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Coupled Modified KP Hierarchy and Its Dispersionless Limit*

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Abstract. We define the coupled modified KP hierarchy and its dispersionless limit. This integrable hierarchy is a generalization of the "half" of the Toda lattice hierarchy as well as an extension of the mKP hierarchy. The solutions are parametrized by a fibered flag manifold. The dispersionless counterpart interpolates several versions of dispersionless mKP hierarchy.

Key words: cmKP hierarchy; fibered flag manifold; dcmKP hierarchy

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Dedicated to the memory of Vadim Kuznetsov.

1 Introduction

Since 1980's many integrable systems with infinitely many degrees of freedom have been studied by means of infinite dimensional homogeneous spaces. Well-known examples are: the KP hierarchy and the Sato–Grassmann manifold ([10, 12, 11, 3] etc.), the Toda lattice hierarchy and "$GL(\infty)$" ([23, 13, 15, 16] etc.), the modified KP (mKP) hierarchy and the flag manifold ([5, 6, 2, 7, 17] etc.). In this paper we add one more example to this series: the *coupled modified KP* (cmKP) hierarchy and the fibered flag manifold.

The modified KP hierarchy is defined in [2] and [17] as a system consisting of two sets of equations: the Lax equations for continuous variables $t = (t_1, t_2, \dots)$ and a set of difference equations for the discrete variable s. The cmKP hierarchy has the same description but the normalization of the operators is different. By this difference the moduli space of solutions of the mKP hierarchy (= the flag manifold) is enlarged.

Actually a special case of the cmKP hierarchy has been known since [23], in which the Toda lattice hierarchy was introduced. A half of the Toda lattice hierarchy without dependence on half of time variables is a cmKP hierarchy (See Appendix B). Therefore the cmKP hierarchy can be considered as the mKP hierarchy coupled to the Toda field.

In this special case the solution space is parametrized by the basic affine space $GL(\infty)/N$ where N is the subgroup of infinite upper triangular matrices with unity on the diagonal. In other words it is a product of the full flag manifold and $(\mathbb{C}^\times)^{\mathbb{Z}}$. The solution space of our cmKP hierarchy is (partial flag manifold) $\times \prod_{s\in S'} (\mathbb{C}^{m_s} \setminus \{0\})$ in general. (See Corollary 1.)

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The dispersionless (quasi-classical) limit of the cmKP hierarchy is taken in the same way as the dispersionless KP and Toda hierarchies. (See [21] and references therein.) We call the resulting system the *dispersionless cmKP hierarchy* (the *dcmKP hierarchy* in short). In fact the dcmKP hierarchy was first introduced by one of the authors [18] (in a slightly different form) as a system which interpolates two versions of the dispersionless mKP hierarchies, one by [8] and [1] and the other by [17]. Hence the name "dispersionless coupled mKP hierarchy" has another interpretation: It connects variants of the dispersionless mKP hierarchy.

This paper is organized as follows: The part on the cmKP hierarchy (Section 2) follows standard recipe. We start from the Lax representation similar to that of the mKP hierarchy [17] and introduce the dressing operator and the wave function as solutions of linear problems. We show existence of the τ functions and construct them explicitly, using the DJKM free fermions.

The dispersionless counterpart is discussed in Section 3, following the strategy of [21]. The basic objects are a formal power series $\mathcal{L}$ and a polynomial $\mathcal{P}$. The hierarchy is defined by the Lax equations. Then we introduce the dressing function, the Orlov–Schulman function, the S function and the τ function. We also discuss the relation with the dispersionless mKP hierarchy and the characterization of the τ function.

In Section 3.8 the cmKP hierarchy and the dcmKP hierarchy, so far discussed independently in principle, are related via the WKB analysis.

Equivalent formulations of the cmKP hierarchy are discussed in the appendices.

2 Coupled modified KP hierarchy

2.1 Definition of the cmKP hierarchy

In this section we define the *cmKP hierarchy* with discrete parameters $\{n_s\}_{s\in S} \subset \mathbb{Z}$, where S is a set of consecutive integers (e.g., $S = \mathbb{Z}$, $S = \{0, 1, \dots, n\}$ etc.) as in [17]. The dispersionless limit can be taken only when $S = \mathbb{Z}$, $n_s = Ns$. Set $S' = S \setminus \{\text{maximum element of } S\}$ if there exists a maximum element of S and $S' = S$ otherwise.

The independent variables of the cmKP hierarchy are the discrete variable $s \in S$ and the set of continuous variables $t = (t_1, t_2, \dots)$. The dependent variables are encapsulated in the following operators with respect to x:

$$L(s; x, t) = \partial + u_1(s, x, t) + u_2(s, x, t)\partial^{-1} + \cdots \\ = \sum_{n=0}^{\infty} u_n(s, x, t)\partial^{1-n}, \tag{2.1}$$

$$P(s; x, t) = p_0(s, x, t)\partial^{m_s} + \cdots + p_{m_s-1}(s, x, t)\partial \\ = \sum_{n=0}^{m_s-1} p_n(s, x, t)\partial^{m_s-n}. \tag{2.2}$$

where $\partial = \partial_x$, $u_0 = 1$, $p_0 \neq 0$, $m_s := n_{s+1} - n_s$. $P(s; x, t)$ is defined only for $s \in S'$. We often write $L(s)$, $P(s)$ instead of $L(s; x, t)$, $P(s; x, t)$. The notation $(L(s), P(s))_{s\in S}$ stands for a pair of sequences $((L(s))_{s\in S}, (P(s))_{s\in S'})$.

The cmKP hierarchy is the following system of differential and difference equations:

$$\frac{\partial L(s)}{\partial t_n} = [B_n(s), L(s)], \tag{2.3}$$

$$L(s+1)P(s) = P(s)L(s), \tag{2.4}$$

$$\left(\frac{\partial}{\partial t_n} - B_n(s+1)\right) P(s) = P(s)\left(\frac{\partial}{\partial t_n} - B_n(s)\right), \tag{2.5}$$

where $B_n(s) = B_n(s; x, t) = \big(L(s; x, t)^n\big)_{>0}$. The projections like $(\cdot)_{>0}$ are defined as follows: for $A(x, \partial) = \sum_{n\in\mathbb{Z}} a_n(x)\partial^n$,

$$A_{>0} := \sum_{n>0} a_n(x)\partial^n, \qquad A_{\geq 0} := \sum_{n\geq 0} a_n(x)\partial^n,$$
$$A_{<0} := \sum_{n<0} a_n(x)\partial^n, \qquad A_{\leq 0} := \sum_{n\leq 0} a_n(x)\partial^n.$$

The last equation (2.5) can be written in the form

$$\frac{\partial P(s)}{\partial t_n} = B_n(s+1)\,P(s) - P(s)\,B_n(s), \tag{2.6}$$

as well. Since $B_1(s) = \partial$, equations (2.3) and (2.6) for $n = 1$ imply that x and t_1 always appear in the combination $x + t_1$.

Note that the cmKP hierarchy is almost the same as the mKP hierarchy in [2] or [17] but the forms of $L(s)$, $P(s)$ and $B_n(s)$ are different.

Remark 1. We can start from $P(s)$ with the 0-th order terms, but if $(L(s), P(s))_{s\in\mathbb{Z}}$ satisfies the cmKP hierarchy, we can gauge away such terms. See Appendix A for details.

By the well-known argument (cf. [3, § 1], [23, Theorem 1.1]) we can prove that the Lax equations (2.3) is equivalent to the Zakharov–Shabat (or zero-curvature) equations,

$$\left[\frac{\partial}{\partial t_m} - B_m(s), \frac{\partial}{\partial t_n} - B_n(s)\right] = 0, \tag{2.7}$$

or

$$\left[\frac{\partial}{\partial t_m} - B^c_m(s), \frac{\partial}{\partial t_n} - B^c_n(s)\right] = 0, \tag{2.8}$$

where $B^c_n(s) := -(L(s)^n)_{\leq 0} = B_n(s) - L(s)^n$.

2.2 Dressing operator, wave function

Similarly to the mKP hierarchy, we can show the existence of the dressing operator.

Proposition 1. *For any solution $(L(s), P(s))_{s\in S}$ there exists an operator $W(s) = W(s; x, t; \partial)$ of the form*

$$W(s; x, t; \partial) = \big(w_0(s; x, t) + w_1(s; x, t)\partial^{-1} + \cdots\big)\partial^{n_s}, \qquad w_0(s; x, t) \neq 0, \tag{2.9}$$

satisfying equations

$$L(s)W(s) = W(s)\partial, \qquad P(s)W(s) = W(s+1), \qquad \frac{\partial W(s)}{\partial t_n} = B^c_n(s)W(s). \tag{2.10}$$

In fact, equations (2.3), (2.8), (2.4) and (2.5) are compatibility conditions for the linear system (2.10).

We call $W(s; x, t; \partial)$ the *dressing operator*.

The *wave function* $w(s; \lambda) = w(s; x, t; \lambda)$ is defined by

$$\begin{aligned} w(s; x, t; \lambda) &:= W(s; x, t; \partial)e^{\xi(x+t,\lambda)} \\ &= \left(\sum_{j=0}^{\infty} w_j(s; x, t)\lambda^{-j}\right) \lambda^{n_s} e^{\xi(x+t,\lambda)}, \end{aligned} \tag{2.11}$$

where $x+t := (x+t_1, t_2, t_3, \dots)$, $\xi(x+t,\lambda) = x\lambda + \sum_{n=1}^{\infty} t_n\lambda^n$. It is subject to the equations:

$$
\begin{aligned}
&L(s)w(s;\lambda) = \lambda w(s;\lambda),\\
&\frac{\partial}{\partial t_n} w(s;\lambda) = B_n(s)w(s;\lambda),\\
&P(s)w(s;\lambda) = w(s+1;\lambda).
\end{aligned}
\tag{2.12}
$$

Recall that the mKP hierarchy in [17] is defined by the equations (2.3)–(2.5) for operators $L(s) = L^{\mathrm{mKP}}(s)$, $B_n(s) = B_n^{\mathrm{mKP}}(s)$, $P(s) = P^{\mathrm{mKP}}(s)$ normalized as

$$L^{\mathrm{mKP}}(s;x,t) = \partial + u_2^{\mathrm{mKP}}(s,x,t)\partial^{-1} + u_3^{\mathrm{mKP}}(s,x,t)\partial^{-2} + \cdots, \tag{2.13}$$

$$B_n^{\mathrm{mKP}}(s;x,t) := (L^{\mathrm{mKP}}(s;x,t)^n)_{\geq 0}, \tag{2.14}$$

$$P^{\mathrm{mKP}}(s;x,t) = \partial^{m_s} + q_1(s,x,t)\partial^{m_s-1} + \cdots + q_{m_s}(s,x,t). \tag{2.15}$$

Its dressing operator $W(s) = W^{\mathrm{mKP}}(s)$ is normalized as

$$W^{\mathrm{mKP}}(s) = (1 + w_1^{\mathrm{mKP}}(s)\partial^{-1} + \cdots)\partial^{n_s}, \tag{2.16}$$

and satisfies the same linear equations (2.10) as the cmKP hierarchy, where $B_n^c(s) = B_n^{\mathrm{mKP},c}(s) = -(L^{\mathrm{mKP}}(s)^n)_{<0} = B_n^{\mathrm{mKP}}(s) - L^{\mathrm{mKP}}(s)^n$.

Proposition 2. (i) *Let $(L(s), P(s))_{s\in S}$ be a solution of the cmKP hierarchy and $W(s)$ be the corresponding dressing operator of the form (2.9). Then $(L^{\mathrm{mKP}}(s), P^{\mathrm{mKP}}(s))_{s\in S}$ defined by*

$$L^{\mathrm{mKP}}(s) := w_0(s)^{-1}L(s)w_0(s), \qquad P^{\mathrm{mKP}}(s) := w_0(s+1)^{-1}P(s)w_0(s)$$

is a solution of the mKP hierarchy and $W^{\mathrm{mKP}}(s) := w_0(s)^{-1}W(s)$ is the corresponding dressing operator.

(ii) *Conversely, if a sequence $\{(f^{(0)}(s), \dots, f^{(m_s-1)}(s))\}_{s\in S}$ of non-zero constant vectors and a solution of the mKP hierarchy $(L^{\mathrm{mKP}}(s), P^{\mathrm{mKP}}(s))_{s\in S}$ are given, there exists a unique function $f(s) = f(s;x,t)$ such that*

$$\partial^k f(s;0,0) = f^{(k)}(s) \qquad \text{for all} \quad s\in S, 0 \leqq k < m_s, \tag{2.17}$$

and $(L(s) := f(s)^{-1}L^{\mathrm{mKP}}(s)f(s), P(s) := f(s+1)^{-1}P^{\mathrm{mKP}}(s)f(s))_{s\in S}$ is a solution of the cmKP hierarchy. ($m_{\max(S)} = \infty$.)

Proof. (i) Note that the linear equations (2.10) imply that

$$\frac{\partial}{\partial x} w_0(s;x,t) = -u_1(s;x,t)w_0(s;x,t), \tag{2.18}$$

$$w_0(s;x,t)p_0(s;x,t) = w_0(s+1;x,t), \tag{2.19}$$

$$\frac{\partial}{\partial t_n} w_0(s;x,t) = -\big(L(s)^n\big)_0 w_0(s;x,t). \tag{2.20}$$

Equation (2.18) and (2.19) mean that $L^{\mathrm{mKP}}(s) = w_0(s)^{-1}L(s)w_0(s)$ and $P^{\mathrm{mKP}}(s) = w_0(s+1)^{-1}P(s)w_0(s)$ have the required form (2.13) and (2.15). It follows from equation (2.20) that

$$w_0(s)^{-1}\left(\frac{\partial}{\partial t_n} - B_n(s)\right)w_0(s) = \frac{\partial}{\partial t_n} - B_n^{\mathrm{mKP}}(s),$$

where $B_n^{\mathrm{mKP}}(s)$ is defined by (2.14) from $L^{\mathrm{mKP}}(s)$. It is easy to see that $(L^{\mathrm{mKP}}(s), P^{\mathrm{mKP}}(s))_{s\in S}$ satisfies the system (2.3)–(2.5).

