Symmetry, Integrability and Geometry: Methods and Applications Vol. 2 (2006), Paper 075, 15 pages

Prolongation Loop Algebras for a Solitonic System of Equations

Maria A. AGROTIS

Department of Mathematics and Statistics, University of Cyprus, Nicosia 1678, Cyprus
E-mail: *agrotis@ucy.ac.cy*

Received September 13, 2006, in final form November 01, 2006; Published online November 08, 2006
Original article is available at http://www.emis.de/journals/SIGMA/2006/Paper075/

Abstract. We consider an integrable system of reduced Maxwell–Bloch equations that describes the evolution of an electromagnetic field in a two-level medium that is inhomogeneously broadened. We prove that the relevant Bäcklund transformation preserves the reality of the n-soliton potentials and establish their pole structure with respect to the broadening parameter. The natural phase space of the model is embedded in an infinite dimensional loop algebra. The dynamical equations of the model are associated to an infinite family of higher order Hamiltonian systems that are in involution. We present the Hamiltonian functions and the Poisson brackets between the extended potentials.

Key words: loop algebras; Bäcklund transformation; soliton solutions

2000 Mathematics Subject Classification: 37K10; 37N20; 35A30; 35Q60; 78A60

1 Introduction

Integrable systems are closely related to the inverse scattering method that serves as a means of integrating the initial value problem, and also to infinite dimensional Lie algebras, else known as Kac–Moody Lie algebras or loop algebras. Studying the underlying loop algebra reveals encoded properties of the equations that are inherited from the integrable character of the model. The Adler–Kostant–Symes (AKS) theorem gives the Lie algebraic formulation of the dynamics of the system.

In [1], several nonlinear partial differential equations were realized as the compatibility condition of two linear matrix systems of the form,

$$\vec{v}_x = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix} \vec{v}, \qquad \vec{v}_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \vec{v}, \tag{1}$$

where q and r are the potentials satisfying a certain nonlinear evolution equation and A, B, C are functions of x and t. The Zakharov–Shabat dressing transformation [2] was then employed to produce the soliton solutions of the system. The matrices appearing on the right-hand-side of (1) are 2×2, traceless matrices that can be viewed as elements of the finite dimensional Lie algebra $\mathfrak{sl}(2)$. $\mathfrak{sl}(2)$ may be prolonged to an infinite dimensional Kac–Moody Lie algebra with the aid of the spectral parameter λ, and lead to an infinite number of systems in involution. This is made exact in the context of the AKS theorem. Briefly, the theorem states that if one starts with a set of commuting functions on a Lie algebra, then the corresponding Hamiltonian systems are of course trivial. However, if we project those functions to appropriate subalgebras then the resulting Hamiltonian systems need not be trivial and continue to be in involution.

In [3, 4, 5, 6, 7, 8, 9], several integrable equations have been studied in the context of the inverse scattering technique and the AKS theorem. For example, the Toda system and the

Korteweg–de Vries equation, serving as representatives of the ordinary and partial differential evolution equations respectively, have been associated to Lax pair equations with one degree of freedom represented by the spectral parameter. The phase space of the relevant model was extended to include infinite dimensional loop algebras. The loop algebra was then decomposed into a vector space direct sum of subalgebras, and with the aid of the trace functional it was identified with its dual. In that way, the systems obtain the Kostant–Kirillov symplectic structure, and an application of the AKS theorem revealed an infinite number of integrable Hamiltonian systems in involution.

In this paper we consider a reduction of the Maxwell–Bloch equations that models the optical pulse propagation of an electric field through a two-level medium in the presence of an external constant electric field. The optical resonance line of the medium is inhomogeneously broadened. Following the terminology used by McCall and Hahn [10] and Lamb [11], we refer to inhomogeneous broadening as the phenomenon that occurs when the atoms of the medium possess different resonant frequencies due to microscopic interactions between them. In such a case, the induced electric dipole polarization is represented as a continuum, and the resulting optical resonance line is inhomogeneously broadened. In solids, such a broadening could be caused by a distribution of static crystalline electric and magnetic fields and in gases by the distribution of Doppler frequencies.

Since the late sixties and seventies with the papers [10, 11, 12], among others, the Maxwell–Bloch equations have undergone several treatments. Recently, various reductions of the equations have been studied both analytically and numerically. Lax pair operators, Darboux transformations and soliton solutions were constructed and analyzed [13, 14, 15, 16] and interesting applications in crystal acoustics have emerged [17, 18].

Our scope in this paper is the study of the integrable structure of a reduced Maxwell–Bloch system and the connections that arise with Kac–Moody Lie algebras. In particular, we prove that the Bäcklund transformation preserves the reality of the n-soliton potentials $\forall\, n \in \mathbb{N}$, and establish their pole structure with respect to the broadening parameter. The solitonic phase space of the model is embedded in an infinite dimensional loop algebra and an application of the AKS theorem allows us to view the system as a member of an infinite family of systems in involution. We present the higher order Hamiltonian functions and flows, as well as the Poisson brackets between the extended potentials.

2 Phase space

The optical equations we shall consider are the ones presented in [19]. They model the propagation of an electric field in a two-level quantized medium, where the optical resonance line of the medium has been inhomogeneously broadened. The classical wave equation of Maxwell (2) is used for the evolution of a unidirectional electric field and is coupled with the quantum mechanical Bloch equations (3)–(5), that describe the behavior of the induced polarization field,

$$\frac{\partial e}{\partial \zeta} + \frac{\partial e}{\partial \tau} = \langle \omega S_\omega \rangle_g \,, \tag{2}$$

$$\frac{\partial R_\omega}{\partial \tau} = (\beta - \gamma e) S_\omega, \tag{3}$$

$$\frac{\partial S_\omega}{\partial \tau} = -(\beta - \gamma e) R_\omega + \frac{1}{2}\omega U_\omega, \tag{4}$$

$$\frac{\partial U_\omega}{\partial \tau} = -2\omega S_\omega. \tag{5}$$

$\langle f\rangle_g = \int_\infty^\infty f(\omega)g(\omega)d\omega$, and denotes the weighted average of the function $f(\omega)$ with respect to the distribution function,

$$g(\omega) = \frac{\sigma}{\pi((\omega-\omega_0)^2+\sigma^2)}.$$

For a physical interpretation of the model see [19]. In this paper we shall study the system (2)–(5) from a Lie algebraic point of view.

The system is completely integrable and admits a Lax pair representation. We define the differential operators L and A whose commutativity, $[L, A] := LA - AL = 0$, is equivalent to equations (2)–(5)

$$A = -\partial_\tau + Q^{(0)}, \qquad L = \partial_\zeta + Q^{(1)},$$

where,

$$\begin{aligned} Q^{(0)} &= \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0\mathcal{E}, \\ Q^{(1)} &= \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0\mathcal{E} + \int_{-\infty}^{\infty} \frac{1}{(\omega^2-\lambda^2)}[\lambda(h_1\mathcal{H} + f_1\mathcal{F}) + e_1\mathcal{E}]d\omega, \end{aligned}$$

and

$$\begin{array}{ll} h_0 = \frac{1}{2}, & h_1 = -\frac{1}{2}\gamma\omega g(\omega)R_\omega, \\ f_0 = 0, & f_1 = -\frac{1}{2}\gamma\omega g(\omega)S_\omega, \\ e_0 = -\frac{1}{2}(\beta - \gamma e), & e_1 = \frac{1}{4}\gamma\omega^2 g(\omega)U_\omega. \end{array} \tag{6}$$

$\mathcal{H}$, $\mathcal{F}$ and $\mathcal{E}$ form a basis of the semi-simple Lie algebra $\mathfrak{su}(2)$, and are given as follows,

$$\mathcal{H} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{E} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We call h_j, f_j, e_j for $j = 0, 1$, the potentials and $Q^{(0)}$, $Q^{(1)}$ loop elements because they can be considered as elements of an infinite dimensional loop algebra that we will define in Section 3. We note that the potentials depend on the solutions e, R_ω, S_ω, U_ω of the inhomogeneously broadened reduced Maxwell–Bloch (ib-rMB) equations (2)–(5). The commutation of the differential operators L and A gives rise to the following Lax pair equation

$$\frac{\partial Q^{(0)}}{\partial\zeta} + \frac{\partial Q^{(1)}}{\partial\tau} = [Q^{(0)}, Q^{(1)}], \tag{7}$$

which is equivalent to the ib-rMB system.

