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Construction of the Bethe State for the $E_{\tau,\eta}(so_3)$ Elliptic Quantum Group*

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Abstract. Elliptic quantum groups can be associated to solutions of the star-triangle relation of statistical mechanics. In this paper, we consider the particular case of the $E_{\tau,\eta}(so_3)$ elliptic quantum group. In the context of algebraic Bethe ansatz, we construct the corresponding Bethe creation operator for the transfer matrix defined in an arbitrary representation of $E_{\tau,\eta}(so_3)$.

Key words: elliptic quantum group; algebraic Bethe ansatz

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1 Introduction

In this article, we explain the first step towards the application of *algebraic* Bethe ansatz to the elliptic (or dynamical) quantum group $E_{\tau,\eta}(so_3)$. The elliptic quantum group is the algebraic structure associated to elliptic solutions of the star-triangle relation which appears in interaction-round-a-face models in statistical mechanics. As it was shown by Felder [4], this structure is also related to Knizhnik–Zamolodchikov–Bernard equation of conformal field theory on tori. In fact, to each solution of the (see [9]) star-triangle relation a dynamical R-matrix can be associated. This R-matrix, in turn, will define an algebra similar to quantum groups appearing in the quantum inverse scattering method (QISM). Despite all the differences, this new structure preserves a prominent feature of quantum groups: a tensor product of representations can be defined.

The adjective dynamical refers to the fact that the R-matrix appearing in these structures contains a parameter which in the classical limit will be interpreted as the position coordinate on the phase space of a classical system and the resulting classical r-matrix will depend on it. In the quantum setting, apart from the appearance of this extra parameter the Yang–Baxter equation (YBE) is also deformed. At the technical level, the main difference between usual quantum groups and the one we are about to describe lies not so much in the elliptic nature of the appearing functions as rather in the introduction of the extra “dynamical” parameter and the corresponding deformation the YBE.

In QISM, the physically interesting quantity is the transfer matrix. The Hamiltonian of the model and other observables are derived from it. The knowledge of its spectrum is thus essential. Different kinds of methods under the federating name of Bethe ansatz have been developed to calculate the eigenvalues of the transfer matrix [3, 10, 11]. The question whether the algebraic Bethe ansatz (ABA) technique can be applied to transfer matrices appearing in the context of

*This paper is a contribution to the Proceedings of the Workshop on Geometric Aspects of Integrable Systems (July 17–19, 2006, University of Coimbra, Portugal). The full collection is available at http://www.emis.de/journals/SIGMA/Coimbra2006.html dynamical quantum groups has received an affirmative answer from Felder and Varchenko [7, 5]. They showed how to implement ABA for the elliptic quantum group $E_{\tau,\eta}(sl_2)$, they also showed its applications to IRF models and Lamé equation. Later, for the $E_{\tau,\eta}(sl_n)$ elliptic quantum group the nested Bethe ansatz method was used [2, 8] and a relation to Ruijsenaars–Schneider [8] and quantum Calogero–Moser Hamiltonians was established [1].

In the first section we introduce the basic definitions of dynamical R-matrix, Yang–Baxter equation, representations, operator algebra and commuting transfer matrices. We define elements Φ_n in the operator algebra which have the necessary symmetry properties to be the creation operators of the corresponding Bethe states. As it turns out, the creation operators are not simple functions of the Lax matrix entries, unlike in [5], but they are complicated polynomials of three generators $A_1(u)$, $B_1(u)$, $B_2(u)$ in the elliptic operator algebra. We give the recurrence relation which defines the creation operators. Moreover, we give explicit formulas of the eigenvalues and Bethe equations for $n = 1, 2$. This strongly suggests that for higher n these are correct choice of creation operators. Derivation of the eigenvalues and the corresponding Bethe equations for general n (from the usual cancelation of the unwanted terms) will be published elsewhere.

2 Representations of $E_{\tau,\eta}(so_3)$ and transfer matrices

2.1 Definitions

Let us first recall the basic definitions which will enter our construction. First, we fix two complex numbers τ, η such that $\mathrm{Im}(\tau) > 0$. The central object in this paper is the R-matrix $R(q, u)$ which depends on two arguments $q, u \in \mathbb{C}$: the first one is referred to as the dynamical parameter, the second one is called the spectral parameter. The elements of the R-matrix are written in terms of Jacobi's theta function:

$$\vartheta(u) = -\sum_{j\in\mathbb{Z}} \exp\left(\pi i\left(j+\frac{1}{2}\right)^2\tau + 2\pi i\left(j+\frac{1}{2}\right)\left(u+\frac{1}{2}\right)\right).$$