(ii) is proved in almost the same way as the fact mentioned in Remark 1, so we prove it in Appendix A. ■

It was shown in [17] that dressing operators of the mKP hierarchy are parametrized by the flag manifold: Let $\mathcal{V}$ be an infinite dimensional linear space $\bigoplus_{\nu\in\mathbb{Z}}\mathbb{C}e_\nu$ with basis $\{e_\nu\}_{\nu\in\mathbb{Z}}$ and $V^\varnothing$ be its subspace defined by $V^\varnothing=\bigoplus_{\nu\geq 0}\mathbb{C}e_\nu$. (Actually we have to take completion of $\mathcal{V}$, but details are omitted.) The *Sato–Grassmann manifold* of charge n, $SGM^{(n)}$, is defined by

$$SGM^{(n)}=\{U\subset\mathcal{V}\mid\ \text{index of } U\to\mathcal{V}/V^\varnothing \text{ is } n\}. \tag{2.21}$$

The set of dressing operators of the KP hierarchy is $SGM^{(0)}$ as is shown in [10, 12] or [11] and the set of dressing operators of the mKP hierarchy is the flag manifold

$$\mathit{Flag}:=\{(U_s)_{s\in S}\mid U_s\in SGM^{(n_s)},\ U_s\subset U_{s+1}\}. \tag{2.22}$$

See Proposition 1.3 of [17]. Hence the set of the dressing operators of the cmKP hierarchy is described as follows.

Corollary 1. *The dressing operator of the cmKP hierarchy $(W(s))_{s\in S}$ is parametrized by $\mathit{Flag}\times\prod\limits_{s\in S'}(\mathbb{C}^{m_s}\setminus\{0\})$.*

Schematically, this space is an infinite dimensional homogeneous space $GL(\infty)/Q$, where Q is a subgroup of the group $GL(\infty)$ of invertible $\mathbb{Z}\times\mathbb{Z}$ matrices defined as follows: $g=(g_{ij})_{i,j\in\mathbb{Z}}\in Q$ if and only if

$$g_{ij}=\begin{cases}0, & i>n_s, j\leqq n_s \text{ or } i>n_s+1, j\leqq n_s+1 \text{ for some } s\in S,\\ 1, & i=j=n_s+1 \text{ for some } s\in S.\end{cases} \tag{2.23}$$

In fact, if we consider an intermediate parabolic subgroup P defined by

$$g=(g_{ij})_{i,j\in\mathbb{Z}}\in P\Longleftrightarrow g_{ij}=0 \qquad (i>n_s, j\leqq n_s \text{ for some } s\in S), \tag{2.24}$$

$GL(\infty)/Q$ is considered as the fiber bundle

$$GL(\infty)/Q\to GL(\infty)/P,$$

over $GL(\infty)/P=\mathit{Flag}$. A point on a fiber $(U_s)_{s\in S}\in\mathit{Flag}$ specifies a series of non-zero vectors in U_{s+1}/U_s $(s\in S)$. ($U_{\max(S)+1}=\mathcal{V}$.)

We do not go into details of infinite dimensional homogeneous spaces. In this picture it is clear that the group $GL(\infty)$ acts on the space of solutions transitively. The action is explicitly described in terms of the fermionic description of the τ functions. See the end of Section 2.5.

2.3 Bilinear identity

Recall that the wave function and the adjoint wave function of the mKP hierarchy have the form

$$\begin{aligned}&w^{\mathrm{mKP}}(s;x,t;\lambda):=W^{\mathrm{mKP}}(s)e^{\xi(x+t;\lambda)}=\hat{w}^{\mathrm{mKP}}(s;x,t;\lambda)\lambda^{n_s}e^{\xi(x+t;\lambda)},\\ &\hat{w}^{\mathrm{mKP}}(s;x,t;\lambda):=1+w_1^{\mathrm{mKP}}(s;x,t)\lambda^{-1}+\cdots,\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}&w^{\mathrm{mKP},*}(s;x,t;\lambda):=((W(s)^{\mathrm{mKP}})^*)^{-1}e^{-\xi(x+t;\lambda)}=\hat{w}^{\mathrm{mKP},*}(s;x,t;\lambda)\lambda^{-n_s}e^{-\xi(x+t;\lambda)},\\ &\hat{w}^{\mathrm{mKP},*}(s;x,t;\lambda):=1+w_1^{\mathrm{mKP},*}(s;x,t)\lambda^{-1}+\cdots,\end{aligned} \tag{2.26}$$

where A^* for an operator A denotes its formal adjoint: $x^* = x$, $\partial^* = -\partial$, $(AB)^* = B^* A^*$. These functions are characterized by the bilinear residue identity:

$$\mathrm{Res}_{\lambda=\infty} w^{\mathrm{mKP}}(s; x, t; \lambda)\, w^{\mathrm{mKP},*}(s'; x', t'; \lambda)\, d\lambda = 0, \tag{2.27}$$

for any x, t, x', t' and $s' \leqq s$. See § 1 of [17] for details.

The wave function of the cmKP hierarchy is also characterized by a bilinear identity as follows:

Proposition 3. (i) *The wave function of the cmKP hierarchy satisfies the following identity:*

$$\mathrm{Res}_{\lambda=\infty} w(s; x, t; \lambda)\tilde{w}(s'; x', t'; \lambda)\, d\lambda = \begin{cases} 0, & s' < s, \\ 1, & s' = s. \end{cases} \tag{2.28}$$

Here $\tilde{w}(s, t; \lambda)$ is the adjoint wave function defined by

$$\tilde{w}(s; x, t; \lambda) := \tilde{W}(s; x, t; \partial) e^{-\xi(x+t,\lambda)}, \tag{2.29}$$

$$\tilde{W}(s, t; \partial) = -\partial^{-1}(W(s, t; \partial)^{-1})^*. \tag{2.30}$$

(ii) *Conversely, let $w(s; x, t; \lambda)$ be a function of the form* (2.11) *and $\tilde{w}(s; x, t; \lambda)$ be a function of the form* (2.29)*, where the operator $\tilde{W}(s)$ has the form*

$$\tilde{W}(s; x, t; \partial) = (\tilde{w}_0(s; x, t) + \tilde{w}_1(s; x, t)(-\partial)^{-1} + \cdots)(-\partial)^{-n_s-1}. \tag{2.31}$$

If the pair $(w(s; x, t; \lambda), \tilde{w}(s; x, t; \lambda))$ satisfies the equation (2.28)*, then $w(s; x, t; \lambda)$ is a wave function of the cmKP hierarchy and $\tilde{w}(s; x, t; \lambda)$ is its adjoint.*

Proof. (i) When $s' = s$, we have only to show

$$\mathrm{Res}_{\lambda=\infty} \frac{\partial^\alpha w}{\partial t^\alpha}(s; x, t; \lambda)\tilde{w}(s; x', t; \lambda)\, d\lambda = \begin{cases} 1, & \alpha = (0, 0, \dots), \\ 0, & \text{otherwise} \end{cases} \tag{2.32}$$

for each multi-index $\alpha = (\alpha_1, \alpha_2, \dots)$.

Let us recall DJKM's lemma (Lemma 1.1 of [3]): For any operators $A(x, \partial_x)$ and $B(x, \partial_x)$, we have

$$\mathrm{Res}_{\lambda=\infty} A(x, \partial_x) e^{x\lambda} B(x', \partial_{x'}) e^{-x'\lambda} d\lambda = f(x, x'), \tag{2.33}$$

where $f(x, x')$ is determined by

$$f(x, x')\partial^{-1}\delta(x - x') = (A(x, \partial_x) B^*(x, \partial_x))_{<0}\delta(x - x'). \tag{2.34}$$

Since $W(s; \partial)\tilde{W}(s; \partial)^* = \partial^{-1}$ by the definition (2.30) of $\tilde{W}(s; \partial)$, we have (2.32) for $\alpha = (0, 0, \dots)$ thanks to (2.33).

When $\alpha \neq (0, 0, \dots)$, we can prove by induction that

$$\frac{\partial^\alpha}{\partial t^\alpha} w(s; x, t; \lambda) = \sum_{i \geq 1} c_i^{(\alpha)}(s; x, t)\partial^i w(s; x, t; \lambda),$$

where $c_i^{(\alpha)}$'s are differential polynomials of coefficients of $B_n(s)$'s. Hence the left hand side of (2.32) vanishes due to (2.33) because

$$\left(\sum_{i \geq 1} c_i^{(\alpha)}(s; x, t)\partial^i\right) W(s; x, t; \partial)\tilde{W}(s; x, t; \partial)^* = \sum_{i \geq 1} c_i^{(\alpha)}(s; x, t)\partial^{i-1} \tag{2.35}$$

is a differential operator.

When $s' < s$, the bilinear residue identity (2.28) is equivalent to the vanishing of its Taylor coefficients:

$$\operatorname{Res}_{\lambda=\infty} \frac{\partial^\alpha w}{\partial t^\alpha}(s;x,t;\lambda)\tilde{w}(s';x',t;\lambda)\,d\lambda = 0, \tag{2.36}$$

for each multi-index α. When $\alpha = (0,0,\dots)$, this follows directly from (2.33) since

$$\begin{aligned} W(s;\partial)\tilde{W}(s';\partial)^* &= W(s;\partial)W(s';\partial)^{-1}\partial^{-1} \\ &= P(s-1)P(s-2)\cdots P(s'+1)P(s')\partial^{-1}, \end{aligned}$$

which is a differential operator due to (2.2). For $\alpha \neq (0,0,\dots)$, the proof is similar to the case $s' = s$.

(ii) When $s' = s$ and $t = t'$, the bilinear residue identity (2.28) is equivalent to

$$(W(s;x,t;\partial)\tilde{W}(s;x,t;\partial)^*)_{<0} = \partial^{-1},$$

which means

$$W(s;x,t;\partial)\tilde{W}(s;x,t;\partial)^* = \partial^{-1},$$

because $W(s;x,t;\partial)\tilde{W}(s;x,t;\partial)^*$ is of order -1. Hence

$$\tilde{W}(s;x,t;\partial) = -\partial^{-1}(W(s;x,t;\partial)^{-1})^*. \tag{2.37}$$

Putting $s' = s-1$ and $t = t'$, we have

$$(W(s;x,t;\partial)\tilde{W}(s-1;x,t;\partial)^*)_{<0} = 0.$$

from (2.28). This means

$$W(s;x,t;\partial)W(s-1;x,t;\partial)^{-1}\partial^{-1} = (\text{differential operator}),$$

by (2.37). Hence we obtain

$$\begin{aligned} P(s-1;\partial) &:= W(s,t;\partial)W(s-1,t;\partial)^{-1} \\ &= (\text{differential operator of order } m_{s-1} \text{ divisible by } \partial). \end{aligned} \tag{2.38}$$

Finally, put $s' = s$ and differentiate (2.28) with respect to t_n. Then we have

$$\operatorname{Res}_{\lambda=\infty} \frac{\partial}{\partial t_n} w(s,t;\lambda)\tilde{w}(s,t;\lambda)d\lambda = 0,$$

namely,

$$\left(\left(\frac{\partial W(s)}{\partial t_n} + W(s)\partial^n\right)\tilde{W}(s)^*\right)_{<0} = 0.$$

Using (2.37), we can rewrite this equation as

$$\frac{\partial W(s)}{\partial t_n}W(s)^{-1} = -(W(s)\partial^n W(s)^{-1})_{\leq 0}.$$

Thus we have recovered the linear equations (2.10). ■

Corollary 2. *The function $w^{\mathrm{mKP}}(s;x,t;\lambda) := w(s;x,t;\lambda)/w_0(s;x,t)$ is a wave function of the mKP hierarchy. Its adjoint wave function is*

$$w^{\mathrm{mKP},*}(s;x,t;\lambda) := -w_0(s;x,t)\partial(\tilde{w}(s;x,t;\lambda)).$$

Proof. This can be directly deduced from Proposition 2. Alternatively we derive it from Proposition 3 here. Functions $w(s;x,t;\lambda)/w_0(s;x,t)$ and $-w_0(s;x,t)\partial(\tilde{w}(s;x,t;\lambda))$ are expanded with respect to λ as in (2.25) and in (2.26) respectively. Hence differentiating (2.28) with respect to x', we obtain the bilinear residue identity (2.27) for the mKP hierarchy. ■

2.4 τ function

In this subsection we prove that the wave functions of the cmKP hierarchy are ratios of the τ functions. In contrast to the (m)KP hierarchy, we need two series of τ functions to express the wave function, unless $m_s = 1$.

