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Polynomials Associated with Dihedral Groups

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Abstract. There is a commutative algebra of differential-difference operators, with two parameters, associated to any dihedral group with an even number of reflections. The intertwining operator relates this algebra to the algebra of partial derivatives. This paper presents an explicit form of the action of the intertwining operator on polynomials by use of harmonic and Jacobi polynomials. The last section of the paper deals with parameter values for which the formulae have singularities.

Key words: intertwining operator; Jacobi polynomials

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1 Introduction

The dihedral group of type $I_2(2s)$ acts on $\mathbb{R}^2$, contains $2s$ reflections and the rotations through angles of $\frac{m\pi}{s}$ for $1 \le m \le 2s-1$, and is of order $4s$, where s is a positive integer. It is the symmetry group of the regular $2s$-gon and has two conjugacy classes of reflections (the mirrors passing through midpoints of pairs of opposite edges and those joining opposite vertices). There is an associated commutative algebra of differential-difference ("*Dunkl*") operators with two parameters, denoted by κ_0, κ_1. It is convenient to use complex coordinates for $\mathbb{R}^2$, that is, $z = x_1 + \mathrm{i}x_2$, $\overline{z} = x_1 - \mathrm{i}x_2$. Notations like $f(z)$ will be understood as functions of z, $\overline{z}$; except that $f(\overline{z}, z)$ will be used to indicate the result of interchanging z and $\overline{z}$. Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$ denote the sets of positive integers, nonnegative integers and rational numbers, respectively. Let $\omega = e^{\mathrm{i}\pi/s}$, then the reflections in the group are $(z, \overline{z}) \mapsto (\overline{z}\omega^m, z\omega^{-m})$, $0 \le m < 2s$ and the rotations are $(z, \overline{z}) \mapsto (z\omega^m, \overline{z}\omega^{-m},)$, $1 \le m < 2s$. Note that $f(\overline{z}\omega^m)$ is the abbreviated form of $f(\overline{z}\omega^m, z\omega^{-m})$. The differential-difference operators are defined by

$$Tf(z) := \frac{\partial}{\partial z} f(z) + \kappa_0 \sum_{j=0}^{s-1} \frac{f(z) - f\left(\overline{z}\omega^{2j}\right)}{z - \overline{z}\omega^{2j}} + \kappa_1 \sum_{j=0}^{s-1} \frac{f(z) - f\left(\overline{z}\omega^{2j+1}\right)}{z - \overline{z}\omega^{2j+1}},$$

$$\overline{T}f(z) := \frac{\partial}{\partial \overline{z}} f(z) - \kappa_0 \sum_{j=0}^{s-1} \frac{f(z) - f\left(\overline{z}\omega^{2j}\right)}{z - \overline{z}\omega^{2j}} \omega^{2j} - \kappa_1 \sum_{j=0}^{s-1} \frac{f(z) - f\left(\overline{z}\omega^{2j+1}\right)}{z - \overline{z}\omega^{2j+1}} \omega^{2j+1},$$

for polynomials $f(z)$. (The second formula implicitly uses the relation $-\frac{\omega^m}{z - \overline{z}\omega^m} = \frac{1}{\overline{z} - z\omega^{-m}}$.) The key fact is that T and $\overline{T}$ commute. The explicit action of T and $\overline{T}$ on monomials is given by

$$Tz^a\overline{z}^b = az^{a-1}\overline{z}^b + s \sum_{j=0}^{\lfloor (a-b-1)/s \rfloor} \left(\kappa_0 + (-1)^j \kappa_1\right) z^{a-1-js}\overline{z}^{b+js}, \tag{1.1}$$

$$\overline{T}z^a\overline{z}^b = bz^a\overline{z}^{b-1} - s \sum_{j=1}^{\lfloor (a-b)/s \rfloor} \left(\kappa_0 + (-1)^j \kappa_1\right) z^{a-js}\overline{z}^{b-1+js}, \tag{1.2}$$

for $a \geq b$; the relations remain valid when both $(z, \overline{z})$ and $\left(T, \overline{T}\right)$ are interchanged. The Laplacian is $4T\overline{T}$. These results are from [2, Section 3]. The harmonic polynomials and formulae (1.1) and (1.2) also appear in Berenstein and Burman [1, Section 2]. The aim of this paper is to find an explicit form of the intertwining operator V. This is the unique linear transformation that maps homogeneous polynomials to homogeneous polynomials of the same degree and satisfies

$$TVf(z) = V\frac{\partial}{\partial z}f(z), \qquad \overline{T}Vf(z) = V\frac{\partial}{\partial \overline{z}}f(z), \qquad V1 = 1.$$

The operator was defined for general finite reflection groups in [4]. Rösler [8] proved that V is a positive operator when $\kappa_0, \kappa_1 > 0$; this roughly means that if a polynomial f satisfies $f(y) \geq 0$ for all y with $\|y\| < R$ (for some R) then $Vf(y) \geq 0$ on the same set. The present paper does not shed light on the positivity question since the formulae are purely algebraic. In Section 5 the special case $-(\kappa_0 + \kappa_1) \in \mathbb{N}$ is considered in more detail. These values of (κ_0, κ_1) are apparent singularities in the expressions for $Vz^a\overline{z}^b$ which are found in Section 4. The book by Y. Xu and the author [7] is a convenient reference for the background of this paper.

In a way, to find $Vz^a\overline{z}^b$ only requires to solve a set of equations involving $Vz^j\overline{z}^k$ for $0 \leq j \leq a$, $0 \leq k \leq b$. This can be implemented in computer algebra for small a, b but it is not really an explicit description. For example, by direct computation we find that

$$Vz^2 = \frac{(\kappa_0 + \kappa_1 + 1)\,z^2 + (\kappa_0 - \kappa_1)\,\overline{z}^2}{(2\kappa_0 + 1)(2\kappa_1 + 1)(2\kappa_0 + 2\kappa_1 + 1)}, \qquad s = 2,$$

$$Vz^2 = \frac{2z^2}{(s\kappa_0 + s\kappa_1 + 1)(s\kappa_0 + s\kappa_1 + 2)}, \qquad s > 2.$$

The idea is to find the harmonic expansion of $Vz^a\overline{z}^b$; suppose $f(z)$ is (real-) homogeneous of degree n then there is a unique expansion $f(z) = \sum\limits_{j=0}^{\lfloor n/2 \rfloor} (z\overline{z})^j f_{n-2j}(z)$ where f_{n-2j} is homogeneous of degree $n-2j$ and is harmonic, that is, $T\overline{T}f_{n-2j} = 0$, for $0 \leq j \leq n/2$. There is some more information easily available for the expansion of $Vz^a\overline{z}^b$. Let $n = a + b$ and suppose $Vz^a\overline{z}^b = \sum\limits_{j=0}^{n} c_j z^{n-j}\overline{z}^j$ for certain coefficients c_j. Because V commutes with the action of the group we deduce that

$$V\left((\omega z)^a (\overline{\omega z})^b\right) = \omega^{a-b}\sum_{j=0}^{n} c_j z^{n-j}\overline{z}^j = \sum_{j=0}^{n} \omega^{n-2j} c_j z^{n-j}\overline{z}^j;$$

thus $c_j \neq 0$ implies $n - 2j \equiv a - b \bmod (2s)$ or $j \equiv b \bmod s$. Further

$$V\left(\overline{z}^a z^b\right) = \sum_{j=0}^{n} c_j \overline{z}^{n-j} z^j,$$

so it will suffice to determine $Vz^a\overline{z}^b$ for $a \geq b$. We will use the Poisson kernel to calculate the polynomials denoted $K_n(x, y) := V^x\left(\frac{1}{n!}(x_1y_1 + x_2y_2)^n\right)$ (see [5, p. 1219]), where $y \in \mathbb{R}^2$ and V^x acts on the variable x. Thus $Vx_1^{n-j}x_2^j$ is $j!\,(n-j)!$ times the coefficient of $y_1^{n-j}y_2^j$ in $K_n(x, y)$. This is adapted to complex coordinates by setting $w = y_1 + \mathrm{i}y_2$, in which case $x_1y_1 + x_2y_2 = \frac{1}{2}(z\overline{w} + \overline{z}w)$.

2 The Poisson kernel

Actually it is only the series expansion of this kernel that is used. For now we assume $\kappa_0, \kappa_1 \geq 0$. The measure on the circle $\mathbb{T} := \left\{e^{\mathrm{i}\theta} : -\pi < \theta \leq \pi\right\}$ associated to the group $I_2(2s)$ and the operators T, $\overline{T}$ is

$$d\mu(e^{\mathrm{i}\theta}) := \frac{1}{2B\left(\kappa_0+\frac{1}{2},\kappa_1+\frac{1}{2}\right)}\left(\sin^2 s\theta\right)^{\kappa_0}\left(\cos^2 s\theta\right)^{\kappa_1} d\theta.$$

Suppose g is a function of $t=\cos 2s\theta$ then

$$\int_{\mathbb{T}} g\left(t\left(\theta\right)\right) d\mu(e^{\mathrm{i}\theta}) = \frac{2^{-\kappa_0-\kappa_1}}{B\left(\kappa_0+\frac{1}{2},\kappa_1+\frac{1}{2}\right)}\int_{-1}^{1} g\left(t\right)\left(1-t\right)^{\kappa_0-1/2}\left(1+t\right)^{\kappa_1-1/2} dt.$$

The inner product in $L^2\left(\mathbb{T},\mu\right)$ is

$$\langle f,g\rangle := \int_{\mathbb{T}} f\left(z\right)\overline{g\left(z\right)}d\mu\left(z\right)$$

and $\|f\| := \langle f,f\rangle^{1/2}$. Throughout the polynomials under consideration have real coefficients so that $\overline{g\left(z,\overline{z}\right)} = g\left(\overline{z},z\right)$. By the group invariance of μ the integral $\int_{\mathbb{T}} z^a\overline{z}^b d\mu\left(z\right)$is real-valued when $a\equiv b \bmod \left(2s\right)$ and vanishes otherwise. There is an orthogonal decomposition $L^2\left(\mathbb{T},\mu\right) = \sum_{n=0}^{\infty}\oplus\mathcal{H}_n$; for $n>0$ each $\mathcal{H}_n$ is of dimension two and consists of the polynomials in $z,\overline{z}$ (real-) homogeneous of degree n and annihilated by $T\overline{T}$ (the harmonic property), while $\mathcal{H}_0$ consists of the constant functions. The Poisson kernel is the reproducing kernel for harmonic polynomials (for more details see [3, 5]). Xu [10] investigated relationships between harmonic polynomials, the intertwining operator and the Poisson kernel for the general reflection group. The paper of Scalas [9] concerns boundary value problems for the dihedral groups. The projection of the kernel onto $\mathcal{H}_n$ is denoted by $P_n\left(z,w\right)$ and satisfies

$$\int_{\mathbb{T}} P_n\left(z,w\right) g\left(w\right) d\mu\left(w\right) = g\left(z\right)$$

for each polynomial $g\in\mathcal{H}_n$. There is a formula for P_n in terms of $\left\{K_{n-2j}: 0\le j\le\frac{n}{2}\right\}$ (see [5, p. 1224]) which can be inverted. In the present case

$$P_n\left(z,w\right) = \sum_{j=0}^{\lfloor n/2\rfloor}\frac{\left(\gamma_0\right)_n}{j!\left(2-n-\gamma_0\right)_j}2^{n-2j}\left(z\overline{z}w\overline{w}\right)^j K_{n-2j}\left(z,w\right),$$

where $\gamma_0 = s\kappa_0+s\kappa_1+1$. The inverse relation is

$$K_n\left(z,w\right) = 2^{-n}\sum_{j=0}^{\lfloor n/2\rfloor}\frac{1}{j!\left(\gamma_0\right)_{n-j}}\left(z\overline{z}w\overline{w}\right)^j P_{n-2j}\left(z,w\right). \tag{2.1}$$

