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Deformation Quantization in White Noise Analysis*

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Abstract. We define and present an example of a deformation quantization product on a Hida space of test functions endowed with a Wick product.

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1 Introduction

This work follows the work of Dito–Léandre [7] which was using tools of the Malliavin Calculus in order to define a Moyal product on a Wiener space.

Let us consider a finite-dimensional symplectic manifold M. It inherits from the symplectic form ω a Poisson structure $\{\cdot,\cdot\}$ whose matrix is the inverse of the matrix of the symplectic structure. The deformation quantization program of a Poisson manifold was initiated by Bayen–Flato–Fronsdal–Lichnerowicz–Sternheimer [1, 2, 8, 17, 23]. The simplest case to study is the case of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ endowed with its constant natural symplectic structure. This leads in particular to the notion of Moyal product.

In the case of a Hilbert space H, Dito [6] defines deformation quantization on a Hilbert space $H \oplus H^*$ endowed with its constant canonical symplectic structure and he defines a Moyal product on it on an appropriate algebra of functions. The main remark is that the constant matrix of the associated Poisson structure is still bounded.

This permits Dito–Léandre [7] to define the Moyal product on $W \oplus W^*$ where W is an abstract Wiener space. The constant symplectic form is the standard one on $H \oplus H^*$, the underlying Hilbert space of $W \oplus W^*$, such that the constant matrix of Poisson structure is still bounded. In that work, they consider the algebra of functionals smooth in the Malliavin sense [15, 18, 19, 20, 22] in order to define the Moyal product on $W \oplus W^*$.

We consider in this work the case where the underlying Hilbert space of the theory is a Sobolev Hilbert space of maps from the circle into $\mathbb{R}^n$ endowed with a constant symplectic structure. We do not consider the standard 2-form in order to do the quantization, but another form which is still interesting to consider. The constant matrix of the involved Poisson structure is unbounded so that we cannot use the construction of [7]. This leads to some modifications:

- We replace the algebra of functionals of Malliavin type by a Hida test functional space [3, 11, 12, 21].
- We replace the Wiener product by the normalized Wick product [11].

The behaviour of our theory is completely different of the behaviour in field theory of the classical Garding–Wightman result [10] for Canonical Commutation Relations in infinite dimension.

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This is related to the fact that the Hida test algebra is a space of continuous functionals on an abstract Wiener space associated to the Hilbert space of the theory. The Hida test algebra is so small that quantities as white noise behaves as if we were in finite dimension. Nevertheless, there is a tentative to interpret some quantum fields by using tools of white noise analysis [13, 14].

White noise analysis was created in order to understand very singular objects as, for instance, the speed of the Brownian motion. This explain that we get equivalences of some deformation quantization in white noise analysis, which were inequivalent in the theory of [6].

Our motivation comes from field theory [4, 5, 9, 24]. We choose a simple model in this approach, but it should be possible to consider more complicated Gaussian models. We have chosen this simple model in order to get simple computations. On the other hand, the free loop space is a well-known object of conformal field theory and string theory.

2 Deformation quantization in white noise analysis

Let $H(S^1;\mathbb{R}^n)$ be the Hilbert space of maps γ from the circle into $\mathbb{R}^n$ endowed with its canonical Hilbert structure such that

$$\int_0^1 |\gamma(s)|^2 ds + \int_0^1 |\gamma'(s)|^2 ds = \|h\|^2 < \infty.$$

We get by Fourier expansion an orthonormal basis $\gamma_{k,i}$ of this Hilbert space:

$$\gamma_{k,i}(s) = \left(\sqrt{C_1 k^2 + 1}\right)^{-1} e_i \cos(2\pi k s)$$

if $k \geq 0$ and if $k < 0$

$$\gamma_{k,i}(s) = \left(\sqrt{C_1 k^2 + 1}\right)^{-1} e_i \sin(2\pi k s),$$

where e_i is the canonical basis of $\mathbb{R}^n$.

