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Orthogonality within the Families of C-, S-, and E-Functions of Any Compact Semisimple Lie Group

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Abstract. The paper is about methods of discrete Fourier analysis in the context of Weyl group symmetry. Three families of class functions are defined on the maximal torus of each compact simply connected semisimple Lie group G. Such functions can always be restricted without loss of information to a fundamental region $\check{F}$ of the affine Weyl group. The members of each family satisfy basic orthogonality relations when integrated over $\check{F}$ (continuous orthogonality). It is demonstrated that the functions also satisfy discrete orthogonality relations when summed up over a finite grid in $\check{F}$ (discrete orthogonality), arising as the set of points in $\check{F}$ representing the conjugacy classes of elements of a finite Abelian subgroup of the maximal torus $\mathbb{T}$. The characters of the centre Z of the Lie group allow one to split functions f on $\check{F}$ into a sum $f = f_1 + \cdots + f_c$, where c is the order of Z, and where the component functions f_k decompose into the series of C-, or S-, or E-functions from one congruence class only.

Key words: orbit functions; Weyl group; semisimple Lie group; continuous orthogonality; discrete orthogonality

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1 Introduction

The purpose of this paper is to prove continuous and discrete orthogonality relations for three families of special functions that arise in connection with each compact simple simply connected Lie group G of rank n. These functions are defined on $\mathbb{R}^n$ (more precisely on the Lie algebra of a maximal torus of G). They are periodic and have invariance properties with respect to the affine Weyl group W_{aff} of G. Following [1] we call them C-, S-, and E-functions to underline the fact that they are generalizations of the cosine, sine, and the exponential functions.

Group transforms, similar to familiar Fourier and cosine transforms, motivate this paper. The C-, S-, and E-functions serve as fundamental functions into which functions with suitable invariance properties can be expanded. Their orthogonality properties make the expansions easy to perform.

It is discrete transforms that primarily interest us here because they have a number of practically useful properties [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. In particular, continuous extension of the discrete transforms smoothly interpolate digital data in any dimension and for any lattice symmetry afforded by the structure of the given Lie group G. Many examples show that relative to the amount of available data, these transforms provide much smoother interpolation than the conventional Fourier transform.

At first encounter with such a theory, one might suspect that there are two major limitations to its applicability: first that it involves symmetry that probably does not exist in most of problems, and second that it involves data grids which reflect this symmetry and that these may be difficult to find and laborious to create. In fact neither of these is the case.

First regarding the symmetry, suppose that some compact domain D in $\mathbb{R}^n$ is given and that we are to analyze certain function f defined on D. After enclosing D inside the fundamental region $\check{F}$ of the affine Weyl group W_{aff}, we may extend f immediately to a W_{aff}-invariant function $\tilde{f}$ on all of $\mathbb{R}^n$. Thus $\tilde{f}$ is completely periodic and also carries full W-symmetry. Analysis of $\tilde{f}$ yields information about f by restriction to $\check{F}$. Second, with regards to grids, the finite subgroups of the torus $\mathbb{T}$ of G provide us with infinitely many different W_{aff}-invariant grids. There is an elegant and rapidly computable method of labeling the points of such grids that lie in the fundamental region $\check{F}$ [3]. An additional feature of the connection with the Lie groups is that we may use the central elements to effect basic splitting of functions, similar to the familiar decomposition of a function into its even and odd parts.

The purpose of this paper is to record for future reference the exact form of orthogonality relations and central splittings of the various classes of the functions (C-, S-, E-) in the context of discrete subgroups of $\mathbb{T}$ of G. The special functions considered here are new in their intended use though the C- and S-functions are well known in Lie theory. More precisely, C-functions, which have also been called Weyl orbit sums or Weyl orbit functions, are constituents of the irreducible characters of G. The discrete version of the C-transform has been described in general in our [3]. Its various mathematical applications are found in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. For other applications, see [15, 16, 17, 18, 19, 20]. Existence of the continuous version of the transform is implied there as well. A review of properties of C-functions is in [13].

The S-functions appear in the Weyl character formula [21], where each character is expressed as the ratio of two particular S-functions. In this paper the functions are put to a different use. A detailed description of the four special cases of S-functions depending on two variables (semisimple Lie groups of rank two), is found in [11]. A general review of properties of S-functions is in [14]. The E-functions are not found in the literature except [1]. Their definition is rather natural, once the even subgroup of the Weyl group is invoked.

The names C-, S-, and E-functions make allusion to the fact that, for rank $n = 1$, the functions become cosine, sine, and the exponential functions. The underlying compact Lie groups are $SU(2)$ for the C- and S-functions, and either $SU(2)$ or $U(1)$ for the E-functions, because $W^e = 1$ in the $SU(2)$ case. From a general perspective, our functions are special cases of functions symmetrized by summing group dependent terms over all elements of a finite group. In our cases the finite groups are the Weyl group W for C and S, and its even subgroup W^e for the E-functions.

In Section 2 we recall pertinent facts from the Lie theory. In Section 3 orthogonality properties of common exponential function are invoked, when the function is integrated over n-dimensional torus, or when it is summed up over a suitable finite subgroup of the torus. In the subsequent Sections 4–6, the families of C-, S-, and E-functions are dealt with respectively. The three families are defined for any compact semisimple Lie group. The rank of the group is equal to the number of independent variables. In each case the definition is followed by the proof of the continuous and discrete orthogonality relations. In Section 7, it is shown how the general problem of decomposition of a function into series of C-, or S-functions can be split into c subproblems, where c is the order of the centre of the Lie group. Central splitting is illustrated on the examples of the groups $SU(3)$, $Sp(4)$, and $SU(2) \times SU(2)$. Concluding remarks are in the last section.

