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Clifford Algebra Derivations of Tau-Functions for Two-Dimensional Integrable Models with Positive and Negative Flows*

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Abstract. We use a Grassmannian framework to define multi-component tau functions as expectation values of certain multi-component Fermi operators satisfying simple bilinear commutation relations on Clifford algebra. The tau functions contain both positive and negative flows and are shown to satisfy the $2n$-component KP hierarchy. The hierarchy equations can be formulated in terms of pseudo-differential equations for $n \times n$ matrix wave functions derived in terms of tau functions. These equations are cast in form of Sato–Wilson relations. A reduction process leads to the AKNS, two-component Camassa–Holm and Cecotti–Vafa models and the formalism provides simple formulas for their solutions.

Key words: Clifford algebra; tau-functions; Kac–Moody algebras; loop groups; Camassa–Holm equation; Cecotti–Vafa equations; AKNS hierarchy

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In memory of Vadim Kuznetsov. *One of us (JvdL) first met Vadim at a seminar on Quantum Groups held at the Korteweg–de Vries Institute in Amsterdam in 1993. Vadim was then still a post-Doc. Later meetings at several other conferences, e.g. in Cambridge, Oberwolfach and Montreal provided better opportunities to learn Vadim's kind personality and his wonderful sense of humor. Both authors of this paper have the best recollection of Vadim from meeting at the conference "Classical and Quantum Integrable Systems" organized in 1998 in Oberwolfach by Werner Nahm and Pierre van Moerbeke. A memory of one pleasant evening spent with him in the setting of a beautiful conference center library clearly stands out. Vadim joined us and Boris Konopelchenko after a (successful) play of pool game against Boris Dubrovin. The library has a wonderful wine cellar and we were all having a good time. Vadim was in great mood and filled the conversation with jokes and funny anecdotes. When thinking of him, we will always remember in particular that most enjoyable evening.*

1 Introduction

Tau-functions are the building blocks of integrable models. The Grassmannian techniques have been shown in the past to be very effective in theory of tau-functions. In this paper we exploit

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue "Integrable Systems and Related Topics". The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html a Grassmannian approach to constructing tau functions in terms of expectation values of certain Fermi operators constructed using boson-fermion correspondence. This formalism provides a systematic way of constructing multi-component tau functions for multi-dimensional Toda models, which, in this paper, embed both positive and negative flows.

The set of Hirota equations for the tau functions is obtained by taking expectation value of both sides of bilinear commutation relation

$$A \otimes A\Omega = \Omega A \otimes A, \qquad \Omega = \sum_{\mathfrak{l},i} \psi_{\mathfrak{l}}^{+\,(i)} \otimes \psi_{-\mathfrak{l}}^{-\,(i)}$$

defined on a Clifford algebra with Fermi operators $\psi_{\mathfrak{l}}^{\pm\,(i)}$, satisfying the relations

$$\psi_{\mathfrak{l}}^{\lambda\,(i)} \psi_{\mathfrak{k}}^{\mu\,(j)} + \psi_{\mathfrak{k}}^{\mu\,(j)} \psi_{\mathfrak{l}}^{\lambda\,(i)} = \delta_{i,j}\delta_{\lambda,\mu}\delta_{\mathfrak{l},-\mathfrak{k}},$$

for $\mathfrak{l}, k \in \mathbb{Z} + 1/2$, $i, j = 1, \dots, n$, $\lambda, \mu = +, -$.

We rewrite Hirota equations in terms of formal pseudo-differential operators acting on matrix wave functions derived from the tau functions. This method gives rise to a general set of Sato–Wilson equations.

Next, we impose a set conditions on the tau function which define a reduction process. Under this reduction process the pseudo-differential equations for the wave functions describe flows of dressing matrices of the multi-dimensional Toda model. In the case of 2×2 matrix wave functions these equations embed the AKNS model and the two-component version of the Camassa–Holm (CH) model for, respectively, positive and negative flows of the multi-dimensional 2×2 Toda model.

Section 2 is meant as an informal review of semi-infinite wedge space and Clifford algebra. In this setup, in Section 3, we formulate the multi-component tau functions as expectation values of operators satisfying the bilinear identity. This formalism contains the Toda lattice hierarchy as a special case. Section 4 shows how to rewrite the formalism in terms of pseudo-differential operators acting an wave function. We arrive in this way in general equations of Sato–Wilson type. The objective of the next Section 5 is to introduce a general reduction process leading to a multi-dimensional Toda model with positive and negative flows acting on $n \times n$ matrix wave functions explicitly found in terms of components of the tau functions from Section 3. The Grassmannian method provides in this section explicit construction of matrices solving the Riemann–Hilbert factorization problem for GL_n.

As shown in Section 6 the model obtained in Section 5 embeds both AKNS and the 2-component Camassa–Holm equations. Further reduction uses an automorphism of order 4 and as described in Section 7 reduces flow equations to Cecotti–Vafa equations. In Section 8, we use the Virasoro algebra constraint to further reduce the model by imposing homogeneity relations on matrices satisfying Cecotti–Vafa equations.

The particular advantage of our construction is that it leads to solutions of the AKNS and the 2-component Camassa–Holm equations and Cecotti–Vafa equations in terms of relatively simple correlation functions involving Fermi operators defined according to the Fermi–Bose correspondence. These solutions are constructed in Section 9 and 10, respectively.

2 Semi-infinite wedge space and Clifford algebra

Following [9], we introduce the semi-infinite wedge space $F = \Lambda^{\frac{1}{2}\infty}\mathbb{C}^\infty$ as the vector space with a basis consisting of all semi-infinite monomials of the form $v_{\mathfrak{i}_1} \wedge v_{\mathfrak{i}_2} \wedge v_{\mathfrak{i}_3} \cdots$, with $\mathfrak{i}_j \in 1/2 + \mathbb{Z}$, where $\mathfrak{i}_1 > \mathfrak{i}_2 > \mathfrak{i}_3 > \cdots$ and $\mathfrak{i}_{\ell+1} = \mathfrak{i}_\ell - 1$ for $\ell \gg 0$. Define the wedging and contracting operators $\psi_{\mathfrak{j}}^{+}$ and $\psi_{\mathfrak{j}}^{-}$ $(\mathfrak{j}\in\mathbb{Z}+1/2)$ on F by

$$\psi_{\mathfrak{j}}^{+}(v_{\mathfrak{i}_1}\wedge v_{\mathfrak{i}_2}\wedge\cdots)=\begin{cases}0 & \text{if } -\mathfrak{j}=\mathfrak{i}_s \text{ for some } s,\\ (-1)^s v_{\mathfrak{i}_1}\wedge v_{\mathfrak{i}_2}\cdots\wedge v_{\mathfrak{i}_s}\wedge v_{-\mathfrak{j}}\wedge v_{\mathfrak{i}_{s+1}}\wedge\cdots & \text{if } \mathfrak{i}_s>-\mathfrak{j}>\mathfrak{i}_{s+1},\end{cases}$$

$$\psi_{\mathfrak{j}}^{-}(v_{\mathfrak{i}_1}\wedge v_{\mathfrak{i}_2}\wedge\cdots)=\begin{cases}0 & \text{if } \mathfrak{j}\neq\mathfrak{i}_s \text{ for all } s,\\ (-1)^{s+1} v_{\mathfrak{i}_1}\wedge v_{\mathfrak{i}_2}\wedge\cdots\wedge v_{\mathfrak{i}_{s-1}}\wedge v_{\mathfrak{i}_{s+1}}\wedge\cdots & \text{if } \mathfrak{j}=\mathfrak{i}_s.\end{cases}$$

These operators satisfy the following relations $(\mathfrak{i},\mathfrak{j}\in\mathbb{Z}+1/2,\ \lambda,\mu=+,-)$:

$$\psi_{\mathfrak{i}}^{\lambda}\psi_{\mathfrak{j}}^{\mu}+\psi_{\mathfrak{j}}^{\mu}\psi_{\mathfrak{i}}^{\lambda}=\delta_{\lambda,-\mu}\delta_{\mathfrak{i},-\mathfrak{j}},\tag{2.1}$$

hence they generate a Clifford algebra, which we denote by $\mathcal{C}\ell$.

Introduce the following elements of F $(m\in\mathbb{Z})$:

$$|m\rangle=v_{m-\frac{1}{2}}\wedge v_{m-\frac{3}{2}}\wedge v_{m-\frac{5}{2}}\wedge\cdots.$$

It is clear that F is an irreducible $\mathcal{C}\ell$-module such that

$$\psi_{\mathfrak{j}}^{\pm}|0\rangle=0\quad\text{for}\quad \mathfrak{j}>0.$$

Think of the adjoint module F^* in the following way, it is the vector space with a basis consisting of all semi-infinite monomials of the form $\cdots\wedge v_{\mathfrak{i}_3}\wedge v_{\mathfrak{i}_2}\wedge v_{\mathfrak{i}_1}$, where $\mathfrak{i}_1<\mathfrak{i}_2<\mathfrak{i}_3<\cdots$ and $\mathfrak{i}_{\ell+1}=\mathfrak{i}_\ell+1$ for $\ell\gg 0$. The operators $\psi_{\mathfrak{j}}^{+}$ and $\psi_{\mathfrak{j}}^{-}$ $(\mathfrak{j}\in\mathbb{Z}+1/2)$ also act on F^* by contracting and wedging, but in a different way, viz.,

$$(\cdots\wedge v_{\mathfrak{i}_2}\wedge v_{\mathfrak{i}_1})\psi_{\mathfrak{j}}^{+}=\begin{cases}0 & \text{if } \mathfrak{j}\neq\mathfrak{i}_s \text{ for all } s,\\ (-1)^{s+1}\cdots\wedge v_{\mathfrak{i}_{s+1}}\wedge v_{\mathfrak{i}_{s-1}}\wedge\cdots v_{\mathfrak{i}_2}\wedge v_{\mathfrak{i}_1} & \text{if } \mathfrak{i}_s=\mathfrak{j},\end{cases}$$

$$(\cdots\wedge v_{\mathfrak{i}_2}\wedge v_{\mathfrak{i}_1})\psi_{\mathfrak{j}}^{-}=\begin{cases}0 & \text{if } -\mathfrak{j}=\mathfrak{i}_s \text{ for some } s,\\ (-1)^{s}\cdots\wedge v_{\mathfrak{i}_{s+1}}\wedge v_{\mathfrak{j}}\wedge v_{\mathfrak{i}_{s}}\wedge\cdots v_{\mathfrak{i}_2}\wedge v_{\mathfrak{i}_1} & \text{if } \mathfrak{i}_s<-\mathfrak{j}<\mathfrak{i}_{s+1}.\end{cases}$$

We introduce the element $\langle m|$ by $\langle m|=\cdots\wedge v_{m+\frac{5}{2}}\wedge v_{m+\frac{3}{2}}\wedge v_{m+\frac{1}{2}}$, such that $\langle 0|\psi_{\mathfrak{j}}^{\pm}=0$ for $\mathfrak{j}<0$. We define the vacuum expectation value by $\langle 0|0\rangle=1$, and denote $\langle A\rangle=\langle 0|A|0\rangle$.

Note that $(\psi_{\mathfrak{k}}^{\pm})^*=\psi_{-\mathfrak{k}}^{\mp}$ and that

$$\begin{aligned}|V(\mathfrak{i}_1,\dots,\mathfrak{i}_k,\mathfrak{j}_1,\dots,\mathfrak{j}_\ell)\rangle&=\psi_{\mathfrak{i}_1}^{+}\psi_{\mathfrak{i}_2}^{+}\cdots\psi_{\mathfrak{i}_k}^{+}\psi_{\mathfrak{j}_1}^{-}\psi_{\mathfrak{j}_2}^{-}\cdots\psi_{\mathfrak{j}_\ell}^{-}|0\rangle,\\ \langle V(\mathfrak{i}_1,\dots,\mathfrak{i}_k,\mathfrak{j}_1,\dots,\mathfrak{j}_\ell)|&=\langle 0|\psi_{-\mathfrak{j}_\ell}^{+}\psi_{-\mathfrak{j}_{\ell-1}}^{+}\cdots\psi_{-\mathfrak{j}_1}^{+}\psi_{-\mathfrak{i}_k}^{-}\psi_{-\mathfrak{i}_{k-1}}^{-}\cdots\psi_{-\mathfrak{i}_1}^{-},\end{aligned}\tag{2.2}$$

with $\mathfrak{i}_1<\mathfrak{i}_2<\cdots<\mathfrak{i}_k<0$ and $\mathfrak{j}_1<\mathfrak{i}_2<\cdots<\mathfrak{j}_\ell<0$ form dual basis of F and F^*, i.e.,

$$\langle V(\mathfrak{r}_1,\dots,\mathfrak{r}_m,\mathfrak{s}_1,\dots,\mathfrak{s}_q)|V(\mathfrak{i}_1,\dots,\mathfrak{i}_k,\mathfrak{j}_1,\dots,\mathfrak{j}_\ell)\rangle=\delta_{(\mathfrak{r}_1,\dots,\mathfrak{r}_m),(\mathfrak{i}_1,\dots,\mathfrak{i}_k)}\delta_{(\mathfrak{s}_1,\dots,\mathfrak{s}_q),(\mathfrak{j}_1,\dots,\mathfrak{j}_\ell)}.\tag{2.3}$$

We relabel the basis vectors v_i and with them the corresponding fermionic operators (the wedging and contracting operators). This relabeling can be done in many different ways, see e.g. [10], the simplest one is the following $(j=1,2,\dots,n)$:

$$v_{\mathfrak{k}}^{(j)}=v_{n\mathfrak{k}-\frac{1}{2}(n-2j+1)},$$

and correspondingly:

$$\psi_{\mathfrak{k}}^{\pm(j)}=\psi_{n\mathfrak{k}\pm\frac{1}{2}(n-2j+1)}^{\pm}.$$

Notice that with this relabeling we have:

$$\psi_{\mathfrak{k}}^{\pm(j)}|0\rangle = 0 \quad \text{for} \quad \mathfrak{k} > 0.$$

Define *partial charges* and *partial energy* by

$$\begin{aligned} \text{charge}_j\ \psi_{\mathfrak{k}}^{\pm(i)} &= \pm\delta_{ij}, \qquad \text{charge}_j\ |0\rangle = 0,\\ \text{energy}_j\ \psi_{\mathfrak{k}}^{\pm(i)} &= -\delta_{ij}k, \qquad \text{energy}_j\ |0\rangle = 0. \end{aligned}$$

Total charge and energy is defined as the sum of partial charges, respectively the sum of partial energy.

