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Bäcklund–Darboux Transformation for Non-Isospectral Canonical System and Riemann–Hilbert Problem*

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Abstract. A GBDT version of the Bäcklund–Darboux transformation is constructed for a non-isospectral canonical system, which plays essential role in the theory of random matrix models. The corresponding Riemann–Hilbert problem is treated and some explicit formulas are obtained. A related inverse problem is formulated and solved.

Key words: Bäcklund–Darboux transformation; canonical system; random matrix theory

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1 Introduction

We shall consider a non-isospectral system

$$w_x(x,z) = i\lambda J H(x) w(x,z), \qquad \lambda = (z-x)^{-1}, \tag{1.1}$$

where $w_x = \frac{d}{dx}w$, and J and $H(x)$ are $m \times m$ matrices:

$$H(x) = H(x)^*, \qquad J = J^* = J^{-1}.$$

When the Hamiltonian $H \geq 0$, and the spectral parameter λ does not depend on x, the system above is a classical canonical system. A version of the Bäcklund–Darboux transformation (BDT) for the classical canonical system have been constructed in [15]. In our case (1.1) the spectral parameter $\lambda = (z-x)^{-1}$ depends on x, and here we construct BDT for this case.

BDT is a fruitful approach to obtain solutions of the linear differential equations and systems. It is also widely used to construct explicit solutions of integrable nonlinear systems. For that purpose BDT is applied simultaneously to two auxiliary linear systems of the integrable one. BDT is closely related to the symmetry properties. Since the original works of Bäcklund and Darboux, a much deeper understanding of this transformation has been achieved and various interesting versions of the Bäcklund–Darboux transformation have been introduced (see, for instance, [1, 2, 6, 8, 11, 12, 13, 21, 23]). Important works on the Bäcklund–Darboux transformation both in the continuous and discrete cases have been written by V.B. Kusnetzov and his coauthors (see [9, 10] and references therein).

We apply BDT to construct explicitly new solutions of the Riemann–Hilbert problem on the interval $[0, l]$:

$$W_+(s) = W_-(s) R(s)^2, \qquad 0 \leq s \leq a, \tag{1.2}$$

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue 'Integrable Systems and Related Topics'. The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html where $W(z)$ is analytic for $z \notin [0, a]$, and $W(z) \to I_m$, when $z \to \infty$, I_m is the $m \times m$ identity matrix. For important classes of R the solution of problem (1.2) takes the form

$$W(z) = w(l,z), \qquad W_+(s) = \lim_{\eta \to +0} w(l, s+i\eta), \qquad W_-(s) = \lim_{\eta \to +0} w(l, s-i\eta), \tag{1.3}$$

where the $m \times m$ fundamental solution w of (1.1) is normalized by the condition

$$w(0,z) = I_m. \tag{1.4}$$

The necessary and sufficient conditions for (1.3) are given in [20, p. 209] (see also [16, 19]). It is useful to obtain explicit formulas for H and R.

The problem (1.2) is of interest in the random matrix theory: the Markov parameters appearing in the series representation $w(l,z) = I_m + z^{-1}M_1(l) + z^{-2}M_2(l) + \cdots$ are essential for the random matrices problems [3, 4]. In particular, in the bulk scaling limit of the Gaussian unitary ensemble of Hermitian matrices the probability that an interval of length l contains no eigenvalues is given by the function $P(l)$, which satisfies the equality $\frac{d}{dl} \log P(l) = i\big(m_{22}(l) - m_{11}(l)\big)$, where m_{11} and m_{22} are the corresponding entries of the 2×2 matrix M_1. Notice that $M_1(l) = \int_0^l JH(x)dx$.

When $J = I_m$, system (1.2) is essential in the prediction theory [22].

We construct a Bäcklund–Darboux transformation for system (1.1) in the next Section 2. Section 3 is dedicated to explicit solutions, and Section 4 is dedicated to an inverse problem.

