# Bethe Ansatz Solutions of the Bose–Hubbard Dimer*

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**Abstract.** The Bose–Hubbard dimer Hamiltonian is a simple yet effective model for describing tunneling phenomena of Bose–Einstein condensates. One of the significant mathematical properties of the model is that it can be exactly solved by Bethe ansatz methods. Here we review the known exact solutions, highlighting the contributions of V.B. Kuznetsov to this field. Two of the exact solutions arise in the context of the Quantum Inverse Scattering Method, while the third solution uses a differential operator realisation of the $su(2)$ Lie algebra.

*Key words:* Bose–Hubbard dimer; Bethe ansatz

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## 1 Introduction

The experimental realisation of Bose–Einstein condensation using atomic alkali gases has provided the means to study macroscopic tunneling in systems with tunable interation parameters [9]. From the theoretical perspective, the Bose–Hubbard dimer model (see equation (1) below), also known as the *discrete self-trapping dimer* [4, 5, 6] or the *canonical Josephson Hamiltonian* [9], has been extremely useful in understanding this tunneling phenomena in the context of a bosonic Josephson junction. Despite its apparent simplicity, the Hamiltonian captures the essence of competing linear and non-linear interactions which lead to non-trivial dynamical behaviour and ground-state properties (e.g. [7, 12, 14, 15, 16]). In particular the model predicts macroscopic self-trapping and the collapse and revival of Rabi oscillations, features which have been directly observed experimentally in a single bosonic Josephson junction [1, 11].

The Bose–Hubard dimer Hamiltonian is given by

$$H = \frac{k}{8}(N_1 - N_2)^2 - \frac{\mu}{2}(N_1 - N_2) - \frac{\mathcal{E}}{2}(b_1^\dagger b_2 + b_2^\dagger b_1), \tag{1}$$

where $b_1^\dagger$, $b_2^\dagger$ denote the single-particle creation operators for two bosonic modes and $N_1 = b_1^\dagger b_1$, $N_2 = b_2^\dagger b_2$ are the corresponding boson number operators. The coupling $k$ provides the strength of the scattering interaction between bosons, $\mu$ is the external potential and $\mathcal{E}$ is the coupling for the tunneling. The change $\mathcal{E} \to -\mathcal{E}$ corresponds to the unitary transformation $b_1 \to b_1$, $b_2 \to -b_2$, while $\mu \to -\mu$ corresponds to $b_1 \leftrightarrow b_2$. The total boson number $N = N_1 + N_2$ is conserved and consequently the model is integrable as it has only two degrees of freedom and two conserved operators, viz. $H$ and $N$. Mathematically the Hamiltonian is of interest because, related to its integrability, it admits exact Bethe ansatz solutions. This property opens avenues

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to rigorously analyse the model. For example, the Bethe ansatz solution can be used to study the ground-state crossover from a delocalised state to a "Schrödinger cat" state in the attractive case [3], as well as facilitating the calculation of form factors [10].

The first Bethe ansatz solution of the Hamiltonian was given by Enol'skii et al. [4, 5] using the machinery of the Quantum Inverse Scattering Method. A key ingredient in this approach was the use of a bosonic realisation of the Yang–Baxter algbera, which was developed in the work of Kuznetsov and Tsiganov [8]. For zero external potential an alternative application of the Quantum Inverse Scattering Method, using the Gaudin algebra formulation, was given by Enol'skii, Kuznetsov and Salerno [6]. We remark this method of solution for the model has also been recently discussed in [13, 14]. In this approach a connection was made with confluent Heun polynomials. It was also observed in their work [6] that this connection could be established using an $su(2)$ realisation of the Hamiltonian (see also [17]). This property provides a direct route to a third Bethe ansatz solution using elementary properties of second-order ordinary differential eigenvalue equations with polynomial solutions.

## 2 Exact Bethe ansatz solution I

In this section we review the Quantum Inverse Scattering Method and associated algebraic Bethe ansatz. The notational conventions we adopt follow those of [10], which also contains the full details for the following calculations. Then we will apply this approach to derive the exact Bethe ansatz solution of (1), as was originally described in [4, 5].

