Generalized Ellipsoidal and Sphero-Conal Harmonics*

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Abstract. Classical ellipsoidal and sphero-conal harmonics are polynomial solutions of the Laplace equation that can be expressed in terms of Lamé polynomials. Generalized ellipsoidal and sphero-conal harmonics are polynomial solutions of the more general Dunkl equation that can be expressed in terms of Stieltjes polynomials. Niven's formula connecting ellipsoidal and sphero-conal harmonics is generalized. Moreover, generalized ellipsoidal harmonics are applied to solve the Dirichlet problem for Dunkl's equation on ellipsoids.

Key words: generalized ellipsoidal harmonic; Stieltjes polynomials; Dunkl equation; Niven formula

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1 Introduction

The theory of ellipsoidal and sphero-conal harmonics is a beautiful achievement of classical mathematics. It is not by accident that the well-known treatise "A Course in Modern Analysis" by Whittaker and Watson [20] culminates in the final chapter "Ellipsoidal Harmonics and Lamé's Equation".

An ellipsoidal harmonic is a polynomial $u(x_0, x_1, \ldots, x_k)$ in $k+1$ variables $x_0, x_1, \ldots, x_k$ which satisfies the Laplace equation

$$\Delta u := \sum_{j=0}^{k} \frac{\partial^2 u}{\partial x_j^2} = 0 \tag{1.1}$$

and assumes the product form

$$u(x_0, x_1, \ldots, x_k) = E(t_0)E(t_1) \cdots E(t_k) \tag{1.2}$$

in ellipsoidal coordinates $(t_0, t_1, \ldots, t_k)$ with E denoting a Lamé quasi-polynomial.

A sphero-conal harmonic is a polynomial $u(x_0, x_1, \ldots, x_k)$ which satisfies the Laplace equation (1.1) and assumes the product form

$$u(x_0, x_1, \ldots, x_k) = r^m E(s_1)E(s_2) \cdots E(s_k) \tag{1.3}$$

in sphero-conal coordinates $(r, s_1, s_2, \ldots, s_k)$. Again, E is a Lamé quasi-polynomial.

In most of the literature, for example, in the books by Hobson [6] and Whittaker and Watson [20], ellipsoidal and sphero-conal harmonics are treated as polynomials of only three variables. The generalization to any number of variables is straight-forward. Since we plan to work

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue "Integrable Systems and Related Topics". The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html in arbitrary dimension we will employ ellipsoidal and sphero-conal coordinates in algebraic form; see Sections 3 and 4. In $\mathbb{R}^3$ we may uniformize these coordinates by using Jacobian elliptic functions. Sphero-conal harmonics are special instances of spherical harmonics. Spherical harmonics in any dimension can be found in books by Hochstadt [7] and Müller [15].

We will extend the theory of ellipsoidal and sphero-conal harmonics by replacing the Laplace equation by the equation

$$\Delta_{\boldsymbol{\alpha}} u := \sum_{j=0}^{k} \mathcal{D}_j^2 u = 0, \qquad \boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_k) \tag{1.4}$$

introduced by Dunkl [2]. In (1.4) we use the generalized partial derivatives

$$\mathcal{D}_j u(\boldsymbol{x}) := \frac{\partial}{\partial x_j} u(\boldsymbol{x}) + \alpha_j \frac{u(\boldsymbol{x}) - u(\sigma_j \boldsymbol{x})}{x_j}, \qquad \boldsymbol{x} = (x_0, x_1, \ldots, x_k), \tag{1.5}$$

where σ_j is the reflection at the jth coordinate plane:

$$\sigma_j(x_0, x_1, \ldots, x_k) = (x_0, x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_k).$$

Equation (1.4) contains real parameters $\alpha_0, \alpha_1, \ldots, \alpha_k$. If $\alpha_j = 0$ for all j then the equation reduces to the Laplace equation.

A *generalized ellipsoidal harmonic* is a polynomial $u(x_0, x_1, \ldots, x_k)$ which satisfies Dunkl's equation (1.4) and assumes the product form (1.2) in ellipsoidal coordinates but with E now denoting a Stieltjes quasi-polynomial. Generalized ellipsoidal harmonics will be treated in Section 3 while Stieltjes quasi-polynomials are introduced in Section 2.