This function has two essential properties. It is quasiperiodic:

$$\vartheta(u+1) = -\vartheta(u), \qquad \vartheta(u+\tau) = -e^{-i\tau-2iu}\vartheta(u)$$

and it verifies the identity:

$$\begin{aligned}\vartheta(u+x)\vartheta(u-x)\vartheta(v+y)\vartheta(v-y) &= \vartheta(u+y)\vartheta(u-y)\vartheta(v+x)\vartheta(v-x)\\ &\quad + \vartheta(u+v)\vartheta(u-v)\vartheta(x+y)\vartheta(x-y).\end{aligned}$$

The entries of the R-matrix are written in terms of the following functions

$$\begin{aligned}
g(u) &= \frac{\vartheta(u-\eta)\vartheta(u-2\eta)}{\vartheta(\eta)\vartheta(2\eta)},\\
\alpha(q_1,q_2,u) &= \frac{\vartheta(\eta-u)\vartheta(q_{12}-u)}{\vartheta(\eta)\vartheta(q_{12})},\\
\beta(q_1,q_2,u) &= \frac{\vartheta(\eta-u)\vartheta(u)\vartheta(q_{12}-2\eta)}{\vartheta(-2\eta)\vartheta(\eta)\vartheta(q_{12})},\\
\varepsilon(q,u) &= \frac{\vartheta(\eta+u)\vartheta(2\eta-u)}{\vartheta(\eta)\vartheta(2\eta)} - \frac{\vartheta(u)\vartheta(\eta-u)}{\vartheta(\eta)\vartheta(2\eta)}\left(\frac{\vartheta(q+\eta)\vartheta(q-2\eta)}{\vartheta(q-\eta)\vartheta(q)} + \frac{\vartheta(q-\eta)\vartheta(q+2\eta)}{\vartheta(q+\eta)\vartheta(q)}\right),\\
\gamma(q_1,q_2,u) &= \frac{\vartheta(u)\vartheta(q_1+q_2-\eta-u)\vartheta(q_1-2\eta)\vartheta(q_2+\eta)}{\vartheta(\eta)\vartheta(q_1+q_2-2\eta)\vartheta(q_1+\eta)\vartheta(q_2)},
\end{aligned}$$

$$\delta(q,u) = \frac{\vartheta(u-q)\vartheta(u-q+\eta)}{\vartheta(q)\vartheta(q-\eta)}.$$

The R-matrix itself will act on the tensor product $V \otimes V$ where V is a three-dimensional complex vector space with the standard basis $\{e_1, e_2, e_3\}$. The matrix units E_{ij} are defined in the usual way: $E_{ij}e_k = \delta_{jk}e_i$. We will also need the following diagonal matrix later on $h = E_{11} - E_{33}$.

Now we are ready to write the explicit form of the R-matrix. The matrix is obtained via a gauge transformation from the solution of the star-triangle relation which is associated to the vector representation of B_1 [9]. According to a remark in [9], that solution can also be derived as a symmetric tensor product (i.e. fusion) of the A_1 solution

$$\begin{aligned} R(q,u) = {} & g(u)E_{11}\otimes E_{11} + g(u)E_{33}\otimes E_{33} + \varepsilon(q,u)E_{22}\otimes E_{22} \\ & + \alpha(\eta,q,u)E_{12}\otimes E_{21} + \alpha(q,\eta,u)E_{21}\otimes E_{12} + \alpha(-q,\eta,u)E_{23}\otimes E_{32} \\ & + \alpha(\eta,-q,u)E_{32}\otimes E_{23} \\ & + \beta(\eta,q,u)E_{11}\otimes E_{22} + \beta(q,\eta,u)E_{22}\otimes E_{11} + \beta(-q,\eta,u)E_{22}\otimes E_{33} \\ & + \beta(\eta,-q,u)E_{33}\otimes E_{22} \\ & + \gamma(-q,q,u)E_{11}\otimes E_{33} + \gamma(-q,\eta,u)E_{12}\otimes E_{32} - \gamma(\eta,q,u)E_{21}\otimes E_{23} \\ & + \gamma(q,-q,u)E_{33}\otimes E_{11} + \gamma(q,\eta,u)E_{32}\otimes E_{12} - \gamma(\eta,-q,u)E_{23}\otimes E_{21} \\ & + \delta(q,u)E_{31}\otimes E_{13} + \delta(-q,u)E_{13}\otimes E_{31}. \end{aligned} \tag{1}$$

This R-matrix also enjoys the unitarity property:

$$R_{12}(q,u)R_{21}(q,-u) = g(u)g(-u)$$

and it is of zero weight:

$$[h\otimes \quad + \quad \otimes h, R_{12}(q,u)] = 0 \qquad (h \in \mathfrak{h}).$$