This is a consequence of the following:

Proposition 1. *Suppose there are two sequences* $\left\{\xi_n : n\in\mathbb{N}_0\right\}$ *and* $\left\{\eta_n : n\in\mathbb{N}_0\right\}$ *in a vector space over* $\mathbb{Q}\left(\gamma_0\right)$ *where* γ_0 *is transcendental, then*

$$\xi_n = \sum_{j=0}^{\lfloor n/2\rfloor}\frac{\left(\gamma_0\right)_n}{j!\left(2-n-\gamma_0\right)_j}\eta_{n-2j}, \qquad n\in\mathbb{N}_0,$$

if and only if

$$\eta_n = \sum_{j=0}^{\lfloor n/2\rfloor}\frac{1}{j!\left(\gamma_0\right)_{n-j}}\xi_{n-2j}, \qquad n\in\mathbb{N}_0.$$

Proof. Consider the matrices A and B defined by $\xi_n = \sum_j A_{jn}\eta_j$, $\eta_n = \sum_j B_{jn}\xi_j$; these matrices are triangular and the diagonal entries are nonzero, hence they are nonsingular. It suffices to show B is a one-sided inverse of A; this is actually finite-dimensional linear algebra, since one can truncate to the range $0 \le n, j \le M$ for any $M \in \mathbb{N}$. Indeed

$$\begin{aligned}
&\sum_{j=0}^{\lfloor n/2\rfloor} \frac{(\gamma_0)_n}{j!\,(2-n-\gamma_0)_j} \sum_{i=0}^{\lfloor n/2-j\rfloor} \frac{1}{i!\,(\gamma_0)_{n-2j-i}}\xi_{n-2j-2i} \\
&\quad= \sum_{k=0}^{\lfloor n/2\rfloor} \xi_{n-2k}\frac{(\gamma_0)_n}{k!\,(\gamma_0)_{n-k}} \sum_{j=0}^{k} \frac{(-k)_j\,(1-n-\gamma_0+k)_j}{j!\,(2-n-\gamma_0)_j} \\
&\quad= \sum_{k=0}^{\lfloor n/2\rfloor} \xi_{n-2k}\frac{(\gamma_0)_n\,(1-k)_k}{k!\,(\gamma_0)_{n-k}\,(2-n-\gamma_0)_k} = \xi_n;
\end{aligned}$$

using the substitution $i = k-j$ we obtain $(\gamma_0)_{n-2j-i} = (\gamma_0)_{n-k-j} = \frac{(-1)^j(\gamma_0)_{n-k}}{(1-n-\gamma_0+k)_j}$ and $\frac{1}{i!} = (-1)^j\,\frac{(-k)_j}{k!}$; the sum over j is found by the Chu–Vandermonde formula. ■

Set $\xi_n = \frac{P_n(z,w)}{(z\overline{z}w\overline{w})^{n/2}}$ and $\eta_n = \frac{2^n K_n(z,w)}{(z\overline{z}w\overline{w})^{n/2}}$ for $n \in \mathbb{N}_0$ to prove equation (2.1).

Suppose for each $n \in \mathbb{N}$ there exist a basis $\{h_{n1}, h_{n2}\}$ and a biorthogonal basis $\{g_{n1}, g_{n2}\}$ for $\mathcal{H}_n$ with real coefficients in $z, \overline{z}$ (so $\overline{h_{n1}(z,\overline{z})} = h_{n1}(\overline{z}, z)$, for example). Thus $\langle h_{ni}, g_{nj}\rangle = \delta_{ij}/\lambda_{ni}$,with structural constants λ_{ni}. Then

$$P_n(z,w) = \sum_{i=1}^{2} \lambda_{ni} h_{ni}(z,\overline{z})\, g_{ni}(\overline{w}, w). \tag{2.2}$$

Once this is made sufficiently explicit we can compute $K_n(z,w)$ and $Vz^{n-j}\overline{z}^j$. The description of harmonic polynomials is in terms of the case $s=1$ (corresponding to the group $I_2(2) = \mathbb{Z}_2\times\mathbb{Z}_2$). In terms of Jacobi polynomials the polynomials annihilated by $\overline{T}$ are:

$$f_{2n}\left(re^{\mathrm{i}\theta}\right) := r^{2n}P_n^{\left(\kappa_0-\frac12,\kappa_1-\frac12\right)}(\cos 2\theta) + \frac{\mathrm{i}}{2}\left(r^2\sin 2\theta\right) r^{2n-2}P_{n-1}^{\left(\kappa_0+\frac12,\kappa_1+\frac12\right)}(\cos 2\theta), \tag{2.3}$$

$$\begin{aligned}
f_{2n+1}\left(re^{\mathrm{i}\theta}\right) &:= \left(n+\kappa_0+\frac12\right) r\cos\theta\, r^{2n}P_n^{\left(\kappa_0-\frac12,\kappa_1+\frac12\right)}(\cos 2\theta) \\
&\quad + \mathrm{i}\left(n+\kappa_1+\frac12\right) r\sin\theta\, r^{2n}P_n^{\left(\kappa_0+\frac12,\kappa_1-\frac12\right)}(\cos 2\theta);
\end{aligned} \tag{2.4}$$

where the subscript indicates the degree of homogeneity, (clearly f_n is a polynomial with real coefficients in z, $\overline{z}$; $\cos 2\theta = \left(z^2+\overline{z}^2\right)/(2z\overline{z})$ and $\frac{\mathrm{i}}{2}\left(r^2\sin 2\theta\right) = \frac14\left(z^2-\overline{z}^2\right)$). The real and imaginary parts form a basis for the harmonic polynomials. Specifically let

$$f_n^0(z) := \operatorname{Re} f_n(z), \qquad f_n^1(z) := \mathrm{i}\operatorname{Im} f_n(z).$$

This implies that both f_n^0 and f_n^1 have real coefficients in $z, \overline{z}$ and $f_n^0(\overline{z}, z) = f_n^0(z,\overline{z})$, $f_n^1(\overline{z}, z) = -f_n^1(z,\overline{z})$. When $s > 1$ and $1 \le t < s$ it is known [2, p. 182] that $\left\{z^t f_n(z^s), \overline{z}^t f_n(\overline{z}^s)\right\}$ is an orthogonal basis for $\mathcal{H}_{ns+t}$ for $n \ge 0$. Henceforth we denote $h_{ns+t,1}(z) = g_{ns+t,1}(z) = z^t f_n(z^s) = \overline{h_{ns+t,2}} = \overline{g_{ns+t,2}}$ and $\lambda_{ns+t,1} = \lambda_{ns+t,2} = \|f_n\|^{-2}$. The integral $\left\langle z^t f_n(z^s), z^t f_n(z^s)\right\rangle$ reduces to the case $s=1$ and $t=0$. When $s \ge 1$ $\left\{f_n^0(z^s), f_n^1(z^s)\right\}$ is an orthogonal basis for $\mathcal{H}_{ns}$ and $\overline{z}^s f_{n-1}(\overline{z}^s)$ is orthogonal to $f_n(z^s)$. By orthogonality $\|f_n\|^2 = \left\|f_n^0\right\|^2 + \left\|f_n^1\right\|^2$ and the latter two norms are standard Jacobi polynomial facts. The associated structural constants are denoted by labeled λ's. Thus

$$\lambda_{2n}^{0} := \left\|f_{2n}^{0}\right\|^{-2} = \frac{n!\left(\kappa_0+\kappa_1+1\right)_n\left(\kappa_0+\kappa_1+2n\right)}{\left(\kappa_0+\frac{1}{2}\right)_n\left(\kappa_1+\frac{1}{2}\right)_n\left(\kappa_0+\kappa_1+n\right)}, \tag{2.5}$$

$$\lambda_{2n}^{1} := \left\|f_{2n}^{1}\right\|^{-2} = \frac{(n-1)!\left(\kappa_0+\kappa_1+1\right)_n\left(\kappa_0+\kappa_1+2n\right)}{\left(\kappa_0+\frac{1}{2}\right)_n\left(\kappa_1+\frac{1}{2}\right)_n}, \tag{2.6}$$

$$\lambda_{2n} := \left\|f_{2n}\right\|^{-2} = \frac{n!\left(\kappa_0+\kappa_1+1\right)_n}{\left(\kappa_0+\frac{1}{2}\right)_n\left(\kappa_1+\frac{1}{2}\right)_n}; \tag{2.7}$$

and

$$\lambda_{2n+1}^{0} := \left\|f_{2n+1}^{0}\right\|^{-2} = \frac{n!\left(\kappa_0+\kappa_1+1\right)_n\left(\kappa_0+\kappa_1+2n+1\right)}{\left(n+\kappa_0+\frac{1}{2}\right)\left(\kappa_0+\frac{1}{2}\right)_{n+1}\left(\kappa_1+\frac{1}{2}\right)_{n+1}}, \tag{2.8}$$

$$\lambda_{2n+1}^{1} := \left\|f_{2n+1}^{1}\right\|^{-2} = \frac{n!\left(\kappa_0+\kappa_1+1\right)_n\left(\kappa_0+\kappa_1+2n+1\right)}{\left(n+\kappa_1+\frac{1}{2}\right)\left(\kappa_0+\frac{1}{2}\right)_{n+1}\left(\kappa_1+\frac{1}{2}\right)_{n+1}}, \tag{2.9}$$

$$\lambda_{2n+1} := \left\|f_{2n+1}\right\|^{-2} = \frac{n!\left(\kappa_0+\kappa_1+1\right)_n}{\left(\kappa_0+\frac{1}{2}\right)_{n+1}\left(\kappa_1+\frac{1}{2}\right)_{n+1}}. \tag{2.10}$$

From this point on we no longer need the measure μ on the circle. Only the algebraic expressions are used. The condition $\kappa_0, \kappa_1 \geq 0$ is replaced by the requirement that none of $-\kappa_0+\frac{1}{2}$, $-\kappa_1+\frac{1}{2}$, $-s\left(\kappa_0+\kappa_1\right)$ equal a positive integer. The exceptional case $-\left(\kappa_0+\kappa_1\right) \in \mathbb{N}$ is taken up in the last section. In the next section we compute the structural constants for the biorthogonal bases $\left\{f_n\left(z^s\right), f_n\left(\overline{z}^s\right)\right\}$ and $\left\{z^s f_{n-1}\left(z^s\right), \overline{z}^s f_{n-1}\left(\overline{z}^s\right)\right\}$ for $\mathcal{H}_{ns}$ (see [3, p. 461]. It is easier to carry this out with material developed in the next section.