2 Preliminaries

The uniformity of the theory of semisimple Lie groups furnishes us with many standard concepts for any compact simply connected simple Lie group G of rank n. We assume that the reader is familiar with the basic theory of such groups but we outline some of it here in order to establish the notation. For more details see [6] and [4].

We denote by G a compact simply connected simple Lie group of rank n; $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ is its complexification. Let $\mathbb{T}$ be a maximal torus of G, $\dim_{\mathbb{R}} \mathbb{T} = n$. Let $\mathfrak{t}'$ be the Lie algebra of $\mathbb{T}$. Then $\mathfrak{t} := i\mathfrak{t}' = \sqrt{-1}\mathfrak{t}'$ is an n-dimensional real space. We have the exponential map

$$\exp 2\pi i(\cdot): \quad \mathfrak{t} \longrightarrow \mathbb{T}.$$

The kernel $\check{Q}$ of $\exp 2\pi i(\cdot)$ is a lattice of rank n in $\mathfrak{t}$ called the *co-root lattice*.

Let $\mathfrak{t}^*$ be the dual space of $\mathfrak{t}$ and let

$$\langle \cdot, \cdot \rangle: \quad \mathfrak{t}^* \times \mathfrak{t} \longrightarrow \mathbb{R}$$

be the natural pairing of $\mathfrak{t}^*$ and $\mathfrak{t}$.

For any finite dimensional (smooth complex) representation

$$\pi_V: \quad G \longrightarrow GL(V),$$

V decomposes into $\mathbb{T}$-eigenspaces

$$V = \oplus_{\lambda \in \Omega(V)} V^{\lambda},$$

where $\Omega(V) \subset \mathfrak{t}^*$ is a subset of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$. Then on each weight space V^{λ}, $\pi_V(\exp 2\pi i x)$ is simply a multiplication by $e^{2\pi i \langle \lambda, x \rangle}$, for all $x \in \mathfrak{t}$. The set $\Omega(V)$ is the set of weights of π_V. Since

$$\check{Q} = \ker(\exp 2\pi i(\cdot)) \text{ one has } e^{2\pi i \langle \lambda, x \rangle} = 1 \text{ for all } \lambda \in \Omega(V) \text{ and for all } x \in \check{Q}.$$

Thus $\lambda \in \Omega(V)$ implies that λ is in the $\mathbb{Z}$-dual lattice P to $\check{Q}$. This is a rank n lattice in $\mathfrak{t}^*$ called the *weight lattice*. Every $\lambda \in P$ occurs in $\Omega(V)$ for some representation π_V.

For the adjoint representation, the eigenspace decomposition is

$$\mathfrak{g}_{\mathbb{C}} = t_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_C^{\alpha}, \tag{1}$$

i.e. $\{0\} \cup \Delta = \Omega(\text{adjoint rep.})$ and Δ is called the *root system* of G relative to $\mathbb{T}$. The $\mathbb{Z}$-span Q of Δ is an n-dimensional lattice in $\mathfrak{t}^*$, called the *root lattice*. Its $\mathbb{Z}$-dual in $\mathfrak{t}$ is the *co-weight lattice* $\check{P}$.

Let us summarize relations between the lattices:

$$\begin{array}{ccc} \mathfrak{t}^* & & \\ \cup & & \\ P & \overset{\mathbb{Z}\text{-dual}}{\longleftrightarrow} & \check{Q} \\ \cup & & \cap \\ Q & \overset{\mathbb{Z}\text{-dual}}{\longleftrightarrow} & \check{P} \\ & & \cap \\ & & \mathfrak{t} \end{array}$$

We have the index of connection:

$$|P/Q| = |\check{P}/\check{Q}|$$

which is the order of the centre of G.

Let

$$(\cdot\,,\cdot): \quad \mathfrak{t}^* \times \mathfrak{t}^* \longrightarrow \mathbb{R}$$

be the transpose of the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{t}$. Then we can introduce the following four bases:

- $\{\alpha_1, \ldots, \alpha_n\}$, basis of Q formed by a set of simple roots,
- $\{\check{\omega}_1, \ldots, \check{\omega}_n\}$, basis of $\check{P}$ dual to the basis $\{\alpha_1, \ldots, \alpha_n\}$,
- $\{\check{\alpha}_1, \ldots, \check{\alpha}_n\}$, basis of $\check{Q} \subset \mathfrak{t}$ of simple co-roots. They are defined by

$$\langle \phi, \check{\alpha} \rangle = \frac{2(\phi, \alpha)}{(\alpha, \alpha)} \qquad \text{for all} \quad \phi \in \mathfrak{t}^*,$$

- $\{\omega_1, \ldots, \omega_n\}$, basis of P, the fundamental weights dual to $\{\check{\alpha}_1, \ldots, \check{\alpha}_n\}$.

For each $M \in \mathbb{Z}_{>0}$, we introduce two finite subgroups of $\mathbb{T}$. For $x \in \mathfrak{t}$, such that

$$(\exp 2\pi i x)^M = 1$$

we have

$$\exp(2\pi i M x) = 1 \quad \iff \quad Mx \in \check{Q} \quad \iff \quad x \in \tfrac{1}{M}\check{Q}.$$

Then $\frac{1}{M}\check{Q}/\check{Q}$ is isomorphic to the group of all elements of $\mathbb{T}$ whose order divides M. This is our first point group.

Then we have the group $\frac{1}{M}\check{P}/\check{Q}$, of all elements with the property

$$\exp(2\pi i x)^M|_{\text{adj. rep.}} = 1.$$

This is the second finite group.