Introduce the fermionic fields ($0 \neq z \in \mathbb{C}$):

$$\psi^{\pm(j)}(z) = \sum_{\mathfrak{k}\in\mathbb{Z}+1/2} \psi_{\mathfrak{k}}^{\pm(j)} z^{-\mathfrak{k}-\frac{1}{2}}.$$

Next, we introduce bosonic fields ($1 \leq i, j \leq n$):

$$\alpha^{(ij)}(z) \equiv \sum_{k\in\mathbb{Z}} \alpha_k^{(ij)} z^{-k-1} =: \psi^{+(i)}(z)\psi^{-(j)}(z) :,$$

where : : stands for the *normal ordered product* defined in the usual way ($\lambda, \mu = +$ or $-$):

$$: \psi_{\mathfrak{k}}^{\lambda(i)} \psi_{\mathfrak{l}}^{\mu(j)} := \begin{cases} \psi_{\mathfrak{k}}^{\lambda(i)} \psi_{\mathfrak{l}}^{\mu(j)} & \text{if } \mathfrak{l} > 0,\\ -\psi_{\mathfrak{l}}^{\mu(j)} \psi_{\mathfrak{k}}^{\lambda(i)} & \text{if } \mathfrak{l} < 0. \end{cases}$$

One checks (using e.g. the Wick formula) that the operators $\alpha_k^{(ij)}$ satisfy the commutation relations of the affine algebra $gl_n(\mathbb{C})^\wedge$ with the central charge 1, i.e.:

$$[\alpha_p^{(ij)}, \alpha_q^{(k\ell)}] = \delta_{jk}\alpha_{p+q}^{(i\ell)} - \delta_{i\ell}\alpha_{p+q}^{(kj)} + p\delta_{i\ell}\delta_{jk}\delta_{p,-q},$$

and that

$$\alpha_k^{(ij)}|m\rangle = 0 \quad \text{if} \quad k > 0 \quad \text{or} \quad k = 0 \text{ and } i < j.$$

The operators $\alpha_k^{(i)} \equiv \alpha_k^{(ii)}$ satisfy the canonical commutation relation of the associative oscillator algebra, which we denote by $\mathfrak{a}$:

$$[\alpha_k^{(i)}, \alpha_\ell^{(j)}] = k\delta_{ij}\delta_{k,-\ell},$$

and one has

$$\alpha_k^{(i)}|m\rangle = 0 \quad \text{for} \quad k > 0, \qquad \langle m|\alpha_k^{(i)} = 0 \quad \text{for} \quad k < 0.$$

Note that $\alpha_0^{(j)}$ is the operator that counts the j-th charge. The j-th energy is counted by the operator

$$- \sum_{\mathfrak{k}\in\frac{1}{2}+\mathbb{Z}} \mathfrak{k} : \psi_{\mathfrak{k}}^{+(j)} \psi_{-\mathfrak{k}}^{-(j)} : .$$

The complete energy is counted by the sum over all j of such operators. In (8.1) we will define another operator L_0, which will also count the complete energy. In order to express the fermionic fields $\psi^{\pm(i)}(z)$ in terms of the bosonic fields $\alpha^{(ii)}(z)$, we need some additional operators Q_i, $i = 1, 2, \dots, n$, on F. These operators are uniquely defined by the following conditions:

$$Q_i|0\rangle = \psi^{+(i)}_{-\frac{1}{2}}|0\rangle, \qquad Q_i\psi^{\pm(j)}_{\mathfrak{k}} = (-1)^{\delta_{ij}+1}\psi^{\pm(j)}_{\mathfrak{k}\mp\delta_{ij}}Q_i. \tag{2.4}$$

They satisfy the following commutation relations:

$$Q_iQ_j = -Q_jQ_i \quad \text{if} \quad i \neq j, \qquad [\alpha^{(i)}_k, Q_j] = \delta_{ij}\delta_{k0}Q_j.$$

We shall use below the following notation

$$|k_1, k_2, \dots, k_n\rangle = Q_1^{k_1}Q_2^{k_2}\cdots Q_n^{k_n}|0\rangle, \qquad \langle k_1, k_2, \dots, k_n| = \langle 0|Q_n^{-k_n}\cdots Q_2^{-k_2}Q_1^{-k_1},$$

such that

$$\langle k_1, k_2, \dots, k_n|k_1, k_2, \dots, k_n\rangle = \langle 0|0\rangle = 1.$$

One easily checks the following relations:

$$[\alpha^{(i)}_k, \psi^{\pm(j)}_{\mathfrak{m}}] = \pm\delta_{ij}\psi^{\pm(j)}_{k+\mathfrak{m}}$$

and

$$Q_i^{\pm1}|k_1, k_2, \dots, k_n\rangle = (-)^{k_1+k_2+\cdots+k_{i-1}}|k_1, k_2, \dots, k_{i-1}, k_i \pm 1, k_{i+1}, \dots, k_n\rangle,$$
$$\langle k_1, k_2, \dots, k_n|Q_i^{\pm1} = (-)^{k_1+k_2+\cdots+k_{i-1}}\langle k_1, k_2, \dots, k_{i-1}, k_i \mp 1, k_{i+1}, \dots, k_n|.$$

These formula's lead to the following vertex operator expression for $\psi^{\pm(i)}(z)$. Given any sequence $s = (s_1, s_2, \dots)$, define

$$\Gamma^{(j)}_{\pm}(s) = \exp\left(\sum_{k=1}^{\infty} s_k\alpha^{(j)}_{\pm k}\right),$$

then

Theorem 1 ([6, 8]).

$$\psi^{\pm(i)}(z) = Q_i^{\pm1}z^{\pm\alpha^{(i)}_0}\exp\left(\mp\sum_{k<0}\frac{1}{k}\alpha^{(i)}_k z^{-k}\right)\exp\left(\mp\sum_{k>0}\frac{1}{k}\alpha^{(i)}_k z^{-k}\right)$$
$$= Q_i^{\pm1}z^{\pm\alpha^{(i)}_0}\Gamma^{(i)}_-(\pm[z])\Gamma^{(i)}_+(\mp[z^{-1}]),$$

where $[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots\right)$.

Note,

$$\Gamma^{(j)}_+(s)\,|k_1, k_2, \dots, k_n\rangle = |k_1, k_2, \dots, k_n\rangle, \qquad \langle k_1, k_2, \dots, k_n|\Gamma^{(j)}_-(s) = \langle k_1, k_2, \dots, k_n|.$$

Also observe that $(\Gamma^{(j)}_{\pm})^* = \Gamma^{(j)}_{\mp}$ and

$$\Gamma^{(j)}_+(s)\,\Gamma^{(k)}_-(s') = \gamma(s, s')^{\delta jk}\,\Gamma^{(k)}_-(s')\,\Gamma^{(j)}_+(s),$$

where

$$\gamma(s, s') = e^{\sum n\, s_n s'_n}.$$

We have

$$\Gamma^{(j)}_{\pm}(s)\,\psi^{+(k)}(z) = \gamma(s, [z^{\pm1}])^{\delta_{jk}}\psi^{+(k)}(z)\,\Gamma^{(j)}_{\pm}(s),$$
$$\Gamma^{(j)}_{\pm}(s)\,\psi^{-(k)}(z) = \gamma(s, -[z^{\pm1}])^{\delta_{jk}}\psi^{-(k)}(z)\,\Gamma^{(j)}_{\pm}(s).$$

Note that

$$\gamma(t, [z]) = \exp\left(\sum_{n\geq1} t_n\, z^n\right).$$

3 Tau functions as matrix elements and bilinear identities

Let A be an operator on F such that

$$[A \otimes A, \Omega] = 0, \qquad \Omega = \sum \psi^+_{\mathfrak{k}} \otimes \psi^-_{-\mathfrak{k}}. \tag{3.1}$$

Note that if $A \in GL_\infty$ then this A satisfies (3.1). So in the n-component case

$$\Omega = \sum_{\mathfrak{k},i} \psi^{+(i)}_{\mathfrak{k}} \otimes \psi^{-(i)}_{-\mathfrak{k}} = \sum_i \operatorname*{Res}_z \psi^{+(i)}(z) \otimes \psi^{-(i)}(z).$$

Here $\mathrm{Res}_z \sum_i f_i z^i = f_{-1}$. Define the following functions

$$\tau^{m_1,m_2,\dots,m_n}_{k_1,k_2,\dots,k_n}(t,s) = \langle k_1,k_2,\dots,k_n|\widetilde{A}\,|m_1,m_2,\dots,m_n\rangle,$$
$$\widetilde{A} = \prod_{i=1}^n \Gamma^{(i)}_+(t^{(i)})\, A \prod_{j=1}^n \Gamma^{(j)}_-(-s^{(j)}). \tag{3.2}$$

Instead of $\tau^{m_1,m_2,\dots,m_n}_{k_1,k_2,\dots,k_n}(t,s)$ we shall write $\tau^\beta_\alpha(t,s)$ for $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_n)$ and a similar expression for β. We will also use $|\alpha| = \alpha_1+\alpha_2+\dots+\alpha_n$, $|\alpha|_i = \alpha_1+\alpha_2+\dots+\alpha_i$, $|\alpha|_0 = 0$ and $\epsilon_j = (0,\dots,0,1,0,\dots,0)$, a 1 on the $j^{\rm th}$ place.

If the operator A satisfies (3.1), then so does $\widetilde{A}$. Following Okounkov [14] we calculate in two ways

$$\langle\alpha|\prod_{i=1}^n \Gamma^{(i)}_+(t^{(i)}) \otimes \langle\gamma|\prod_{k=1}^n \Gamma^{(k)}_+(s^{(k)})(A\otimes A)\,\Omega \prod_{j=1}^n \Gamma^{(j)}_-(-t'^{(j)})|\beta\rangle \otimes \prod_{\ell=1}^n \Gamma^{(\ell)}_-(-s'^{(\ell)})|\delta\rangle$$
$$= \langle\alpha|\prod_{i=1}^n \Gamma^{(i)}_+(t^{(i)}) \otimes \langle\gamma|\prod_{k=1}^n \Gamma^{(k)}_+(s^{(k)})\Omega(A\otimes A) \prod_{j=1}^n \Gamma^{(j)}_-(-t'^{(j)})|\beta\rangle \otimes \prod_{\ell=1}^n \Gamma^{(\ell)}_-(-s'^{(\ell)})|\delta\rangle. \tag{3.3}$$

Clearly

$$\psi^{+(m)}(z)\prod_{j=1}^n \Gamma^{(j)}_-(-t'^{(j)})|\beta\rangle = Q_m z^{\alpha^{(m)}_0}\Gamma^{(m)}_-([z])\Gamma^{(m)}_+(-[z^{-1}])\prod_{j=1}^n \Gamma^{(j)}_-(-t'^{(j)})|\beta\rangle$$
$$= (-)^{|\beta|_{m-1}} z^{\beta_m}\gamma([z^{-1}],t'^{(m)})\Gamma^{(m)}_-([z])\prod_{j=1}^n \Gamma^{(j)}_-(-t'^{(j)})|\beta+\epsilon_m\rangle$$
$$= (-)^{|\beta|_{m-1}} z^{\beta_m}\gamma([z^{-1}],t'^{(m)})\Gamma^{(m)}_-(-t'^{(m)}+[z])\prod_{j\neq m}^n \Gamma^{(j)}_-(-t'^{(j)})|\beta+\epsilon_m\rangle$$

and in a similar way

$$\psi^{-(m)}(z)\prod_{\ell=1}^n \Gamma^{(\ell)}_-(-s'^{(1)})|\delta\rangle$$
$$= (-)^{|\delta|_{m-1}} z^{-\delta_m}\gamma([z^{-1}],-s'^{(m)})\Gamma^{(m)}_-(-s'^{(m)}-[z])\prod_{\ell\neq m}\Gamma^{(\ell)}_-(-s'^{(\ell)})|\delta-\epsilon_m\rangle.$$

Also

$$\langle\alpha|\prod_{i=1}^n \Gamma^{(i)}_+(t^{(i)})\psi^{+(m)}(z) = \langle\alpha|\prod_{i=1}^n \Gamma^{(i)}_+(t^{(i)})Q_m z^{\alpha^{(m)}_0}\Gamma^{(m)}_-([z])\Gamma^{(m)}_+(-[z^{-1}])$$

$$
\begin{aligned}
&= (-)^{|\alpha|_{m-1}} z^{\alpha_m - 1} \gamma([z], t^{(m)}) \langle \alpha - \epsilon_m | \prod_{i=1}^{n} \Gamma_+^{(i)}(t^{(i)}) \Gamma_+^{(m)}(-[z^{-1}]) \\
&= (-)^{|\alpha|_{m-1}} z^{\alpha_m - 1} \gamma([z], t^{(m)}) \langle \alpha - \epsilon_m | \Gamma_+^{(m)}(t^{(m)} - [z^{-1}]) \prod_{i \neq m} \Gamma_+^{(i)}(t^{(i)})
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle \gamma | \prod_{k=1}^{n} \Gamma_+^{(k)}(s^{(k)}) \psi^{-(m)}(z) \\
&\qquad = (-)^{|\gamma|_{m-1}} z^{-\gamma_m - 1} \gamma(s^{(m)}, -[z]) \langle \gamma + \epsilon_m | \Gamma_+^{(m)}(s^{(m)} + [z^{-1}]) \prod_{k \neq m} \Gamma_+^{(k)}(s^{(k)}).
\end{aligned}
$$

Using this we rewrite (3.3):

$$
\begin{aligned}
&\operatorname*{Res}_z \sum_{m=1}^{n} (-)^{|\beta+\delta|_{m-1}} z^{\beta_m - \delta_m} \gamma([z^{-1}], t'^{(m)} - s'^{(m)}) \tau_\alpha^{\beta+\epsilon_m}(t^{(i)}, t'^{(j)} - \delta_{jm}[z]) \\
&\quad \times \tau_\gamma^{\delta-\epsilon_m}(s^{(k)}, s'^{(\ell)} + \delta_{\ell m}[z]) = \operatorname*{Res}_z \sum_{m=1}^{n} (-)^{|\alpha+\gamma|_{m-1}} z^{\alpha_m - \gamma_m - 2} \gamma([z], t^{(m)} - s^{(m)}) \\
&\quad \times \tau_{\alpha-\epsilon_m}^{\beta}(t^{(i)} - \delta_{im}[z^{-1}], t'^{(j)}) \tau_{\gamma+\epsilon_m}^{\delta}(s^{(k)} + \delta_{km}[z^{-1}], s'^{(\ell)}).
\end{aligned} \tag{3.4}
$$

For $n = 1$ this is the Toda lattice hierarchy of Ueno and Takasaki [15].

We now rewrite the left-hand side of (3.4) to obtain a more familiar form. For this we replace z by z^{-1} and write u, resp. v for t', resp. s', we thus obtain the following bilinear identity.