2 Bäcklund–Darboux transformation

To construct Bäcklund–Darboux transformation we use the methods developed in [14, 15] for non-isospectral problems and canonical system, respectively. For this purpose fix integer $n > 0$ and $n \times n$ parameter matrices $A(0)$, $S(0) = S(0)^*$. Fix also $n \times m$ parameter matrix $\Pi(0)$ so that the matrix identity

$$A(0)S(0) - S(0)A(0)^* = i\Pi(0)J\Pi(0)^* \tag{2.1}$$

holds. Introduce now matrix functions $A(x)$, $S(x)$ and $\Pi(x)$ by their values at $x = 0$ and equations

$$A_x = A^2, \qquad \Pi_x = -iA\Pi JH, \tag{2.2}$$

$$S_x = \Pi JHJ^*\Pi^* - (AS + SA^*). \tag{2.3}$$

Then it can be checked by direct differentiation that the matrix identity

$$AS - SA^* = i\Pi J\Pi^* \tag{2.4}$$

holds for each x. Notice that the equation $A_x = A^2$ is motivated by the similar equation $\lambda_x = \lambda^2$ for the spectral parameter λ because A can be viewed as a generalized spectral parameter (see [14]). In the points of invertibility of S we can introduce a transfer matrix function in the Lev Sakhnovich form [17, 18, 19]

$$w_A(x,z) = I_m - iJ\Pi(x)^*S(x)^{-1}(A - \lambda I_n)^{-1}\Pi(x). \tag{2.5}$$

This transfer matrix function has an important J-property [17]:

$$w_A(x,\overline{z})^*Jw_A(x,z) = J, \qquad \text{i.e.,} \qquad w_A(x,z)^{-1} = Jw_A(x,\overline{z})^*J. \tag{2.6}$$

Put

$$v(x, z) = w_0(x)^{-1} w_A(x, z), \tag{2.7}$$

where matrix function w_0 is defined by the relations

$$\frac{d}{dx} w_0(x) = \widetilde{G}_0(x) w_0(x), \qquad w_0(0)^* J w_0(0) = J, \tag{2.8}$$

$$\widetilde{G}_0 = -J(i\Pi^* S^{-1}\Pi - HJ\Pi^* S^{-1}\Pi + \Pi^* S^{-1}\Pi J^* H) \tag{2.9}$$

up to J-unitary initial value $w_0(0)$. (We omit sometimes argument x in the formulas for brevity.)

Theorem 1. *Suppose w is the fundamental solution of system* (1.1) *and relations* (2.1)–(2.3) *are valid. Then in the points of invertibility of $S(x)$ the matrix function*

$$\widetilde{w}(x, z) = v(x, z) w(x, z) \tag{2.10}$$

is well defined and satisfies the transformed system

$$\frac{d}{dx}\widetilde{w} = i\lambda J \widetilde{H} \widetilde{w}, \tag{2.11}$$

where

$$\widetilde{H}(x) = w_0(x)^* H(x) w_0(x). \tag{2.12}$$

Moreover, if $\det S(x) \neq 0$ $(0 \leq x \leq l)$ *then the fundamental solution of system* (2.11) *is given by the formula*

$$\widetilde{w}(x, z) = v(x, z) w(x, z) v(0, z)^{-1}, \qquad 0 \leq x \leq l. \tag{2.13}$$

Proof. The proof is based on the equation for the transfer matrix function

$$\frac{d}{dx} w_A(x, z) = \widetilde{G}(x, z) w_A(x, z) - i\lambda w_A(x, z) J H(x), \tag{2.14}$$

where

$$\widetilde{G}(x, z) = i\lambda J H - J(i\Pi^* S^{-1}\Pi - HJ\Pi^* S^{-1}\Pi + \Pi^* S^{-1}\Pi J^* H). \tag{2.15}$$

To prove (2.14) consider first $\frac{d}{dx} J\Pi^* S^{-1}$. By (2.2) we have

$$\frac{d}{dx} J\Pi^* S^{-1} = iJHJ\Pi^* A^* S^{-1} - J\Pi^* S^{-1} S_x S^{-1}. \tag{2.16}$$

Use now (2.3) and (2.16) to get

$$\frac{d}{dx} J\Pi^* S^{-1} = (iJHJ + J)\Pi^* A^* S^{-1} + J\Pi^* S^{-1} A - J\Pi^* S^{-1}\Pi JHJ\Pi^* S^{-1}. \tag{2.17}$$

Rewrite identity (2.4) as $A^* S^{-1} = S^{-1} A - iS^{-1}\Pi J\Pi^* S^{-1}$ and substitute this equality into (2.17) to obtain

$$\frac{d}{dx} J\Pi^* S^{-1} = (iJHJ + 2J)\Pi^* S^{-1} A$$
$$+ J\big((HJ - iI_m)\Pi^* S^{-1}\Pi J - \Pi^* S^{-1}\Pi JHJ\big)\Pi^* S^{-1}. \tag{2.18}$$