Theorem 1. (i) *Let $w(s;x,t;\lambda)$ be a wave function of the cmKP hierarchy and $\tilde{w}(s;x,t;\lambda)$ be its adjoint. Then there exist functions $\tau_0(s;t)$ and $\tau_1(s;t)$ such that*

$$w(s;x,t;\lambda) := \frac{\tau_0(s;x+t-[\lambda^{-1}])}{\tau_1(s;x+t)} \lambda^{n_s} e^{\xi(x+t;\lambda)}, \tag{2.39}$$

$$\tilde{w}(s;x,t;\lambda) := \frac{\tau_1(s;x+t+[\lambda^{-1}])}{\tau_0(s;x+t)} \lambda^{-n_s-1} e^{-\xi(x+t;\lambda)}, \tag{2.40}$$

where $[\lambda^{-1}] := (\lambda^{-1}, \lambda^{-2}/2, \lambda^{-3}/3, \dots)$. The τ functions *$\tau_0(s;t)$ and $\tau_1(s;t)$ are determined only up to multiplication by an arbitrary function of s.*

(ii) *If $m_s = n_{s+1} - n_s = 1$, we can choose τ functions so that $\tau_1(s;t) = \tau_0(s+1;t)$.*

(iii) *The τ functions are characterized by the following bilinear residue identity:*

$$\begin{aligned}
&\operatorname{Res}_{\lambda=\infty} \tau_0(s;t-[\lambda^{-1}])\tau_1(s';t'+[\lambda^{-1}]) e^{\xi(t,\lambda)-\xi(t',\lambda)} \lambda^{n_s - n_{s'}-1}\, d\lambda \\
&\qquad = \begin{cases} 0, & s' < s, \\ \tau_1(s;t)\, \tau_0(s;t'), & s' = s. \end{cases}
\end{aligned} \tag{2.41}$$

Proof. (i) Let us denote the non-trivial parts of the wave function (2.11) and the adjoint wave function (2.29) as follows:

$$\begin{aligned}
\hat{w}(s;x,t;\lambda) &:= w_0(s;x,t) + w_1(s;x,t)\lambda^{-1} + w_2(s;x,t)\lambda^{-2} + \cdots, \\
\hat{\tilde{w}}(s;x,t;\lambda) &:= \tilde{w}_0(s;x,t) + \tilde{w}_1(s;x,t)\lambda^{-1} + \tilde{w}_2(s;x,t)\lambda^{-2} + \cdots.
\end{aligned}$$

Namely,

$$\begin{aligned}
w(s;x,t;\lambda) &= \hat{w}(s;x,t;\lambda)\lambda^{n_s} e^{\xi(x+t;\lambda)}, \\
\tilde{w}(s;x,t;\lambda) &= \hat{\tilde{w}}(s;x,t;\lambda)\lambda^{-n_s-1} e^{-\xi(x+t;\lambda)}.
\end{aligned}$$

Putting $s = s'$, $x = x' = 0$, replacing t_n by $t_n + \zeta^{-n}/n$, t'_n by t_n in the bilinear identity (2.28), we have

$$\begin{aligned}
1 &= \operatorname{Res}_{\lambda=\infty} \left(\hat{w}(s;t+[\zeta^{-1}];\lambda)\hat{\tilde{w}}(s;t;\lambda) \frac{\lambda^{-1}}{1-\lambda\zeta^{-1}} \right) d\lambda \\
&= \hat{w}(s;t+[\zeta^{-1}];\zeta)\hat{\tilde{w}}(s;t;\zeta).
\end{aligned} \tag{2.42}$$

In the limit $\zeta^{-1} \to 0$ we have

$$w_0(s;t)\tilde{w}_0(s;t) = 1. \tag{2.43}$$

Since $w_0(s;t)^{-1} w(s;t;\lambda)$ is a wave function of the mKP hierarchy (cf. Corollary 2), there exists a tau function $\tau_0(s;t)$ such that

$$\frac{\hat{w}(s;t;\lambda)}{w_0(s,t)} = \frac{\tau_0(s;t-[\lambda^{-1}])}{\tau_0(s;t)}. \tag{2.44}$$

Define the function $\tau_1(s;t)$ by

$$\tau_1(s;t) := \frac{\tau_0(s;t)}{w_0(s;t)}. \tag{2.45}$$

Equation (2.39) follows from (2.44) and (2.45). Equation (2.40) follows from (2.42).

(ii) From (2.40) and (2.42), we can see that dependence of $\tau_0(s;t)$ on t_n ($n \geq 1$) is determined uniquely by the equation

$$\frac{\partial \log \tau_0(s;t)}{\partial t_n} = A_n(s;t), \tag{2.46}$$

where

$$A_n(s;t) = -\operatorname{Res}_{\lambda=\infty}\left(\lambda^n\left(\sum_{j=1}^{\infty}\lambda^{-j-1}\frac{\partial}{\partial t_j} + \frac{\partial}{\partial\lambda}\right)\log\hat{\tilde{w}}(s;t;\lambda)\right) d\lambda.$$

See § 1.6 of [3] for detailed arguments.

When $n_{s+1} = n_s + 1$, by putting $s' = s-1$ and replacing t_n by $t_n + \zeta^{-n}/n$, t'_n by t_n in the bilinear identity (2.28), we have

$$\begin{aligned}
0 &= \operatorname{Res}_{\lambda=\infty}\left(\hat{w}(s;t+[\zeta^{-1}];\lambda)\hat{\tilde{w}}(s-1;t;\lambda)\frac{1}{1-\lambda\zeta^{-1}}\right) d\lambda \\
&= \hat{w}(s;t+[\zeta^{-1}];\zeta)\hat{\tilde{w}}(s-1;t;\zeta) - w_0(s;t+[\zeta^{-1}])\tilde{w}_0(s-1;t).
\end{aligned}$$

Using (2.42) and (2.43), and putting $\zeta = \lambda$, we obtain

$$\frac{\hat{\tilde{w}}(s-1;t;\lambda)}{\hat{\tilde{w}}(s;t;\lambda)} = \frac{w_0(s;t+[\lambda^{-1}])}{w_0(s-1;t)}$$

Therefore, we have from definition (2.46),

$$\frac{\partial \log \tau_0(s;t)}{\partial t_n} - \frac{\partial \log \tau_0(s-1;t)}{\partial t_n} = -\frac{\partial \log w_0(s-1;t)}{\partial t_n}.$$

Hence, we can fix the dependence of $\tau_0(s;t)$ on s by

$$\log \tau_0(s;t) - \log \tau_0(s-1;t) = -\log w_0(s-1;t).$$

Comparing this with the definition (2.45), we find that $\tau_1(s;t) = \tau_0(s+1;t)$, as desired.

Statement (iii) is a direct consequence of (i) and the bilinear residue identity for the wave functions, (2.28). ■

2.5 Construction of τ function

In this subsection we construct τ functions of the cmKP hierarchy in terms of the free fermions or, in other words, the Clifford algebra as in the case of the KP hierarchy [3] or of the Toda lattice hierarchy [15, 16].

Let ψ_n and ψ_n^* ($n \in \mathbb{Z}$) be free fermion operators, i.e., generators of a Clifford algebra $\mathcal{A}$ which satisfy the canonical anti-commutation relations:

$$[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0, \qquad [\psi_m, \psi_n^*]_+ = \delta_{mn}, \tag{2.47}$$

where $[A, B]_+ := AB+BA$. The Fock space $\mathcal{F}$ and the dual Fock space $\mathcal{F}^\vee$ are generated by the vacuum vector $|\mathrm{vac}\rangle$ and its dual $\langle\mathrm{vac}|$ over $\mathcal{A}$ respectively. $\mathcal{F}$ and $\mathcal{F}^\vee$ contain states of charge k, $|k\rangle$ and $\langle k|$ respectively, which are characterized by

$$\psi_n|k\rangle = \begin{cases} 0, & n < k, \\ |k+1\rangle, & n = k, \end{cases} \qquad \psi^*_m|k\rangle = \begin{cases} 0, & m \geqq k, \\ |k-1\rangle, & m = k-1, \end{cases} \tag{2.48}$$

$$\langle k|\psi_n = \begin{cases} 0, & n \geqq k, \\ \langle k-1|, & n = k-1, \end{cases} \qquad \langle k|\psi^*_m = \begin{cases} 0, & m < k, \\ \langle k+1|, & m = k. \end{cases} \tag{2.49}$$

In fact, $|\mathrm{vac}\rangle = |0\rangle$ and $\langle\mathrm{vac}| = \langle 0|$. The pairing of $\mathcal{F}$ and $\mathcal{F}^\vee$ is naturally defined by $\langle k|k\rangle = 1$.

We define the operators $J(t)$, $\psi(\lambda)$ and $\psi^*(\lambda)$ as follows:

$$J(t) := \sum_{n=1}^{\infty} t_n \sum_{k\in\mathbb{Z}} \psi_k \psi^*_{k+n}, \tag{2.50}$$

$$\psi(\lambda) := \sum_{n\in\mathbb{Z}} \psi_n \lambda^n, \qquad \psi^*(\lambda) := \sum_{n\in\mathbb{Z}} \psi^*_n \lambda^{-n}. \tag{2.51}$$

We quote important formulae from § 2.6 of [3]:

$$\langle m|e^{J(t)}\psi(\lambda) = \lambda^{m-1} e^{\xi(t,\lambda)} \langle m-1|e^{J(t-[\lambda^{-1}])}, \tag{2.52}$$

$$\langle m|e^{J(t)}\psi^*(\lambda) = \lambda^{-m} e^{-\xi(t,\lambda)} \langle m+1|e^{J(t+[\lambda^{-1}])}. \tag{2.53}$$

The bilinear identity comes from the following intertwining relation [3, § 2.1]:

$$\sum_{n\in\mathbb{Z}} \psi_n g \otimes \psi^*_n g = \sum_{n\in\mathbb{Z}} g\psi_n \otimes g\psi^*_n, \tag{2.54}$$

where g is an arbitrary element of the Clifford group generated by ψ_n's and ψ^*_n's.

Putting this equation between $\langle m+1|e^{J(t)} \otimes \langle m'-1|e^{J(t')}$ and $|m\rangle \otimes |m'\rangle$, we have

$$\begin{aligned} &\mathrm{Res}_{\lambda=\infty} \langle m+1|e^{J(t)}\psi(\lambda)g|m\rangle\langle m'-1|e^{J(t')}\psi^*(\lambda)g|m'\rangle \frac{d\lambda}{\lambda} \\ &\quad = \sum_{n\in\mathbb{Z}} \langle m+1|e^{J(t)}\psi_n g|m\rangle\langle m'-1|e^{J(t')}\psi^*_n g|m'\rangle \\ &\quad = \sum_{n\in\mathbb{Z}} \langle m+1|e^{J(t)}g\psi_n|m\rangle\langle m'-1|e^{J(t')}g\psi^*_n|m'\rangle. \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\mathrm{Res}_{\lambda=\infty} \langle m+1|e^{J(t)}\psi(\lambda)g|m\rangle\langle m'-1|e^{J(t')}\psi^*(\lambda)g|m'\rangle \frac{d\lambda}{\lambda} \\ &\quad = \begin{cases} 0, & m' \leqq m, \\ \langle m+1|e^{J(t)}g|m+1\rangle\langle m|e^{J(t')}g|m\rangle, & m' = m+1 \end{cases} \end{aligned} \tag{2.55}$$

due to (2.48).

For a Clifford group element g we define τ functions by

$$\tau_0(s;t) := \langle n_s|e^{J(t)}g|n_s\rangle, \qquad \tau_1(s;t) := \langle n_s+1|e^{J(t)}g|n_s+1\rangle. \tag{2.56}$$

The bilinear residue identity (2.41) holds because of (2.52), (2.53) and (2.55). Namely, we have constructed a pair of τ functions of the cmKP hierarchy for each g in the Clifford group.

The action of $GL(\infty)$ mentioned at the end of Section 2.2 is realized as the action of the Clifford group, $g \mapsto g'g$ ($g' \in GL(\infty)$) in (2.56). Hence the above construction exhausts all the solutions of the cmKP hierarchy.

The vertex operator description of the $gl(\infty)$ symmetry is the same as that for the KP hierarchy in [3].

3 Dispersionless modified KP hierarchy

3.1 Definition of the dcmKP hierarchy

Let N be a positive integer. When $n_s = Ns$, we can introduce the parameter $\hbar$ into the cmKP hierarchy and take the dispersionless limit, as is done for the mKP hierarchy in [17]. Now for the dcmKP hierarchy, the independent variables are the continuous variables s, x and $t = (t_1, t_2, \ldots)$. The dependent variables are encapsulated in $\mathcal{L}(k; s, t)$ and $\mathcal{P}(k; s, t)$, which are respectively formal power series and polynomial of k having the following form:

$$\mathcal{L}(s) = \mathcal{L}(k; s, t) = k + u_1(s, t) + u_2(s, t)k^{-1} + \cdots = \sum_{n=0}^{\infty} u_n(s, t)k^{1-n}, \tag{3.1}$$

$$\mathcal{P}(s) = \mathcal{P}(k; s, t) = p_0(s, t)k^N + \cdots + p_{N-1}(s, t)k = \sum_{n=0}^{N-1} p_n(s, t)k^{N-n}. \tag{3.2}$$

Here $u_0 = 1$, $p_0 \neq 0$. We do not write the dependence on x explicitly for the reason we are going to see later.

The N-dcmKP hierarchy is the following system of equations:

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \qquad \mathcal{B}_n := (\mathcal{L}^n)_{>0}, \tag{3.3}$$

$$\frac{\partial \mathcal{L}}{\partial s} = \{\log \mathcal{P}, \mathcal{L}\}, \tag{3.4}$$

$$\frac{\partial \log \mathcal{P}}{\partial t_n} = \frac{\partial \mathcal{B}_n}{\partial s} - \{\log \mathcal{P}, \mathcal{B}_n\}, \tag{3.5}$$

where now the projection $(\cdot)_{>0}$ is with respect to k, $\log \mathcal{P}$ is formally understood as

$$\log \mathcal{P} = \log p_0 + \log k^N + \sum_{n=1}^{\infty} \mathfrak{p}_n k^{-n},$$

and $\{\cdot, \cdot\}$ is the Poisson bracket

$$\{f(k, x), g(k, x)\} = \frac{\partial f}{\partial k}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial k}.$$

As usual, the $n = 1$ case of equations (3.3) and (3.5) implies that the dependence on x and t_1 appears in the combination $x + t_1$. As a result, we usually omit x and identify t_1 with x.

