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N-Wave Equations with Orthogonal Algebras: $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ Reductions and Soliton Solutions[⋆]

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Abstract. We consider N-wave type equations related to the orthogonal algebras obtained from the generic ones via additional reductions. The first $\mathbb{Z}_2$-reduction is the canonical one. We impose a second $\mathbb{Z}_2$-reduction and consider also the combined action of both reductions. For all three types of N-wave equations we construct the soliton solutions by appropriately modifying the Zakharov–Shabat dressing method. We also briefly discuss the different types of one-soliton solutions. Especially rich are the types of one-soliton solutions in the case when both reductions are applied. This is due to the fact that we have two different configurations of eigenvalues for the Lax operator L: doublets, which consist of pairs of purely imaginary eigenvalues, and quadruplets. Such situation is analogous to the one encountered in the sine-Gordon case, which allows two types of solitons: kinks and breathers. A new physical system, describing Stokes-anti Stokes Raman scattering is obtained. It is represented by a 4-wave equation related to the $\mathbf{B}_2$ algebra with a canonical $\mathbb{Z}_2$ reduction.

Key words: solitons; Hamiltonian systems

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1 Introduction

The N-wave equation related to a semisimple Lie algebra $\mathfrak{g}$ is a matrix system of nonlinear differential equations of the type

$$i[J, Q_t(x,t)] - i[I, Q_x(x,t)] + [[I, Q(x,t)], [J, Q(x,t)]] = 0, \tag{1.1}$$

where the squared brackets denote the commutator of matrices and the subscript means a partial derivative with respect to independent variables x and t. The constant matrices I and J are regular and real elements of the Cartan subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$. The matrix-valued function $Q(x,t) \in \mathfrak{g}$ can be expanded as follows

$$Q(x,t) = \sum_{\alpha\in\Delta} Q_\alpha(x,t) E_\alpha,$$

where Δ denotes the root system and E_α are elements of the Weyl basis of $\mathfrak{g}$ parametrized by roots of $\mathfrak{g}$. It is also assumed that $Q(x,t)$ satisfies a vanishing boundary condition, i.e. $\lim\limits_{x\to\pm\infty} Q(x,t) = 0$.

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The N-wave equation is an example of an S-integrable evolutionary equation. Such type of equations appear in nonlinear optics and describes the propagation of N wave packets in nonlinear media [2].

Another application of the N-wave systems is in differential geometry. Ferapontov [4] showed that N-wave equations naturally occurred when one studied isoparametric hypersurfaces in spheres.

Our aim in this paper is two-fold. First, we outline the derivation of N-wave equations with $\mathbb{Z}_2$-reductions and calculate their soliton solutions by applying one of the basic methods in theory of integrable systems – Zakharov–Shabat dressing procedure. One of the considered examples has a physical interpretation – it describes a Stokes-anti-Stokes Raman scattering in nonlinear optics. Secondly, we further reduce the N-wave equations by imposing a second $\mathbb{Z}_2$-reduction. As a result we derive a special class of N-wave equations which, like the sine-Gordon equation (SG), possess breather type solutions. Next we obtain their soliton solutions. The additional symmetries of the nonlinear equations have been taken into account when we chose a proper dressing factor. The soliton solutions of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-reduced N-wave equation are of two types. The solitons of the first type are connected with a couple of purely imaginary discrete eigenvalues of the Lax operator L. These solutions correspond to the kinks (or topological solitons) of SG. The solitons of the second type, associated with quadruplets of discrete eigenvalues L which are symmetrically located with respect to the coordinate axis in the complex λ-plane, are analogues of the breather solutions of SG. Recently the Darboux-dressing transformations have been applied to the Lax pair associated with systems of coupled nonlinear wave equations: vector nonlinear Schrödinger-type equations [3]. Solutions with boomeronic and trapponic behaviour were investigated.

The problem for classification and investigation of all admissible reductions of an integrable equation is one of the fundamental problems in the theory of integrable systems. N-wave equations with canonical $\mathbb{Z}_2$ symmetries have been discussed for the first time by Zakharov and Manakov in [23] for $\mathfrak{g} \simeq \mathfrak{sl}(n)$. More recently the $\mathbb{Z}_2$-reductions of the N-wave equations related to the low-rank simple Lie algebras were analyzed and classified [8, 9].

Since we shall deal with the inverse scattering transform we begin with a reminder of all necessary facts concerning that theory. For more detailed information we recommend the monographs [21] and [24].

2 General formalism

As we mentioned above the N-wave system (1.1) is an integrable one. It admits a Lax representation with Lax operators

$$
\begin{aligned}
&L\psi(x,t,\lambda) = (i\partial_x + U(x,t,\lambda))\,\psi(x,t,\lambda) = 0,\\
&M\psi(x,t,\lambda) = (i\partial_t + V(x,t,\lambda))\,\psi(x,t,\lambda),
\end{aligned}
\tag{2.1}
$$

where λ is an auxiliary (so-called spectral) parameter and the potentials $U(x,t,\lambda)$, $V(x,t,\lambda)$ are elements of $\mathfrak{g}$ and they are linear functions of λ defined by

$$
\begin{aligned}
&U(x,t,\lambda) = U^0(x,t) - \lambda J = [J, Q(x,t)] - \lambda J,\\
&V(x,t,\lambda) = V^0(x,t) - \lambda I = [I, Q(x,t)] - \lambda I.
\end{aligned}
$$

The nonlinear evolution equation itself is equivalent to the compatibility condition of the differential operators L and M

$$
[L, M] = 0 \quad \Leftrightarrow \quad i[J, Q(x,t)_t] - i[I, Q(x,t)_x] + [[I, Q(x,t)], [J, Q(x,t)]] = 0.
$$

Since L and M commute they have the same eigenfunctions and the system (2.1) can be presented by

$$\begin{aligned} &L\psi(x,t,\lambda) = (i\partial_x + U(x,t,\lambda))\,\psi(x,t,\lambda) = 0, \\ &M\psi(x,t,\lambda) = (i\partial_t + V(x,t,\lambda))\,\psi(x,t,\lambda) = \psi(x,t,\lambda)C(\lambda), \end{aligned} \tag{2.2}$$

where $C(\lambda)$ is a constant matrix with respect to x and t. The fundamental solutions $\psi(x,t,\lambda)$ of the auxiliary linear system (2.2) take values in the Lie group G which corresponds to the Lie algebra $\mathfrak{g}$.

In order to find the soliton solutions we need the so-called fundamental analytic solutions (FAS) derived for the first time in [24] for $G \simeq SL(n)$; for other simple Lie algebras see [6, 7]. There is a standard algorithm to construct these solutions by using another class of fundamental solutions of the linear problem (2.2) – Jost solutions. The Jost solutions $\psi_\pm(x,t,\lambda)$ are determined by their asymptotics at infinity, i.e.

$$\lim_{x\to\pm\infty} \psi_\pm(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1}.$$

Remark. This definition is correct provided we fixed up the matrix-valued function $C(\lambda)$ by

$$C(\lambda) = \lim_{x\to\pm\infty} V(x,t,\lambda) = -\lambda I,$$

i.e. the asymptotics of $\psi_\pm(x,t,\lambda)$ are t-independent. $C(\lambda)$ is directly related to the dispersion law of the nonlinear equation. Thus the dispersion law of the N-wave equation is a linear function of the spectral parameter λ.

The Jost solutions $\psi_\pm(x,t,\lambda)$ are linearly related, which means that there exists a matrix $T(t,\lambda)$ such that

$$\psi_-(x,t,\lambda) = \psi_+(x,t,\lambda)T(t,\lambda).$$

The matrix-valued function $T(t,\lambda)$ is called a scattering matrix. Its time evolution is determined by the second equation of (2.2), i.e.

$$i\partial_t T(t,\lambda) - \lambda[I, T(t,\lambda)] = 0.$$

Consequently

$$T(t,\lambda) = e^{-i\lambda It}T(0,\lambda)e^{i\lambda It}. \tag{2.3}$$

The inverse scattering transform (IST) allows one to solve the Cauchy problem for the nonlinear evolution equation, i.e. finding a solution $Q(x,t)$ when its initial condition $Q_{\mathrm{in}}(x) = Q(x,0)$ is given. The idea of IST is illustrated in the following diagram

$$Q_{\mathrm{in}}(x) \to U(x,0,\lambda) \xrightarrow{\text{DSP}} T(0,\lambda) \longrightarrow T(t,\lambda) \xrightarrow{\text{ISP}} U(x,t,\lambda) \to Q(x,t).$$

The first step consists in constructing the scattering matrix at some initial moment $t = 0$ by using the potential at the same moment (or equivalently by the solution $Q_{\mathrm{in}}(x)$ at that moment). This is a direct scattering problem (DSP). The evolution of the scattering matrix (data) is already known and it is given by (2.3). The third step is recovering of the potential $U(x,t,\lambda)$ and respectively the solution of the nonlinear equation $Q(x,t)$ at an arbitrary moment from the scattering data at that moment – this is an inverse scattering problem (ISM). That step is actually the only nontrivial one. Thus following all steps we can solve the Cauchy problem for the nonlinear evolution equation. Since we know the evolution of scattering data we can easily determine the time dependence of fundamental solutions, solutions of nonlinear problem etc.