The Lax pair can be used to construct a Bäcklund transformation (BT) that iteratively produces the soliton solutions of equations (2)–(5).

We consider the spectral problem,

$$\partial_\tau\Psi = Q^{(0)}\Psi, \qquad \partial_\zeta\Psi = -Q^{(1)}\Psi,$$

and aim to find a new eigenfunction Ψ and the corresponding new loop element $Q^{(1)}$ that satisfy the spectral problem. The loop elements are functions of h_j, f_j, e_j, for $j = 0, 1$ and will, in turn, give rise to the new solutions of the ib-rMB system via expressions (6). This transformation theory leads to an analogue of superposition formulas that allows one to construct multi-soliton solutions starting from single solitons by algebraic means [21, 22, 23, 24, 15]. We briefly describe the procedure and quote the relevant theorem from reference [19].

One begins with a constant solution to equations (2)–(5), which in turn determines potentials (6) and the corresponding loop element, call it Q_0. We then find a simultaneous, fundamental solution Ψ_1 to the Lax pair system $L\Psi = 0, A\Psi = 0$ and define $\vec{\Phi}_1 = \begin{pmatrix} \phi_1^1 \\ \phi_2^1 \end{pmatrix} := \Psi_1(\lambda = \nu_1)\vec{c_1}$, where $\vec{c_1} = \begin{pmatrix} c_1^1 \\ ic_2^1 \end{pmatrix}$ is a constant vector with $c_1^1, c_2^1 \in \mathbb{R}$, and the matrix N_1 as,

$$N_1 = \begin{pmatrix} \phi_1^1 & -\overline{\phi_2^1} \\ \phi_2^1 & \overline{\phi_1^1} \end{pmatrix}.$$

The BT matrix function is constructed as:

$$G(\nu_1, \vec{c_1}; \lambda) = N_1 \begin{pmatrix} \lambda - \nu_1 & 0 \\ 0 & \lambda - \overline{\nu_1} \end{pmatrix} N_1^{-1}.$$

Applying G to Ψ_1 yields a new fundamental solution: $\Psi_2(\nu_1, \vec{c_1}; \lambda) = G(\nu_1, \vec{c_1}; \lambda)\Psi_1(\nu_1, \vec{c_1}; \lambda)$, and the procedure is iterated. The formula for the loop element after n iterations of the BT, call it Q_n, in terms of the previous one Q_{n-1} and the matrix N_n is the context of the next theorem.

Theorem 1.

$$\begin{aligned} Q_n(\lambda) = {} & \lambda h_0^{n-1}\mathcal{H} + m_n h_0^{n-1}[\mathcal{H}, N_n\mathcal{H}N_n^{-1}] + e_0^{n-1}\mathcal{E} + \int_{-\infty}^{\infty} \frac{1}{(\omega^2 - \lambda^2)} \frac{1}{(\omega^2 + m_n^2)} \\ & \times \big\{\lambda \big[\omega^2(h_1^{n-1}\mathcal{H} + f_1^{n-1}\mathcal{F}) - (m_n)^2(N_n\mathcal{H}N_n^{-1})(h_1^{n-1}\mathcal{H} + f_1^{n-1}\mathcal{F})(N_n\mathcal{H}N_n^{-1}) \\ & + m_n e_1^{n-1}[\mathcal{E}, N_n\mathcal{H}N_n^{-1}]\big] + m_n\omega^2 \left(h_1^{n-1}[\mathcal{H}, N_n\mathcal{H}N_n^{-1}] + f_1^{n-1}[\mathcal{F}, N_n\mathcal{H}N_n^{-1}]\right) \\ & + \omega^2 e_1^{n-1}\mathcal{E} - (m_n)^2 e_1^{n-1}(N_n\mathcal{H}N_n^{-1})\mathcal{E}(N_n\mathcal{H}N_n^{-1})\big\}\, d\omega. \end{aligned} \tag{8}$$

We have taken the specific value of the spectral parameter to be purely imaginary, $\nu_n = im_n \in i\mathbb{R}$. The general form of the n-soliton loop is given by,

$$Q_n(\lambda) = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0^n\mathcal{E} + \int_{-\infty}^{\infty} \frac{1}{(\omega^2 - \lambda^2)}[\lambda(h_1^n\mathcal{H} + f_1^n\mathcal{F}) + e_1^n\mathcal{E}]d\omega, \tag{9}$$

where the upper index of e_0^n, h_1^n, f_1^n, e_1^n and the lower index of Q_n, Ψ_n, $\vec{\Phi}_n$, N_n indicate the level of the Bäcklund transform. We note that h_0 and f_0 are constant functions of space and time and are invariants of the level of the BT. To ensure the reality of the potentials and consequently of the solutions e, R_ω, S_ω, U_ω, we compare (8) and (9) and impose the following conditions:

$$a)\ \ [\mathcal{H}, N_n\mathcal{H}N_n^{-1}], [\mathcal{F}, N_n\mathcal{H}N_n^{-1}], (N_n\mathcal{H}N_n^{-1})\mathcal{E}(N_n\mathcal{H}N_n^{-1}) \in \operatorname{span}\{\mathcal{E}\}, \tag{10}$$

$$b)\ \ (N_n\mathcal{H}N_n^{-1})\mathcal{H}(N_n\mathcal{H}N_n^{-1}), (N_n\mathcal{H}N_n^{-1})\mathcal{F}(N_n\mathcal{H}N_n^{-1}), [\mathcal{E}, N_n\mathcal{H}N_n^{-1}] \in \operatorname{span}\{\mathcal{H}, \mathcal{F}\}. \tag{11}$$

By definition

$$N_n = \operatorname{Re}(\phi_1^n)\mathcal{I} + \operatorname{Im}(\phi_1^n)\mathcal{H} - \operatorname{Re}(\phi_2^n)\mathcal{F} + \operatorname{Im}(\phi_2^n)\mathcal{E},$$

where $\mathcal{I}$ is the 2×2 identity matrix. One can find that conditions (10)–(11) are satisfied if and only if

$$\operatorname{Im}(\phi_1^n)\operatorname{Im}(\phi_2^n) + \operatorname{Re}(\phi_1^n)\operatorname{Re}(\phi_2^n) = 0. \tag{12}$$

We can construct ϕ_1^n, ϕ_2^n such that one of the following two cases holds: a) $\phi_1^n \in \mathbb{R}$, $\phi_2^n \in i\mathbb{R}$, or b) $\phi_1^n \in i\mathbb{R}$, $\phi_2^n \in \mathbb{R}$. This is the context of the following proposition.