Proposition 1. *The tau functions satisfy the following identity:*

$$
\begin{aligned}
&\operatorname*{Res}_z \sum_{m=1}^{n} (-)^{|\beta+\delta|_{m-1}} z^{\delta_m - \beta_m - 2} \gamma([z], u^{(m)} - v^{(m)}) \tau_\alpha^{\beta+\epsilon_m}(t^{(i)}, u^{(j)} - \delta_{jm}[z^{-1}]) \\
&\quad \times \tau_\gamma^{\delta-\epsilon_m}(s^{(k)}, v^{(\ell)} + \delta_{\ell m}[z^{-1}]) = \operatorname*{Res}_z \sum_{m=1}^{n} (-)^{|\alpha+\gamma|_{m-1}} z^{\alpha_m - \gamma_m - 2} \gamma([z], t^{(m)} - s^{(m)}) \\
&\quad \times \tau_{\alpha-\epsilon_m}^{\beta}(t^{(i)} - \delta_{im}[z^{-1}], u^{(j)}) \tau_{\gamma+\epsilon_m}^{\delta}(s^{(k)} + \delta_{km}[z^{-1}], v^{(\ell)}).
\end{aligned} \tag{3.5}
$$

From the commutation relations (2.1) one easily deduces the following

Proposition 2. *Let $\lambda = +, -$, and $a_j \in \mathbb{C}$, for $1 \le j \le n$.*

(a) *The operator $A = \sum_{j=1}^{n} a_j \psi^{\lambda(j)}(z)$ satisfies* (3.1)*.*

(b) *Let $f_j(z) = \sum_i f_j^i z^i$, for $1 \le j \le n$, then the operator $A = \sum_{j=1}^{n} \operatorname{Res}_z f_j(z) \psi^{\lambda(j)}(z)$ satisfies* (3.1)*.*

Another way of obtaining solutions is as follows. We sketch the case $n = 2$ (see e.g. [7]) which is related to a two matrix model. Let $d\mu(x, y)$ be a measure (in general complex), supported either on a finite set of products of curves in the complex x and y planes or, alternatively, on a domain in the complex z plane, with identifications $x = z$, $y = \bar{z}$. Then, for each $1 \le j, k \le n$ and $\lambda, \nu = +, -$ the operator $A = e^B$ with

$$
B = \int \psi^{\lambda(j)}(x) \psi^{\nu(k)}(y) \, d\mu(x, y)
$$

satisfy (3.1). If one chooses $j = 1$, $k = 2$ and $\lambda = +$, $\nu = -$ and defines for the above A

$$\begin{aligned}\tau_{\alpha_1,\alpha_2}^{\alpha_1+N,\alpha_2-N}(t,u) &= \langle \alpha_1,\alpha_2|\Gamma_+^{(1)}(t^{(1)})\Gamma_+^{(2)}(t^{(2)})e^B\Gamma_-^{(1)}(-u^{(1)})\Gamma_-^{(2)}(-u^{(2)})|\alpha_1+N,\alpha_2-N\rangle\\ &= \frac{1}{N!}\langle \alpha_1,\alpha_2|\Gamma_+^{(1)}(t^{(1)})\Gamma_+^{(2)}(t^{(2)})B^N\,\Gamma_-^{(1)}(-u^{(1)})\Gamma_-^{(2)}(-u^{(2)})|\alpha_1+N,\alpha_2-N\rangle,\end{aligned}$$

then these tau-functions satisfy (3.5).

4 Wave functions and pseudo-differential equations

We will now rewrite the equations (3.5) in another form. Note that for

$$|\beta-\alpha| = |\delta-\gamma| = j \qquad \text{for fixed} \quad j\in\mathbb{Z} \tag{4.1}$$

these are the equations of the $2n$-component KP hierarchy, see [9]. One obtains equation (66) of [9] if one chooses $t^{(m+n)} = u^{(m)}$, $s^{(m+n)} = v^{(m)}$ and $\tau_\alpha^\beta(t) = (i)^{|\alpha|^2}\tau_{(\alpha,-\beta)}(t)$, where

$$\tau_{(\alpha,-\beta)}(t) = \tau_{(\alpha_1,\alpha_2,\dots,\alpha_n,-\beta_1,-\beta_2,\dots,-\beta_n)}(t)$$

are the $2n$-component KP tau-functions, viz for $(\alpha,-\beta)=\rho$ and $(\gamma,-\delta)=\sigma$ they satisfy the $2n$-component KP equations:

$$\begin{aligned}&\operatorname*{Res}_z \sum_{m=1}^{2n}(-)^{|\rho+\sigma|_{m-1}}z^{\rho_m-\sigma_m-2}\gamma([z],t^{(m)}-s^{(m)})\tau_{\rho-\epsilon_m}(t^{(i)}-\delta_{im}[z^{-1}])\\ &\quad\times\tau_{\sigma+\epsilon_m}(s^{(k)}+\delta_{km}[z^{-1}]) = 0.\end{aligned}$$

In [9] one showed that one can rewrite these equations to get $2n\times 2n$ matrix wave functions. Here we want to obtain two $n\times n$ matrix wave functions. *We assume from now on that* (4.1) *holds for* $j=0$. Denote ($s=0,1$):

$$\begin{aligned}&P^{\pm(s)}(\alpha,\beta,t,u,\pm z) = (P^{\pm(s)}(\alpha,\beta,t,u,\pm z)_{k\ell})_{1\le k,\ell\le n},\\ &P^{\pm(0)}(\alpha,\beta,t,u,\pm z)_{k\ell} = (-)^{|\epsilon_k|_{\ell-1}}z^{\delta_{k\ell}-1}\frac{\tau^\beta_{\alpha\pm(\epsilon_k-\epsilon_\ell)}(t^{(i)}\mp\delta_{i\ell}[z^{-1}],u^{(j)})}{\tau_\alpha^\beta(t^{(i)},u^{(j)})},\\ &P^{\pm(1)}(\alpha,\beta,t,u,\pm z)_{k\ell} = z^{-1}\frac{\tau^{\beta\pm\epsilon_\ell}_{\alpha\pm\epsilon_k}(t^{(i)},u^{(j)}\mp\delta_{j\ell}[z^{-1}])}{\tau_\alpha^\beta(t^{(i)},u^{(j)})},\\ &R^{\pm(s)}(\alpha,\beta,\pm z) = \operatorname{diag}(R^{\pm(s)}(\alpha,\beta,\pm z)_\ell),\\ &R^{\pm(0)}(\alpha,\beta,\pm z)_\ell = (-)^{|\alpha|_{\ell-1}}z^{\pm\alpha_\ell},\\ &R^{\pm(1)}(\alpha,\beta,\pm z)_\ell = (-)^{|\beta|_{\ell-1}}z^{\mp\beta_\ell},\\ &Q^\pm(t,\pm z) = \operatorname{diag}(\gamma([z],\pm t^{(1)}),\gamma([z],\pm t^{(2)}),\dots,\gamma([z],\pm t^{(n)})).\end{aligned} \tag{4.2}$$

Replace $t_1^{(a)}$, $s_1^{(a)}$, $u_1^{(a)}$, $v_1^{(a)}$ by $t_1^{(a)}+x_0$, $s_1^{(a)}+y_0$, $u_1^{(a)}+x_1$, $v_1^{(a)}+y_1$ then some of the above functions also depend on x_0 and x_1 we will add these variables to these functions and write e.g. $P^{\pm(0)}(\alpha,\beta,x,t,u,\pm z)$ for $P^{\pm(0)}(\alpha,\beta,t_i^{(j)}+\delta_{i1}x_0,u,\pm z)$. Introduce for $k=0,1$ the following differential symbols $\partial_k = \partial/\partial x_k$ $\partial'_k = \partial/\partial y_k$. Introduce the wave functions, here x is short hand notation for $x=(x_0,x_1)$

$$\begin{aligned}\Psi^{\pm(0)}(\alpha,\beta,x,t,u,z) &= P^{\pm(0)}(\alpha,\beta,x,t,u,\pm z)R^{\pm(0)}(\alpha,\beta,\pm z)Q^\pm(t,\pm z)e^{\pm x_0 z}\\ &= P^{\pm(0)}(\alpha,\beta,x,t,u,\partial_0)R^{\pm(0)}(\alpha,\beta,\partial_0)Q^\pm(t,\partial_0)e^{\pm x_0 z},\end{aligned}$$

$$\Psi^{\pm(1)}(\alpha,\beta,x,t,u,z) = P^{\pm(1)}(\alpha,\beta,x,t,u,\pm z)R^{\pm(1)}(\alpha,\beta,\pm z)Q^{\pm}(u,\pm z)e^{\pm x_1 z}$$
$$= P^{\pm(1)}(\alpha,\beta,x,t,u,\partial_1)R^{\pm(1)}(\alpha,\beta,\partial_1)Q^{\pm}(u,\partial_1)e^{\pm x_1 z}. \tag{4.3}$$

Then (3.5) leads to

$$\operatorname*{Res}_z \Psi^{+(0)}(\alpha,\beta,x,t,u,z)\Psi^{-(0)}(\gamma,\delta,y,s,v,z)^T$$
$$= \operatorname*{Res}_z \Psi^{+(1)}(\alpha,\beta,x,t,u,z)\Psi^{-(1)}(\gamma,\delta,y,s,v,z)^T. \tag{4.4}$$

One can also deduce the following 6 equations, for a proof see the appendix A:

$$P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,\partial_0)^{-1} = P^{-(0)}(\alpha,\beta,x_0,y_1,s,v,\partial_0)^*, \tag{4.5}$$

$$P^{+(1)}(\alpha,\beta,x_0,x_1,t,u,\partial_1)^{-1} = S(\partial_1)P^{-(1)}\left(\alpha+\sum_{i=1}^n \epsilon_i,\beta+\sum_{i=1}^n \epsilon_i,x_0,x_1,t,u,\partial_1\right)^* S(\partial_1), \tag{4.6}$$

$$\frac{\partial P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,\partial_0)}{\partial t_j^{(a)}}$$
$$= -(P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,\partial_0)E_{aa}\partial_0^j P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,\partial_0)^{-1})_-$$
$$\times P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,\partial_0), \tag{4.7}$$

$$\frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)}{\partial u_j^{(a)}} = -(P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)\partial_1^j E_{aa}P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)^{-1})_{<1}$$
$$\times P^{+(1)}(\alpha,\beta,x,t,u,\partial_1). \tag{4.8}$$

And

$$\sum_{j=1}^\infty \frac{\partial P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)}{\partial u_j^{(a)}} z^{-j-1}$$
$$= P^{+(1)}(\alpha,\beta,x,t,u,z)E_{aa}\partial_0^{-1}P^{-(1)}(\alpha,\beta,x,t,u,-z)^T P^{+(0)}(\alpha,\beta,x,t,u,\partial_0), \tag{4.9}$$

$$\left(E_{aa}+\sum_{j=1}^\infty \frac{\partial}{\partial t_j^{(a)}} z^{-j}\right)(P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)) = P^{+(0)}(\alpha,\beta,x,t,u,z)E_{aa}S(\partial_1)^{-1}$$
$$\times P^{-(0)}\left(\alpha+\sum_{i=1}^n \epsilon_i,\beta+\sum_{i=1}^n \epsilon_i,x,s,v,-z\right)^T S(\partial_1)P^{+(1)}(\alpha,\beta,x,t,u,\partial_1), \tag{4.10}$$

where

$$S(\partial) = \sum_{i=1}^n (-)^{i+1}E_{ii}\partial.$$

Recall that $(P(x)\partial^k)^* = (-\partial)^k \cdot P(x)^T$.

5 First reduction and the generalized AKNS model

Assume from now on that our tau functions satisfy

$$\tau_{\alpha-\sum_j \epsilon_j}^{\beta-\sum_j \epsilon_j} = (-)^{\sum_{i=1}^{n-1}|\beta-\alpha|_i} c\tau_\alpha^\beta, \qquad \text{where} \quad 0 \neq c \in \mathbb{C}.$$

This holds e.g. when

$$Q_1 Q_2 \cdots Q_n A = cA Q_1 Q_2 \cdots Q_n. \tag{5.1}$$

Substituting this in (4.2), gives

$$P^{\pm(i)}\left(\alpha - \sum_j \epsilon_j, \beta - \sum_j \epsilon_j, t, u, \partial_i\right) = \sum_{k=1}^{n} (-)^k E_{kk} P^{\pm(i)}(\alpha, \beta, t, u, \partial_i) \sum_{k=1}^{n} (-)^k E_{kk}. \tag{5.2}$$

Now from the second equation of (A.3) and (4.7) we deduce that

$$\sum_{j=1}^{n} \frac{\partial P^{+(0)}(\alpha, \beta, x_0, x_1, t, u, \partial_0)}{\partial t_1^{(j)}} = 0. \tag{5.3}$$

Hence this implies that

$$[\partial_0, P^{+(0)}(\alpha, \beta, x_0, x_1, t, u, \partial_0)] = 0. \tag{5.4}$$

In a similar way we deduce from (A.6) and (4.8) that

$$\sum_{j=1}^{n} \frac{\partial P^{+(1)}(\alpha, \beta, x_0, x_1, t, u, \partial_1)}{\partial u_1^{(j)}} = 0. \tag{5.5}$$

Hence also

$$[\partial_1, P^{+(1)}(\alpha, \beta, x_0, x_1, t, u, \partial_1)] = 0. \tag{5.6}$$

Remark. Note that the above does not imply that both

$$[\partial_1, P^{+(0)}(\alpha, \beta, x_0, x_1, t, u, \partial_0)] = 0 \qquad \text{and} \qquad [\partial_0, P^{+(1)}(\alpha, \beta, x_0, x_1, t, u, \partial_1)] = 0.$$

Next, using this reduction and (A.4) then (A.9) turns into

$$\begin{aligned} &\frac{\partial P^{+(1)}(\alpha, \beta, x, t, u, \partial_1)}{\partial x_0} \\ &\quad = \left(P_1^{+(0)}(\alpha, \beta, x, t, u)\partial_1^{-1} - \partial_1^{-1} P_1^{+(0)}(\alpha, \beta, x, t, u)\right) \partial_1 P^{+(1)}(\alpha, \beta, x, t, u, \partial_1) \\ &\quad = \sum_{j=1}^{\infty} (-)^{j+1} \frac{\partial^j P_1^{+(0)}(\alpha, \beta, x, t, u)}{\partial x_1^j} P^{+(1)}(\alpha, \beta, x, t, u, \partial_1)\partial_1^{-j}. \end{aligned} \tag{5.7}$$