We consider a multiindex $I = ((k_1, i_1), \dots, (k_{|I|}, i_{|I|}))$. We introduce the Hida weight:

$$w_r(I) = \prod \left(\sqrt{C_1 k_i^2 + 1}\right)^r$$

associated to this multiindex. F^I denotes the normalized symmetric tensor product of the γ_{k_i,e_i} associated to this multiindex.

We consider the weighted Fock space $W \cdot N_{r,C}$ of series

$$\sum b_I F^I = F$$

such that

$$\|F\|_{r,C}^2 = \sum |b_I|^2 w_r(I) C^{|I|} < \infty,$$

where $b_I \in \mathbb{C}$. In order to avoid some redundancies, we order the multiindices by lexicographic order such that after this choice F is written in a unique way.

Definition 1. The Hida test function $W \cdot N_{\infty-}$ space is the intersection of all $W \cdot N_{r,C}$ for $r > 0$ and $C > 0$ endowed with its natural topology.

It is a Fréchet space. Since the system of norms $\|\cdot\|_{r,C}$ increase with r and C, it is a projective limit of Hilbert spaces. Moreover if r and C are large, the imbedding of $W \cdot N_{r,C}$ in $W \cdot N_{r',C'}$ is Hilbert–Schmidt for large r' and C'. This follows from the classical fact that

$$\sum w_r(I)^{-1}C^{|I|} < \infty$$

if r is large enough and C small. This shows that $W \cdot N_{\infty-}$ is a nuclear space.

We define the Wick product $:F^{I_1}F^{I_2}:$ as the normalized symmetric tensor product of all the $\gamma_{k,i}$ in the concatenation of the multiindices I_1 and I_2.

Theorem 1. *$W \cdot N_{\infty-}$ is a commutative algebra for the Wick product.*

Proof. Let

$$F_1 = \sum b_I^1 F^I, \qquad F_2 = \sum b_I^2 F^I.$$

Therefore

$$F_3 =: F_1F_2: = \sum b_I^3 F^I,$$

where

$$a_I^3 = \sum_{I_1,I_2} b_{I_1}^1 b_{I_2}^2, \tag{1}$$

where the sum runs over all considered multiindices I_1 and I_2 whose concatenation is I. If I_1 and I_2 are such multiindices, we have clearly

$$w_r(I) = w_r(I_1)w_r(I_2).$$

Moreover there are at most $C^{|I|} = C^{|I_1|}C^{|I_2|}$ terms in the sum (1) such that

$$|b_I^3|^2 \le C \sum_{I_1,I_2} \left(|b_{I_1}^1|^2 C^{|I_1|}\right)\left(|b_{I_2}^2|^2 C^{|I_2|}\right). \qquad \blacksquare$$

(We refer to [16] for an analogous statement.)

Definition 2. A Poisson structure on $W \cdot N_{\infty-}$ $\{\cdot,\cdot\}$ is given by a $\mathbb{C}$-bilinear map $\{\cdot,\cdot\}$ from $WN_{\infty-} \times WN_{\infty-}$ into $W \cdot N_{\infty-}$ such that:

(i) $\{\cdot,\cdot\}$ is antisymmetric, satisfies the Jacobi derivation and is a derivation with respect of the Wick product in each argument;

(ii) if $F_1 = 1$, $\{F_1, F_2\} = 0$;

(iii) for all r and C, there exist r_1 and C_1 such that

$$\|\{F_1,F_2\}\|_{r,C} \le K\|F_1\|_{r_1,C_1}\|F_2\|_{r_1,C_1}.$$

In this formalism we can easily consider the deformation quantization of [1, 2, 8, 17, 23].

We consider the set of formal series $W \cdot N_{\infty-}[[h]]$ with coefficients in the Hida test functional space.