We begin with the $su(2)$-invariant $R$-matrix $R(u) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, depending on the spectral parameter $u \in \mathbb{C}$:

$$R(u) = \left( \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ \hline 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \tag{2}$$

with $b(u) = u/(u+\eta)$ and $c(u) = \eta/(u+\eta)$. Above, $\eta$ is an arbitrary parameter. It is easy to check that $R(u)$ satisfies the Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{3}$$

on $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$. Above $R_{jk}(u)$ denotes the matrix acting non-trivially on the $j$-th and $k$-th spaces and as the identity on the remaining space. Next we define the Yang–Baxter algebra with *monodromy matrix* $T(u)$,

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \tag{4}$$

subject to the constraint

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v). \tag{5}$$

Given a representation $\pi$ of the monodromy matrix, the transfer matrix is defined

$$t(u) = \pi(A(u) + D(u)) \tag{6}$$

which satisfies $[t(u), t(v)] = 0$ for any choice of $u$ and $v$ as a result of (3). If there exists a pseudovacuum state $|\chi\rangle$ which satisfies

$$\pi(A(u))\,|\chi\rangle = a(u)\,|\chi\rangle\,, \qquad \pi(B(u))\,|\chi\rangle = 0,$$

$$\pi(C(u))\,|\chi\rangle \neq 0, \qquad \pi(D(u))\,|\chi\rangle = d(u)\,|\chi\rangle$$

the transfer matrix has eigenvalues

$$\Lambda(u) = a(u)\prod_{k=1}^{M}\frac{u-v_k+\eta}{u-v_k} + d(u)\prod_{k=1}^{M}\frac{u-v_k-\eta}{u-v_k}. \tag{7}$$

Provided the Bethe ansatz equations

$$\frac{a(v_k)}{d(v_k)} = \prod_{j\neq k}^{M}\frac{v_k-v_j-\eta}{v_k-v_j+\eta}, \qquad k=1,\ldots,M \tag{8}$$

are satisfied.

We may choose the following realization for the Yang–Baxter algebra, with arbitrary $\omega \in \mathbb{C}$,

$$\pi(T(u)) = L_1^b(u+\omega)L_2^b(u-\omega) \tag{9}$$

written in terms of the bosonic realisation of the Lax operator given by Kuznetsov and Tsiganov [8]:

$$L_i^b(u) = \begin{pmatrix} u+\eta N_i & b_i \\ b_i^\dagger & \eta^{-1}\end{pmatrix}, \qquad i=1,2. \tag{10}$$

Since $L(u)$ satisfies the relation

$$R_{12}(u-v)L_{i1}^b(u)L_{i2}^b(v) = L_{i2}^b(v)L_{i1}^b(u)R_{12}(u-v), \qquad i=1,2 \tag{11}$$

it is easy to check that the relations of the Yang–Baxter algebra (5) are obeyed. Specifically, the realisation of the generators of the Yang–Baxter algebra is

$$\begin{aligned}
\pi(A(u)) &= (u^2-\omega^2)I + \eta u N + \eta^2 N_1N_2 - \eta\omega(N_1-N_2) + b_2^\dagger b_1,\\
\pi(B(u)) &= (u+\omega+\eta N_1)b_2 + \eta^{-1}b_1,\\
\pi(C(u)) &= b_1^\dagger(u-\omega+\eta N_2) + \eta^{-1}b_2^\dagger,\\
\pi(D(u)) &= b_1^\dagger b_2 + \eta^{-2}I,
\end{aligned}$$

It is straightforward to verify the Hamiltonian (1) is related with the transfer matrix (6) through

$$H = -\rho\left(t(u) - \frac{1}{4}(t'(0))^2 - ut'(0) - \eta^{-2} + \omega^2 - u^2\right),$$

where the following identification has been made for the coupling constants:

$$\frac{k}{8} = \frac{\rho\eta^2}{4}, \qquad \frac{\mu}{2} = -\rho\eta\omega, \qquad \frac{\mathcal{E}}{2} = \rho.$$