A *generalized sphero-conal harmonic* is a polynomial $u(x_0, x_1, \ldots, x_k)$ which satisfies Dunkl's equation (1.4) and assumes the product form (1.3) in sphero-conal coordinates. Again, E is a Stieltjes quasi-polynomial. Generalized sphero-conal harmonics will be considered in Section 4.

It is very pleasing to see Stieltjes quasi-polynomials taking over the role of Lamé quasi-polynomials. Stieltjes polynomials have been considered for a long time but they did not appear in the context of separated solutions of the Laplace equation. Therefore, our paper shows how Stieltjes polynomials become part of the theory of "Special Functions".

It is quite remarkable that all the known results for classical ellipsoidal and sphero-conal harmonics carry over to their generalizations. In Section 5 we generalize formulas due to Hobson [6, Chapter 4]. In Section 6, as a consequence, we prove a generalization of Niven's formula [20, Chapter 23] connecting ellipsoidal and sphero-conal harmonics. In Section 7 we apply generalized ellipsoidal harmonics in order to solve a Dirichlet problem for (1.4) on ellipsoids. This generalizes the classical result that ellipsoidal harmonics may be used to find the harmonic function which has prescribed values on the boundary of an ellipsoid. Finally, we give some examples in Section 8.

We point out that parts of Section 4 overlap with the author's paper [19]. The contents of the present paper are also related to the book by Dunkl and Xu [4], and the papers by Liamba [14] and Xu [21], however, these works do not involve Stieltjes polynomials. We also refer to papers by Kalnins and Miller [8, 9, 10]. The paper [10] addresses Niven's formula from a different perspective. Kutznetsov [13] and Kutznetsov and Komarov [12] have also worked in related areas. Kutznetsov jointly with Sleeman wrote the chapter on Heun functions for the Digital Library of Mathematical Functions. Stieltjes polynomials appear in this chapter.

2 Stieltjes quasi-polynomials

We consider the Fuchsian differential equation

$$\prod_{j=0}^{k}(t-a_j)\left[v''+\sum_{j=0}^{k}\frac{\alpha_j+\frac{1}{2}}{t-a_j}v'\right]+\left[-\frac{1}{2}\sum_{j=0}^{k}\frac{p_j\alpha_j A_j}{t-a_j}+\sum_{i=0}^{k-1}\lambda_i t^i\right]v=0 \tag{2.1}$$

for the function $v(t)$ where the prime denotes differentiation with respect to t. This differential equation contains four sets of real parameters:

$$a_0<a_1<\cdots<a_k, \tag{2.2}$$
$$\alpha_0,\alpha_1,\ldots,\alpha_k\in(-\tfrac{1}{2},\infty), \tag{2.3}$$
$$p_0,p_1,\ldots,p_k\in\{0,1\}, \tag{2.4}$$
$$\lambda_0,\lambda_1,\ldots\lambda_{k-1}\in\mathbb{R}, \tag{2.5}$$

and A_j is an abbreviation:

$$A_j:=\prod_{\substack{i=0\\ i\neq j}}^{k}(a_j-a_i). \tag{2.6}$$

Usually, the first three sets of parameters are given while the λ's play the role of eigenvalue parameters.

Equation (2.1) has regular singularities at infinity and at each a_j, $j=0,1,\ldots,k$. The exponents at a_j are $\nu_j=\frac{p_j}{2}$ and $\mu_j=\frac{1-p_j}{2}-\alpha_j$. If ν_{k+1}, μ_{k+1} denote the exponents at infinity then

$$\lambda_{k-1}=\nu_{k+1}\mu_{k+1} \tag{2.7}$$

and

$$\sum_{j=0}^{k+1}(\nu_j+\mu_j)=k. \tag{2.8}$$

The accessory parameters $\lambda_0,\lambda_1,\ldots,\lambda_{k-2}$ are unrelated to the exponents.