Definition 3. A star-product on $W \cdot N_{\infty-}[[h]]$ is a $\mathbb{C}[[h]]$-bilinear product $*_h$ on $WN_{\infty-}[[h]] \times W \cdot N_{\infty-}[[h]]$ with values in $W \cdot N_{\infty-}[[h]]$ given by

$$F_1 *_h F_2 = \sum_{l\ge 0} h^l P_l(F_1,F_2)$$

for F_1 and F_2 belonging to the Hida test functional space. The star-product is extended by $\mathbb{C}[[h]]$-bilinearity to $W \cdot N_{\infty-}[[h]]$and satisfies if F_1, F_2, F_3 belong to $W \cdot N_{\infty-}$:

(i) $P_0(F_1, F_2) =: F_1F_2:$;

(ii) $P_1(F_1, F_2) - P_1(F_2, F_1) = 2\{F_1, F_2\}$;

(iii) for all r, C, l, there exist r_1, C_1 such that

$$\|P_l(F_1, F_2)\|_{r,C} \leq K\|F_1\|_{r_1,C_1}\|F_2\|_{r_1,C_1}$$

and P_l vanishes on constants.

(iv) $F_1 *_h (F_2 *_h F_3) = (F_1 *_h F_2) *_h F_3$.

3 Example: the Hida star product

Let $\omega = \omega_{i,j}$ be a nondegenerate antisymmetric bilinear form on $\mathbb{R}^n$ (n is even). Without loss of generality, we can write ω as:

$$\omega_{2i,2i+1} = -\omega_{2i+1,2i} = 1 \tag{2}$$

and $\omega_{i,j} = 0$ elsewhere. Namely, this diagonalization do not change the Hida space of test functionals we have considered in this paper. This comes from the fact that a linear transformation on $\mathbb{R}^n$ induces a linear transformation on $(\mathbb{R}^n)^{\hat{\otimes}r}$, the symmetric tensor product of $\mathbb{R}^n$ of length r of norm bounded by C^k. If we perform a linear change of coordinates on $\mathbb{R}^n$, this induces on the system of F_I $|I| = r$ a linear transformation. But it is a linear transformation on each block realized by the F^I $I = (k_I, i_I)$, k_I being fixed. Each block is identified with $(\mathbb{R}^n)^{\hat{\otimes}r}$, where we can look at the induced linear transformation. But, we could repeat the following considerations without using this diagonalization of ω.

We introduce the antisymmetric bilinear form on $H(S^1; \mathbb{R}^n)$

$$\Omega(\gamma^1, \gamma^2) = \int_0^1 \omega(\gamma^1(s), \gamma^2(s))ds, \qquad \Omega = \Omega_{(k_1,i_1),(k_2,i_2)},$$

where

$$\Omega_{(k,2i),(k,2i+1)} = (Ck^2 + 1)^{-1} = -\Omega_{(k,2i+1),(k,2i)}$$

and other components of Ω elsewhere vanish. This antisymmetric bilinear form leads to a singular Poisson structure whose matrix form is Ω^{-1} which is not bounded: therefore the theory of [7] is not suitable to describe the deformation quantization theory related to this symplectic structure.

Let $a_{(k,i)}$ be the annihilation operator on the symmetric Fock space associated to $\gamma_{(k,i)}$. We have that:

$$a_{(k_1,i_1)} \cdots a_{(k_l,i_l)} F^I = C(I_1, (k_1, i_1), \ldots, (k_l, i_l))F^{I_1},$$

where in I_1 we have removed $(k_1, i_1), \ldots, (k_l, i_l)$ if it is possible (in the other case the expression vanishes). We have the bound

$$|C(I_1, (k_1, i_1), \ldots, (k_l, i_l))| \leq C_l^{|I|}. \tag{3}$$

The constant $C(I_1, (k_1, i_1), \ldots, (k_l, i_l))$ comes from the fact that we are considering the Bosonic Fock space: if we consider the same Boson at the power n, the associated annihilation operator transforms it in n-times the Boson at the power $n - 1$.