The Jost solutions are defined for real values of λ only (they do not possess analytic properties for $\lambda \notin \mathbb{R}$). For our purpose it is necessary to construct fundamental solutions which admit analytic continuation beyond the real axes. It can be shown that there exist fundamental solutions $\chi^+(x,t,\lambda)$ and $\chi^-(x,t,\lambda)$ analytic in the upper half-plane $\mathbb{C}_+$ and in the lower half-plane $\mathbb{C}_-$ of the spectral parameter respectively. They can be obtained from the Jost solutions by a simple algebraic procedure proposed by Shabat [19, 20], see also [24]. The procedure uses a Gauss decomposition of the scattering matrix $T(t,\lambda)$, namely

$$\chi^\pm(x,t,\lambda) = \psi_-(x,t,\lambda)S^\pm(t,\lambda) = \psi_+(x,t,\lambda)T^\mp(t,\lambda)D^\pm(\lambda),$$

where matrices $S^\pm(t,\lambda)$, $T^\pm(t,\lambda)$ and $D^\pm(\lambda)$ are Gauss factors of the matrix $T(t,\lambda)$, i.e.

$$T(t,\lambda) = T^\mp(t,\lambda)D^\pm(\lambda)(S^\pm(t,\lambda))^{-1}.$$

The matrices $S^+(t,\lambda)$ and $T^+(t,\lambda)$ (resp. $S^-(t,\lambda)$ and $T^-(t,\lambda)$) are upper (resp. lower) triangular with unit diagonal elements and taking values in G. Their time dependence is given by:

$$i\partial_t S^\pm(t,\lambda) - \lambda[I, S^\pm(t,\lambda)] = 0, \qquad i\partial_t T^\pm(t,\lambda) - \lambda[I, T^\pm(t,\lambda)] = 0.$$

The matrices $D^+(\lambda)$ and $D^-(\lambda)$ are diagonal and allow analytic extension in λ for $\mathrm{Im}\lambda > 0$ and $\mathrm{Im}\lambda < 0$. They do not depend on time and provide the generating functionals of the integrals of motion of the nonlinear evolution equation [24, 6, 8, 9], see also the review paper [7].

A powerful method for obtaining solutions to nonlinear differential equations is Bäcklund transformation (see [15] and [18, 17] for more detailed information). A Bäcklund transformation maps a solution of a differential equation into a solution of another differential equation. If both equations coincide one speaks of an auto-Bäcklund transformation. A very important particular case of an auto-Bäcklund transformation is the dressing method proposed by Zakharov and Shabat [25]. Its basic idea consists in constructing a new solution $Q(x,t)$ starting from a known solution $Q_0(x,t)$ taking into account the existence of the auxiliary linear system (2.2).

Let $\psi_0(x,t,\lambda)$ satisfy the linear problem

$$L_0\psi_0(x,t,\lambda) = i\partial_x\psi_0(x,t,\lambda) + ([J, Q_0(x,t)] - \lambda J)\psi_0(x,t,\lambda) = 0. \tag{2.4}$$

We construct a function $\psi(x,t,\lambda)$ by introducing a gauge transformation $g(x,t,\lambda)$ – dressing procedure of the solution $\psi_0(x,t,\lambda)$

$$\psi_0(x,t,\lambda) \to \psi(x,t,\lambda) = g(x,t,\lambda)\psi_0(x,t,\lambda)$$

such that the linear system (2.4) is covariant under the action of that gauge transformation. Thus the dressing factor has to satisfy

$$i\partial_x g(x,t,\lambda) + [J, Q(x,t)]g(x,t,\lambda) - g(x,t,\lambda)[J, Q_0(x,t)] - \lambda[J, g(x,t,\lambda)] = 0. \tag{2.5}$$

If we choose a dressing factor which is a meromorphic function of the spectral parameter λ as follows

$$g(x,t,\lambda) = \mathbb{1} + \frac{A(x,t)}{\lambda - \lambda^+} + \frac{B(x,t)}{\lambda - \lambda^-}, \tag{2.6}$$

where $\lambda^\pm \in \mathbb{C}_\pm$, we obtain the following relation between $Q(x,t)$ and $Q_0(x,t)$

$$[J, Q(x,t)] = [J, Q_0(x,t) + A(x,t) + B(x,t)].$$

As a result we are able to construct new solutions if we know the functions $A(x,t)$ and $B(x,t)$. We will show later how this can be done.

The simplest class of solutions are the so-called reflectionless potentials. A soliton solution is obtained by dressing the trivial solution $Q_0(x,t) \equiv 0$. Then the fundamental solution of the linear problem is just a plane wave $\psi_0(x,t,\lambda) = e^{-i\lambda(Jx+It)}$ and the one-soliton solution itself is given by

$$[J, Q_{1s}(x,t)] = [J, A_{1s}(x,t) + B_{1s}(x,t)].$$

As we said above the dressing procedure maps a solution of the linear problem to another solution of a linear problem with a different potential. In particular the Jost solutions $\psi_{0,\pm}(x,t,\lambda)$ are transformed into

$$\psi_\pm(x,t,\lambda) = g(x,t,\lambda)\psi_{0,\pm}(x,t,\lambda)g_\pm^{-1}(\lambda),$$

where the factors $g_\pm(\lambda)$ ensure the proper asymptotics of the dressed solutions and are defined by

$$g_\pm(\lambda) = \lim_{x\to\pm\infty} g(x,t,\lambda).$$

Hence the dressed scattering matrix $T(t,\lambda)$ reads

$$T(t,\lambda) = g_+(\lambda)T_0(t,\lambda)g_-^{-1}(\lambda).$$

It can be proven that the FAS $\chi_0^\pm(x,t,\lambda)$ transform into

$$\chi^\pm(x,t,\lambda) = g(x,t,\lambda)\chi_0^\pm(x,t,\lambda)g_-^{-1}(\lambda). \tag{2.7}$$

The spectral properties of L are determined by the behaviour of its resolvent operator. The resolvent $R(t,\lambda)$ is an integral operator (see [5, 7] for more details) given by

$$R(t,\lambda)f(x) = \int_{-\infty}^{\infty} \mathcal{R}(x,y,t,\lambda)f(y)dy,$$

where the kernel $\mathcal{R}(x,y,t,\lambda)$ must be a piece-wise analytic function of λ satisfying the equation

$$L\mathcal{R}(x,y,t,\lambda) = \delta(x-y)\mathbb{1}.$$

The kernel $\mathcal{R}(x,y,t,\lambda)$ can be constructed by using the fundamental analytical solutions as follows

$$\mathcal{R}(x,y,t,\lambda) = \begin{cases} \mathcal{R}^+(x,y,t,\lambda), & \mathrm{Im}\lambda > 0, \\ \mathcal{R}^-(x,y,t,\lambda), & \mathrm{Im}\lambda < 0, \end{cases}$$

where

$$\begin{aligned} &\mathcal{R}^\pm(x,y,t,\lambda) = \pm i\chi^\pm(x,t,\lambda)\Theta^\pm(x-y)(\chi^\pm(y,t,\lambda))^{-1}, \\ &\Theta^\pm(x-y) = \theta(\mp(x-y))\Pi - \theta(\pm(x-y))(1-\Pi), \\ &\Pi = \sum_{p=1}^{a} E_{pp}\ , \qquad (E_{pq})_{rs} = \delta_{pr}\delta_{qs}, \end{aligned}$$

where θ is the standard Heaviside function and a is the number of positive eigenvalues of J. Due to the fact that we have chosen J to be a real matrix with

$$J_1 > J_2 > \cdots > J_a > 0 > J_{a+1} > \cdots > J_n,$$

the resolvent $R(t,\lambda)$ is a bounded integral operator for $\mathrm{Im}\lambda \neq 0$. For $\mathrm{Im}\lambda = 0$ $R(t,\lambda)$ is an unbounded integral operator, which means that the continuous spectrum of L fills up the real axes $\mathbb{R}$. Since the discrete part of the spectrum of L is determined by the poles of $R(t,\lambda)$ it coincides with the poles and zeroes of $\chi^{\pm}(x,t,\lambda)$.

From (2.7) and from the explicit form of $\mathcal{R}(t,\lambda)$ it follows that the dressed kernel is related to the bare one by

$$\mathcal{R}^{\pm}(x,y,t,\lambda) = g(x,t,\lambda)\mathcal{R}_0^{\pm}(x,y,t,\lambda)g^{-1}(y,t,\lambda).$$

If we assume that the "bare" operator L_0 has no discrete eigenvalues then the poles of $g(x,t,\lambda)$ determine the discrete eigenvalues of L

$$L_0 \xrightarrow{g} L \quad \Leftrightarrow \quad \mathrm{spec}(L_0) \quad \rightarrow \quad \mathrm{spec}(L) = \mathrm{spec}(L_0) \cup \left\{\lambda^+, \lambda^-\right\}.$$

Many classical integrable systems correspond to Lax operators with potentials possessing additional symmetries. That is why it is of particular interest to consider the case when certain symmetries are imposed on the potential $U(x,t,\lambda)$ (resp. on the solution $Q(x,t)$).