Proposition 1. *At any given level of the Bäcklund transformation the reality of the n-soliton potentials can be secured by an appropriate choice of the transformation data.*

Proof. A constant set of solutions to equations (2)–(5) is given by $e = \frac{\beta}{\gamma}$, $S_\omega = 0$, $U_\omega = 0$ and $R_\omega = R^{\text{init}}$ a nonzero constant. The corresponding potentials become $h_0 = \frac{1}{2}$, $h_1 = -\frac{1}{2}\gamma\omega g(\omega)R^{\text{init}}$, $f_0 = f_1 = e_0 = e_1 = 0$. Following the procedure of the Bäcklund transform we find, $\phi_1^1 = \text{Re}(c_1^1)e^{x_1} + i\text{Im}(c_1^1)e^{x_1}$, $\phi_2^1 = \text{Re}(c_2^1)e^{-x_1} + i\text{Im}(c_2^1)e^{-x_1}$, where $x_1 = 2((v_1 + m_1h_0)\zeta - m_1h_0\tau) \in \mathbb{R}$, and v_1 is an expression independent of ω, λ, ζ and τ. We choose $c_1^1 \in \mathbb{R}$ and $c_2^1 \in i\mathbb{R}$ so that $\phi_1^1 \in \mathbb{R}$, $\phi_2^1 \in i\mathbb{R}$ and condition (12) holds for $n = 1$. The proof that the condition is satisfied at any given level of the BT lies on the following:

$N_{2n-1} \in \text{span}\{\mathcal{I}, \mathcal{E}\}$, and $N_{2n} \in \text{span}\{\mathcal{H}, \mathcal{F}\}$. This yields $G_n(\lambda) \in \text{span}\{\lambda I, \mathcal{H}, \mathcal{F}\}$, $\forall n \in \mathbb{N}$. Using the definition $\Psi_n = G_n(\lambda = im_n)\Psi_{n-1}(\lambda = im_n)$ we deduce that $\Psi_{2n-1} \in \left\{\begin{pmatrix} \mathbb{R} & i\mathbb{R} \\ i\mathbb{R} & \mathbb{R} \end{pmatrix}\right\}$, and $\Psi_{2n} \in \left\{\begin{pmatrix} i\mathbb{R} & \mathbb{R} \\ \mathbb{R} & i\mathbb{R} \end{pmatrix}\right\}$. Therefore, if we choose $\vec{c}_n \in \left\{\begin{pmatrix} \mathbb{R} \\ i\mathbb{R} \end{pmatrix}\right\}$, and use $\vec{\Phi}_n = \vec{\Psi}_n\vec{c}_n$, we obtain that $\vec{\Phi}_{2n-1} \in \left\{\begin{pmatrix} \mathbb{R} \\ i\mathbb{R} \end{pmatrix}\right\}$, and $\vec{\Phi}_{2n} \in \left\{\begin{pmatrix} i\mathbb{R} \\ \mathbb{R} \end{pmatrix}\right\}$ as desired. ■

We aim to give an integral-free representation of $Q_n(\lambda)$ so that its λ-structure becomes apparent. We begin with $Q_1(\lambda)$ and then generalize the construction for $Q_n(\lambda)$. The general form of $Q_1(\lambda)$ is the following,

$$Q_1(\lambda) = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0^1\mathcal{E} + \int_{-\infty}^{\infty} \frac{1}{(\omega^2 - \lambda^2)}\left[\lambda(h_1^1\mathcal{H} + f_1^1\mathcal{F}) + e_1^1\mathcal{E}\right]d\omega.$$

The one soliton potentials are given as,

$$e_0^1 = 2m_1h_0\,\text{sech}(x_1), \qquad e_1^1 = \frac{2m_1\omega^2h_1}{(\omega^2 + m_1^2)}\,\text{sech}(x_1),$$
$$f_1^1 = \frac{2m_1^2h_1}{(\omega^2 + m_1^2)}\,\text{sech}(x_1)\tanh(x_1), \qquad h_1^1 = \frac{h_1}{(\omega^2 + m_1^2)}\left[\omega^2 + m_1^2 - 2m_1^2\,\text{sech}^2(x_1)\right],$$

where $x_1 = 2((v_1 + m_1h_0)\zeta - m_1h_0\tau) + \ln(c_1^1/c_2^1)$, $h_1 = -\frac{1}{2}\gamma\omega g(\omega)R^{\text{init}}$ and $v_1 = -\frac{1}{2}\frac{\gamma R^{init}m_1\omega_0}{((m_1+\sigma)^2+\omega_0^2)}$. We write the last three in the following more convenient form,

$$e_1^1 = \frac{\beta_1\omega^3}{(\omega^2 + m_1^2)((\omega - \omega_0)^2 + \sigma^2)}, \tag{13}$$
$$f_1^1 = \frac{\beta_2\omega}{(\omega^2 + m_1^2)((\omega - \omega_0)^2 + \sigma^2)}, \tag{14}$$
$$h_1^1 = \frac{\beta_3\omega^3 + \beta_4\omega}{(\omega^2 + m_1^2)((\omega - \omega_0)^2 + \sigma^2)}, \tag{15}$$

where β_1, β_2, β_3, β_4 are functions of ζ and τ, but do not depend on ω or λ. We will use contour integration and Cauchy's integral formula to compute Q_1. We define the complex-valued function,

$$h(z) = \frac{z}{(z^2 - \lambda^2)(z^2 + m_1^2)((z - \omega_0)^2 + \sigma^2)}\left\{\lambda[(\beta_3z^2 + \beta_4)\mathcal{H} + \beta_2\mathcal{F}] + \beta_1z^2\mathcal{E}\right\},$$

which has three poles in the upper half complex plane at $z_1 = \lambda$, $z_2 = im_1$, and $z_3 = \omega_0 + i\sigma$. We integrate around a simple contour that consists of a semicircle of radius R in the upper-half complex plane, call it C_R, and the segment on the real axis from $-R$ to R. We choose R big enough to include all the poles of $h(z)$ that appear in the upper-half complex plane. It is not hard to see that $\lim\limits_{R\mapsto\infty}\int_{C_R} h(z)dz = 0$, and thus using the Cauchy integral formula we obtain,

$$Q_1 = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0^1\mathcal{E} + 2\pi i \sum_{k=1}^{3} \mathrm{Res}_{z=z_k} h(z),$$

which can be written as,

$$\begin{aligned} Q_1(\lambda) &= \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0^1\mathcal{E} + \frac{1}{(\lambda^2 + m_1^2)}[\lambda(\delta_1\mathcal{H} + \delta_2\mathcal{F}) + \delta_3\mathcal{E}] \\ &\quad + \frac{\lambda(\tilde{\delta}_1\mathcal{H} + \tilde{\delta}_2\mathcal{F}) + \tilde{\delta}_3\mathcal{E} + \lambda^3(\tilde{\tilde{\delta}}_1\mathcal{H} + \tilde{\tilde{\delta}}_2\mathcal{F}) + \lambda^2\tilde{\tilde{\delta}}_3\mathcal{E}}{((\omega_0^2 - \sigma^2 - \lambda^2)^2 + 4\omega_0^2\sigma^2)}, \end{aligned} \tag{16}$$

where δ_j, $\tilde{\delta}_j$, $\tilde{\tilde{\delta}}_j$, $j = 1, 2, 3$ are independent of λ and ω. Let $S_1 = \{\delta_1, \delta_2, \delta_3\}$, $S_2 = \{\tilde{\delta}_1\tilde{\delta}_2, \tilde{\delta}_3\}$ and $S_3 = \{\tilde{\tilde{\delta}}_1\tilde{\tilde{\delta}}_2, \tilde{\tilde{\delta}}_3\}$. The elements of each set S_1, S_2 or S_3 are functions of the one-soliton potentials and they obey the ib-rMB equations.