We consider finite sums $F_1 = \sum b_I^1 F^I$ and $F_2 = \sum b_I^2 F^I$. In a traditional way, we can put

$$\{F_1, F_2\} = \sum \Omega^{(k_1,i_1),(k_2,i_2)} : a_{(k_1,i_1)}F_1 a_{(k_2,i_2)}F_2 : .$$

(We consider normalized Wick products.) $\Omega^{(k_1,i_1),(k_2,i_2)}$ are the generic elements of the inverse of the symplectic form Ω and are not bounded. Therefore $\{\cdot,\cdot\}$ do not act on the Sobolev spaces of the Malliavin Calculus unlike the Poisson structure studied in [7], and we have to consider different functional spaces if we want to extend the previous formula from finite sums to series. We have:

Proposition 1. $\{\cdot,\cdot\}$ *defines a Poisson structure in the sense of Definition* 3 *on* $W \cdot N_{\infty-}$*.*

Proof. Let us show first (iii) in Definition 3. We have

$$\{F_1, F_2\} = \sum b_I^3 F^I,$$

where

$$b_I^3 = \sum (Ck^2+1) b^1_{I_1\cup(k,2i)} b^2_{I_2\cup(k,2i+1)} C(I_1,(k,2i)) C(I_2,(k,2i+1)) + A, \tag{4}$$

where A is a similar term and where we sum over all k, i, I_1, I_2 so that the concatenation $I_1 \cup I_2$ of I_1, I_2 is equal to I. We apply Cauchy–Schwartz inequality in (k,i), we use the bound of $C(I_1,(k,2i))$, $C(I_2,(k,2i+1))$ given in (3) to get

$$\begin{aligned}\omega_r(I)|b_I^3|^2 \le K \sum_{k,k',i,i'} &\left(\omega_{r_1}(I_1\cup(k,2i))|b^1_{I_1\cup(k,2i)}|^2 C_1^{|I_1|+1}\right)\\ &\times \left(\omega_{r_1}(I_2\cup(k',2i'+1))|b^2_{I_2\cup(k',2i'+1)}|^2 C_1^{|I_2|+1}\right)\end{aligned}$$

for r_1 and C_1 large enough, where we sum on the set of multiindices I_1 and I_2 is such that $I_1 \cup I_2 = I$. We do the same for A in (4). We use for that

$$\sum \left(Ck^2+1\right)^{-r'} < \infty$$

if $r' > 1$. We deduce from the previous inequality that

$$\|\{F_1,F_2\}\|_{r,C} \le K \left(\sum \omega_{r_1}(I)|I| C_1^{|I|} |b_I^2|^2\right)\left(\sum \omega_{r_1}(I)|I| C_1^{|I|}|b_I^2|^2\right).$$

Therefore we deduce that

$$\|\{F_1,F_2\}\|_{r,C} \le K \|F_1\|_{r_1,C_1'} \|F_2\|_{r_1,C_1'} \tag{5}$$

for r_1, C_1' large enough. This shows (iii).

The algebraic properties of the Poisson product arise from the fact that the family of annihilation operators commute and an annihilation operator is a derivation for the Wick product. ■

If F_1 and F_2 are finite sums, we can define as usual by using the Wick product:

$$\begin{aligned}F_1 *_h F_2 = \sum_{l\ge 0} (-h/2)^l l!^{-1} \sum \Omega^{(k_1,i_1),(k_1',i_1')} \dots \Omega^{(k_l,i_l),(k_l',i_l')}\\ \times : a_{(k_1,i_1)} \cdots a_{(k_l,i_l)} F_1 a_{(k_1',i_1')} \cdots a_{(k_l',i_l')} F_2 : .\end{aligned}$$

The sum is in fact finite since F_1 and F_2 are finite sums. It is the exponential of the Poisson Bracket. Let us stress the difference with [7]: in [7], we were considering the canonical symplectic form on $W \oplus W^*$ whose inverse is **bounded**, and so the Moyal product of [7] was acting on the **big space** of Malliavin test functionals. Here it is not the case. Let us recall the main difference between the Malliavin test algebra and the Hida test algebra. The Hilbert space $H(S^1;\mathbb{R}^n)$ induces a Gaussian measure on the Banach space B of continuous functions from S^1

into $\mathbb{R}^n$. The ordinary Fock space $W\cdot N_{0,1}$ coincides with the L^2 of this Gaussian measure. The Malliavin test algebra is constituted of functionals almost surely defined on B and the Hida test algebra is constituted of continuous functionals on B. In order to stress the difference, let us consider chaos of length 1, $F=\sum_{|I|=1} b_I F^I$. These chaoses of length 1 belong to the Malliavin algebra if and only if $\sum |b_I|^2<\infty$ because the L^p norms and the L^2 norms are equivalent on an abstract Wiener space for Wiener chaos of bounded norm. It is therefore clear that our Poisson structure does not act on the restriction of the Malliavin algebra constituted of chaoses of length 1.