We shall apply (2.18) as well as the second relation in (2.2) to differentiate $w_A(x,z)$:

$$\frac{d}{dx}w_A = -i\Big((iJHJ+2J)\Pi^*S^{-1}\big((A-\lambda I_n)+\lambda I_n\big)(A-\lambda I_n)^{-1}\Pi$$
$$+J\big((HJ-iI_m)\Pi^*S^{-1}\Pi J-\Pi^*S^{-1}\Pi JHJ\big)\Pi^*S^{-1}(A-\lambda I_n)^{-1}\Pi\Big)$$
$$+iJ\Pi^*S^{-1}(A-\lambda I_n)^{-1}(A_x-\lambda_x I_n)(A-\lambda I_n)^{-1}\Pi$$
$$-iJ\Pi^*S^{-1}(A-\lambda I_n)^{-1}\big(-i\big((A-\lambda I_n)+\lambda I_n\big)\Pi JH\big). \tag{2.19}$$

Use substitutions $(A-\lambda I_n)^{-1}(A-\lambda I_n)=I_n$ and

$$(A-\lambda I_n)^{-1}(A_x-\lambda_x I_n)(A-\lambda I_n)^{-1}=I_n+2\lambda(A-\lambda I_n)^{-1},$$

and collect terms to rewrite (2.19) in the form (2.14).

According to formulas (2.7)–(2.9), (2.14), and (2.15) we have

$$\frac{d}{dx}v(x,z)=w_0(x)^{-1}\big(\widetilde{G}(x,z)-\widetilde{G}_0(x)\big)w_A(x,z)-i\lambda v(x,z)JH(x)$$
$$=i\lambda w_0(x)^{-1}JH(x)w_0(x)v(x,z)-i\lambda v(x,z)JH(x). \tag{2.20}$$

Taking into account (2.8) we get

$$w_0(x)^*Jw_0(x)=w_0(0)^*Jw_0(0)=J. \tag{2.21}$$

Thus we rewrite (2.20) as

$$\frac{d}{dx}v(x,z)=i\lambda J\widetilde{H}(x)v(x,z)-i\lambda v(x,z)JH(x), \tag{2.22}$$

where $\widetilde{H}$ is given by (2.12). From (1.1) and (2.22) it follows that (2.11) is true for $\widetilde{w}$ of the form (2.10). In view of (1.4) one can see that normalization (2.13) yields $\widetilde{w}(0,z)=I_m$. ■

Our next proposition provides conditions for invertibility of S.

Proposition 1. *Suppose matrix functions $H(x)\geq 0$ and $A(x)$ are summable on the interval $[0,\,l]$, and $S(0)>0$. Then $S(x)>0$ for $0\leq x\leq l$, and so $S(x)$ is invertible.*

Proof. Put

$$Q(x)=V(x)S(x)V(x)^*, \qquad \text{where} \qquad V_x=VA, \qquad V(0)=I_n. \tag{2.23}$$

Then in view of (2.3) and (2.23) we have

$$Q_x=V(S_x+AS+SA^*)V^*=V\Pi JHJ^*\Pi^*V^*\geq 0, \qquad Q(0)=S(0).$$

It follows that

$$Q(x)>0, \qquad S(x)=V(x)^{-1}Q(x)\big(V(x)^*\big)^{-1}>0.$$ ■

In view of the first equality in (2.2) invertible matrix function A is of the form $A=(B-xI_n)^{-1}$. Further we shall suppose that A is defined and both A and S are invertible on some interval $[0,l]$.

Remark 1. Suppose $A(x)$ and $S(x)$ are invertible on the interval $[0,l]$. Using (2.17) we can differentiate

$$w_A(x,\infty) := I_m - iJ\Pi^* S^{-1} A^{-1}\Pi.$$

In this way similarly to (2.14) we can show that the matrix function w_0, which satisfies (2.8), (2.9) and initial condition $w_0(0) = U$, admits representation

$$w_0(x) = w_A(x,\infty)\widetilde{U}, \qquad \widetilde{U} = w_A(0,\infty)^{-1}U, \qquad 0 \le x \le l. \tag{2.24}$$

Notice that by (2.12) and (2.21) the equality $HJH \equiv 0$ yields $\widetilde{H}J\widetilde{H} \equiv 0$, i.e., if JH is nilpotent, then $J\widetilde{H}$ is nilpotent too.