The following result defines *Stieltjes quasi-polynomials* $E_{\boldsymbol{n},\boldsymbol{p}}$. Let parameter sets (2.2) and (2.3) be given. For every multi-index $\boldsymbol{n}=(n_1,n_2,\ldots,n_k)\in\mathbb{N}^k$ of nonnegative integers and $\boldsymbol{p}=(p_0,p_1,\ldots,p_k)\in\{0,1\}^k$ there exist uniquely determined values of the parameters $\lambda_0,\ldots,\lambda_{k-1}$ such that (2.1) admits a solution of the form

$$E_{\boldsymbol{n},\boldsymbol{p}}(t)=\left(\prod_{j=0}^{k}|t-a_j|^{p_j/2}\right)\tilde{E}_{\boldsymbol{n},\boldsymbol{p}}(t),\qquad t\in\mathbb{R}, \tag{2.9}$$

where $\tilde{E}_{\boldsymbol{n},\boldsymbol{p}}$ is a polynomial with exactly n_j zeros in the open interval (a_{j-1},a_j) for each $j=1,\ldots,k$. The polynomial $\tilde{E}_{\boldsymbol{n},\boldsymbol{p}}(t)$ is uniquely determined up to a constant factor and has the degree $|\boldsymbol{n}|=n_1+\cdots+n_k$. We normalize $\tilde{E}_{\boldsymbol{n},\boldsymbol{p}}$ so that its leading coefficient is unity. Then we may write $E_{\boldsymbol{n},\boldsymbol{p}}$ in the form

$$E_{\boldsymbol{n},\boldsymbol{p}}(t)=\prod_{j=0}^{k}|t-a_j|^{p_j/2}\prod_{\ell=1}^{|\boldsymbol{n}|}(t-\theta_\ell), \tag{2.10}$$

where

$$\theta_1 < \theta_2 < \cdots < \theta_{|\boldsymbol{n}|}.$$

Then $\theta_1, \ldots \theta_{n_1}$ lie in (a_0, a_1), $\theta_{n_1+1}, \ldots, \theta_{n_1+n_2}$ lie in (a_1, a_2) and so on.

If $\boldsymbol{p} = \boldsymbol{0}$ then $E_{\boldsymbol{n},\boldsymbol{0}}$ is a polynomial introduced by Stieltjes [17] whose work was influenced by Heine [5, Part III]. A proof of existence and uniqueness of the polynomials $E_{\boldsymbol{n},\boldsymbol{0}}$ can be found in Szegö [18, Section 6.8]. For general $\boldsymbol{p}$ a computation shows that $\tilde{E}_{\boldsymbol{n},\boldsymbol{p}}$ is the Stieltjes polynomial $E_{\boldsymbol{n},\boldsymbol{0}}$ with α_j replaced by $\alpha_j + p_j$. Therefore, the proof of existence and uniqueness in the general case can be reduced to the special case $\boldsymbol{p} = \boldsymbol{0}$.

The value of λ_{k-1} associated with $E_{\boldsymbol{n},\boldsymbol{p}}$ can be computed. One of the exponents at infinity must be

$$\nu_{k+1} = |\boldsymbol{n}| + \tfrac{1}{2}|\boldsymbol{p}|, \qquad |\boldsymbol{p}| := \sum_{j=0}^{k} p_j.$$

Using (2.7), (2.8), we obtain

$$\lambda_{k-1} = -\tfrac{1}{2}m\left(\tfrac{1}{2}m + |\boldsymbol{\alpha}| + \tfrac{k-1}{2}\right), \qquad m := 2|\boldsymbol{n}| + |\boldsymbol{p}|, \qquad |\boldsymbol{\alpha}| := \sum_{j=0}^{k} \alpha_j. \tag{2.11}$$

No formulas are known for the corresponding values of the accessory parameters.

If $\alpha_j = 0$ for all j then Stieltjes quasi-polynomials reduce to Lamé quasi-polynomials in arbitrary dimension. If we work in $\mathbb{R}^3$ there are eight possible choices of the parameters (2.4) giving us the familiar eight types of classical Lamé quasi-polynomials; see Arscott [1].

