Noncommutative Geometry: Fuzzy Spaces, the Groenewold–Moyal Plane*

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Abstract. In this talk, we review the basics concepts of fuzzy physics and quantum field theory on the Groenewold–Moyal Plane as examples of noncommutative spaces in physics. We introduce the basic ideas, and discuss some important results in these fields. At the end we outline some recent developments in the field.

Key words: noncommutative geometry; quantum algebra; quantum field theory

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1 Introduction

Noncommutative geometry is a branch of mathematics due to Gel'fand, Naimark, Connes, Rieffel and many others [1, 2, 3]. Physicists in a very short time adopted it and nowadays use this phrase whenever spacetime algebra is noncommutative.

There are two such particularly active fields in physics at present

1. Fuzzy Physics,
2. Quantum Field Theory (QFT) on the Groenewald–Moyal Plane.

Item 1 is evolving into a tool to regulate QFT's, and for numerical work. It is an alternative to lattice methods. Item 2 is more a probe of Planck-scale physics. This introductory talk will discuss both items 1 and 2.

2 History

The Groenewold–Moyal (G-M) plane is associated with noncommutative spacetime coordinates:

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}.$$

It is an example where spacetime coordinates do not commute.

The idea that spatial coordinates may not commute first occurs in a letter from Heisenberg to Peierls [4, 5]. Heisenberg suggested that an uncertainty principle such as

$$\Delta x_\mu \Delta x_\nu \geq \frac{1}{2}|\theta_{\mu\nu}|, \qquad \theta_{\mu\nu} = \text{const}$$

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can provide a short distance cut-off and regulate quantum field theories (qft's). In this letter, he apparently complains about his lack of mathematical skills to study this possibility. Peierls communicated this idea to Pauli, Pauli to Oppenheimer and finally Oppenheimer to Snyder. Snyder wrote the first paper on the subject [6]. This was followed by a paper of Yang [7].

In mid-90's, Doplicher, Fredenhagen and Roberts [8, 9] systematically constructed unitary quantum field theories on the G-M plane and its generalizations, even with time-space noncommutativity.

Later string physics encountered these structures.

3 What is noncommutative geometry

According to Connes [1, 2, 3], noncommutative geometry is a spectral triple,

$$(\mathcal{A}, D, \mathcal{H}),$$

where $\mathcal{A}$ = a C^*-algebra, possibly noncommutative, D = a Dirac operator, $\mathcal{H}$ = a Hilbert space on which they are represented.

If $\mathcal{A}$ is a commutative C^*-algebra, we can recover a Hausdorff topological space on which $\mathcal{A}$ are functions,using theorems of Gel'fand and Naimark. But that is not possible if $\mathcal{A}$ is not commutative. But it is still possible to formulate qft's using the spectral triple. A class of examples of noncommutative geometry with $\mathcal{A}$ noncommutative is due to Connes and Landi [10].

If some of the strict axioms are not enforced then the examples include $SU(2)_q$, fuzzy spaces, the G-M plane, and many more.

The introduction of noncommutative geometry has introduced a conceptual revolution. Manifolds are being replaced by their "duals", algebras, and these duals are being "quantized", much as in the passage from classical to quantum mechanics.

4 Fuzzy physics

In what follows, we sketch the contents of "fuzzy physics". Reference [11] contains a detailed survey. For pioneering work on fuzzy physics, see [12, 13, 14].

4.1 What is fuzzy physics [11]

We explain the basic ideas of fuzzy physics by a two-dimensional example: S^2_F.

Consider the two-sphere S^2. We quantize it to regularize by introducing a short distance cut-off. For example in classical mechanics, the number of states in a phase space volume

$$\Delta V = d^3p d^3q$$

is infinite. But we know since Planck and Bose that on quantization, it becomes

$$\frac{\Delta V}{h^3} = \text{finite}.$$

This is the idea behind fuzzy regularization.

In detail, this regularization works as follows on S^2. We have

$$S^2 = [\vec{x} \in \mathbb{R}^3 : \ \vec{x} \cdot \vec{x} = r^2].$$

Now consider angular momentum L_i:

$$[L_i, L_j] = i\epsilon_{ijk} L_k, \qquad \vec{L}^2 = l(l+1).$$

Set

$$\hat{x}_i = r\frac{L_i}{\sqrt{l(l+1)}} \qquad \Rightarrow$$
$$\hat{x}\cdot\hat{x} = r^2, \qquad [\hat{x}_i, \hat{x}_j] = \frac{r}{\sqrt{l(l+1)}} i\epsilon_{ijk}\hat{x}_k,$$

where $\hat{x}_i \in \mathrm{Mat}_{2l+1} \equiv$ space of $(2l+1)\times(2l+1)$ matrices. As $l \to \infty$, they become commutative. They give the fuzzy sphere S_F^2 of radius r and dimensions $2l+1$.

4.2 Why is this space fuzzy

As $\hat{x}_i$, $\hat{x}_j$ $(i \neq j)$ do not commute, we cannot sharply localize $\hat{x}_i$. Roughly in a volume $4\pi r^2$ there are $(2l+1)$ states.