Thus

$$\tfrac{1}{M}\check{P}/\check{Q} \supset \tfrac{1}{M}\check{Q}/\check{Q} \qquad \text{and} \qquad \left[\tfrac{1}{M}\check{P}/\check{Q}\right] \;:\; \tfrac{1}{M}\check{Q}/\check{Q}] = [\check{P} : \check{Q}] \tag{2}$$

For each of these two groups we have natural dual, in the sense of duality in Abelian groups. Namely, we have the map

$$\begin{array}{lcl} P \times \frac{1}{M}\check{Q} & \longrightarrow & M\text{-th roots of } 1 \\ (\lambda\ ,\ a) & \mapsto & e^{2\pi i \langle \lambda, a \rangle} \end{array} \tag{3}$$

or equivalently,

$$\begin{array}{lcl} P/MP \times \frac{1}{M}\check{Q}/\check{Q} & \longrightarrow & U(1) \\ (\lambda, a) & \mapsto & e^{2\pi i \langle \lambda, a \rangle}, \end{array} \tag{4}$$

which is a dual pairing. Likewise

$$P/MQ \times \tfrac{1}{M}\check{P}/\check{Q} \quad \longrightarrow \quad U(1), \tag{5}$$

is a dual pairing.

Of course the critical object in this theory is the Weyl group W which describes the inherent symmetry of the situation. It acts on $\mathbb{T}$, $\mathfrak{t}$, and $\mathfrak{t}^*$.

All the objects (2)–(4) are W-invariant. In particular the finite groups $\frac{1}{M}\check{Q}/\check{Q}$, $\frac{1}{M}\check{P}/\check{Q}$, P/MQ, P/MP play a fundamental role in the introduction of discrete methods into computational problems involving class functions on G or $\mathfrak{t}$.

We let P^+ denote the set of dominant weights, and P^{++} the set of strictly dominant weights.

In the sequel Γ will denote some W-invariant finite subgroup of $\mathbb{T}$. Mostly we have in mind the groups $\frac{1}{M}\check{Q}/\check{Q}$ and $\frac{1}{M}\check{P}/\check{Q}$, but this is not assumed unless explicitly stated so. Usually we will denote the elements of Γ by elements of $\mathfrak{t}$ via the exponential map.

On $\mathfrak{t}$, W can be combined with the translation group defined by $\check{Q}$ to give the affine Weyl group $W_{\rm aff} = W \ltimes \check{Q}$.

A convenient fundamental region $\check{F}$ for the affine Weyl group of a simple compact Lie group G of rank n is the simplex which is specified by the $n+1$ vertices,

$$\check{F} \quad \text{is the convex hull of} \quad \left\{0, \tfrac{\check{\omega}_1}{q_1}, \tfrac{\check{\omega}_2}{q_2}, \dots, \tfrac{\check{\omega}_n}{q_n}\right\} \subset \mathfrak{t},$$

where $q_1, \dots, q_n$ are the coefficients of the highest root of G relative to the basis of simple roots. By definition $\check{F}$ is closed. In case G is semisimple but not simple, its fundamental region is the cartesian product of fundamental regions of its simple components.

In the case of $W^e_{\rm aff}$, the affine W^e, its fundamental region is the union of (any) two adjacent copies of $\check{F}$ of the corresponding affine Weyl group $W_{\rm aff}$.

3 Basic orthogonality relations

Let us fix an n-dimensional torus $\mathbb{T}$ of a compact simple Lie group G. For $\lambda, \mu \in P$ one has the well known basic continuous orthogonality relation of the elementary functions:

$$\int_{\mathbb{T}} e^{2\pi i\langle\lambda,t\rangle}\overline{e^{2\pi i\langle\mu,t\rangle}}dt = \int_{\mathbb{T}} e^{2\pi i\langle\lambda-\mu,t\rangle}dt = \delta_{\lambda,\mu}\,, \tag{6}$$

assuming $\int_{\mathbb{T}} 1dt = 1$.

One has the basic discrete orthogonality relation on the finite Abelian subgroup Γ of $\mathbb{T}$ for $\lambda, \mu \in P$,

$$\sum_{a\in\Gamma} e^{2\pi i\langle\lambda,a\rangle}\overline{e^{2\pi i\langle\mu,a\rangle}} = \begin{cases} |\Gamma| & \text{if} \quad \lambda|_\Gamma = \mu|_\Gamma, \\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

For instance consider (7). For any $\lambda \in P$ (in particular for $\lambda - \mu$ in (7)),

$$\sum_{a\in\Gamma} e^{2\pi i\langle\lambda,a\rangle} \in \mathbb{C}$$

and for any $b \in \Gamma$,

$$e^{2\pi i\langle\lambda,b\rangle}\sum_{a\in\Gamma} e^{2\pi i\langle\lambda,a\rangle} = \sum_{a\in\Gamma} e^{2\pi i\langle\lambda,a+b\rangle} = \sum_{a\in\Gamma} e^{2\pi i\langle\lambda,a\rangle}.$$

So either $\sum\limits_{a\in\Gamma} e^{2\pi i\langle\lambda,a\rangle} = 0$ or $e^{2\pi i\langle\lambda,b\rangle} = 1$. If the former is false, $e^{2\pi i\langle\lambda,b\rangle} = 1$ for all $b \in \Gamma$. Therefore

$$\sum_{b\in\Gamma} e^{2\pi i\langle\lambda,b\rangle} = |\Gamma|\,.$$

The proof of (6) is even simpler.