We will use induction to obtain the general form of the n-soliton potentials h_1^n, f_1^n, e_1^n.

Proposition 2. *The pole structure with respect to the broadening parameter ω of the general n-soliton potentials of the ib-rMB system is the following:*

$$h_1^n = \frac{\omega\sum\limits_{k=0}^{n}\gamma_k\omega^{2k}}{((\omega-\omega_0)^2+\sigma^2)\prod\limits_{k=1}^{n}(\omega^2+m_k^2)}, \qquad f_1^n = \frac{\omega\sum\limits_{k=0}^{n}\delta_k\omega^{2k}}{((\omega-\omega_0)^2+\sigma^2)\prod\limits_{k=1}^{n}(\omega^2+m_k^2)},$$

$$e_1^n = \frac{\omega\sum\limits_{k=0}^{n}\epsilon_k\omega^{2k}}{((\omega-\omega_0)^2+\sigma^2)\prod\limits_{k=1}^{n}(\omega^2+m_k^2)}.$$

We note that $\gamma_k, \delta_k, \epsilon_k$, $k = 1, \ldots, n$, are analytic functions of ζ and τ, and do not depend on ω.

Proof. The proposition holds for $n = 1$ as can be seen by (13)–(15). We assume that the proposition holds for $n - 1$ and show that it holds for n. Using the induction hypothesis in Theorem 1 and the fact that $N_k\mathcal{H}N_k^{-1} \in \mathrm{span}\{\mathcal{H}, \mathcal{F}\}$ for any $k \in \mathbb{N}$ we get,

$$\begin{aligned} Q_n(\lambda) &= \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0^n\mathcal{E} + \int_{-\infty}^{\infty} \frac{\omega}{(\omega^2-\lambda^2)((\omega-\omega_0)^2+\sigma^2)\prod\limits_{k=1}^{n}(\omega^2+m_k^2)} \\ &\quad \times \left\{\lambda\left[\sum_{k=0}^{n}\gamma_k\omega^{2k}\mathcal{H} + \sum_{k=0}^{n}\delta_k\omega^{2k}\mathcal{F}\right] + \sum_{k=0}^{n}\epsilon_k\omega^{2k}\mathcal{E}\right\} d\omega. \end{aligned}$$

By definition

$$Q_n(\lambda) = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0^n\mathcal{E} + \int_{-\infty}^{\infty} \frac{1}{(\omega^2-\lambda^2)}[\lambda(h_1^n\mathcal{H} + f_1^n\mathcal{F}) + e_1^n\mathcal{E}]d\omega. \tag{17}$$

Therefore, by equating the last two expressions of $Q_n(\lambda)$ we prove the proposition. We note that some of the coefficients γ_k, δ_k, ϵ_k may be equal to zero. For example, for $n = 1$, $\delta_1 = 0$ and $\epsilon_0 = 0$. ■

The form of the n-soliton potentials can be used to identify the λ-structure of a general n-soliton loop element $Q_n(\lambda)$.

Proposition 3.

$$Q_n(\lambda) = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0^n\mathcal{E} + \frac{\lambda(\tilde{\delta}_1\mathcal{H} + \tilde{\delta}_2\mathcal{F}) + \tilde{\delta}_3\mathcal{E} + \lambda^3(\tilde{\tilde{\delta}}_1\mathcal{H} + \tilde{\tilde{\delta}}_2\mathcal{F}) + \lambda^2\tilde{\tilde{\delta}}_3\mathcal{E}}{((\omega_0^2 - \sigma^2 - \lambda^2)^2 + 4\omega_0^2\sigma^2)}$$
$$+ \sum_{k=1}^{n} \frac{1}{(\lambda^2 + m_k^2)}[\lambda(\delta_1^k\mathcal{H} + \delta_2^k\mathcal{F}) + \delta_3^k\mathcal{E}]. \tag{18}$$

The proof follows the same idea that was used to derive the integral-free form (16) of $Q_1(\lambda)$ and is omitted.

Let $S_k = \{\delta_1^k, \delta_2^k, \delta_3^k\}$, $k = 1, \dots, n$, $S_{n+1} = \{\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3\}$ and $S_{n+2} = \{\tilde{\tilde{\delta}}_1, \tilde{\tilde{\delta}}_2, \tilde{\tilde{\delta}}_3\}$. The elements of each set S_k, $k = 1, \dots, n+2$ are functions of the m-soliton potentials for $m = 1, \dots, k$ and they satisfy the ib-rMB equations.

3 Embedding

We call $\alpha_k = im_k$, $k = 1, \dots, n$, $\alpha_{n+1} = \omega_0 - i\sigma$ and $\alpha_{n+2} = \omega_0 + i\sigma$. Then $Q_n(\lambda)$ of (18) can be rewritten as

$$Q_n(\lambda) = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0\mathcal{E} + \sum_{k=1}^{n+2} \frac{X_k^-}{\lambda - \alpha_k} + \frac{X_k^+}{\lambda + \alpha_k},$$

where X_k^-, X_k^+ are linear combinations of the basis elements of the Lie algebra $\mathfrak{su}(2)$, $\{\mathcal{H}, \mathcal{F}, \mathcal{E}\}$.

We define the following infinite dimensional Lie algebra Q^{ext} that extends loops of the form (18),

$$Q^{\text{ext}} = \left\{ X : X = \sum_{j=0}^{\infty}\sum_{k=1}^{n+2} \frac{1}{(\lambda^2 - \alpha_k^2)^j}[\lambda(h_j^k\mathcal{H} + f_j^k\mathcal{F}) + e_j^k\mathcal{E}] \right\}. \tag{19}$$

We note that the coefficients h_j^k, f_j^k, e_j^k of (19) are not the same as those of (17) where the upper index denotes the level of the BT. In (19) X is an element that extends the natural solitonic phase space of the ib-rMB equations and the upper index indicates the relevant pole α_k, whereas the lower index indicates the order of the pole α_k^2.

We embed this Lie algebra into a larger one,

$$Q = \left\{ X : X = \sum_{j=1}^{\infty} \left(X_{j-1}\lambda^{j-1} + \sum_{k=1}^{2(n+2)} \frac{X_j^k}{(\lambda - \alpha_k)^j} \right) \right\}, \tag{20}$$

where we have set $\alpha_{k+n+2} = -\alpha_k$, for $k = 1, \dots, n+2$. We note that for Q to be a Lie algebra a finiteness condition needs to imposed. Namely, $X_j = 0$ and $X_j^k = 0$, $\forall\, k = 1, \dots, 2(n+2)$ and $j \geq j_0$ for some $j_0 \in \mathbb{N}$.

4 Application of the Adler–Kostant–Symes theorem

Following the ideas in the theorem of Adler, Kostant and Symes [5, 6, 7, 8], we decompose the infinite dimensional loop algebra Q into a direct sum of two subalgebras a and b, and define an ad-invariant, non-degenerate inner product on Q. The perpendicular complements $a^\perp$, $b^\perp$ with respect to the inner product serve as another direct sum decomposition of Q. Using the Riesz representation theorem and the non-degenerate inner product one may define an isomorphism between $a^{\perp}$ and b^*, the dual of the Lie subalgebra b. The canonical Lie–Poisson bracket that exists on b^* is then represented on $a^{\perp}$. If necessary, one may translate $a^{\perp}$ by any element $\beta \in b^{\perp}$ that satisfies $(\beta, [x, y]) = 0 \ \forall\, x, y \in a$, so that the translated space $\beta + a^{\perp}$ includes the natural phase space of the relevant system [20].