We can apply the algebraic Bethe ansatz method, using the Fock vacuum $|0\rangle$ as the pseudovacuum $|\chi\rangle$, giving

$$a(u) = u^2-\omega^2, \qquad d(u) = \eta^{-2}.$$

For this case the Bethe ansatz equations are

$$\eta^2(v_k^2-\omega^2) = \prod_{j\neq k}^{M}\frac{v_k-v_j-\eta}{v_k-v_j+\eta}, \qquad k=1,\ldots,M, \tag{12}$$

where $M$ is the eigenvalue of the total number operator $N$. The energies of the Hamiltonian are

$$\begin{aligned} E = -\rho \Bigg( & \eta^{-2} \prod_{i=1}^{M} \left(1 + \frac{\eta}{v_i - u}\right) - \frac{\eta^2 M^2}{4} - u\eta M - u^2 \\ & - \eta^{-2} + \omega^2 + (u^2 - \omega^2) \prod_{i=1}^{M} \left(1 - \frac{\eta}{v_i - u}\right)\Bigg). \end{aligned}$$

This last expression is independent of the spectral parameter $u$, which can be chosen arbitrarily.

# 3 Exact Bethe ansatz solution II

The second Bethe ansatz solution of (1) described by Enol'skii, Kuznetsov and Salerno [6] applies only when $\mu = 0$, i.e. for the Hamiltonian

$$H = \frac{k}{8}(N_1 - N_2)^2 - \frac{\mathcal{E}}{2}(b_1^\dagger b_2 + b_2^\dagger b_1). \qquad (13)$$

To obtain this solution, first we introduce new operators through a transformation

$$\begin{aligned} b_1 &= \frac{1}{\sqrt{2}}(a_1 - ia_2), \qquad & b_1^\dagger &= \frac{1}{\sqrt{2}}(a_1^\dagger + ia_2^\dagger), \\ b_2 &= \frac{1}{\sqrt{2}}(a_1 + ia_2), \qquad & b_2^\dagger &= \frac{1}{\sqrt{2}}(a_1^\dagger - ia_2^\dagger) \end{aligned}$$

such that the canonical commutation relations $[a_j, a_k^\dagger] = \delta_{jk} I$ etc. hold. Under the above transformation the Hamiltonian (13) becomes

$$H = \frac{k}{8}\left(\frac{1}{2}(2n_1 + I)(2n_2 + I) - (a_1^\dagger)^2 a_2^2 - (a_2^\dagger)^2 a_1^2 - \frac{1}{2} I\right) + \frac{\mathcal{E}}{2}\left(n_2 - n_1\right), \qquad (14)$$

where $n_j = a_j^\dagger a_j$ and $N = n_1 + n_2$.

The next step is to write (14) in terms of an $su(2)$ realisation. The $su(2)$ algebra has generators $\{S^z, S^\pm\}$ with relations

$$[S^z, S^\pm] = \pm S^\pm, \qquad [S^+, S^-] = 2S^z. \qquad (15)$$

It may be shown that

$$S^+ = -\frac{1}{2}(a^\dagger)^2, \qquad S^- = \frac{1}{2}a^2, \qquad S^z = \frac{1}{4}(2N + I)$$

is an $su(2)$ realisation preserving the commutation relations (15). It follows that we may write

$$H = \frac{k}{2}\left(2S_1^z S_2^z + S_1^+ S_2^- + S_1^- S_2^+ - \frac{1}{8} I\right) + \mathcal{E}\left(S_2^z - S_1^z\right). \qquad (16)$$

To derive the Bethe ansatz solution for (16), one takes

$$\mathfrak{g} = \begin{pmatrix} \exp(\eta\alpha) & 0 \\ 0 & \exp(-\eta\alpha) \end{pmatrix},$$

with $\alpha \in \mathbb{C}$, and constructs the monodromy matrix

$$\pi(T(u)) = \mathfrak{g} L_1^S(u + \beta) L_2^S(u - \beta),$$

where $\beta \in \mathbb{C}$ and

$$L_i^S(u) = \frac{1}{u}\begin{pmatrix} uI + \eta S_i^z & \eta S_i^- \\ \eta S_i^+ & uI - \eta S_i^z \end{pmatrix}, \qquad i = 1, 2.$$