Now, use (4.6) to see that (A.12) turns into

$$\begin{aligned} &\frac{\partial P^{+(0)}(\alpha, \beta, x, t, u, \partial_0)}{\partial x_1} \\ &\quad = P_1^{+(1)}(\alpha, \beta, x, t, u)\partial_0^{-1} P_1^{(1)}(\alpha, \beta, x, t, u,)^{-1} P^{+(0)}(k_1, k_2, x, t, u, \partial_0). \end{aligned} \tag{5.8}$$

Taking the coefficient of ∂_0^{-1} of this equation we thus get

$$\frac{\partial P_1^{+(0)}(\alpha, \beta, x, t, u)}{\partial x_1} = I$$

and hence (5.7) turns into

$$\frac{\partial P^{+(1)}(\alpha, \beta, x, t, u, \partial_1)}{\partial x_0} = P^{+(1)}(\alpha, \beta, x, t, u, \partial_1)\partial_1^{-1}.$$

In particular

$$\frac{\partial P_1^{+(1)}(\alpha,\beta,x,t,u)}{\partial x_0}=0.$$

Substituting this in (5.8) we obtain

$$\frac{\partial P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)}{\partial x_1}=P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)\partial_0^{-1}. \tag{5.9}$$

Note first that from (5.9) one deduces that

$$\begin{aligned}&\frac{\partial P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)^{-1}}{\partial x_1}\\ &\quad=-P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)^{-1}\frac{\partial P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)}{\partial x_1}P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)^{-1}\partial_0^{-1}\\ &\quad=-P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)^{-1}P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)\partial_0^{-1}P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)^{-1}\\ &\quad=-P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)^{-1}\partial_0^{-1}.\end{aligned}$$

Clearly also

$$\frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)^{-1}}{\partial x_0}=-P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)^{-1}\partial_1^{-1}.$$

Using (4.5) we rewrite (A.8):

$$\begin{aligned}&\frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)}{\partial t_j^{(a)}}=\operatorname*{Res}_z P^{+(0)}(\alpha,\beta,x,t,u,z)z^{j-1}E_{aa}\\ &\quad\times\partial_1^{-1}P^{+(0)}(\alpha,\beta,x,t,u,z)^{-1}\partial_1P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)\\ &\quad=\operatorname*{Res}_z P^{+(0)}(\alpha,\beta,x,t,u,z)z^{j-1}E_{aa}\\ &\quad\times\sum_{i=0}^{\infty}(-)^i\frac{\partial^iP^{+(0)}(\alpha,\beta,x,t,u,z)^{-1}}{\partial x_1^i}\partial_1^{-i}P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)\\ &\quad=\operatorname*{Res}_z P^{+(0)}(\alpha,\beta,x,t,u,z)E_{aa}\\ &\quad\times P^{+(0)}(\alpha,\beta,x,t,u,z)^{-1}\sum_{i=0}^{\infty}z^{j-i-1}P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)\partial_1^{-i}\\ &\quad=\operatorname*{Res}_z\sum_{i=0}^{j}z^{j-i-1}P^{+(0)}(\alpha,\beta,x,t,u,z)E_{aa}\\ &\quad\times P^{+(0)}(\alpha,\beta,x,t,u,z)^{-1}P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)\partial_1^{-i}.\end{aligned} \tag{5.10}$$

N.B. This summation starts with 0.

In particular

$$\frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)}{\partial t_j^{(1)}}+\frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)}{\partial t_j^{(2)}}=P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)\partial_1^{-j}. \tag{5.11}$$

In a similar way we deduce, using (4.6), from (A.10):

$$\frac{\partial P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)}{\partial u_j^{(a)}}=\operatorname*{Res}_z\sum_{i=1}^{j}z^{j-i-1}P^{+(1)}(\alpha,\beta,x,t,u,z)E_{aa}$$

$$\times P^{+(1)}(\alpha,\beta,x,t,u,z)^{-1}P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)\partial_0^{-i}. \tag{5.12}$$

N.B. This summation starts with 1.

In particular

$$\sum_{k=1}^{n}\frac{\partial P^{+(0)}(\alpha,\beta_2,x,t,u,\partial_0)}{\partial u_j^{(k)}} = P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)\partial_0^{-j}. \tag{5.13}$$

We will now combine (4.7), (4.8), (5.3)–(5.6) and (5.10)–(5.13). For this purpose we replace ∂_0 by the loop variable z and ∂_1 by the loop variable z^{-1}. We write

$$\begin{aligned} &P^{(0)}(\alpha,\beta,x,t,u,z) = P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,z) \qquad \text{and}\\ &P^{(1)}(\alpha,\beta,x,t,u,z) = z^{-1}P^{+(1)}(\alpha,\beta,x_0,x_1,t,u,z^{-1}) \end{aligned} \tag{5.14}$$

and thus obtain for fixed α and β ($P^{(j)}(x,t,u,z) = P^{(j)}(\alpha,\beta,x,t,u,z)$):

$$\begin{aligned} &\frac{\partial P^{(0)}(x,t,u,z)}{\partial x_0} = 0, \qquad \frac{\partial P^{(0)}(x,t,u,z)}{\partial x_1} = P^{(0)}(x,t,u,z)z^{-1},\\ &\frac{\partial P^{(1)}(x,t,u,z)}{\partial x_0} = P^{(1)}(x,t,u,z)z, \qquad \frac{\partial P^{(1)}(x,t,u,z)}{\partial x_1} = 0,\\ &\frac{\partial P^{(0)}(x,t,u,z)}{\partial t_j^{(a)}} = -(P^{(0)}(x,t,u,z)E_{aa}P^{(0)}(x,t,u,z)^{-1}z^j)_-P^{(0)}(x,t,u,z),\\ &\frac{\partial P^{(0)}(x,t,u,z)}{\partial u_j^{(a)}} = (P^{(1)}(x,t,u,z)E_{aa}P^{(1)}(x,t,u,z)^{-1}z^{-j})_-P^{(0)}(x,t,u,z),\\ &\frac{\partial P^{(1)}(x,t,u,z)}{\partial t_j^{(a)}} = (P^{(0)}(x,t,u,z)E_{aa}P^{(0)}(x,t,u,z)^{-1}z^j)_+P^{(1)}(x,t,u,z),\\ &\frac{\partial P^{(1)}(x,t,u,z)}{\partial u_j^{(a)}} = -(P^{(1)}(x,t,u,z)E_{aa}P^{(1)}(x,t,u,z)^{-1}z^{-j})_+P^{(1)}(x,t,u,z). \end{aligned} \tag{5.15}$$

Which are the generalized AKNS equations (2.5)–(2.10) of [1]. Write

$$P^{(0)}(x,t,u,z) = I + \sum_{i=1}^{\infty}P_i(x,t,u)z^{-i}, \qquad P^{(1)}(x,t,u,z) = \sum_{i=0}^{\infty}M_i(x,t,u)z^i$$

and from now on in this section

$$\partial_j = \frac{\partial}{\partial t_1^{(j)}}, \qquad \partial_{-j} = \frac{\partial}{\partial u_1^{(j)}}.$$

We thus obtain the following equations:

$$\begin{aligned} &\partial_jP_1 = [P_1,E_{jj}]P_1 + [E_{jj},P_2], \qquad \partial_{-j}P_1 = M_0E_{jj}M_0^{-1},\\ &\partial_jM_0 = [P_1,E_{jj}]M_0, \qquad \partial_jM_1 = E_{jj}M_0 + [P_1,E_{jj}]M_1,\\ &\partial_{-j}M_0 = M_0[E_{jj},M_0^{-1}M_1], \qquad \partial_{-j}M_1 = M_0[E_{jj},M_0^{-1}M_2]. \end{aligned} \tag{5.16}$$

From this one easily deduces the following equations for M_0:

$$\partial_i(M_0^{-1}\partial_{-j}M_0) = [E_{jj},M_0^{-1}E_{ii}M_0],$$

$$\partial_{-i}(M_0 E_{jj} M_0^{-1}) = \partial_{-j}(M_0 E_{ii} M_0^{-1}),$$
$$\partial_i(M_0^{-1} E_{jj} M_0) = \partial_j(M_0^{-1} E_{ii} M_0).$$

Note that if we define $\bar{P}_i = M_0^{-1} M_i$, then from (5.16) we get:

$$\begin{aligned} &\partial_j P_1 = [P_1, E_{jj}]P_1 + [E_{jj}, P_2], \qquad \partial_{-j} P_1 = M_0 E_{jj} M_0^{-1}, \qquad \partial_j \bar{P}_1 = M_0^{-1} E_{jj} M_0, \\ &\partial_j \bar{P}_1 = [\bar{P}_1, E_{jj}]\bar{P}_1 + [E_{jj}, \bar{P}_2], \qquad \partial_j M_0 = [P_1, E_{jj}]M_0, \qquad \partial_{-j} M_0 = M_0[E_{jj}, \bar{P}_1]. \end{aligned} \tag{5.17}$$

Note that for $x_0 = x_1 = 0$ we have

$$\begin{aligned} &(M_0(\alpha,\beta,t,u))_{kl} = \frac{\tau_{\alpha+\epsilon_k}^{\beta+\epsilon_\ell}(t,u)}{\tau_\alpha^\beta(t,u)}, \qquad (M_1(\alpha,\beta,t,u))_{kl} = -\frac{\partial_{-\ell}(\tau_{\alpha+\epsilon_k}^{\beta+\epsilon_\ell}(t,u))}{\tau_\alpha^\beta(t,u)}, \\ &(M_2(\alpha,\beta,t,u))_{kl} = \frac{1}{2}\frac{(\partial_{-\ell}^2 - \partial_{u_2^{(\ell)}})(\tau_{\alpha+\epsilon_k}^{\beta+\epsilon_\ell}(t,u))}{\tau_\alpha^\beta(t,u)}, \\ &(P_1(\alpha,\beta,t,u))_{kl} = \begin{cases} (-)^{|\epsilon_k|_{\ell-1}} \dfrac{\tau_{\alpha+\epsilon_k-\epsilon_\ell}^{\beta}(t,u)}{\tau_\alpha^\beta(t,u)} & \text{if} \quad k \neq \ell, \\ -\dfrac{\partial_\ell(\tau_\alpha^\beta(t,u))}{\tau_\alpha^\beta(t,u)} & \text{if} \quad k = \ell, \end{cases} \\ &(P_2(\alpha,\beta,t,u))_{kl} = \begin{cases} -(-)^{|\epsilon_k|_{\ell-1}} \dfrac{\partial_\ell\left(\tau_{\alpha+\epsilon_k-\epsilon_\ell}^{\beta}(t,u)\right)}{\tau_\alpha^\beta(t,u)} & \text{if} \quad k \neq \ell, \\ \dfrac{1}{2}\dfrac{\left(\partial_\ell^2 - \partial_{t_2^\ell}\right)\left(\tau_\alpha^\beta(t,u)\right)}{\tau_\alpha^\beta(t,u)} & \text{if} \quad k = \ell. \end{cases} \end{aligned} \tag{5.18}$$

6 AKNS and the two-component Camassa–Holm model

We still assume that A satisfies (5.1), hence that we have the reduction of the previous section. We consider the case $n = 2$ and define

$$y = \frac{t_1^{(1)} - t_1^{(2)}}{2}, \qquad \bar{y} = \frac{t_1^{(1)} + t_1^{(2)}}{2}, \qquad s = 2u_1^{(1)} - 2u_1^{(2)}, \qquad \bar{s} = 2u_1^{(1)} + 2u_1^{(2)}, \tag{6.1}$$

then

$$\frac{\partial}{\partial_y} = \partial_1 - \partial_2, \qquad \frac{\partial}{\partial_s} = \frac{1}{4}\left(\partial_{-1} - \partial_{-2}\right).$$

Now Let $E = E_{11} - E_{22}$ and define

$$\psi(z) = P^{(0)}(0,t,u,z)\begin{pmatrix} e^{yz} & 0 \\ 0 & e^{-yz} \end{pmatrix}$$

then (5.15) turns into:

$$\begin{aligned} \frac{\partial\psi(z)}{\partial y} &= (zE + [P_1, E])\psi(z) = \left(\begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\right)\psi(z), \\ \frac{\partial\psi(z)}{\partial s} &= \frac{z^{-1}}{4} M_0 E M_0^{-1}\psi(z) = z^{-1}\begin{pmatrix} A & B \\ C & -A \end{pmatrix}\psi(z), \end{aligned}$$

where

$$q = -2(P_1)_{12}, \qquad r = 2(P_1)_{21}, \qquad A = \frac{1}{4\det M_0}((M_0)_{11}(M_0)_{22} + (M_0)_{12}(M_0)_{21}),$$
$$B = \frac{-1}{2\det M_0}(M_0)_{11}(M_0)_{12}, \qquad C = \frac{1}{2\det M_0}(M_0)_{22}(M_0)_{21}.$$

Now (5.16) turns into:

$$\frac{\partial P_1}{\partial y} = [P_1, E]P_1 + [E, P_2] = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} P_1 + [E, P_2], \quad \frac{\partial P_1}{\partial s} = \frac{1}{4} M_0 E M_0^{-1} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},$$
$$\frac{\partial M_0}{\partial y} = [P_1, E]M_0 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} M_0, \quad \frac{\partial M_1}{\partial y} = EM_0 + [P_1, E]M_1 = EM_0 + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} M_1,$$
$$\frac{\partial M_0}{\partial s} = \frac{1}{4} M_0 [E, M_0^{-1} M_1], \qquad \frac{\partial M_1}{\partial s} = \frac{1}{4} M_0 [E, M_0^{-1} M_2]. \tag{6.2}$$

We will now describe transition from the matrix equations (6.2) to the 2-component Camassa–Holm (CH) model. The matrix P_1 is parametrized by $q = -2(P_1)_{12}$, $r = -2(P_1)_{21}$ and $\operatorname{Tr}(P_1 E) = -(\ln \tau_0^0)_y$, where r and q are variables of the AKNS model and τ_0^0 is its tau function. Furthermore, $A^2 + BC = 1/16$. Calculating $\operatorname{Tr}((P_1)_y E)$ using the first equation of (6.2) we get

$$rq = -(\ln \tau_0^0)_{yy}. \tag{6.3}$$

Next, from the second equation of (6.2) we find

$$q_s = -2B, \qquad r_s = 2C \tag{6.4}$$

and by taking $\operatorname{Tr}((P_1)_s E)$:

$$A = -\frac{1}{2}(\ln \tau_0^0)_{ys}. \tag{6.5}$$

By comparing eqs. (6.3) and (6.5) we deduce that $A_y = (rq)_s/2$.

