Para-Grassmann Variables and Coherent States[⋆]

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Abstract. The definitions of para-Grassmann variables and q-oscillator algebras are recalled. Some new properties are given. We then introduce appropriate coherent states as well as their dual states. This allows us to obtain a formula for the trace of a operator expressed as a function of the creation and annihilation operators.

Key words: para-Grassmann variables; q-oscillator algebra; coherent states

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1 Introduction

The study of different generalisations of Grassmann variables and their applications has attracted a great deal of interest in the last decades (see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and references therein).

Our approach is motivated by the fact that generalised Grassmann variables provide a natural framework for the description of particles obeying generalised statistics. We thus focus on the q-oscillator algebra (introduced in [15, 16]) which is particularly appealing for our purpose for two distinct reasons. First, the nilpotency property of the creation and annihilation operator is in direct connection with the maximal occupation number of the studied particles. Second, for the case of multi-particle states, the wave function acquires a nontrivial phase when two particles are interchanged (one may recall that this phase is trivial in the case of bosons and is -1 in the case of fermions). One can note also that in [17] the authors have discussed many-body states and the algebra of creation and annihilation operators for particles obeying exclusion statistics.

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This paper is structured as follows. We first review the definition and basic properties of the para-Grassmann variables. We then re-examine the q-oscillator algebra and introduce appropriate coherent states. New properties of the coherent states are given. Finally we find a representation for the trace of any operator, as an integration over para-Grassmann variables. We show that the trace can be represented as a para-Grassmann integral of the matrix element of the respective operator on the coherent state. This result is the natural generalisation of the usual formula for the trace of an operator in the case of bosons or fermions (see for example [18]). In the last section, some perspectives are briefly outlined. Let us mention here that this work presents some partial results of a future publication [19].

2 One-particle states

2.1 Para-Grassmann variables

Consider the non-commutative variables θ and $\bar{\theta}$:

$$\theta^{p+1} = 0, \qquad \bar{\theta}^{p+1} = 0, \qquad \theta\bar{\theta} = q^2\bar{\theta}\theta, \qquad \text{where} \qquad q^2 = e^{\frac{2\pi i}{p+1}} \tag{1}$$

with p some non-zero integer number. Note that in [3] these variables are referred to as *classical* $(p+1)$*-variables*. Moreover, the name "para-Grassmann" was used also for different other definitions, see for example [1], where some different variables, in connection with para-statistics, were defined. Finally let us mention that in [9, 10], q-deformed classical variables and different techniques were introduced.

We will use here the conventions of [3] for the definitions of a differential and integral calculus appropriate for these variables. Thus [3, 20]

$$\int d\theta\, \theta^n = \delta_p^n \sqrt{[p]!},$$

where

$$[X] = \frac{q^{2X} - 1}{q^2 - 1} \tag{2}$$

for any given c-number or operator X and

$$[n]! = [1] \cdots [n]$$

for any given number n. Of special importance is the *q-deformed exponential*

$$e_q^x = \sum_{n=0}^{p} \frac{x^n}{[n]!}.$$

2.2 q-oscillator algebra

Consider the **q-boson oscillator** [15, 16]

$$aa^\dagger - qa^\dagger a = q^{-N}, \qquad aa^\dagger - q^{-1}a^\dagger a = q^N, \tag{3}$$

where $q \neq -1$ is a complex number. Note that we deal here with some generalisation of *bosons* and not of *fermions*.

Following the conventions of [3] we define

$$[\![X]\!] = \frac{q^X - q^{-X}}{q - q^{-1}},$$

for any c-number or operator X. If q is a unit phase,

$$[\![X]\!] = [\![X]\!]^*,$$

where by * we understand complex conjugation. We have

$$a^\dagger a = [\![N]\!].$$

In particular, if $q^2 = e^{\frac{2\pi i}{p+1}}$ we can write

$$N = \frac{p+1}{\pi} \arcsin\left(a^\dagger a \sin\frac{\pi}{p+1}\right).$$

Occupation number representation: Introducing a vacuum vector $|0\rangle$ (s.t. $a|0\rangle = 0$) we define

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[\![n]\!]!}}|0\rangle.$$

Using again the commutation relations (3) we get

$$N|n\rangle = n|n\rangle, \qquad a|n\rangle = \sqrt{[\![n]\!]}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{[\![n+1]\!]}|n+1\rangle.$$

Furthermore

$$[N, a] = -a, \qquad [N, a^\dagger] = a^\dagger \tag{4}$$

and if $q = e^{\frac{2\pi i}{p+1}}$ it can be proved that the creation and annihilation operators are $(p+1)$-nilpotent, $a^{p+1} = 0 = (a^\dagger)^{p+1}$ (see [2]). Moreover, using (4) we have that for any c-number λ

$$q^{\lambda N} a^\dagger = q^\lambda a^\dagger q^{\lambda N}, \qquad q^{\lambda N} a = q^{-\lambda} a q^{\lambda N}.$$

(see [3]).