We write Q as the vector space direct sum of the Lie subalgebras

$$a = \left\{ X \in Q: \ X = \sum_{j=1}^{\infty} \sum_{k=1}^{2(n+2)} \frac{X_j^k}{(\lambda - \alpha_k)^j} \right\}, \qquad b = \left\{ X \in Q: \ X = \sum_{j=0}^{\infty} X_j \lambda^j \right\}.$$

A non-degenerate inner product is defined on Q using the trace map. Namely,

$$Q \times Q \longrightarrow \mathbb{C}, \qquad (X, Y) \mapsto \mathrm{Tr}(XY)_0,$$

where $(XY)_0$ denotes the matrix coefficient of λ^0 in the product XY. We note that the inner product is ad-invariant. That is, $(X, \mathrm{ad}_Y Z) + (\mathrm{ad}_Y X, Z) = 0$, where $\mathrm{ad}_X Y = [X, Y]$. The perpendicular complements of the subalgebras a and b with respect to this inner product take the form,

$$a^{\perp} = \left\{ X \in Q: \ X = X_0 + \sum_{j=1}^{\infty} \sum_{k=1}^{2(n+2)} \frac{X_j^k}{(\lambda - \alpha_k)^j} \right\}, \qquad b^{\perp} = \left\{ X \in Q: \ X = \sum_{j=1}^{\infty} X_j \lambda^j \right\}.$$

The natural phase space of the ib-rMB equations, as can be seen in (18), contains elements that belong in $a^{\perp}$ as well as terms of order λ^1. Therefore we translate the space $a^{\perp}$ by $\beta = \lambda X_1 \in b^{\perp}$. We note that $(\beta, [X, Y]) = 0 \ \forall\, X, Y \in a$, which is a necessary condition for the AKS theorem. We consider the set,

$$a^{\perp} + \beta = \left\{ X \in Q: \ X = X_0 + \lambda X_1 + \sum_{j=1}^{\infty} \sum_{k=1}^{2(n+2)} \frac{X_j^k}{(\lambda - \alpha_k)^j} \right\}, \tag{21}$$

which includes the phase space of the ib-rMB equations, and define a Lie–Poisson bracket on $a^{\perp} + \beta$. If Φ and Ψ are functions on $a^{\perp} + \beta$ we first compute their gradients $\nabla\Phi$, $\nabla\Psi$ in the full Lie algebra Q and then define

$$\{\Phi, \Psi\}(X) = -\left(X, \left[\prod_b \nabla\Phi(X), \prod_b \nabla\Psi(X) \right] \right), \qquad \forall\, X \in a^{\perp} + \beta. \tag{22}$$

We note that $\prod_b$ denotes projection on the Lie subalgebra b. $\nabla H(X)$ denotes the gradient of a function H on Q and is defined as follows,

$$(Y, \nabla H(X)) = \lim_{\epsilon \longrightarrow 0} \frac{H(X + \epsilon Y) - H(X)}{\epsilon}. \tag{23}$$

We quote the AKS theorem and show how it can be applied in the case of the ib-rMB equations.

Theorem 2. *If H, F are invariant functions on $Q^* \cong Q$, and we denote by H^{proj}, F^{proj} their projection to the subspace $a^{\perp} + \beta$ then $\{H^{\mathrm{proj}}, F^{\mathrm{proj}}\} = 0$, and the Hamiltonian system associated with the invariant function H is given as,*

$$\frac{d}{dt}(\beta + \alpha) = -\mathrm{ad}^*_{\prod_a \nabla H(\beta+\alpha)}(\beta + \alpha), \qquad \forall\, \beta + \alpha \in \beta + a^{\perp}. \tag{24}$$

Remark 1. ad^* is negative the dual of the ad map.

Remark 2. The definition of H being an invariant function is $\mathrm{ad}^*_{\nabla H(x)}x = 0$, $\forall\, x \in Q$.

Remark 3. The existence of an invariant inner product allows one to identify the ad-map with ad^*, because on one hand by definition $(\mathrm{ad}_x y, z) = -(y, \mathrm{ad}^*_x z)$, and on the other a simple calculation using the definition of the inner product shows that $(\mathrm{ad}_x y, z) = -(y, \mathrm{ad}_x z)$.

If a set of invariant functions is given, the theorem guarantees their commutativity in the canonical Lie–Poisson bracket and gives the form of the Hamiltonian system associated with the invariant function. To apply the theorem we construct a set of invariant functions $\{\Phi_k\}_{k\in\mathbb{N}}$ via the following operator,

$$M_k(X) = \prod_{i=1}^{2(n+2)} (\lambda - \alpha_i)^k X.$$

The invariant functions are defined as $\Phi_k(X) = \frac{1}{2}(M_k(X), X)$. We compute the gradient of the $\Phi_k(X)$. By definition,

$$(Y, \nabla\Phi_k(X)) = \lim_{\epsilon\to 0} \frac{\Phi_k(X + \epsilon Y) - \Phi_k(X)}{\epsilon} = \left(\prod_{i=1}^{2(n+2)} (\lambda - \alpha_i)^k X, Y \right) = (Y, M_k(X)).$$

Thus $\nabla\Phi_k(X) = M_k(X)$. According to Remarks 2 and 3, to actually demonstrate the invariance of Φ_k we must show that $[\nabla\Phi_k(X), X] = 0$. We have that $[\nabla\Phi_k(X), X] = \prod_{i=1}^{2(n+2)} (\lambda - \alpha_i)^k [X, X] = 0$. Therefore the functions Φ_k, $k \in \mathbb{N}$ are invariant and can be used in the context of the AKS theorem, which reads as follows,

$$\frac{dX}{dt_k} = -\left[\prod_a \nabla\Phi_k(X), X \right] = \left[\prod_b \nabla\Phi_k(X), X \right],$$

since $0 = [\nabla\Phi_k(X), X] = [\prod_a \nabla\Phi_k(X), X] + [\prod_b \nabla\Phi_k(X), X]$. We used the subscript k for the time variable t_k to distinguish between the different dynamical evolutions of the systems associated with the Hamiltonian functions Φ_k. For each $k \in \mathbb{N}$ the following systems are in involution:

$$\frac{dX}{dt_k} = \left[\prod_b M_k(X), X \right]. \tag{25}$$

For $k = 0$ and a truncated X of the form (19) where $h^i_j, f^i_j, e^i_j = 0$ for $j \geq 2$ and $i = 1, \ldots, n+2$,

$$X = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0\mathcal{E} + \sum_{i=1}^{n+2} \frac{1}{(\lambda^2 - \alpha_i^2)} [\lambda(h^i_1\mathcal{H} + f^i_1\mathcal{F}) + e^i_1\mathcal{E}], \tag{26}$$

the Hamiltonian system (25) becomes

$$\frac{dQ^{(1)}}{dt_0} = [Q^{(0)}, Q^{(1)}], \tag{27}$$

which is the zero curvature representation of the Lax pair equation (7) for the ib-rMB equations. We have thus identified the ib-rMB system as a member of the infinite family of systems (25), that commute with respect to the canonical Lie–Poisson bracket (22).