From the third equation of (6.2) we derive:

$$(P_1)_{sy} = \frac{1}{4}\left(M_0 E M_0^{-1}\right)_y = \begin{pmatrix} A_y & B_y \\ C_y & -A_y \end{pmatrix} = \left[\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\right]$$

or in components:

$$B_y = -2qA, \qquad C_y = 2rA, \qquad A_y = -\frac{1}{2}(\ln \tau_0^0)_{syy} = qC - rB \tag{6.6}$$

and

$$r_{sy} = 4rA, \qquad q_{sy} = 4qA.$$

In view of the fact that the determinant of the matrix $(M_0 E M_0^{-1})/4$ is equal to the constant, $-1/16$, we choose to parametrize this matrix in terms of two parameters, A and f, which enter expressions for B and C as follows:

$$B = e^f\left(A - \frac{1}{4}\right), \qquad C = -e^{-f}\left(A + \frac{1}{4}\right).$$

Recalling equation (6.4) we easily find

$$r_s e^f - q_s e^{-f} = 2(Ce^f + Be^{-f}) = -1, \qquad r_s e^f + q_s e^{-f} = 2(Ce^f - Be^{-f}) = -4A. \tag{6.7}$$

Using the first two identities of equation (6.6) and the fact that

$$C_y e^f = f_y\left(A+\frac{1}{4}\right) - A_y, \qquad B_y e^{-f} = f_y\left(A-\frac{1}{4}\right) + A_y$$

we derive expressions for $re^f \pm qe^{-f}$ as follows:

$$re^f - qe^{-f} = \frac{C_y}{2A}e^f + \frac{B_y}{2A}e^{-f} = \frac{1}{2A}\left(f_y\left(A+\frac{1}{4}\right) - A_y + f_y\left(A-\frac{1}{4}\right) + A_y\right) = f_y \tag{6.8}$$

and

$$re^f + qe^{-f} = \frac{C_y}{2A}e^f - \frac{B_y}{2A}e^{-f} = \frac{1}{2A}\left(f_y\left(A+\frac{1}{4}\right) - A_y - f_y\left(A-\frac{1}{4}\right) - A_y\right) = \frac{f_y}{4A} - \frac{A_y}{A}. \tag{6.9}$$

Taking a derivative of relation $re^f - qe^{-f} = f_y$ with respect to variable s we find, in view of (6.7), that:

$$f_{ys} = r_s e^f - q_s e^{-f} + f_s(re^f + qe^{-f}) = -1 + f_s(re^f + qe^{-f})$$

or

$$re^f + qe^{-f} = 2g, \tag{6.10}$$

where we defined:

$$g = \frac{1}{2f_s}(1 + f_{sy}). \tag{6.11}$$

Comparing two expressions (6.9) and (6.10) for the quantity $re^f + qe^{-f}$ we find the following relation

$$2gA = \frac{f_y}{4} - A_y. \tag{6.12}$$

By adding and subtracting (6.10) and (6.8) we get

$$2re^f = 2g + f_y, \qquad 2qe^{-f} = 2g - f_y$$

or

$$q = e^f\left(g - \frac{f_y}{2}\right), \qquad r = e^{-f}\left(g + \frac{f_y}{2}\right).$$

Plugging these expressions into the second relation in eq. (6.7) yields:

$$\begin{aligned} -4A = r_s e^f + q_s e^{-f} &= \left(-f_s r + e^{-f}\left(g_s + \frac{f_{sy}}{2}\right)\right)e^f + \left(f_s q + e^f\left(g_s - \frac{f_{sy}}{2}\right)\right)e^{-f} \\ &= -f_s(re^f - qe^{-f}) + 2g_s = -f_s f_y + 2g_s. \end{aligned}$$

Thus,

$$A = \frac{1}{4}(f_s f_y - 2g_s) = \frac{1}{4}\left(f_s f_y - \left(\frac{1}{f_s}\right)_s - \left(\frac{f_{sy}}{f_s}\right)_s\right). \tag{6.13}$$

Plugging definition (6.11) of g into relation (6.12) leads to:

$$A = \frac{1}{4}f_s f_y - f_s A_y - A f_{sy} = \frac{1}{4}f_s f_y - (f_s A)_y.$$

Inserting on the left hand side of the above identity the value of A from equation (6.13) and multiplying by -4 yields

$$\begin{aligned}\left(\frac{1}{f_s}\right)_s &= -\left(\frac{f_{sy}}{f_s}\right)_s + 4(f_sA)_y = \left(-\frac{f_{ss}}{f_s} + 4f_sA\right)_y \\ &= \left(-\frac{f_{ss}}{f_s} + f_s^2 f_y - f_s\left(\frac{1+f_{sy}}{f_s}\right)_s\right)_y = \left(f_s^2 f_y - f_{ssy} + \frac{f_{ss}f_{sy}}{f_s}\right)_y, \end{aligned} \tag{6.14}$$

where we again used value of A from equation (6.13). Note that equation (6.14) is written solely in terms of f. For a quantity u defined as:

$$u = f_s^2 f_y - f_{ssy} + \frac{f_{ss}f_{sy}}{f_s} - \frac{1}{2}\kappa, \tag{6.15}$$

with κ being an integration constant, it holds from relation (6.14) that

$$u_y = \left(\frac{1}{f_s}\right)_s. \tag{6.16}$$

Let us now denote the product $f_s^2 f_y$ by m. Then from relations (6.15) and (6.16) we derive

$$m = f_s^2 f_y = u + f_{ssy} - \frac{f_{ss}f_{sy}}{f_s} + \frac{1}{2}\kappa = u - f_s\left(f_s\left(\frac{1}{f_s}\right)_s\right)_y + \frac{1}{2}\kappa = u - f_s(f_su_y)_y + \frac{1}{2}\kappa.$$

Taking a derivative of $m = f_s^2 f_y$ with respect to s yields

$$m_s = 2f_yf_sf_{ss} + f_s^2 f_{sy} = 2m\frac{f_{ss}}{f_s} + f_s^2 f_{sy} = -2mf_s\left(\frac{1}{f_s}\right)_s + f_s^2 f_{sy} = -2mf_su_y + f_s^2 f_{sy}. \tag{6.17}$$

In terms of the basic quantities u and $\rho = f_s$ of the two-component Camassa–Holm model equations (6.16) and (6.17) take the following form

$$\rho_s = -\rho^2 u_y, \tag{6.18}$$

$$m_s = -2m\rho u_y + \rho^2 \rho_y \tag{6.19}$$

for

$$m = u - \rho(\rho u_y)_y + \frac{1}{2}\kappa. \tag{6.20}$$

Performing an inverse reciprocal transformation $(y, s) \mapsto (x, t)$ defined by relations:

$$F_x = \rho\, F_y, \qquad F_t = F_s - \rho\, u\, F_y$$

for an arbitrary function F, we find that equations (6.18), (6.19) and (6.20) become

$$\rho_t = -(u\rho)_x, \tag{6.21}$$

$$m_t = -2mu_x - m_xu + \rho\rho_x, \tag{6.22}$$

$$m = u - u_{xx} + \frac{1}{2}\kappa \tag{6.23}$$

in terms of the (x, t) variables. Equations (6.21)–(6.23) were introduced by Liu and Zhang in [13] and are called the two-component Camassa–Holm equations (see also [4]).

The relation (6.14) is equivalent to the following condition

$$\frac{f_{ss}}{2f_s^3} + f_{sy}f_y + \frac{1}{2}f_s f_{yy} - \frac{f_{ssyy}}{2f_s} + \frac{f_{ssy}f_{sy}}{2f_s^2} + \frac{f_{ss}f_{syy}}{2f_s^2} - \frac{f_{ss}f_{sy}^2}{2f_s^3} = 0,$$

which first appeared in [4].

Comparing equations (6.13) and (6.15) we find that

$$4f_sA = u + \frac{1}{2}\kappa + \frac{f_{ss}}{f_s} = u - u_x + \frac{1}{2}\kappa$$

where

$$4f_sA = -\frac{r_{ss}}{2r_s}\left(A + \frac{1}{4}\right) + \frac{q_{ss}}{2q_s}\left(A - \frac{1}{4}\right) = -\frac{r_{ss}q_s}{2\left(A - \frac{1}{4}\right)} + \frac{q_{ss}r_s}{2\left(A + \frac{1}{4}\right)}$$

or

$$4f_sA = -2f_s(\ln\tau_0^0)_{sy} = -2(\ln\tau_0^0)_{sx}.$$

These relations give u, f_s in terms of the AKNS quantities r, q, τ_0^0.

7 A second reduction and the Cecotti–Vafa equations

In order to obtain the Cecotti–Vafa equations we define an automorphism of order 4 on the Clifford algebra by

$$\omega(\psi_{\mathfrak{k}}^{\pm(j)}) = (-)^{\mathfrak{k}+\frac{1}{2}} i\psi_{\mathfrak{k}}^{\mp(j)}$$

then $\omega(\psi^{\pm(j)}(z)) = i\psi^{\mp(j)}(-z)$ and

$$\omega(\alpha_m^{(j)}) = -(-)^m\alpha_m^{(j)}, \qquad \omega(Q_j^{\pm 1}) = iQ_j^{\mp 1}(-)^{\alpha_0^{(j)}}.$$

For the derivation of the last equation see [11]. Next, let

$$\omega(|0\rangle) = |0\rangle, \qquad \text{and} \qquad \omega(\langle 0|) = \langle 0|,$$

then this induces also an automorphism on the representation spaces F and F^*. It is straightforward to check that

$$\omega(|\alpha\rangle) = (-)^{\sum_{j=1}^n \frac{1}{2}|\alpha_j|(|\alpha_j|-1)} i^{\sum_{j=1}^n |\alpha_j|} |-\alpha\rangle,$$
$$\omega(\langle\alpha|) = (-)^{\sum_{j=1}^n \frac{1}{2}|\alpha_j|(|\alpha_j|+1)} i^{\sum_{j=1}^n |\alpha_j|} \langle -\alpha|.$$

Since

$$\omega(\psi_{\mathfrak{k}}^{\lambda(j)}\psi_{-\mathfrak{k}}^{-\lambda(j)}) = \psi_{\mathfrak{k}}^{-\lambda(j)}\psi_{-\mathfrak{k}}^{\lambda(j)}$$

one easily deduces that from (2.2) and (2.3) that

$$\begin{aligned}&\langle\omega(\langle V(\mathfrak{r}_1,\dots,\mathfrak{r}_m,\mathfrak{s}_1,\dots,\mathfrak{s}_q)|)|\omega(|V(\mathfrak{i}_1,\dots,\mathfrak{i}_k,\mathfrak{j}_1,\dots,\mathfrak{j}_\ell)\rangle)\rangle\\ &\qquad = \langle V(\mathfrak{r}_1,\dots,\mathfrak{r}_m,\mathfrak{s}_1,\dots,\mathfrak{s}_q)|V(\mathfrak{i}_1,\dots,\mathfrak{i}_k,\mathfrak{j}_1,\dots,\mathfrak{j}_\ell)\rangle.\end{aligned} \tag{7.1}$$

Assume for the second reduction that our A, which commutes with Ω, also satisfies

$$\omega(A) = A, \tag{7.2}$$

then one deduces from (7.1) that

$$\omega(\tau_\alpha^\beta(t,u)) = \tau_\alpha^\beta(t,u).$$

On the other hand if we use the definition (3.2) for the tau-functions one deduces

$$\omega(\tau_\alpha^\beta(t_m^{(j)}, u_p^{(q)})) = (-)^{\sum_{j=1}^n \frac{1}{2}|\alpha_j|(|\alpha_j|+1)+\frac{1}{2}|\beta_j|(|\beta_j|-1)} i^{\sum_{j=1}^n |\alpha_j|+|\beta_j|} \tau_{-\alpha}^{-\beta}(-(-)^m t_m^{(j)}, -(-)^p u_p^{(q)}).$$

From now on lets write $\tilde{t}_m^{(j)} = -(-)^m t_m^{(j)}$, then combining the above two equations, one has when (7.2) holds:

$$\tau_\alpha^\beta(t,u) = (-)^{\sum_{j=1}^n \frac{1}{2}|\alpha_j|(|\alpha_j|+1)+\frac{1}{2}|\beta_j|(|\beta_j|-1)} i^{\sum_{j=1}^n |\alpha_j|+|\beta_j|} \tau_{-\alpha}^{-\beta}(\tilde{t},\tilde{u}).$$

In particular

$$\tau_0^0(t,u) = \tau_0^0(\tilde{t},\tilde{u}), \qquad \tau_{\epsilon_k-\epsilon_\ell}^0(t,u) = -\tau_{\epsilon_\ell-\epsilon_k}^0(\tilde{t},\tilde{u}), \qquad \tau_{\epsilon_k}^{\epsilon_\ell}(t,u) = \tau_{-\epsilon_\ell}^{-\epsilon_k}(\tilde{t},\tilde{u}).$$

Now substituting this in (4.3) for $\alpha = \beta = 0$ we obtain that

$$\psi^{+(0)}(0,0,x,t,u,z) = \psi^{-(0)}(0,0,x,\tilde{t},\tilde{u},-z),$$
$$\psi^{+(1)}(0,0,x,t,u,z) = -\psi^{-(1)}(0,0,x,\tilde{t},\tilde{u},-z).$$

Assume from now on that A satisfies (5.1) *and* (7.2)*, viz. that both reductions hold*, then from (4.5), (4.6), (5.2) we deduce that

$$P^{+(0)}(0,0,x,t,u,z)^{-1} = P^{+(0)}(0,0,x,\tilde{t},\tilde{u},-z)^T,$$
$$P^{+(1)}(0,0,x,t,u,z)^{-1} = -P^{+(1)}(0,0,x,\tilde{t},\tilde{u},-z)^T z^2,$$

then using the definition (5.14) one finally obtains

$$P^{(0)}(0,0,x,t,u,z)^{-1} = P^{(0)}(0,0,x,\tilde{t},\tilde{u},-z)^T,$$
$$P^{(1)}(0,0,x,t,u,z)^{-1} = P^{(1)}(0,0,x,\tilde{t},\tilde{u},-z)^T.$$

Now putting all $t_{2m}^{(j)} = u_{2m}^{(j)} = 0$, this gives the following equations for P_1, M_0, M_1 and $\bar{P}_1$:

$$P_1^T = P_1, \qquad M_0^T = M_0^{-1}, \qquad M_1^T = M_0^T M_1 M_0^T, \qquad \bar{P}_1^T = \bar{P}_1. \tag{7.3}$$