If instead, we assume the nilpotency condition of creation and annihilation operators

$$a^{p+1} = 0 = (a^\dagger)^{p+1}. \tag{5}$$

without imposing any condition on q (here we require that the exponent $p+1$ is the minimal exponent of nilpotency, so $a^r \neq 0$ and $(a^\dagger)^r \neq 0$ if $r \leq p$), we get

$$a(a^\dagger)^i = (1 + q^2 + \cdots + (q^2)^{i-1}) q^{-N} (a^\dagger)^{i-1} + q^i (a^\dagger)^i a.$$

If $q^2 \neq 1$ this becomes

$$a(a^\dagger)^i = \frac{1-(q^2)^i}{1-q^2} q^{-N} (a^\dagger)^{i-1} + q^i (a^\dagger)^i a.$$

Taking now $i = p+1$ one has

$$a(a^\dagger)^{p+1} = \frac{1-(q^2)^{p+1}}{1-q^2} q^{-N} (a^\dagger)^p + q^{p+1} (a^\dagger)^{p+1} a.$$

Now, using (5) we derive that $(q^2)^{p+1} = 1$.

From the discussion above we conclude that for the q-boson oscillator algebra (3) the conditions: q^2 is a primitive $p+1$ root of unity and a and $a^\dagger$ are $(p+1)$-nilpotent, are equivalent.

Finally, let us mention here that the operators a and $a^\dagger$ are hermitian conjugates and they generate a unitary representation [21].

Consider now the change of variables

$$b = q^{\frac{N}{2}} a, \qquad \bar{b} = a^\dagger q^{\frac{N}{2}}.$$

In these new variables the relations (3) reads

$$b\bar{b} - q^2 \bar{b} b = 1, \qquad b\bar{b} - \bar{b} b = q^{2N}, \tag{6}$$

and thus

$$\bar{b} b = [N],$$

where $[N]$ was defined in (2).

We can also express the occupation number states in terms of $\bar{b}$ and b as follows

$$|n\rangle = \frac{(\bar{b})^n}{\sqrt{[n]!}} |0\rangle, \qquad b|n\rangle = \sqrt{[n]} |n-1\rangle, \qquad \bar{b}|n\rangle = \sqrt{[n+1]} |n+1\rangle.$$

Furthermore

$$[N, b] = -b, \qquad [N, \bar{b}] = \bar{b}.$$

Unlike the operators a and $a^\dagger$, the operators b and $\bar{b}$ are not hermitian conjugates ($b^\dagger \neq \bar{b}$) so in order to define the dual vectors we introduce the operators $b^\dagger$ and $\bar{b}^\dagger$, the hermitian conjugate of b and $\bar{b}$ respectively. One has (see also [8])

$$b^\dagger = \bar{b} q^{-N}, \qquad \bar{b}^\dagger = q^{-N} b.$$

Thus, up to a phase, $\bar{b}$ coincides with $b^\dagger$ and b with $\bar{b}^\dagger$. We then have

$$\langle n| = \langle 0| \frac{b^n}{\sqrt{[n]!}}, \qquad \langle n| b = \langle n+1| \sqrt{[n+1]}, \qquad \langle n| \bar{b} = \langle n-1| \sqrt{[n]}.$$

Before going further let us mention that different q-deformed algebraic structures with similar properties exist in the literature, like the para-Grassmann algebra (see [4, 5, 7]) or the quon-algebra (see [23, 24] and references therein).

Commutation relations between para-Grassmannians and creation/annihilation operators. We complete the set of commutation relations given in equations (1) and (6) with

$$\theta b = q^2 b \theta, \qquad \theta \bar{b} = q^{-2} \bar{b} \theta, \qquad \bar{\theta}\, \bar{b} = q^2 \bar{b}\, \bar{\theta}, \qquad \bar{\theta} b = q^{-2} b \bar{\theta} \tag{7}$$

(notice that instead of the set (7), in some papers [13] regular commutation relations are assumed).