Let now $w(l,z)$ satisfy Riemann–Hilbert equation (1.2) where $W_\pm(s) = \lim_{\eta\to+0} w(l, s \pm i\eta)$. Suppose all conditions of Theorem 1 are fulfilled. Then, putting

$$\widetilde{W}_\pm(s) = \lim_{\eta\to+0} \widetilde{w}(l, s \pm i\eta),$$

we get

$$\widetilde{W}_\pm(s) = v(l,s)W_\pm(s)v(0,s)^{-1}. \tag{2.25}$$

In view of (1.2) and (2.25) we obtain

$$\widetilde{W}_+(s) = \widetilde{W}_-(s)v(0,s)R(s)^2 v(0,s)^{-1} = \widetilde{W}_-(s)\widetilde{R}(s)^2,$$

where

$$\widetilde{R}(s) = v(0,s)R(s)v(0,s)^{-1}. \tag{2.26}$$

The subcase of nilpotent matrix function $\widetilde{R}(s) - I_m$ is important [20]. According to (2.26) we have

$$\widetilde{R}(s) - I_m = v(0,s)\big(R(s) - I_m\big)v(0,s)^{-1}. \tag{2.27}$$

Hence, we get a corollary.

Corollary 1. *If $R(s) - I_m$ is nilpotent, then $\widetilde{R}(s) - I_m$ is nilpotent too.*

3 Explicit solutions

If we know A, S, Π, then using the results of the previous section we can construct explicit expressions for $\widetilde{H}$ and $\widetilde{R}$. Consider the simplest case

$$m = 2, \qquad H = \beta^*\beta, \qquad \beta \equiv [1 \quad i], \qquad 0 \le x \le l, \qquad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.1}$$

Then in formula (1.2) we have

$$R(s) \equiv I_2 + \pi J\beta^*\beta, \qquad 0 < x < l. \tag{3.2}$$

Indeed, in view of (1.1) and (3.1) we get

$$\beta w_x(x,z) = 0, \qquad \beta J w_x(x,z) = 2i(z-x)^{-1}\beta w(x,z), \qquad \text{i.e.,}$$

$$\beta w(x,z) = \beta w(0,z) = \beta, \qquad \beta J w(x,z) = -2i\big(\ln(z-x)\big)\beta + \text{const}, \tag{3.3}$$

where const means some constant (vector). In the first relation in (3.3) we use normalization condition (1.4). Taking into account (1.4) again, from the second relation in (3.3) we derive

$$\beta J w(x,z) = 2i\left(\ln \frac{z}{z-x}\right)\beta + \beta J. \tag{3.4}$$

Equalities (3.3) and (3.4) imply that

$$\beta W_+(s) = \beta W_-(s) = \beta, \qquad \beta J W_+(s) = 2i\left(\left(\ln\left|\frac{s}{s-l}\right|\right) - i\pi\right)\beta + \beta J,$$

$$\beta J W_-(s) = 2i\left(\left(\ln\left|\frac{s}{s-l}\right|\right) + i\pi\right)\beta + \beta J. \tag{3.5}$$

From (3.5) it follows that

$$TW_+(s) - TW_-(s) = \begin{bmatrix} 0 \\ 4\pi\beta \end{bmatrix}, \qquad T := \begin{bmatrix} \beta \\ \beta J \end{bmatrix}. \tag{3.6}$$

Notice that

$$TJT^* = 2J. \tag{3.7}$$

So according to (3.6) we have

$$W_+(s) - W_-(s) = \frac{1}{2} JT^*J \begin{bmatrix} 0 \\ 4\pi\beta \end{bmatrix} = 2\pi J\beta^*\beta. \tag{3.8}$$

Moreover, formula (3.5) implies that

$$W_-(s) = \frac{1}{2} JT^*J \begin{bmatrix} \beta \\ 2i\left(\left(\ln\left|\frac{s}{s-l}\right|\right) + i\pi\right)\beta + \beta J \end{bmatrix}.$$

Hence, we obtain

$$J\beta^* = W_-(s)J\beta^*. \tag{3.9}$$

Substitute (3.9) into (3.8) to see that $R^2 = I_2 + 2\pi J\beta^*\beta$, i.e., we can assume (3.2).