Now denote

$$P_1 = (\beta_{ij})_{1\le i,j\le n}, \qquad M_0 = (m_{ij})_{1\le i,j\le n} \qquad \bar{P}_1 = (\bar{\beta}_{ij})_{1\le i,j\le n},$$

then the equations (7.3) and (5.17) give the following system of Cecotti–Vafa equations (see e.g. [1, 3, 5]):

$$\begin{aligned}
&\beta_{ij} = \beta_{ji}, \quad \bar{\beta}_{ij} = \bar{\beta}_{ji}, \quad \partial_j\beta_{ik} = \beta_{ij}\beta_{jk}, \quad \partial_{-j}\bar{\beta}_{ik} = \bar{\beta}_{ij}\bar{\beta}_{jk}, \qquad i,j,k \quad \text{distinct},\\
&\sum_{j=1}^n \partial_j\beta_{ik} = \sum_{j=1}^n \partial_j\bar{\beta}_{ik} = \sum_{j=1}^n \partial_{-j}\beta_{ik} = \sum_{j=1}^n \partial_{-j}\bar{\beta}_{ik} = 0, \qquad i \ne k,\\
&\partial_{-j}\beta_{ik} = m_{ij}m_{kj}, \qquad \partial_j\bar{\beta}_{ik} = m_{ji}m_{jk} \qquad i \ne k,\\
&\sum_{j=1}^n \partial_j m_{ik} = \sum_{j=1}^n \partial_{-j} m_{ik} = 0, \qquad \partial_j m_{ik} = \beta_{ij}m_{jk}, \qquad \partial_{-j} m_{ik} = m_{ij}\bar{\beta}_{jk}.
\end{aligned} \tag{7.4}$$

8 Homogeneity

Sometimes one wants to obtain solutions of the Cecotti–Vafa equations that satisfy certain homogeneity condition (see e.g. [5]). For this we introduce the L_0 element of a Virasoro algebra. The most natural definition in our construction of the Clifford algebra is the one given in terms of the oscillator algebra.

$$L_0 = \sum_{j=1}^{n} \frac{1}{2}(\alpha_0^{(j)})^2 + \sum_{k=1}^{\infty} \alpha_{-k}^{(j)}\alpha_k^{(j)}. \tag{8.1}$$

It is straightforward to check that

$$[L_0, \alpha_k^{(j)}] = -k\alpha_k^{(j)}$$

and

$$\langle\beta|L_0 = \frac{1}{2}\sum_{j=1}^{n}|\beta_j|^2\langle\beta|, \qquad L_0|\beta\rangle = \frac{1}{2}\sum_{j=1}^{n}|\beta_j|^2|\beta\rangle.$$

Moreover, one also has

$$\left[L_0, \psi_{\mathfrak{k}}^{\pm(j)}\right] = -\mathfrak{k}\psi_{\mathfrak{k}}^{\pm(j)}.$$

Assume from now on that our operator A that commutes with Ω is homogeneous of degree p with respect to L_0, i.e.,

$$[L_0, A] = pA.$$

We then calculate

$$\langle\alpha|L_0\tilde{A}|\beta\rangle = \langle\alpha|L_0\prod_{j=1}^{n}\Gamma_+^{(j)}(t^{(j)})A\prod_{k=1}^{n}\Gamma_-^{(k)}(-u^{(k)})|\beta\rangle.$$

It is straightforward to see that this is equal to

$$\frac{1}{2}\sum_{j=1}^{n}|\alpha_j|^2\langle\alpha|L_0\tilde{A}|\beta\rangle.$$

On the other hand using

$$[L_0, \Gamma_+^{(j)}(t^{(j)})] = -\sum_{k=1}^{\infty} kt_k^{(j)}\alpha_k^{(j)}\Gamma_+^{(j)}(t^{(j)}) = -\sum_{k=1}^{\infty} kt_k^{(j)}\frac{\partial}{\partial t_k^{(j)}}\Gamma_+^{(j)}(t^{(j)})$$

and

$$[L_0, \Gamma_-^{(j)}(-u^{(j)}] = \sum_{k=1}^{\infty} ku_k^{(j)}\frac{\partial}{\partial u_k^{(j)}}\Gamma_-^{(j)}(-u^{(j)})$$

we also have:

$$\langle\alpha|L_0\tilde{A}|\beta\rangle = \left(p + \frac{1}{2}\sum_{j=1}^{n}|\beta_j|^2 + \sum_{j=1}^{n}\sum_{k=1}^{\infty} ku_k^{(j)}\frac{\partial}{\partial u_k^{(j)}} - kt_k^{(j)}\frac{\partial}{\partial t_k^{(j)}}\right)\langle\alpha|\tilde{A}|\beta\rangle.$$

Now let

$$\mathcal{E} = \sum_{k=1}^{\infty} kt_k^{(j)} \frac{\partial}{\partial t_k^{(j)}} - ku_k^{(j)} \frac{\partial}{\partial u_k^{(j)}},$$

then

$$\mathcal{E}\tau_\alpha^\beta(t,u) = \left(p + \frac{1}{2}\sum_{j=1}^{n} |\beta_j|^2 - |\alpha_j|^2\right)\tau_\alpha^\beta(t,u),$$

in particular

$$\mathcal{E}\tau_0^0(t,u) = p\tau_0^0(t,u), \qquad \mathcal{E}\tau^0_{\epsilon_k-\epsilon_\ell}(t,u) = (p-1)\tau^0_{\epsilon_k-\epsilon_\ell}(t,u), \qquad \mathcal{E}\tau^{\epsilon_\ell}_{\epsilon_k}(t,u) = p\tau^{\epsilon_\ell}_{\epsilon_k}(t,u).$$

Assume now that we also have imposed the first and second reduction, then one has

$$\mathcal{E}P_1 = -P_1, \qquad \mathcal{E}P_0 = 0, \qquad \mathcal{E}M_1 = M_1$$

and thus also

$$\mathcal{E}\bar{P}_1 = \bar{P}_1.$$

Putting all $t_m^{(j)} = u_m^{(j)} = 0$ for all $m > 1$ and all $1 \le j \le n$ one has that the β_{ij}, m_{ij} and $\bar{\beta}_{ij}$ not only satisfy (7.4), but also

$$\sum_{j=1}^{n}(t^j\partial_j - u^j\partial_{-j})\beta_{ij} = -\beta_{ij}, \qquad \sum_{j=1}^{n}(t^j\partial_j - u^j\partial_{-j})m_{ij} = 0,$$
$$\sum_{j=1}^{n}(t^j\partial_j - u^j\partial_{-j})\bar{\beta}_{ij} = \bar{\beta}_{ij}, \tag{8.2}$$

for $t^j = t_1^{(j)}$ and $u^j = u_1^{(j)}$.

Note that in Section 10 we will construct explicit solutions of (7.4). These solutions are however not homogeneous, so they do not satisfy (8.2). We will construct such solutions in a forthcoming publication.

9 Explicit construction of solutions in the AKNS case

We will construct an operator A that satisfies (3.1) and (5.1) in the Camassa–Holm case, i.e., the case that $n = 2$. Now using Proposition 2, we see that the element

$$A_k = (a_1^{(1)}\psi^{\lambda_1(1)}(z_1) + a_1^{(2)}\psi^{\lambda_1(2)}(z_1))(a_2^{(1)}\psi^{\lambda_2(1)}(z_2) + a_2^{(2)}\psi^{\lambda_2(2)}(z_2))\cdots$$
$$\cdots(a_k^{(1)}\psi^{\lambda_k(1)}(z_k) + a_k^{(2)}\psi^{\lambda_k(2)}(z_k)),$$

satisfies condition (3.1). Using (2.4) and the definition of the fermionic fields, we see that

$$Q_1Q_2A_k = z_1^{\lambda_1}z_2^{\lambda_2}\cdots z_k^{\lambda_k}A_kQ_1Q_2.$$

Thus A_k also satisfies (5.1). Since we want $\tau_0^0 \neq 0$, we take A_{2k} and will assume that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_k = + \qquad \text{and} \qquad \lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{2k} = -.$$

We now want to calculate $\tau^0_{\epsilon_i-\epsilon_j}$ and $\tau^{\epsilon_j}_{\epsilon_i}$, for $1 \le i, j \le 2$ in order to get some solutions related to the Camassa–Holm equation. Let us start with τ^0_0:

$$\begin{aligned}
\tau^0_0(t,u) &= \langle 0| \prod_{i=1}^{2} \Gamma^{(i)}_+(t^{(i)})\, A_{2k} \prod_{j=1}^{2} \Gamma^{(j)}_-(-u^{(j)})|0\rangle \\
&= \sum_{\ell_1+\cdots \ell_k=\ell_{k+1}+\cdots \ell_{2k}} a_1^{(\ell_1)} a_2^{(\ell_2)} \cdots a_{2k}^{(\ell_{2k})} \langle 0| \prod_{i=1}^{2} \Gamma^{(i)}_+(t^{(i)})\, \psi^{+(\ell_1)}(z_1)\psi^{+(\ell_2)}(z_2)\cdots \\
&\quad \cdots \psi^{+(\ell_k)}(z_k)\psi^{-(\ell_{k+1})}(z_{k+1})\cdots \psi^{-(\ell_{2k})}(z_{2k}) \prod_{j=1}^{2} \Gamma^{(j)}_-(-u^{(j)})|0\rangle \\
&= \gamma(t^{(1)}, -u^{(1)})\gamma(t^{(2)}, -u^{(2)}) \sum_{\ell_1+\cdots \ell_k=\ell_{k+1}+\cdots \ell_{2k}} \prod_{i=1}^{k} a_i^{(\ell_i)} a_{i+k}^{(\ell_{i+k})} \gamma(t^{(\ell_i)}, [z_i]) \\
&\quad \times \gamma(-t^{(\ell_{k+i})}, [z_{k+i}])\gamma(u^{(\ell_i)}, [z_i^{-1}])\gamma(-u^{(\ell_{k+i})}, [z_{k+i}^{-1}]) \\
&\quad \times \langle 0|\psi^{+(\ell_1)}(z_1)\psi^{+(\ell_2)}(z_2)\cdots\cdots \psi^{+(\ell_k)}(z_k)\psi^{-(\ell_{k+1})}(z_{k+1})\cdots \psi^{-(\ell_{2k})}(z_{2k})|0\rangle.
\end{aligned}$$

So we have to calculate explicitly:

$$\begin{aligned}
&\langle 0|\psi^{+(\ell_1)}(z_1)\psi^{+(\ell_2)}(z_2)\cdots \psi^{+(\ell_k)}(z_k)\psi^{-(\ell_{k+1})}(z_{k+1})\cdots \psi^{-(\ell_{2k})}(z_{2k})|0\rangle \\
&\quad = \sigma(\ell_1, \ell_2, \dots \ell_{2k}) \frac{\prod_{1\le i<j\le k} (z_i - z_j)^{\delta_{\ell_i,\ell_j}} (z_{k+i} - z_{k+j})^{\delta_{\ell_{k+i},\ell_{k+j}}}}{\prod_{1\le i,j\le k} (z_i - z_{k+j})^{\delta_{\ell_i,\ell_{k+j}}}},
\end{aligned}$$

where

$$\sigma(\ell_1, \ell_2, \dots, \ell_{2k}) = \langle 0|Q_{\ell_1} Q_{\ell_2} \cdots Q_{\ell_k} Q^{-1}_{\ell_{k+1}} \cdots Q^{-1}_{\ell_{2k}}|0\rangle.$$

Note that the above expression takes only values -1, 0 or 1. Thus

$$\begin{aligned}
\tau^0_0(t,u) &= \gamma(t^{(1)}, -u^{(1)})\gamma(t^{(2)}, -u^{(2)}) \\
&\quad \times \sum_{\ell_1+\cdots \ell_k=\ell_{k+1}+\cdots \ell_{2k}} \sigma(\ell_1, \ell_2, \dots, \ell_{2k}) \frac{\prod_{1\le i<j\le k} (z_i - z_j)^{\delta_{\ell_i,\ell_j}} (z_{k+i} - z_{k+j})^{\delta_{\ell_{k+i},\ell_{k+j}}}}{\prod_{1\le i,j\le k} (z_i - z_{k+j})^{\delta_{\ell_i,\ell_{k+j}}}} \\
&\quad \times \prod_{i=1}^{k} a_i^{(\ell_i)} a_{i+k}^{(\ell_{i+k})} \gamma(t^{(\ell_i)}, [z_i])\gamma(-t^{(\ell_{k+i})}, [z_{k+i}])\gamma(u^{(\ell_i)}, [z_i^{-1}])\gamma(-u^{(\ell_{k+i})}, [z_{k+i}^{-1}]).
\end{aligned}$$

Now $\tau^0_{\epsilon_i-\epsilon_j}$ for $i \ne j$ has the same expression, except that we have to take the summation over

$$j + \ell_1 + \ell_2 + \cdots + \ell_k = i + \ell_{k+1} + \ell_{k+2} + \cdots + \ell_{2k}$$

and replace the σ by

$$(-)^i \sigma(j, \ell_1, \ell_2, \dots, \ell_{2k}, i).$$

For $\tau^{\epsilon_j}_{\epsilon_i}$, where i can be equal to j, we find

$$\tau^{\epsilon_j}_{\epsilon_i}(t,u) = (-)^{1-\delta_{ij}} \gamma(t^{(1)}, -u^{(1)})\gamma(t^{(2)}, -u^{(2)}) \sum_{j+\ell_1+\cdots \ell_k=i+\ell_{k+1}+\cdots \ell_{2k}} \sigma(j, \ell_1, \ell_2, \dots, \ell_{2k}, i)$$

$$\times z_1^{\delta_{j\ell_1}} z_2^{\delta_{j\ell_2}} \cdots z_k^{\delta_{j\ell_k}} z_{k+1}^{-\delta_{j\ell_{k+1}}} \cdots z_{2k}^{-\delta_{j\ell_{2k}}} \frac{\prod_{1\le i<j\le k} (z_i - z_j)^{\delta_{\ell_i,\ell_j}} (z_{k+i} - z_{k+j})^{\delta_{\ell_{k+i},\ell_{k+j}}}}{\prod_{1\le i,j\le k} (z_i - z_{k+j})^{\delta_{\ell_i,\ell_{k+j}}}}$$

$$\times \prod_{i=1}^k a_i^{(\ell_i)} a_{i+k}^{(\ell_{i+k})} \gamma(t^{(\ell_i)}, [z_i]) \gamma(-t^{(\ell_{k+i})}, [z_{k+i}]) \gamma(u^{(\ell_i)}, [z_i^{-1}]) \gamma(-u^{(\ell_{k+i})}, [z_{k+i}^{-1}]).$$