3 Generalized ellipsoidal harmonics

We say that a function $u : \mathbb{R}^{k+1} \to \mathbb{R}$ has parity $\boldsymbol{p} = (p_0, \ldots, p_k) \in \{0,1\}^{k+1}$ if

$$u(\boldsymbol{x}) - u(\sigma_j \boldsymbol{x}) = 2p_j u(\boldsymbol{x}) \qquad \text{for} \quad j = 0, 1, \ldots, k.$$

Equation (1.4) can be written in the form

$$\Delta u(\boldsymbol{x}) + \sum_{j=0}^{k} \frac{2\alpha_j}{x_j}\frac{\partial}{\partial x_j}u(\boldsymbol{x}) - \sum_{j=0}^{k} \frac{\alpha_j}{x_j^2}(u(\boldsymbol{x}) - u(\sigma_j \boldsymbol{x})) = 0. \tag{3.1}$$

If u has parity $\boldsymbol{p}$ then (3.1) becomes the partial differential equation

$$\Delta u(\boldsymbol{x}) + \sum_{j=0}^{k} \frac{2\alpha_j}{x_j}\frac{\partial}{\partial x_j}u(\boldsymbol{x}) - \sum_{j=0}^{k} \frac{2p_j\alpha_j}{x_j^2}u(\boldsymbol{x}) = 0. \tag{3.2}$$

In order to introduce ellipsoidal coordinates, fix the parameters (2.2). For every $(x_0, \ldots, x_k)$ in the positive cone of $\mathbb{R}^{k+1}$

$$x_0 > 0, \ldots, x_k > 0, \tag{3.3}$$

its ellipsoidal coordinates $t_0, t_1, \ldots, t_k$ lie in the intervals

$$a_k < t_0 < \infty, \qquad a_{i-1} < t_i < a_i, \qquad i = 1, \ldots, k, \tag{3.4}$$

and satisfy

$$\sum_{j=0}^{k} \frac{x_j^2}{t_i - a_j} = 1 \qquad \text{for} \quad i = 0, 1, \ldots, k. \tag{3.5}$$

Conversely, for given t_i in the intervals (3.4), we have

$$x_j^2 = \frac{\prod_{i=0}^{k} (t_i - a_j)}{\prod_{\substack{i=0 \\ i \neq j}}^{k} (a_i - a_j)}. \tag{3.6}$$

These coordinates provide a bijective mapping between the positive cone (3.3) and the cube (3.4).

We now transform the partial differential equation (3.2) for functions $u(\boldsymbol{x})$ defined on the cone (3.3) to ellipsoidal coordinates, and then we apply the method of separation of variables. We obtain $k+1$ times the Fuchsian equation (2.1) coupled by the separation constants $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$. We do not carry out the details of these known calculations. A good reference is Schmidt and Wolf [16]. Therefore, if $v_j(t_j)$, $j = 0, 1 \ldots, k$, are solutions of (2.1) with t_j ranging in the intervals (3.4) then the function

$$u(x_0, \ldots, x_k) = v_0(t_0) \cdots v_k(t_k) \tag{3.7}$$

satisfies (3.2). Of course, the values of the parameter sets (2.2)–(2.5) must be the same in each equation (2.1).

As a special case choose v_j as the Stieltjes quasi-polynomial $E_{\boldsymbol{n},\boldsymbol{p}}$ for each j. Then we know that

$$F_{\boldsymbol{n},\boldsymbol{p}}(x_0, x_1, \ldots, x_k) := E_{\boldsymbol{n},\boldsymbol{p}}(t_0) E_{\boldsymbol{n},\boldsymbol{p}}(t_1) \cdots E_{\boldsymbol{n},\boldsymbol{p}}(t_k) \tag{3.8}$$

solves (3.2). This function $F_{\boldsymbol{n},\boldsymbol{p}}$ is our generalized ellipsoidal harmonic.