Let $\lambda, \mu \in P$. We say that they satisfy the *separation condition* on Γ, if one has

$$w\lambda|_\Gamma = w'\mu|_\Gamma \quad \Longleftrightarrow \quad w\lambda = w'\mu, \qquad \text{for any} \quad w, w' \in W. \tag{8}$$

(Since Γ is W-invariant, $w\lambda|_\Gamma$ is unambiguously defined.)

Remark 1. The separation condition (8) avoids the problem that is usually called aliasing in the discrete transform literature. A given finite group Γ can only separate a finite number of weights, just as in the discrete Fourier analysis where a finite number of division points can only be used with band width limited functions.

In our case Γ is typically $\frac{1}{M}\check{Q}/\check{Q}$ or $\frac{1}{M}\check{P}/\check{Q}$ and Γ can then separate all the weights of P mod MP or all the weights of P mod MQ respectively.

4 C-functions

Since $\mathbb{T} = \mathfrak{t}/\check{Q}$ is a compact Abelian group, its Fourier analysis is expressed in terms of objects coming from the lattice dual to $\check{Q}$, namely P. Thus Fourier expansions of $\mathbb{T}$ are of the form

$$\check{x} \;\mapsto\; \sum_{\lambda \in P} a_\lambda e^{2\pi i \langle \lambda, \check{x} \rangle}.$$

where $\check{x} \in \mathfrak{t}$. These are literally functions on $\mathfrak{t}$ but are effectively functions on $\mathbb{T}$ since

$$e^{2\pi i \langle \lambda, \check{x} + \check{Q} \rangle} = e^{2\pi i \langle \lambda, \check{x} \rangle} \qquad \text{for all} \quad \lambda \in P.$$

4.1 Definition of C-functions

Orbit sums, equivalently C-functions, $C_\lambda(t)$ are the finite sums

$$C_\lambda(x) := \sum_{\lambda' \in W\lambda} e^{2\pi i \langle \lambda', x \rangle}, \qquad \lambda \in \mathbb{T}, \quad x \in \check{\mathbb{T}}. \tag{9}$$

We can assume that λ is dominant since $C_\lambda(\check{x})$ depends on the Weyl group orbit $W\lambda$, not on λ.

As is well-known, the characters of finite dimensional representations are linear combinations of C-functions (9). Coefficients of such a linear combination are the multiplicities of the dominant weights in the weight system of the representation. In general, it is a laborious problem to calculate the multiplicities. For simple Lie groups of ranks $n < 9$ a limited help is offered by the tabulation [22].

4.2 Continuous orthogonality of C-functions

For λ, μ dominant, we have the continuous orthogonality relations:

$$\int_{\mathbb{T}} C_\lambda(\check{x})\overline{C_\mu(\check{x})}d\check{x} = |W| \int_{\check{F}} C_\lambda(\check{x})\overline{C_\mu(\check{x})}d\check{x} = \begin{cases} |W\lambda| & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Here overline denotes complex conjugation, and $|W|$, $|W\lambda|$ respectively stand for the size of the Weyl group and for the size of the orbit containing λ.

4.3 Discrete orthogonality of C-functions

Let Γ be a W-invariant finite subgroup of $\mathbb{T}$. Let

$$\check{F} \cap \Gamma = \{\check{u}_1, \dots, \check{u}_N\},$$

which we view as a set of elements of $\mathfrak{t}$.

Let $\lambda, \mu \in P$ and assume that they satisfy the separation condition (8) on Γ. We wish to prove the orthogonality relation

$$\sum_{j=1}^{N} |W\check{u}_j| C_\lambda(\check{u}_j) \overline{C_\mu(\check{u}_j)} = \begin{cases} |W\lambda||\Gamma| & \text{if} \quad W\lambda = W\mu, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

To see this, note that

$$\sum_{\check{a}\in\Gamma} C_\lambda(\check{a}) \overline{C_\mu(\check{a})} = \sum_{j=1}^{N} |W\check{u}_j| C_\lambda(\check{u}_j) \overline{C_\mu(\check{u}_j)} = \sum_{\lambda'\in W\lambda} \sum_{\mu'\in W\mu} \sum_{\check{a}\in\Gamma} e^{2\pi i\langle \lambda'-\mu', \check{a}\rangle}. \tag{12}$$

We have from (7) and (8) that for $\lambda' \in W\lambda$ and $\mu' \in W\mu$,

$$\sum_{\check{a}\in\Gamma} e^{2\pi i\langle \lambda'-\mu', \check{a}\rangle} = \begin{cases} |\Gamma| & \text{if } \lambda' = \mu', \\ 0 & \text{otherwise,} \end{cases}$$

from which (11) follows.

5 S-functions

5.1 Definition of S-functions

$$S_\lambda(\check{x}) := \sum_{\lambda'\in W\lambda} \varepsilon(\lambda') e^{2\pi i\langle \lambda', \check{x}\rangle}, \tag{13}$$

where

$$\varepsilon(\lambda') = (-1)^{l(w)} \qquad \text{if} \quad \lambda' = w\lambda.$$

Here $l(w)$ is the length of w as word in the elementary reflections of W.

Remark 2. Note that $S_\lambda(\check{x})$ is well-defined only if $\mathrm{Stab}_W(\lambda) = \{1\}$, otherwise it is ambiguous. Thus each sum (13) has a unique term $\exp 2\pi i\langle \lambda', \check{x}\rangle$ with λ' strictly dominant ($\lambda' \in P^{++}$) and hence the S_λ are parametrized by P^{++}.

The W-skew-invariance of S_λ also implies that S_λ vanishes on the boundaries of affine chambers of $\mathfrak{t}$.