We now give the simplest of such expressions. We take $k = 1$ and put all $t_j^{(i)} = u_j^{(i)} = 0$ for $j > 2$, then

$$\tau_0^0(t,u) = \frac{T(t,u)}{z_1 - z_2}(T_{11}(t,u) + T_{22}(t,u)), \qquad \tau^0_{\epsilon_i-\epsilon_j}(t,u) = (-)^{i+1} T(t,u) T_{ij}(t,u),$$
$$\tau_{\epsilon_i}^{\epsilon_j}(t,u) = z_2^{-1} T(t,u) T_{ij}(t,u), \qquad \text{for} \qquad i \ne j,$$
$$\tau_{\epsilon_i}^{\epsilon_i}(t,u) = \frac{T(t,u)}{z_1 - z_2}\left(\left(\frac{z_1}{z_2}\right)^{\delta_{1i}} T_{11}(t,u) + \left(\frac{z_1}{z_2}\right)^{\delta_{2i}} T_{22}(t,u)\right),$$

where

$$T(t,u) = e^{-t_1^{(1)}u_1^{(1)} - t_1^{(2)}u_1^{(2)} - 2t_2^{(1)}u_1^{(2)} - 2t_2^{(2)}u_2^{(2)}},$$
$$T_{ij}(t,u) = T_{ij}(t,u,z_1,z_2)$$
$$= a_1^{(i)} a_2^{(j)} e^{t_1^{(i)} z_1 + u_1^{(i)} z_1^{-1} + t_2^{(i)} z_1^2 + u_2^{(i)} z_1^{-2} - (t_1^{(j)} z_2 + u_1^{(j)} z_2^{-1} + t_2^{(j)} z_2^2 + u_2^{(j)} z_2^{-2}}. \tag{9.1}$$

In order to determine M_0, M_1 and P_1, see (5.18), we also calculate

$$\partial_j \tau_0^0(t,u) = -u_1^{(j)} \frac{T(t,u)}{z_1 - z_2}(T_{11}(t,u) + T_{22}(t,u)) + T(t,u) T_{jj}(t,u),$$
$$\partial_{-j} \tau_{\epsilon_i}^{\epsilon_j}(t,u) = -(z_2^{-1} t_1^{(j)} + z_2^{-2}) T(t,u) T_{ij}(t,u), \qquad \text{for} \qquad i \ne j,$$
$$\partial_{-i} \tau_{\epsilon_i}^{\epsilon_i}(t,u) = -t_1^{(i)} \frac{T(t,u)}{z_1 - z_2}\left(\left(\frac{z_1}{z_2}\right)^{\delta_{1i}} T_{11}(t,u) + \left(\frac{z_1}{z_2}\right)^{\delta_{2i}} T_{22}(t,u)\right) - z_2^{-2} T(t,u) T_{ii}(t,u).$$

Now make the change of variables (6.1) and choose $t_2^{(i)} = u_2^{(i)} = 0$. One thus obtains

$$t_1^{(1)} = \bar{y} + y, \qquad t_1^{(2)} = \bar{y} - y, \qquad u_1^{(1)} = \frac{\bar{s} + s}{4}, \qquad u_1^{(2)} = \frac{\bar{s} - s}{4} \tag{9.2}$$

and

$$T(y,s) = e^{-\frac14(\bar{y}+y)(\bar{s}+s) - \frac14(\bar{y}-y)(\bar{s}-s)},$$
$$T_{ij}(y,s,z_1,z_2) = T_{ij}(y,s) = a_1^{(i)} a_2^{(j)} e^{(\bar{y}-(-)^i y)z_1 + \frac14(\bar{s}-(-)^i s)z_1^{-1} - ((\bar{y}-(-)^j y)z_2 + \frac14(\bar{s}-(-)^j s)z_2^{-1})}. \tag{9.3}$$

Note that in the calculation of M_0, M_1 and P_1, see (5.18), $T(y,s)$ drops out. Thus

$$M_0 = \left(\delta_{ij} + \frac{(z_1 z_2^{-1} - 1) T_{ij}(y,s)}{T_{11}(y,s) + T_{22}(y,s)}\right)_{1\le i,j\le 2},$$
$$M_1 = \left(\delta_{ij}(\bar{y} - (-)^j y) + \frac{(z_1 z_2^{-1} - 1)(z_2^{-1} + \bar{y} - (-)^j y) T_{ij}(y,s)}{T_{11}(y,s) + T_{22}(y,s)}\right)_{1\le i,j\le 2},$$
$$P_1 = \left(\delta_{ij} \frac{\bar{s} - (-)^j s}{4} - \frac{(z_1 - z_2) T_{ij}(y,s)}{T_{11}(y,s) + T_{22}(y,s)}\right)_{1\le i,j\le 2}.$$

One has $\det M_0 = z_1/z_2$ and M_0, M_1 and P_1 satisfy (6.2). The above solutions should in principle be closely related to the ones obtained in [2] using vertex method. Finally, we give the Camassa–Holm function f for this case:

$$f(y,s) = \log \frac{a_1^{(1)} a_2^{(1)} z_1 e^{\frac{s}{2z_1}+2yz_1} + a_1^{(2)} a_2^{(2)} z_2 e^{\frac{s}{2z_2}+2yz_2}}{(z_2 - z_1) a_1^{(2)} a_2^{(1)}}.$$

We give expression for tau functions for $k = 2$. Again we set $t_j^{(i)} = u_j^{(i)} = 0$ for $i, j > 2$ and use $T(t,u)$ and $T_{ij}(t,u,z_k,z_l)$ with $i, j = 1, 2$, $(k,\ell) = (1,3)$, $(k,\ell) = (2,4)$. For brevity we will denote

$$T = T(t,u), \qquad T_{ij}^{k,\ell} = T_{ij}(t,u,z_k,z_\ell).$$

Then, the τ_0^0 function is given by

$$\begin{aligned}\tau_0^0 = T\Bigg[&\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_3)(z_1-z_4)(z_2-z_4)}\big(T_{11}^{1,3}T_{11}^{2,4} + T_{22}^{1,3}T_{22}^{2,4}\big)\\ &- \frac{1}{(z_1-z_3)(z_2-z_4)}\big(T_{11}^{1,3}T_{22}^{2,4} + T_{22}^{1,3}T_{11}^{2,4}\big) + \frac{1}{(z_1-z_4)(z_2-z_3)}\big(T_{12}^{1,3}T_{21}^{2,4} + T_{21}^{1,3}T_{12}^{2,4}\big)\Bigg].\end{aligned}$$

In the following formulas for remaining tau functions we choose indices $i, j = 1, 2$ so that $i \neq j$. Then

$$\begin{aligned}\tau_{\epsilon_i-\epsilon_j} &= (-1)^{i+1} T\Bigg[\frac{(z_1-z_2)}{(z_1-z_3)(z_2-z_3)} T_{ii}^{1,3}T_{ij}^{2,4} - \frac{(z_1-z_2)}{(z_1-z_4)(z_2-z_4)} T_{ij}^{1,3}T_{ii}^{2,4}\\ &\quad + \frac{(z_3-z_4)}{(z_2-z_3)(z_2-z_4)} T_{ij}^{1,3}T_{jj}^{2,4} - \frac{(z_3-z_4)}{(z_1-z_3)(z_1-z_4)} T_{jj}^{1,3}T_{ij}^{2,4}\Bigg],\\ \tau_{\epsilon_j}^{\epsilon_i} &= -T\Bigg[\frac{z_1(z_3-z_4)}{z_3z_4(z_1-z_3)(z_1-z_4)} T_{ii}^{1,3}T_{ji}^{2,4} - \frac{z_2(z_3-z_4)}{z_3z_4(z_2-z_3)(z_2-z_4)} T_{ji}^{1,3}T_{ii}^{2,4}\\ &\quad + \frac{(z_1-z_2)}{z_3(z_1-z_4)(z_2-z_4)} T_{ji}^{1,3}T_{jj}^{2,4} - \frac{(z_1-z_2)}{z_4(z_1-z_3)(z_2-z_3)} T_{jj}^{1,3}T_{ji}^{2,4}\Bigg],\\ \tau_{\epsilon_i}^{\epsilon_i} &= T\Bigg[\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_1-z_4)(z_2-z_3)(z_2-z_4)}\left(\frac{z_1z_2}{z_3z_4} T_{ii}^{1,3}T_{ii}^{2,4} + T_{jj}^{1,3}T_{jj}^{2,4}\right)\\ &\quad - \frac{1}{(z_1-z_3)(z_2-z_4)}\left(\frac{z_1}{z_3} T_{ii}^{1,3}T_{jj}^{2,4} + \frac{z_2}{z_4} T_{jj}^{1,3}T_{ii}^{2,4}\right)\\ &\quad + \frac{1}{(z_1-z_4)(z_2-z_3)}\left(\frac{z_1}{z_4} T_{ij}^{1,3}T_{ji}^{2,4} + \frac{z_2}{z_3} T_{ji}^{1,3}T_{ij}^{2,4}\right)\Bigg].\end{aligned}$$

If we make the change of variables (9.2) then the formula's (9.1) change into (9.3) and the same formula's hold for the tau functions.

Another way to construct solutions is as follows. We take an arbitrary element of the loop algebra of $sl_2(\mathbb{C})$. Such an element is given by

$$\begin{aligned}g = \operatorname*{Res}_z\, & a(z)\big(\psi^{+(1)}(z)\psi^{-(1)}(z) - \psi^{+(2)}(z)\psi^{-(2)}(z)\big) + b(z)\psi^{+(1)}(z)\psi^{-(2)}(z)\\ &+ c(z)\psi^{+(2)}(z)\psi^{-(2)}(z),\end{aligned}$$

where $a(z)$, $b(z)$ and $c(z)$ are arbitrary functions. Then the element $A = e^g$ commutes with the action of Ω and satisfies (5.1). Hence the corresponding tau-functions will satisfy the equations of the AKNS hierarchy.

10 Explicit construction of solutions in the Cecotti–Vafa case

We will construct an operator A that satisfies (3.1), (5.1) and (7.2). We generalize the construction of the previous section. Using Proposition 2, we see that the element

$$A_k = \left(\sum_{j_1=1}^n a_1^{(j_1)}\psi^{+(j_1)}(z_1)\right)\left(\sum_{j_1=1}^n a_1^{(j_1)}\psi^{-(j_1)}(-z_1)\right)\left(\sum_{j_1=1}^n a_2^{(j_2)}\psi^{+(j_2)}(z_2)\right) \times\left(\sum_{j_1=1}^n a_2^{(j_2)}\psi^{-(j_2)}(-z_2)\right)\cdots\left(\sum_{j_k=1}^n a_k^{(j_k)}\psi^{+(j_k)}(z_k)\right)\left(\sum_{j_k=1}^n a_k^{(j_k)}\psi^{-(j_k)}(-z_k)\right),$$

satisfies condition (3.1). Moreover, if we assume that

$$\sum_{j=1}^n (a_\ell^{(j)})^2 = 0 \qquad \text{for all} \quad 1\le \ell\le k, \tag{10.1}$$

then

$$\left(\sum_{j=1}^n a_\ell^{(j)}\psi^{+(j)}(z)\right)\left(\sum_{k=1}^n a_\ell^{(k)}\psi^{-(k)}(y)\right) = -\left(\sum_{k=1}^n a_\ell^{(k)}\psi^{-(k)}(y)\right)\left(\sum_{j=1}^n a_\ell^{(j)}\psi^{+(j)}(z)\right).$$

Thus

$$\omega\left(\left(\sum_{j=1}^n a_\ell^{(j)}\psi^{+(j)}(z)\right)\left(\sum_{k=1}^n a_\ell^{(k)}\psi^{-(k)}(-z)\right)\right) = -\left(\sum_{j=1}^n a_\ell^{(j)}\psi^{-(j)}(-z)\right)\left(\sum_{k=1}^n a_\ell^{(k)}\psi^{+(k)}(z)\right) = \left(\sum_{k=1}^n a_\ell^{(k)}\psi^{+(k)}(z)\right)\left(\sum_{j=1}^n a_\ell^{(j)}\psi^{-(j)}(-z)\right)$$

and hence A_k also satisfies (7.2). Let us calculate for $k=1$ the corresponding tau functions. One finds

$$\tau_0^0 = \frac{T}{2z}\sum_{j=1}^n T_{jj}(z), \qquad \tau^0_{\epsilon_i-\epsilon_j} = (-)^{|\epsilon_j|_i}T(z)T_{ij}(z), \qquad \tau^{\epsilon_j}_{\epsilon_i} = -\frac{T}{z}T_{ij}(z) \qquad \text{for} \qquad i\ne j,$$

$$\tau^{\epsilon_i}_{\epsilon_i} = \frac{T}{2z}\sum_{j=1}^n (-)^{\delta_{ij}}T_{jj}(z),$$

where

$$T = \prod_{i=1}^n \gamma(t^{(i)}, -u^{(i)}),$$

$$T_{ij}(z_k) = a_k^{(i)}a_k^{(j)}\gamma(t^{(i)},[z_k])\gamma(-t^{(j)},[-z_k])\gamma(u^{(i)},[z_k^{-1}])\gamma(-u^{(j)},[-z_k^{-1})]).$$

If we only keep $t_1^{(i)}$ and $u_1^{(i)}$ and put all higher times equal to zero, we find that

$$\beta_{ij} = zm_{ij} = z^2\bar\beta_{ij}, \qquad \text{for} \qquad i\ne j$$

and that

$$m_{ij} = \delta_{ij} - \frac{2a_1^{(i)}a_1^{(j)}e^{(t_1^{(i)}+t_1^{(j)})z+(u_1^{(i)}+u_1^{(j)})z^{-1}}}{\sum_{i=1}^n (a_1^{(i)})^2 e^{2t_1^{(i)}z+2u_1^{(i)}z^{-1}}},$$

where (10.1) still holds. These β_{ij}, m_{ij} and $\bar\beta_{ij}$ satisfy (7.4).