3 Expressions for coefficients

This is a detailed study of the coefficients of $f_n(z)$ in terms of powers of z, $\overline{z}$. The expressions are in the form of a single sum of hypergeometric ${}_3F_2$-type, and can not be simplified any further. For a polynomial f in z, $\overline{z}$ define $c(f; a, b)$ to be the coefficient of $z^a\overline{z}^b$ in f, that is,

$$f(z, \overline{z}) = \sum_{a,b \geq 0} c(f; a, b)\, z^a \overline{z}^b.$$

Since we restrict to polynomials with real coefficients the equation $c\big(\overline{f(z)}; a, b\big) = c(f; b, a)$ is valid. Further $c\left(f\left(z^s\right); as, bs\right) = c(f; a, b)$. Recall

$$K_n(z, w) := \frac{1}{2^n n!} V^z\left(\left(z\overline{w} + \overline{z}w\right)^n\right),$$

thus $V z^{n-j}\overline{z}^j$ is $2^n j!\,(n-j)!$ times the coefficient of $w^j\overline{w}^{n-j}$ in $K_n(z, w)$. To adapt the notation from equation (2.2) for P_0 set $h_{01} = g_{01} = \lambda_{01} = 1$ and $h_{02} = g_{02} = \lambda_{02} = 0$. Then

$$K_n(z, w) = 2^{-n} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{j!\left(s\kappa_0 + s\kappa_1 + 1\right)_{n-j}} \times \left(z\overline{z}w\overline{w}\right)^j \sum_{i=1}^{2} \lambda_{n-2j,i} h_{n-2j,i}(z, \overline{z})\, g_{n-2j,i}(\overline{w}, w). \tag{3.1}$$

Proposition 2. *For $0 \leq m \leq n$,*

$$V\left(z^{n-m}\overline{z}^{m}\right) = m!\,(n-m)!\sum_{j=0}^{\lfloor n/2\rfloor}\frac{1}{j!\,(s\kappa_0+s\kappa_1+1)_{n-j}} \tag{3.2}$$
$$\times (z\overline{z})^j\sum_{i=1}^{2}\lambda_{n-2j,i}c\left(g_{n-2j,i};n-m-j,m-j\right)h_{n-2j,i}\left(z,\overline{z}\right).$$

The nonzero terms appear at increments (in j) of $2s$. We start by finding $c\left(f_n^0;n-j,j\right)$ and $c\left(f_n^1;n-j,j\right)$. This is straightforward and will serve as motivation for introducing a specific useful ${}_3F_2$-series. Consider $f_{2n}^0\left(z\right)$ and recall that

$$P_n^{(\alpha,\beta)}\left(t\right) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-t}{2}\right).$$

When $z = re^{\mathrm{i}\theta}$ we have $\frac{1}{2}\left(1-\cos 2\theta\right) = -\left(z-\overline{z}\right)^2/\left(4z\overline{z}\right)$ so

$$\begin{aligned} r^{2n}&P_n^{\left(\kappa_0-\frac{1}{2},\kappa_1-\frac{1}{2}\right)}\left(\cos 2\theta\right)\\ &= \frac{\left(\kappa_0+\frac{1}{2}\right)_n}{n!}\sum_{l=0}^{n}\sum_{i=0}^{2l}\frac{(-n)_l\,(n+\kappa_0+\kappa_1)_l\,(2l)!}{l!\left(\kappa_0+\frac{1}{2}\right)_l i!\,(2l-i)!}2^{-2l}z^{n+l-i}\overline{z}^{n-l+i}\left(-1\right)^{l+i}\\ &= \frac{\left(\kappa_0+\frac{1}{2}\right)_n}{n!}\sum_{j=-n}^{n}\left(-1\right)^j z^{n+j}\overline{z}^{n-j}\sum_{i=\max(-2j,0)}^{n-j}\frac{(-n)_{j+i}\,(n+\kappa_0+\kappa_1)_{j+i}\left(\frac{1}{2}\right)_{j+i}}{i!\left(\kappa_0+\frac{1}{2}\right)_{j+i}(2j+i)!};\end{aligned}$$

(substituting $l = i+j$, so $0 \leq i+j \leq n$ and $0 \leq i \leq 2i+2j$ are the ranges of the summation) by the $(z,\overline{z})$-symmetry it suffices to consider $j \geq 0$. Thus

$$\begin{aligned} c\left(f_{2n}^0;n+j,n-j\right) &= \frac{\left(\kappa_0+\frac{1}{2}\right)_n(-n)_j\,(n+\kappa_0+\kappa_1)_j\left(\frac{1}{2}\right)_j}{\left(\kappa_0+\frac{1}{2}\right)_j(2j)!n!}\left(-1\right)^j\\ &\times\sum_{i=0}^{n-j}\frac{(j-n)_i\,(n+\kappa_0+\kappa_1+j)_i\left(\frac{1}{2}+j\right)_i}{i!\left(\kappa_0+\frac{1}{2}+j\right)_i(2j+1)_i}\\ &= \frac{(n+\kappa_0+\kappa_1)_j\left(\kappa_0+\frac{1}{2}+j\right)_{n-j}}{2^{2j}j!\,(n-j)!}\,{}_3F_2\left(\begin{matrix}j-n,n+\kappa_0+\kappa_1+j,j+\frac{1}{2}\\ \kappa_0+\frac{1}{2}+j,2j+1\end{matrix};1\right);\end{aligned}$$

this used $(2j)! = 2^{2j}j!\left(\frac{1}{2}\right)_j$ and $\left(\kappa_0+\frac{1}{2}\right)_n/\left(\kappa_0+\frac{1}{2}\right)_j = \left(\kappa_0+\frac{1}{2}+j\right)_{n-j}$. The sum, which appears to be a mysterious combination of the parameters, actually has a nice form revealing more useful information.

Definition 1. For $n \in \mathbb{N}_0$ and parameters a, b, c_1, c_2 let

$$\begin{aligned} E_n\left(a,b;c_1,c_2\right) &:= \frac{(a)_n\,(c_2)_n}{n!\,(c_1+c_2)_n}\,{}_3F_2\left(\begin{matrix}-n,b,c_1\\ 1-n-a,1-c_2-n\end{matrix};1\right)\\ &= \frac{1}{n!\,(c_1+c_2)_n}\sum_{j=0}^{n}\frac{(-n)_j}{j!}\,(a)_{n-j}\,(b)_j\,(c_1)_j\,(c_2)_{n-j}\,.\end{aligned}$$

Observe the symmetry $E_n\left(a,b;c_1,c_2\right) = \left(-1\right)^n E_n\left(b,a;c_2,c_1\right)$. This follows from manipulations such as $(a)_{n-j} = (-1)^j\,(a)_n/\left(1-n-a\right)_j$. The following transformation is relevant to the calculation of coefficients.

Proposition 3. *For $n \in \mathbb{N}_0$ and parameters a, b, c_1, c_2*

$$E_n(a, b; c_1, c_2) = \frac{(a + c_1)_n}{n!} \, {}_3F_2\left(\begin{matrix} -n, n + a + b + c_1 + c_2 - 1, c_1 \\ a + c_1, c_1 + c_2 \end{matrix}; 1\right).$$

Proof. Use the transformation

$${}_3F_2\left(\begin{matrix} -n, A, B \\ C, D \end{matrix}; 1\right) = \frac{(D - B)_n}{(D)_n} \, {}_3F_2\left(\begin{matrix} -n, C - A, B \\ C, 1 + B - D - n \end{matrix}; 1\right).$$

First set $A = b, B = c_1, C = 1 - n - a, D = 1 - n - c_2$ then

$$\frac{(a)_n (c_2)_n}{n! (c_1 + c_2)_n} \, {}_3F_2\left(\begin{matrix} -n, b, c_1 \\ 1 - n - a, 1 - c_2 - n \end{matrix}; 1\right) = \frac{(a)_n}{n!} \, {}_3F_2\left(\begin{matrix} -n, 1 - n - a - b, c_1 \\ 1 - n - a, c_1 + c_2 \end{matrix}; 1\right).$$

Set $A = 1 - n - a - b$, $B = c_1$, $C = c_1 + c_2$, $D = 1 - n - a$ to obtain the stated formula. In the calculation the reversal such as $(1 - n - a)_n = (-1)^n (a)_n$ is used several times. ■

We arrive at a pleasing formula:

$$c\left(f_{2n}^0; n + j, n - j\right) = \frac{(n + \kappa_0 + \kappa_1)_j}{2^{2j} j!} E_{n-j}\left(\kappa_0, \kappa_1; j + \frac{1}{2}, j + \frac{1}{2}\right).$$

It is useful because it clearly displays the result of setting one or both parameters equal to zero (or a negative integer). That is $E_n(\kappa_0, 0; c_1, c_2) = \frac{(\kappa_0)_n (c_2)_n}{n! (c_1 + c_2)_n}$ and $n \geq 1$ implies $E_n(0, 0; c_1, c_2) = 0$. (When $\kappa_0 = \kappa_1 = 0$ the polynomial f_{2n}^0 is a multiple of the Chebyshev polynomial of the first kind, that is $f_{2n}^0(z) = \frac{(n)_n}{n! 2^{2n}} \left(z^{2n} + \overline{z}^{2n}\right)$, a fact obvious from the definition of f_{2n}^0.) The remaining basis polynomials can all be expressed in terms of the function E.

Proposition 4. *For $n \in \mathbb{N}$*

$$\begin{aligned} f_{2n}^0(z) &= \sum_{j=1}^{n} \left(z^{n+j} \overline{z}^{n-j} + z^{n-j} \overline{z}^{n+j}\right) \frac{1}{2^{2j} j!} \\ &\quad \times (n + \kappa_0 + \kappa_1)_j \; E_{n-j}\left(\kappa_0, \kappa_1; j + \frac{1}{2}, j + \frac{1}{2}\right) + z^n \overline{z}^n E_n\left(\kappa_0, \kappa_1; \frac{1}{2}, \frac{1}{2}\right), \\ f_{2n}^1(z) &= \sum_{j=1}^{n} \left(z^{n+j} \overline{z}^{n-j} - z^{n-j} \overline{z}^{n+j}\right) \frac{1}{2^{2j} (j - 1)!} \\ &\quad \times (n + \kappa_0 + \kappa_1 + 1)_{j-1} \; E_{n-j}\left(\kappa_0, \kappa_1; j + \frac{1}{2}, j + \frac{1}{2}\right). \end{aligned}$$

Proof. The expansion for f_{2n}^0 has already been determined. Next $\frac{\mathrm{i}}{2} r^2 \sin 2\theta = \frac{1}{4}\left(z^2 - \overline{z}^2\right)$ so

$$\begin{aligned} &\left(\frac{\mathrm{i}}{2} r^2 \sin 2\theta\right) r^{2n-2} P_{n-1}^{\left(\kappa_0 + \frac{1}{2}, \kappa_1 + \frac{1}{2}\right)}(\cos 2\theta) = \frac{\left(\kappa_0 + \frac{3}{2}\right)_{n-1}}{(n - 1)!} \\ &\quad \times \sum_{l=0}^{n-1} \frac{(1 - n)_l (n + \kappa_0 + \kappa_1 + 1)_l}{l! \left(\kappa_0 + \frac{3}{2}\right)_l} 2^{-2l-2} (-1)^l (z + \overline{z}) (z - \overline{z})^{2l+1} (z\overline{z})^{n-1-l}, \end{aligned}$$

and

$$\begin{aligned} &2^{-2l-2} (z + \overline{z}) (z - \overline{z})^{2l+1} (z\overline{z})^{n-1-l} \\ &\quad = 2^{-2l-2} \sum_{i=0}^{2l+2} \frac{(2l + 1)! (2l + 2 - 2i)}{i! (2l + 2 - i)!} (-1)^i z^{n+l+1-i} \overline{z}^{n-l-1+i} \end{aligned}$$

$$= \sum_{i=0}^{2l+2} \frac{l! \left(\frac{1}{2}\right)_{l+1} (l+1-i)}{i! \, (2l+2-i)!} (-1)^i z^{n+l+1-i} \overline{z}^{n-l-1+i};$$

substitute $l = j+i-1$. By the symmetry $\overline{f_{2n}^1 (z)} = -f_{2n}^1 (z)$ it suffices to find $c\left(f_{2n}^1; n+j, n-j\right)$ for $1 \leq j \leq n$. Indeed

$$\begin{aligned} c\left(f_{2n}^1; n+j, n-j\right) &= \frac{j \left(\kappa_0 + \frac{3}{2}\right)_{n-1} (1-n)_{j-1} (n+\kappa_0+\kappa_1+1)_{j-1} \left(\frac{1}{2}\right)_j}{\left(\kappa_0 + \frac{3}{2}\right)_{j-1} (2j)! \, (n-1)!} (-1)^{j-1} \\ &\quad \times \sum_{i=0}^{n-j} \frac{(j-n)_i (n+\kappa_0+\kappa_1+j)_i \left(\frac{1}{2}+j\right)_i}{i! \left(\kappa_0 + \frac{1}{2} + j\right)_i (2j+1)_i} \\ &= \frac{(n+\kappa_0+\kappa_1+1)_{j-1}}{2^{2j} (j-1)!} E_{n-j}\left(\kappa_0, \kappa_1; j+\frac{1}{2}, j+\frac{1}{2}\right). \end{aligned}$$