5.2 Continuous orthogonality of S-functions

Let λ, μ be strictly dominant integral weights. Then

$$\begin{aligned} \int_{\mathbb{T}} S_\lambda(\check{x}) \overline{S_\mu(\check{x})} d\check{x} &= \sum_{\lambda'\in W\lambda} \varepsilon(\lambda') \sum_{\mu'\in W\mu} \varepsilon(\mu') \int_{\mathbb{T}} e^{2\pi i\langle \lambda'-\mu', \check{x}\rangle} d\check{x} \\ &= \sum_{\lambda'\in W\lambda} \varepsilon(\lambda') \sum_{\mu'\in W\mu} \varepsilon(\mu')\, \delta_{\lambda',\mu'} = \begin{cases} |W| & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{14}$$

5.3 Discrete orthogonality of *S*-functions

Let Γ be a finite W-invariant subgroup of $\mathbb{T}$, and let $\lambda, \mu \in P^{++}$. Assume the separation condition (8).

We prove that

$$\sum_{\check{u}_j \in \Gamma \cap \operatorname{int}(\check{F})} S_\lambda(\check{u}_j)\overline{S_\mu(\check{u}_j)} = \begin{cases} |\Gamma| & \text{if} \quad \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

To see this we note first that if $\check{u}_j$ is on the boundary of $\check{F}$, then there is an reflection r such that $r\check{u}_j = \check{u}_j$ and

$$S_\lambda(\check{u}_j) = S_\lambda(r\check{u}_j) = -S_\lambda(\check{u}_j) = 0,$$

so there is no contribution from this $\check{u}_j$.

Now,

$$\begin{aligned} \sum_{j=1}^{n} |W\check{u}_j| S_\lambda(\check{u}_j)\overline{S_\mu(\check{u}_j)} &= \sum_{\check{u}_j \in \Gamma \cap \operatorname{int}(\check{F})} |W\check{u}_j| S_\lambda(\check{u}_j)\overline{S_\mu(\check{u}_j)} \\ &= |W| \sum_{\check{u}_j \in \Gamma \cap \operatorname{int}(\check{F})} S_\lambda(\check{u}_j)\overline{S_\mu(\check{u}_j)}. \end{aligned} \tag{16}$$

At the same time,

$$\sum_{j=1}^{n} |W\check{u}_j| S_\lambda(\check{u}_j)\overline{S_\mu(\check{u}_j)} = \sum_{\check{a} \in \Gamma} S_\lambda(\check{a})\overline{S_\mu(\check{a})} \tag{17}$$

and

$$\begin{aligned} \sum_{a \in \Gamma} S_\lambda(\check{a})\overline{S_\mu(\check{a})} &= \sum_{\check{a} \in \Gamma} \sum_{\lambda' \in W\lambda} \varepsilon(\lambda') e^{2\pi i \langle \lambda', \check{a} \rangle} \sum_{\mu' \in W\mu} \varepsilon(\mu') e^{-2\pi i \langle \mu', \check{a} \rangle} \\ &= \sum_{\lambda' \in W\lambda} \sum_{\mu' \in W\mu} \varepsilon(\lambda')\varepsilon(\mu') \sum_{\check{a} \in \Gamma} e^{2\pi i \langle \lambda' - \mu', \check{a} \rangle} \\ &= \begin{cases} |W||\Gamma| & \text{if} \quad \lambda = \mu, \\ 0 & \text{if} \quad \lambda \neq \mu. \end{cases} \end{aligned} \tag{18}$$

Putting together (16) and (18), we obtain (15).

6 *E*-functions

6.1 Definition of *E*-functions

Let $W^e \subset W$ be the even subgroup of W, and W^e_{aff} the even subgroup of W_{aff},

$$W^e := \{w \in W \mid (-1)^{l(w)} = 1\}, \qquad W^e_{\mathrm{aff}} := W^e \ltimes \check{Q}.$$

Let r be any fixed simple affine reflection, and set

$$\check{F}^e := \check{F} \cup r\check{F}.$$

Then F^e is a fundamental region for $W^e_{\rm aff}$, the affine even Weyl group. Let

$$P^+_e := P^+ \cup rP^+.$$

For $\lambda \in P$,

$$E_\lambda(\breve{x}) := \sum_{\lambda' \in W^e\lambda} e^{2\pi i \langle \lambda', \breve{x}\rangle}. \tag{19}$$

Since $E_\lambda(\breve{x})$ depends on $W^e\lambda$, not on λ, we can suppose $\lambda \in P^+_e$.

$$C_\lambda = \begin{cases} E_\lambda + E_{r\lambda} & \text{if } \lambda \neq r\lambda, \\ E_\lambda & \text{if } \lambda = r\lambda, \end{cases} \tag{20}$$

which also shows that $E_{r\lambda}$ depends only on λ, not on the choice of the reflection r.

Let Γ be as above. Put

$$\Gamma \cap \breve{F}^e = \{\breve{u}^e_1, \breve{u}^e_2, \dots, \breve{u}^e_m\}.$$

6.2 Continuous orthogonality of $\boldsymbol{E}$-functions

Let $\lambda, \mu \in P^+_e$,

$$\begin{aligned}(E_\lambda, E_\mu) &= \int_{\mathbb{T}} E_\lambda(\breve{x})\overline{E_\mu(\breve{x})} d\breve{x} = \sum_{\lambda' \in W^e\lambda} \sum_{\mu' \in W^e\mu} \int_{\mathbb{T}} e^{2\pi i \langle \lambda' - \mu', \breve{x}\rangle} d\breve{x} \\ &= \sum_{\lambda' \in W^e\lambda} \sum_{\mu' \in W^e\mu} \delta_{\lambda',\mu'} = \begin{cases} |W^e\lambda| & \text{if} \quad \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}\end{aligned} \tag{21}$$