For the case that $k = 2$ we describe the tau functions

$$\tau_0^0 = T\bigg(\frac{(z_1 - z_2)^2}{4z_1 z_2(z_1 + z_2)^2}\sum_{i=1}^{n} T_{ii}(z_1)T_{ii}(z_2) + \sum_{i\neq j}\frac{1}{4z_1 z_2}T_{ii}(z_1)T_{jj}(z_2) - \sum_{i\neq j}\frac{1}{(z_1 + z_2)^2}T_{ij}(z_1)T_{ji}(z_2)\bigg),$$

$$\tau_{\epsilon_k}^{\epsilon_k} = T\bigg(\frac{(z_1 - z_2)^2}{4z_1 z_2(z_1 + z_2)^2}\sum_{i=1}^{n} T_{ii}(z_1)T_{ii}(z_2) + \sum_{i\neq j}(-)^{\delta_{ik}+\delta_{jk}}\frac{1}{4z_1 z_2}T_{ii}(z_1)T_{jj}(z_2) - \sum_{i\neq j}\Big(-\frac{z_1}{z_2}\Big)^{\delta_{ik}}\Big(-\frac{z_2}{z_1}\Big)^{\delta_{jk}}\frac{1}{(z_1 + z_2)^2}T_{ij}(z_1)T_{ji}(z_2)\bigg),$$

$$\tau_{\epsilon_i-\epsilon_j}^0 = T\bigg(\frac{z_2 - z_1}{2(z_1 + z_2)}\Big(\frac{1}{z_2}T_{ij}(z_1)(T_{ii}(z_2) + T_{jj}(z_2)) - \frac{1}{z_1}(T_{jj}(z_1) + T_{ii}(z_1))T_{ij}(z_2)\Big) + \sum_{k\neq i,j}\Big(\frac{1}{z_1 + z_2}(T_{ik}(z_1)T_{kj}(z_2) + T_{ik}(z_2)T_{kj}(z_1)) - \frac{1}{2z_2}T_{ij}(z_1)T_{kk}(z_2) - \frac{1}{2z_1}T_{kk}(z_1)T_{ij}(z_2)\Big)\bigg),$$

$$\tau_{\epsilon_i}^{\epsilon_j} = T\bigg(\frac{z_2 - z_1}{2z_1 z_2(z_1 + z_2)}(T_{ij}(z_1)(T_{ii}(z_2) - T_{jj}(z_2)) + (T_{jj}(z_1) - T_{ii}(z_1))T_{ij}(z_2)) + \sum_{k\neq i,j}\Big(\frac{1}{z_1 + z_2}\Big(\frac{1}{z_2}T_{ik}(z_1)T_{kj}(z_2) + \frac{1}{z_1}T_{ik}(z_2)T_{kj}(z_1)\Big) - \frac{1}{2z_1 z_2}(T_{ij}(z_1)T_{kk}(z_2) - T_{kk}(z_1)T_{ij}(z_2))\Big)\bigg).$$

A Appendix

In this appendix we want to proof the equations (4.5)–(4.10). For the proof of these equations we need the following lemma:

Lemma 1.

$$\mathrm{Res}_z P(x_i, t, \partial_i)e^{x_i z}(Q(x_i', t', \partial_i')e^{-x_i' z})^T = \sum_j R_j(x_i, t)S_j(x_i', t')$$

if and only if

$$\big(P(x_i, t, \partial_i)Q(x_i, t', \partial_i)^*\big)_- = \sum_j R_j(x_i, t)\partial_i^{-1}S_j(x_i, t')$$

This Lemma is a consequence of Lemma 4.1 of [9]. We rewrite (4.4):

$$\operatorname*{Res}_z P^{+(0)}(\alpha, \beta, x, t, u, z)R^{+(0)}(\alpha, \beta, z)Q^+(t, z)e^{x_0 z}(P^{-(0)}(\gamma, \delta, y, s, v, -z) \times R^{-(0)}(\gamma, \delta, -z)Q^-(s, -z)e^{-y_0 z})^T = \operatorname*{Res}_z P^{+(1)}(\alpha, \beta, x, t, u, z)R^{+(1)}(\alpha, \beta, z) \times Q^+(u, z)e^{x_1 z}(P^{-(1)}(\gamma, \delta, y, s, v, -z)R^{-(1)}(\gamma, \delta, -z)Q^-(v, -z)e^{-y_1 z})^T. \tag{A.1}$$

First, applying the Lemma to (A.1) gives

$$(P^{+(0)}(\alpha, \beta, x_0, x_1, t, u, \partial_0)R^{+(0)}(\alpha, \beta, \partial_0)Q^+(t, \partial_0)(P^{-(0)}(\gamma, \delta, x_0, y_1, s, v, \partial_0)$$

$$\times R^{-(0)}(\gamma,\delta,\partial_0)Q^-(s,\partial_0))^*)_- = \operatorname*{Res}_z P^{+(1)}(\alpha,\beta,x_0,x_1,t,u,z)R^{+(1)}(\alpha,\beta,z)Q^+(u,z)$$
$$\times e^{x_1 z}\partial_0^{-1}(P^{-(1)}(\gamma,\delta,x_0,y_1,s,v,-z)R^{-(1)}(\gamma,\delta,-z)Q^-(v,-z)e^{-y_1 z})^T.$$

Now putting $s=t$, $u=v$, $x_1=y_1$, we obtain

$$(P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,\partial_0)P^{-(0)}(\alpha,\beta,x_0,y_1,s,v,\partial_0)^*)_- = 0, \tag{A.2}$$

$$\left(P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,\partial_0)S(\partial_0)P^{-(0)}(\alpha-\sum_{j=1}^n \epsilon_j,\beta-\sum_{j=1}^n \epsilon_j,x_0,y_1,s,v,\partial_0)^*\right)_- = 0. \tag{A.3}$$

Since

$$P^{\pm(0)}(\alpha,\beta,x_0,x_1,t,u,\partial_0) = I + \sum_{j=1}^\infty P_j^{\pm(0)}(\alpha,\beta,x_0,x_1,t,u)\partial_0^{-j}$$
$$P^{\pm(1)}(\alpha,\beta,x_0,x_1,t,u,\partial_1) = \sum_{j=1}^\infty P_j^{\pm(1)}(\alpha,\beta,x_0,x_1,t,u)\partial_1^{-j},$$

this implies (4.5), one thus also has:

$$P_1^{-(0)}(\alpha,\beta,x_0,x_1,s,v)^T = P_1^{+(0)}(\alpha,\beta,x_0,x_1,s,v). \tag{A.4}$$

If we apply the Lemma in the other way we obtain

$$(P^{+(1)}(\alpha,\beta,x_0,x_1,t,u,\partial_1)R^{+(1)}(\alpha,\beta,\partial_1)Q^+(u,\partial_1)(P^{-(1)}(\gamma,\delta,y_0,x_1,s,v,\partial_1)$$
$$\times R^{-(1)}(\gamma,\delta,\partial_1)Q^-(v,\partial_1))^*)_- = \operatorname*{Res}_z P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,z)R^{+(0)}(\alpha,\beta,z)$$
$$\times Q^+(t,z)e^{x_0 z}\partial_1^{-1}(P^{-(0)}(\gamma,\delta,y_0,x_1,s,v,-z)R^{-(0)}(\gamma,\delta,-z)Q^-(s,-z)e^{-y_0 z})^T. \tag{A.5}$$

Now taking $s=t$, $u=v$ and $y_0=x_0$, we find

$$(P^{+(1)}(\alpha,\beta,x_0,x_1,t,u,\partial_1)R^{+(1)}(\alpha-\gamma,\beta-\delta,\partial_1)P^{-(1)}(\gamma,\delta,x_0,x_1,t,u,\partial_1)^*)_-$$
$$= \operatorname*{Res}_z P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,z)R^{+(0)}(\alpha-\gamma,\beta-\delta,z)\partial_1^{-1}P^{-(0)}(\gamma,\delta,x_0,x_1,t,u,-z)^T. \tag{A.6}$$

Now choose $\gamma-\alpha=\sum_{i=1}^n \epsilon_i$, then

$$R^{+(0)}(\alpha-\gamma,\beta-\delta,z) = \sum_{i=1}^n (-)^{i+1}z^{-1}E_{ii} = S(z)^{-1}.$$

Since we have assumed that (4.1) holds for $j=0$

$$|\delta-\beta| = |\gamma-\alpha| = n.$$

Now choose $\delta-\beta=\sum_{i=1}^n \epsilon_i$, then (A.5) turns into

$$P^{+(1)}(\alpha,\beta,x_0,x_1,t,u,\partial_1)S(\partial_1)$$
$$\times P^{-(1)}\left(\alpha+\sum_{i=1}^n \epsilon_i,\beta+\sum_{i=1}^n \epsilon_i,x_0,x_1,t,u,\partial_1\right)^* = S(\partial_1)^{-1},$$

thus we obtain (4.6).

Differentiate the bilinear identity (A.1) to $t_j^{(a)}$, one obtains

$$\begin{aligned}
&\operatorname*{Res}_z\left(\frac{\partial P^{+(0)}(\alpha,\beta,x,t,u,z)}{\partial t_j^{(a)}}+P^{+(0)}(\alpha,\beta,x,t,u,z)z^jE_{aa}\right)R^{+(0)}(\alpha,\beta,z)\\
&\quad\times Q^+(t,z)e^{x_0z}(P^{-(0)}(\gamma,\delta,y,s,v,-z)R^{-(0)}(\gamma,\delta,-z)Q^-(s,-z)e^{-y_0z})^T\\
&\quad=\operatorname*{Res}_z\frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,z)}{\partial t_j^{(a)}}R^{+(1)}(\alpha,\beta,z)Q^+(u,z)e^{x_1z}\\
&\quad\times(P^{-(1)}(\gamma,\delta,y,s,v,-z)R^{-(1)}(\gamma,\delta,-z)Q^-(v,-z)e^{-y_1z})^T. \qquad (A.7)
\end{aligned}$$

Now using the Lemma and choosing $\alpha=\gamma$, $\beta=\delta$, $s=t$, $u=v$ and $x_1=y_1$ this gives the familiar Sato–Wilson equations (4.7). Taking $j=1$ one obtains

$$\sum_{i=1}^n\frac{\partial P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,\partial_0)}{\partial t_1^{(i)}}=[\partial_0,P^{+(0)}(\alpha,\beta,x_0,x_1,t,u,\partial_0)].$$

Return to (A.7) and use the Lemma in the opposite way and choosing, $\gamma-\alpha=\delta-\beta=\sum\limits_{i=1}^n\epsilon_i$, $s=t$, $u=v$ and $x_0=y_0$ this gives:

$$\begin{aligned}
&\operatorname*{Res}_z P^{+(0)}(\alpha,\beta,x,t,u,z)z^jE_{aa}S(z)^{-1}\partial_1^{-1}P^{-(0)}\left(\alpha+\sum_{i=1}^n\epsilon_i,\beta+\sum_{i=1}^n\epsilon_i,x,s,v,-z\right)^T\\
&\quad=\frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)}{\partial t_j^{(a)}}S(\partial_1)P^{-(1)}\left(\alpha+\sum_{i=1}^n\epsilon_i,\beta+\sum_{i=1}^n\epsilon_i,y,s,v,-z\right)^*
\end{aligned}$$

using (4.6) one deduces:

$$\begin{aligned}
&\frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)}{\partial t_j^{(a)}}=\operatorname*{Res}_z z^{j-1}P^{+(0)}(\alpha,\beta,x,t,u,z)E_{aa}S(\partial_1)^{-1}\\
&\quad\times P^{-(0)}(\alpha+\sum_{i=1}^n\epsilon_i,\beta+\sum_{i=1}^n\epsilon_i,x,s,v,-z)^TS(\partial_1)P^{+(1)}(\alpha,\beta,x,t,u,\partial_1). \qquad (A.8)
\end{aligned}$$

Note that we can rewrite (A.8) as (4.10). Since we have replaced in the wave matrices $t_1^{(a)}$, by $t_1^{(a)}+x_0$, (A.8) for $j=1$ implies

$$\begin{aligned}
&\frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)}{\partial x_0}=\operatorname*{Res}_z P^{+(0)}(\alpha,\beta,x,t,u,z)S(\partial_1)^{-1}\\
&\quad\times P^{-(0)}(\alpha+\sum_{i=1}^n\epsilon_i,\beta+\sum_{i=1}^n\epsilon_i,x,s,v,-z)^TS(\partial_1)P^{+(1)}(\alpha,\beta,x,t,u,\partial_1). \qquad (A.9)
\end{aligned}$$

Differentiating the bilinear identity to $u_j^{(a)}$ gives that

$$\begin{aligned}
&\operatorname*{Res}_z\frac{\partial P^{+(0)}(\alpha,\beta,x,t,u,z)}{\partial u_j^{(a)}}R^{+(0)}(\alpha,\beta,z)Q^+(t,z)e^{x_0z}\\
&\quad\times(P^{-(0)}(\gamma,\delta,y,s,v,-z)R^{-(0)}(\gamma,\delta,-z)Q^-(s,-z)e^{-y_0z})^T\\
&\quad=\operatorname*{Res}_z\left(\frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,z)}{\partial u_j^{(a)}}+P^{+(1)}(\alpha,\beta,x,t,u,z)z^jE_{aa}\right)R^{+(1)}(\alpha,\beta,z)
\end{aligned}$$

$$\times Q^{+}(u,z)e^{x_1 z}(P^{-(1)}(\gamma,\delta,y,s,v,-z)R^{-(1)}(\gamma,\delta,-z)Q^{-}(v,-z)e^{-y_1 z})^T.$$

Using the Lemma and next choosing, $\alpha=\gamma$, $\beta=\delta$, $s=t$, $u=v$ and $x_1=y_1$ we obtain

$$\frac{\partial P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)}{\partial u_j^{(a)}}P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)^{-1} = \operatorname*{Res}_z z^j P^{+(1)}(\alpha,\beta,x,t,u,z)E_{aa}\partial_0^{-1}P^{-(1)}(\alpha,\beta,x,t,u,-z)^T. \tag{A.10}$$

In particular taking $j=1$ one obtains

$$\frac{\partial P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)}{\partial u_1^{(a)}}P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)^{-1} = -P_1^{+(1)}(\alpha,\beta,x,t,u)E_{aa}\partial_0^{-1}P_1^{-(1)}(\alpha,\beta,x,t,u,)^T. \tag{A.11}$$

Since we have replaced in the wave matrices $u_1^{(a)}$, by $u_1^{(a)}+x_1$, (A.11) implies

$$\frac{\partial P^{+(0)}(\alpha,\beta,x,t,u,\partial_0)}{\partial x_1} = -P_1^{+(1)}(\alpha,\beta,x,t,u)\partial_0^{-1}P_1^{-(1)}(\alpha,\beta,x,t,u,)^T P^{+(0)}(\alpha,\beta,x,t,u,\partial_0) \tag{A.12}$$

which is equivalent to (4.9).

Using the Lemma in the other way one deduces for $\gamma-\alpha=\delta-\beta=\sum_{i=1}^n \epsilon_i$, that

$$\left(\left(\frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)}{\partial u_j^{(a)}}+P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)\partial_1^j E_{aa}\right)S(\partial_1) \times P^{-(1)}\left(\alpha+\sum_{i=1}^n\epsilon_i,\beta+\sum_{i=1}^n\epsilon_i,x,t,u,\partial_1\right)^*\right)_- = 0.$$

Now, using (4.6) one deduces the Sato–Wilson equations (4.8). A special case of (4.8), viz $j=1$ states that

$$\sum_{i=1}^n \frac{\partial P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)}{\partial u_1^{(i)}} = [\partial_1, P^{+(1)}(\alpha,\beta,x,t,u,\partial_1)].$$

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