The R-matrix also obeys the dynamical quantum Yang–Baxter equation (DYBE) in $\mathrm{End}(V\otimes V\otimes V)$:

$$\begin{aligned} & R_{12}(q-2\eta h_3, u_{12})R_{13}(q,u_1)R_{23}(q-2\eta h_1, u_2) \\ & \quad = R_{23}(q,u_2)R_{13}(q-2\eta h_2, u_1)R_{12}(q,u_{12}), \end{aligned}$$

where the "dynamical shift" notation has the usual meaning:

$$R_{12}(q-2\eta h_3, u)\cdot v_1\otimes v_2\otimes v_3 = (R_{12}(q-2\eta\lambda, u)v_1\otimes v_2)\otimes v_3, \tag{2}$$

whenever $hv_3 = \lambda v_3$. Shifts on other spaces are defined in an analogous manner. Notice that the notion and notation of "dynamical shift" can be extended to different situations as well, even if the appearing (possibly different) vector spaces V_i are not 3-dimensional. For this, one only needs to verify two conditions: that an action of h is defined on each V_i, and that each V_i is a direct sum of the weight subspaces $V_i[\lambda]$ defined by that action of h. It is easy to see then that equation (2) makes sense. Furthermore, along these lines the notion of dynamical quantum Yang–Baxter equation and of the corresponding algebraic structures can be extended to the case where h is replaced by a higher rank Abelian Lie algebra $\mathfrak{h}$. However, in this paper we only deal with the a special rank-one case, so from now on $\mathfrak{h} = \mathbb{C}h$. It will be clear from the context how to generalize the relevant notions to the higher rank case.

Let us also describe a more intuitive way of looking at this shift. Define first the shift operator acting on functions of the dynamical parameter:

$$\exp(2\eta\partial_q)f(q) = f(q+2\eta)\exp(2\eta\partial_q).$$

Then equation (2) can also be written in the following form:

$$R_{12}(q-2\eta h_3,u)=\exp(-2\eta h_3\partial_q)R_{12}(q,u)\exp(2\eta h_3\partial_q)$$

in the sequel we will use whichever definition is the fittest for the particular point in our calculation.

2.2 Representation, operator algebra

Now we describe the notion of representation of (or module over) $E_{\tau,\eta}(so_3)$. It is a pair $(\mathcal{L}(q,u),W)$ where W is a diagonalizable $\mathfrak{h}$-module, that is, W is a direct sum of the weight subspaces $W=\oplus_{\lambda\in\mathbb{C}}W[\lambda]$ and $\mathcal{L}(q,u)$ is an operator in $\mathrm{End}(V\otimes W)$ obeying:

$$\begin{aligned}&R_{12}(q-2\eta h_3,u_{12})\mathcal{L}_{13}(q,u_1)\mathcal{L}_{23}(q-2\eta h_1,u_2)\\ &\qquad=\mathcal{L}_{23}(q,u_2)\mathcal{L}_{13}(q-2\eta h_2,u_1)R_{12}(q,u_{12}).\end{aligned}\tag{3}$$

$\mathcal{L}(q,u)$ is also of zero weight

$$[h_V\otimes\ \ +\ \ \otimes h_W,\mathcal{L}_{V,W}(q,u)]=0\qquad(h\in\mathfrak{h}),$$

where the subscripts remind the careful reader that in this formula h might act in a different way on spaces W and V.

An example is given forthwith by $W=V$ and $\mathcal{L}(q,u)=R(q,u-z)$ which is called the fundamental representation with evaluation point z. A tensor product of representations can also be defined which corresponds to the existence of a coproduct-like structure at the abstract algebraic level. Let $(\mathcal{L}(q,u),X)$ and $(\mathcal{L}'(q,u),Y)$ be two $E_{\tau,\eta}(so_3)$ modules, then $\mathcal{L}_{1X}(q-2\eta h_Y,u)\mathcal{L}_{1Y}(q,u)$, $X\otimes Y$ is a representation of $E_{\tau,\eta}(so_3)$ on $X\otimes Y$ endowed, of course, with the tensor product $\mathfrak{h}$-module structure.

The operator $\mathcal{L}$ is reminescent of the quantum Lax matrix in the FRT formulation of the quantum inverse scattering method, although it obeys a different exchange relation, therefore we will also call it a Lax matrix. This allows us to view the $\mathcal{L}$ as a matrix with operator-valued entries.

Inspired by that interpretation, for any module over $E_{\tau,\eta}(so_3)$ we define the corresponding *operator algebra*. Let us take an arbitrary representation $\mathcal{L}(q,u)\in\mathrm{End}(V\otimes W)$. The elements of the operator algebra corresponding to this representation will act on the space $\mathrm{Fun}(W)$ of meromorphic functions of q with values in W. Namely let $L\in\mathrm{End}(V\otimes\mathrm{Fun}(W))$ be the operator defined as

$$L(u)=\begin{pmatrix}A_1(u)&B_1(u)&B_2(u)\\C_1(u)&A_2(u)&B_3(u)\\C_2(u)&C_3(u)&A_3(u)\end{pmatrix}=\mathcal{L}(q,u)e^{-2\eta h\partial_q}.$$

We can view it as a matrix with entries in $\mathrm{End}(\mathrm{Fun}(W))$. It follows from equation (3) that $\tilde{L}$ verifies:

$$R_{12}(q-2\eta h,u_{12})L_{1W}(q,u_1)L_{2W}(q,u_2)=L_{2W}(q,u_2)L_{1W}(q,u_1)\tilde{R}_{12}(q,u_{12})\tag{4}$$

with $\tilde{R}_{12}(q,u):=\exp(2\eta(h_1+h_2)\partial_q)R_{12}(q,u)\exp(-2\eta(h_1+h_2)\partial_q)$.