4.3 Field theory on fuzzy sphere

A scalar field on fuzzy sphere is defined as a polynomial in $\hat{x}_i$, i.e.,

$$\text{A scalar field } \Phi \;=\; \text{A polynomial in } \hat{x}_i \;=\; \text{A } (2l+1) - \text{dimensional matrix.}$$

Differentiation is given by infinitesimal rotation:

$$\mathcal{L}_i\Phi = [L_i, \Phi].$$

A simple rotationally invariant scalar field action is given by

$$S(\Phi) = \mu\,\mathrm{Tr}\,[L_i, \Phi]^\dagger [L_i, \Phi] + \frac{m}{2}\,\mathrm{Tr}\,(\Phi^\dagger\Phi) + \lambda\,\mathrm{Tr}\,(\Phi^\dagger\Phi)^2.$$

Simulations have been performed [15, 16] on the partition function $\mathcal{Z} = \int d\Phi e^{-S(\Phi)}$ of this model and the major findings include the following:

- Continuum limit exists.
- If

  $$\Phi = \sum c_{lm}\hat{Y}_{lm}, \qquad \hat{Y}_{lm} = \text{spherical tensor},$$

  then there are three phases:

  1. Disordered : $\langle \sum |c_{lm}|^2 \rangle = 0$.
  2. Uniform ordered: $\langle |c_{00}|^2 \rangle \neq 0$, $\langle |c_{lm}|^2 \rangle = 0$ for $l \neq 0$.
  3. Non-uniform ordered: $\langle |c_{1m}|^2 \rangle \neq 0$, $\langle |c_{lm}^2 \rangle = 0$ for $l \neq 1$.

  The last one is an analogue of the Gupser–Sondhi phase [17, 18, 19].

4.3.1 Dirac operator

S_F^2 has a Dirac operator including instantons and with no fermion doubling [20, 21, 22, 23, 24, 25, 26].

Also S_F^2 can nicely describe topological features. Hence it seems better suited for preserving symmetries than lattice approximations.

4.3.2 Supersymmetry

If we replace $SU(2)$ by $OSp(2,1)$, the fuzzy sphere becomes the $N = 1$ supersymmetric fuzzy sphere and can be used to simulate supersymmetry [20, 21, 27, 28, 29, 30, 31, 32]. Simulations in this regard are already starting.

4.3.3 Strings [33]

If N D-branes are close, the transverse coordinates Φ_i become $N \times N$ matrices with the action given by

$$\mathrm{S} = \lambda \mathrm{Tr}\,[\Phi_i, \Phi_j]^\dagger [\Phi_i, \Phi_j] + i f_{ijk} \Phi_i \Phi_j \Phi_k,$$

where f_{ijk} are totally antisymmetric.

The equations of motion

$$[\Phi_i, \Phi_j] \cong i f_{ijk} \Phi_k$$

give solutions when f_{ijk} are structure constants of a simple compact Lie group. Thus we can have

$$\Phi_i = cL_i, \qquad f_{ijk} = c\epsilon_{ijk}, \qquad c = \text{const}, \qquad L_i = \text{angular momentum operators.}$$

If L_i form an irreducible set, then we have

$$\vec{L} \cdot \vec{L} = l(l+1), \qquad (2l+1) = N,$$

and we have one fuzzy sphere. Or we can have a direct sum of irreducible representations:

$$L_i = \oplus L_i^{l_k}, \qquad \vec{L}^{l_k} \cdot \vec{L}^{l_k} = l_k(l_k+1), \qquad \sum (2l_k + 1) = N.$$

Then we have many fuzzy spheres.

Stability analysis of these solutions including numerical studies has been done by many groups.

5 The G-M Plane

5.1 Quantum gravity and spacetime noncommutativity: heuristics

The following arguments were described by Doplicher, Fredenhagen and Robert in their work in support of the necessity of noncommutative spacetime at Planck scale.

5.1.1 Space-space noncommutativity

In order to probe physics at the Planck scale L, the Compton wavelength $\frac{\hbar}{Mc}$ of the probe must fulfill

$$\frac{\hbar}{Mc} \leq L \qquad \text{or} \qquad M \geq \frac{\hbar}{Lc} \simeq \text{Planck mass.}$$

Such high mass in the small volume L^3 will strongly affect gravity and can cause black holes to form. This suggests a fundamental length limiting spatial localization.

5.1.2 Time-space noncommutativity

Similar arguments can be made about time localization. Observation of very short time scales requires very high energies. They can produce black holes and black hole horizons will then limit spatial resolution suggesting

$$\Delta t \Delta |\vec{x}| \geq L^2, \qquad L = \text{a fundamental length.}$$

The G-M plane *models* above spacetime uncertainties.

5.2 What is the G-M plane

The Groenewold–Moyal plane $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ consists of functions $\alpha, \beta, \ldots$ on $\mathbb{R}^{d+1}$ with the $*$-product

$$\alpha * \beta = \alpha e^{\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} \beta.$$

For spacetime coordinates, this implies,

$$[x_\mu, x_\nu]_* = x_\mu * x_\nu - x_\nu * x_\mu = i\theta_{\mu\nu}.$$

Conversely these coordinate commutators imply the general $*$-product up to certain equivalencies.