The elements of the monodromy matrix are found to be

$$\pi(A(u)) = \exp(\eta\alpha)\left\{\left(I + \frac{\eta}{u+\beta}S_1^z\right)\left(I + \frac{\eta}{u-\beta}S_2^z\right) + \frac{\eta^2}{u^2-\beta^2}S_1^-S_2^+\right\},$$
$$\pi(B(u)) = \exp(\eta\alpha)\left\{\frac{\eta}{u+\beta}S_1^-\left(I - \frac{\eta}{u-\beta}S_2^z\right) + \frac{\eta}{u-\beta}S_2^-\left(I + \frac{\eta}{u+\beta}S_1^z\right)\right\},$$
$$\pi(C(u)) = \exp(-\eta\alpha)\left\{\frac{\eta}{u+\beta}S_1^+\left(I + \frac{\eta}{u-\beta}S_2^z\right) + \frac{\eta}{u-\beta}S_2^+\left(I - \frac{\eta}{u+\beta}S_1^z\right)\right\},$$
$$\pi(D(u)) = \exp(-\eta\alpha)\left\{\left(I - \frac{\eta}{u+\beta}S_1^z\right)\left(I - \frac{\eta}{u-\beta}S_2^z\right) + \frac{\eta^2}{u^2-\beta^2}S_1^+S_2^-\right\}$$

from which we can construct the transfer matrix (6). For the Bethe ansatz solution, the pseudo-vacuum state $|\chi\rangle$ can be chosen to be the vacuum state $|0\rangle$, either of the one-particle states $a_1^\dagger|0\rangle$ or $a_2^\dagger|0\rangle$, or the two particle state $a_1^\dagger a_2^\dagger|0\rangle$, since for all cases

$$\pi(B(u))|\chi\rangle = 0$$

and

$$\pi(A(u))|\chi\rangle = a(u)|\chi\rangle, \qquad \pi(D(u))|\chi\rangle = d(u)|\chi\rangle.$$

In this manner the form of the Bethe ansatz solution depends on whether the total particle number is even or odd. We find

$$a(u) = \exp(\eta\alpha)\left(1 + \frac{\eta\kappa_1}{u+\beta}\right)\left(1 + \frac{\eta\kappa_2}{u-\beta}\right), \tag{17}$$

$$d(u) = \exp(-\eta\alpha)\left(1 - \frac{\eta\kappa_1}{u+\beta}\right)\left(1 - \frac{\eta\kappa_2}{u-\beta}\right), \tag{18}$$

where $\kappa_1 = \kappa_2 = 1/4$ or $\kappa_1 = \kappa_2 = 3/4$ for the even case, and $\kappa_1 = 3/4$, $\kappa_2 = 1/4$ or $\kappa_1 = 1/4$, $\kappa_2 = 3/4$ for the odd case. It can now be shown that $\tau_1$, $\tau_2$ defined by

$$\tau_1 = \lim_{\eta\to 0}\lim_{u\to-\beta}\left(\frac{u+\beta}{\eta^2}\right)t(u) = 2\alpha S_1^z - \frac{1}{2\beta}\left(2S_1^zS_2^z + S_1^+S_2^- + S_1^-S_2^+\right),$$
$$\tau_2 = \lim_{\eta\to 0}\lim_{u\to\beta}\left(\frac{u-\beta}{\eta^2}\right)t(u) = 2\alpha S_2^z + \frac{1}{2\beta}\left(2S_1^zS_2^z + S_1^+S_2^- + S_1^-S_2^+\right)$$

are related to the Hamiltonian (16) and the total number operator through

$$H = \tau_2 - \tau_1 - \frac{k}{16}I, \qquad N = \frac{2}{\mathcal{E}}(\tau_1 + \tau_2) - I$$

with

$$\beta = \frac{2}{k}, \qquad \alpha = \frac{\mathcal{E}}{2}.$$

To make the Bethe ansatz solution of the model explicit it is a matter of substituting (17), (18) into (8) and taking the limit $\eta \to 0$ to obtain

$$\alpha + \frac{\kappa_1}{v_k+\beta} + \frac{\kappa_2}{v_k-\beta} = \sum_{j\neq k}^{M}\frac{1}{v_j - v_k}, \qquad k = 1, \ldots, M. \tag{19}$$