Theorem 2. *Formula* (5) *can be extended in a star-product in the sense of Definition* 3. *We call it the Hida star product associated to the symplectic structure given by* Ω.

Proof. The algebra is the same as in the classical case [6]. Only the analysis is different. We put

$$P_l(F_1,F_2)=\sum b_I^3F^I,$$

where b_I^3 is a sum of a bounded terms of the following type:

$$\begin{aligned}A=\sum &(Ck_1^2+1)\cdots(Ck_l^2+1)b^1_{I_1\cup(k_1,2i_1)\cup\cdots\cup(k_l,2i_l)}\\ &\times C(I_1,(k_1,2i_1),\dots,(k_l,2i_l))b^2_{I_2\cup(k'_1,2i'_1+1)\cup\cdots\cup(k'_l,2i'_l+1)}\\ &\times C(I_2,(k'_1,2i'_1+1),\dots,(k'_l,2i'_l+1)),\end{aligned}$$

where we sum on all k_i, i_l, i'_l and all multiindices I_1 and I_2 such that their concatenation $I_1\cup I_2$ equals I. By doing as before and using the estimates (3), we deduce that

$$\begin{aligned}\omega_r(I)|A|^2\le K\sum &\left(\omega_{r_1}(I_1\cup(k_1,2i_1)\cup\cdots\cup(k_l,2i_l))C_1^{|I_1|+l}|b^1_{I_1\cup(k_1,2i_1)\cup\cdots\cup(k_l,2i_l)}|^2\right)\\ &\times\left(\omega_{r_1}(I_2\cup(k'_1,2i'_1+1)\cup\cdots\cup(k'_l,2i'_l+1))C_1^{|I_2|+l}|b^2_{I_2\cup(k'_1,2i'_1)\cup\cdots\cup(k'_l,2i'_l+1)}|^2\right),\end{aligned}$$

where we sum on all $(k_l,2i_l)$, all $(k_{l'},2i_{l'}+1)$ and all multiindices I_1 and I_2 such that $I_1\cup I_2=I$.

We deduce that:

$$\|P_l(F_1,F_2)\|^2_{r,C}\le K\left(\sum\omega_{r_1}(I)C_1^{|I|}|I|^l\ |b_I^1|^2\right)\left(\sum\omega_{r_1}(I)C_1^{|I|}|I|^l|b_I^2|^2\right)$$

for r_1 and C_1 large enough such that

$$\|P_l(F_1,F_2)\|_{r,C}\le K\|F_1\|_{r_1,C'_1}\|F_2\|_{r_1,C'_1}$$

from which the result follows. ■

4 Equivalence of deformations

The main difference between this work and [6, 7] is that the space of Hida test functionals is very small, hence the space of allowed deformation is very big. This implies that some inequivalent deformations in the theory of [6] are here equivalent. In order to stress the difference, we will take the model of [7].

H is the Hilbert space of maps from $[0,1]$ into $\mathbb{R}$ such that

$$\|\gamma\|_0^2=\int_0^1|d/ds\gamma(s)|^2ds<\infty.$$

We consider $H \oplus H^* = H_t$ endowed with its canonical symplectic form. We define $W_{\infty-}$ to be the space of maps such that $\int_0^1 |d^r/ds^r \gamma(s)|^2 ds < \infty$ for all r. It is a Fréchet space. We choose a convenient Hilbert basis of H: if $n > 0$, $\gamma_n(s) = \frac{\sin[2\pi n s]}{C_1 n}$ and if $n < 0$, $\gamma_n(s) = \frac{\cos[2\pi n s]-1}{C_1 n}$. Associated to this Hilbert space, and by using the convenient Hida weights associated to this basis, we can define the Hida test algebra $W \cdot N_{\infty-}$. We can define a Poisson structure $\{\cdot,\cdot\}$ associated to this symplectic form on H_t which acts continuously on $W \cdot N_{\infty-} \times W \cdot N_{\infty-}$. Computations are similar to the Part III, but simpler since the matrix of the Poisson structure is bounded.