Also we can set

$$A = (B - xI_n)^{-1}, \qquad B = \mathrm{diag}\{b_1, b_2, \ldots, b_n\}.$$

From $\Pi_x = -iA\Pi JH$ and (3.1) we get

$$\Pi J\beta^* = g = \{g_k\}_{k=1}^n \equiv \text{const}. \tag{3.10}$$

We also have

$$\frac{d}{dx}\Pi(x)\beta^* = -2i(B - xI_n)^{-1}\Pi(x)J\beta^*.$$

It follows that

$$\Pi(x)\beta^* = 2\big(\{ig_k \ln(b_k - x)\}_{k=1}^n + h\big), \qquad h \equiv \text{const}. \tag{3.11}$$

Formulas (3.10) and (3.11) give us Π. We shall assume that $b_k \notin [0,\infty)$, and so Π is well-defined on $[0,\infty)$. Taking into account (3.7) we also get

$$\begin{aligned}\Pi(x)J\Pi(x)^* &= \frac{1}{2}\Pi(x)T^*JT\Pi(x)^* \\ &= \big(\{ig_k \ln(b_k-x)\}_{k=1}^n + h\big)g^* + g\big(\{ig_k \ln(b_k-x)\}_{k=1}^n + h\big)^*. \end{aligned}\tag{3.12}$$

The matrix function S is easily derived from the identity $AS - SA^* = i\Pi J\Pi^*$.

Finally, in view of (2.24) and (3.10) we get

$$\beta w_0(x) = \big(\beta - ig^*S(x)^{-1}(B - xI_n)\Pi(x)\big)\widetilde{U},$$

which, taking into account (2.12) and (3.1), implies

$$\widetilde{H}(x) = \widetilde{U}^*\big(\beta - ig^*S(x)^{-1}(B-xI_n)\Pi(x)\big)^*\big(\beta - ig^*S(x)^{-1}(B-xI_n)\Pi(x)\big)\widetilde{U}. \tag{3.13}$$

From (2.10) it follows that $v(x,s)$ is J-unitary and we rewrite (2.27) as

$$\widetilde{R}(s) = I_2 + v(0,s)\big(R(s) - I_2\big)Jv(0,s)^*J.$$

Now by (2.5), (2.7), (3.2), and (3.10) we get

$$\begin{aligned}\widetilde{R}(s) = I_2 &+ \pi J w_0(0)^*\big(\beta^* - is\Pi(0)^*S(0)^{-1}B(sI_n - B)^{-1}g\big) \\ &\times \big(\beta^* - is\Pi(0)^*S(0)^{-1}B(sI_n-B)^{-1}g\big)^* w_0(0).\end{aligned}\tag{3.14}$$

Thus matrix functions $\widetilde{H}(x)$ and $\widetilde{R}(s)$ are given by formulas (3.13) and (3.14), respectively.

Example 1. Consider the simplest case $n = 1$. Put $b_1 = b$ and assume $b \notin \mathbb{R}$. Rewrite (3.12) as

$$\Pi(x)J\Pi(x)^* = i|g|^2\big(\ln(b-x) - \overline{\ln(b-x)}\big) + h\overline{g} + g\overline{h}.$$

Here $\overline{g}$ is the complex number conjugated to g. Hence, in view of (2.4) we get

$$S(x) = \big(|g|^2\big(\ln(b-x) - \overline{\ln(b-x)}\big) - ih\overline{g} - ig\overline{h}\big)\frac{(b-x)(\overline{b}-x)}{b-\overline{b}}.$$

Put now $h = 0$ to derive

$$S(x) = \frac{2i}{b-\overline{b}}|g|^2\big(\arg(b-x)\big)(b-x)(\overline{b}-x) \neq 0. \tag{3.15}$$

Rewrite (3.14) as

$$\widetilde{R}(s) = I_2 + \pi JU^*r(s)^*r(s)U, \tag{3.16}$$

where $U = w_0(0)$, $r(s) = \beta + i\,\overline{b}\,\overline{g}\big(\overline{S(0)}\big)^{-1}s(s-\overline{b})^{-1}\Pi(0)$. By (3.7), (3.10), and (3.11) we have

$$\Pi(x) = \frac{1}{2}\Pi(x)T^*JTJ = \frac{1}{2}g\Big[i\big(1+2\ln(b-x)\big) \quad 1 - 2\ln(b-x)\Big]. \tag{3.17}$$

Formulas (3.15) and (3.17) imply

$$r(s) = \beta + \frac{b-\overline{b}}{4b\arg\, b}\frac{s}{s-\overline{b}}\Big[i\big(1+2\ln\, b\big) \quad 1 - 2\ln\, b\Big]. \tag{3.18}$$