The zero weight condition on L yields the relations:

$$\begin{aligned}&[h,A_i]=0,\qquad [h,B_j]=-B_j\quad(j=1,3),\qquad [h,B_2]=-2B_2,\\ &[h,C_j]=C_j\quad(j=1,3),\qquad [h,C_2]=2C_2,\end{aligned}$$

so B_i's act as lowering and C_i's as raising operators.

And finally the following theorem shows how to associate a family of commuting quantities to a representation of the elliptic quantum group.

Theorem 1. *Let W be a representation of $E_{\tau,\eta}(so_3)$. Then the transfer matrix defined by $t(u) = \operatorname{Tr} \tilde{L}(u) \in \operatorname{End}(\operatorname{Fun}(W))$ preserves the subspace $\operatorname{Fun}(W)[0]$ of functions with values in the zero weight subspace of W. When restricted to this subspace, they commute at different values of the spectral parameter:*

$$[t(u), t(v)] = 0.$$

Proof. The proof is analogous to references [6, 1]. ■

The eigenvalues of the transfer matrix can be found by means of the algebraic Bethe ansatz. In the next section we develop the first steps in this direction.

3 Construction of the Bethe state

3.1 The framework of the algebraic Bethe ansatz

The above theorem tells us how to associate the transfer matrix to an arbitrary representation of the dynamical quantum group. Our aim is to determine the spectrum of such a transfer matrix in the usual sense of the Bethe ansatz techniques, i.e. to write the Bethe equations fixing the eigenvalues.

In order for the algebraic Bethe ansatz to work, this representation must be a highest weight representation, that is possess a highest weight vector $|0\rangle$ (also called pseudovacuum) which is annihilated by the raising operators and is an eigenvector of the diagonal elements of the quantum Lax matrix

$$C_i(u)|0\rangle = 0, \qquad A_i(u)|0\rangle = a_i(q,u)|0\rangle, \qquad i = 1,2,3.$$

Actually, any vector of the form $|\Omega\rangle = f(q)|0\rangle$ is also a highest weight vector of the representation in question. This freedom in choosing the highest weight vector will prove essential in the sequel, so we do not fix the arbitrary function $f(q)$ for the moment. The preceding relations are modified as follows:

$$C_i(u)|\Omega\rangle = 0, \quad i = 1,2,3, \qquad A_1(u)|\Omega\rangle = a_1(q,u)\frac{f(q-2\eta)}{f(q)}|\Omega\rangle,$$

$$A_2(u)|\Omega\rangle = a_2(q,u)|\Omega\rangle, \qquad A_3(u)|\Omega\rangle = a_3(q,u)\frac{f(q+2\eta)}{f(q)}|\Omega\rangle.$$

The representations obtained by tensorising the fundamental vector representation possesses this highest weight vector and its transfer matrix is the transfer matrix of an IRF (interaction-round-a-face) model with Boltzmann weights derived from the dynamical R-matrix (1). In this case we also have the property that $a_1(q,u)$ does not depend on q, this is what we will assume in the sequel. Other representation are related to Lamé equation or Ruijsenaars–Schneider Hamiltonians (see [5] for the $E_{\tau,\eta}(sl_2)$ case). This is expected to happen in the $E_{\tau,\eta}(so_3)$ case, too, and we hope to report on progress in representations and related models soon.

Once the pseudovacuum is fixed, one looks for eigenvalues in the form:

$$\Phi_n(u_1,\ldots,u_n)|\Omega\rangle$$

under some simple (symmetry) assumptions on the lowering operator Φ_n. In the XXZ model, or for $E_{\tau,\eta}(sl_2)$, Φ_n is a simple product of the only lowering operator $B(u)$. We will explain later, in analogy with the Izergin–Korepin model, why Φ_n is not that simple in the $E_{\tau,\eta}(so_3)$ case.

The main result of this paper is the construction of Φ_n (the Bethe state) for the $E_{\tau,\eta}(so_3)$ dynamical quantum group under simple assumptions.