This completes the proof. ∎

Proposition 5. *For* $n \in \mathbb{N}_0$

$$\begin{aligned} f_{2n+1}^0 (z) &= \left(n+\kappa_0+\frac{1}{2}\right) \sum_{j=0}^{n} \left(z^{n+1+j}\overline{z}^{n-j} + z^{n-j}\overline{z}^{n+1+j}\right) \frac{1}{2^{2j+1} j!} \\ &\quad \times (n+\kappa_0+\kappa_1+1)_j \; E_{n-j}\left(\kappa_0, \kappa_1; j+\frac{1}{2}, j+\frac{3}{2}\right), \\ f_{2n+1}^1 (z) &= \left(n+\kappa_1+\frac{1}{2}\right) \sum_{j=0}^{n} \left(z^{n+1+j}\overline{z}^{n-j} - z^{n-j}\overline{z}^{n+1+j}\right) \frac{1}{2^{2j+1} j!} \\ &\quad \times (n+\kappa_0+\kappa_1+1)_j \; E_{n-j}\left(\kappa_0, \kappa_1; j+\frac{3}{2}, j+\frac{1}{2}\right). \end{aligned}$$

Proof. The second equation is straightforward:

$$\begin{aligned} & f_{2n+1}^1 (z) = \frac{1}{2}\left(n+\kappa_1+\frac{1}{2}\right) (z-\overline{z}) \, r^{2n} P_n^{\left(\kappa_0+\frac{1}{2}, \kappa_1-\frac{1}{2}\right)} (\cos 2\theta) = \left(n+\kappa_1+\frac{1}{2}\right) \frac{\left(\kappa_0+\frac{3}{2}\right)_n}{n!} \\ & \times \sum_{l=0}^{n} \sum_{i=0}^{2l+1} \frac{(-n)_l (n+\kappa_0+\kappa_1+1)_l (2l+1)!}{l! \left(\kappa_0+\frac{3}{2}\right)_l i! \, (2l+1-i)!} 2^{-2l-1} z^{n+1+l-i} \overline{z}^{n-l+i} (-1)^{l+i} \\ & = \left(n+\kappa_1+\frac{1}{2}\right) \frac{\left(\kappa_0+\frac{3}{2}\right)_n}{n!} \sum_{j=0}^{n} \left(z^{n+1+j}\overline{z}^{n-j} - z^{n-j}\overline{z}^{n+1-j}\right) \\ & \times (-1)^j \frac{(-n)_j (n+\kappa_0+\kappa_1+1)_j \left(\frac{1}{2}\right)_{j+1}}{\left(\kappa_0+\frac{3}{2}\right)_j (2j+1)!} \sum_{i=0}^{n-j} \frac{(j-n)_i (n+\kappa_0+\kappa_1+j+1)_i \left(\frac{3}{2}+j\right)_i}{i! \left(\kappa_0+\frac{3}{2}+j\right)_i (2j+2)_i}, \end{aligned}$$

(substituting $l = j+i$ for $0 \leq j \leq n$) thus

$$\begin{aligned} & c\left(f_{2n+1}^1; n+1+j, n-j\right) = -c\left(f_{2n+1}^1; n-j, n+1+j\right) \\ & \quad = \left(n+\kappa_1+\frac{1}{2}\right) \frac{(n+\kappa_0+\kappa_1+1)_j}{j! 2^{2j+1}} E_{n-j}\left(\kappa_0, \kappa_1; j+\frac{3}{2}, j+\frac{1}{2}\right). \end{aligned}$$

Note that $(2j+1)! = 2^{2j+1} j! \left(\frac{1}{2}\right)_{j+1}$ and $\left(\kappa_0+\frac{3}{2}\right)_n / \left(\kappa_0+\frac{3}{2}\right)_j = \left(\kappa_0+\frac{3}{2}+j\right)_{n-j}$. For f_{2n+1}^0 reverse the parameters, that is,

$$P_n^{\left(\kappa_0-\frac{1}{2}, \kappa_1+\frac{1}{2}\right)} (\cos 2\theta) = (-1)^n P_n^{\left(\kappa_1+\frac{1}{2}, \kappa_0-\frac{1}{2}\right)} (-\cos 2\theta),$$

and note $\frac{1}{2}(1+\cos 2\theta) = (z+\overline{z})^2/(4z\overline{z})$ and $r\cos\theta = \frac{1}{2}(z+\overline{z})$. Thus

$$\begin{aligned}
f^0_{2n+1}(z) &= (-1)^n \left(n+\kappa_0+\frac{1}{2}\right) \frac{\left(\kappa_1+\frac{3}{2}\right)_n}{n!} \\
&\quad\times \sum_{l=0}^{n} \sum_{i=0}^{2l+1} \frac{(-n)_l (n+\kappa_0+\kappa_1+1)_l (2l+1)!}{l! \left(\kappa_1+\frac{3}{2}\right)_l i! (2l+1-i)!} 2^{-2l-1} z^{n+1+l-i} \overline{z}^{n-l+i} \\
&= (-1)^n \left(n+\kappa_0+\frac{1}{2}\right) \frac{\left(\kappa_1+\frac{3}{2}\right)_n}{n!} \sum_{j=0}^{n} \left(z^{n+1+j}\overline{z}^{n-j} + z^{n-j}\overline{z}^{n+1-j}\right) \\
&\quad\times \frac{(-n)_j (n+\kappa_0+\kappa_1+1)_j \left(\frac{1}{2}\right)_{j+1}}{\left(\kappa_1+\frac{3}{2}\right)_j (2j+1)!} \sum_{i=0}^{n-j} \frac{(j-n)_i (n+\kappa_0+\kappa_1+j+1)_i \left(\frac{3}{2}+j\right)_i}{i! \left(\kappa_1+\frac{3}{2}+j\right)_i (2j+2)_i},
\end{aligned}$$

thus

$$\begin{aligned}
&c\left(f^0_{2n+1}; n+1+j, n-j\right) = c\left(f^0_{2n+1}; n-j, n+1+j\right) \\
&\quad= \left(n+\kappa_0+\frac{1}{2}\right) \frac{(n+\kappa_0+\kappa_1+1)_j}{j! 2^{2j+1}} (-1)^{n-j} E_{n-j}\left(\kappa_1, \kappa_0; j+\frac{3}{2}, j+\frac{1}{2}\right).
\end{aligned}$$

The symmetry relation $E_m(b,a;c_2,c_1) = (-1)^m E_m(a,b;c_1,c_2)$ finishes the computation. ■

To find the coefficients of f_n we use contiguity relations satisfied by E_m.

Lemma 1. *For $m \in \mathbb{N}_0$ and parameters a, b, c*

$$\begin{aligned}
&(m+a+c) E_m(a,b;c,c+1) - (m+b+c) E_m(a,b;c+1,c) \\
&\quad= 2(m+1) E_{m+1}(a,b;c,c),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&(m+a+c) E_m(a,b;c,c+1) + (m+b+c) E_m(a,b;c+1,c) \\
&\quad= \frac{m+2c+1}{2c+1} (m+a+b+2c) E_m(a,b;c+1,c+1).
\end{aligned} \tag{3.4}$$

Proof. We compute the coefficient of $(b)_j$ for $0 \le j \le m+1$ in the two identities. Note that $(m+b+c)(b)_j = (b)_{j+1} + (m+c-j)(b)_j$, then replace j by $j-1$ for the first term. The coefficient of $(b)_j$ in $(m+b+c) E_m(a,b;c+1,c)$ is

$$\begin{aligned}
&\frac{1}{m! (2c+1)_m j!} \\
&\times\left\{ (-m)_j (m+c-j) (a)_{m-j} (c)_{m-j} (c+1)_j + j(-m)_{j-1} (a)_{m+1-j} (c)_{m+1-j} (c+1)_{j-1} \right\} \\
&= \frac{(-m)_{j-1}}{m! (2c+1)_m j!} (a)_{m-j} (c)_{m+1-j} (c+1)_{j-1} \{(-m+j-1)(c+j) + j(a+m-j)\}.
\end{aligned}$$

The coefficient of $(b)_j$ in the left side of (3.3) is

$$\begin{aligned}
&\frac{(-m)_{j-1}}{m! (2c+1)_m j!} (a)_{m-j} (c)_j (c+1)_{m-j} \\
&\quad\times \{(m+a+c)(-m+j-1) - (-m+j-1)(c+j) - j(a+m-j)\} \\
&\quad= \frac{(-m)_{j-1}}{m! (2c+1)_m j!} (a)_{m-j} (c)_j (c+1)_{m-j} (a+m-j)(-m-1) \\
&\quad= \frac{2(-1-m)_j}{m! (2c)_{m+1} j!} (a)_{m+1-j} (c)_j (c)_{m+1-j}.
\end{aligned}$$

This proves equation (3.3). For the right side of (3.4) the coefficient of $(b)_j$ is found similarly as before ($(m+a+b+2c)(b)_j = (m+a+2c-j)(b)_j + (b)_{j+1}$, and so on). The coefficient of $(b)_j$ in the left side is

$$\frac{(-m)_{j-1}}{m!\,(2c+1)_m\, j!}(a)_{m-j}(c)_j(c+1)_{m-j}\left\{(m+a+2c)(-m+j-1)+j(a-1)\right\},$$

and in the right side

$$\begin{aligned}
&\frac{m+2c+1}{m!\,(2c+1)(2c+2)_m}\left\{\frac{(-m)_j}{j!}(m+a+2c-j)(a)_{m-j}(c+1)_j(c+1)_{m-j}\right.\\
&\quad\left.+\frac{(-m)_{j-1}}{j!}j\,(a)_{m+1-j}(c+1)_{j-1}(c+1)_{m+1-j}\right\}\\
&\quad=\frac{(-m)_{j-1}}{m!\,(2c+1)_m\, j!}(a)_{m-j}(c+1)_{j-1}(c+1)_{m-j}\\
&\quad\times\left\{(-m+j-1)(m+a+2c-j)(c+j)+j(a+m-j)(c+m+1-j)\right\};
\end{aligned}$$

the expression in $\{\cdot\}$ equals $c(m+a+2c)(-m+j-1)+cj(a-1)$ which proves (3.4). ■

Proposition 6. *For $n\in\mathbb{N}_0$*

$$\begin{aligned}
f_{2n}(z) &= \sum_{j=1}^{n}\left((n+\kappa_0+\kappa_1+j)\,z^{n+j}\overline{z}^{n-j}+(n+\kappa_0+\kappa_1-j)\,z^{n-j}\overline{z}^{n+j}\right)\\
&\quad\times\frac{1}{2^{2j}j!}(n+\kappa_0+\kappa_1+1)_{j-1}\; E_{n-j}\left(\kappa_0,\kappa_1;j+\frac{1}{2},j+\frac{1}{2}\right)\\
&\quad+E_n\left(\kappa_0,\kappa_1;\frac{1}{2},\frac{1}{2}\right)z^n\overline{z}^n,\\
f_{2n+1}(z) &= \sum_{j=1}^{n+1}\left((n+1+j)\,z^{n+j}\overline{z}^{n+1-j}+(n+1-j)\,z^{n-j}\overline{z}^{n+1+j}\right)\\
&\quad\times\frac{1}{2^{2j}j!}(n+\kappa_0+\kappa_1+1)_j\; E_{n+1-j}\left(\kappa_0,\kappa_1;j+\frac{1}{2},j+\frac{1}{2}\right)\\
&\quad+(n+1)\,E_{n+1}\left(\kappa_0,\kappa_1;\frac{1}{2},\frac{1}{2}\right)z^n\overline{z}^{n+1}.
\end{aligned}$$

