The 6 Vertex Model and Schubert Polynomials*

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Abstract. We enumerate staircases with fixed left and right columns. These objects correspond to ice-configurations, or alternating sign matrices, with fixed top and bottom parts. The resulting partition functions are equal, up to a normalization factor, to some Schubert polynomials.

Key words: alternating sign matrices; Young tableaux; staircases; Schubert polynomials; integrable systems

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To the memory of Vadim Kuznetsov

1 Introduction

The 6 vertex model is supposed to tell something about ice, though not of the kind used in cold drinks. One has a planar array filled with water molecules having 6 possible types of orientation. One chooses 6 functions of the two coordinates. Each molecule is weighted by one these functions, according to its type. The weight of a given 'square ice' is the product of the weights of the molecules composing it.

In our case, the weights will be x_i/y_j, or $x_i/y_j - 1$ for two types of molecules, i, j being the coordinates, the other four types having weight 1.

There are many other combinatorial objects equivalent to square-ice configurations: alternating sign matrices (ASM), totally symmetric self-complementary plane partitions(TSSCPP), monotone triangles, staircases (subfamily of Young tableaux). We choose this last object, because it is the most compact, and because we are going to relate the combinatorics of weights to the usual combinatorics of tableaux. We shall refer to Bressoud [1] for a description of the different approaches to square-ice configurations.

There is a huge literature concerned with the enumeration of plane partitions, and of different related combinatorial objects, starting from the work of MacMahon (the book of Bressoud [1] provides many references). We are only concerned here with the enumeration of ASM's, which started with the work of Robbins and Rumsey [15, 17], with notable contributions of Zeilberger [18] and Kuperberg [8].

One can put different weights on ice configurations or ASM. The weight chosen by physicists allows, by specialization, the enumeration of ASM, after the evaluation of the Izergin–Korepin determinant [6]. Kirillov and Smirnov [7] obtained determinants which correspond to more general partition functions. Gaudin [3, Appendix B, p.72] had previously given for the Bethe model a determinant similar to the Izergin–Korepin determinant.

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Okada [16], Hamel–King [4, 5] obtained sums over sets of ASM, with another weight involving one coordinate only. Their formulas generalize the interpretation of a Schur function as a sum of Young tableaux of a given shape.

The weights that we have chosen involve the two coordinates, and generalize the *inversions* of a permutation; we already used them to obtain *Grothendieck polynomials* from sets of ASM [10]. The sets that we take in this text are simpler. They are sets of ASM having top and bottom part fixed (the case where one fixes the top row an ASM already appears in [15, 17]). For these sets the 'partition function' is proportional to some Schubert polynomial, or to some determinant generalizing Schur functions.

One remarkable feature of the functions that we obtain is a symmetry property in the x-variables. This property is different from the symmetry related to the Yang-Baxter equation that the Izergin–Korepin determinant displays (see also [9]).

The weight that we have chosen cannot be specialized to 1 (and thus, does not allow plain enumeration of ASM). On the other hand, specializing all the x_i's to 2, and y_i's to 1 amounts to weigh an ASM by 2^k, where k is the number of -1 entries. The '2'-enumeration of all ASM, or to ASM having a top row fixed, is due to [15], our study gives it for the ASM having fixed top and bottom parts of any size.

2 Reminder

Divided differences ∂_σ are operators on polynomials in $x_1, x_2, \ldots$ indexed by permutations σ. They are products of *Newton divided differences* $\partial_i := \partial_{s_i}$ corresponding to the case of simple transpositions s_i, and acting as

$$\partial_i \colon f \to \frac{f - f^{s_i}}{x_i - x_{i+1}}$$

denoting by f^{s_i} the image of f under the exchange of x_i and x_{i+1}.

In the case where $\sigma = \omega := [n, \ldots, 1]$ is the maximal permutation of the symmetric group $\mathfrak{S}_n$ on n letters, then ∂_ω, apart from being expressible as a product of ∂_i's, can also be written

$$f \to \sum_{w \in \mathfrak{S}_n} (-1)^{\ell(w))} f^w \prod_{1 \le i < j \le n} (x_i - x_j)^{-1},$$

where $\ell(w)$ is the *length* of w.

The *code* $\mathfrak{c}(\sigma)$ of the permutation σ is the sequence of numbers $c_i := \#(j \colon j > i, \sigma_j < \sigma_i)$.

Schubert polynomials are polynomials in two sets of variables $\mathbf{x}$, $\mathbf{y}$ indexed by permutations or by their codes. We use the symbol $\mathbb{X}$ in the first case, $\mathbb{Y}$ in the second:

$$\mathbb{X}_\sigma(\mathbf{x}, \mathbf{y}) = \mathbb{Y}_{\mathfrak{c}(\sigma)}(\mathbf{x}, \mathbf{y}).$$

A *dominant permutation* is a permutation with weakly decreasing code. A *Graßmannian permutation* is a permutation with only one descent, i.e. such that there exists r:

$$\sigma_1 < \cdots < \sigma_r, \quad \sigma_{r+1} < \sigma_{r+2}, \quad \ldots$$

(r is the *descent* of σ).

Dominant Schubert polynomials $\mathbb{Y}_v(\mathbf{x}, \mathbf{y})$, (for $v_1 \ge v_2 \ge v_3 \ge \cdots$), are equal to

$$\mathbb{Y}_v(\mathbf{x}, \mathbf{y}) := \prod_i \prod_{j=1}^{v_i} (x_i - y_j).$$

General Schubert polynomials are by definition all the non-zero images of dominant Schubert polynomials under divided differences [11, 13].