Finally, one calculates the action of the transfer matrix on the Bethe state. This will yield 3 kinds of terms. The first part (usually called wanted terms in the literature) will tell us the eigenvalue of the transfer matrix, the second part (called unwanted terms) must be annihilated by a careful choice of the spectral parameters u_i in $\Phi_n(u_1, \dots, u_n)$; the vanishing of these unwanted terms is ensured if the u_i are solutions to the so called Bethe equation. The third part contains terms ending with a raising operator acting on the pseudovacuum and thus vanishes. We hope to report soon on the form of the Bethe equations and eigenvalues, too.

Right now, we propose to develop step 2 and write the recurrence relation defining Φ_n. We thus assume that a representation with highest weight vector pseudovacuum already exists.

3.2 The creation operators

We explicitly write the commutation relations coming from the RLL relations (4) which will be used in the construction of the Bethe state

$$B_1(u_1)B_1(u_2) = \omega_{21}\left(B_1(u_2)B_1(u_1) - \frac{1}{y_{21}(q)}B_2(u_2)A_1(u_1)\right) + \frac{1}{y_{12}(q)}B_2(u_1)A_1(u_2), \tag{5}$$

$$A_1(u_1)B_1(u_2) = z_{21}(q)B_1(u_2)A_1(u_1) - \frac{\alpha_{21}(\eta, q)}{\beta_{21}(q, \eta)}B_1(u_1)A_1(u_2), \tag{6}$$

$$\begin{aligned} A_1(u_1)B_2(u_2) = \frac{1}{\gamma_{21}(q, -q)}&\left(g_{21}B_2(u_2)A_1(u_2) + \gamma_{21}(\eta, -q)B_1(u_1)B_1(u_2)\right. \\ &\left. - \delta_{21}(-q)B_2(u_1)A_1(u_1)\right), \end{aligned} \tag{7}$$

$$B_1(u_2)B_2(u_1) = \frac{1}{g_{21}}\left(\beta_{21}(-q, \eta)B_2(u_1)B_1(u_2) + \alpha_{21}(\eta, -q)B_1(u_1)B_2(u_2)\right), \tag{8}$$

$$B_2(u_2)B_1(u_1) = \frac{1}{g_{21}}\left(\beta_{21}(\eta, -q)B_1(u_1)B_2(u_2) + \alpha_{21}(-q, \eta)B_2(u_1)B_1(u_2)\right), \tag{9}$$

where

$$\omega(q, u) = \frac{\varepsilon(q, -u)\gamma(q, -q, -u) + \gamma(q, \eta, -u)\gamma(\eta, -q, -u)}{g(-u)\gamma(q, -q, -u)},$$

$$y(q, u) = \frac{\gamma(q, -q, u)}{\gamma(q, \eta, u)}, \qquad z(q, u) = \frac{g(u)}{\beta(q, \eta, u)}$$

and as usual

$$y_{12}(q) = y(q, u_1 - u_2) \quad \text{etc.}$$

Remark 1. Furthermore, the function $\omega(q, u)$ is actually independent of q, a property which will prove important later on, and takes the following simple form:

$$\omega(u) = \frac{\vartheta(u + \eta)\vartheta(u - 2\eta)}{\vartheta(u - \eta)\vartheta(u + 2\eta)}. \tag{10}$$

This identity can be proved by looking at transformation properties under $u \to u+1$, $u \to u+\tau$ of both sides of (10).

Remark 2. Notice also that $\omega(u)\omega(-u) = 1$.

Now we turn to the construction of the Bethe state. In the application of algebraic Bethe ansatz to the $E_{\tau,\eta}(sl_2)$ elliptic quantum group the algebra contains a generator (usually also denoted by $B(u)$) which acts as a creation operator. It also enjoys the property $B(u)B(v) = B(v)B(u)$. This allows for the straightforward construction of the creation operators Φ_n as

$$\Phi_n(u_1, \dots, u_n) = B(u_1)B(u_2)\cdots B(u_n),$$

since we immediately have the property

$$\Phi_n(u_1,\ldots,u_n)=\Phi_n(u_1,\ldots,u_{i-1},u_{i+1},u_i,u_{i+2},\ldots,u_n),\qquad i=1,2,\ldots,n-1.$$

As it turns out, in the $E_{\tau,\eta}(so_3)$ case the creation operators are not simple functions of the Lax matrix entries but they are complicated functions of three generators $A_1(u)$, $B_1(u)$, $B_2(u)$ in the elliptic operator algebra. This situation is analogous to that of the Izergin–Korepin model as described by Tarasov in [13].

We give the following definition for the creation operator.

Definition 1. Let Φ_n be defined be the recurrence relation for $n\geq 0$:

$$\begin{aligned}\Phi_n(u_1,\ldots,u_n)&=B_1(u_1)\Phi_{n-1}(u_2,\ldots,u_n)\\ &\quad-\sum_{j=2}^{n}\frac{\prod_{k=2}^{j-1}\omega_{jk}}{y_{1j}(q)}\prod_{\substack{k=2\\k\neq j}}^{n}z_{kj}(q+2\eta)\ B_2(u_1)\Phi_{n-2}(u_2,\ldots,\widehat{u_j},\ldots,u_n)A_1(u_j),\end{aligned}$$

where $\Phi_0=1$, $\Phi_1(u_1)=B_1(u_1)$ and the hat means that that parameter is omitted.