Theorem 1. *The generalized ellipsoidal harmonic $F_{\boldsymbol{n},\boldsymbol{p}}$ is a polynomial in $x_0, x_1, \ldots, x_k$ which satisfies Dunkl's equation* (1.4)*. It is of total degree $2|\boldsymbol{n}| + |\boldsymbol{p}|$ and has parity $\boldsymbol{p}$.*

Proof. If $t_0, \ldots, t_k$ denote ellipsoidal coordinates of $x_0, \ldots, x_k$, then

$$\prod_{j=0}^{k} (t_j - \theta) = \left(\prod_{i=0}^{k} (a_i - \theta) \right) \left(1 - \sum_{j=0}^{k} \frac{x_j^2}{\theta - a_j} \right) \tag{3.9}$$

for every θ different from each a_j. In fact, both sides of (3.9) are polynomials in θ of degree $k+1$ with leading coefficient $(-1)^{k+1}$. Moreover, both sides of the equation vanish at $\theta = t_0, t_1, \ldots, t_k$ by definition (3.5). So equation (3.9) follows.

By (2.10), (3.6), (3.8) and (3.9), the function $F_{\boldsymbol{n},\boldsymbol{p}}$ can be written as

$$F_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{x}) = c_{\boldsymbol{n},\boldsymbol{p}} \boldsymbol{x}^{\boldsymbol{p}} \prod_{\ell=1}^{|\boldsymbol{n}|} \left(\sum_{j=0}^{k} \frac{x_j^2}{\theta_\ell - a_j} - 1 \right), \tag{3.10}$$

where

$$\boldsymbol{x}^{\boldsymbol{p}} := x_0^{p_0} \cdots x_k^{p_k} \qquad \text{for} \quad \boldsymbol{x} = (x_0, \ldots, x_k), \quad \boldsymbol{p} = (p_0, \ldots, p_k),$$

and $c_{\boldsymbol{n},\boldsymbol{p}}$ is the constant

$$c_{\boldsymbol{n},\boldsymbol{p}} := (-1)^{|\boldsymbol{n}|}\left(\prod_{j=0}^{k}|A_j|^{p_j/2}\right)\left(\prod_{\ell=1}^{|\boldsymbol{n}|}\prod_{i=0}^{k}(a_i-\theta_\ell)\right) \tag{3.11}$$

with A_j according to (2.6). This shows that $F_{\boldsymbol{n},\boldsymbol{p}}$ is a polynomial of total degree $2|\boldsymbol{n}|+|\boldsymbol{p}|$. We know that $F_{\boldsymbol{n},\boldsymbol{p}}$ solves (3.2) on the cone (3.3) and since it has parity $\boldsymbol{p}$ it solves (1.4) on $\mathbb{R}^{k+1}$. ■

4 Generalized sphero-conal harmonics

In order to introduce sphero-conal coordinates, fix the parameters (2.2). Let $(x_0, x_1, \dots, x_k)$ be in the positive cone (3.3) of $\mathbb{R}^{k+1}$. Its sphero-conal coordinates $r, s_1, \dots, s_k$ are determined in the intervals

$$r > 0, \qquad a_{i-1} < s_i < a_i, \qquad i = 1, \dots, k \tag{4.1}$$

by the equations

$$r^2 = \sum_{j=0}^{k} x_j^2 \tag{4.2}$$

and

$$\sum_{j=0}^{k} \frac{x_j^2}{s_i - a_j} = 0 \qquad \text{for} \quad i = 1, \dots, k. \tag{4.3}$$

This defines a bijective map from the positive cone in $\mathbb{R}^{k+1}$ to the set of points $(r, s_1, \dots, s_k)$ satisfying (4.1). The inverse map is given by

$$x_j^2 = r^2 \frac{\prod_{i=1}^{k}(s_i - a_j)}{\prod_{\substack{i=0\\ i\neq j}}^{k}(a_i - a_j)}. \tag{4.4}$$

We now transform the partial differential equation (3.2) for functions $u(x_0, x_1, \dots, x_k)$ defined on the cone (3.3) to sphero-conal coordinates and then we apply the method of separation of variables [16]. For the variable r we obtain the Euler equation

$$v_0'' + \frac{k+2|\alpha|}{r} v_0' + \frac{4\lambda_{k-1}}{r^2} v_0 = 0 \tag{4.5}$$

while for the variables $s_1, s_2, \dots, s_k$ we obtain the Fuchsian equation (2.1). More precisely, if $\lambda_0, \dots, \lambda_{k-1}$ are any given numbers (separation constants), if $v_0(r)$, $r > 0$, solves (4.5) and $v_i(s_i)$, $a_{i-1} < s_i < a_i$, solve (2.1) for each $i = 1, \dots, k$, then

$$u(x_0, x_1, \dots, x_k) = v_0(r) v_1(s_1) v_2(s_2) \cdots v_k(s_k)$$

solves (3.2).