Thus, the structure we study further on consists of the nilpotent operators b and $\bar{b}$ obeying the q-boson algebra (6), and the para-Grassmann variables θ and $\bar{\theta}$ obeying the commutation relations given in equations (1) and (7). We also set the value of q to $q = e^{\frac{\pi i}{p+1}}$. Notice that, because of the commutation relations (7), the vectors $|n\rangle$ do not commute with θ. Indeed, if we impose $\theta|0\rangle = |0\rangle\theta$ it follows that $\theta|n\rangle = q^{-2n}|n\rangle\theta$.

Coherent states. To find a coherent state $|\theta\rangle$ we write generically

$$|\theta\rangle = \sum_{n=0}^{p} c_n |n\rangle.$$

Imposing now

$$b|\theta\rangle = \theta|\theta\rangle$$

we get

$$|\theta\rangle = \sum_{n=0}^{p} \frac{q^{n(n+1)}}{\sqrt{[n]!}} \theta^n |n\rangle$$

which can be written as

$$|\theta\rangle = e_q^{\bar{b}\theta}|0\rangle.$$

The action of $\bar{b}$ over the state θ is given by

$$\bar{b}|\theta\rangle = q^{-2} \sum_{n=1}^{p} q^{n(n+1)} \frac{[n]}{\sqrt{[n]!}} \theta^{n-1}|n\rangle.$$

Finally one has the scalar product

$$\langle n|\theta\rangle = \frac{q^{-n(n-1)}}{\sqrt{[n]!}} \theta^n. \tag{8}$$

In analogy with the definition of $|\theta\rangle$ we define a dual state $\langle\bar{\theta}|$ through the relation

$$\langle\bar{\theta}|\bar{b} = \langle\bar{\theta}|\bar{\theta}.$$

We have

$$\langle\bar{\theta}| = \sum_{n=0}^{p} \frac{q^{n(n-1)}}{\sqrt{[n]!}} \bar{\theta}^n \langle n|$$

or, expressed in terms of b,

$$\langle\bar{\theta}| = \langle 0|e_q^{\bar{\theta}b}.$$

Finally, we can compute the scalar product:

$$\langle\bar{\theta}|n\rangle = \frac{q^{n(n-1)}}{\sqrt{[n]!}}\, \bar{\theta}^n. \tag{9}$$

Let us stress that the scalar product (9) *is not* the complex conjugate of the scalar product (8). First, the para-Grassmannians θ and $\bar{\theta}$ cannot be complex conjugated to each other (this is incompatible with the commutation relations (1)) and second, $[n]$ is not a real number.

Let us mention here that different definitions of coherent states have been proposed for different algebraic structures in some of the references cited. Thus, the definition we give is different by some phase (see equations (8) and (9)) of the one proposed in [3] (also for the q-boson oscillator algebra). Another example is given by the definition of [12, 13]; here also the analytical difference is given by some phase, but in [12, 13] the coherent states are defined for a different algebraic structure.

The matrix elements of the identity operator can be written as

$$\langle\bar{\theta}|\,\mathrm{Id}\,|\theta\rangle = \langle\bar{\theta}|\theta\rangle = \sum_{n=0}^{p} \frac{1}{[n]!}\, \bar{\theta}^n \theta^n.$$

We can compute explicitly the matrix element $\langle\bar\theta|\mathcal{O}|\theta\rangle$ for any operator $\mathcal{O}$ expressed as a function of b and $\bar b$. If in the case of bosons and fermions this matrix element has a compact form, independent of the form of $\mathcal{O}$ (see for example [18]), this does not hold anymore for para-Grassmannians.

Let us now look for a resolution of the identity

$$\mathrm{Id} = \int d\bar\theta d\theta \mu(\bar\theta,\theta)|\theta\rangle\langle\bar\theta|, \tag{10}$$

where $\mu(\bar\theta,\theta) = \sum_{n=0}^{p} \mu_n \bar\theta^n\theta^n$ is a weight factor to be determined (μ_n being some complex number coefficient). Equation (10) is equivalent to

$$\delta_{mn} = \langle n|m\rangle = \langle n|\,\mathrm{Id}\,|m\rangle = \int d\bar\theta d\theta\mu(\bar\theta,\theta)\langle n|\theta\rangle\langle\bar\theta|m\rangle = \int d\bar\theta d\theta\, \mu(\bar\theta,\theta)\frac{\theta^n\bar\theta^m}{\sqrt{[n]![m]!}}, \tag{11}$$

where we have used expressions (8) and (9) for the scalar products $\langle n|\theta\rangle$ and $\langle\bar\theta|m\rangle$. Notice that the q-factors involved in these scalar products cancel each other, also since $\mu(\bar\theta,\theta)$ only involves powers of $\bar\theta\theta$, it commutes with $\langle n|$.