5 Extended flow

We consider a general element of the Lie algebra Q of the form,

$$X = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0\mathcal{E} + \sum_{j=1}^{\infty}\sum_{i=1}^{n+2}\frac{1}{(\lambda^2-\alpha_i^2)^j}[\lambda(h_j^i\mathcal{H} + f_j^i\mathcal{F}) + e_j^i\mathcal{E}], \tag{28}$$

or equivalently,

$$X = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0\mathcal{E} + \sum_{j=1}^{\infty}\frac{\sum\limits_{m=0}^{j(n+1)}\lambda^{2m+1}(h_j^{2m+1}\mathcal{H} + f_j^{2m+1}\mathcal{F}) + \lambda^{2m}e_j^{2m}\mathcal{E}}{\prod\limits_{i=1}^{n+2}(\lambda^2-\alpha_i^2)^j}. \tag{29}$$

We remark that h_j^i, f_j^i, e_j^i in (28) are not the same as the ones in (29). The latter ones are linear combinations of the former. However, to avoid introducing yet another symbol we use h_j^{2m+1}, f_j^{2m+1}, e_j^{2m} in (29), where j indicates the order of the pole at $\lambda^2 = \alpha_i^2$ and $2m+1$ or $2m$ the power of λ multiplying $\mathcal{H}$, $\mathcal{F}$ or $\mathcal{E}$ respectively in the numerator.

To examine the extended flow associated with the ib-rMB equations we set $k = 0$ in the involutive systems (25). The relevant system takes the form,

$$\frac{dX}{dt_0} = \left[\prod_b \nabla\Phi_0(X), X\right],$$

where $\nabla\Phi_0(X) = M_0(X) = X$. We write X as follows:

$$X = \sum_{j=0}^{\infty}\frac{P_j}{\prod\limits_{i=1}^{n+2}(\lambda^2-\alpha_i^2)^j},$$

where P_j is a polynomial in λ given as

$$P_j(\lambda) = \sum_{m=0}^{j(n+1)}\lambda^{2m+1}(h_j^{2m+1}\mathcal{H} + f_j^{2m+1}\mathcal{F}) + \lambda^{2m}e_j^{2m}\mathcal{E}, \qquad j = 1, 2, \ldots,$$

and $P_0 = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0\mathcal{E}$. The degree of $P_j(\lambda)$ is given by, $\deg(P_j(\lambda)) = 2j(n+1)+1$. Projecting $\nabla\Phi_0(X)$ to the subalgebra b is equivalent to keeping the polynomial part of X, which is P_0. Thus the Hamiltonian flow for $k = 0$ takes the form,

$$\frac{dX}{dt_0} = \sum_{j=1}^{\infty}\left[P_0, \frac{P_j}{\prod\limits_{i=1}^{n+2}(\lambda^2-\alpha_i^2)^j}\right],$$

which unravels to

$$\frac{dX}{dt_0} = \sum_{j=1}^{\infty}\left(\sum_{m=0}^{j(n+1)}\lambda^{2m+1}(dh_{0,j}^m\mathcal{H} + df_{0,j}^m\mathcal{F}) + \lambda^{2m+2}de_{0,j}^m\mathcal{E}\right)\Big/\left(\prod_{i=1}^{n+2}(\lambda^2-\alpha_i^2)^j\right). \tag{30}$$

The coefficients of the matrices $\mathcal{H}$, $\mathcal{F}$, and $\mathcal{E}$ appearing in (30) are defined as follows,

$$dh_{0,j}^m = 2\big(f_0e_j^{2m} - e_0f_j^{2m+1}\big), \qquad df_{0,j}^m = 2\big(e_0h_j^{2m+1} - h_0e_j^{2m}\big),$$

$$de_{0,j}^{m} = 2\big(h_0 f_j^{2m+1} - f_0 h_j^{2m+1}\big).$$

We observe that the expressions multiplying $\mathcal{H}$ and $\mathcal{F}$ in (30) have no polynomial part since the degree of the numerator that equals $2j(n+1)+1$ is strictly smaller that the degree of the denominator that equals $2j(n+2)$, for $j \geq 1$. However, the expression multiplying $\mathcal{E}$ carries a polynomial term. In particular the evolution equation (30) can be written as

$$\begin{aligned}\frac{dX}{dt_0} = {} & \sum_{j=1}^{\infty}\left(\sum_{m=0}^{j(n+1)} \lambda^{2m+1}(dh_{0,j}^{m}\mathcal{H} + df_{0,j}^{m}\mathcal{F}) + \sum_{m=1}^{j(n+1)} \lambda^{2m} de_{0,j}^{m-1}\mathcal{E}\right.\\ & \left. + \sum_{m=(j-1)(n+1)}^{j(n+1)} C_{m-(j-1)(n+1)}\lambda^{2m} de_{0,j}^{j(n+1)}\mathcal{E}\right) \Big/ \left(\prod_{i=1}^{n+2}(\lambda^2-\alpha_i^2)^j\right)\\ & + \sum_{j=2}^{\infty}\lambda^{2(j-1)(n+1)} de_{0,j}^{j(n+1)}\mathcal{E} \Big/ \left(\prod_{i=1}^{n+2}(\lambda^2-\alpha_i^2)^{j-1}\right) + de_{0,1}^{n+1}\mathcal{E},\end{aligned}$$

where $C_{n+1-m} = (-1)^m \sum\limits_{i_{l_1}\neq i_{l_2}}^{n+2} \alpha_{i_1}^2 \cdots \alpha_{i_{m+1}}^2$, $i_{l_p} \in \{1,\dots,n+2\}$. On the other hand by the definition of X we have that,

$$\begin{aligned}\frac{dX}{dt_0} = {} & \lambda\Big(\frac{dh_0}{dt_0}\mathcal{H} + \frac{df_0}{dt_0}\mathcal{F}\Big) + \frac{de_0}{dt_0}\mathcal{E}\\ & + \sum_{j=1}^{\infty}\left(\lambda^{2m+1}\Big(\frac{dh_j^{2m+1}}{dt_0}\mathcal{H} + \frac{df_j^{2m+1}}{dt_0}\mathcal{F}\Big) + \lambda^{2m}\frac{de_j^{2m}}{dt_0}\mathcal{E}\right) \Big/ \left(\prod_{i=1}^{n+2}(\lambda^2-\alpha_i^2)^j\right).\end{aligned}$$

By equating the different powers of λ we disclose the system induced by the Hamiltonian function Φ_0. As previously noted, for a fixed $j \in \mathbb{N}$, the elements of the sets $S_m = \{h_j^{2m+1}, f_j^{2m+1}, e_j^{2m}\}$ satisfy the same system of equations for any $m = 0, 1, \dots, j(n+1)$. Therefore it suffices to consider only one such set. Without loss of generality we choose $S_{j(n+1)}$. The evolution equations take the form:

$$\begin{aligned}& \frac{dh_0}{dt_0} = \frac{df_0}{dt_0} = 0, \qquad \frac{de_0}{dt_0} = 2\big(h_0 f_1^{2n+3} - f_0 h_1^{2n+3}\big),\\ & \frac{dh_j^{2j(n+1)+1}}{dt_0} = 2\big(f_0 e_j^{2j(n+1)} - e_0 f_j^{2j(n+1)+1}\big),\\ & \frac{df_j^{2j(n+1)+1}}{dt_0} = 2\big(e_0 h_j^{2j(n+1)+1} - h_0 e_j^{2j(n+1)}\big),\\ & \frac{de_j^{2j(n+1)}}{dt_0} = 2\big(h_0 f_j^{2j(n+1)-1} - f_0 h_j^{2j(n+1)-1}\big) + 2\left(\sum_{i=1}^{n+2} a_i^2\right)\big(h_0 f_j^{2j(n+1)+1} - f_0 h_j^{2j(n+1)+1}\big)\\ & \qquad\qquad + 2\big(h_0 f_{j+1}^{2(j+1)(n+1)+1} - f_0 h_{j+1}^{2(j+1)(n+1)+1}\big).\end{aligned}$$