The Graßmannian Schubert polynomials $\mathbb{X}_\sigma(\mathbf{x}, \mathbf{y})$, σ having descent in r, are the images under ∂_{ω_r} of the dominant polynomials $\mathbb{Y}_v$, $v \in \mathbb{N}^r$. They have a determinantal expression. Let $v = [\sigma_1 - 1, \ldots, \sigma_r - 1]$. Then

$$\mathbb{X}_\sigma(\mathbf{x}, \mathbf{y}) = S_{v_1-0,\ldots,v_r-r+1}(\mathbf{x}^r - \mathbf{y}^{v_1}, \ldots, \mathbf{x}^r - \mathbf{y}^{v_r}) = \det\left|S_{v_j+1-i}(\mathbf{x}^r - \mathbf{y}^{v_j})\right|_{i,j=1,\ldots,r},$$

where $\mathbf{x}^r = [x_1, \ldots, x_r]$, $\mathbf{y}^k = [y_1, \ldots, y_k]$, and the *complete function* $S_m(\mathbf{x}^r - \mathbf{y}^k)$ is defined as the coefficient of z^m in

$$\prod_1^k (1 - zy_i) \prod_1^r (1 - zx_i)^{-1}.$$

Multiplication of such a polynomial by $x_1 \cdots x_r$ is easy, starting from the appropriate dominant polynomial. It reduces to a uniform increase of the index of the Schur function:

$$x_1 \cdots x_r \mathbb{X}_\sigma(\mathbf{x}, \mathbf{y}) = S_{v_1+1,\ldots,v_r-r+2}(\mathbf{x}^r - \mathbf{y}^{v_1}, \ldots, \mathbf{x}^r - \mathbf{y}^{v_r}). \tag{1}$$

The Graßmannian Schubert polynomials specialized to $\mathbf{y} = \mathbf{0}$ are exactly the Schur functions. When specialized to $\mathbf{y} = [0, 1, 2, \ldots]$, they coincide with the *factorial Schur functions*.

We shall need more general Schur-like functions.

Given $u, v \in \mathbb{N}^n$, and $2n + 1$ alphabets $\mathbf{z}, \mathbf{y}_1, \ldots, \mathbf{y}_n, \mathbf{x}_1, \ldots, \mathbf{x}_n$, then

$$S_{v/u}(\mathbf{z} - \mathbf{y}_1, \ldots, \mathbf{z} - \mathbf{y}_n // \mathbf{x}_1, \ldots, \mathbf{x}_n) := \det\left|S_{v_j - u_i + j - i}(\mathbf{z} + \mathbf{x}_i - \mathbf{y}_j)\right|.$$

Notice that if $\mathbf{x}_n, \mathbf{x}_{n-1}, \ldots, \mathbf{x}_1$ are of cardinality majorized respectively by $0, 1, \ldots, n - 1$, then [11, Lemma 1.4.1]

$$\begin{aligned} S_{v/u}(\mathbf{z} - \mathbf{y}_1, \ldots, \mathbf{z} - \mathbf{y}_n // \mathbf{x}_1, \ldots, \mathbf{x}_n) &= S_{v/u}(\mathbf{z} - \mathbf{y}_1, \ldots, \mathbf{z} - \mathbf{y}_n) \\ &:= S_{v/u}(\mathbf{z} - \mathbf{y}_1, \ldots, \mathbf{z} - \mathbf{y}_n // \mathbf{0}, \ldots, \mathbf{0}). \end{aligned} \tag{2}$$

3 Staircases

A *column* u is a strictly decreasing sequence of integers, its length is denoted $\ell(u)$. A *staircase* is a sequence of columns of lengths $k, k - 1, \ldots, k - r$, such that, writing them in the Cartesian plane as a Young tableau, aligning their bottoms, then rows are weakly increasing, diagonals are weakly decreasing (this last condition is added to the usual definition of a Young tableau, *staircases are special Young tableaux*):

$$\begin{array}{|c|}\hline c \\ \hline a \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline c & \multicolumn{1}{|c}{} \\ \hline \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{b} \\ \cline{2-2} \end{array} \qquad \textit{example} \qquad \begin{array}{|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 5 & 6 \\ \hline 2 & 4 & 5 \\ \hline 1 & 1 & 2 \\ \hline \end{array}$$

conditions

with $c > a$, $a \le b \le c$.

Staircases are fundamental in the description of the Ehresmann–Bruhat order of the symmetric group [12].

Given two columns u, v with $\ell(u) > \ell(v)$, let $\mathcal{E}(u, v)$ be the set of all staircases with first column u, last column v. We write $\mathcal{E}(n, v)$ when $u = [n, \ldots, 1]$.

Given a staircase written in the plane as a tableau t, with entries denoted $t[i, j]$, we give a weight to each entry of t as follows:

- $t[i,j] = b$ has weight $x_i y^{-b} - 1$ if $t[i-1,j] = b$;
- $t[i,j] = b$ has weight $x_i y^{-b}$ if $t[i-1,j] < b < t[i-1,j+1]$;
- in all other cases, $t[i,j]$ has weight 1;
- the weight of the staircase is the product of all these elementary weights.

In other words, entries in the first column have weight 1, and there are three possible configurations for the other entries, according to their left neighbors, with $a < b < c$:

$$\begin{array}{lcccccc} & c & & c & & b & \\ & b & b & a & b & a & b \\ \textit{weight} & \frac{x_i}{y_b} - 1 & & \frac{x_i}{y_b} & & 1 & \end{array}$$

Given two columns u, v, the sum of weights of the staircases in $\mathcal{E}(u,v)$ (resp. $\mathcal{E}(n,v)$) is denoted $F(u,v)$ (resp. $F(n,v)$). We call it the *partition function* of the set of staircases. When taking the alphabet $z_1, z_2, \ldots$ instead of $x_1, x_2, \ldots$, we write $F(u,v; z_1, z_2, \ldots)$.