Let $E_{\boldsymbol{n},\boldsymbol{p}}$ be a Stieltjes quasi-polynomial. It follows from (2.11) that $v_0(r) = r^m$ is a solution of (4.5), where $m := 2|\boldsymbol{n}| + |\boldsymbol{p}|$. Therefore,

$$G_{\boldsymbol{n},\boldsymbol{p}}(x_0, x_1, \dots, x_k) := r^m E_{\boldsymbol{n},\boldsymbol{p}}(s_1) E_{\boldsymbol{n},\boldsymbol{p}}(s_2) \cdots E_{\boldsymbol{n},\boldsymbol{p}}(s_k) \tag{4.6}$$

is a solution of (3.2). This function $G_{\boldsymbol{n},\boldsymbol{p}}$ is our generalized sphero-conal harmonic.

Theorem 2. *The generalized sphero-conal harmonic $G_{\boldsymbol{n},\boldsymbol{p}}$ is a polynomial in $x_0, x_1, \dots, x_k$, it is homogeneous of degree $2|\boldsymbol{n}| + |\boldsymbol{p}|$, it has parity $\boldsymbol{p}$ and it solves Dunkl's equation* (1.4).

Proof. Let $(x_0, \dots, x_k)$ be a point with $x_j > 0$ for all j, and let $(r, s_1, \dots, s_k)$ denote its corresponding sphero-conal coordinates. We claim that

$$r^2(s_1 - \theta) \dots (s_k - \theta) = \left(\prod_{i=0}^{k} (a_i - \theta) \right) \sum_{j=0}^{k} \frac{x_j^2}{a_j - \theta} \tag{4.7}$$

for all θ which are different from each a_j. Both sides of (4.7) are polynomials in θ of degree k with leading coefficient $(-1)^k r^2$. Moreover, both sides vanish at $\theta = s_1, \dots, s_k$ because of definition (4.3). Equation (4.7) is established.

We write the Stieltjes quasi-polynomial $E_{\boldsymbol{n},\boldsymbol{p}}$ in the form (2.10). Using (4.4), (4.6) and (4.7), we obtain

$$G_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{x}) = c_{\boldsymbol{n},\boldsymbol{p}} \boldsymbol{x}^{\boldsymbol{p}} \prod_{\ell=1}^{|\boldsymbol{n}|} \sum_{j=0}^{k} \frac{x_j^2}{\theta_\ell - a_j}, \tag{4.8}$$

where $c_{\boldsymbol{n},\boldsymbol{p}}$ is given by (3.11). This shows that $G_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{x})$ is a polynomial in $x_0, x_1, \dots, x_k$, it is homogeneous of degree $2|\boldsymbol{n}| + |\boldsymbol{p}|$, and it has parity $\boldsymbol{p}$. We know that $G_{\boldsymbol{n},\boldsymbol{p}}$ solves (3.2) on the cone (3.3) and since it has parity $\boldsymbol{p}$ it solves (1.4) on $\mathbb{R}^{k+1}$. ■

A *generalized spherical harmonic* is a homogeneous polynomial u in the variables $x_0, x_1, \dots, x_k$ which solves Dunkl's equation (1.4). For a given set of parameters (2.3) we let $\mathcal{H}_m$ denote the finite dimensional linear space of all generalized spherical harmonics of degree m. If $\alpha_j = 0$ for each j then we obtain the classical spherical harmonics.