Let G_R be a discrete group acting in G by group automorphisms and in $\mathbb{C}$ by conformal transformations

$$\kappa : \lambda \rightarrow \kappa(\lambda).$$

Therefore we have an induced action of G_R in the space of functions $f(x,t,\lambda)$ taking values in G as follows

$$\mathcal{K} : f(x,t,\lambda) \rightarrow \tilde{f}(x,t,\lambda) = \tilde{K}\left(f\left(x,t,\kappa^{-1}(\lambda)\right)\right), \qquad \tilde{K} \in \mathrm{Aut}(G).$$

This action in turn induces another action on the differential operators L and M

$$\tilde{L}(\lambda) = \mathcal{K}L(\kappa^{-1}(\lambda))\mathcal{K}^{-1},$$
$$\tilde{M}(\lambda) = \mathcal{K}M(\kappa^{-1}(\lambda))\mathcal{K}^{-1}.$$

The group action is consistent with Lax representation which is equivalent to invariance of Lax representation under the action of G_R, i.e. if $[L,M]=0$ then also

$$[\tilde{L},\tilde{M}] = 0.$$

The requirement of G_R-invariance of the set of fundamental solutions $\{\psi(x,t,\lambda)\}$ leads to the following symmetry condition

$$\tilde{K}U(x,t,\kappa^{-1}(\lambda))\tilde{K}^{-1} = U(x,t,\lambda).$$

In other words the potential $U(x,t,\lambda)$, as well as $Q(x,t)$ are reduced. This fact motivates the name of the group G_R – a reduction group, proposed by Mikhailov in [16].

One can prove that the dressing factor $g(x,t,\lambda)$ ought to be invariant under the action of the reduction group G_R

$$\tilde{K}\left(g(x,t,\kappa^{-1}(\lambda))\right) = g(x,t,\lambda). \tag{2.8}$$

3 N-wave equations with a $\mathbb{Z}_2$ reduction

In this chapter we are going to demonstrate an algorithm to derive soliton solutions for a N-wave equation related to the orthogonal algebra (i.e. $\mathfrak{g} \equiv \mathfrak{so}(n,\mathbb{C})$) with a $\mathbb{Z}_2$ reduction imposed on it. We pay special attention to the particular case of $\mathbf{B}_2 \simeq \mathfrak{so}(5,\mathbb{C})$ algebra: we display the explicit form of the 4-wave system and calculate by components its one-soliton solutions. One of the 4-wave systems under consideration enjoys a physical application in nonlinear optics. We are following the ideas presented by Zakharov and Mikhailov in [22].

3.1 Physical applications of 4-wave equation with a $\mathbb{Z}_2$ reduction

From now on we shall deal with the orthogonal group $SO(n,\mathbb{C})$ and its Lie algebra $\mathfrak{so}(n,\mathbb{C})$. That is why we remind the reader several basic facts concerning this topic without proving them, for more detailed information we recommend the book [13].

The orthogonal group is, by definition, a matrix group which consists of all isometries acting on $\mathbb{C}^n$, i.e.

$$SO(n,\mathbb{C}) = \left\{C \in GL(n,\mathbb{C}) |\, C^T S C = S\right\},$$

where S stands for the metric in $\mathbb{C}^n$. It determines a scalar product by the formula

$$(u,v) = u^T S v, \qquad u,v \in \mathbb{C}^n.$$

$\mathfrak{so}(n,\mathbb{C})$ is the matrix Lie algebra of all infinitesimal isometries on $\mathbb{C}^n$, i.e.

$$\mathfrak{so}(n,\mathbb{C}) = \left\{\mathfrak{c} \in \mathfrak{gl}(n,\mathbb{C}) |\, \mathfrak{c}^T S + S\mathfrak{c} = 0\right\}.$$

The orthogonal algebra is a semisimple Lie algebra. We remind that its maximal commutative subalgebra is called a Cartan subalgebra $\mathfrak{h}$. It is more convenient to work with a basis in $\mathbb{C}^n$ such that the matrix of S in that basis reads

$$S = \sum_{k=1}^{n} (-1)^{k-1}(E_{k,2n+1-k} + E_{2n+1-k,k}) + (-1)^n E_{nn}, \qquad \text{for} \quad \mathbf{B}_n \simeq \mathfrak{so}(2n+1,\mathbb{C})$$

and

$$S = \sum_{k=1}^{n} (-1)^{k-1}(E_{k,2n+1-k} + E_{2n+1-k,k}), \qquad \text{for} \quad \mathbf{D}_n \simeq \mathfrak{so}(2n,\mathbb{C}).$$

This choice of a basis ensures that the corresponding Cartan subalgebras consist of diagonal matrices. More precisely, Cartan basis $\{H_k\}_{k=1}^n$, i.e. the basis which spans $\mathfrak{h}$, has the form

$$\begin{aligned} &H_k = E_{kk} - E_{2n+2-k,2n+2-k}, &&\text{for} \quad \mathbf{B}_n,\\ &H_k = E_{kk} - E_{2n+1-k,2n+1-k}, &&\text{for} \quad \mathbf{D}_n. \end{aligned}$$

The root systems of the series $\mathbf{B}_n$ and $\mathbf{D}_n$ consist of the roots

$$\begin{aligned} &\pm e_i \pm e_j, \quad e_i, \quad i \neq j, \qquad \text{for} \quad \mathbf{B}_n,\\ &\pm e_i \pm e_j, \quad i \neq j, \qquad \text{for} \quad \mathbf{D}_n, \end{aligned}$$

where $\{e_i\}_{i=1}^n$ stands for the standard orthonormal basis in $\mathbb{C}^n$. The positive roots are as follows

$$\begin{aligned} &e_i \pm e_j, \quad i < j, \qquad e_i, \qquad \text{for} \quad \mathbf{B}_n,\\ &e_i \pm e_j, \quad i < j, \qquad \text{for} \quad \mathbf{D}_n. \end{aligned}$$

The set of all simple roots reads

$$\begin{aligned} &\alpha_k = e_k - e_{k+1}, \qquad k = 1,\dots,n-1, \qquad \alpha_n = e_n \qquad \text{for} \quad \mathbf{B}_n,\\ &\alpha_k = e_k - e_{k+1}, \qquad \alpha_n = e_{n-1} + e_n, \qquad \text{for} \quad \mathbf{D}_n. \end{aligned}$$

Let us illustrate these general results by a physical example related to the $\mathbf{B}_2$ algebra. This algebra has two simple roots $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2$, and two more positive roots: $\alpha_1 + \alpha_2 = e_1$ and $\alpha_1 + 2\alpha_2 = e_1 + e_2 = \alpha_{\max}$. When they come as indices, e.g. in $Q_\alpha(x,t)$ we will replace them by sequences of two positive integers: $\alpha \to kn$ if $\alpha = k\alpha_1 + n\alpha_2$; if $\alpha = -(k\alpha_1 + n\alpha_2)$ we will use $\overline{kn}$. The reduction which extracts the real forms of $\mathbf{B}_2$ is

$$K_1 U^\dagger(x,t,\lambda^*) K_1^{-1} = U(x,t,\lambda),$$

where K_1 is an element of the Cartan subgroup: $K_1 = \operatorname{diag}(s_1, s_2, 1, s_2, s_1)$ with $s_k = \pm 1$, $k = 1, 2$. This means that $J_i = J_i^*$, $i = 1, 2$ and $Q_\alpha(x,t)$ must satisfy:

$$Q_{\overline{10}}(x,t) = -s_1 s_2 Q^*_{10}(x,t), \qquad Q_{\overline{11}}(x,t) = -s_1 Q^*_{11}(x,t),$$
$$Q_{\overline{12}}(x,t) = -s_1 s_2 Q^*_{12}(x,t), \qquad Q_{\overline{01}}(x,t) = -s_2 Q^*_{01}(x,t).$$

and the matrix $Q(x,t)$ is defined by

$$\begin{aligned} Q(x,t) &= Q_{10}(x,t) E_{e_1 - e_2} + Q_{12}(x,t) E_{e_1 + e_2} + Q_{11}(x,t) E_{e_1} + Q_{01}(x,t) E_{e_2} \\ &\quad + Q_{\overline{10}}(x,t) E_{-(e_1 - e_2)} + Q_{\overline{12}}(x,t) E_{-(e_1+e_2)} + Q_{\overline{11}}(x,t) E_{-e_1} + Q_{\overline{01}}(x,t) E_{-e_2}. \end{aligned}$$