Proof. Recall $f_n = f_n^0 + f_n^1$. For $0\le j\le n$ from Proposition 4 we find

$$\begin{aligned}
c(f_{2n};n+j,n-j) &= c\left(f_{2n}^0;n+j,n-j\right)+c\left(f_{2n}^1;n+j,n-j\right)\\
&= \frac{(n+\kappa_0+\kappa_1+1)_{j-1}}{2^{2j}j!}((n+\kappa_0+\kappa_1)+j)\,E_{n-j}\left(\kappa_0,\kappa_1;j+\frac{1}{2},j+\frac{1}{2}\right),\\
c(f_{2n};n-j,n+j) &= c\left(f_{2n}^0;n+j,n-j\right)-c\left(f_{2n}^1;n+j,n-j\right)\\
&= \frac{(n+\kappa_0+\kappa_1+1)_{j-1}}{2^{2j}j!}((n+\kappa_0+\kappa_1)-j)\,E_{n-j}\left(\kappa_0,\kappa_1;j+\frac{1}{2},j+\frac{1}{2}\right).
\end{aligned}$$

It remains to compute $c(f_{2n+1};n+1+j,n-j)$ and $c(f_{2n+1};n-j,n+1-j)$ for $0\le j\le n$. Write the arguments as $\left(n+\frac{1}{2}+\varepsilon\left(j+\frac{1}{2}\right),n+\frac{1}{2}-\varepsilon\left(j+\frac{1}{2}\right)\right)$ with $\varepsilon=\pm1$. Then, by Proposition 5,

$$c\left(f_{2n+1};n+\frac{1}{2}+\varepsilon\left(j+\frac{1}{2}\right),n+\frac{1}{2}-\varepsilon\left(j+\frac{1}{2}\right)\right)$$

$$= c\left(f_{2n+1}^0; n+1+j, n-j\right) + \varepsilon c\left(f_{2n+1}^1; n+1+j, n-j\right)$$
$$= \frac{(n+\kappa_0+\kappa_1+1)_j}{j!2^{2j+1}} \left\{ \left(n+\kappa_0+\frac{1}{2}\right) E_{n-j}\left(\kappa_0,\kappa_1; j+\frac{1}{2}, j+\frac{3}{2}\right)\right.$$
$$\left. + \varepsilon\left(n+\kappa_1+\frac{1}{2}\right) E_{n-j}\left(\kappa_0,\kappa_1; j+\frac{3}{2}, j+\frac{1}{2}\right)\right\}.$$

When $\varepsilon = 1$ by (3.4) we obtain

$$c\left(f_{2n+1}; n+1+j, n-j\right) = \frac{(n+\kappa_0+\kappa_1+1)_{j+1}}{(j+1)!2^{2j+2}} (n+j+2) E_{n-j}\left(\kappa_0,\kappa_1; j+\frac{3}{2}, j+\frac{3}{2}\right),$$

and when $\varepsilon = -1$ by (3.3) we obtain

$$c\left(f_{2n+1}; n-j, n+1-j\right) = \frac{(n+\kappa_0+\kappa_1+1)_j}{j!2^{2j}} (n-j+1) E_{n-j+1}\left(\kappa_0,\kappa_1; j+\frac{1}{2}, j+\frac{1}{2}\right).$$

The stated formula for f_{2n+1} uses $c\left(f_{2n+1}; n+j, n+1-j\right)$ explicitly (j is shifted by 1). ■

For $\mathcal{H}_{ns}$ with $n > 0$ we intend to use both the orthogonal basis $\left\{f_n^0\left(z^s\right), f_n^1\left(z^s\right)\right\}$ as well as the biorthogonal bases $\left\{f_n\left(z^s\right), \overline{f_n\left(z^s\right)}\right\}$ and $\left\{z^s f_{n-1}\left(z^s\right), \overline{z}^s\overline{f_{n-1}\left(z^s\right)}\right\}$. For the latter we need the value of $\nu_n := \langle f_n\left(z^s\right), z^s f_{n-1}\left(z^s\right)\rangle$. Instead of doing the integral directly we use the two formulae for $P_{ns}(z,w)$, that is,

$$P_{ns}(z,w) = \lambda_n^0 f_n^0\left(z^s\right) f_n^0\left(w^s\right) + \lambda_n^1 f_n^1\left(z^s\right) \overline{f_n^1\left(w^s\right)}$$
$$= \nu_n^{-1}\left(f_n\left(z^s\right) \overline{w}^s f_{n-1}\left(\overline{w}^s\right) + f_n\left(\overline{z}^s\right) w^s f_{n-1}\left(w^s\right)\right).$$

From the coefficients of $\overline{w}^{ns}$ in the equation we obtain

$$\lambda_n^0 c\left(f_n^0; 0, n\right) f_n^0\left(z^s\right) + \lambda_n^1 c\left(f_n^1; n, 0\right) f_n^1\left(z^s\right) = \nu_n^{-1} c\left(f_{n-1}; n-1, 0\right) f_n\left(z^s\right).$$

But $f_n = f_n^0 + f_n^1$ so by the linear independence of $\left\{f_n^0, f_n^1\right\}$ there are two equations for c_n (one is redundant). Thus

$$\nu_n = \frac{c\left(f_{n-1}; n-1, 0\right)}{\lambda_n^0 c\left(f_n^0; 0, n\right)} = \frac{c\left(f_{n-1}; n-1, 0\right)}{\lambda_n^1 c\left(f_n^1; n, 0\right)}.$$

The calculation has two cases depending on n being even or odd:

$$\nu_{2n} = \frac{2\left(\kappa_0+\frac{1}{2}\right)_n\left(\kappa_1+\frac{1}{2}\right)_n}{(n-1)!\left(\kappa_0+\kappa_1+1\right)_{n-1}\left(\kappa_0+\kappa_1+2n\right)}, \qquad n \geq 1, \tag{3.5}$$
$$\nu_{2n+1} = \frac{2\left(\kappa_0+\frac{1}{2}\right)_{n+1}\left(\kappa_1+\frac{1}{2}\right)_{n+1}}{n!\left(\kappa_0+\kappa_1+1\right)_n\left(\kappa_0+\kappa_1+2n+1\right)}, \qquad n \geq 0. \tag{3.6}$$

4 The intertwining operator

We describe $Vz^a\overline{z}^b$ for $a \geq b$. It is helpful to consider the representations of $I_2(2s)$ since V commutes with the group action on polynomials. Since $z\overline{z}$ is invariant it suffices to consider $\left(z\overline{z}\right)^b z^{a-b}$, or z^m. The residue of $m \bmod 2s$ is the determining factor. Suppose $m \equiv j \bmod 2s$ and $j \neq 0, s$. The representation of $I_2(2s)$ on span$\left\{z^m, \overline{z}^m\right\}$ is irreducible and isomorphic to the one on span$\left\{z^j, \overline{z}^j\right\}$ if $1 \leq j < s$, and to the one on span$\left\{\overline{z}^{2s-j}, z^{2s-j}\right\}$ if $s < j < 2s$. If $m \equiv 0 \bmod 2s$ then span$\left\{z^m, \overline{z}^m\right\}$ is the direct sum of the identity and determinant representations (on $\mathbb{C}1$ and

$\mathbb{C}\left(z^{2s}-\overline{z}^{2s}\right)$ respectively). If $m\equiv s \bmod 2s$ then $\operatorname{span}\{z^m,\overline{z}^m\}$ is the direct sum of the two representations realized on $\mathbb{C}\left(z^s-\overline{z}^s\right)$ and $\mathbb{C}\left(z^s+\overline{z}^s\right)$ (these are relative invariants). Recall $P_m(z,w)=\sum_{i=1}^{2}\lambda_{mi}h_{mi}(z,\overline{z})\,g_{mi}(\overline{w},w)$ and equation (3.2) shows that the nonzero terms in the expansion of $Vz^a\overline{z}^b$ occur only when the condition

$$c\left(g_{a+b-2j,i};a-j,b-j\right)\neq 0 \tag{4.1}$$

is satisfied. If $m\equiv 0\bmod s$ then $g_{mi}(\overline{w},w)$ is a polynomial in $w^s,\overline{w}^s$ thus (4.1) is equivalent to $a-j\equiv b-j\equiv 0\bmod s$, in particular $a\equiv b\bmod s$. In this case suppose $a=us+r\geq b=vs+r$ with $0\leq r<s$. Set $b-j=(v-k)s$ then $j=ks+r$, $a-j=(u-k)s$, $a+b-2j=(u+v-2k)s$, and $0\leq k\leq v\leq u$. We see that the nonzero terms occur for $P_{(u+v-2k)s}$ with $0\leq k\leq v$.

If $m\equiv t\bmod s$ and $1\leq t<s$ then $g_{m1}(w,\overline{w})=w^tf_{(m-t)/s}(w^s,\overline{w}^s)$ and (4.1) implies $a-j\equiv t\bmod s$, $b-j\equiv 0\bmod s$; further $g_{m2}(w,\overline{w})=g_{m1}(\overline{w},w)$ and (4.1) implies $a-j\equiv 0\bmod s$, $b-j\equiv t\bmod s$.

Theorem 1. *Suppose $a-b\equiv t\bmod s$, $1\leq t<s$ and $a>b$. Let $b=vs+r$ with $v\geq 0$ and $0\leq r<s$ and $a=us+r+t$, then*

$$\begin{aligned}
V\left(z^a\overline{z}^b\right)&=a!b!\sum_{k=0}^{v}\frac{1}{(ks+r)!\,(s\kappa_0+s\kappa_1+1)_{a+(v-k)s}}\lambda_{u+v-2k}\\
&\quad\times c\left(f_{u+v-2k};u-k,v-k\right)(z\overline{z})^{ks+r}\,z^tf_{u+v-2k}\left(z^s\right)\\
&\quad+a!b!\sum_{k=1-\lfloor (r+t)/s\rfloor}^{v}\frac{1}{((k-1)s+r+t)!\,(s\kappa_0+s\kappa_1+1)_{b+(u-k+1)s}}\lambda_{u+v+1-2k}\\
&\quad\times c\left(f_{u+v+1-2k};v-k,u-k+1\right)(z\overline{z})^{(k-1)s+r+t}\,\overline{z}^{s-t}f_{u+v+1-2k}\left(\overline{z}^s\right).
\end{aligned}$$

Proof. Since $0<a-b=(u-v)s+t$ we have $u\geq v$. For the first part of the series, corresponding to $i=1$ in P_{ns+t} let $b-j=(v-k)s$ with $k\leq v$; then $j=b-(v-k)s=ks+r$, implying $k\geq 0$. Further $a-j=(u-k)s+t$, $a+b-2j=(u+v-2k)s+t=a+b-2r-2ks$ (and $a+b-j=a+(b-j)=a+(v-k)s$). Also $c\left(z^tf_{u+v-2k}(z^s);a-j,b-j\right)=c\left(f_{u+v-2k};u-k,v-k\right)$. This proves the first part. For the second part, with $i=2$ in P_{ns-t} let $b-j=(v-k)s+(s-t)$, thus $k\leq v$. Then $j=(k-1)s+r+t$. The requirement $j\geq 0$ implies $1-k\leq\frac{r+t}{s}$, that is $k\geq 1-\left\lfloor\frac{r+t}{s}\right\rfloor$ (if $0\leq r+t<s$ then $k\geq 1$, otherwise $s\leq r+t<2s$ and $k\geq 0$). Also $a-j=(u-k+1)s$ and $a+b-2j=(u+v+1-2k)s+(s-t)$ (and $a+b-j=b+(a-j)=b+(u-k+1)s$). In this case we use $c\left(\overline{z}^{s-t}f_{u+v+1-2k}(\overline{z}^s);a-j,b-j\right)=c\left(f_{u+v+1-2k};v-k,u-k+1\right)$. ■