By the standard argument, equation (3.3) is equivalent to the Zakharov–Shabat (or zero-curvature) equations

$$\frac{\partial \mathcal{B}_n}{\partial t_m} - \frac{\partial \mathcal{B}_m}{\partial t_n} + \{\mathcal{B}_n, \mathcal{B}_m\} = 0, \qquad \text{or} \qquad \frac{\partial \mathcal{B}_n^c}{\partial t_m} - \frac{\partial \mathcal{B}_m^c}{\partial t_n} + \{\mathcal{B}_n^c, \mathcal{B}_m^c\} = 0, \tag{3.6}$$

where $\mathcal{B}_n^c = -(\mathcal{L}^n)_{\leq 0} = \mathcal{B}_n - \mathcal{L}^n$.

Remark 2. As in the case of the cmKP hierarchy (cf. Remark 1), even when we start from polynomial $\mathcal{P}$ with a constant term, $\mathcal{P}(s) = p_0(s)k^N + \cdots + p_N(s)$, we can gauge away $p_N(s)$. See Appendix A.

In general, we can let $\mathcal{P}$ be a power series with leading term $p_0 k^N$, and have infinitely many negative power terms, i.e., $\mathcal{P} = \sum\limits_{n=0}^{\infty} p_n k^{N-n}$. In particular, if $\mathcal{P} = \mathcal{L}^N$, then equation (3.4) says that there is no dependence on s, and (3.5) is equivalent to (3.3). This is the dmKP hierarchy considered by Kupershmidt [8], Chang and Tu [1].

Remark 3. Suppose $(\mathcal{L},\mathcal{P})$ is a solution of N-dcmKP hierarchy (3.3)–(3.5) and $Q(\zeta)=a_m\zeta^m+\cdots+a_0$ is a polynomial with coefficients $a_0,\dots,a_m$ independent of s, x and t. If $Q(\mathcal{L}(k))\mathcal{P}(k)$ is a polynomial of k without constant term[1], then it is easy to see that $(\mathcal{L},\mathcal{P}Q(\mathcal{L}))$ is a solution of $(N+m)$-dcmKP hierarchy. We say that this solution is equivalent to the solution $(\mathcal{L},\mathcal{P})$.

3.2 Dressing operator

As in [21], we can show the existence of a dressing operator.

Proposition 4. *For any solution $(\mathcal{L}(s),\mathcal{P}(s))$ of the dcmKP hierarchy, there exists an operator* $\exp\operatorname{ad}\phi(s)$ *that satisfies*

$$\mathcal{L}=(\exp\operatorname{ad}\phi)k,$$
$$\nabla_{t_n,\phi}\phi=\mathcal{B}_n^c,\qquad \nabla_{s,\phi}\phi=\log\mathcal{P}-\log\mathcal{L}^N, \tag{3.7}$$

where $\phi(s)$ is a power series of the form $\phi(s)=\sum_{n=0}^{\infty}\phi_n(s;t)k^{-n}$, $(\operatorname{ad}f)g=\{f,g\}$ and

$$\nabla_{u,\psi}\phi=\sum_{m=0}^{\infty}\frac{1}{(m+1)!}(\operatorname{ad}\psi)^m\frac{\partial\phi}{\partial u}$$

for series ψ, ϕ and variable u.

Comparing the coefficients of k^0 on both sides of equations in (3.7), we have

Corollary 3. *The function $\phi_0(s,t)$ satisfies*

$$\frac{\partial\phi_0}{\partial t_n}=-(\mathcal{L}^n)_0,\qquad \frac{\partial\phi_0}{\partial s}=\log p_0.$$

3.3 Orlov–Schulman function

Using the dressing operator ϕ, we can construct the Orlov–Schulman function $\mathcal{M}$ by

$$\begin{aligned}\mathcal{M}&=e^{\operatorname{ad}\phi}\left(\sum_{n=1}^{\infty}nt_nk^{n-1}+x+Nsk^{-1}\right)\\&=\sum_{n=1}^{\infty}nt_n\mathcal{L}^{n-1}+x+\frac{Ns}{\mathcal{L}}+\sum_{n=1}^{\infty}v_n\mathcal{L}^{-n-1},\end{aligned} \tag{3.8}$$

where v_n are functions of s, t. $\mathcal{M}$ has the property that it forms a canonical pair with $\mathcal{L}$, namely

$$\{\mathcal{L},\mathcal{M}\}=1. \tag{3.9}$$

Using Lemma A.1 in Appendix A of [21] and Proposition 4, we find that

$$\frac{\partial\mathcal{M}}{\partial t_n}=\{\mathcal{B}_n,\mathcal{M}\},\qquad \frac{\partial\mathcal{M}}{\partial s}=\{\log\mathcal{P},\mathcal{M}\}. \tag{3.10}$$

As in [20], we can show by using the equations (3.3)–(3.5), (3.10), (3.9) and Corollary 3 that the expansion of $\mathcal{B}_n$ and $\log\mathcal{P}$ with respect to $\mathcal{L}$ can be expressed through the functions v_n. More precisely,

[1]We do not need to impose this condition when we consider the generalized cmKP hierarchy as in Remark 2.

Proposition 5. *We have the following relations:*

$$\mathcal{B}_n = \mathcal{L}^n + \frac{\partial\phi_0}{\partial t_n} - \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial v_m}{\partial t_n}\mathcal{L}^{-m}, \tag{3.11}$$

$$\log\mathcal{P} = \log\mathcal{L}^N + \frac{\partial\phi_0}{\partial s} - \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial v_m}{\partial s}\mathcal{L}^{-m}. \tag{3.12}$$

3.4 Fundamental two form and S function

The fundamental two form ω is defined by

$$\omega = dk\wedge dx + \sum_{n=1}^{\infty} d\mathcal{B}_n\wedge dt_n + d\log\mathcal{P}\wedge ds.$$

The exterior derivative d is taken with respect to the independent variables k, x, s and t. From definition, ω is closed

$$d\omega = 0,$$

and it follows from the zero-curvature equation (3.6) and equation (3.5) that

$$\omega\wedge\omega = 0.$$

$(\mathcal{L},\mathcal{M})$ is a pair of functions that play the role of Darboux coordinates. Namely

$$d\mathcal{L}\wedge d\mathcal{M} = \omega.$$

In fact, we can prove as Proposition 2 in [20] that

Proposition 6. *The system of equations* (3.3)–(3.5), (3.10) *and* (3.9) *are equivalent to*

$$d\mathcal{L}\wedge d\mathcal{M} = dk\wedge dx + \sum_{n=1}^{\infty} d\mathcal{B}_n\wedge dt_n + d\log\mathcal{P}\wedge ds. \tag{3.13}$$

This formula implies that there exists a function $S(\mathcal{L}; s, t)$ such that

$$dS = \mathcal{M}d\mathcal{L} + kdx + \sum_{n=1}^{\infty}\mathcal{B}_n dt_n + \log\mathcal{P}ds,$$

or equivalently,

$$\frac{\partial S}{\partial\mathcal{L}} = \mathcal{M}, \qquad \frac{\partial S}{\partial x} = \frac{\partial S}{\partial t_1} = k, \qquad \frac{\partial S}{\partial t_n} = \mathcal{B}_n, \qquad \frac{\partial S}{\partial s} = \log\mathcal{P}.$$

From the formula (3.8) and Proposition 5, it is easy to see that

Proposition 7. *The S function is given explicitly by*

$$S = \sum_{n=1}^{\infty} t_n\mathcal{L}^n + x\mathcal{L} + s\log\mathcal{L}^N - \sum_{n=1}^{\infty}\frac{v_n}{n}\mathcal{L}^{-n} + \phi_0.$$

3.5 Tau function

We introduce the power series $\mathsf{k}(z;s,t)$ as the (formal) inverse of $\mathcal{L}(k;s,t)$ with respect to k, i.e. $\mathcal{L}(\mathsf{k}(z;s,t);s,t)=z$ and $\mathsf{k}(\mathcal{L}(k;s,t);s,t)=k$. Define the Grunsky coefficients b_{mn}, $m,n>0$ and $b_{n0}=b_{0n}$, $n>0$ of $\mathsf{k}(z)=\mathsf{k}(z;s,t)$ (cf. [4, 9, 19]) by the expansions

$$\log\frac{\mathsf{k}(z_1)-\mathsf{k}(z_2)}{z_1-z_2}=-\sum_{m=1}^{\infty}b_{mn}z_1^{-m}z_2^{-n}, \tag{3.14}$$

$$\log\frac{\mathsf{k}(z)}{z}=-\sum_{n=1}^{\infty}b_{n0}z^{-n}. \tag{3.15}$$

Obviously, b_{mn} are symmetric. In terms of the Grunsky coefficients, we have (cf. [4, 9, 19])

$$(\mathcal{L}^n)_0=nb_{n,0},\qquad \mathcal{B}_n=\mathcal{L}^n-nb_{n,0}+n\sum_{m=1}^{\infty}b_{nm}\mathcal{L}^{-m}. \tag{3.16}$$

Comparing with (3.11), we find that

$$\frac{\partial v_m}{\partial t_n}=-nmb_{nm},\qquad \frac{\partial\phi_0}{\partial t_n}=-nb_{n,0}. \tag{3.17}$$

Therefore by the symmetry of Grunsky coefficients, the first equation gives

$$\frac{\partial v_m}{\partial t_n}=\frac{\partial v_n}{\partial t_m}. \tag{3.18}$$

Consequently, we have

Proposition 8. *There exists a tau function $\tau_{\mathrm{dcmKP}}(s;t)$, determined up to a function of s, such that*

$$\frac{\partial\log\tau_{\mathrm{dcmKP}}}{\partial t_n}=v_n. \tag{3.19}$$

Define $\mathcal{F}=\log\tau_{\mathrm{dcmKP}}$. It is called the free energy. Using equations (3.19) and (3.17), we can rewrite the equations (3.14), (3.15) as

$$\begin{aligned} z_1\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial\phi_0}{\partial t_n}z_1^{-n}\right)-z_2\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial\phi_0}{\partial t_n}z_2^{-n}\right)\\ =(z_1-z_2)\exp\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_m\partial t_n}z_1^{-m}z_2^{-n}\right). \end{aligned} \tag{3.20}$$

Comparing the coefficients of z_2^0 on both sides, we have

$$z\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial\phi_0}{\partial t_n}z^{-n}\right)=z+\frac{\partial\phi_0}{\partial t_1}-\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial^2\mathcal{F}}{\partial t_n\partial t_1}z^{-n}. \tag{3.21}$$

On the other hand, we can formulate a partial converse of Proposition 8 as:

Proposition 9. *If $\tau_{\mathrm{dcmKP}}(s,t)$ and $\phi_0(s,t)$ are functions that satisfy the equation* (3.20)*, then the pair of functions $(\mathcal{L},\mathcal{P})$, where $\mathcal{L}(k)=\mathcal{L}(k;s,t)$ is defined by taking the inverse of the formal power series*

$$\mathsf{k}(z)=\mathsf{k}(z;s,t)=z\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial\phi_0}{\partial t_n}z^{-n}\right)$$

and $\mathcal{P}(k) = \mathcal{P}(k; s, t)$ is defined so that its composition with $\mathsf{k}(z)$ is given by

$$\mathcal{P}(\mathsf{k}(z)) = \mathcal{P}(\mathsf{k}(z; s, t); s, t) = \exp\left(\frac{\partial \phi_0}{\partial s}\right) z^N \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^2 \log \tau_{\mathrm{dcmKP}}}{\partial s \partial t_n} z^{-n}\right), \tag{3.22}$$

satisfy the dcmKP hierarchy (3.3)–(3.5)*, in the generalized sense as Remark 2.*

Proof. From (3.21), we have

$$\mathsf{k}(z) = z + \frac{\partial \phi_0}{\partial t_1} - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^2 \mathcal{F}}{\partial t_n \partial t_1} z^{-n}.$$

Therefore, it follows immediately from the definition of $\log \mathcal{P}$ that

$$\left.\frac{\partial \log \mathcal{P}}{\partial t_1}\right|_{\mathcal{L}\ \text{fixed}} = \left.\frac{\partial \mathsf{k}(\mathcal{L})}{\partial s}\right|_{\mathcal{L}\ \text{fixed}}.$$

As in the proof of Proposition 3.2 in [19], this implies equation (3.4).