On the k-dimensional unit sphere $\boldsymbol{S}^k$ we introduce the inner product

$$\langle f, g \rangle_w := \int_{\boldsymbol{S}^k} w(\boldsymbol{x}) f(\boldsymbol{x}) g(\boldsymbol{x}) \, dS(\boldsymbol{x}), \tag{4.9}$$

and norm

$$\|f\|_w := \langle f, f \rangle_w^{1/2}, \tag{4.10}$$

where the weight function w is defined by

$$w(x_0, x_1, \dots, x_k) := |x_0|^{2\alpha_0} |x_1|^{2\alpha_1} \cdots |x_k|^{2\alpha_k}. \tag{4.11}$$

The surface measure on the sphere is normalized so that $\int_{\boldsymbol{S}^k} dS(\boldsymbol{x})$ equals the surface area of the sphere $\boldsymbol{S}^k$. The condition $\alpha_j > -\frac{1}{2}$ ensures that $\langle f, g \rangle_w$ is well-defined if f and g are continuous on $\boldsymbol{S}^k$.

Theorem 3. *Let $m \in \mathbb{N}$. The system of all generalized sphero-conal harmonics $G_{\boldsymbol{n},\boldsymbol{p}}$ of degree m forms an orthogonal basis for $\mathcal{H}_m$ with respect to the inner product* (4.9).

Proof. We consider the system of all sphero-conal harmonics $G_{\boldsymbol{n},\boldsymbol{p}}$, where $\boldsymbol{n}, \boldsymbol{p}$ satisfy $m = 2|\boldsymbol{n}| + |\boldsymbol{p}|$. By Theorem 2, $G_{\boldsymbol{n},\boldsymbol{p}}$ belongs to $\mathcal{H}_m$. The dimension of the linear space of generalized spherical harmonics of degree m which have parity $\boldsymbol{p}$ is

$$\binom{\frac{1}{2}m - \frac{1}{2}|\boldsymbol{p}| + k - 1}{k - 1} \tag{4.12}$$

if $m - |\boldsymbol{p}|$ is a nonnegative even integer and zero otherwise. This can be proved as in Hochstadt [7, p. 170] or it follows from Dunkl [3, Proposition 2.6] where a basis of $\mathcal{H}_m$ in terms of Jacobi polynomials is constructed. The dimension (4.12) agrees with the number of multi-indices $\boldsymbol{n} = (n_1, n_2, \dots, n_k)$ for which $m = 2|\boldsymbol{n}| + |\boldsymbol{p}|$. We conclude that the number of pairs $\boldsymbol{n}, \boldsymbol{p}$ with $m = 2|\boldsymbol{n}| + |\boldsymbol{p}|$ agrees with the dimension of $\mathcal{H}_m$. Therefore, in order to complete the proof of the theorem, we have to show that $G_{\boldsymbol{n},\boldsymbol{p}}$ is orthogonal to $G_{\boldsymbol{n}',\boldsymbol{p}'}$ provided $(\boldsymbol{n}, \boldsymbol{p}) \neq (\boldsymbol{n}', \boldsymbol{p}')$. If $\boldsymbol{p} \neq \boldsymbol{p}'$ this is clear because the weight function (4.11) is an even function. If $\boldsymbol{p} = \boldsymbol{0}$ and $\boldsymbol{n} \neq \boldsymbol{n}'$ then orthogonality was shown in [19, Theorem 3.3]. The proof of orthogonality in the remaining cases is analogous and is omitted. ■

Extending the method of proof of [19, Theorem 3.3] we also establish the following theorem.

Theorem 4. *The system of all generalized sphero-conal harmonics $G_{\boldsymbol{n},\boldsymbol{p}}$, $\boldsymbol{n} \in \mathbb{N}^k$, $\boldsymbol{p} \in \{0, 1\}^{k+1}$, when properly normalized, forms an orthonormal basis of $L^2_w(\boldsymbol{S}^k)$.*

No explicit formula is known for the norm of $G_{\boldsymbol{n},\boldsymbol{p}}$ in $L^2_w(\boldsymbol{S}^k)$. However, the norm of a polynomial can be computed using the formula

$$\int_{\boldsymbol{S}^k} |x_0|^{2\beta_0 - 1} |x_1|^{2\beta_1 - 1} \cdots |x_k|^{2\beta_k - 1}\, dS(\boldsymbol{x}) = \frac{2 \prod_{j=0}^{k} \Gamma(\beta_j)}{\Gamma(\beta_0 + \beta_1 + \cdots + \beta_k)} \tag{4.13}$$

which holds whenever $\beta_j > 0$, $j = 0, 1, \dots, k$.