Note that the degrees of f_m have the same parity as $u+v$ in the first sum, and the opposite in the second sum. By Proposition 6 we can find the coefficients explicitly. If $u+v$ is even then

$$\begin{aligned}
c\left(f_{u+v-2k};u-k,v-k\right)&=\frac{1}{2^{u-v}\left(\frac{u-v}{2}\right)!}\left(\frac{u+v}{2}-k+\kappa_0+\kappa_1\right)_{\frac{u-v}{2}}\\
&\quad\times E_{v-k}\left(\kappa_0,\kappa_1;\frac{u-v+1}{2},\frac{u-v+1}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
c\left(f_{u+v+1-2k};v-k,u-k+1\right)&=\frac{(v-k+1)}{2^{u-v}\left(\frac{u-v}{2}\right)!}\left(\frac{u+v}{2}-k+\kappa_0+\kappa_1+1\right)_{\frac{u-v}{2}}\\
&\quad\times E_{v-k+1}\left(\kappa_0,\kappa_1;\frac{u-v+1}{2},\frac{u-v+1}{2}\right).
\end{aligned}$$

If $u+v$ is odd then

$$c\left(f_{u+v-2k};u-k,v-k\right)=\frac{(u-k+1)}{2^{u-v+1}\left(\frac{u-v+1}{2}\right)!}\left(\frac{u+v+1}{2}-k+\kappa_0+\kappa_1\right)_{\frac{u-v+1}{2}}$$
$$\times E_{v-k}\left(\kappa_0,\kappa_1;\frac{u-v}{2}+1,\frac{u-v}{2}+1\right),$$

and

$$c\left(f_{u+v+1-2k};v-k,u-k+1\right)=\frac{(v-k+1)}{2^{u-v+1}\left(\frac{u-v+1}{2}\right)!}\left(\frac{u+v+3}{2}-k+\kappa_0+\kappa_1\right)_{\frac{u-v-1}{2}}$$
$$\times E_{v-k}\left(\kappa_0,\kappa_1;\frac{u-v}{2}+1,\frac{u-v}{2}+1\right).$$

Theorem 2. *Suppose $a\equiv b \bmod s$, and $a\geq b$. Let $a=us+r\geq b=vs+r$ with $0\leq r<s$ and $v\geq 0$. If $a>b$ then*

$$V\left(z^a\overline{z}^b\right)=a!b!\sum_{k=0}^{v}\frac{1}{(ks+r)!\left(s\kappa_0+s\kappa_1+1\right)_{b+(u-k)s}}\nu_{u+v-2k}^{-1}$$
$$\times\left(z\overline{z}\right)^{ks+r}\left\{c\left(f_{u+v-2k-1};u-k-1,v-k\right)f_{u+v-2k}\left(z^s\right)\right.$$
$$\left.+c\left(f_{u+v-2k-1};v-k-1,u-k\right)f_{u+v-2k}\left(\overline{z}^s\right)\right\},$$
$$V\left(\frac{1}{2}\left(z^a\overline{z}^b-z^b\overline{z}^a\right)\right)=a!b!\sum_{k=0}^{v}\frac{1}{(ks+r)!\left(s\kappa_0+s\kappa_1+1\right)_{b+(u-k)s}}\lambda_{u+v-2k}^{1}$$
$$\times c\left(f_{u+v-2k}^1;u-k,v-k\right)\left(z\overline{z}\right)^{ks+r}f_{u+v-2k}^1\left(z^s\right).$$

If $a\geq b$ then

$$V\left(\frac{1}{2}\left(z^a\overline{z}^b+z^b\overline{z}^a\right)\right)=a!b!\sum_{k=0}^{v}\frac{1}{(ks+r)!\left(s\kappa_0+s\kappa_1+1\right)_{b+(u-k)s}}\lambda_{u+v-2k}^{0}$$
$$\times c\left(f_{u+v-2k}^0;u-k,v-k\right)\left(z\overline{z}\right)^{ks+r}f_{u+v-2k}^0\left(z^s\right).$$

Proof. The three different expansions for $z^a\overline{z}^b$, $\frac{1}{2}\left(z^a\overline{z}^b-z^b\overline{z}^a\right)$ and $\frac{1}{2}\left(z^a\overline{z}^b+z^b\overline{z}^a\right)$ use the bases $\left\{f_j,\overline{f_j}\right\}$, $\left\{f_j^1\right\}$ and $\left\{f_j^0\right\}$ respectively. Suppose $a=us+r\geq b=vs+r$ with $0\leq r<s$. Set $b-j=(v-k)s$ then $j=ks+r$, $a-j=(u-k)s$, $a+b-2j=(u+v-2k)s$, and $0\leq k\leq v\leq u$. Consider the case $a>b$, that is, $u>v$. For arbitrary $m\geq 1$ the basis $\left\{f_m\left(z^s,\overline{z}^s\right),f_m\left(\overline{z}^s,z^s\right)\right\}$ for $\mathcal{H}_{sm}$ has the biorthogonal set $\left\{z^sf_{m-1}\left(z^s,\overline{z}^s\right),\overline{z}^sf_{m-1}\left(\overline{z}^s,z^s\right)\right\}$ and

$$c\left(z^sf_{n_1+n_2-1}\left(z^s,\overline{z}^s\right);n_1s,n_2s\right)=c\left(f_{n_1+n_2-1};n_1-1,n_2\right),$$
$$c\left(\overline{z}^sf_{n_1+n_2-1}\left(\overline{z}^s,z^s\right);n_1s,n_2s\right)=c\left(f_{n_1+n_2-1};n_2-1,n_1\right).$$

The constants ν_m are given in equations (3.5) and (3.6). This demonstrates the first series. The remaining two follow from Proposition 3.2. ■

Observe that in the series for $V\left(z^a\overline{z}^b\right)$ the lowest-degree term with $k=v<u$ reduces to one summand since $c\left(f_{u-v-1};-1,u-v\right)=0$. Each term in $V\left(z^a\overline{z}^b-z^b\overline{z}^a\right)$ is of the same representation type, $\mathbb{C}\left(z^s-\overline{z}^s\right)$ when $a-b\equiv s \bmod 2s$ or $\mathbb{C}\left(z^{2s}-\overline{z}^{2s}\right)$ when $a-b\equiv 0 \bmod 2s$. Similarly each term in $V\left(z^a\overline{z}^b+z^b\overline{z}^a\right)$ is of the representation type $\mathbb{C}\left(z^s+\overline{z}^s\right)$ or $\mathbb{C}1$ (depending on the parity of $\frac{a-b}{s}$). The coefficients can be found from Propositions 4 and 5.

For $a>b$ consider $z^a\overline{z}^b$ as $(z\overline{z})^b$ times the (ordinary) harmonic polynomial z^{a-b}. The fact that $V\left(z^a\overline{z}^b\right)$ is $L^2\left(\mathbb{T},\mu\right)$-orthogonal to $\mathcal{H}_n$ for $n<a-b$, equivalently, that the above series for $V\left(z^a\overline{z}^b\right)$ contain no terms involving $\mathcal{H}_n$ with $n<a-b$ (that is, a term like $c_n\left(z\overline{z}\right)^{(a+b-n)/2}p_n\left(z\right)$ with $p_n\in\mathcal{H}_n$), is a special case of a result of Xu [10]. This paper also has formulae for Vz^{2m} when $s=2$, that is, the group $I_2\left(4\right)$.

5 Singular values

The term "singular values" refers to the set K^s of pairs $(\kappa_0,\kappa_1)\in\mathbb{C}^2$ for which V is not defined on all polynomials in $z,\overline{z}$. Let

$$K_0:=\left\{(\kappa_0,\kappa_1)\in\mathbb{C}^2:\{\kappa_0,\kappa_1\}\cap\left(-\tfrac{1}{2}-\mathbb{N}_0\right)\neq\varnothing\right\},$$

(at least one of κ_0,κ_1 is in $\left\{-\frac{1}{2},-\frac{3}{2},\dots\right\}$). It was shown by de Jeu, Opdam and the author [6, p. 248] that $K^s=K_0\cup\left\{(\kappa_0,\kappa_1):\kappa_0+\kappa_1=-\frac{j}{s},j\in\mathbb{N},\frac{j}{s}\notin\mathbb{N}\right\}$. To illustrate how the singular values appear in the formulae for V consider Vz^{2ns+1} (for $s>1,n\geq1$) which has only one term in the formula from Theorem 1. In particular

$$c\left(Vz^{2ns+1};2ns+1,0\right)=\frac{(2ns+1)!\left(\kappa_0+\kappa_1+1\right)_{2n}\left(n+\kappa_0+\kappa_1+1\right)_n}{2^{4n}n!\left(\kappa_0+\frac{1}{2}\right)_n\left(\kappa_1+\frac{1}{2}\right)_n\left(s\kappa_0+s\kappa_1+1\right)_{2ns+1}}.$$

The denominator vanishes for $\kappa_0,\kappa_1=-\frac{1}{2},-\frac{3}{2},\dots,-\frac{2n-1}{2}$ and $\kappa_0+\kappa_1=-\frac{k}{s}$ for $1\leq k\leq 2ns+1$. There appear to be singularities at $\kappa_0+\kappa_1=-k$ for $1\leq k\leq 2n$ but the term $\left(\kappa_0+\kappa_1+1\right)_{2n}$ in the numerator cancels these zeros. The same cancellation occurs for arbitrary $V\left(z^a\overline{z}^b\right)$ in a more complicated way. The formula for $K_n\left(z,w\right)$ has the factors $\left(s\left(\kappa_0+\kappa_1\right)+1\right)_{n-j}$ in the denominators thus the individual terms can have simple poles at $\kappa_0+\kappa_1=-\frac{k}{s}$ for $k\in\mathbb{N}$. We will show directly that the singularities at $\kappa_0+\kappa_1=-m$ are removable when $K_n\left(z,w\right)$ is expressed as a quotient of polynomials in κ_0, κ_1. It turns out that the terms with poles can be paired in such a way that the sum of each pair has a removable singularity. The pairs correspond to $\{P_k,P_{2sm-k}\}$ for certain values of k.

Throughout we assume that $(\kappa_0,\kappa_1)\notin K_0$.

We use an elementary algebraic result: suppose a rational function $F\left(\alpha,\beta\right)$ (with coefficients in the ring $\mathbb{Q}\left[z,\overline{z},w,\overline{w}\right]$) vanishes for a countable set of values $\{\alpha=0,\beta=r_n:n\in\mathbb{N}_0\}$ (which are not poles) then $F\left(\alpha,\beta\right)$ is divisible by α; indeed the numerator of $F\left(0,\beta\right)$ is a polynomial in β vanishing at all $\beta=r_n$ hence is zero. This result will be applied with $\alpha=\kappa_0+\kappa_1+m$, $\beta=\kappa_0-\kappa_1$.

Most of the section concerns the proof of the following result: let $\kappa_0+\kappa_1=-m$ then $P_N\left(z,w\right)=0$ for $N>2sm$ and $P_N\left(z,w\right)+\left(z\overline{z}w\overline{w}\right)^{N-sm}P_{2sm-N}\left(z,w\right)=0$ for $0\leq N\leq 2sm$. The Poisson kernels P_n were described in equation (2.2). There are a number of cases, roughly corresponding to the representations of $I_2\left(2s\right)$.