Given an ASM, one builds a new matrix by replacing each row by the sum of all rows above. The successive columns of the staircase record the positions of the 1's in the successive rows of this new matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{array}{|c|c|c|c|c|c|} \hline 6 \\ \cline{1-1} 5 & 6 \\ \cline{1-2} 4 & 5 & 6 \\ \cline{1-3} 3 & 4 & 5 & 6 \\ \cline{1-4} 2 & 2 & 3 & 4 & 5 \\ \cline{1-5} 1 & 1 & 1 & 2 & 3 & 4 \\ \hline \end{array} \,.$$

Of course, ASM having the same first (resp. last) k rows correspond to staircases having the same k right (resp. left) columns. Moreover, given a staircase in the letters $1, \ldots, n$, one can complete it in a canonical manner into a staircase with columns of lengths $n, \ldots, 1$. The partition function of a set of staircases having two fixed columns u, v factorizes into $F(n,u)F(u,v)F(v,[\,])$. The middle part can be interpreted as the partition function (Theorem 2) of a set of ASM having a fixed top and bottom, up to a factor due to the fixed top and bottom parts.

Instead of staircases, one can use *ribbon tableaux* (a *ribbon* is a skew diagram which does not contain a 2×2 sub-diagram, a *ribbon tableau* is an increasing sequence of diagrams of partitions which differ by a ribbon). Indeed, given n and a column u of length $n-k$ such that $u_1 \leq n$, let $\widetilde{u} = \{1, \ldots, n\} \setminus u$ (sorted increasingly). Define

$$\mathfrak{p}(u,n) := [\widetilde{u}_1 - 1, \ldots, \widetilde{u}_k - k].$$

It is immediate to translate the condition that a sequence of columns is a staircase in terms of diagrams of partitions.

Lemma 1. *Given n, a column u, $k = n - \ell(u)$. Then the map*

$$v \to \mu = \mathfrak{p}(v,n)$$

is a bijection between the set of columns v, $v_1 \leq n$, such that vu is a staircase and the set of ribbons $\big(\mathfrak{p}(u,n) + 1^k\big)/\mu$, $\mu \in \mathbb{N}^{k-1}$.

For example, for $u = [5,3,2]$, $n = 6$, then

$$\mathfrak{p}(u,6) = [1-1, 4-2, 6-3] = [0,2,3],$$

hence $\mathfrak{p}(u,6) + 1^3 = [1,3,4]$. There are eleven ribbons $[1,3,4]/\mu$, $\mu \in \mathbb{N}^2$, in bijection with the staircases $[v,u]$:

$$\begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 3 & 5 \\ \hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \begin{array}{cccc} \heartsuit \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \hline 3 & 5 \\ \hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \begin{array}{cccc} \heartsuit \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \heartsuit \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 4 & 5 \\ \hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \begin{array}{cccc} \heartsuit \\ \cdot & \cdot & \heartsuit \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

$$\begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \hline 4 & 5 \\ \hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \begin{array}{cccc} \heartsuit \\ \cdot & \cdot & \heartsuit \\ \cdot & \cdot & \cdot & \heartsuit \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 4 & 5 \\ \hline 3 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \begin{array}{cccc} \heartsuit \\ \cdot & \heartsuit & \heartsuit \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \hline 4 & 5 \\ \hline 3 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \begin{array}{cccc} \heartsuit \\ \cdot & \heartsuit & \heartsuit \\ \cdot & \cdot & \cdot & \heartsuit \end{array}$$

$$\begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 4 & 5 \\ \hline 3 & 3 \\ \hline 2 & 2 \\ \hline \end{array} \begin{array}{cccc} \heartsuit \\ \heartsuit & \heartsuit & \heartsuit \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \hline 4 & 5 \\ \hline 3 & 3 \\ \hline 2 & 2 \\ \hline \end{array} \begin{array}{cccc} \heartsuit \\ \heartsuit & \heartsuit & \heartsuit \\ \cdot & \cdot & \cdot & \heartsuit \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \hline 5 & 5 \\ \hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \begin{array}{cccc} \heartsuit \\ \cdot & \cdot & \heartsuit \\ \cdot & \cdot & \heartsuit & \heartsuit \end{array}$$

$$\begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \hline 5 & 5 \\ \hline 3 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \begin{array}{cccc} \heartsuit \\ \cdot & \heartsuit & \heartsuit \\ \cdot & \cdot & \heartsuit & \heartsuit \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \hline 5 & 5 \\ \hline 3 & 3 \\ \hline 2 & 2 \\ \hline \end{array} \begin{array}{cccc} \heartsuit \\ \heartsuit & \heartsuit & \heartsuit & \cdot \\ \cdot & \cdot & \heartsuit & \heartsuit \end{array}$$

We shall now give a weight to a ribbon ζ/μ. Number boxes of the diagram of ζ uniformly in each diagonal, by $1, 2, 3, \ldots$, starting from the top leftmost box. For each box $\square$ of ζ/μ, denote $c(\square)$ this number (this is a shifted *content* [14, p. 11]). A box of a ribbon is *terminal* if it is the rightmost in its row.