Let $\gamma_1 \oplus \gamma_2$ belong to $W_{\infty-} \oplus W_{\infty-}$. We consider the Wick exponential Φ_{γ_1,γ_2}

$$h_1 \oplus h_2 \to : \exp[\langle h_1, \gamma_1\rangle_0 + \langle h_2, \gamma_2\rangle_0] : .$$

The Wick exponentials are dense in $W \cdot N_{\infty-}$.

We consider an operator $A : \gamma_n \to \lambda_n \gamma_n$ with $|\lambda_n| \le C|n|^\alpha$. According to [6], we put

$$E_A[F_1, F_2] = \sum : a_i^1 F_1 \lambda_i a_i^2 F_2 : + \sum : a_i^1 F_2 \lambda_i a_i^2 F_1 : ,$$

where a_i^1 are the standard annihilation operators in the direction of H associated to γ_i and a_i^2 are standard annihilation operators in the direction of $H^* \sim H$. Since $|\lambda_i|$ are bounded by $C|i|^\alpha$ and we are considering the same Hida weight as in the first part, but with this new orthogonal basis, it follows that:

Theorem 3. *E_A is continuous from $W \cdot N_{\infty-} \times W \cdot N_{\infty-}$ into $W \cdot N_{\infty-}$.*

We put according to [6],

$$C_1^A[F_1, F_2] = \{F_1, F_2\} + E_A[F_1, F_2]$$

and $C_r^A[F_1, F_2] = (C_1^A)^r[F_1, F_2]$ in the sense of bidifferential operators. C_r^A is still continuous from $W \cdot N_{\infty-} \times W \cdot N_{\infty-}$ into $W \cdot N_{\infty-}$ (the proof is very similar to the proof of Theorem 1).

Definition 4. We put

$$F_1 *_h^A F_2 =: FG : + \sum_{r \ge 1} \frac{h^r}{r!} C_r^A(F_1, F_2).$$

As in Theorem 2, due to the polynomial growth of the λ_i, $*_h^A$ defines a quantization by deformation in Hida sense of $\{\cdot,\cdot\}$. But unlike in [6], we have:

Proposition 2. *$*_h^A$ and $*_h$ are equivalent on the Hida test functional space.*

Proof. We put as in [6]

$$T_1 F = -\sum \lambda_i a_i^1 a_i^2 F.$$

Due to the polynomial growth of the λ_i, $T' = \exp[hT_1]$ is continuous on $W \cdot N_{\infty-}[[h]]$ Moreover, let us recall that by [6] formula (2)

$$\Phi_{\gamma_1,\gamma_2} *_h^A \Phi_{\gamma_1',\gamma_2'} = \exp[h(\langle \gamma_2', (A+\mathbb{I})\gamma_1\rangle_0 + \rangle \gamma_2, (A+\mathbb{I})\gamma_1'\rangle_0)]\Phi_{\gamma_1+\gamma_1',\gamma_2+\gamma_2'}.$$

We conclude as in [6], by remarking that

$$T'(\Phi_{\gamma_1,\gamma_2} *_h^A \Phi_{\gamma_1',\gamma_2'}) = T'\Phi_{\gamma_1,\gamma_2} *_h T'\Phi_{\gamma_1',\gamma_2'}.$$

This proves the theorem since the Wick exponentials are dense in the Hida space. ∎

Remark 1. If $A = \mathbb{I}$, we get the normal product. Let us stress the difference with the theory of [6]. In [6], $*_h^A$ and $*_h$ were equivalent if and only if A is a Hilbert–Schmidt operator, then the Moyal product and the normal product ($A = \mathbb{I}$) were inequivalent. In the case of $\mathbb{R}^n$, the Moyal product and the normal product are equivalent. For the Hida Calculus, deformation theory behaves more or less as in finite dimension. This comes from that the Hida test functional space is so small that all algebraic considerations in finite dimension, where we were considering finite sums, remain true in this context.

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