On the other hand, let b_{mn}, $m, n \geq 0$ be the Grunsky coefficients of $\mathsf{k}(z)$ defined as in (3.14), (3.15). Comparing the equations (3.20) with (3.14), (3.15), we find from (3.16) that

$$\mathcal{B}_n = \mathcal{L}^n + \frac{\partial \phi_0}{\partial t_n} - \sum_{n=1}^{\infty} \frac{1}{m} \frac{\partial^2 \log \tau_{\mathrm{dcmKP}}}{\partial t_m \partial t_n} \mathcal{L}^{-m}.$$

Therefore,

$$\left.\frac{\partial \mathcal{B}_n}{\partial t_1}\right|_{\mathcal{L}\ \text{fixed}} = \left.\frac{\partial \mathsf{k}(\mathcal{L})}{\partial t_n}\right|_{\mathcal{L}\ \text{fixed}}, \qquad \left.\frac{\partial \mathcal{B}_n}{\partial s}\right|_{\mathcal{L}\ \text{fixed}} = \left.\frac{\partial \log \mathcal{P}}{\partial t_n}\right|_{\mathcal{L}\ \text{fixed}}. \tag{3.23}$$

The first equation implies equation (3.3). On the other hand, by using the second equation in (3.23) and equations (3.3) and (3.4), we have

$$\begin{aligned}
&\frac{\partial \log \mathcal{P}}{\partial t_n} - \frac{\partial \mathcal{B}_n}{\partial s} + \{\log \mathcal{P}, \mathcal{B}_n\} \\
&\quad = \left.\frac{\partial \log \mathcal{P}}{\partial t_n}\right|_{\mathcal{L}\ \text{fixed}} + \frac{\partial \log \mathcal{P}}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial t_n} - \left.\frac{\partial \mathcal{B}_n}{\partial s}\right|_{\mathcal{L}\ \text{fixed}} - \frac{\partial \mathcal{B}_n}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial s} \\
&\quad + \frac{\partial \log \mathcal{P}}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial k} \left.\frac{\partial \mathcal{B}_n}{\partial t_1}\right|_{\mathcal{L}\ \text{fixed}} - \left.\frac{\partial \log \mathcal{P}}{\partial t_1}\right|_{\mathcal{L}\ \text{fixed}} \frac{\partial \mathcal{B}_n}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial k} = 0.
\end{aligned}$$

This gives equation (3.5). ∎

3.6 Relation with the dmKP hierarchy

The dmKP hierarchy in [17] is defined by the system of equations

$$\frac{\partial \mathcal{L}^{\mathrm{dmKP}}}{\partial t_n} = \{\mathcal{B}_n^{\mathrm{dmKP}}, \mathcal{L}^{\mathrm{dmKP}}\}, \tag{3.24}$$

$$\frac{\partial \mathcal{L}^{\mathrm{dmKP}}}{\partial s} = \{\log \mathcal{P}^{\mathrm{dmKP}}, \mathcal{L}^{\mathrm{dmKP}}\}, \tag{3.25}$$

$$\frac{\partial \log \mathcal{P}^{\mathrm{dmKP}}}{\partial t_n} = \frac{\partial \mathcal{B}_n^{\mathrm{dmKP}}}{\partial s} - \{\log \mathcal{P}^{\mathrm{dmKP}}, \mathcal{B}_n^{\mathrm{dmKP}}\}, \tag{3.26}$$

for the power series

$$\mathcal{L}^{\mathrm{dmKP}}(k;s,t) = k + u_2^{\mathrm{dmKP}}(s,t)k^{-1} + u_3^{\mathrm{dmKP}}k^{-2} + \cdots, \tag{3.27}$$
$$\mathcal{B}_n^{\mathrm{dmKP}}(k;s,t) = (\mathcal{L}^{\mathrm{dmKP}}(k;s,t)^n)_{\geq 0},$$
$$\mathcal{P}^{\mathrm{dmKP}}(k;s,t) = k^N + q_1(s,t)k^{N-1} + \cdots + q_N(s,t). \tag{3.28}$$

We have

Proposition 10. *If $(\mathcal{L}(s), \mathcal{P}(s))$ is a solution of the dcmKP hierarchy, then the pair $(\mathcal{L}^{\mathrm{dcmKP}}(s), \mathcal{P}^{\mathrm{dcmKP}}(s))$, where*

$$\mathcal{L}^{\mathrm{dmKP}}(s) = \mathcal{L}^{\mathrm{dmKP}}(k;s,t) = \mathcal{L}(k - u_1(s,t); s,t), \tag{3.29}$$
$$\mathcal{P}^{\mathrm{dmKP}}(s) = \mathcal{P}^{\mathrm{dmKP}}(k;s,t) = p_0(s,t)^{-1}\mathcal{P}(k - u_1(s,t); s,t),$$

is a solution of the dmKP hierarchy.

Proof. It is easy to see that $\mathcal{L}^{\mathrm{dmKP}}$ and $\mathcal{P}^{\mathrm{dmKP}}$ defined by (3.29) has the form required by (3.27) and (3.28). Let $\exp\left(\operatorname{ad}\sum_{n=0}^{\infty}\phi_n(s,t)k^{-n}\right)$ be the dressing operator of the solution $(\mathcal{L}(s), \mathcal{P}(s))$. Using Corollary 3, it is easy to check that $\mathcal{L}^{\mathrm{dmKP}}$, $\log\mathcal{P}^{\mathrm{dmKP}}$ can be written as

$$\mathcal{L}^{\mathrm{dmKP}} = e^{-\operatorname{ad}\phi_0}\mathcal{L}, \qquad \log\mathcal{P}^{\mathrm{dmKP}} = e^{-\operatorname{ad}\phi_0}\log\mathcal{P} - \frac{\partial\phi_0}{\partial s}. \tag{3.30}$$

Since

$$e^{-\operatorname{ad}\phi_0}(\mathcal{L}^n)_{>0} + (\mathcal{L}^n)_0 = e^{-\operatorname{ad}\phi_0}(\mathcal{L}^n)_{\geq 0} = (e^{-\operatorname{ad}\phi_0}\mathcal{L}^n)_{\geq 0},$$

we have

$$\mathcal{B}_n^{\mathrm{dmKP}} = e^{-\operatorname{ad}\phi_0}\mathcal{B}_n - \frac{\partial\phi_0}{\partial t_n}. \tag{3.31}$$

Using this relation, equation (3.30), equations (3.3)–(3.5) and Lemma A.1 in [21], it is a direct computation to verify that $(\mathcal{L}^{\mathrm{dmKP}}, \mathcal{P}^{\mathrm{dmKP}})$ satisfy equations (3.24)–(3.26). ■

The map (3.29) is called a dispersionless Miura map, corresponding to the Miura map between a solution of KdV hierarchy and a solution of modified KdV hierarchy.

3.7 The special case $\mathcal{P} = p_0 k$

In the special case where $N = 1$ and $\mathcal{P} = p_0 k$, we have from Corollary 3 and equation (3.15), (3.17),

$$\log\mathcal{P} = \log p_0 + \log k = \frac{\partial\phi_0}{\partial s} + \log\mathcal{L} + \sum_{n=1}^{\infty}\frac{\partial\phi_0}{\partial t_n}\mathcal{L}^{-n}.$$

Comparing with (3.12), we find that

$$\frac{\partial^2\log\tau_{\mathrm{dcmKP}}}{\partial t_n\partial s} = \frac{\partial v_n}{\partial s} = -\frac{\partial\phi_0}{\partial t_n}.$$

Therefore we can fix the dependence of τ_{dcmKP} on s by the equation

$$\frac{\partial\log\tau_{\mathrm{dcmKP}}}{\partial s} = -\phi_0$$

and equation (3.20) can be written as

$$z_1 \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^2 \mathcal{F}}{\partial s \partial t_n} z_1^{-n}\right) - z_2 \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^2 \mathcal{F}}{\partial s \partial t_n} z_2^{-n}\right)$$
$$= (z_1 - z_2) \exp\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} z_1^{-m} z_2^{-n}\right), \tag{3.32}$$

which we call the dispersionless Hirota equation for dcmKP hierarchy with $\mathcal{P} = p_0 k$. The counterpart of Proposition 9 becomes

Proposition 11. *If $\tau_{\mathrm{dcmKP}}(s,t)$ is a function that satisfies the dispersionless Hirota equation* (3.32)*, then the pair of functions $(\mathcal{L}, \mathcal{P})$, where $\mathcal{L}(k) = \mathcal{L}(k; s, t)$ is defined by inverting*

$$\mathsf{k}(z) = \mathsf{k}(z; s, t) = z \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^2 \log \tau_{\mathrm{dcmKP}}}{\partial t_n \partial s} z^{-n}\right)$$

and

$$\mathcal{P}(k) = \mathcal{P}(k; s, t) = k \exp\left(-\frac{\partial^2 \log \tau_{\mathrm{dcmKP}}}{\partial s^2}\right)$$

satisfy the dcmKP hierarchy (3.3)–(3.5)*.*

Proof. This can be directly deduced from Proposition 9. ■

Comparing with Proposition 3.1 in [19] and its following discussion, we find that if $(\mathcal{L}(k; s, t), \mathcal{P}(k; s, t) = p_0(s, t)k)$ is a solution of dcmKP hierarchy, $\mathcal{L}(k; s, t)$ is a solution of the hierarchy

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{(\mathcal{L}^n)_{\geq 0}, \mathcal{L}\}_T;$$

$\mathcal{L}_1(k; s, t) = \mathcal{L}(p_0^{-1} k; s, t)$ is a solution of the hierarchy

$$\frac{\partial \mathcal{L}_1}{\partial t_n} = \{(\mathcal{L}_1^n)_{>0}, \mathcal{L}_1\}_T, \tag{3.33}$$

and $\mathcal{L}_{1/2}(k; s, t) = \mathcal{L}(p_0^{-1/2} k; s, t)$ is a solution of the hierarchy

$$\frac{\partial \mathcal{L}_{1/2}}{\partial t_n} = \left\{(\mathcal{L}_{1/2}^n)_{>0} + \frac{1}{2}(\mathcal{L}_{1/2}^n)_0,\ \mathcal{L}_{1/2}\right\}_T. \tag{3.34}$$

Here $\{\cdot, \cdot\}$ is the Poisson bracket of dToda hierarchy:

$$\{f(k, s), g(k, s)\}_T = k \frac{\partial f}{\partial k} \frac{\partial g}{\partial s} - k \frac{\partial f}{\partial s} \frac{\partial g}{\partial k}.$$

(3.33) and (3.34) can be considered as gauge equivalent form of the dcmKP hierarchy with $\mathcal{P} = p_0 k$ and with gauge parameter 1 and 1/2 respectively. For the cmKP version of (3.33) and (3.34) we refer to Appendices B, C and D. The form (3.34) was used in the work [22].

3.8 Quasi-classical limit of the cmKP hierarchy

The dispersionless KP hierarchy and the dispersionless Toda hierarchy are obtained as quasi-classical limit of corresponding "dispersionful" hierarchies. See [21]. The dcmKP hierarchy is also quasi-classical limit of the cmKP hierarchy. In this subsection we briefly explain the correspondence.

Let us define the *order* as in [21], § 1.7.1:

$$\operatorname{ord}^{\hbar}\left(\sum a_{n,m}(t)\hbar^n\partial^m\right) := \max\{m-n \mid a_{n,m}(t) \neq 0\}. \tag{3.35}$$

In particular, $\operatorname{ord}^{\hbar}(\hbar) = -1$, $\operatorname{ord}^{\hbar}(\partial) = 1$. The *principal symbol* (resp. the *symbol of order* l) of an operator $A = \sum a_{n,m}\hbar^n\partial^m$ is

$$\sigma^{\hbar}(A) := \hbar^{-\operatorname{ord}^{\hbar}(A)} \sum_{m-n=\operatorname{ord}^{\hbar}(A)} a_{n,m}k^m,$$

respectively

$$\sigma^{\hbar}_l(A) := \hbar^{-l} \sum_{m-n=l} a_{n,m}k^m.$$

Let us redefine the cmKP hierarchy with a small parameter $\hbar$ as follows. Fix a positive integer N. The discrete independent variable s runs in $S := \hbar\mathbb{Z}$. Operators L, P are of the form

$$L(s;x,t) = \sum_{n=0}^{\infty} u_n(\hbar,s,x,t)(\hbar\partial)^{1-n}, \tag{3.36}$$

$$P(s;x,t) = \sum_{n=0}^{N-1} p_n(\hbar,s,x,t)(\hbar\partial)^{N-n}, \tag{3.37}$$

where $u_0 = 1$, $p_0 \neq 0$ and all coefficients $u_n(\hbar,s,x,t)$ and $p_n(\hbar,s,x,t)$ are regular in $\hbar$. Namely they do not contain negative powers of $\hbar$. The cmKP hierarchy is rewritten as

$$\hbar\frac{\partial L(s)}{\partial t_n} = [B_n(s), L(s)], \tag{3.38}$$

$$(L(s+\hbar) - L(s))P(s) = [P(s), L(s)], \tag{3.39}$$

$$(B_n(s) - B_n(s+\hbar))\,P(s) = \left[P(s), \hbar\frac{\partial}{\partial t_n} - B_n(s)\right]. \tag{3.40}$$

where $B_n(s) = B_n(s;x,t)$ is defined as before. The principal symbols of (3.38), (3.39) and (3.40) give equations (3.3), (3.4) and (3.5) of the dcmKP hierarchy respectively, where $\mathcal{L}(s) = \sigma^{\hbar}_0(L(s))$ and $\mathcal{P}(s) = \sigma^{\hbar}_0(P(s))$.

The dressing operator $W(s;x,t;\partial)$ (2.9) should have the form

$$W(s;x,t;\hbar;\partial) = \exp(\hbar^{-1}X(\hbar,x,t;\partial))(\hbar\partial)^{Ns},$$

$$X(\hbar,x,t;\partial) = \sum_{n=0}^{\infty} \chi_n(\hbar,x,t)(\hbar\partial)^{-n},$$

where $\chi_n(\hbar,x,t)$ is regular in $\hbar$. The principal symbol of X is the function ϕ in Proposition 4.

Remark 4. Solutions of the dispersionless KP and Toda hierarchies can be lifted up to solutions of the KP and Toda hierarchies (with $\hbar$) respectively by lifting the dressing operator. See Corollary 1.7.6 and Corollary 2.7.6 of [21].

As for the dcmKP hierarchy, we conjecture that any solution of the dcmKP hierarchy would be lifted up to a solution of the cmKP hierarchy but there is difficulty coming from the form of $P(s)$. Naive lift of $\phi(s,t)$ would cause a tail of $P(s)$ which has non-positive order as a micro-differential operator and negative order in the sense of (3.35). We can correct this by inductively modifying $X(s)$ to get $P(s)$ of the form (3.37), but this inductive procedure might make $X(s)$ behave wildly with respect to s.