The G-M plane also emerges in quantum Hall effect and string physics.

5.3 How the G-M plane emerges from quantum Hall effect and strings

5.3.1 Quantum Hall effect (the Landau problem) [34]

Consider an electron in 1–2 plane and an external magnetic field $\vec{B} = (0, 0, B)$ perpendicular to the plane. Then the Lagrangian for the system is

$$L = \frac{1}{2} m \dot{x}_a^2 + e \dot{x}_a^2 A_a,$$

where

$$A_a = -\frac{B}{2} \epsilon_{ab} x^b, \qquad a, b = 1, 2,$$

is the electromagnetic potential and x_a are the coordinates of the electron.

Now if $eB \to \infty$, then

$$L \ \sim \ \frac{eB}{2} (\dot{x}_1 x_2 - \dot{x}_2 x_1).$$

This means that on quantization we will have

$$[\hat{x}_a, \hat{x}_b] = \frac{i}{eB} \epsilon_{ab}$$

which defines a G-M plane.

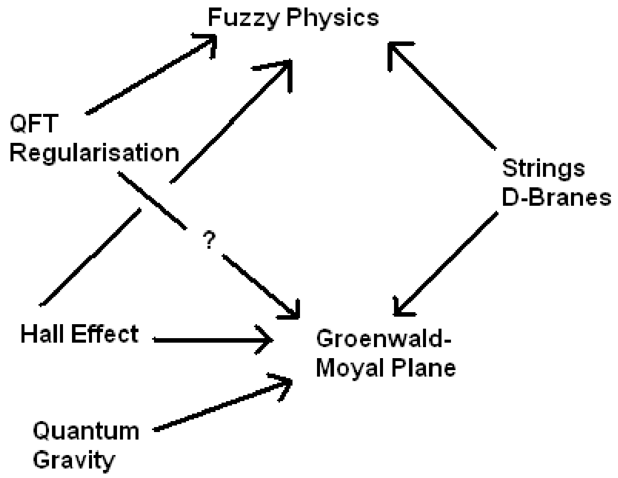

Figure 1. The tangled web: emergence of noncommutative spaces from different fields.

5.3.2 Strings [35]

Consider open strings ending on Dp-Branes. If there is a background two-form Neveu–Schwarz field given by the constants $B_{ij} = -B_{ji}$, then the action is given by

$$S_\Sigma = \frac{1}{4\pi\alpha'}\int_\Sigma \left[g_{ij}\partial_a x^i \partial_a x^j - 2\pi\alpha' B_{ij}\partial_a x^i \partial_b x^j \epsilon^{ab} + \text{spinor terms}\right] d\sigma dt.$$

As $B \to \infty$ or equivalently $g_{ij} \to 0$,

$$S_\Sigma = -\frac{2\pi e}{4\pi}\int_\Sigma B_{ij} dx^i \wedge dx^j = \left[\int_{\partial\Sigma^0} - \int_{\partial\Sigma^1}\right] eB_{ij}x^i \frac{dx^j}{dt}$$

$$\Rightarrow \quad e[B_{ij}\hat{x}^j, \hat{x}^k] = i\delta_{ik} \qquad \text{or} \qquad [\hat{x}_j, \hat{x}_k] = \frac{i}{e}(B^{-1})_{jk}$$

which is just a G-M plane.

Fig. 1 indicates different sources wherefrom fuzzy physics and the G-M plane emerge. The question mark is to indicate that the G-M plane may not regularize qft's.

5.4 Prehistory (before 2004/2005)

Until 2004/2005, much work was done on

- QFT's on the G-M plane and its renormalization theory, uncovering the phenomenon of UV/IR mixing [36].
- Phenomenology, including the study of the effects of noncommutativity on Lorentz invariance violation (from $\theta_{\mu\nu}$ in $[x_\mu, x_\nu]_* = i\theta_{\mu\nu}$), C, CP and CPT.

5.5 Modern era

In 2004/2005, Chaichian et al. [37, 38] and Aschieri et al. [39, 40] applied the Drinfel'd twist [41] which restores full diffeomorphism invariance (with a twist in the "coproduct") despite the presence of constants $\theta_{\mu\nu}$ in $[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$. This twist also twists statistics [42, 43][1].

Much of this was known to Majid [47], Oeckl [48], Fiore and Schupp [49, 50, 51] and Watts [52, 53]. So the Drinfel'd twist twists both

[1]There are claims to the contrary, see [44, 45, 46] for the debate.

1. action of diffeomorphisms, and
2. exchange statistics.

This brings into question much of the prehistory-analysis. Examples include the following new results:

1. The Pauli principle can be violated on the G-M plane.
2. (Twisted) Lorentz invariance need not be violated even if $\theta_{\mu\nu} \neq 0$.
3. There need be no ultraviolet-infrared (UV-IR) mixing in the absence of gauge fields [54].

There is also a striking, clean separation of matter from gauge fields due to the Drinfel'd twist [55], (in the sense that they have to be treated differently) reminiscent of the distinction between particles and waves in the classical theory.

Literature should be consulted for details of these developments.

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