$ is finite.

In conclusion, we have developed a new method to solve linear differential equations of arbitrary order, both without and with source term. Using this method, we worked out a few examples for the case of second order linear differential equations. We obtained known, as well as, the new representations for the corresponding solutions. The same approach is also applicable to QES models based on Heun equation and its generalizations. In particular a new potential of QES type is constructed through this approach. We intend to analyze the symmetry properties of the Heun equation through the present formalism.

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