Proposition 7. *Suppose* $-\left(\kappa_0+\kappa_1\right)=m\in\mathbb{N}$ *then*

$$f_{2n}^0\left(z\right)=\frac{\left(\kappa_0+\frac{1}{2}\right)_n(m-n)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}n!}\left(z\overline{z}\right)^{2n-m}f_{2m-2n}^0\left(z\right),\qquad 0\leq n\leq m,$$

$$f_{2n}^1\left(z\right)=\frac{\left(\kappa_0+\frac{1}{2}\right)_n(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}(n-1)!}\left(z\overline{z}\right)^{2n-m}f_{2m-2n}^1\left(z\right),\qquad 1\leq n\leq m-1.$$

Proof. The argument uses the Jacobi polynomials directly. Recall $z=re^{\mathrm{i}\theta}$. Then for $0\leq n\leq m$

$$f_{2n}^0\left(z\right)=r^{2n}\frac{\left(\kappa_0+\frac{1}{2}\right)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,n-m\\ \kappa_0+\frac{1}{2}\end{matrix};\frac{1-\cos2\theta}{2}\right),$$

$$f_{2m-2n}^{0}(z)=r^{2m-2n}\frac{\left(\kappa_0+\frac{1}{2}\right)_{m-n}}{(m-n)!}\,{}_2F_1\left(\begin{matrix}-(m-n),(m-n)-m\\ \kappa_0+\frac{1}{2}\end{matrix};\frac{1-\cos 2\theta}{2}\right),$$

while for $1\leq n\leq m-1$

$$f_{2n}^{1}(z)=\mathrm{i}r^{2n}\sin 2\theta\frac{\left(\kappa_0+\frac{3}{2}\right)_{n-1}}{(n-1)!}\,{}_2F_1\left(\begin{matrix}1-n,n-m+1\\ \kappa_0+\frac{3}{2}\end{matrix};\frac{1-\cos 2\theta}{2}\right),$$

$$f_{2m-2n}^{1}(z)=\mathrm{i}r^{2m-2n}\sin 2\theta\frac{\left(\kappa_0+\frac{3}{2}\right)_{m-n-1}}{(m-n-1)!}\,{}_2F_1\left(\begin{matrix}-(m-n-1),1-n\\ \kappa_0+\frac{1}{2}\end{matrix};\frac{1-\cos 2\theta}{2}\right).$$

This proves the formulae. ■

Proposition 8. *Suppose* $-(\kappa_0+\kappa_1)=m\in\mathbb{N}$ *and* $0\leq n<m$ *then*

$$f_{2n+1}^{0}(z)=\frac{\left(\kappa_0+\frac{1}{2}\right)_{n+1}(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}n!}(z\overline{z})^{2n-m+1}f_{2m-2n-1}^{0}(z),$$

$$f_{2n+1}^{1}(z)=\frac{\left(\kappa_0+\frac{1}{2}\right)_{n}(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n-1}n!}(z\overline{z})^{2n-m+1}f_{2m-2n-1}^{1}(z).$$

Proof. Similarly to the even case we have

$$f_{2n+1}^{0}(z)=r^{2n+1}\cos\theta\frac{\left(\kappa_0+\frac{1}{2}\right)_{n+1}}{n!}\,{}_2F_1\left(\begin{matrix}-n,n-m+1\\ \kappa_0+\frac{1}{2}\end{matrix};\frac{1-\cos 2\theta}{2}\right),$$

$$f_{2m-2n-1}^{0}(z)=r^{2m-2n-1}\cos\theta\frac{\left(\kappa_0+\frac{1}{2}\right)_{m-n}}{(m-n-1)!}\,{}_2F_1\left(\begin{matrix}-(m-n-1),-n\\ \kappa_0+\frac{1}{2}\end{matrix};\frac{1-\cos 2\theta}{2}\right),$$

and

$$f_{2n+1}^{1}(z)=\mathrm{i}r^{2n+1}\sin\theta\frac{\left(\kappa_1+n+\frac{1}{2}\right)\left(\kappa_0+\frac{3}{2}\right)_{n}}{n!}\,{}_2F_1\left(\begin{matrix}-n,n-m+1\\ \kappa_0+\frac{3}{2}\end{matrix};\frac{1-\cos 2\theta}{2}\right),$$

$$\begin{aligned}f_{2m-2n-1}^{1}(z)&=\mathrm{i}r^{2m-2n-1}\sin\theta\frac{\left(\kappa_1+m-n-\frac{1}{2}\right)\left(\kappa_0+\frac{3}{2}\right)_{m-n-1}}{(m-n-1)!}\\&\quad\times\,{}_2F_1\left(\begin{matrix}-(m-n-1),-n\\ \kappa_0+\frac{3}{2}\end{matrix};\frac{1-\cos 2\theta}{2}\right).\end{aligned}$$

Thus

$$\begin{aligned}\frac{f_{2n+1}^{1}(z)}{f_{2m-2n-1}^{1}(z)}&=r^{4n-2m+2}\frac{(m-n-1)!\left(\kappa_0+\frac{1}{2}\right)_{n+1}\left(-m-\kappa_0+n+\frac{1}{2}\right)}{n!\left(\kappa_0+\frac{1}{2}\right)_{m-n}\left(-\kappa_0-n-\frac{1}{2}\right)}\\&=r^{4n-2m+2}\frac{(m-n-1)!\left(\kappa_0+\frac{1}{2}\right)_{n}}{n!\left(\kappa_0+\frac{1}{2}\right)_{m-n-1}}.\end{aligned}$$ ■

Proposition 9. *Suppose* $-(\kappa_0+\kappa_1)=m\in\mathbb{N}$ *and* $0\leq n<m$ *then*

$$f_{2n}(z)=\frac{\left(\kappa_0+\frac{1}{2}\right)_{n}(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}n!}(z\overline{z})^{2n-m}\,\overline{z}f_{2m-2n-1}(\overline{z}).$$

Proof. We use the expressions from Proposition 6. First we show for $0\leq j\leq\min(n,m-n)$ that

$$E_{n-j}\left(\kappa_0,\kappa_1;j+\frac{1}{2},j+\frac{1}{2}\right)=\frac{\left(\kappa_0+\frac{1}{2}\right)_{m-n}(n-j)!}{\left(\kappa_0+\frac{1}{2}\right)_{n}(m-n-j)!}E_{m-n-j}\left(\kappa_0,\kappa_1;j+\frac{1}{2},j+\frac{1}{2}\right).$$

Indeed by Proposition 3

$$\left(\kappa_0+\frac{1}{2}\right)_j E_{n-j}\left(\kappa_0,\kappa_1;j+\frac{1}{2},j+\frac{1}{2}\right)=\frac{\left(\kappa_0+\frac{1}{2}\right)_n}{(n-j)!}\,{}_3F_2\left(\begin{matrix}j-n,n-m+j,j+\frac{1}{2}\\ \kappa_0+j+\frac{1}{2},2j+1\end{matrix};1\right),$$

and

$$\left(\kappa_0+\frac{1}{2}\right)_j E_{m-n-j}\left(\kappa_0,\kappa_1;j+\frac{1}{2},j+\frac{1}{2}\right)=\frac{\left(\kappa_0+\frac{1}{2}\right)_{m-n}}{(m-n-j)!}\,{}_3F_2\left(\begin{matrix}n-m+j,j-n,j+\frac{1}{2}\\ \kappa_0+j+\frac{1}{2},2j+1\end{matrix};1\right).$$

Let

$$g_1(z):=\frac{n!}{\left(\kappa_0+\frac{1}{2}\right)_n}f_{2n}(z),\qquad g_2(z):=\frac{(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}}(z\overline{z})^{2n-m}\,\overline{z}f_{2m-2n-1}(\overline{z}),$$

and for $j\geq 0$ let

$$b_j:=\frac{1}{2^{2j}j!}\,{}_3F_2\left(\begin{matrix}n-m+j,j-n,j+\frac{1}{2}\\ \kappa_0+j+\frac{1}{2},2j+1\end{matrix};1\right).$$

Then

$$g_1(z)=\sum_{j=-n}^{n}(n-m+1)_{|j|-1}\frac{n!}{(n-|j|)!}b_{|j|}(n-m+j)\,z^{n+j}\overline{z}^{n-j},$$

$$g_2(z)=\sum_{j=n-m}^{m-n}(-n)_{|j|}\frac{(m-n-1)!}{(m-n-|j|)!}b_{|j|}(m-n+j)\,\overline{z}^{n+j}z^{n-j}.$$

Thus $c(g_1;n,n)=b_0=c(g_2;n,n)$. Suppose $|j|\geq 1$, then

$$c(g_1;n+j,n-j)=(n-m+1)_{|j|-1}(-n)_{|j|}(-1)^j(n-m+j)\,b_{|j|}$$

for $|j|\leq n$, and the equation remains valid if $n<|j|\leq m-n$ because $(-n)_{|j|}=0$ for $|j|>n$. Also $c(g_2;n+j,n-j)=(-n)_{|j|}(n+1-m)_{|j|-1}(-1)^{j-1}(m-n-j)\,b_{|j|}$, and the equation remains valid if $m-n<|j|\leq n$ (that is $n-m+|j|-1\geq 0$). Thus $c(g_1;n+j,n-j)=c(g_2;n+j,n-j)$ for $|j|\leq\max(n,m-n)$ and $g_1=g_2$. ∎

Note that if $k=0,1,2,\dots$ then $(-k)_j=0$ for $j>k$. Recall the structural constants for the Poisson kernel P_n from equations (2.5)–(2.10). These are rational functions of κ_0, κ_1 defined for all $(\kappa_0,\kappa_1)\notin K_0$.

Proposition 10. *Suppose* $-(\kappa_0+\kappa_1)=m\in\mathbb{N}$, $1\leq n\leq m-1$, *and* $n\neq\frac{m}{2}$ *then*

$$\frac{\lambda^0_{2m-2n}}{\lambda^0_{2n}}=-\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_n(m-n)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}n!}\right)^2,\qquad \frac{\lambda^1_{2m-2n}}{\lambda^1_{2n}}=-\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_n(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}(n-1)!}\right)^2,$$

$$\lambda^0_{2m}=-\left(\frac{m!}{\left(\kappa_0+\frac{1}{2}\right)_m}\right)^2,\qquad \lambda^1_{2m}=0.$$

Proof. Recall $\lambda^0_{2n}=\frac{n!(\kappa_0+\kappa_1+1)_{n-1}(\kappa_0+\kappa_1+2n)}{\left(\kappa_0+\frac{1}{2}\right)_n\left(\kappa_1+\frac{1}{2}\right)_n}$ (for $n\in\mathbb{N}$). Also $\left(\kappa_1+\frac{1}{2}\right)_n=\left(-m-\kappa_0+\frac{1}{2}\right)_n=(-1)^n\left(\kappa_0+\frac{1}{2}+m-n\right)_n=(-1)^n\frac{\left(\kappa_0+\frac{1}{2}\right)_m}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}}$, and similarly $\left(\kappa_1+\frac{1}{2}\right)_{m-n}=(-1)^{m-n}\frac{\left(\kappa_0+\frac{1}{2}\right)_m}{\left(\kappa_0+\frac{1}{2}\right)_n}$. Thus

$$\frac{\lambda^0_{2m-2n}}{\lambda^0_{2n}}=(-1)^m\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_n}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}}\right)^2\frac{(m-n)!\,(1-m)_{m-n-1}(m-2n)}{n!\,(1-m)_{n-1}(-m+2n)}.$$

But $\frac{(1-m)_{m-n-1}}{(1-m)_{n-1}} = (-1)^m \frac{(m-1)!(m-n)!}{(m-1)!n!}$ (note $(-k)_j = (-1)^j \frac{k!}{(k-j)!}$ for $k \in \mathbb{N}_0$). Next $\lambda^1_{2n} = \frac{(n-1)!(\kappa_0+\kappa_1+1)_n(\kappa_0+\kappa_1+2n)}{\left(\kappa_0+\frac{1}{2}\right)_n\left(\kappa_1+\frac{1}{2}\right)_n}$. Similarly we find

$$\frac{\lambda^1_{2m-2n}}{\lambda^1_{2n}} = (-1)^m \left(\frac{\left(\kappa_0+\frac{1}{2}\right)_n}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}}\right)^2 \frac{(m-n-1)!\,(1-m)_{m-n}\,(m-2n)}{(n-1)!\,(1-m)_n\,(-m+2n)},$$

and $\frac{(1-m)_{m-n}}{(1-m)_n} = (-1)^m \frac{(m-1)!(m-n-1)!}{(m-1)!(n-1)!}$. The special case λ^0_{2m} follows from setting $n=0$ in the first formula. The term $(\kappa_0+\kappa_1+1)_m$ shows $\lambda^1_{2m}=0$. ■

The following two propositions are proven by similar calculations.