As a result we get the following 4-wave system

$$\begin{aligned} & i(J_1 - J_2) Q_{10,t}(x,t) - i(I_1 - I_2) Q_{10,x}(x,t) - k s_2 Q_{11}(x,t) Q^*_{01}(x,t) = 0, \\ & i J_1 Q_{11,t}(x,t) - i I_1 Q_{11,x}(x,t) - k(Q_{10} Q_{01} + s_2 Q_{12} Q^*_{01})(x,t) = 0, \\ & i(J_1 + J_2) Q_{12,t}(x,t) - i(I_1 + I_2) Q_{12,x}(x,t) - k Q_{11}(x,t) Q_{01}(x,t) = 0, \\ & i J_2 Q_{01,t}(x,t) - i I_2 Q_{01,x}(x,t) - k s_1 (Q^*_{11} Q_{12} + s_2 Q^*_{10} Q_{11})(x,t) = 0, \end{aligned}$$

where $k = J_1 I_2 - J_2 I_1$ and its Hamiltonian $H(t) = H_0(t) + H_{\rm int}(t)$ is

$$\begin{aligned} H_0(t) &= \frac{i}{2} \int_{-\infty}^{\infty} dx \, \big[(I_1 - I_2)(Q_{10}(x,t) Q^*_{10,x}(x,t) - Q_{10,x}(x,t) Q^*_{10}(x,t)) \\ &\quad + I_2 (Q_{01}(x,t) Q^*_{01,x}(x,t) - Q_{01,x}(x,t) Q^*_{01}(x,t)) + I_1 (Q_{11}(x,t) Q^*_{11,x}(x,t) \\ &\quad - Q_{11,x}(x,t) Q^*_{11}(x,t)) + (I_1 + I_2)(Q_{12}(x,t) Q^*_{12,x}(x,t) - Q_{12,x}(x,t) Q^*_{12}(x,t)) \big], \\ H_{\rm int}(t) &= 2 k s_1 \int_{-\infty}^{\infty} dx \, \left[s_2 (Q_{12}(x,t) Q^*_{11}(x,t) Q^*_{01}(x,t) + Q^*_{12}(x,t) Q_{11}(x,t) Q_{01}(x,t)) \right. \\ &\quad \left. + (Q_{11}(x,t) Q^*_{01}(x,t) Q^*_{10}(x,t) + Q^*_{11}(x,t) Q_{01}(x,t) Q_{10}(x,t)) \right], \end{aligned}$$

and the symplectic form:

$$\begin{aligned} \Omega^{(0)} &= i \int_{-\infty}^{\infty} dx \, \big[(J_1 - J_2) \delta Q_{10}(x,t) \wedge \delta Q^*_{10}(x,t) + J_2 \delta Q_{01}(x,t) \wedge \delta Q^*_{01}(x,t) \\ &\quad + J_1 \delta Q_{11}(x,t) \wedge \delta Q^*_{11}(x,t) + (J_1 + J_2) \delta Q_{12}(x,t) \wedge \delta Q^*_{12}(x,t) \big]. \end{aligned}$$

The invariance condition (2.8) leads to the following form of the dressing factor

$$g(x,t,\lambda) = \mathbb{1} + \frac{A(x,t)}{\lambda - \lambda^+} + \frac{K_1 S A^*(x,t)(K_1 S)^{-1}}{\lambda - (\lambda^+)^*}, \tag{3.1}$$

i.e. comparing with (2.6) we see that

$$B(x,t) = K_1 S A^*(x,t) (K_1 S)^{-1}, \qquad \lambda^- = (\lambda^+)^*.$$

By taking the limit $\lambda \to \infty$ in equation (2.5) and taking into account the explicit formula (3.1) one can derive the following relation between the bare solution $Q_0(x,t)$ and the dressed one $Q(x,t)$

$$[J, Q(x,t)] = [J, Q_0(x,t) + A(x,t) + K_1 S A^*(x,t) S K_1].$$

Thus the one soliton solution of the N-wave equation is determined by one matrix-valued function $A(x,t)$. We will obtain $A(x,t)$ in two steps by deriving certain algebraic and differential relations. First of all recall that the dressing factor $g(x,t,\lambda)$ must belong to the orthogonal group $\mathrm{SO}(n,\mathbb{C})$, hence

$$g^{-1}(x,t,\lambda) = S^{-1}g^T(x,t,\lambda)S.$$

Besides, the equality $gg^{-1} = 1$ must hold identically with respect to λ, therefore $A(x,t)$ satisfies the algebraic restrictions

$$A(x,t)SA^T(x,t) = 0, \qquad A(x,t)S\omega^T(x,t) + \omega(x,t)SA^T(x,t) = 0, \tag{3.2}$$

where

$$\omega(x,t) = \mathbb{1} + \frac{K_1SA^*(x,t)SK_1}{2i\nu}, \qquad \lambda^+ = \mu + i\nu.$$

From the first equality in (3.2) it follows that the matrix $A(x,t)$ is a degenerate one and it can be decomposed

$$A(x,t) = X(x,t)F^T(x,t),$$

where $X(x,t)$ and $F(x,t)$ are $n\times k$ $(1 \leq k < n)$ matrices of maximal rank k. The equalities (3.2) can be rewritten in terms of $X(x,t)$ and $F(x,t)$ as follows

$$F^T(x,t)SF(x,t) = 0, \qquad X(x,t)F^T(x,t)S\omega^T(x,t) + \omega(x,t)SFX^T(x,t) = 0$$

or introducing a $k\times k$ skew symmetric matrix $\alpha(x,t)$ the latter restriction reads

$$\left(\mathbb{1} + \frac{K_1SX^*(x,t)F^\dagger(x,t)SK_1}{2i\nu}\right)SF(x,t) = X(x,t)\alpha(x,t). \tag{3.3}$$

Another type of restrictions concerning the matrix-valued functions $F(x,t)$ and $\alpha(x,t)$ comes form the λ-independence of the potential $[J,Q(x,t)]$. If we express the potential in the equation (2.5) we get

$$[J,Q(x,t)] = -i\partial_x gg^{-1}(x,t,\lambda) + g[J,Q_0(x,t)]g^{-1}(x,t,\lambda) + \lambda\left(J - gJg^{-1}(x,t,\lambda)\right). \tag{3.4}$$

Annihilation of residues in (3.4) leads to the following linear differential equations

$$i\partial_x F^T(x,t) - F^T(x,t)([J,Q_0(x,t)] - \lambda^+J) = 0$$

and

$$i\partial_x\alpha(x,t) + F^T(x,t)JSF(x,t) = 0.$$

After integration we obtain that

$$F(x,t) = S\chi_0^+(x,t,\lambda^+)SF_0,$$

where the constant matrix F_0 obeys the following equality

$$F_0^TSF_0 = 0$$

and the matrix $\alpha(x,t)$ reads

$$\alpha(x,t) = F_0^T(\chi_0^+(x,t,\lambda^+))^{-1}\partial_\lambda\chi_0^+(x,t,\lambda^+)SF_0 + \alpha_0.$$

In the soliton case the fundamental solution is just a plane wave. Therefore functions $F(x,t)$ and $\alpha(x,t)$ get the form

$$F(x,t) = e^{i\lambda^+(Jx+It)}F_0, \qquad \alpha(x,t) = iF_0^T(Jx+It)SF_0 + \alpha_0.$$

It remains to find the factor $X(x,t)$. For this purpose we solve the linear equation (3.3) and find that

$$X(x,t) = 2\nu(iK_1F^* - 2\nu SF(F^\dagger K_1F)^{-1}\alpha^*)\big(F^TK_1F^* - 4\nu^2\alpha(F^\dagger K_1F)^{-1}\alpha^*\big)^{-1}.$$

In particular, in the simplest case when $\mathrm{rank}X(x,t) = \mathrm{rank}F(x,t) = 1$ we have $\alpha(x,t) \equiv 0$ and therefore

$$X(x,t) = \frac{2i\nu}{F^\dagger(x,t)K_1F(x,t)}K_1F^*(x,t).$$

The soliton solution obtains the form

$$Q_{ij}(x,t) = \begin{cases} \dfrac{2i\nu}{F^\dagger(x,t)K_1F(x,t)}\big((K_1F^*(x,t))_iF_j(x,t) \\ \qquad + (-1)^{i+j+1}F_{6-i}(x,t))(K_1F^*(x,t))_{6-j}\big), & i \neq j, \\ 0, & i = j. \end{cases}$$

Turning back to the $\mathbf{B}_2$ case this result can be written by components as follows

$$\begin{aligned}
Q_{10}(x,t) &= \frac{2i\nu}{F^\dagger K_1F}\left(s_1F_{0,1}^*F_{0,2}e^{i\left(\lambda^+z_2-\lambda^{+*}z_1\right)} + s_2F_{0,4}^*F_{0,5}e^{i\left(\lambda^{+*}z_2-\lambda^+z_1\right)}\right), \\
Q_{11}(x,t) &= \frac{2i\nu}{F^\dagger K_1F}\left(s_1F_{0,1}^*F_{0,3}e^{-i\lambda^{+*}z_1} - F_{0,3}^*F_{0,5}e^{-i\lambda^+z_1}\right), \\
Q_{12}(x,t) &= \frac{2i\nu}{F^\dagger K_1F}\left(s_1F_{0,1}^*F_{0,4}e^{-i\left(\lambda^{+*}z_1+\lambda^+z_2\right)} + s_2F_{0,2}^*F_{0,5}e^{-i\left(\lambda^+z_1+\lambda^{+*}z_2\right)}\right), \\
Q_{01}(x,t) &= \frac{2i\nu}{F^\dagger K_1F}\left(s_2F_{0,2}^*F_{0,3}e^{-i\lambda^{+*}z_2} + F_{0,3}^*F_{0,4}e^{-i\lambda^+z_2}\right), \\
F^\dagger K_1F &= s_1|F_{0,1}|^2e^{-2\nu z_1} + s_2|F_{0,2}|^2e^{-2\nu z_2} + |F_{0,3}|^3 + s_2|F_{0,4}|^2e^{2\nu z_2} + s_1|F_{0,5}|^2e^{2\nu z_1}, \\
z_\sigma &= J_\sigma x + I_\sigma t, \qquad \sigma = 1,2.
\end{aligned}$$