Given two partitions ζ, μ such that ζ/μ is a ribbon, we shall weigh the ribbon by giving a weight to each of its boxes as follows:

- a box $\square$ which is not terminal is weighted $x - y_{c(\square)}$;
- a box which is terminal has weight $y_{c(\square)}$ if it is above another box, or weight x if not;
- $\theta(\zeta/\mu)$ is the product of these elementary weights (this is a polynomial in $x, y_1, y_2, \ldots$).

For example, for $\zeta = [3,3,3,5,5]$, $\mu = [1,2,2,3,5]$, one has the following weights:

$$\begin{array}{ccccc} \cdot & x-y_2 & y_3 \\ \cdot & \cdot & y_4 \\ \cdot & \cdot & x \\ \cdot & \cdot & \cdot & x-y_7 & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

The link between the weight of ribbons and the weight of staircases will appear in the proof of the next theorem.

4 Right truncated staircases

Given a column $u = [u_1, \ldots, u_r]$, write the numbers $u_1, \ldots, 1$ inside a ribbon, passing to a new level for each value belonging to u, u_1 being at level 0. The sequence of levels of $1, 2, \ldots, u_1$ is denoted $\langle u \rangle$.

$$u = [\mathbf{5}, \mathbf{3}, \mathbf{2}] \quad \Rightarrow \begin{array}{|c|c|} \hline \mathbf{5} & \multicolumn{1}{c}{} \\ \hline 4 & \mathbf{3} \\ \hline \multicolumn{1}{c|}{} & \mathbf{2} \\ \cline{2-2} \multicolumn{1}{c|}{} & 1 \\ \cline{2-2} \end{array} \quad \Rightarrow \begin{array}{|c|c|} \hline 0 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2} \multicolumn{1}{c|}{} & 3 \\ \cline{2-2} \end{array} \quad \Rightarrow \langle u \rangle = [3, 2, 1, 1, 0].$$

Theorem 1. *Let n be a positive integer, let u be a column of length $n - r$, such that $u_1 \le n$. Then $F(n, u) = x^{\rho_r} y^{-\langle \widetilde{u} \rangle} \mathbb{X}_{\widetilde{u}, u^\omega}(\mathbf{x}, \mathbf{y})$, with $u^\omega = [u_{n-r}, \ldots, u_1]$, $\widetilde{u}$ the (increasing) complement of u in $[1, \ldots, n]$, and $\rho_r = [r-1, \ldots, 1, 0]$.*

Proof. We shall decompose the set $\mathcal{E}(n, u)$ according to the ante-penultimate column v, assuming the theorem true for columns of length $n - r + 1$. This translates into the equality

$$F(n, u; \mathbf{x}^r) = \sum_v F(n, v; \mathbf{x}^{r-1}) F(v, u; x),$$

where the sum is over all columns v of length $n - r + 1$, writing x for x_r.

In terms of Schubert polynomials, one therefore is reduced to show that

$$(x_1 \cdots x_{r-1}) y^{-\langle \widetilde{u} \rangle} \mathbb{X}_{\widetilde{u}, u^\omega}(\mathbf{x}^r, \mathbf{y}) = \sum_v y^{-\langle \widetilde{v} \rangle} \mathbb{X}_{\widetilde{v}, u^\omega}(\mathbf{x}^{r-1}, \mathbf{y}) F(v, u; x).$$

Proposition 1 gives the expansion of $(x_1 \cdots x_r) \mathbb{X}_{\widetilde{u}, u^\omega}(\mathbf{x}^r, \mathbf{y})$, the coefficients being weights of ribbons. We have therefore to compare these weights, multiplied by a monomial in y, and divided by $x = x_r$, to the appropriate $F(v, u; x)$ to be able to conclude.

In more details, given v, u, then $\zeta = [\widetilde{u}_1, \widetilde{u}_2 - 1, \ldots, \widetilde{u}_{n-r} - n + r + 1]$, $\mu = [\widetilde{v}_1 - 1, \ldots, \widetilde{v}_{n-r-1} - n + r + 1]$, and the ribbon is ζ / μ. One notices that $y^{\langle v \rangle} / y^{\langle u \rangle} = y^{\langle \widetilde{u} \rangle} / y^{\langle \widetilde{v} \rangle}$ is a monomial without multiplicity. The i's such that y_i appear in it are the contents of the boxes which are not terminal boxes at the bottom of their column. But these values are exactly those which correspond to a factor x / y_i or $(x / y_i - 1)$ in $F(v, u; x)$. Moreover, it is clear that the contents of boxes which are not terminal boxes are the numbers which appear in u and v at the same level. Therefore the expansion of the Schubert polynomial $\mathbb{X}_{\widetilde{u}, u^\omega}(\mathbf{x}^r, \mathbf{y})$, multiplied by $x_1 \cdots x_{r-1} x$, and the expansion of $F(n, u)$ according to the last variable $x = x_r$ coincide, once normalized, and the theorem is proved. ■

For example, for $v = [16, 15, 12, 10, 9, 5, 3, 2, 1]$, $u = [15, 14, 10, 9, 5, 4, 3, 1]$, and $n = 17$, one has $\widetilde{v} = [4, 6, 7, 8, 11, 13, 14, 17]$, $\mu = [3, 4, 4, 4, 6, 7, 7, 9]$, $\widetilde{u} = [2, 6, 7, 8, 11, 12, 13, 17, 17]$, $\zeta = [2, 5, 5, 5, 7, 7, 7, 9, 9]$. Moreover, $\langle v \rangle = [8, 7, 6, 6, 5, 5, 5, 5, 4, 3, 3, 2, 2, 2, 1]$, $\langle u \rangle = [7, 7, 6, 5, 4, 4, 4, 4, 3, 2, 2, 2, 2, 1]$ and $y^{\langle v \rangle - \langle u \rangle} = y_1 y_4 y_5 y_6 y_7 y_8 y_9 y_{10} y_{11} y_{14} y_{15}$.