Equalities (3.16) and (3.18) define $\widetilde{R}(s)$ determined by the parameters b and g and J-unitary matrix U. According to (2.6), (2.24), and (3.13) the corresponding matrix function $\widetilde{H}(x)$ is of the form

$$\widetilde{H}(x) = \widetilde{U}^* h(x)^* h(x) \widetilde{U},$$

where $h(x) = \beta - i\,\overline{g}\,(b-x)S(x)^{-1}\Pi(x)$, and

$$\widetilde{U} = Jw_A(0,\infty)^* JU = \left(I_2 + \frac{i\,\overline{b}}{S(0)} J\Pi(0)^*\Pi(0) \right) U.$$

Finally, using (3.15) and (3.17) rewrite h in the explicit form:

$$h(x) = \beta - \frac{b-\overline{b}}{4(\overline{b}-x)\arg(b-x)} \Big[i\big(1 + 2\ln(b-x)\big) \quad 1 - 2\ln(b-x) \Big].$$

4 Inverse problem: explicit solutions

In view of (2.6) it is immediate that formula (3.14) can be written in the form (3.16): $\widetilde{R}(s) = I_2 + \pi JU^* r(s)^* r(s) U$, where vector function

$$r(s) = \beta J w_A(0,s)^* J = \beta + isg^*(sI_n - B^*)^{-1} B^* S(0)^{-1} \Pi(0) \tag{4.1}$$

is rational and satisfies the following properties

$$r(s) = [r_1(s) \quad r_2(s)] \in \mathbb{C}^2, \qquad r(s)Jr(s)^* = 0, \qquad r(0) = \beta. \tag{4.2}$$

Here J is defined in (3.1). Introduce matrices K and j:

$$K := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad j := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad K^* = K^{-1}, \qquad KjK^* = J. \tag{4.3}$$

Consider function

$$u(s) = \big(\overline{r_1(s^{-1})} + \overline{r_2(s^{-1})}\big)\big(\overline{r_1(s^{-1})} - \overline{r_2(s^{-1})}\big)^{-1}. \tag{4.4}$$

From (4.2) and (4.3) we get $rKjK^*r^* = 0$, and so

$$|u| \equiv 1, \qquad u(\infty) = (1-i)/(1+i). \tag{4.5}$$

Rational function u satisfying (4.5) admits [17] a so called minimal realization

$$u(s) = c^2\big(1 + i\theta^* S_0^{-1}(sI_n - \alpha)^{-1}\theta\big), \qquad c = (1-i)/\sqrt{2}, \tag{4.6}$$

where

$$\alpha S_0 - S_0 \alpha^* = i\theta\theta^*. \tag{4.7}$$

Using (4.6) one recovers $\widetilde{H}$ (explicitly, though not necessarily uniquely) from the given function u.

Theorem 2. *Let J-unitary matrix U and rational function u satisfying* (4.5) *be given. Consider realization* (4.6) *and choose vectors θ_1 and θ_2 so that*

$$c\,\theta_1 + \overline{c}\,\theta_2 = \theta, \qquad \det\big(\alpha - i\,c\,\theta\theta_2^* S_0^{-1}\big) \neq 0. \tag{4.8}$$

Supposing relations (4.8) *are valid, put*

$$A(0) := \alpha - i\,c\,\theta\theta_2^* S(0)^{-1}, \qquad S(0) := S_0, \qquad \Pi(0) := \Lambda K^*, \qquad \Lambda := [\theta_1 \quad \theta_2]. \tag{4.9}$$

Introduce now matrix-functions S *and* Π *by* (4.9) *and equations*

$$S_x = gg^* - (AS + SA^*), \qquad \Pi(x)J\beta^* = g, \qquad \Pi_x(x)\beta^* = -2i(B - xI_n)^{-1}g, \tag{4.10}$$

where

$$B = A(0)^{-1}, \qquad A(x) = (B - xI_n)^{-1}, \qquad g = \theta. \tag{4.11}$$

Then on the intervals $[0,\, l]$*, where* $\det(B - xI_n) \neq 0$ *and* $\det\, S(x) \neq 0$*, matrix function* $\widetilde{R}$ *is well-defined by* (3.16)*,* (4.1)*,* (4.9)*, and* (4.11)*. Moreover, equality* (4.4) *is true and* $\widetilde{H}$ *corresponding to* $\widetilde{R}$ *is given by* (3.13)*,* (4.9)*–*(4.11)*, and the second relation in* (2.24)*.*