It may be useful to give explicitly the first three creation operators

$$\begin{aligned}&\Phi_1(u_1)=B_1(u_1),\\ &\Phi_2(u_1,u_2)=B_1(u_1)B_1(u_2)-\frac{1}{y_{12}(q)}B_2(u_1)A_1(u_2),\\ &\Phi_3(u_1,u_2,u_3)=B_1(u_1)B_1(u_2)B_1(u_3)-\frac{1}{y_{23}(q)}B_1(u_1)B_2(u_2)A_1(u_3)\\ &\qquad-\frac{z_{32}(q+2\eta)}{y_{12}(q)}B_2(u_1)B_1(u_3)A_1(u_2)-\frac{\omega_{32}z_{23}(q+2\eta)}{y_{13}(q)}B_2(u_1)B_1(u_2)A_1(u_3).\end{aligned}$$

The Bethe vector is then not completely symmetric under the interchange of two neighboring spectral parameters but verifies the following property instead:

$$\begin{aligned}&\Phi_2(u_1,u_2)=\omega_{21}\Phi_2(u_2,u_1),\\ &\Phi_3(u_1,u_2,u_3)=\omega_{21}\Phi_3(u_2,u_1,u_3)=\omega_{32}\Phi_3(u_1,u_3,u_2).\end{aligned}$$

For general n we prove the following theorem.

Theorem 2. Φ_n *verifies the following symmetry property:*

$$\Phi_n(u_1,\ldots,u_n)=\omega_{i+1,i}\Phi_n(u_1,\ldots,u_{i-1},u_{i+1},u_i,u_{i+2},\ldots,u_n),\qquad i=1,2,\ldots,n-1.\tag{11}$$

Proof. The proof is by induction on n. The symmetry property is immediately proved for $i\neq 1$. To verify for $i=1$, we have to expand Φ_n by one more induction step:

$$\begin{aligned}\Phi_n(u_1,\ldots,u_n)&=B_1(u_1)B_1(u_2)\Phi_{n-2}(u_3,\ldots,u_n)\\ &-\frac{\prod_{k=3}^{n}z_{k2}(q+2\eta)}{y_{12}(q)}B_2(u_1)\Phi_{n-2}(u_3,\ldots,u_n)A_1(u_2)\\ &-\sum_{j=3}^{n}\frac{\prod_{k=3}^{j-1}\omega_{jk}}{y_{2j}(q)}\prod_{\substack{k=3\\k\neq j}}^{n}z_{kj}(q+2\eta)B_1(u_1)B_2(u_2)\Phi_{n-3}(u_3,\ldots,\widehat{u_j},\ldots,u_n)A_1(u_j)\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=3}^{n} \frac{\omega_{j2} z_{2j}(q) \prod_{k=3}^{j-1} \omega_{jk}}{y_{1j}(q)} \prod_{\substack{k=3 \\ k\neq j}}^{n} z_{kj}(q+2\eta) B_2(u_1) B_1(u_2) \Phi_{n-3}(u_3, \ldots, \widehat{u_j}, \ldots, u_n) A_1(u_j) \\
&+ \sum_{3\leq l \leq j \leq n} \left[\frac{\omega_{j2} z_{2j}(q+2\eta) z_{lj}(q+2\eta)}{y_{1j}(q) y_{2l}(q+2\eta)} + \omega_{lj} \frac{\omega_{l2} z_{2l}(q+2\eta) z_{jl}(q+2\eta)}{y_{1l}(q) y_{2j}(q+2\eta)} \right] \\
&\times \prod_{k=3}^{j-1} \omega_{jk} \prod_{k=3}^{l-1} \omega_{lk} \prod_{\substack{k=3 \\ k\neq j,l}}^{n} z_{kj}(q+4\eta) z_{kl}(q+4\eta) \frac{\vartheta(q+\eta)^2}{\vartheta(q-\eta)\vartheta(q+3\eta)} \\
&\times B_2(u_1) B_2(u_2) \Phi_{n-4}(u_3, \ldots, \widehat{u_l}, \ldots, \widehat{u_j}, \ldots, u_n) A_1(u_l) A_1(u_j)
\end{aligned}
$$

then substitute into (11) and bring the right-hand-side to normal order of the spectral parameters by using relations (5)–(9). We find then that property (11) is fulfilled provided the following identities hold true:

$$
-\frac{\omega_{12} g_{21}}{y_{23}(q) \beta_{21}(\eta, -q)} + \frac{\alpha_{21}(\eta, -q)}{\beta_{21}(\eta, -q) y_{13}(q)} = -\frac{\omega_{31} z_{13}(q+2\eta)}{y_{23}(q)} - \frac{\alpha_{31}(\eta, q+2\eta)}{\beta_{31}(q+2\eta, \eta) y_{21}(q)}
$$