$$\begin{array}{|c|c|} \hline 16 & \multicolumn{1}{c}{} \\ \hline 15 & 15 \\ \hline 12 & 14 \\ \hline 10 & 10 \\ \hline 9 & 9 \\ \hline 5 & 5 \\ \hline 3 & 4 \\ \hline 2 & 3 \\ \hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{ccccccccc} \boxed{1} & x & & & & & & & \\ \cdot & \cdot & \cdot & \boxed{5} & 6 & & & & \\ \cdot & \cdot & \cdot & \cdot & 7 & & & & \\ \cdot & \cdot & \cdot & \cdot & 8 & & & & \\ \cdot & \cdot & \cdot & \cdot & \boxed{9} & \boxed{10} & 11 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \boxed{15} & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

The weights are represented, writing i for xy_i^{-1} and $\boxed{i}$ for $xy_i^{-1}-1$.

As a special case of the theorem, one can filter the complete staircases according to their column of length 1.

Corollary 1. *Let n, b be two positive integers, $b \le n$. Then*

$$F(n,[b]) = x^{\rho} y^{-\rho} y^{-[0^{b-1},1^{n-b}]} \mathbb{Y}_{[0^{b-1},1^{n-b}]}(\mathbf{x},\mathbf{y}),$$

with $\rho = [n-2,\ldots,0]$, $\mathbf{x} = [x_1,\ldots,x_{n-1}]$.

The Schubert polynomials appearing in the corollary specialize, for $\mathbf{y}=\mathbf{0}$, to the elementary symmetric functions in $x_1,\ldots,x_{n-1}$.

For example, for $n=3$, 2 staircases contribute to $F(3,[1]) = x_1/y_1 y^{-[1,1]}\mathbb{Y}_{11}(\mathbf{x},\mathbf{y})$, 3 staircases contribute to $F(3,[2]) = x_1/y_1 y^{-[0,1]}\mathbb{Y}_{01}(\mathbf{x},\mathbf{y})$, and the last two contribute to $F(3,[3]) = x_1/y_1 y^{-[0,0]}\mathbb{Y}_{00}(\mathbf{x},\mathbf{y})$.

In our opinion, the most fundamental property shown by the above theorem is the symmetry, in the variables x_i's, of the function $F(n,u)x^{-\rho}$.

5 General staircases

Let n, k, r be three positive integers, and v be a column of length $n-k-r \ge 0$. Put $\mathbf{x} = [x_1,\ldots,x_k]$, $\mathbf{z} = [x_{k+1},\ldots,x_{k+r}]$. Then

$$F(n,v;\mathbf{x},\mathbf{z}) = x^{\rho_k}(x_1\cdots x_k)^r z^{\rho_r} y^{-\langle\widetilde{u}\langle} \mathbb{X}_{\widetilde{u},u^{\omega}}(\mathbf{x},\mathbf{y}) = \sum_u F(n,u;\mathbf{x})F(u,v;\mathbf{z}),$$

sum over all columns u of length $n-k$.

Since Schubert polynomials in $\mathbf{x}$ are linearly independent (and so are their products by a fixed x^{ρ}), then $F(u,v;\mathbf{z})$ can be characterized as the coefficient of $\mathbb{X}_{\widetilde{u},u^{\omega}}(\mathbf{x},\mathbf{y})$ in

$$x^{-\rho_k}F(n,v;\mathbf{x},\mathbf{z})y^{\langle\widetilde{v}\rangle} = (x_1\cdots x_k)^r z^{\rho_r}\mathbb{X}_{\widetilde{v},v^{\omega}}(\mathbf{x},\mathbf{z};\mathbf{y}).$$

Therefore it is the specialization $x_1=y_1, x_2=y_2,\ldots$ of the image of this polynomial under ∂_σ, with $\sigma = [\widetilde{u},u^{\omega}]$, thanks to Lemma 2. Let $\beta = [\widetilde{v}_1-1,\ldots,\widetilde{v}_{k+r}-1]$, $\alpha = [\beta_1+r,\ldots,\beta_{k+r}-k+1]$. Writing the Schubert polynomial $\mathbb{X}_{\widetilde{v},v^{\omega}}$ as a determinant

$$S_{\beta_1-0,\ldots,\beta_{r+k}-r-k+1}(\mathbf{x}+\mathbf{z}-\mathbf{y}^{\beta_1},\ldots,\mathbf{x}+\mathbf{z}-\mathbf{y}^{\beta_{r+k}}),$$

then its product by $x_1^r\cdots x_k^r z_1^r\cdots z_r^r$ is equal to

$$S_{\alpha}(\mathbf{x}+\mathbf{z}-\mathbf{y}^{\beta_1},\ldots,\mathbf{x}+\mathbf{z}-\mathbf{y}^{\beta_{r+k}}).$$

Thanks to (2), the value of this determinant is not changed by replacing, in rows $1,2,\ldots,n{+}k$, $\mathbf{x}=\mathbf{x}^k$ by $\mathbf{0},\ldots,\mathbf{0},\mathbf{x}^1,\ldots,\mathbf{x}^k$ respectively.

Now, one can easily compute the image of this last determinant under ∂_σ. The transformation boils down to decreasing, in each row the indices of complete functions by some quantity, and increasing at the same time the exponent of $\mathbf{x}^j$ by the same amount.