In the context of the WKB analysis of the linear equations (2.12) the S function introduced in Section 3.4 is the phase function:

$$w(s;x,t;\lambda) = \exp\big(\hbar^{-1}(S + O(\hbar))\big), \tag{3.41}$$

$$S = \sum_{n=1}^{\infty} t_n\lambda^n + x\lambda + Ns\log\lambda - \sum_{n=1}^{\infty}\frac{v_n}{n}\lambda^{-n} + \phi_0. \tag{3.42}$$

By replacing λ with $\mathcal{L}$, we obtain the S function in Proposition 7.

If the conjecture in Remark 4 is true, the form of $\mathcal{P}(s)$ of the dcmKP hierarchy reduces drastically by the following proposition.

Proposition 12. *If a solution $(\mathcal{L}(s),\mathcal{P}(s))_{s\in S}$ of the dcmKP hierarchy is the quasi-classical limit of a solution $(L(s),P(s))_{s\in S}$ of the cmKP hierarchy with $\hbar$. Then $\mathcal{P}(s) = p_0(s,t)k^N$.*

Proof. Let $\tau_0(s;t)$ and $\tau_1(s;t)$ be the tau function of $(L(s),P(s))_{s\in S}$. They are expressed by the Clifford algebra as in (2.56): for $s\in\mathbb{Z}\hbar$

$$\begin{aligned}\tau_0(\hbar;s;t) &= \langle Ns\hbar^{-1}|e^{J(t)\hbar^{-1}}g|Ns\hbar^{-1}\rangle,\\ \tau_1(\hbar;s;t) &= \langle Ns\hbar^{-1}+1|e^{J(t)\hbar^{-1}}g|Ns\hbar^{-1}+1\rangle.\end{aligned} \tag{3.43}$$

Defining

$$\tau(s;t) := \langle s\hbar^{-1}|e^{J(t)\hbar^{-1}}g|s\hbar^{-1}\rangle \tag{3.44}$$

we have a solution $(\tilde{L}(s),\tilde{P}(s))_{s\in\mathbb{Z}}$ of the cmKP hierarchy with $N=1$ whose τ functions are $\tilde{\tau}_0(s;t) = \tau(s;t)$, $\tilde{\tau}_1(s;t) = \tau(s+\hbar;t)$. It is easy to see that the dressing operators $W(s)$ of $(L(s),P(s))_{s\in S}$ and $\tilde{W}(s)$ of $(\tilde{L}(s),\tilde{P}(s))_{s\in\mathbb{Z}}$ are related by $W(s) = \tilde{W}(Ns)$.

Hence $L(s) = \tilde{L}(Ns)$ and $P(s) = \tilde{P}(Ns+(N-1)\hbar)\cdots\tilde{P}(Ns)$. The symbol of order 0 of the last equation gives

$$\mathcal{P}(s) = \sigma_0^\hbar(P(s)) = \sigma_0^\hbar(\tilde{P}(Ns+(N-1)\hbar))\cdots\sigma_0^\hbar(\tilde{P}(Ns)),$$

because $\sigma_0^\hbar(AB) = \sigma_0^\hbar(A)\sigma_0^\hbar(B)$ for any operators A, B, $\mathrm{ord}^\hbar(A) = \mathrm{ord}^\hbar(B) = 0$. Since $\tilde{P}(s)$ is of the form $\tilde{p}_0(s,t)\hbar\partial$, $\mathcal{P}(s)$ is of the form $p_0(s,t)k^N$. ∎

A proof of the above statement without lifting up to the cmKP hierarchy is desirable.

If $\mathcal{P}(s) = p_0(s,t)k^N$, we have $\operatorname{Res}\mathcal{L}^n d_k\log\mathcal{P}(s) = N(\mathcal{L}^n)_0$, which is equivalent to

$$\frac{\partial v_n}{\partial s} = -N\frac{\partial\phi_0}{\partial t_n}, \tag{3.45}$$

because of (3.12) and Corollary 3. This equation together with (3.18) is a compatibility condition of equations (3.19) and

$$\frac{\partial\log\tau_{\mathrm{dcmKP}}}{\partial s} = -N\phi_0, \tag{3.46}$$

which fixes the s-dependence of $\log\tau_{\mathrm{dcmKP}}$.

In fact, this is consistent with the quasi-classical limit. We express the τ functions of $(L(s), P(s))_{s\in S}$ as (3.43) and define $\tau(s;t)$ by (3.44). As in [21], that τ function behaves as

$$\tau(s;t) = e^{\hbar^{-2}F(\hbar,s,t)}$$

and therefore τ_0 and τ_1 behave as

$$\tau_0(\hbar;s;t) = e^{\hbar^{-2}F(\hbar,Ns,t)}, \qquad \tau_1(\hbar;s;t) = e^{\hbar^{-2}F(\hbar,Ns+\hbar,t)}.$$

Substituting this into (2.39) and comparing the result with (3.41) and (3.42), we have

$$\frac{\partial F(\hbar,Ns,t)}{\partial t_n} = v_n(s,t) + O(\hbar), \qquad -\frac{\hbar}{N}\frac{\partial F(\hbar,Ns,t)}{\partial s} = \hbar\phi_0(s,t) + O(\hbar^2).$$

Hence $\log \tau_{\mathrm{dcmKP}}(s;t) = F(\hbar,Ns,t)|_{\hbar=0}$, which satisfies (3.19) and (3.46).

A Form of $P(s)$, $\mathcal{P}(s)$ and proof of Proposition 2 (ii)

In the main text we assumed that operator $P(s)$ of the cmKP hierarchy does not have the 0-th order term as in (2.2). We also put similar requirement (3.2) to $\mathcal{P}(s)$ of the dcmKP hierarchy. At first glance, these assumptions might seem artificial but in fact they are not restriction as we show in this appendix.

Assume that $L(s)$ is of the form (2.1) and that $P(s)$ has the form

$$\begin{aligned} P(s;x,t) &:= p_0(s,x,t)\partial^{m_s} + \cdots + p_{m_s-1}(s,t)\partial + p_{m_s}(s,x,t) \\ &= \sum_{n=0}^{m_s} p_n(s,x,t)\partial^{m_s-n}, \qquad p_0(s,x,t) \neq 0, \end{aligned} \tag{A.1}$$

instead of the form (2.2). Assume further that $(L(s), P(s))_{s\in S}$ satisfies the system (2.3), (2.4) and (2.5). We show that there exists a function $f(s) = f(s,x,t)$ which satisfies

- The pair $(\tilde{L}(s) := f(s)^{-1}L(s)f(s), \tilde{P}(s) := f(s+1)^{-1}P(s)f(s))_{s\in S}$ is a solution of the cmKP hierarchy.
- $\tilde{P}(s)$ does not have the 0-th order term:

$$\tilde{P}(s) = \tilde{p}_0(s)\partial^{m_s} + \cdots + \tilde{p}_{m_s-1}(s)\partial + \tilde{p}_{m_s}(s), \qquad \tilde{p}_{m_s}(s) = 0. \tag{A.2}$$

In this sense we can assume without loss of generality that $p_{m_s}(s) = 0$ in (2.2).

The following is the basic lemma:

Lemma 1. *Let* $Q = \sum_{j=0}^{N} q_j(x)\partial^{N-j}$ *be a differential operator and* $f(x)$ *is a function. Then the* 0*-th order term of the composition* $Q \circ f$ *is the function* $Q(f)$ *obtained by applying* Q *on* f*.*

This is a direct consequence of the Leibniz rule: $\partial^k \circ f = \sum_{r=0}^{k} \binom{k}{r} f^{(k-r)}\partial^r$.

Hence the second condition (A.2) for the function $f(s,t)$ is equivalent to

$$P(s)(f(s)) = 0. \tag{A.3}$$

Let us introduce an operator $C_n(s)$ by

$$f(s)^{-1}(\partial_{t_n} - B_n(s))f(s) = \partial_{t_n} - C_n(s),$$

$$\text{i.e.,}\quad C_n(s) = f(s)^{-1}B_n(s)f(s) - f(s)^{-1}\frac{\partial f(s)}{\partial t_n}. \tag{A.4}$$

Then, it follows from (2.3), (2.4) and (2.5) that the pair $(\tilde{L}(s), \tilde{P}(s))_{s\in S}$ satisfies the following equations:

$$[\tilde{L}(s), \partial_{t_n} - C_n(s)] = 0, \tag{A.5}$$

$$\tilde{L}(s+1)\tilde{P}(s) = \tilde{P}(s)\tilde{L}(s), \tag{A.6}$$

$$(\partial_{t_n} - C_n(s+1))\tilde{P}(s) = \tilde{P}(s)(\partial_{t_n} - C_n(s)). \tag{A.7}$$

Hence if $C_n(s) = (\tilde{L}(s)^n)_{>0}$, then $(\tilde{L}(s), \tilde{P}(s))$ is a solution of the cmKP hierarchy. By Lemma 1 we have

$$C_n(s) = (f(s)^{-1}B_n(s)f(s))_{>0} + f(s)^{-1}B_n(s)(f(s)) - f(s)^{-1}\frac{\partial f(s)}{\partial t_n}.$$

Therefore the condition $C_n(s) = (\tilde{L}(s)^n)_{>0}$ is equivalent to

$$(\partial_{t_n} - B_n(s))(f(s)) = 0. \tag{A.8}$$

So, we have to find a function $f(s)$ satisfying (A.3) and (A.8). This is done inductively as follows. First solve equation (A.3) for $t_1 = t_2 = \cdots = 0$ but for arbitrary x. We denote the solution by $f_0(s, x)$:

$$P(s; x, t_1 = t_2 = \cdots = 0)f_0(s, x) = 0, \qquad f_0(s, 0) = 1. \tag{A.9}$$

Function $f_1(s, x, t_1) = f_0(s, x+t_1)$ satisfies (A.3) as well as (A.8) with $n = 1$ for $t_2 = t_3 = \cdots = 0$.

Suppose we have function $f_m(s, x, t_1, \dots, t_m)$ which satisfies (A.3) and (A.8) with $n = 1, \dots, m$ and $t_k = 0$ $(k > m)$. We can solve the Cauchy problem

$$\begin{aligned}
&\frac{\partial}{\partial t_{m+1}} f_{m+1}(s, x, t_1, \dots, t_{m+1}) \\
&\qquad - B_{m+1}(s, x, t_1, \dots, t_{m+1}, 0, 0, \dots) f_{m+1}(s, x, t_1, \dots, t_{m+1}) = 0, \\
&f_{m+1}(s, x, t_1, \dots, t_m, 0) = f_m(s, x, t_1, \dots, t_m),
\end{aligned} \tag{A.10}$$

with respect to t_{m+1}. By (2.7) and (2.5) for $n = m+1$, the solution f_{m+1} of (A.10) satisfies (A.3) and (A.8) for $n = 1, \dots, m+1$ and $t_k = 0$ $(k > m+1)$.

The desired function $f(s, t) = f(s, x, t_1, t_2, \dots)$ is defined by the inductive limit of the sequence $f_n(s, x, t_1, t_2, \dots, t_n)$.

The second statement of Proposition 2 is proved in the same way. Suppose that a solution $(L^{\mathrm{mKP}}(s), P^{\mathrm{mKP}}(s))_{s\in S}$ of the mKP hierarchy and a sequence $\{(f^{(0)}(s), \dots, f^{(m_s-1)}(s))\}_{s\in S}$ of non-zero constant vectors are given. Replace $L(s)$, $P(s)$ and $B_n(s)$ in the above argument by $L^{\mathrm{mKP}}(s)$, $P^{\mathrm{mKP}}(s)$ and $B_n^{\mathrm{mKP}}(s)$ respectively. (See (2.13), (2.15) and (2.14).) If we solve equation (A.9) under the initial condition

$$\partial^k f_0(s, 0) = f^{(k)}(s), \qquad k = 0, \dots, m_s - 1, \quad (s \in S'),$$

we obtain a function $f(s) = f(s, x, t)$ such that

$$\partial^k f_0(s; 0, 0) = f^{(k)}(s) \qquad \text{for all} \quad s \in S, \quad 0 \leqq k < m_s,$$

and $(L(s) := f(s)^{-1}L^{\mathrm{mKP}}(s)f(s), P(s) := f(s+1)^{-1}P^{\mathrm{mKP}}(s)f(s))_{s\in S}$ is a solution of the cmKP hierarchy. ∎

We proceed to the case of the dcmKP hierarchy.

Lemma 2. *Let $(\mathcal{L}(s), \mathcal{P}(s))$, where*

$$\begin{aligned}
&\mathcal{L}(s) = \mathcal{L}(k; s, t) = k + \sum_{n=0}^{\infty} u_{n+1}(s, t) k^{-n}, \\
&\mathcal{P}(s) = \mathcal{P}(k; s, t) = p_0(s, t) k^N + \cdots + p_N(s, t),
\end{aligned} \tag{A.11}$$

be a solution of the dcmKP hierarchy. If $\varphi(s, t)$ is a function that satisfies the system of equations[2]

$$\frac{\partial \varphi(s, t)}{\partial t_n} = -\mathcal{B}_n \left(-\frac{\partial \varphi(s, t)}{\partial x}; s, t \right), \qquad n \geq 1, \tag{A.12}$$

then the pair $(\tilde{\mathcal{L}}(s), \tilde{\mathcal{P}}(s))$, where

$$\begin{aligned}
&\tilde{\mathcal{L}}(k; s, t) = e^{\operatorname{ad} \varphi(s,t)} \mathcal{L}(k; s, t) = \mathcal{L} \left(k - \frac{\partial \varphi(s, t)}{\partial x}; s, t \right), \\
&\tilde{\mathcal{P}}(k; s, t) = e^{\partial \varphi(s,t)/\partial s} e^{\operatorname{ad} \varphi(s,t)} \mathcal{L}(k; s, t) = e^{\partial \varphi(s,t)/\partial s} \mathcal{P} \left(k - \frac{\partial \varphi(s, t)}{\partial x}; s, t \right),
\end{aligned}$$

is also a solution of the dcmKP hierarchy.