A natural choice for the matrix which determines the action of the reduction group G_R is $K_1 = \mathbb{1}$. Let us also require the symmetry conditions

$$F_{0,1} = F_{0,5}^*, \qquad F_{0,2} = F_{0,4}^*, \qquad F_{0,3} = F_{0,3}^*.$$

As a result the one-soliton solution simplifies significantly

$$\begin{aligned}
Q_{10}(x,t) &= \frac{i\nu}{\Delta_1}\sinh 2\theta_0\cosh(\nu(z_1+z_2))e^{-i\mu(z_1-z_2+\phi_1-\phi_2)}, \\
Q_{11}(x,t) &= -\frac{2\sqrt{2}i\nu}{\Delta_1}\sinh\theta_0\sinh(\nu z_1)e^{-i(\mu z_1+\phi_1)}, \\
Q_{12}(x,t) &= \frac{i\nu}{\Delta_1}\sinh(2\theta_0)\cosh(\nu(z_1-z_2))e^{-i\mu(z_1+z_2+\phi_1+\phi_2)}, \\
Q_{01}(x,t) &= \frac{2\sqrt{2}i\nu}{\Delta_1}\cosh\theta_0\cosh(\nu z_2)e^{-i\mu(z_2+\phi_2)},
\end{aligned}$$

where we have used the representation

$$F_{0,1} = \frac{F_{0,3}}{\sqrt{2}}\sinh\theta_0e^{i\phi_1}, \qquad F_{0,2} = \frac{F_{0,3}}{\sqrt{2}}\cosh\theta_0e^{i\phi_2},$$

$$\Delta_1(x,t) = 2\left(\sinh^2\theta_0 \sinh^2(\nu z_1) + \cosh^2\theta_0 \cosh^2(\nu z_2)\right),$$

where θ_0 is an arbitrary real constant.

If we apply this dressing procedure to the one-soliton solution we obtain a two-soliton solution. Iterating this process we can generate multisoliton solutions, i.e.

$$0 \xrightarrow{g_{1\mathrm{s}}} Q_{1\mathrm{s}}(x,t) \xrightarrow{g_{2\mathrm{s}}} Q_{2\mathrm{s}}(x,t) \longrightarrow \cdots \xrightarrow{g_{m\mathrm{s}}} Q_{m\mathrm{s}}(x,t).$$

The particular case $s_1 = s_2 = 1$ leads to the compact real form $\mathfrak{so}(5,0) \simeq \mathfrak{so}(5,\mathbb{R})$ of the $\mathbf{B}_2$-algebra. The choice $s_1 = -s_2 = -1$ leads to the noncompact real form $\mathfrak{so}(2,3)$ and $s_1 = s_2 = -1$ gives another noncompact one – the Lorentz algebra $\mathfrak{so}(1,4)$. If $s_1 = s_2 = 1$ and we identify

$$Q_{01}(x,t) = -\frac{i}{\kappa} Q_{\mathrm{pol}}(x,t), \qquad Q_{10}(x,t) = -\frac{i}{\kappa} E_{\mathrm{s}}(x,t),$$
$$Q_{11}(x,t) = -\frac{i}{\kappa} E_{\mathrm{p}}(x,t), \qquad Q_{12}(x,t) = -\frac{i}{\kappa} E_{\mathrm{a}}(x,t),$$

then we obtain the system similar to that studied in [1, 11] which describes Stokes-anti-Stokes wave generation. Here $Q_{\mathrm{pol}}(x,t)$ is the normalized effective polarization of the medium and $E_{\mathrm{p}}(x,t)$, $E_{\mathrm{s}}(x,t)$ and $E_{\mathrm{a}}(x,t)$ are the normalized pump, Stokes and anti-Stokes wave amplitudes respectively. In this case the matrix $Q(x,t)$ is defined by

$$Q(x,t) = -\frac{i}{\kappa}\begin{pmatrix} 0 & E_{\mathrm{s}} & E_{\mathrm{p}} & E_{\mathrm{a}} & 0 \\ E_{\mathrm{s}}^* & 0 & Q_{\mathrm{pol}} & 0 & E_{\mathrm{a}} \\ E_{\mathrm{p}}^* & Q_{\mathrm{pol}}^* & 0 & Q_{\mathrm{pol}} & -E_{\mathrm{p}} \\ E_{\mathrm{a}}^* & 0 & Q_{\mathrm{pol}}^* & 0 & E_{\mathrm{s}} \\ 0 & E_{\mathrm{a}}^* & -E_{\mathrm{p}}^* & E_{\mathrm{s}}^* & 0 \end{pmatrix},$$

and for 4-wave system we have

$$\begin{aligned}
&(J_1 - J_2)E_{\mathrm{s},t}(x,t) - (I_1 - I_2)E_{\mathrm{s},x}(x,t) - E_{\mathrm{p}}(x,t)Q_{\mathrm{pol}}^*(x,t) = 0,\\
&J_2 Q_{\mathrm{pol},t}(x,t) - I_2 Q_{\mathrm{pol},x}(x,t) - E_{\mathrm{p}}^*(x,t)E_{\mathrm{a}}(x,t) - E_{\mathrm{p}}(x,t)E_{\mathrm{s}}^*(x,t) = 0,\\
&J_1 E_{\mathrm{p},t}(x,t) - I_1 E_{\mathrm{p},x}(x,t) - E_{\mathrm{a}}(x,t)Q_{\mathrm{pol}}^*(x,t) + E_{\mathrm{s}}(x,t)Q_{\mathrm{pol}}(x,t) = 0,\\
&(J_1 + J_2)E_{\mathrm{a},t}(x,t) - (I_1 + I_2)E_{\mathrm{a},x}(x,t) + E_{\mathrm{p}}(x,t)Q_{\mathrm{pol}}(x,t) = 0.
\end{aligned}$$

Finally we can rewrite it in the form:

$$\begin{aligned}
&\frac{1}{v_{-1}}\frac{\partial E_{\mathrm{s}}(x,t)}{\partial t} + \frac{\partial E_{\mathrm{s}}(x,t)}{\partial x} - \kappa_{-1}E_{\mathrm{p}}(x,t)Q_{\mathrm{pol}}^*(x,t) = 0,\\
&\frac{\partial Q_{\mathrm{pol}}(x,t)}{\partial t} - \kappa_{pol}(E_{\mathrm{p}}^*(x,t)E_{\mathrm{a}}(x,t) + E_{\mathrm{p}}(x,t)E_{\mathrm{s}}^*(x,t)) = 0,\\
&\frac{1}{v_0}\frac{\partial E_{\mathrm{p}}(x,t)}{\partial t} + \frac{\partial E_{\mathrm{p}}(x,t)}{\partial x} - \kappa_0(E_{\mathrm{a}}(x,t)Q_{\mathrm{pol}}^*(x,t) - E_{\mathrm{s}}(x,t)Q_{\mathrm{pol}}(x,t)) = 0,\\
&\frac{1}{v_1}\frac{\partial E_{\mathrm{a}}(x,t)}{\partial t} + \frac{\partial E_{\mathrm{a}}(x,t)}{\partial x} + \kappa_1 E_{\mathrm{p}}(x,t)Q_{\mathrm{pol}}(x,t) = 0,
\end{aligned} \tag{3.5}$$

where

$$v_{-1} = \frac{d\omega_{-1}}{dk_{-1}} = -\frac{I_1 - I_2}{J_1 - J_2}, \qquad \kappa_{-1} = -\frac{1}{I_1 - I_2},$$
$$v_0 = \frac{d\omega_0}{dk_0} = -\frac{I_1}{J_1}, \qquad \kappa_0 = -\frac{1}{I_1}, \qquad I_2 = 0,$$

$$v_1 = \frac{d\omega_1}{dk_1} = -\frac{I_1 + I_2}{J_1 + J_2}, \qquad \kappa_1 = -\frac{1}{I_1 + I_2}, \qquad \kappa_{\rm pol} = \frac{1}{J_2}.$$

The particular case $s_1 = s_2 = 1$, $I_2 = 0$, the one-soliton solution obeys (3.5). Finally for the simplest one soliton solution of (3.5) the modulus squared of one soliton amplitudes are given by

$$\begin{aligned}
|E_{\rm s}(x,t)|^2 &= \frac{\kappa^2\nu^2}{\Delta_1^2} \sinh^2 2\theta_0 \cosh^2 \nu[(J_1 + J_2)x + I_1 t], \\
|E_{\rm p}(x,t)|^2 &= \frac{8\kappa^2\nu^2}{\Delta_1^2} \sinh^2 \theta_0 \sinh^2 \nu(J_1 x + I_1 t), \\
|E_{\rm a}(x,t)|^2 &= \frac{\kappa^2\nu^2}{\Delta_1^2} \sinh^2 2\theta_0 \cosh^2 \nu[(J_1 - J_2)x + I_1 t], \\
|Q_{\rm pol}(x,t)|^2 &= \frac{8\kappa^2\nu^2}{\Delta_1^2} \cosh^2 \theta_0 \cosh^2 \nu J_2 x,
\end{aligned}$$

where

$$\kappa = -J_2 I_1, \qquad \Delta_1(x,t) = 2\left(\sinh^2\theta_0 \sinh^2(\nu(J_1 x + I_1 t)) + \cosh^2\theta_0 \cosh^2(\nu J_2 x)\right).$$

Note that the canonical reduction ensures that $\Delta_1(x,t)$ is positive for all x and t, so the solitons of this model can have no singularities.