Integrating (11) we get (see [3])

$$\mu_n = \frac{(-1)^n}{[n]!}q^{n(n-1)}$$

so we finally obtain

$$\mu(\bar\theta,\theta) = \sum_{n=0}^{p}\frac{(-1)^n}{[n]!}q^{n(n-1)}\bar\theta^n\theta^n = \sum_{n=0}^{p}\frac{(-1)^n}{[n]!}\left(\bar\theta\theta\right)^n = e_q^{-\bar\theta\theta}.$$

Hence, we have the following resolution of the identity

$$\mathrm{Id} = \int d\bar\theta d\theta\, e_q^{-\bar\theta\theta}|\theta\rangle\langle\bar\theta|$$

thus allowing us to check the definition of a coherent state (see for example [25]).

Trace of an operator. Let us consider an operator $\mathcal{O}$ expressed as a function of b and $\bar b$. We want to express its trace in the form

$$\mathrm{Tr}\,\mathcal{O} = \int d\bar\theta d\theta\rho(\theta,\bar\theta)\langle\bar\theta|\mathcal{O}|\theta\rangle \tag{12}$$

with $\rho(\theta,\bar\theta)$ some function to be determined. We propose the following ansatz (that we will justify later)

$$\rho(\theta,\bar\theta) = \sum_{n=0}^{p}\rho_n\theta^n\bar\theta^n. \tag{13}$$

Equation (12) can be written as

$$\begin{aligned}\mathrm{Tr}\,\mathcal{O} &= \sum_{m,n=0}^{p}\int d\bar\theta d\theta\rho(\theta,\bar\theta)\langle\bar\theta|n\rangle\langle n|\mathcal{O}|m\rangle\langle m|\theta\rangle\\ &= \sum_{m,n=0}^{p}\langle n|\mathcal{O}|m\rangle\int d\bar\theta d\theta\rho(\theta,\bar\theta)\langle\bar\theta|n\rangle\langle m|\theta\rangle\end{aligned}$$

so we have

$$\int d\bar{\theta}d\theta\rho(\theta,\bar{\theta})\langle\bar{\theta}|n\rangle\langle m|\theta\rangle = \delta_{nm}.$$

Since only terms with $n = m$ are nonzero, the function $\rho(\theta,\bar{\theta})$ can only have terms with the same powers of θ and $\bar{\theta}$, in agreement with our ansatz (13). A straightforward computation gives

$$\rho_n = \frac{(-1)^n}{[n]!}q^{(n+1)(n+2)}$$

so we get

$$\mathrm{Tr}\,\mathcal{O} = \int d\bar{\theta}d\theta\sum_{n=0}^{p}\frac{(-1)^n}{[n]!}q^{(n+1)(n+2)}\theta^n\bar{\theta}^n\langle\bar{\theta}|\mathcal{O}|\theta\rangle. \tag{14}$$

(In the framework of the para-Grassmann algebra mentioned above, some related calculations have been performed in [7].)

The importance of formula (14) comes from the fact that it is a direct generalisation of the trace formula for boson and fermion coherent states (see for example [18]). Following the same line of reasoning it is more useful to use this formula rather than the trace expressed in terms of occupation states for the computation of some specific quantities (like for example the partition function or the occupation number). Furthermore, this would represent a direct generalisation of the calculations performed in the case of bosons or fermions.

3 Perspectives

In this paper we have studied para-Grassmann variables and the q-oscillator boson algebra. Appropriate coherent states were defined and some new properties studied. Finally we obtained a trace formula for any operator $\mathcal{O}$ expressed as a function of the creation and annihilation operators. This formula is expressed as an integral over para-Grassmann variables of the coherent-state matrix elements of the operator $\mathcal{O}$.

The next step in the direction of work we propose here is to generalise these results to multi-particle states. Once one has the equivalent of the trace formula (14) for multi-particle states, one can calculate several physical quantities, like the partition function and occupation number. The results can then be compared with the behaviour of particles obeying generalised statistics. We will report on these issues in a future paper [19].

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