6.3 Discrete orthogonality of $\boldsymbol{E}$-functions

Assume $\lambda, \mu \in P^+_e$, and assume the separation condition (8) for the group W^e. Since E_λ, E_μ are W^e-invariant,

$$\begin{aligned}\sum_{j=1}^m |W^e\breve{u}_j| E_\lambda(\breve{u}^e_j)\overline{E_\mu(\breve{u}^e_j)} &= \sum_{a \in \Gamma} E_\lambda(\breve{a})\overline{E_\mu(\breve{a})} \\ &= \sum_{\lambda' \in W^e\lambda} \sum_{\mu' \in W^e\mu} \sum_{\breve{a} \in \Gamma} e^{2\pi i \langle \lambda' - \mu', \breve{a}\rangle} = |\Gamma| \sum_{\lambda' \in W^e\lambda} \sum_{\mu' \in W^e\mu} \delta_{\lambda',\mu'} \\ &= \begin{cases} |\Gamma||W^e\lambda| & \text{if} \quad \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}\end{aligned} \tag{22}$$

7 Central splitting of the functions

All but three compact simply connected simple Lie groups, G_2, F_4, E_8, have the centre Z of order $|Z| = c > 1$. Consider such a group with $c > 1$. Then $Z \subset \mathbb{T}$ and

$$Z = \{\breve{z}_1, \dots, \breve{z}_c\}, \quad \text{where} \quad c = \left|\breve{P}/\breve{Q}\right| \quad \text{(index of connection)},$$

and again we identify the elements of Z with elements of $\mathfrak{t}$.

Let $\chi_1, \dots, \chi_c$, be the irreducible characters of Z, i.e. the homomorphisms

$$\chi : \quad Z \longrightarrow U(1).$$

Each $\lambda \in P$ determines an irreducible character on Z,

$$\chi_\lambda : \quad \check{z} \mapsto e^{2\pi i\langle\lambda,\check{z}\rangle} \tag{23}$$

so,

$$\chi_\lambda = \chi_j \quad \text{for some} \quad 1 \le j \le c, \tag{24}$$

and j is called the congruence class of λ. It depends only on $\lambda \mod Q$. Hence it is constant on the W- and W^e-orbits of λ.

The following arguments can be made for C-, S-, and E-functions. Using just the C-functions, we have

$$C_\lambda(\check{x}+\check{z}) = \sum_{\lambda'\in W\lambda} e^{2\pi i\langle\lambda',\check{x}+\check{z}\rangle} = \chi_j(\check{z})C_\lambda(\check{x}) \qquad \text{for all} \quad \check{z} \in Z, \tag{25}$$

where j is the congruence class of λ. Thus for any class k we have

$$\sum_{\check{z}\in Z} \overline{\chi_k(\check{z})} C_\lambda(x+\check{z}) = \left(\sum_{\check{z}\in Z} \overline{\chi_k(\check{z})}\chi_j(\check{z})\right) C_\lambda(x) = c\,\delta_{kj} C_\lambda(x).$$

From this, if f is a linear combination of C- or S-functions, we can determine its splitting into the sum of congruence class components (central splitting) as follows,

$$f(x) = f_1(x) + \cdots + f_c(x), \tag{26}$$

where

$$f_k(x) = \tfrac{1}{c}\sum_{\check{z}\in Z} \overline{\chi_k(\check{z})} f(x+\check{z}), \qquad 1 \le k \le c. \tag{27}$$

7.1 Example $SU(3)$

The centre Z has 3 elements, $\{\check{z}_0, \check{z}_1, \check{z}_2\}$ (cyclic group of 3 elements). Its characters are

$$\begin{aligned} \chi_0 : &\quad \check{z} \mapsto 1 \quad \text{(trivial character)}, \\ \chi_1 : &\quad \check{z}_1 \mapsto e^{2\pi i/3}, \\ \chi_2 : &\quad \check{z}_2 \mapsto e^{-2\pi i/3}. \end{aligned} \tag{28}$$

According to (26) a function f on $\check{F}$ is decomposed into the sum of three functions,

$$f = f_0 + f_1 + f_2, \tag{29}$$

where

$$\begin{aligned} f_0(\check{x}) &= \tfrac{1}{3}\{f(\check{x}) + f(\check{x}+\check{z}_1) + f(\check{x}+\check{z}_2)\}, \\ f_1(\check{x}) &= \tfrac{1}{3}\{f(\check{x}) + e^{-2\pi i/3} f(\check{x}+\check{z}_1) + e^{2\pi i/3} f(\check{x}+\check{z}_2)\}, \\ f_2(\check{x}) &= \tfrac{1}{3}\{f(\check{x}) + e^{2\pi i/3} f(\check{x}+\check{z}_1) + e^{-2\pi i/3} f(\check{x}+\check{z}_2)\}. \end{aligned} \tag{30}$$

By substitution of (29) into (30), followed by use of (25), one verifies directly the basic property of the central splitting (29) of $f(\check{x})$:

Each of the components f_0, f_1, f_2 of f decomposes into a linear combination of C- or S-functions from one congruence class only, respectively class 0, 1, *and* 2.

For specific application of (30) using a function $f(\check{x})$ given on $\check{F}$, it is useful to make one more step, namely to express each component function as a sum of three terms, each term being the given function evaluated at particular points in $\check{F}$. For $\check{x} \in \check{F}$, the points $\check{x} + \check{z}_1$ and $\check{x} + \check{z}_2$ need not be in $\check{F}$ and need to be brought there by suitable transformations from the affine Weyl group of A_2.