3.2 Another type of $\mathbb{Z}_2$ reduction

Let us consider the $\mathbb{Z}_2$ reduction

$$\chi^-(x,t,\lambda) = K_2 \left\{[\chi^+(x,t,-\lambda)]^T\right\}^{-1} K_2^{-1}.$$

where $K_2 = \operatorname{diag}(s_1, s_2, 1, s_4, s_5)$ and $s_1 = s_5 = \pm 1$, $s_2 = s_4 = \pm 1$.

Therefore we have the symmetry conditions

$$K_2 U(x,t,-\lambda)^T K_2^{-1} = -U(x,t,\lambda) \quad \Rightarrow \quad Q(x,t) = K_2 Q^T(x,t) K_2^{-1}.$$

In particular if we choose $K_2 = \mathbb{1}$ then $Q(x,t)$ is a symmetric matrix.

The invariance condition implies that the dressing matrix gets the form

$$g(x,t,\lambda) = \mathbb{1} + \frac{A(x,t)}{\lambda - \lambda^+} - \frac{K_2 S A(x,t)(K_2 S)^{-1}}{\lambda + \lambda^+}.$$

Hence here the poles of the dressing factor form a doublet $\{\lambda^+, -\lambda^+\}$ whose residues are related by:

$$B(x,t) = -K_2 S A(x,t)(K_2 S)^{-1}, \qquad \lambda^- = -\lambda^+.$$

The soliton solution is expressed by the residues of the dressing matrix as follows

$$[J, Q(x,t)] = [J, A(x,t) - K_2 S A(x,t) S K_2].$$

Like before one can present the matrix $A(x,t)$ in the following way

$$A(x,t) = X(x,t) F^T(x,t),$$

where $F(x,t) = S\chi_0^+(x,t,\lambda^+) S F_0$. In the soliton case it is just the well known plane wave $F(x,t) = e^{i\lambda^+(Jx+It)} F_0$. The other factor $X(x,t)$ is given by

$$X(x,t) = 2\lambda^+ \left(K_2 F - 2\lambda^+ S F (F^T K_2 F)^{-1}\alpha\right)\left(F^T K_2 F - 4(\lambda^+)^2 \alpha (F^T K_2 F)^{-1}\alpha\right)^{-1}.$$

As a particular case when $\operatorname{rank} X(x,t) = \operatorname{rank} F(x,t) = 1$ and $\alpha(x,t)\equiv 0$ it is simply

$$X(x,t) = \frac{2\lambda^+ K_2 F(x,t)}{F^T(x,t)K_2F(x,t)}, \qquad F^T(x,t)K_2F(x,t) = \sum_{k=1}^{5} s_k e^{2i\lambda^+(J_kx+I_kt)}F_{0,k}^2.$$

Consequently the one-soliton solution reads

$$Q_{ij}(x,t) = \begin{cases} \dfrac{2\lambda^+}{F^T(x,t)K_2F(x,t)}\left((K_2F)_iF_j + (-1)^{i+j+1}F_{6-i}(K_2F)_{6-j}\right), & i\neq j,\\ 0, & i=j. \end{cases} \tag{3.6}$$

In the case of $\mathbf{B}_2$ algebra there are only 4 independent fields as shown below

$$Q(x,t) = \begin{pmatrix} 0 & Q_{10}(x,t) & Q_{11}(x,t) & Q_{12}(x,t) & 0\\ s_1s_2Q_{10}(x,t) & 0 & Q_{01}(x,t) & 0 & Q_{12}(x,t)\\ s_1Q_{11}(x,t) & s_2Q_{01}(x,t) & 0 & Q_{01}(x,t) & -Q_{11}(x,t)\\ s_1s_2Q_{12}(x,t) & 0 & s_2Q_{01}(x,t) & 0 & Q_{10}(x,t)\\ 0 & s_1s_2Q_{12}(x,t) & -s_1Q_{11}(x,t) & s_1s_2Q_{10}(x,t) & 0 \end{pmatrix}.$$

The corresponding 4-wave system reads

$$\begin{aligned} &i(J_1-J_2)Q_{10,t}(x,t) - i(I_1-I_2)Q_{10,x}(x,t) + ks_2Q_{11}(x,t)Q_{01}(x,t) = 0,\\ &iJ_1Q_{11,t}(x,t) - iI_1Q_{11,x}(x,t) + kQ_{01}(x,t)(s_2Q_{12}(x,t) - Q_{10}(x,t)) = 0,\\ &i(J_1+J_2)Q_{12,t}(x,t) - i(I_1+I_2)Q_{12,x}(x,t) - kQ_{11}(x,t)Q_{01}(x,t) = 0,\\ &iJ_2Q_{01,t}(x,t) - iI_2Q_{01,x}(x,t) + ks_1Q_{11}(x,t)(Q_{12}(x,t) + s_2Q_{10}(x,t)) = 0. \end{aligned}$$

Its one-soliton solution when $K_2 = \mathbb{1}$ is presented by the expressions

$$\begin{aligned} Q_{10}(x,t) &= \frac{2\lambda^+}{F^TF}\left(e^{i\lambda^+[(J_1+J_2)x+(I_1+I_2)t]}F_{0,1}F_{0,2} + e^{-i\lambda^+[(J_1+J_2)x+(I_1+I_2)t]}F_{0,4}F_{0,5}\right),\\ Q_{11}(x,t) &= \frac{2\lambda^+F_{0,3}}{F^TF}\left(e^{i\lambda^+(J_1x+I_1t)}F_{0,1} - e^{-i\lambda^+(J_1x+I_1t)}F_{0,5}\right),\\ Q_{12}(x,t) &= \frac{2\lambda^+}{F^TF}\left(e^{i\lambda^+[(J_1-J_2)x+(I_1-I_2)t]}F_{0,1}F_{0,4} + e^{-i\lambda^+[(J_1-J_2)x+(I_1-I_2)t]}F_{0,2}F_{0,5}\right),\\ Q_{01}(x,t) &= \frac{2\lambda^+F_{0,3}}{F^TF}\left(e^{i\lambda^+(J_2x+I_2t)}F_{0,2} + e^{-i\lambda^+(J_2x+I_2t)}F_{0,4}\right), \end{aligned}$$

where $F^T(x,t)F(x,t)$ is obtained from the expression in (3.6) putting $s_k = 1$. After imposing the additional restriction $F_{0,1} = F_{0,5}$ and $F_{0,2} = F_{0,4}$ we get

$$\begin{aligned} Q_{10}(x,t) &= \frac{\lambda^+}{\Delta_2(x,t)}\sinh 2\theta_0 \cos\lambda^+[(J_1+J_2)x + (I_1+I_2)t],\\ Q_{11}(x,t) &= \frac{2\sqrt{2}\,i\lambda^+}{\Delta_2(x,t)}\sinh 2\theta_0 \sin\lambda^+(J_1x + I_1t),\\ Q_{12}(x,t) &= \frac{\lambda^+}{\Delta_2(x,t)}\sinh 2\theta_0 \cos\lambda^+[(J_1-J_2)x + (I_1-I_2)t],\\ Q_{01}(x,t) &= \frac{2\sqrt{2}\lambda^+}{\Delta_2(x,t)}\cosh 2\theta_0 \cos\lambda^+(J_2x + I_2t), \end{aligned}$$

where

$$F_{0,1} = \frac{F_{0,3}}{\sqrt{2}}\sinh\theta_0, \qquad F_{0,2} = \frac{F_{0,3}}{\sqrt{2}}\cosh\theta_0, \tag{3.7}$$

θ_0 is a complex parameter and

$$\Delta_2(x,t) = 2(\cosh^2\theta_0 \cos^2 \lambda^+(J_2x + I_2t) - \sinh^2\theta_0 \sin^2 \lambda^+(J_1x + I_1t)).$$

In its turn the existence of such reduction leads to the existence of a special class of solitons, the so-called breathers, in the case when there are two $\mathbb{Z}_2$ reductions (canonical one and another one of the type mentioned above) applied to the N-wave systems.