Proof. First, observe that

$$\tilde{\mathcal{B}}_n = (\tilde{\mathcal{L}}^n)_{>0} = e^{\operatorname{ad} \varphi} \mathcal{B}_n - \mathcal{B}_n \left(-\frac{\partial \varphi}{\partial x} \right).$$

Therefore, we have

$$\begin{aligned}
\frac{\partial \tilde{\mathcal{L}}}{\partial t_n} &= \left\{ \frac{\partial \varphi}{\partial t_n}, \tilde{\mathcal{L}} \right\} + e^{\operatorname{ad} \varphi} \{ \mathcal{B}_n, \mathcal{L} \} \\
&= \left\{ -\mathcal{B}_n \left(-\frac{\partial \varphi}{\partial x} \right) + e^{\operatorname{ad} \varphi} \mathcal{B}_n, \tilde{\mathcal{L}} \right\} = \{ \tilde{\mathcal{B}}_n, \tilde{\mathcal{L}} \}.
\end{aligned}$$

Similarly,

$$\frac{\partial \tilde{\mathcal{L}}}{\partial s} = \left\{ \frac{\partial \varphi}{\partial s} + e^{\operatorname{ad} \varphi} \log \mathcal{P}, \tilde{\mathcal{L}} \right\} = \{ \log \tilde{\mathcal{P}}, \tilde{\mathcal{L}} \}.$$

Finally,

$$\begin{aligned}
\frac{\partial \log \tilde{\mathcal{P}}}{\partial t_n} &= \frac{\partial^2 \varphi}{\partial t_n \partial s} + \left\{ \frac{\partial \varphi}{\partial t_n}, e^{\operatorname{ad} \varphi} \log \mathcal{P} \right\} + e^{\operatorname{ad} \varphi} \left(\frac{\partial \mathcal{B}_n}{\partial s} - \{ \log \mathcal{P}, \mathcal{B}_n \} \right) \\
&= \left\{ \frac{\partial \varphi}{\partial t_n}, \log \tilde{\mathcal{P}} \right\} + \frac{\partial^2 \varphi}{\partial s \partial t_n} + \frac{\partial}{\partial s} \left(e^{\operatorname{ad} \varphi} \mathcal{B}_n \right) - \left\{ \frac{\partial \varphi}{\partial s}, e^{\operatorname{ad} \varphi} \mathcal{B}_n \right\} \\
&\quad - \left\{ e^{\operatorname{ad} \varphi} \log \mathcal{P}, e^{\operatorname{ad} \varphi} \mathcal{B}_n \right\} \\
&= \frac{\partial \tilde{\mathcal{B}}_n}{\partial s} - \left\{ \log \tilde{\mathcal{P}}, \tilde{\mathcal{B}}_n \right\}.
\end{aligned}$$

■

[2] The equation when $n = 1$ is a tautology.

Lemma 3. *Let $(\mathcal{L}(s), \mathcal{P}(s))$ be as in the lemma above. If $\mathcal{P}(k; s, t)$ has a root $\psi(s,t)$ as a polynomial of k, the system*

$$\frac{\partial \varphi(s,t)}{\partial t_n} = -\mathcal{B}_n(\psi(s,t); s, t), \qquad n \geq 1, \tag{A.13}$$

has a solution, unique up to a function of s.

In particular, (A.13) for $n = 1$ implies that $\psi(s,t) = -\frac{\partial \varphi}{\partial x}$. Hence the function satisfying (A.12) is obtained and we can gauge away the constant term of $\mathcal{P}(s)$ according to Lemma 2.

Proof. Let us factorize $\mathcal{P}(k; s, t)$ as $\mathcal{P}(k; s, t) = p_0(s,t) \prod_{i=1}^{N} (k - \psi_i(s,t))$. Differentiating by t_n, we have

$$\frac{\partial \log \mathcal{P}}{\partial t_n} = \sum_{i=1}^{N} \frac{-\frac{\partial \psi_i}{\partial t_n}}{k - \psi_i} + \frac{\partial}{\partial t_n} \log p_0. \tag{A.14}$$

The left hand side is, due to (3.5),

$$\begin{aligned}
\frac{\partial \log \mathcal{P}}{\partial t_n} &= \frac{\partial \mathcal{B}_n}{\partial s} - \frac{\partial \log \mathcal{P}}{\partial k}\frac{\partial \mathcal{B}_n}{\partial x} + \frac{\partial \log \mathcal{P}}{\partial x}\frac{\partial \mathcal{B}_n}{\partial k} \\
&= \frac{\partial \mathcal{B}_n}{\partial s} - \frac{\partial \mathcal{B}_n}{\partial x} \sum_{i=1}^{N} \frac{1}{k - \psi_i} + \frac{\partial \mathcal{B}_n}{\partial k} \sum_{i=1}^{N} \frac{-\frac{\partial \psi_i}{\partial x}}{k - \psi_i}.
\end{aligned}$$

Substituting this into (A.14), multiplying $\frac{\partial \mathcal{B}_m}{\partial k}$, subtracting the same equation with m and n interchanged, we have

$$\{\mathcal{B}_n, \mathcal{B}_m\}(\psi_i; s, t) = \frac{\partial \mathcal{B}_n}{\partial k}(\psi_i; s, t)\frac{\partial \psi_i(s,t)}{\partial t_m} - \frac{\partial \mathcal{B}_m}{\partial k}(\psi_i; s, t)\frac{\partial \psi_i(s,t)}{\partial t_n}$$

by comparing the residue at $k = \psi_i$. Using this equation, we can check the consistency of the system (A.13) as follows:

$$\begin{aligned}
&\frac{\partial}{\partial t_m}\mathcal{B}_n(\psi(s,t); s, t) - \frac{\partial}{\partial t_n}\mathcal{B}_m(\psi(s,t); s, t) \\
&\quad = \left(\frac{\partial \mathcal{B}_n}{\partial t_m} - \frac{\partial \mathcal{B}_m}{\partial t_n}\right)(\psi(s,t); s, t) \\
&\qquad + \frac{\partial \mathcal{B}_n}{\partial k}(\psi(s,t); s, t)\frac{\partial \psi(s,t)}{\partial t_m} - \frac{\partial \mathcal{B}_m}{\partial k}(\psi(s,t); s, t)\frac{\partial \psi(s,t)}{\partial t_n} \\
&\quad = \left(\frac{\partial \mathcal{B}_n}{\partial t_m} - \frac{\partial \mathcal{B}_m}{\partial t_n} + \{\mathcal{B}_n, \mathcal{B}_m\}\right)(\psi(s,t); s, t) = 0.
\end{aligned} \tag{A.15}$$

Therefore, the system (A.13) has a solution $\varphi(s,t)$ unique up to a function of s. ■

B Difference operator formalism

When the set $\{n_s\}_{s \in S}$ is equal to the whole set of integer numbers $\mathbb{Z}$, we can formulate the cmKP hierarchy in terms of difference operators. In fact in this case the cmKP hierarchy can be thought of as the "half" of the Toda lattice hierarchy of Ueno and Takasaki [23] whose dependence on half of the time variables are suppressed.

In this appendix, we first present the difference operator formalism of the cmKP hierarchy and then show in Appendix D that it is equivalent to the cmKP hierarchy in the main text.

We also introduce a gauge parameter α, $\alpha \neq 0$. (See [14] for the gauge parameter of the Toda lattice hierarchy.)

Let $\boldsymbol{L}$ be a difference operator of the form

$$\boldsymbol{L} = b_0(s,t)e^{\partial_s} + b_1(s,t) + b_2(s,t)e^{-\partial_s} + \cdots = \sum_{j=0}^{\infty} b_j(s,t)e^{(1-j)\partial_s}, \tag{B.1}$$

where $e^{k\partial_s}$ is the k-step shift operator $e^{k\partial_s}f(s) = f(k+s)$ and $t = (t_1, t_2, t_3, \dots)$ is a sequence of continuous variables. We assume that b_0 never vanishes: $b_0(s,t) \neq 0$.

We call the following system the *difference operator formalism of the cmKP hierarchy* with gauge parameter α:

$$\frac{\partial \boldsymbol{L}}{\partial t_n} = [\boldsymbol{B}_n, \boldsymbol{L}]. \tag{B.2}$$

Here $\boldsymbol{B}_n$ is a difference operator defined by

$$\boldsymbol{B}_n := (\boldsymbol{L}^n)_{\geq 0} - \alpha(\boldsymbol{L}^n)_0, \tag{B.3}$$

where $(\cdot)_{\geq 0}$ and $(\cdot)_0$ are projections of difference operators: for $A = \sum_j a_j(s)e^{j\partial_s}$,

$$A_{\geq 0} := \sum_{j\geq 0} a_j(s)e^{j\partial_s}, \qquad A_0 := a_0(s), \qquad A_{<0} := \sum_{j<0} a_j(s)e^{j\partial_s}. \tag{B.4}$$

By the same argument as in [23, § 1], the Lax representation (B.2) of the cmKP hierarchy is equivalent to the Zakharov–Shabat (or zero-curvature) representations:

$$[\partial_{t_m} - \boldsymbol{B}_m, \partial_{t_n} - \boldsymbol{B}_n] = 0, \tag{B.5}$$
$$[\partial_{t_m} - \boldsymbol{B}^c_m, \partial_{t_n} - \boldsymbol{B}^c_n] = 0, \tag{B.6}$$

where

$$\boldsymbol{B}^c_n = \boldsymbol{B}_n - \boldsymbol{L}^n = -(\boldsymbol{L}^n)_{<0} - \alpha(\boldsymbol{L}^n)_0. \tag{B.7}$$

Example 1. Dispersionless limit of the case $\alpha = 1/2$ is (3.34). It is related to the Löwner equation. See [22].

The proof of the following proposition is the same as those of Theorem 1.2 of [23].

Proposition 13. *For each solution of the cmKP hierarchy with gauge parameter α, there exists a difference operator $\hat{\boldsymbol{W}}$ of the following form with coefficients $w_0(s,t) = e^{-\alpha\varphi(s,t)}$ and $w_j(s,t)$:*

$$\hat{\boldsymbol{W}} = e^{-\alpha\varphi(s,t)} + w_1(s,t)e^{-\partial_s} + \cdots = \sum_{j=0}^{\infty} w_j(s,t)e^{-j\partial_s}, \tag{B.8}$$

satisfying the equations

$$\boldsymbol{L} = \hat{\boldsymbol{W}} e^{\partial_s} \hat{\boldsymbol{W}}^{-1}, \qquad \frac{\partial \hat{\boldsymbol{W}}}{\partial t_n} = \boldsymbol{B}^c_n \hat{\boldsymbol{W}}, \tag{B.9}$$

where the operators $\boldsymbol{B}^c_n$ are defined by (B.7).

We call $\hat{\boldsymbol{W}}$ the *wave matrix*.

Proposition 14. *(i) The operator $\hat{\boldsymbol{W}}$ in Proposition* 13 *and the function $\varphi(s,t)$ satisfy the following bilinear equation for arbitrary t and t':*

$$\boldsymbol{W}(t)\boldsymbol{W}(t')^{-1} = e^{(1-\alpha)(\varphi(s,t)-\varphi(s,t'))} + \text{(strictly upper triangular)}, \tag{B.10}$$

where $\boldsymbol{W}(t) = \hat{\boldsymbol{W}}(t)\exp\left(\sum\limits_{n=1}^{\infty} t_n e^{n\partial_s}\right)$ and the "(strictly upper triangular)*" part is an operator of the form $\sum\limits_{n>0} a_n(s)e^{n\partial_s}$.*

(ii) Conversely, if a function $\varphi(s,t)$ and an operator $\hat{\boldsymbol{W}}$ of the form (B.8) *satisfies the equation* (B.10)*, then the operator $\boldsymbol{L}$ defined by $\boldsymbol{L} = \hat{\boldsymbol{W}}e^{\partial_s}\hat{\boldsymbol{W}}^{-1}$ is a solution of the cmKP hierarchy* (B.2)*.*

The proof is essentially the same as the proof of Proposition 1.4 and Theorem 1.5 of [23].

By means of $\varphi(s,t)$ in Proposition 13, we can change the gauge as follows.

Proposition 15. *Let $\boldsymbol{L}$ be a solution of the cmKP hierarchy* (B.2) *with gauge parameter α and φ be the function defined in Proposition* 13*. Then the difference operator defined by*

$$\tilde{\boldsymbol{L}} := e^{(\alpha-\beta)\varphi(s,t)}\boldsymbol{L}e^{-(\alpha-\beta)\varphi(s,t)} \tag{B.11}$$

satisfies equation (B.2) *with gauge parameter β. Here β can be* 0*.*

Simple calculation is sufficient to check (B.2) for $\tilde{\boldsymbol{L}}$. We have only to note that the truncating operations $(\cdot)_{\geq 0}$, $(\cdot)_0$ (cf. (B.4)) commute with the adjoint operation $e^{(\alpha-\beta)\varphi(s,t)}(\cdot)e^{-(\alpha-\beta)\varphi(s,t)}$.