The vertices of $\check{F} = \{0, \check{\omega}_1, \check{\omega}_2\}$ also happen to be the points representing the elements of the centre of $SU(3)$, i.e. $\check{z}_1 = \check{\omega}_1$, $\check{z}_2 = \check{\omega}_2$, and $\check{z}_0 = 0$ standing for the identity element of $SU(3)$. Required transformations are accomplished, for example by the following pairs of affine reflections. Putting $\check{x} = a\check{\omega}_1 + b\check{\omega}_2 = (a, b)$, we have $\check{x} + \check{z}_1 = (a + 1, b)$ and $\check{x} + \check{z}_2 = (a, b + 1)$,

$$R_{\alpha_1+\alpha_2} R_{\alpha_1}(a + 1, b) = (1 - a - b, a) \in \check{F}, \tag{31}$$

$$R_{\alpha_1+\alpha_2} R_{\alpha_2}(a, b + 1) = (b, 1 - a - b) \in \check{F}. \tag{32}$$

Here R_β denotes the affine reflection in the plane orthogonal to β and passing through the point $\frac{1}{2}\beta$.

Finally one has,

$$\begin{aligned}
f_0(a, b) &= \tfrac{1}{3}\{f(a, b) + f(1 - a - b, a) + f(b, 1 - a - b)\},\\
f_1(a, b) &= \tfrac{1}{3}\{f(a, b) + e^{-2\pi i/3} f(1 - a - b, a) + e^{2\pi i/3} f(b, 1 - a - b)\},\\
f_2 a, b) &= \tfrac{1}{3}\{f(a, b) + e^{2\pi i/3} f(1 - a - b, a) + e^{-2\pi i/3} f(b, 1 - a - b)\}.
\end{aligned} \tag{33}$$

7.2 Example $Sp(4)$

The fundamental region is the isosceles triangle $\check{F}$ with vertices

$$\{0, \tfrac{1}{2}\check{\omega}_1, \check{\omega}_2\}$$

the right angle being at $\frac{1}{2}\check{\omega}_1$. The centre Z has 2 elements. Their parameters inside $\check{F}$ are the points $\{0, \check{\omega}_2\}$. According to (26), a function $f(\check{x})$ on the triangle $\check{F}$ is decomposed into the sum of two functions, $f(\check{x}) = f_0(\check{x}) + f_1(\check{x})$, where

$$\begin{aligned}
f_0(\check{x}) &= \tfrac{1}{2}\{f(\check{x}) + f(\check{x} + \check{\omega}_2)\},\\
f_1(\check{x}) &= \tfrac{1}{2}\{f(\check{x}) - f(\check{x} + \check{\omega}_2)\}.
\end{aligned} \tag{34}$$

When $\check{x} \neq 0$, we have $\check{x} + \check{\omega}_2$ outside of $\check{F}$. An appropriate affine reflection brings $\check{x} + \check{\omega}_2$ into $\check{F}$.

A point $\check{x} = x_1\check{\omega}_1 + x_2\check{\omega}_2 = (x_1, x_2)$ is in $\check{F}$, if $0 \le 2x_1 + x_2 \le 1$. Then $\check{x} + \check{\omega}_2$ is reflected into $\check{F}$ as follows:

$$R_{2\check{\omega}_2}(x_1, 1 + x_2) = (x_1, 1 - 2x_1 - x_2).$$

Therefore the component functions can be written with their arguments given explicitly in the $\check{\omega}$-basis:

$$\begin{aligned}
f_0(x_1, x_2) &= \tfrac{1}{2}\{f(x_1, x_2) + f(x_1, 1 - 2x_1 - x_2)\},\\
f_1(x_1, x_2) &= \tfrac{1}{2}\{f(x_1, x_2) - f(x_1, 1 - 2x_1 - x_2)\}.
\end{aligned} \tag{35}$$

The functions (35) have definite symmetry properties with respect to the line connecting the points $\frac{1}{2}\check{\omega}_1$ and $\frac{1}{2}\check{\omega}_2$ on the boundary of $\check{F}$. In case one has $f(\check{x})$ expanded in terms of C-functions (S-functions), $f_0(x_1, x_2)$ is symmetric (antisymmetric), while $f_1(x_1, x_2)$ is antisymmetric (symmetric) with respect to that line.

Equivalently, in case $f(\check{x})$ is expanded in terms of C-functions, then $f_0(x_1, x_2)$ is expanded in terms of C-functions of congruence class 0, while $f_1(x_1, x_2)$ is expanded in terms of C-functions of congruence class 1. Similarly, in case the expansion of $f(\check{x})$ is in terms of S-functions, the previous conclusions about congruence classes pertinent to f_0 and f_1 expansions are interchanged.

Recall that the functions C_λ and also S_λ, where $\lambda = a\omega_1 + b\omega_2 \in P$, are said to be of congruence class 0 or 1, according to the value of $a \mod 2$.

7.3 Example $SU(2) \times SU(2)$

The centre of $SU(2) \times SU(2)$ is the product $Z = Z_2 \times Z_2$ of two cyclic groups of order 2. Its order is $c = 4$. In ω-basis the parameters of the centre elements are the vertices of the square $\check{F}$:

$$z_1 = (0, 0), \quad z_2 = (1, 0), \quad z_3 = (0, 1), \quad z_4 = (1, 1).$$

Suppose a function $f(x, y)$ ('the data') is given within the square $\check{F}$, and that our goal is to consider expansions of $f(x, y)$ into series of C- and S-functions of $SU(2) \times SU(2)$.