4 N-wave equations with a $\mathbb{Z}_2 \times \mathbb{Z}_2$-reduction and their soliton solutions

In this section we consider a $\mathbb{Z}_2 \times \mathbb{Z}_2$-reduced N-wave system associated to the orthogonal algebra. Such systems admit real valued solutions. The soliton solutions can be classified into two major types: "doublet" solitons which are related to two imaginary discrete eigenvalues (a "doublet") of L and solitons which correspond to four eigenvalues (a "quadruplet").

4.1 Doublet solitons

Let the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in the space of fundamental solutions of the linear problem is given by

$$\begin{aligned} \chi^-(x,t,\lambda) &= K_1 \left((\chi^+)^\dagger(x,t,\lambda^*)\right)^{-1} K_1^{-1}, \\ \chi^-(x,t,\lambda) &= K_2 \left((\chi^+)^T(x,t,-\lambda)\right)^{-1} K_2^{-1}, \end{aligned}$$

where $K_{1,2} \in SO(n)$ and $[K_1, K_2] = 0$. Consequently the potential $U(x,t,\lambda)$ satisfies the following symmetry conditions

$$\begin{aligned} &K_1 U^\dagger(x,t,\lambda^*) K_1^{-1} = U(x,t,\lambda), \qquad K_1 J^* K_1^{-1} = J, \\ &K_2 U^T(x,t,-\lambda) K_2^{-1} = -U(x,t,\lambda), \qquad K_2 J K_2^{-1} = J. \end{aligned}$$

In accordance with what we said in previous chapter the dressing factor $g(x,t,\lambda)$ must be invariant under the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e.

$$K_1 \left(g^\dagger(x,t,\lambda^*)\right)^{-1} K_1^{-1} = g(x,t,\lambda), \tag{4.1}$$

$$K_2 \left(g^T(x,t,-\lambda)\right)^{-1} K_2^{-1} = g(x,t,\lambda). \tag{4.2}$$

Let for the sake of simplicity require that $K_1 = K_2 = \mathbb{1}$. As a result we find that the poles of the dressing matrix can be purely imaginary, i.e.

$$\lambda^\pm = \pm i\nu, \qquad \nu > 0.$$

Thus the invariance condition implies that the dressing matrix gets the form

$$g(x,t,\lambda) = \mathbb{1} + \frac{A(x,t)}{\lambda - i\nu} + \frac{SA^*(x,t)S}{\lambda + i\nu}, \qquad A^*(x,t) = -A(x,t).$$

Following already discussed procedures we derive in the simplest case that the explicit form of $A(x,t)$ is

$$A(x,t) = \frac{2i\nu}{F^T(x,t)F(x,t)} F(x,t)F^T(x,t),$$

where the vector $F(x,t) = e^{-\nu(Jx+It)}F_0$ is real. Consequently the soliton solution written in a standard matrix notation is

$$Q_{kl}(x,t) = \begin{cases} 0, & \text{if } k = l, \\ \dfrac{2i\nu}{F^T(x,t)F(x,t)}\big(F_k(x,t)F_l(x,t) \\ \qquad + (-1)^{k+l+1}F_{6-k}(x,t)F_{6-l}(x,t)\big), & \text{if } k \neq l. \end{cases} \tag{4.3}$$

In the case of $\mathbf{B}_2$ algebra we obtain the following $\mathbb{Z}_2 \times \mathbb{Z}_2$-reduced 4-wave system

$$\begin{aligned}
&(J_1 - J_2)q_{10,t}(x,t) - (I_1 - I_2)q_{10,x}(x,t) + kq_{11}(x,t)q_{01}(x,t) = 0, \\
&J_1 q_{11,t}(x,t) - I_1 q_{11,x}(x,t) + k(q_{12}(x,t) - q_{10}(x,t))q_{01}(x,t) = 0, \\
&(J_1 + J_2)q_{12,t}(x,t) - (I_1 + I_2)q_{12,x}(x,t) - kq_{11}(x,t)q_{01}(x,t) = 0, \\
&J_2 q_{01,t}(x,t) - I_2 q_{01,x}(x,t) + k(q_{10}(x,t) + q_{12}(x,t))q_{11}(x,t) = 0,
\end{aligned}$$

where $q_{10}(x,t)$, $q_{11}(x,t)$, $q_{12}(x,t)$ and $q_{01}(x,t)$ are real valued fields and their indices are associated with the basis of simple roots of $\mathbf{B}_2$ introduced in the previous section, i.e.

$$\begin{aligned}
&Q_{10}(x,t) = iq_{10}(x,t), \qquad Q_{11}(x,t) = iq_{11}(x,t), \\
&Q_{12}(x,t) = iq_{12}(x,t), \qquad Q_{01}(x,t) = iq_{01}(x,t).
\end{aligned}$$

The "bonding" constant k coincides with the one in the previous examples

$$k = J_1 I_2 - J_2 I_1.$$

Therefore the solution (4.3) turns into

$$\begin{aligned}
q_{10}(x,t) &= \frac{2\nu}{F^T F}\left(e^{-\nu[(J_1+J_2)x+(I_1+I_2)t]}F_{0,1}F_{0,2} + e^{\nu[(J_1+J_2)x+(I_1+I_2)t]}F_{0,5}F_{0,4}\right), \\
q_{11}(x,t) &= \frac{2\nu}{F^T F}\left(e^{-\nu(J_1x+I_1t)}F_{0,1}F_{0,3} - e^{\nu(J_1x+I_1t)}F_{0,5}F_{0,3}\right), \\
q_{12}(x,t) &= \frac{2\nu}{F^T F}\left(e^{-\nu[(J_1-J_2)x+(I_1-I_2)t]}F_{0,1}F_{0,4} + e^{\nu[(J_1-J_2)x+(I_1-I_2)x]}F_{0,5}F_{0,2}\right), \\
q_{01}(x,t) &= \frac{2\nu}{F^T F}\left(e^{-\nu(J_2x+I_2t)}F_{0,2}F_{0,3} + e^{\nu(J_2x+I_2t)}F_{0,4}F_{0,3}\right).
\end{aligned}$$

These solutions can be rewritten in terms of hyperbolic functions as follows

$$\begin{aligned}
q_{10}(x,t) &= \frac{4\nu}{F^T F}N_1N_2\cosh\{\nu[(J_1+J_2)x + (I_1+I_2)t] + \delta_1 + \delta_2\}, \\
q_{11}(x,t) &= -\frac{4\nu}{F^T F}N_1F_{0,3}\sinh[\nu(J_1x + I_1t) + \delta_1], \\
q_{12}(x,t) &= \frac{4\nu}{F^T F}N_1N_2\cosh\{\nu[(J_1-J_2)x + (I_1-I_2)t] + \delta_1 - \delta_2\}, \\
q_{01}(x,t) &= \frac{4\nu}{F^T F}N_2F_{0,3}\cosh[\nu(J_2x + I_2t) + \delta_2], \\
F^T(x,t)F(x,t) &= 2N_1^2\cosh 2(\nu(J_1x+I_1t)+\delta_1) + 2N_2^2\cosh 2(\nu(J_2x+I_2t)+\delta_2) + F_{0,3}^2,
\end{aligned}$$

where we have implied that $F_{0,k} > 0$ for $k = 1, 2, 4, 5$ and therefore the following expressions

$$\delta_1 = \frac{1}{2}\ln\frac{F_{0,5}}{F_{0,1}}, \qquad \delta_2 = \frac{1}{2}\ln\frac{F_{0,4}}{F_{0,2}}, \qquad N_1 = \sqrt{F_{0,1}F_{0,5}}, \qquad N_2 = \sqrt{F_{0,2}F_{0,4}}$$

make sense.

In particular when $F_{0,1} = F_{0,5}$ and $F_{0,2} = F_{0,4}$ or in other words $\delta_1 = \delta_2 = 0$ we can apply the same representation as in (3.7) and as a result we obtain

$$\begin{aligned}
q_{10}(x,t) &= \frac{\nu}{\Delta_D} \sinh(2\theta_0) \cosh \nu[(J_1 + J_2)x + (I_1 + I_2)t], \\
q_{11}(x,t) &= -\frac{2\sqrt{2}\nu}{\Delta_D} \sinh \theta_0 \sinh \nu(J_1 x + I_1 t), \\
q_{12}(x,t) &= \frac{\nu}{\Delta_D} \sinh(2\theta_0) \cosh \nu[(J_1 - J_2)x + (I_1 - I_2)t], \\
q_{01}(x,t) &= \frac{2\sqrt{2}\nu}{\Delta_D} \cosh \theta_0 \cosh \nu(J_2 x + I_2 t),
\end{aligned}$$

where $\theta_0 \in \mathbb{R}$ and

$$\Delta_D(x,t) = 2\left(\sinh^2\theta_0 \sinh^2 \nu(J_1 x + I_1 t) + \cosh^2\theta_0 \cosh^2 \nu(J_2 x + I_2 t)\right).$$