Letting $\lambda_j$ denote the eigenvalue of $\tau_j$, it follows from (7) that

$$\lambda_1 = \lim_{\eta\to 0}\lim_{u\to-\beta}\left(\frac{u+\beta}{\eta^2}\right)\Lambda(u) = 2\kappa_1\left(\alpha - \frac{\kappa_2}{2\beta} - \sum_{j=1}^{M}\frac{1}{v_j+\beta}\right),$$

$$\lambda_2 = \lim_{\eta\to 0}\lim_{u\to\beta}\left(\frac{u-\beta}{\eta^2}\right)\Lambda(u) = 2\kappa_2\left(\alpha + \frac{\kappa_1}{2\beta} - \sum_{j=1}^{M}\frac{1}{v_j-\beta}\right).$$

The eigenvalues of the Hamiltonian are given by

$$E = \mathcal{E}(\kappa_2-\kappa_1) + \frac{kN^2}{8} + \frac{k\mathcal{E}}{2}\sum_{j=1}^{M}v_j,$$

where those of the number operator are

$$N = 2M + 2(\kappa_1+\kappa_2) - 1.$$

It is apparent that the Bethe ansatz equations (12) with $\omega = 0$, which are in multiplicative form, take on a different form to those given by (19) which are additive. Moreover, the Bethe ansatz equations (12) are associated with a single reference state whereas (19) are dependent on the choice of reference state. In this latter case there are four forms of the Bethe ansatz equations associated with the choices of $\kappa_1$, $\kappa_2$ which can take values 1/4 or 3/4. In the following it will be shown how a unified system of Bethe ansatz equations can be derived in the additive form. This approach does not use the Quantum Inverse Scattering Method.

## 4 Exact Bethe ansatz solution III

We again follow the work of Enol'skii, Kuznetsov and Salerno [6] (see also [17]) and start with the Jordan-Schwinger realisation of the $su(2)$ algebra (15):

$$S^+ = b_1^\dagger b_2, \qquad S^- = b_2^\dagger b_1, \qquad S^z = \frac{1}{2}(N_1 - N_2)$$

which is $(N+1)$-dimensional when the constraint of fixed particle number $N = N_1 + N_2$ is imposed. In terms of this realisation the Hamiltonian (1) may be written as

$$H = \frac{k}{2}(S^z)^2 - \mu S^z - \frac{\mathcal{E}}{2}\left(S^+ + S^-\right). \tag{20}$$

The same $(N+1)$-dimensional representation of $su(2)$ is given by the mapping to differential operators

$$S^z = u\frac{\mathrm{d}}{\mathrm{d}u} - \frac{N}{2}, \qquad S^+ = Nu - u^2\frac{\mathrm{d}}{\mathrm{d}u}, \qquad S^- = \frac{\mathrm{d}}{\mathrm{d}u}$$

acting on the $(N+1)$-dimensional space of polynomials with basis $\{1, u, u^2, \ldots, u^N\}$. We can then equivalently represent (20) as the second-order differential operator

$$\begin{aligned} H &= \frac{k}{2}\left(u^2\frac{\mathrm{d}^2}{\mathrm{d}u^2} + (1-N)u\frac{\mathrm{d}}{\mathrm{d}u} + \frac{N^2}{4}\right) - \mu\left(u\frac{\mathrm{d}}{\mathrm{d}u} - \frac{N}{2}\right) - \frac{\mathcal{E}}{2}\left(Nu + (1-u^2)\frac{\mathrm{d}}{\mathrm{d}u}\right) \\ &= \frac{ku^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \frac{1}{2}\left((k(1-N)-2\mu)u + \mathcal{E}(u^2-1)\right)\frac{\mathrm{d}}{\mathrm{d}u} + \frac{kN^2}{8} + \frac{\mu N}{2} - \frac{\mathcal{E}Nu}{2}. \end{aligned} \tag{21}$$