Proof. First notice that equations (4.7) and (4.9) and the first relation in (4.8) imply identities

$$A(0)S(0) - S(0)A(0)^* = i\Lambda j\Lambda^* = i\Pi(0)J\Pi(0)^*.$$

The correspondence between $\widetilde{R}$ and $\widetilde{H}$ follows now from the results of Section 3. It remains to prove (4.4). For this purpose notice that by (2.5), (4.1), (4.3), and (4.9) we have

$$jK^* r(s^{-1})^* = W(s)jK^*\beta^* = W(s)\begin{bmatrix} c \\ \overline{c} \end{bmatrix}, \tag{4.12}$$

where 2×2 matrix function W is of the form

$$W(s) = \{W_{kj}(s)\}_{k,j=1}^2 = I_2 - ij\Lambda^* S(0)^{-1}\big(A(0) - sI_n\big)^{-1}\Lambda. \tag{4.13}$$

In view of (4.12) we get

$$\frac{\overline{r_1(s^{-1})} + \overline{r_2(s^{-1})}}{\overline{r_1(s^{-1})} - \overline{r_2(s^{-1})}} = \frac{cW_{11}(s) + \overline{c}W_{12}(s)}{cW_{21}(s) + \overline{c}W_{22}(s)}. \tag{4.14}$$

From the first relation in (4.8) and (4.13) follow the representations

$$cW_{11}(s) + \overline{c}W_{12}(s) = c - i\theta_1^* S(0)^{-1}\big(A(0) - sI_n\big)^{-1}\theta, \tag{4.15}$$

$$cW_{21}(s) + \overline{c}W_{22}(s) = \overline{c} + i\theta_2^* S(0)^{-1}\big(A(0) - sI_n\big)^{-1}\theta. \tag{4.16}$$

Using system theory results on the realization of the inverse matrix function, from the first relation in (4.9) and (4.16) we derive

$$\big(cW_{21}(s) + \overline{c}W_{22}(s)\big)^{-1} = c\big(1 - i\,c\,\theta_2^* S(0)^{-1}\big(\alpha - sI_n\big)^{-1}\theta\big). \tag{4.17}$$

Finally, taking into account that $i(A(0) - \alpha) = c\,\theta\theta_2^* S(0)^{-1}$ from (4.15) and (4.17) we get

$$\frac{cW_{11}(s) + \overline{c}W_{12}(s)}{cW_{21}(s) + \overline{c}W_{22}(s)} = c^2\big(1 + i\theta^* S_0^{-1}(sI_n - \alpha)^{-1}\theta\big). \tag{4.18}$$

Equalities (4.6), (4.14), and (4.18) imply (4.4). ∎

Compare the result with the recovery of the so called pseudo-exponential potentials (see [5, 7] and references therein).

Remark 2. As $|u| = 1$, so matrix α in the realization (4.6) is invertible. Therefore we can choose θ_2 satisfying the first relation in (4.8) and sufficiently small for $\alpha - i\,c\,\theta\theta_2^* S_0^{-1}$ to be invertible too.

5 Summary

The first new result in this paper is the construction of the Bäcklund–Darboux transformation for the non-isospectral canonical system (1.1), which is important both in prediction theory and random matrices theory. The GBDT-version of the Bäcklund–Darboux transformation, constructed in Theorem 1, is more general than iterated BDT and admits parameter matrix A with an arbitrary Jordan structure. (For the applications of GBDT to non-isospectral integrable systems see [14].)

In Section 3, we apply GBDT to the initial system with $H \equiv \text{const}$ to obtain a family of explicit solutions of system (1.1) and of the corresponding Riemann–Hilbert problem (1.2). In particular, we construct the transformed Hamiltonians $\widetilde{H}$ and the transformed jump functions $\widetilde{R}^2$ (see formula (3.13) for $\widetilde{H}$ and formula (3.14) for $\widetilde{R}$). The subcase from Example 1 is treated in greater detail. The interesting case of non-diagonal matrix A and applications to prediction theory will follow elsewhere.

Finally, in Section 4, using the methods of system theory, we recover $\widetilde{H}$ and $\widetilde{R}$ from a partial information on $\widetilde{R}$ similar to the way, in which the Dirac system is recovered explicitly from its Weyl function in [7].

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