4.2 Quadruplet solitons

There is another way to ensure the $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance of the dressing factor. This time we consider dressing factors $g(x,t,\lambda)$ with two more poles. The requirements (4.1)–(4.2) lead to the following dressing matrix

$$\begin{aligned}
g(x,t,\lambda) = \mathbb{1} &+ \frac{A(x,t)}{\lambda - \lambda^+} + \frac{K_1 S A^*(x,t)(K_1 S)^{-1}}{\lambda - (\lambda^+)^*} - \frac{K_2 S A(x,t)(K_2 S)^{-1}}{\lambda + \lambda^+} \\
&- \frac{K_1 K_2 A^*(x,t)(K_1 K_2)^{-1}}{\lambda + (\lambda^+)^*}.
\end{aligned} \tag{4.4}$$

By taking the limit $\lambda \to \infty$ in equation (2.5) and taking into account the explicit formula (4.4) one can derive in the soliton case $Q_0(x,t) \equiv 0$ the following relation

$$[J,Q](x,t) = [J, A + K_1 S A^* S K_1 - K_2 S A S K_2 - K_1 K_2 A^* K_2 K_1](x,t). \tag{4.5}$$

Like in previous considerations we decompose the matrix $A(x,t)$ using two matrix factors $X(x,t)$ and $F(x,t)$ and derive some differential equation for $F(x,t)$ which leads to

$$F(x,t) = e^{i\lambda^+(Jx+It)} F_0.$$

The linear system for $X(x,t)$ in this case is following

$$\begin{aligned}
&\left(\mathbb{1} - \frac{K_2 S A(x,t) S K_2}{2\lambda^+} + \frac{K_1 S A^*(x,t) S K_1}{2i\nu} - \frac{K_1 K_2 A^*(x,t) K_2 K_1}{2\mu}\right) S F(x,t) \\
&\qquad = X(x,t)\alpha(x,t),
\end{aligned} \tag{4.6}$$

where $\alpha(x,t)$ is a linear function of x and t as follows

$$\alpha(x,t) = i F_0^T (Jx + It) S F_0.$$

Starting from the equation (4.6) by multiplying with $K_1 S$, $K_2 S$ and performing complex conjugation if necessary, we can derive the following auxiliary linear system

$$\begin{aligned}
SF(x,t) &= X\alpha + Y\frac{(G,F)}{2\lambda^+} - Z\frac{(H,F)}{2i\nu} + W\frac{(N,F)}{2\mu}, \\
SG(x,t) &= X\frac{(F,G)}{2\lambda^+} + Y\alpha + Z\frac{(H,G)}{2\mu} - W\frac{(N,G)}{2i\nu},
\end{aligned}$$

$$SH(x,t) = X\frac{(F,H)}{2i\nu} + Y\frac{(G,H)}{2\mu} + Z\alpha^* + W\frac{(N,H)}{2(\lambda^+)^*},$$
$$SN(x,t) = X\frac{(F,N)}{2\mu} + Y\frac{(G,N)}{2i\nu} + Z\frac{(H,N)}{2(\lambda^+)^*} + W\alpha^*,$$

where we introduced auxiliary entities

$$\begin{aligned} &Y(x,t) = K_2SX(x,t), && Z(x,t) = K_1SX^*(x,t), && W(x,t) = K_1K_2X^*(x,t),\\ &G(x,t) = K_2SF(x,t), && H(x,t) = K_1SF^*(x,t), && N(x,t) = K_1K_2F^*(x,t),\\ &(F,H) = F^TSH. \end{aligned}$$

In matrix notations this system reads

$$\left(SF,\ \ SG,\ \ SH,\ \ SN \right) = \left(X,\ \ Y,\ \ Z,\ \ W \right) \begin{pmatrix} \alpha & a & b & c \\ a & \alpha & c & b \\ b^* & c^* & \alpha^* & a^* \\ c^* & b^* & a^* & \alpha^* \end{pmatrix},$$

where

$$a(x,t) = \frac{(F(x,t),G(x,t))}{2\lambda^+}, \qquad b(x,t) = \frac{(F(x,t),H(x,t))}{2i\nu}, \qquad c(x,t) = \frac{(F(x,t),N(x,t))}{2\mu}.$$

To calculate $X(x,t)$ we just have to find the inverse matrix of the block matrix shown above. In the simplest case when $\mathrm{rank}X(x,t) = \mathrm{rank}F(x,t) = 1$ and $\alpha \equiv 0$ we have

$$\begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} = \frac{1}{\Delta(x,t)} \begin{pmatrix} 0 & a^* & b & -c \\ a^* & 0 & -c & b \\ -b & -c & 0 & a \\ -c & -b & a & 0 \end{pmatrix} \begin{pmatrix} SF \\ SG \\ SH \\ SN \end{pmatrix},$$

where

$$\Delta(x,t) = |a(x,t)|^2 + b^2(x,t) - c^2.$$

Finally putting the result for $X(x,t)$ in (4.5) we obtain the quadruplet solution

$$\begin{aligned} Q(x,t) = \frac{1}{\Delta} \big[&\left(a^*(x,t)K_2F(x,t) + b(x,t)K_1F^*(x,t) - c(x,t)K_1K_2SF^*(x,t)\right) F^T(x,t) \\ &- K_2S\left(a^*(x,t)K_2F(x,t) + b(x,t)K_1F^*(x,t) - c(x,t)K_1K_2SF^*(x,t)\right) F^T(x,t)SK_2 \\ &+ K_1S\left(a(x,t)K_2F^*(x,t) - b(x,t)K_1F(x,t) - c(x,t)K_1K_2SF(x,t)\right) F^\dagger(x,t)SK_1 \\ &- K_1K_2\left(a(x,t)K_2F^*(x,t) - b(x,t)K_1F(x,t) - c(x,t)K_1K_2SF(x,t)\right) F^\dagger(x,t)K_2K_1 \big]. \end{aligned}$$

Consider the 4-wave system associated with the $\mathbf{B}_2$ algebra. Let $K_1 = K_2 = \mathbb{1}$ then its generic quadruplet (or breather-like) solution is

$$q_{10}(x,t) = \frac{4}{\Delta}\mathrm{Im}\left[a^*N_1\cosh(\varphi_1+\varphi_2) - \frac{imN_1^*}{\mu\nu}\left(\mu\cosh(\varphi_1^*+\varphi_2) - i\nu\cosh(\varphi_1^*-\varphi_2)\right)\right]N_2,$$
$$q_{11}(x,t) = \frac{4}{\Delta}\mathrm{Im}\left[a^*N_1\sinh(\varphi_1) - \frac{im\lambda^+}{\mu\nu}N_1^*\sinh(\varphi_1^*)\right]F_{0,3},$$
$$q_{12}(x,t) = \frac{4}{\Delta}\mathrm{Im}\left[a^*N_1\cosh(\varphi_1-\varphi_2) - \frac{imN_1^*}{\mu\nu}\left(\mu\cosh(\varphi_1^*-\varphi_2) - i\nu\cosh(\varphi_1^*+\varphi_2)\right)\right]N_2,$$

$$q_{01}(x,t) = \frac{4}{\Delta}\mathrm{Im}\left[a^* N_2 \cosh(\varphi_2) - \frac{im\lambda^{+*}}{\mu\nu} N_2^* \cosh(\varphi_2^*)\right] F_{0,3},$$

where

$$a(x,t) = \frac{1}{\mu + i\nu}\left[N_1^2 \cosh 2\varphi_1 + N_2^2 \cosh 2\varphi_2 + \frac{F_{0,3}^2}{2}\right],$$

$$b(x,t) = \frac{m(x,t)}{i\nu}, \qquad c(x,t) = \frac{m(x,t)}{\mu},$$

$$m(x,t) = |N_1|^2 \cosh(2\mathrm{Re}\ \varphi_1) + |N_2|^2 \cosh(2\mathrm{Re}\ \varphi_2) + \frac{|F_{0,3}|^2}{2}, \qquad N_\sigma = \sqrt{F_{0,\sigma} F_{0,6-\sigma}},$$

$$\varphi_\sigma(x,t) = i\lambda^+(J_\sigma x + I_\sigma t) + \frac{1}{2}\log\frac{F_{0,\sigma}}{F_{0,6-\sigma}}, \qquad \sigma = 1, 2.$$

5 Conclusion

New N-wave type equations related to the orthogonal algebras and obtained from the generic ones via additional reductions are analyzed. These new nonlinear equations are solvable by the inverse scattering method. In particular, the systems related to the $\mathfrak{so}(5)$ algebra involve 4 waves. Imposing $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ reductions on them we obtain 4-wave systems of physical importance. The 4-wave system with a canonical $\mathbb{Z}_2$ reduction describes Stokes-anti-Stokes Raman scattering. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-reduced N wave system possesses real valued solutions while other types of N wave equations have complex valued solutions. This determines its mathematical importance.

The soliton solutions of these integrable systems are parametrized by two types of parameters: by the discrete eigenvalues of L and by the "polarization" vector F_0. Therefore the problem of classifying all types of one-soliton solutions of the N-wave equations is equivalent to that of classifying all possible types of discrete eigenvalues and polarization vectors. When both reductions are applied we have two different configurations of eigenvalues for the Lax operator L: doublets and quadruplets. This situation is analogous to one encountered in the sine-Gordon case, which was the reason to call quadruplet solitons 'breather'-like solutions [12].

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