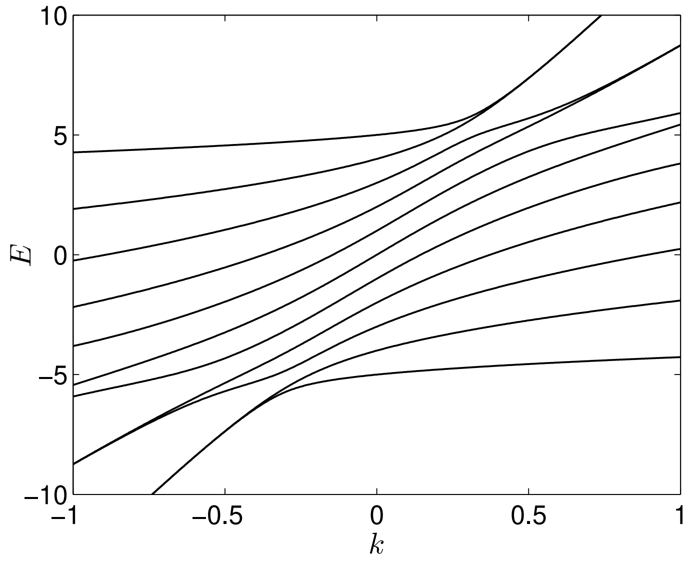


**Figure 1.** Energy levels $E$ versus coupling $k$ of the Hamiltonian (1) for $N = 10$, $\mu = 0$, and $\mathcal{E} = 1$.

Solving for the spectrum of the Hamiltonian (1) is then equivalent to solving the eigenvalue equation

$$HQ(u) = EQ(u), \tag{22}$$

where $H$ is given by (21) and $Q(u)$ is a polynomial function of $u$ of order $N$.

From this point, it is little effort to obtain a third Bethe ansatz solution for the Hamiltonian (1) (cf. [3]). First express $Q(u)$ in terms of its roots $\{v_j\}$:

$$Q(u) = \prod_{j=1}^{N} (u - v_j).$$

Evaluating (22) at $u = v_l$ for each $l$ leads to the set of Bethe ansatz equations

$$\frac{\mathcal{E}v_l^2 + (k(1-N) - 2\mu)v_l - \mathcal{E}}{kv_l^2} = \sum_{j \neq l}^{N} \frac{2}{v_j - v_l}, \qquad l = 1, \ldots, N. \tag{23}$$

Writing the asymptotic expansion $Q(u) \sim u^N - u^{N-1} \sum_{j=1}^{N} v_j$ and by considering the terms of order $N$ in (22), the energy eigenvalues are found to be

$$E = \frac{kN^2}{8} - \frac{\mu N}{2} + \frac{\mathcal{E}}{2} \sum_{j=1}^{N} v_j. \tag{24}$$

In the above manner a single form of additive Bethe ansatz equations (23) is obtained. As far as we are aware, the mapping of the solution (23), (24) to (12), (2) remains an unsolved problem.

Fig. 1 shows the energy levels for the model with $\mu = 0$ and $N = 10$, obtained from the solution (23), (24). Even for such low particle number it is clearly seen the ground state becomes *quasi-degenerate* in the attractive regime. This property underlies the validity of using spontaneous symmetry breaking based on a mean-field approximation, as discussed in [2], to distinguish the quantum phases of the Bose–Hubbard dimer Hamiltonian (1). Alternatively, associated with the Bethe ansatz solution (23), (24) there is a mapping of the spectrum of the Hamiltonian into the low energy spectrum of a one-dimensional Schrödinger equation. This facilitates a different approach for determining quantum phases of the Hamiltonian where the crossover is identified with a bifurcation of the Schrödinger equation potential [3].

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