A function $f(x, y)$ on $\check{F}$ has $0 \le x, y \le 1$. We decompose $f(x, y)$ into the sum of four component functions which we label by two integers $\mod 2$,

$$f(x, y) = f_{00}(x, y) + f_{10}(x, y) + f_{01}(x, y) + f_{11}(x, y), \tag{36}$$

where

$$\begin{aligned} f_{00}(x, y) &= \tfrac{1}{4}\{f(x, y) + f(x + 1, y) + f(x, y + 1) + f(x + 1, y + 1)\},\\ f_{10}(x, y) &= \tfrac{1}{4}\{f(x, y) - f(x + 1, y) + f(x, y + 1) - f(x + 1, y + 1)\},\\ f_{01}(x, y) &= \tfrac{1}{4}\{f(x, y) + f(x + 1, y) - f(x, y + 1) - f(x + 1, y + 1)\},\\ f_{11}(x, y) &= \tfrac{1}{4}\{f(x, y) - f(x + 1, y) - f(x, y + 1) + f(x + 1, y + 1)\}. \end{aligned} \tag{37}$$

The coefficients in (37) are taken from the rows of the $Z_2 \times Z_2$ character table:

$$\begin{array}{rrrr} 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1 \end{array}$$

Since the displaced points in (37) are outside of $\check{F}$, we bring them back by the appropriate affine reflection. Finally we get

$$\begin{aligned} f_{00}(x, y) &= \tfrac{1}{4}\{f(x, y) + f(1 - x, y) + f(x, 1 - y) + f(1 - x, 1 - y)\},\\ f_{10}(x, y) &= \tfrac{1}{4}\{f(x, y) - f(1 - x, y) + f(x, 1 - y) - f(1 - x, 1 - y)\},\\ f_{01}(x, y) &= \tfrac{1}{4}\{f(x, y) + f(1 - x, y) - f(x, 1 - y) - f(1 - x, 1 - y)\},\\ f_{11}(x, y) &= \tfrac{1}{4}\{f(x, y) - f(1 - x, y) - f(x, 1 - y) + f(1 - x, 1 - y)\}. \end{aligned} \tag{38}$$

In (38) each component function $f_j(x, y)$ is given by the values of $f(x, y)$ at four points in $\check{F}$. Moreover, its C- or S-transforms involve the C- or S-functions of one of the four congruence classes only.

8 Concluding remarks

1. Specific examples of orthogonalities of C-, S-, or E-functions can be found elsewhere in the literature, for example in [16, 9, 10, 11, 12, 13, 14].

2. A practically important distinction between the the C-, S-, and E-transforms comes from the behaviour of the functions of each family at the $(n-1)$-dimensional boundary $\partial\check{F}$ of $\check{F}$. The behaviour is shared by all members of the family and it does not depend on the type of the underlying Lie group. It is easy to from the definitions that C-functions have normal derivative at $\partial\check{F}$ equal to zero; the S-functions are antisymmetric on both sides of $\partial\check{F}$, passing trough 0 at $\partial\check{F}$; while no such property is implied in the case of E-functions.

3. Besides the C-, S-, and E-transforms, we have been considering so far, there are other curious possibilities of their combinations when G is semisimple but not simple. On some occasions such hybrid transforms may prove to be useful.

Suppose $G = G_1 \times G_2$, where G_1 and G_2 are simple, their fundamental regions being $\check{F}_1$ and $\check{F}_2$ respectively. A function $f(\check{x}_1, \check{x}_2)$ on $\check{F}_1 \times \check{F}_2$, with $\check{x}_1 \in \check{F}_1$ and $\check{x}_2 \in \check{F}_2$, can be expanded using, say, C-function on $\check{F}_1$ and any of the three types on $\check{F}_2$. Thus one may have CS-, or CE-, or SE-transforms in such a case, rather than CC-, SS-, or EE-transforms we implied to have in the main body of the paper.

4. It is interesting and useful to know when the C-, S-, E-functions are real valued. Traditional Lie theory has a ready answer to such a question:

When the opposite involution w_{opp} (the longest element in W in terms of the generating reflections $r_{\alpha_1}, \ldots, r_{\alpha_m}$) is the -1 map, every W-orbit contains with every weight μ also $-\mu$. Then the C-functions are real. When $w_{\mathrm{opp}} = -1$ and it is even, the same goes for W^e: every W^e-orbit contains with each weight μ also $-\mu$. Then S- and E-functions are real.

More interesting, perhaps, is when S-functions are purely imaginary. This happens when $w_{\mathrm{opp}} = -1$ and w_{opp} is odd.

$w_{\mathrm{opp}} = -1$ and w_{opp} is even for B_{2k}, C_{2k}, D_{2k}, E_8, F_4, G_2. Hence they have real valued C-, S-, E-functions.

$w_{\mathrm{opp}} = -1$ and w_{opp} is odd for A_1, B_{2k+1}, C_{2k+1}, E_7. They have real valued C-functions, and purely imaginary valued S-functions.

5. A simple straightforward motivation for central splitting of a function $f(\check{x})$ on $\check{F}$ of G is in reducing a larger problem, decomposition of $f(\check{x})$ into its C-, or S-, or E-series, into c smaller problems for each component function. That alone should lead to some computational economy. More ambitiously, one can view the central splitting, described in Section 7, as the first step of a general multidimensional fast transform. This will be described in [23]. In the case $G = SU(2)$, this transform coincides with the fast cosine and sine transforms.

6. For a given group G, the C- and S-functions have a number of other interesting properties, most of which are found in [13, 14]. In addition to that, let us point out also the arithmetic properties described in [4] for the C-functions but apparently never studied for S-functions. As an illustration, let us mention that for any G there are only finitely many points in $\check{F}$, at which the C-functions $C_\lambda(\check{x})$ of G take integer values for all $\lambda \in P$.

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