Proposition 11. *Suppose* $-(\kappa_0+\kappa_1) = m \in \mathbb{N}$, $0 \le n \le m-1$, *and* $n \ne \frac{m-1}{2}$ *then*

$$\frac{\lambda^0_{2m-2n-1}}{\lambda^0_{2n+1}} = -\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_{n+1}(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n} n!}\right)^2,$$

$$\frac{\lambda^1_{2m-2n-1}}{\lambda^1_{2n+1}} = -\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_{n}(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n-1} n!}\right)^2.$$

Proposition 12. *Suppose* $-(\kappa_0+\kappa_1) = m \in \mathbb{N}$, $0 \le n \le m-1$ *then*

$$\frac{\lambda_{2m-2n-1}}{\lambda_{2n}} = -\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_{n}(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n} n!}\right)^2.$$

Proposition 13. *Suppose* $-(\kappa_0+\kappa_1) = m \in \mathbb{N}$ *then* $\lambda^0_m = 0 = \lambda^1_m$. *If* $n > 2m$ *then* $\lambda^0_n = 0 = \lambda^1_n$. *If* $n \ge 2m$ *then* $\lambda_n = 0$.

Proof. Since both λ^0_n and λ^1_n contain the factor $(\kappa_0+\kappa_1+n)$ for n even or odd, it follows that $\lambda^0_m = 0 = \lambda^1_m$. The term $(\kappa_0+\kappa_1+1)_j$ vanishes for $j > m$, and $j = k-1$ for λ^0_{2k}, $j = k$ for each of λ_{2k}, λ_{2k+1}, λ^1_{2k}, λ^0_{2k+1}, λ^1_{2k+1}. ■

Theorem 3. *Suppose* $-(\kappa_0+\kappa_1) = m \in \mathbb{N}$ *then* $P_N(z,w) = 0$ *for* $N > 2sm$ *and* $P_N(z,w) + (z\overline{z}w\overline{w})^{N-sm} P_{2sm-N}(z,w) = 0$ *for* $0 \le N \le 2sm$.

Proof. If $N = sk > 2sm$ then $P_{sk}(z,w) = \lambda^0_k f^0_k(z^s) f^0_k(w^s) + \lambda^1_k f^1_k(z^s) f^1_k(\overline{w}^s)$ and $\lambda^0_k = \lambda^1_k = 0$ by Proposition 13. If $N = sk+t$ with $1 \le t < s$ and $N > 2sm$ then $P_{sk+t}(z,w) = \lambda_k\left(z^t\overline{w}^t f_k(z^s) f_k(\overline{w}^s) + \overline{z}^t w^t f_k(\overline{z}^s) f_k(w^s)\right)$, $k \ge 2sm$ and $\lambda_k = 0$. If $N = sm$ then $\lambda^0_m = \lambda^1_m = 0$. Suppose $N = 2ns$ and $0 < N < 2sm$ (so $0 < n < m$), then

$$\begin{aligned} P_{2ns}(z,w) &= \lambda^0_{2n}\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_n(m-n)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n} n!}\right)^2 (z\overline{z}w\overline{w})^{(2n-m)s} f^0_{2m-2n}(z^s) f^0_{2m-2n}(w^s) \\ &\quad + \lambda^1_{2n}\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_n(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n} (n-1)!}\right)^2 (z\overline{z}w\overline{w})^{(2n-m)s} f^1_{2m-2n}(z^s) f^1_{2m-2n}(\overline{w}^s), \end{aligned}$$

thus by Propositions 7 and 10

$$\begin{aligned} &(z\overline{z}w\overline{w})^{(m-2n)s} P_{2ns}(z,w) + P_{2ms-2ns}(z,w) \\ &\quad = \lambda^0_{2n}\left\{\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_n(m-n)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n} n!}\right)^2 + \frac{\lambda^0_{2m-2n}}{\lambda^0_{2n}}\right\} f^0_{2m-2n}(z^s) f^0_{2m-2n}(w^s) \end{aligned}$$

$$+\lambda_{2n}^{1}\left\{\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_n(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}(n-1)!}\right)^2+\frac{\lambda_{2m-2n}^{1}}{\lambda_{2n}^{1}}\right\}f_{2m-2n}^{1}\left(z^s\right)f_{2m-2n}^{1}\left(\overline{w}^s\right)=0.$$

For the special case $N=2sm$ we have $P_{2sm}(z,w)=\lambda_{2m}^{0}\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_m}{m!}\right)^2(z\overline{z}w\overline{w})^{ms}=-\left(z\overline{z}w\overline{w}\right)^{ms}P_0$ because $\lambda_{2m}^{1}=0, P_0=1$ and $\lambda_{2m}^{0}=-\left(\frac{m!}{\left(\kappa_0+\frac{1}{2}\right)_m}\right)^2$. Similarly by use of Propositions 8 and 11 we show the result holds for $N=(2n+1)\,s$ for $0\leq n<m$ and $2n+1\neq m$. Suppose $N=sk+t$ with $1\leq t<s$ and $0<N<2sm$, then $2sm-N=s\,(2m-k-1)+(s-t)$. One of $k,2m-k-1$ is even so assume $k=2n$ with $0\leq n<m$ (otherwise replace N by $2sm-N$ and t by $s-t$). By Propositions 9 and 12

$$\begin{aligned}&\lambda_{2n}z^t\overline{w}^tf_{2n}\left(z^s\right)f_{2n}\left(\overline{w}^s\right)+\lambda_{2m-2n-1}\left(z\overline{z}w\overline{w}\right)^{(2n-m)s+t}\overline{z}^{s-t}w^{s-t}f_{2m-2n-1}\left(\overline{z}^s\right)f_{2m-2n-1}\left(w^s\right)\\&\quad=\lambda_{2n}\left(z\overline{w}\right)^{(2n-m)s+t}\left(\overline{z}w\right)^{(2n-m+1)s}f_{2m-2n-1}\left(\overline{z}^s\right)f_{2m-2n-1}\left(w^s\right)\\&\quad\times\left\{\left(\frac{\left(\kappa_0+\frac{1}{2}\right)_n(m-n-1)!}{\left(\kappa_0+\frac{1}{2}\right)_{m-n}n!}\right)^2+\frac{\lambda_{2m-2n-1}}{\lambda_{2n}}\right\}=0.\end{aligned}$$

Add this equation to its complex conjugate to show

$$P_{2ns+t}(z,w)+\left(z\overline{z}w\overline{w}\right)^{(2n-m)s+t}P_{(2m-2n)s-t}(z,w)=0. \qquad \blacksquare$$

Theorem 4. *For $n,m\in\mathbb{N}$ equation (3.1) for $K_n(z,w)$ has a removable singularity at $\kappa_0+\kappa_1=-m$.*

Proof. Consider the series

$$K_n(z,w)=2^{-n}\sum_{j=0}^{\lfloor n/2\rfloor}\frac{1}{j!\left(s\kappa_0+s\kappa_1+1\right)_{n-j}}\left(z\overline{z}w\overline{w}\right)^jP_{n-2j}(z,w).$$

The possible poles occur at $n-j\geq sm$ (that is, $(1-sm)_{n-j}=0$) and the multiplicities do not exceed 1. Thus there are no poles if $n<sm$. If $n-2j>2sm$ then $P_{n-2j}(z,w)$ is divisible by $(\kappa_0+\kappa_1+m)$, by Theorem 3, and the singularity is removable. It remains to consider the case $n-2j\leq 2sm$ and $n-j\geq sm$. Suppose $j=j_0$ satisfies these inequalities and let $j_1=n-j_0-sm$. Then $j_1\geq 0$ and $n-2j_1=2sm-n+2j_0\geq 0$, hence $j=j_1$ appears in the sum. But $2sm-(n-2j_0)=n-2j_1$ so Theorem 3 applies. We can assume $j_1\leq j_0$. Consider the following subset of the sum for $K_n(z,w)$:

$$\frac{\left(z\overline{z}w\overline{w}\right)^{j_0}P_{n-2j_0}(z,w)}{j_0!\left(s\kappa_0+s\kappa_1+1\right)_{n-j_0}}+\frac{\left(z\overline{z}w\overline{w}\right)^{j_1}P_{n-2j_1}(z,w)}{j_1!\left(s\kappa_0+s\kappa_1+1\right)_{n-j_1}}=\frac{\left(z\overline{z}w\overline{w}\right)^{j_0}}{j_0!\left(s\kappa_0+s\kappa_1+1\right)_{n-j_0}}C_{n,j_0}.$$

with

$$C_{n,j_0}=P_{n-2j_0}(z,w)+\frac{j_0!\left(z\overline{z}w\overline{w}\right)^{j_1-j_0}P_{n-2j_1}(z,w)}{j_1!\left(s\kappa_0+s\kappa_1+1+n-j_0\right)_{j_0-j_1}}.$$

The expression C_{n,j_0} has no pole at $\kappa_0+\kappa_1=-m$ since $1-sm+n-j_0\geq 1$. Indeed $(1-sm+n-j_0)_{j_0-j_1}=(j_1+1)_{j_0-j_1}=j_0!/j_1!$. In the special case $n-2j_0=ms$, and $j_0=j_1=(n-sm)/2$ we replace C_{n,j_0} by $P_{n-2j_0}(z,w)$. By Theorem 3 $C_{n,j_0}=0$ when $\kappa_0+\kappa_1=-m$, thus C_{n,j_0} is divisible by $(\kappa_0+\kappa_1+m)\,s$. The sum of the two terms ($j=j_0$ and $j=j_1$) has a removable singularity there. $\blacksquare$

The expressions for $V\left(z^a\overline{z}^b\right)$ are derived from the series (3.1) for $K_n\left(z,w\right)$ thus the result about singularities at $\kappa_0+\kappa_1=-m$ being removable by grouping the expansion into certain pairs applies. Note that in the above proof the paired terms are P_{n-2j_0} and P_{n-2j_1} with $j_0+j_1=n-sm$. To analyze $V\left(z^a\overline{z}^b\right)$ it suffices to identify the pairs. For the case $a\equiv b \bmod s$ and $a\geq b$ let $a=us+r$, $b=vs+r$ and $0\leq r<s$. The paired indices in the sum from Theorem 2 consist of $\left\{(k,k'):0\leq k<k'\leq v, k+k'=u+v-m\right\}$. Indeed for k, k' with $0\leq k,k'\leq v$ define j by $a+b-2j=\left(u+v-2k\right)s$ so that $j=ks+r$ and similarly set $j':=k's+r$. The pairing condition $j+j'=a+b-sm$ is equivalent to $k+k'=u+v-m$. Thus k, k' are paired exactly when $k+k'=u+v-m$ and $0\leq k,k'\leq v$.

For the case $a-b\equiv t \bmod s$, and with $a=us+r+t>b=vs+r$, $0\leq r<s$, $1\leq t<s$ the pairing in the formula from Theorem 1 combines terms from the first sum with corresponding terms in the second. For the first sum suppose $0\leq k\leq v$ and $j:=ks+r$ so that $a+b-2j=t+\left(u+v-2k\right)s$. For the second sum let $1-\left\lfloor\frac{r+t}{s}\right\rfloor\leq k'\leq v$ and let $j':=\left(k'-1\right)s+r+t$ so that $a+b-2j'=\left(s-t\right)+\left(u+v-2k'+1\right)s$. The pairing condition $j+j'=a+b-sm$ is equivalent to $k+k'=u+v+1-m$. To remove the singularities at $\kappa_0+\kappa_1=-m$ combine the term in the first sum of index k with the term in the second of index k' for all pairs (k,k') satisfying $k+k'=u+v+1-m$, $0\leq k\leq v$, $1-\left\lfloor\frac{r+t}{s}\right\rfloor\leq k'\leq v$.

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