C The cmKP hierarchy with a gauge parameter

As is naturally expected from Section B, we can introduce a gauge parameter α ($\alpha \neq 0$) in the cmKP hierarchy when $\{n_s\}_{s\in S} = \mathbb{Z}$. In this case operator $L(s)$ has the form as in (2.1) but $P(s)$ has a 0-th order term:

$$L(s) = L(t;s) := \partial + u_1(s,t) + u_2(s,t)\partial^{-1} + u_3(s,t)\partial^{-2} + \cdots, \tag{C.1}$$

$$P(s) = P(s,t) := p_0(s,t)\partial + p_1(s,t), \tag{C.2}$$

which satisfy the condition

$$(1-\alpha)p_0(s,t)u_1(s,t) + \alpha p_1(s,t) = 0. \tag{C.3}$$

This condition is equivalent to saying that $B_1(s)$ defined later is equal to ∂. The cmKP hierarchy in Section 2.1 is recovered when $\alpha = 1$.

We introduce operators $P^{(n)}(s)$ which play the role of the n-step shift operators:

$$P^{(n)}(s) := \begin{cases} P(s+n-1)\cdots P(s+1)P(s), & n>0, \\ 1, & n=0, \\ P(s+n)^{-1}\cdots P(s-2)^{-1}P(s-1)^{-1}, & n<0. \end{cases} \tag{C.4}$$

The fundamental properties of $P^{(n)}(s)$ are the following:

Lemma 4. (i) *Any microdifferential operator Q has a unique expansion of the form*

$$Q = \sum_{\nu\in\mathbb{Z}} a_\nu P^{(\nu)}(s). \tag{C.5}$$

If Q is a n-th order differential operator, the sum is taken over $0 \leqq \nu \leqq n$.

(ii)

$$P^{(m)}(s+n)\,P^{(n)}(s) = P^{(m+n)}(s). \tag{C.6}$$

According to Lemma 4 (i), the operator $L(s)^n$ is expanded as:

$$L(s)^n = \sum_{j=0}^{\infty} b_j^{(n)}(s,t) P^{(n-j)}(s). \tag{C.7}$$

For example, since $L(s) = p_0(s,t)^{-1}P(s) - p_0(s,t)^{-1}p_1(s,t) + u_1(s,t) + \cdots$, we have

$$b_0^{(1)}(s,t) = p_0(s,t)^{-1}, \qquad b_1^{(1)}(s,t) = -p_0(s,t)^{-1}p_1(s,t) + u_1(s,t). \tag{C.8}$$

We define operator B_n by

$$\begin{aligned} B_n(s) &:= \left(L(s)^n\right)_{\geq 0} - \alpha b_n^{(n)}(s,t) \\ &= \sum_{j=0}^{n-1} b_j^{(n)}(s,t) P^{(n-j)}(s) + (1-\alpha) b_n^{(n)}(s,t). \end{aligned} \tag{C.9}$$

Here $(\cdot)_{\geq 0}$ is the projection of a microdifferential operator to the differential operator part. It is easy to see that condition (C.3) is equivalent to $B_1(s) = \partial$ and that $B_n(s) = (L(s)^n)_{>0}$ when $\alpha = 1$.

The definition of the cmKP hierarchy with a gauge parameter α is the same as the usual one, i.e., (2.3), (2.4) and (2.5).

Proposition 16. *The pair $(\tilde{L}(s), \tilde{P}(s))_{s\in\mathbb{Z}}$ of sequences of differential operators*

$$\begin{aligned} \tilde{L}(s) &:= e^{(\alpha-\beta)\varphi(s,t)} L(s) e^{-(\alpha-\beta)\varphi(s,t)}, \\ \tilde{P}(s) &:= e^{(\alpha-\beta)\varphi(s+1,t)} P(s) e^{-(\alpha-\beta)\varphi(s,t)} \end{aligned} \tag{C.10}$$

for $s \in \mathbb{Z}$ is a solution of the system (2.3), (2.4), (2.5) *with gauge parameter β. Here β can be* 0.

Proof. Let us check the condition (C.3) first. The operators $\tilde{L}(s)$ and $\tilde{P}(s)$ have the form

$$\begin{aligned} \tilde{L}(s) &= \partial + \tilde{u}_1(s) + \tilde{u}_2(s)\partial^{-1} + \cdots, \\ \tilde{P}(s) &= \tilde{p}_0(s)\partial + \tilde{p}_1(s), \end{aligned} \tag{C.11}$$

where

$$\begin{aligned} \tilde{u}_1(s) &= u_1(s) - (\alpha-\beta)\varphi'(s,t), \\ \tilde{p}_0(s) &= e^{(\alpha-\beta)\left(\varphi(s+1,t)-\varphi(s,t)\right)} p_0(s), \\ \tilde{p}_1(s) &= e^{(\alpha-\beta)\left(\varphi(s+1,t)-\varphi(s,t)\right)} (p_1(s) - (\alpha-\beta)p_0(s)\varphi'(s,t)). \end{aligned} \tag{C.12}$$

Here $\varphi'(s,t) = \partial\varphi(s,t)/\partial x = \partial\varphi(s,t)/\partial t_1 = b_1^{(1)}(s,t)$. Using the explicit form (C.8), we can check that $(1-\beta)\tilde{p}_0(s,t)\tilde{u}_1(s,t) + \beta\tilde{p}_1(s,t) = 0$.

The operator $P^{(n)}(s)$ defined by (C.4) transforms as

$$P^{(n)}(s) \mapsto \tilde{P}^{(n)}(s) := e^{(\alpha-\beta)\varphi(s+n,t)} P^{(n)}(s) e^{-(\alpha-\beta)\varphi(s,t)}$$

by the transformation (C.10). Hence $(\tilde{L}(s))^n$ is expanded as

$$\begin{aligned} (\tilde{L}(s))^n &= \sum_{j=0}^{\infty} \tilde{b}_j^{(n)}(s) \tilde{P}^{(n-j)}(s), \\ \tilde{b}_j^{(n)}(s) &:= e^{(\alpha-\beta)\left(\varphi(s,t)-\varphi(s+n-j,t)\right)} b_j^{(n)}(s). \end{aligned} \tag{C.13}$$

Particularly, $\tilde{b}_n^{(n)}(s) = b_n^{(n)}(s)$, which implies

$$\begin{aligned}\tilde{B}_n(s) &:= (\tilde{L}(s))^n_{\geq 0} - \beta(\tilde{L}(s))^n_0 \\ &= e^{(\alpha-\beta)\varphi(s,t)} B_n(s) e^{-(\alpha-\beta)\varphi(s,t)} + (\alpha - \beta) b_n^{(n)}(s).\end{aligned}$$

It remains to check (2.3) and (2.5) by straightforward computation. (Equation (2.4) is obvious from the definition of $\tilde{L}(s)$ and $\tilde{P}(s)$, (C.10).) ■

Corollary 4. *The cmKP hierarchies with different gauge parameters are equivalent through the transformation* (C.10)

If $\beta = 0$ in Proposition 16, the resulting system is the mKP hierarchy. Proposition 2 is the case when $(\alpha, \beta) = (1, 0)$. See Section 2.2.

D Equivalence of two formulations

The two formalisms of the cmKP hierarchy discussed in Appendix B and Appendix C are equivalent. The proof is almost straightforward computation but lengthy.

Rewriting the difference operator formalism to the differential operator formalism is essentially the same as the procedure described in § 1.2 of [23], where the KP hierarchy is embedded in the Toda lattice hierarchy. Assume that a solution $\boldsymbol{L} = \sum_{j=0}^{\infty} b_j(s,t) e^{(1-j)\partial_s}$ of the system (B.2) is given. The idea is to interpret the operator $\partial_{t_1} - \boldsymbol{B}_1$ as the operator $b_0(s,t)(P(s) - e^{\partial_s})$. Namely, we define the operator P by

$$P(s) := b_0(s,t)^{-1}(\partial - (1-\alpha) b_1(s,t)), \tag{D.1}$$

where t_1 is replaced by $t_1 + x$. (We do not write the dependence on x explicitly.) Using the operator $P^{(n)}(s)$ defined by (C.4), we define the L operator by

$$\begin{aligned}L(s) &:= \sum_{j=0}^{\infty} b_j(s,t) P^{(1-j)}(s) \\ &= b_0(s,t)P(s) + b_1(s,t) + b_2(s,t)P^{(-1)}(s) + \cdots \\ &= \partial + \alpha b_1(s,t) + \cdots .\end{aligned} \tag{D.2}$$

The condition (C.3) is automatically satisfied, thus we have $B_1(s) = \partial$.

First we prove (2.4). The left hand side of (2.4) is

$$L(s+1)P(s) = \sum_{j=0}^{\infty} b_j(s+1) P^{(2-j)}(s), \tag{D.3}$$

by the definition (C.4). The right hand side of (2.4) is

$$\begin{aligned}P(s)L(s) &= P(s) \sum_{j=0}^{\infty} b_j(s) P^{(1-j)}(s) \\ &= b_0(s)^{-1} \sum_{j=0}^{\infty} \left(\frac{\partial b_j(s)}{\partial x} + b_j(s)\partial - (1-\alpha) b_j(s) b_1(s) \right) P^{(1-j)}(s).\end{aligned} \tag{D.4}$$

Since $\partial b_j(s)/\partial x = \partial b_j(s)/\partial t_1$, the Lax equation (B.2) with $n = 1$ gives information on $\partial b_j(s)/\partial x$. Comparing the coefficients of $e^{(1-j)\partial_s}$ in (B.2), we have

$$\begin{aligned}\frac{\partial b_j(s)}{\partial x} &= b_0(s)b_{j+1}(s+1) - b_0(s-j)b_{j+1}(s)\\ &\quad + (1-\alpha)b_1(s)b_j(s) - (1-\alpha)b_1(s+1-j)b_j(s).\end{aligned} \tag{D.5}$$

Substituting it into (D.4) and using the property (C.6) of $P^{(j)}(s)$, we have

$$P(s)L(s) = \sum_{j=0}^{\infty} b_j(s+1)P^{(2-j)}(s), \tag{D.6}$$

which, together with (D.3), proves (2.4). The following formula is a consequence of (2.4):

$$L(s+m)P^{(m)}(s) = P^{(m)}(s)L(s). \tag{D.7}$$

The proof of (2.5) is almost the same. Note that if we expand $\boldsymbol{L}^n$ as

$$\boldsymbol{L}^n = \sum_{j=0}^{\infty} b_j^{(n)}(s)e^{(n-j)\partial_s}, \tag{D.8}$$

operator $L(s)^n$ is expanded as (C.7) with the same coefficients $b_j^{(n)}(s)$ as in (D.8). This is proved by induction with the help of (2.4) proved above and (C.6).

Hence the coefficients $b_j^{(n)}(s)$ in (C.9) are the same as in (D.8). Using this fact and (B.5) $m = 1$, we can prove (2.6), i.e., (2.5). We omit details which are similar to the above proof of (2.4). The formula

$$(\partial_{t_n} - B_n(s+m))P^{(m)}(s) = P^{(m)}(s)(\partial_{t_n} - B_n(s)), \tag{D.9}$$

$$\text{i.e.,}\quad \frac{\partial P^{(m)}(s)}{\partial t_n} = B_n(s+m)P^{(m)}(s) - P^{(m)}(s)B_n(s), \tag{D.10}$$

derived from (2.5) shall be used in the following step.

Let us proceed to the proof of the Lax equation (2.3). Its left hand side is

$$\begin{aligned}\frac{\partial L(s)}{\partial t_n} &= \sum_{j=0}^{\infty} \frac{\partial b_j(s)}{\partial t_n} P^{(1-j)}(s)\\ &\quad + \sum_{j=0}^{\infty}\sum_{k=0}^{n} b_j(s)b_k^{(n)}(s+1-j)P^{(n+1-j-k)}(s) - L(s)B_n(s).\end{aligned} \tag{D.11}$$

(We used (D.10) and (C.6).) On the other hand, (D.7) and (C.6) imply

$$[B_n(s), L(s)] = \sum_{j=0}^{n}\sum_{k=0}^{\infty} b_j^{(n)}(s)b_k(s+n-j)P^{(n+1-j-k)}(s) - L(s)B_n(s). \tag{D.12}$$

In order to prove that (D.11) and (D.12) are equal, we have only to show

$$\frac{\partial b_l(s)}{\partial t_n} = \sum_{\substack{0\le j\le n, 0\le k\\ j+k=l+n}} \Big(b_j^{(n)}(s)b_k(s+n-j) - b_k(s)b_j^{(n)}(s+1-k)\Big),$$

which is nothing but the coefficient of $e^{(1-l)\partial_s}$ of (B.2). Thus we have proved that a solution of the difference operator equations (B.2) gives a solution of the system (2.3), (2.4) and (2.5).

Conversely, when a solution $(L(s), P(s))_{s\in\mathbb{Z}}$ of the system (2.3), (2.4), (2.5) is given and $L(s)$ is expanded as

$$L(s) = \sum_{j=0}^{\infty} b_j(s) P^{(1-j)}(s), \tag{D.13}$$

then the $\boldsymbol{L}$ operator defined by (B.1) satisfies the system (B.2). Note that, due to condition (C.3), $B_1(s) = \partial$, which means that t_1 and x always appear in the form $t_1 + x$. Hence we can eliminate x by just replacing $t_1 + x$ by t_1. The Lax equation (B.2) for the difference operator is proved by tracing back the above proof of (2.3).

Remark 5. This correspondence holds also for the case $\alpha = 0$. If $p_0(s)$ is normalized to 1, the system (2.3), (2.4), (2.5) is the mKP hierarchy in [2, 17]. Its equivalence to the system (B.2)[3] was proved in [2] by a different method.

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