The resulting determinant is

$$S_{\alpha/[0^r,\gamma]}(\mathbf{z}-\mathbf{y}^{\beta_1},\ldots,\mathbf{z}-\mathbf{y}^{\beta_{r+k}}//\mathbf{0},\ldots,\mathbf{0},\mathbf{x}^{1+\gamma_1},\ldots,\mathbf{x}^{k+\gamma_k}),$$

with $\gamma = [\tilde{u}_1-1,\ldots,\tilde{u}_k-k]$. The last transformation is to replace $\mathbf{x}$ by $\mathbf{y}$, and this produces

$$S_{\alpha/[0^r,\gamma]}(\mathbf{z}-\mathbf{y}^{\beta_1},\ldots,\mathbf{z}-\mathbf{y}^{\beta_{r+k}}//\mathbf{0},\ldots,\mathbf{0},\mathbf{y}^{1+\gamma_1},\ldots,\mathbf{y}^{k+\gamma_k}).$$

In conclusion, one has the following theorem.

Theorem 2. *Let n, k, r be three positive integers, v be a column of length $n-k-r$, u be a column of length $n-k$, $\mathbf{z} = [z_1, \ldots, z_r]$. Let $\beta = [\widetilde{v}_1 - 1, \ldots, \widetilde{v}_{k+r} - 1]$, $\alpha = [\beta_1 + r, \ldots, \beta_{k+r} - k + 1]$, $\gamma = [\tilde{u}_1 - 1, \ldots, \tilde{u}_k - k]$. Then*

$$F(u, v; \mathbf{z}) = z^{\rho_r} y^{\langle v\rangle - \langle u\rangle} S_{\alpha/[0^r,\gamma]}(\mathbf{z} - \mathbf{y}^{\beta_1}, \ldots, \mathbf{z} - \mathbf{y}^{\beta_{r+k}} // \mathbf{0}, \ldots, \mathbf{0}, \mathbf{y}^{1+\gamma_1}, \ldots, \mathbf{y}^{k+\gamma_k}). \qquad (3)$$

For example, for $n = 6$, $u = [6, 5, 3, 1]$, $v = [5, 2]$, one has $\widetilde{v} = [1346]$, $\widetilde{u} = [24]$, $\beta = [0, 2, 3, 5]$, $\alpha = [2, 3, 3, 4]$, $\gamma = [1, 2]$,

$$F(6, [5, 2]) = x^{3210}/y^{33211}\mathbb{X}_{134625} = x^{3210}/y^{33211}\mathbb{Y}_{011200},$$
$$F((6, [6, 5, 3, 1]) = x^{10}/y^{211}\mathbb{X}_{241356} = x^{10}/y^{211}\mathbb{Y}_{120000},$$
$$\mathbb{Y}_{011200} = S_{0112}(\mathbf{x}^4 - \mathbf{y}^0, \mathbf{x}^4 - \mathbf{y}^2, \mathbf{x}^4 - \mathbf{y}^3, \mathbf{x}^4 - \mathbf{y}^5),$$
$$x^{2222}\mathbb{Y}_{011200} = S_{2334}(\mathbf{x}^4 - \mathbf{y}^0, \mathbf{x}^4 - \mathbf{y}^2, \mathbf{x}^4 - \mathbf{y}^3, \mathbf{x}^4 - \mathbf{y}^5).$$

Putting $[x_3, x_4] = \mathbf{z}$, the last determinant is written

$$S_{2334}(\mathbf{z} - \mathbf{y}^0, \mathbf{z} - \mathbf{y}^2, \mathbf{z} - \mathbf{y}^3, \mathbf{z} - \mathbf{y}^5 // \mathbf{0}, \mathbf{0}, \mathbf{x}^1, \mathbf{x}^2).$$

Its image under $\partial_\sigma = \partial_{241356} = \partial_1\big(\partial_3(\partial_2())\big)$ is

$$S_{2334/0012}(\mathbf{z} - \mathbf{y}^0, \mathbf{z} - \mathbf{y}^2, \mathbf{z} - \mathbf{y}^3, \mathbf{z} - \mathbf{y}^5 // \mathbf{0}, \mathbf{0}, \mathbf{x}^{1+1}, \mathbf{x}^{2+2}).$$

In final,

$$F([6, 5, 3, 1], [5, 2]) = x^{10}y^{211}/y^{33211} \cdot \begin{vmatrix} S_2(\mathbf{z}) & S_4(\mathbf{z} - \mathbf{y}^2) & S_5(\mathbf{z} - \mathbf{y}^3) & S_7(\mathbf{z} - \mathbf{y}^5) \\ S_1(\mathbf{z}) & S_3(\mathbf{z} - \mathbf{y}^2) & S_4(\mathbf{z} - \mathbf{y}^3) & S_6(\mathbf{z} - \mathbf{y}^5) \\ 0 & S_1(\mathbf{z} - \mathbf{y}^2 + \mathbf{y}^2) & S_2(\mathbf{z} - \mathbf{y}^3 + \mathbf{y}^2) & S_4(\mathbf{z} - \mathbf{y}^5 + \mathbf{y}^2) \\ 0 & 0 & S_0(\mathbf{z} - \mathbf{y}^3 + \mathbf{y}^4) & S_2(\mathbf{z} - \mathbf{y}^5 + \mathbf{y}^4) \end{vmatrix}.$$

Notice that the expression of $F(u, v; \mathbf{z})z^{-\rho_r}$ given in (3) is symmetrical in $z_1, \ldots, z_r$. It would be interesting to prove directly that $F(u, v; z_1, z_2)z_1^{-1}$, for $\ell(u) = \ell(v)+2$, is a symmetric function in z_1, z_2.