and

$$
\begin{aligned}
&\omega_{12} \left(\frac{\omega_{42} z_{24}(q+2\eta) z_{34}(q+2\eta)}{y_{14}(q) y_{23}(q+2\eta)} + \omega_{34} \frac{\omega_{32} z_{23}(q+2\eta) z_{43}(q+2\eta)}{y_{13}(q) y_{24}(q+2\eta)} \right) \\
&\quad - \left(\frac{\omega_{41} z_{14}(q+2\eta) z_{34}(q+2\eta)}{y_{24}(q) y_{13}(q+2\eta)} + \frac{\omega_{34} \omega_{31} z_{13}(q+2\eta) z_{43}(q+2\eta)}{y_{23}(q) y_{14}(q+2\eta)} \right) \\
&\quad + \frac{\omega_{12}}{y_{12}(q)} \left(\frac{\delta_{42}(-q-2\eta)}{\gamma_{42}(q+2\eta, -q-2\eta) y_{43}(q)} + \frac{z_{42}(q+2\eta) \alpha_{32}(\eta, q+2\eta) \omega_{24}}{\beta_{32}(q+2\eta, \eta) y_{24}(q+2\eta)} \right) \\
&\quad - \frac{1}{y_{21}(q)} \left(\frac{\delta_{41}(-q-2\eta)}{\gamma_{41}(q+2\eta, -q-2\eta) y_{43}(q)} + \frac{z_{41}(q+2\eta) \alpha_{31}(\eta, q+2\eta) \omega_{14}}{\beta_{31}(q+2\eta, \eta) y_{14}(q+2\eta)} \right) = 0.
\end{aligned}
$$

These identities can be verified by tedious calculations using once again quasiperiodicty properties of the ϑ-function. ■

Remark 3. Φ_n contains the more familiar string of $B_1(u_1) \cdots B_1(u_n)$ with coefficient 1.

It is straightforward to check the following relations using the commutation relations (4)

$$
\begin{aligned}
t(u)\Phi_1(u_1) &= z_{1u}(q) B_1(u_1) A_1(u) - \frac{\alpha_{1u}(\eta, q)}{\beta_{1u}(q, \eta)} B_1(u) A_1(u_1) \\
&+ \frac{z_{u1}(q)}{\omega_{u1}} B_1(u_1) A_2(u) - \frac{\alpha_{u1}(q, \eta)}{\beta_{u1}(q, \eta)} B_1(u) A_2(u_1) + \frac{1}{y_{u1}(q)} B_3(u) A_1(u_1) \\
&+ \frac{\beta_{u1}(\eta, -q)}{\gamma_{u1}(q, -q)} B_1(u_1) A_3(u) - \frac{\gamma_{u1}(q, \eta)}{\gamma_{u1}(q, -q)} B_3(u) A_2(u_1),
\end{aligned}
$$