To reveal the extended flow for the ib-rMB equations we set j=1:

$$\begin{aligned}& \frac{dh_0}{dt_0} = \frac{df_0}{dt_0} = 0, \qquad \frac{de_0}{dt_0} = 2\big(h_0 f_1^{2n+3} - f_0 h_1^{2n+3}\big),\\ & \frac{dh_1^{2n+3}}{dt_0} = 2\big(f_0 e_1^{2n+2} - e_0 f_1^{2n+3}\big), \qquad \frac{df_1^{2n+3}}{dt_0} = 2\big(e_0 h_1^{2n+3} - h_0 e_1^{2n+2}\big), \qquad (31)\end{aligned}$$

$$\frac{de_1^{2n+2}}{dt_0} = 2\big(h_0 f_1^{2n+1} - f_0 h_1^{2n+1}\big) + 2\left(\sum_{i=1}^{n+2} a_i^2\right)\big(h_0 f_1^{2n+3} - f_0 h_1^{2n+3}\big)$$
$$+ 2\big(h_0 f_2^{4n+5} - f_0 h_2^{4n+5}\big).$$

We note that the coupling of the above system to the evolution equations satisfied by the higher order potentials that correspond to $j \geq 2$, is captured in the dynamical equation for e_1^{2n+2}. If $h_2^{4n+5} = f_2^{4n+5} = 0$, then the system reduces to the dynamical equations that the n-soliton potentials of the ib-rMB equations satisfy.

6 Hamiltonian functions and Poisson brackets

In this section we aim to write the extended flow of the ib-rMB equations given by system (31) in Section 5, in the canonical Poisson form,

$$\frac{\partial e_0}{\partial t} = \{e_0, \Phi_0\}, \qquad \frac{\partial h_1^{2n+3}}{\partial t} = \{h_1^{2n+3}, \Phi_0\},$$
$$\frac{\partial f_1^{2n+3}}{\partial t} = \{f_1^{2n+3}, \Phi_0\}, \qquad \frac{\partial e_1^{2n+2}}{\partial t} = \{e_1^{2n+2}, \Phi_0\}. \tag{32}$$

The Hamiltonian functions for the systems (25) described in the context of the AKS theorem in Section 4 are defined as

$$\Phi_k(X) = \frac{1}{2}(M_k(X), X) = \frac{1}{2}\mathrm{Tr}\left(\prod_{i=1}^{2(n+2)} (\lambda - \alpha_i)^k X^2\right)_0 .$$

We let $k = 0$ and consider a general X of the form,

$$X = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0\mathcal{E}$$
$$+ \sum_{j=1}^{\infty}\left(\frac{\sum\limits_{m=0}^{j(n+1)} \lambda^{2m+1}\big(h_j^{2m+1}\mathcal{H} + f_j^{2m+1}\mathcal{F}\big) + \lambda^{2m} e_j^{2m}\mathcal{E}}{\prod\limits_{i=1}^{n+2}(\lambda^2 - \alpha_i^2)^j}\right).$$

The Hamiltonian function is found to be:

$$\Phi_0(X) = \frac{1}{2}\mathrm{Tr}\big(X^2\big)_0 = e_0^2 + 2\big(h_0 h_1^{2n+3} + f_0 f_1^{2n+3}\big). \tag{33}$$

We define the following functionals, relevant to the potentials that appear in the Hamiltonian:

$$h_0(X) = h_0 \text{ coefficient of } \lambda\mathcal{H}, \quad h_1^{2n+3}(X) = h_1^{2n+3} \text{ coefficient of } \frac{\lambda^{2n+3}}{\prod\limits_{i=1}^{n+2}(\lambda^2 - \alpha_i^2)}\mathcal{H},$$

$$f_0(X) = f_0 \text{ coefficient of } \lambda\mathcal{F}, \quad f_1^{2n+3}(X) = f_1^{2n+3} \text{ coefficient of } \frac{\lambda^{2n+3}}{\prod\limits_{i=1}^{n+2}(\lambda^2 - \alpha_i^2)}\mathcal{F},$$

$$e_0(X) = e_0 \text{ coefficient of } \mathcal{E}, \qquad e_1^{2n+2}(X) = e_1^{2n+2} \text{ coefficient of } \frac{\lambda^{2n+2}}{\prod\limits_{i=1}^{n+2}(\lambda^2 - \alpha_i^2)}\mathcal{E}.$$

Using definition (23) we compute the gradients of these functionals:

$$
\begin{aligned}
&\nabla h_0(X) = -\frac{1}{2}\frac{\lambda^{2n+3}}{\prod\limits_{i=1}^{n+2}(\lambda^2-\alpha_i^2)}\mathcal{H}, && \nabla h_1^{2n+3}(X) = -\frac{1}{2}\lambda\mathcal{H}, \\
&\nabla f_0(X) = -\frac{1}{2}\frac{\lambda^{2n+3}}{\prod\limits_{i=1}^{n+2}(\lambda^2-\alpha_i^2)}\mathcal{F}, && \nabla f_1^{2n+3}(X) = -\frac{1}{2}\lambda\mathcal{F}, \\
&\nabla e_0(X) = -\frac{1}{2}\mathcal{E}, && \nabla e_1^{2n+2}(X) = -\frac{1}{2}\lambda^2\mathcal{E}.
\end{aligned}
\tag{34}
$$

For example, to obtain $\nabla h_0(X)$ we consider the equation

$$
(Y, \nabla h_0(X)) = \lim_{\epsilon\to 0}\frac{h_0(X+\epsilon Y) - h_0(X)}{\epsilon},
$$

which implies that $\operatorname{Tr}(Y\nabla h_0(X))_0 = h_0(Y)$. Therefore $\nabla h_0(X)$ is the element of Q such that the constant term (with respect to λ) of the product $Y\nabla h_0(X)$ has trace that equals precisely $h_0(Y)$. We note that the matrices $\mathcal{HF}$, $\mathcal{FE}$, $\mathcal{EH}$ are traceless whereas $\mathcal{H}^2 = \mathcal{F}^2 = \mathcal{E}^2 = -\mathcal{I}$. Having that in mind, we find that $\nabla h_0(X) = -\frac{1}{2}\frac{\lambda^{2n+3}}{\prod_{i=1}^{n+2}(\lambda^2-\alpha_i^2)}\mathcal{H}$.

The Poisson brackets between the potentials appearing in the Hamiltonian can be computed using definition (22). For instance,

$$
\begin{aligned}
\{e_0, h_1^{2n+3}\}(X) &= -\left(X, \left[\prod_b \nabla e_0(X), \prod_b \nabla h_1^{2n+3}(X)\right]\right) \\
&= -\left(X, \frac{1}{2}\lambda\mathcal{F}\right) = -\operatorname{Tr}\left(\frac{1}{2}\lambda X\mathcal{F}\right)_0 = f_1^{2n+3}.
\end{aligned}
$$

In a similar fashion we obtain the rest of the Poisson brackets,

$$
\begin{aligned}
&\{e_0, h_1^{2n+3}\} = f_1^{2n+3}, \qquad \{f_1^{2n+3}, e_0\} = h_1^{2n+3}, \qquad \{h_1^{2n+3}, f_1^{2n+3}\} = e_1^{2n+2}, \\
&\{e_1^{2n+2}, h_1^{2n+3}\} = f_1^{2n+1} + \left(\sum_{i=1}^{n+2} a_i^2\right) f_1^{2n+3} + f_2^{4n+5}, \\
&\{f_1^{2n+3}, e_1^{2n+2}\} = h_1^{2n+1} + \left(\sum_{i=1}^{n+2} a_i^2\right) h_1^{2n+3} + h_2^{4n+5}.
\end{aligned}
\tag{35}
$$

Using the Poisson brackets (35), we find that the extended flow for the ib-rMB equations given in (31) can be expressed as the canonical flow (32) associated with the Hamiltonian function Φ_0.