6 Left truncated staircases

Let us now treat the staircases $[u = u_1, u_2, \ldots, u_{r+1} = [\,]]$, with a fixed left column u. Let n be such that $u_1 \le n$.

¿From the preceding section, one knows that $F(u, [\,]; z_1, \ldots, z_r)z^{-\rho_r}$ is the coefficient of $F(n, u)x^{-\rho_{n-r}}$ in the expansion of

$$x_1^r \cdots x_{n-r}^r = \mathbb{X}_{[r+1,\ldots,n,1,\ldots,r]}(\mathbf{x}, \mathbf{0}).$$

This expansion is a special case of Cauchy formula [11, Theorem 10.2.6] for three alphabets, and any permutation σ:

$$\mathbb{X}_\sigma(\mathbf{x}, \mathbf{w}) = \sum_{\sigma', \sigma''} \mathbb{X}_{\sigma'}(\mathbf{y}, \mathbf{w})\mathbb{X}_{\sigma''}(\mathbf{x}, \mathbf{y}),$$

sum over all reduced products $\sigma = \sigma'\sigma''$, in the case where σ is a Graßmannian permutation, and $\mathbf{w} = \mathbf{0} := [0, 0, \ldots]$.

Using that $\mathbb{X}_\tau(\mathbf{y}, \mathbf{0}) = \mathbb{X}_{\tau^{-1}}(\mathbf{0}, \overline{\mathbf{y}})$, for any permutation τ, with $\overline{\mathbf{y}} = [-y_1, -y_2, \ldots]$, going back to the x variables instead of z, and taking into account the value of u_r (which introduces a shift of indices, compared to the case $u_r = 1$), one has the following theorem.

Theorem 3. *Let u be a column of length r, with $u_r = 1$. Then*

$$F(u, []) = x^{\rho_r} y^{-\langle u\rangle} \mathbb{X}_{u^\omega, \widetilde{u}}(\mathbf{0}, \overline{\mathbf{y}}),$$

with $\overline{\mathbf{y}} = [-y_1, -y_2, \ldots]$, $u^\omega = [u_r, \ldots, u_1]$, $\widetilde{u}$ the (increasing) complement of u in $[1, \ldots, n]$, and $\langle u\rangle$ defined Section 4.

If $u_r = k > 1$, then $F(u, [])$ is obtained from $F([u_r{-}k{+}1, \ldots, u_1{-}k{+}1], [])$ by increasing the indices of all y_i by $k - 1$.

For example, if $u = [5, 3, 1]$, $n = 6$, then $\langle u\rangle = [2, 2, 1, 1]$

$$F([5, 3, 1], []) = x^{210} y^{-2,-2,-1,-1} \mathbb{X}_{135246}(\mathbf{0}, \overline{\mathbf{y}}).$$

If $u = [6, 4, 2]$, then

$$F([6, 4, 2], []) = x^{210} y^{0,-2,-2,-1,-1} \mathbb{X}_{135246}(\mathbf{0}, [-y_2, -y_3, \ldots]).$$

7 Appendix

We need some combinatorial properties of Schubert polynomials (Lemma 3, Proposition 1), completing those which can be found in [2, 11, 13].

Schubert polynomials $\mathbb{X}_\sigma(\mathbf{x}, \mathbf{y})$ constitute a linear basis of the ring of polynomials in $x_1, x_2, \ldots$, with coefficients in $y_1, y_2, \ldots$, indexed by permutations in $\mathfrak{S}_\infty$ (permutations fixing all, but a finite number of integers). Their family is stable under divided differences, and they all vanish when $\mathbf{x}$ is specialized to $\mathbf{y}$ (i.e. $x_1 \to y_1$, $x_2 \to y_2, \ldots$), except the polynomial $\mathbb{X}_1(\mathbf{x}, \mathbf{y}) = 1$. In fact, these two properties, added to some normalizations, characterize Schubert polynomials uniquely.

This allows, for example, to expand any polynomial in $\mathbf{x}$ in the basis of Schubert polynomials [11, Theorem 9.6.1]:

Lemma 2. *Given a polynomial $f(x_1, x_2, \ldots)$, then*

$$f(\mathbf{x}) = \sum_{\sigma\in\mathfrak{S}_\infty} \partial_\sigma\big(f(\mathbf{x})\big)\big|_{\mathbf{x}=\mathbf{y}} \mathbb{X}_\sigma(\mathbf{x}, \mathbf{y}).$$

Proof. Take the image of both members under some ∂_σ. The right hand-side is still a sum of Schubert polynomials, with modified indices. Specialize now $\mathbf{x}$ to $\mathbf{y}$. There is only one term which survives, the Schubert polynomial $\mathbb{X}_1$, which comes from $\mathbb{X}_\sigma(\mathbf{x}, \mathbf{y})$. Schubert polynomials being a linear basis, these equations for all σ determine the function $f(x)$. ■

Schubert polynomials can be interpreted in terms of tableaux. This gives a simple way of obtaining the branching rule of these polynomials according to the last variable x_r.

In the special case of a Graßmannian polynomial, with descent in r, using increasing partitions, the branching is

$$\mathbb{Y}_\lambda(\mathbf{x}^r, \mathbf{y}) = \sum_\mu \psi^h\left(\frac{\lambda}{\mu}; r\right) \mathbb{Y}_\mu(\mathbf{x}^{r-1}, \mathbf{y}) \tag{4}$$

sum over all μ such that λ/μ be an horizontal strip, the weight of the strip being

$$\psi^h(\lambda/\mu; r) := \prod_{\square\in\lambda/\mu} (x_r - y_{r+c(\square)}),$$

product over all boxes of the strip, $c(\square)$ being the usual content, i.e. the distance of $\square$ to the main diagonal of the diagram of λ.