for $n = 2$

$$
\begin{aligned}
&t(u)\Phi_2(u_1, u_2) = z_{1u}(q) z_{2u}(q) \Phi_2(u_1, u_2) A_1(u) + \frac{z_{u1}(q) z_{u2}(q-2\eta)}{\omega_{u1}\omega_{u2}} \Phi_2(u_1, u_2) A_2(u) \\
&+ \frac{\beta_{u1}(\eta, -q) \beta_{u2}(\eta, -q)}{\gamma_{u1}(q-2\eta, -q+2\eta) \gamma_{u2}(q-2\eta, -q+2\eta)} \Phi_2(u_1, u_2) A_3(u) \\
&+ \left(-\frac{z_{1u}(q) \alpha_{2u}(\eta, q)}{\beta_{2u}(q, \eta)} + \frac{\alpha_{1u}(\eta, q) \alpha_{21}(\eta, q) \omega_{1u}}{\beta_{1u}(q, \eta) \beta_{21}(q, \eta)} - \frac{\gamma_{1u}(\eta, -q)}{y_{12}(q-2\eta) g a_{1u}(q, -q)} \right) \Phi_2(u_1, u) A_1(u_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{z_{u1}(q)\alpha_{u2}(q-2\eta,\eta)}{\omega_{u1}\beta_{u2}(q-2\eta,\eta)}+\frac{\alpha_{u1}(q,\eta)\alpha_{12}(q-2\eta,\eta)}{\beta_{u1}(q,\eta)\beta_{12}(q-2\eta,\eta)\omega_{u1}}\right)\Phi_2(u_1,u)A_2(u_2)\\
&-\frac{z_{21}(q)\alpha_{1u}(\eta,q)}{\beta_{1u}(q,\eta)}\Phi_2(u,u_2)A_1(u_1)-\frac{\alpha_{u1}(q,\eta)z_{12}(q-2\eta)}{\beta_{u1}(q,\eta)\omega_{12}}\Phi_2(u,u_2)A_2(u_1)\\
&+\left(-\frac{z_{21}(q)\alpha_{1u}(\eta,q)}{y_{u2}(q)\beta_{1u}(q,\eta)}+\frac{\alpha_{1u}(\eta,q)\alpha_{21}(\eta,q)}{\beta_{1u}(q,\eta)\beta_{1u}(q,\eta)y_{u1}(q)}-\frac{\gamma_{1u}(\eta,-q)}{\gamma_{1u}(q,-q)y_{12}(q-2\eta)y_{u1}(q)}\right.\\
&\left.+\frac{\delta_{1u}(-q)}{\gamma_{1u}(q,-q)y_{12}(q-2\eta)}\right)B_2(u)A_1(u_1)A_1(u_2)\\
&\times\left(\frac{z_{u1}(q)\alpha_{u1}(q,\eta)}{\omega_{u1}y_{u2}(q)\beta_{u1}(\eta,-q)}-\frac{\alpha_{u1}(q,\eta)g_{12}}{\beta_{u1}(q,\eta)\omega_{12}y_{u2}(q)\beta_{12}(q+2\eta,\eta)}+\frac{\alpha_{u1}(q,\eta)\alpha_{12}(q+2\eta,\eta)}{\beta_{u1}(q,\eta)y_{u1}(q)\beta_{12}(q+2\eta,\eta)}\right.\\
&\left.-\frac{\alpha_{u1}(q,\eta)\alpha_{u1}(\eta,-q)}{y_{12}(q)\beta_{u1}(\eta,-q)\beta_{u1}(q,\eta)}\right)B_2(u)A_2(u_1)A_1(u_2)\\
&\times\frac{1}{\gamma_{u1}(q,-q)}\left(\frac{\delta_{u1}(q)}{y_{12}(q-2\eta,-q+2\eta)}-\frac{\alpha_{u1}(q,\eta)}{y_{u2}(q-2\eta)}\right)B_2(u)A_2(u_1)A_2(u_2)\\
&\times\left(\frac{g_{u1}}{\beta_{u1}(\eta,-q)\omega_{u1}y_{u2}(q)}-\frac{\alpha_{21}(\eta,q+2\eta)}{y_{u1}(q)\beta_{21}(q+2\eta,\eta)}-\frac{\alpha_{u1}(\eta,-q)}{y_{12}(q)\beta_{u1}(\eta,-q)}\right)B_3(u)B_1(u_1)A_1(u_2)\\
&+\frac{\alpha_{12}(q,\eta)}{y_{u1}(q,-q)\beta_{12}(q,\eta)}B_3(u)B_1(u_1)A_2(u_2)\\
&\frac{z_{21}(q+2\eta)}{y_{u1}(q)}B_3(u)B_1(u_1)A_2(u_1)-\frac{z_{12}(q)}{y_{u1}(q)\omega_{12}}B_3(u)B_1(u_2)A_2(u_1)\\
&+\text{terms ending with } C.
\end{aligned}
$$

The cancelation of the unwanted terms is ensured for $n=2$ if u_1 and u_2 are solutions of the following Bethe equations.

$$
\frac{a_1(u_i)}{a_2(u_i,q)}=\prod_{\substack{j=1\\ j\neq i}}^{2}\frac{\vartheta(u_{ij}-\eta)}{\vartheta(u_{ij}+\eta)}\times\frac{\vartheta(q-3\eta)^2}{\vartheta(q-\eta)\vartheta(q-5\eta)}\times\frac{f(q)}{f(q-2\eta)},\qquad i=1,2.
$$

The role of the function $f(q)$ becomes clear in this context. It has to be chosen so as to eliminate the q-dependence from the Bethe equation.

These results suggest that Φ_n for all n are the correct choice of creation operators of the corresponding Bethe states. The complete proof of the general case will be given elsewhere.

4 Conclusion

In this paper we defined the elliptic quantum group $E_{\tau,\eta}(so_3)$ along the lines described by Felder in [4]. Although dynamical, the R-matrix appearing in the exchange relations has a matrix form similar to that of the Izergin–Korepin model. Lax operator, operator algebra and families of commuting transfer matrices are defined in complete analogy with the $E_{\tau,\eta}(sl_2)$ case.

Our aim was to apply algebraic Bethe ansatz method in this setting. We have obtained a recurrence relation for the creation operators and have proved that these operators have a certain symmetry property under the interchange of two adjacent spectral parameters. Both the form of the recurrence relation and this symmetry property are an elliptic generalization of Tarasov's results in [13]. Finally, we have obtained the Bethe equations and the eigenvalues for the 2-magnon state. The Bethe equations for the general n-magnon state will be published later [12].

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