Higher order functionals can also be defined using a diagonal formation that gradually sweeps all the potentials. In particular, for a general X of the form (29) we define for $N = 0, 1, 2, \ldots$, the following higher order functionals (in bold to distinguish between $\boldsymbol{h}_0$, $\boldsymbol{f}_0$, $\boldsymbol{e}_0$ and h_0, f_0, e_0):

$$
\begin{aligned}
&\boldsymbol{h}_N(X) = \sum_{j=1}^{N+1} h_j^{2j(n+2)-2N-1}, \qquad \boldsymbol{f}_N(X) = \sum_{j=1}^{N+1} f_j^{2j(n+2)-2N-1}, \\
&\boldsymbol{e}_N(X) = \sum_{j=1}^{N+1} e_j^{2j(n+2)-2N-2}.
\end{aligned}
$$

We note that if the upper index of the potentials that appear in the sums is less than zero then the potentials are set to zero. The set $S = \{e_0, \boldsymbol{h}_N, \boldsymbol{f}_N, \boldsymbol{e}_N : N = 0, 1, 2, \ldots\}$ includes all the dynamical quantities that enter the AKS flows (25). The gradients of these higher order functionals can be computed using definition (23). For instance,

$$\nabla \boldsymbol{h}_1(X) = -\frac{1}{2}\lambda^3\mathcal{H} - \left(\sum_{i=1}^{n+2}\alpha_i^2\right)\nabla \boldsymbol{h}_0, \qquad \nabla \boldsymbol{f}_1(X) = -\frac{1}{2}\lambda^3\mathcal{F} - \left(\sum_{i=1}^{n+2}\alpha_i^2\right)\nabla \boldsymbol{f}_0,$$

$$\nabla \boldsymbol{e}_1(X) = -\frac{1}{2}\lambda^4\mathcal{E} - \left(\sum_{i=1}^{n+2}\alpha_i^2\right)\nabla \boldsymbol{e}_0,$$

where $\nabla \boldsymbol{h}_0 = \nabla h_1^{2n+3}$, $\nabla \boldsymbol{f}_0 = \nabla f_1^{2n+3}$, $\nabla \boldsymbol{e}_0 = \nabla e_1^{2n+2}$, and are given in (34).

Working in a similar manner as in the example for $\{e_0, h_1^{2n+3}\}$, one can find the Poisson brackets between the higher order functionals, i.e. $\{e_0, \boldsymbol{h}_1\}(X) = \boldsymbol{f}_1(X)$.

7 Summary

In this paper we have considered an integrable system of reduced Maxwell–Bloch equations, that is inhomogeneously broadened. We show that the relevant Bäcklund transformation preserves the reality of the n-soliton potentials $\forall\, n \in \mathbb{N}$, and establish their pole structure with respect to the broadening parameter. We obtain a representation of the relevant phase space in the spectral parameter λ, which is then embedded in a prolonged loop algebra. The equations satisfied by the n-soliton potentials are associated to an infinite family of higher order Hamiltonian involutive systems. We present the Hamiltonian functions of the higher order flows and the Poisson brackets between the extended potentials.

Acknowledgements

The author would like to thank P. Shipman for useful discussions and the Cyprus Research Promotion Foundation for support through the grant CRPF0504/03.

[1] Ablowitz M.J., Kaup D.J., Newell A.C., Segur H., The inverse scattering transform-Fourier analysis for nonlinear problems, *Stud. Appl. Math.*, 1974, V.53, 249–315.

[2] Zakharov V.E., Shabat A.B., Integration of the nonlinear equations of mathematical physics by the method of the inverse scattering problem, *Funct. Anal. Appl.*, 1979, V.13, N 3, 13–22.

[3] Flaschka H., The Toda lattice I. Existence of integrals, *Phys. Rev. B*, 1974, V.9, 1924–1925.

[4] Flaschka H., On the Toda lattice II. Inverse-scattering solution, *Progr. Theoret. Phys.*, 1974, V.51, 703–716.

[5] Adler M., On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg-de Vries type equations, *Invent. Math.*, 1979, V.50, 219–48.

[6] Adler M., van Moerbeke P., Completely integrable systems, Euclidean Lie algebras, and curves, *Adv. Math.*, 1980, V.38, 267–317.

[7] Kostant B., The solution to a generalized Toda lattice and representation theory, *Adv. Math.*, 1979, V.34, 195–338.

[8] Symes W.W., Systems of Toda type, inverse spectral problems, and representation theory, *Invent. Math.*, 1980, V.59, 13–51.

[9] Flaschka H., Newell A.C., Ratiu T., Kac–Moody Lie algebras and soliton equations, *Phys. D*, 1983, V.9, 300–323.

[10] McCall S.L., Hahn E.L., Self-induced transparency, *Phys. Rev.*, 1969, V.183, 457–486.

[11] Lamb G.L., Analytical description of utrashort optical pulse propagation in a resonant medium, *Rev. Modern Phys.*, 1971, V.43, 99–124.

[12] Eilbeck J.C., Gibbon J.D., Caudrey P.J., Bullough R.K., Solitons in nonlinear optics I. A more accurate description of the 2π pulse in self-induced transparency, *J. Phys. A: Math. Gen.*, 1973, V.6, 1337–1347.

[13] Caputo J., Maimistov A.I., Unidirectional propagation of an ultra-short electromagnetic pulse in a resonant medium with high frequency stark shift, *Phys. Lett. A*, 2002, V.296, 34–42, nlin.SI/0107040.

[14] Elyutin S.O., Dynamics of an extremely short pulse in a stark medium, *JETP*, 2004, V.101, 11–21.

[15] Qing-Chun J.I., Darboux transformation and solitons for reduced Maxwell–Bloch equations, *Commun. Theor. Phys.*, 2005, V.43, 983–986.

[16] Sazonov S.V., Ustinov N.V., Pulsed transparency of anisotropic media with stark level splitting, *Quantum Electronics*, 2005, V.35, 701–704.

[17] Sazonov S.V., Ustinov N.V., Nonlinear acoustic transparency phenomena in strained paramagnetic crystals, *JETP*, 2006, V.102, 741–752.

[18] Bakhar N.V., Ustinov N.V., Dynamics of two-component electromagnetic and acoustic extremely short pulses, *Proceedings of SPIE*, 2006, 61810Q, 10 pages, nlin.SI/0512068.

[19] Glasgow S.A., Agrotis M.A., Ercolani N.M., An integrable reduction of inhomogeneously broadended optical equations, *Phys. D*, 2005, V.212, 82–99.

[20] Agrotis M.A., Hamiltonian flows for a reduced Maxwell–Bloch system with permanent dipole, *Phys. D*, 2003, V.183, 141–158.

[21] Bäcklund A.V., Zur Theorie der Flachentransformationen, *Math. Ann.*, 1881, V.19, 387–422.

[22] Lamb G.L., Elements in soliton theory, Wiley-Interscience Pub., 1980.

[23] Lonngren K., Alwyn S. (Editors), Solitons in action, Academic Press, 1978.

[24] Newell A.C., Solitons in mathematics and physics, *CBMS-NSF Regional Conference Series*, Vol. 48, SIAM Press, 1985.