Given two partitions ζ, μ such that ζ/μ be a vertical strip, define similarly the *weight* of the strip to be

$$\psi^v\left(\frac{\zeta}{\mu}; r\right) = \prod_{\square \in \zeta/\mu} y_{r+c(\square)}.$$

This weight appear in the product of a Graßmannian polynomial by a monomial, as states the following lemma.

Lemma 3. *Let $\nu \in \mathbb{N}^r$ be a partition, $\zeta = \nu + 1^r$. Then*

$$x_1 \cdots x_r \mathbb{Y}_\nu(\mathbf{x}, \mathbf{y}) = \sum_\mu \psi^v\left(\frac{\zeta}{\mu}; r\right) \mathbb{Y}_\mu(\mathbf{x}, \mathbf{y}),$$

sum over all μ such that ζ/μ be a vertical strip.

Proof. Let $\lambda = [\nu_r + r - 1, \ldots, \nu_1 + 0]$. The polynomial $\mathbb{Y}_\nu(\mathbf{x}, \mathbf{y})$ is by definition the image of

$$\mathbb{Y}_\lambda(\mathbf{x}, \mathbf{y}) = \prod_{(i,j) \in \lambda} (x_i - y_j)$$

under ∂_{ω_r}. Since multiplication by $x_1 \cdots x_r$ commutes with ∂_{ω_r}, then $x_1 \cdots x_r \mathbb{Y}_\nu(\mathbf{x}, \mathbf{y})$ is the image of

$$x_1 \cdots x_r \mathbb{Y}_\lambda(\mathbf{x}, \mathbf{y}) = \big((x_1 - y_{\nu_r+r}) + y_{\nu_r+r}\big) \cdots \big((x_r - y_{\nu_1+1}) + y_{\nu_1+1}\big) \mathbb{Y}_\nu(\mathbf{x}, \mathbf{y}).$$

This polynomial expands into a sum of Schubert polynomials, multiplied by some y_j. Taking the image of this sum under ∂_{ω_r} involves only transforming the indices. Explicitly,

$$x_1 \cdots x_r \mathbb{Y}_\nu(\mathbf{x}, \mathbf{y}) = \sum_{\epsilon \in [0,1]^n} y_{\nu_1+1}^{\epsilon_1} \cdots y_{\nu_r+r}^{\epsilon_r} \mathbb{Y}_{\zeta-\epsilon},$$

sum over all ϵ such that $\zeta - \epsilon$ be a partition, say μ. In other words, ζ/μ is a vertical strip and the lemma is proved. ∎

For example, for $\nu = [2, 2, 3]$,

$$\begin{aligned} x_1 x_2 x_3 \mathbb{Y}_{223} &= \sum_\epsilon y_3^{\epsilon_1} y_4^{\epsilon_2} y_6^{\epsilon_3} \mathbb{Y}_{334-\epsilon} \\ &= \mathbb{Y}_{334} + y_3 \mathbb{Y}_{234} + y_6 \mathbb{Y}_{333} + y_3 y_4 \mathbb{Y}_{224} + y_3 y_6 \mathbb{Y}_{233} + y_3 y_4 y_6 \mathbb{Y}_{223}, \end{aligned}$$

discarding the terms $y_4 \mathbb{Y}_{324}$, $y_4 y_6 \mathbb{Y}_{323}$ as being non-conform.

Proposition 1. *Let $\nu \in \mathbb{N}^r$ be a partition, $\zeta = \nu + 1^r$. Then*

$$x_1 \cdots x_r \mathbb{Y}_\nu(\mathbf{x}^r, \mathbf{y})\big|_{x_r = x} = \sum_\mu \theta\left(\frac{\zeta}{\mu}\right) \mathbb{Y}_\mu(\mathbf{x}^{r-1}, \mathbf{y}),$$

sum over all partitions $\mu \in \mathbb{N}^{r-1}$ such that ζ/μ be a ribbon, $\theta(\zeta/\mu)$ being defined in Section 3.

Proof. The polynomial $x_1 \cdots x_r \mathbb{Y}_\nu(\mathbf{x}^r, \mathbf{y})$ is a sum of Schubert polynomials obtained by suppressing vertical strips to ζ, according to Lemma 3. Expanding then according to x_r involves removing horizontal strips. The resulting partitions μ differ from ζ by a ribbon. Given such a μ, the coefficient of $\mathbb{Y}_\mu$ in the final polynomial will the sum

$$\sum_\eta \psi^v\left(\frac{\zeta}{\eta}; r\right)\psi^h\left(\frac{\eta}{\mu}\right) \tag{5}$$

sum over all η such that ζ/η be a vertical strip and η/μ be an horizontal strip.

The non-terminal boxes of ζ/μ are common to all horizontal strips, the terminal boxes which are above another box of the ribbon are common to all vertical strips. Hence, up to a common factor, the sum (5) reduces to

$$\sum_{A' \cup A''} \prod_{\square \in A'} y_{c(\square)} \prod_{\square \in A''} (x - y_{c(\square)}),$$

sum over all decompositions of the remaining set A of boxes of ζ/μ into two disjoint subsets. This sum is clearly equal to x^k, k being the cardinality of A. In total, the factor of $Y_\mu(\mathbf{x}^{r-1}, \mathbf{y})$ in the expansion of $x_1 \cdots x_r \mathbb{Y}_\nu(\mathbf{x}^r, \mathbf{y})$ is precisely $\theta(\zeta/\mu)$. ■

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