In general, ellipsoidal harmonics are not homogeneous polynomials so they are not spherical harmonics. However, they are related to spherical harmonics in the following way.

Theorem 5. *Let $m \in \mathbb{N}$. The system of all generalized ellipsoidal harmonics $F_{\boldsymbol{n},\boldsymbol{p}}$ of total degree $2|\boldsymbol{n}| + |\boldsymbol{p}|$ at most m is a basis for the direct sum*

$$\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m. \tag{4.14}$$

Proof. The Dunkl operator $\Delta_{\boldsymbol{\alpha}}$ maps a homogeneous polynomial of degree q to a homogeneous polynomial of degree $q - 2$. Therefore, if we write a generalized ellipsoidal harmonic as a sum of homogeneous polynomials, then these homogeneous polynomials also satisfy (1.4). Hence every generalized ellipsoidal harmonic of total degree at most m lies in the direct sum (4.14). By comparing (3.10) with (4.8), we see that

$$F_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{x}) = G_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{x}) + \text{terms of lower degree.} \tag{4.15}$$

By Theorem 3, the system of all $G_{\boldsymbol{n},\boldsymbol{p}}$ with $2|\boldsymbol{n}| + |\boldsymbol{p}| \leq m$ is a basis for the direct sum (4.14). The statement of the theorem follows. ■

Of course, the spaces $\mathcal{H}_m$ of generalized spherical harmonics and the direct sum (4.14) depend on the parameters $\alpha_0, \alpha_1, \dots, \alpha_k$. However, it is easy to show that the set of functions on $\boldsymbol{S}^k$ which are restrictions of function in (4.14) is independent of these parameters. In fact, this set consists of all functions that are restrictions of polynomials of total degree at most m to $\boldsymbol{S}^k$.

5 Hobson's formulas

In this section we generalize some formulas given by Hobson [6, p. 124]. These formulas will be applied in the next section to obtain a generalization of Niven's formula.

Lemma 1. *Let $\mathcal{D}$ be the operator given by*

$$\mathcal{D}f(x) := f'(x) + \alpha \frac{f(x) - f(-x)}{x},$$

where α is a constant. Then, for $A(z) := z^\ell$, $\ell, m \in \mathbb{N}$, we have

$$\mathcal{D}^m x^{2\ell} = \sum_{j=0}^{m} 2^{m-2j} A^{(m-j)}\left(x^2\right) \frac{1}{j!} \mathcal{D}^{2j} x^m. \tag{5.1}$$

Proof. If $m > 2\ell$, then both sides of (5.1) are zero. So we assume that $m \leq 2\ell$. We first consider the case that $m = 2n$ is even. The left-hand side of (5.1) is equal to

$$2^m \frac{\ell!}{(\ell-n)!} (-1)^n \left(\tfrac{1}{2} - \ell - \alpha\right)_n x^{2\ell-m}. \tag{5.2}$$

The right-hand side of (5.1) is equal to

$$\sum_{j=0}^{n} 2^{m-2j} \frac{\ell!}{(\ell-m+j)!} \frac{1}{j!} 2^{2j} \frac{n!}{(n-j)!} (-1)^j \left(\tfrac{1}{2} - \alpha - n\right)_j x^{2\ell-m}. \tag{5.3}$$

After some simplifications, equality of (5.2) and (5.3) follows from the Chu-Vandermonde sum

$$\sum_{j=0}^{n} \frac{(a)_j (b)_{n-j}}{j!(n-j)!} = \frac{(a+b)_n}{n!}$$

applied to $a = \frac{1}{2} - \alpha - n$, $b = n - \ell$. This completes the proof of (5.1) if m is even. The similar proof for odd m is omitted. ■

Clearly, in (5.1) it would be enough to let j run from 0 to $\lfloor \frac{m}{2} \rfloor$. Similar remarks apply to other formulas in this section.

In the following lemma, $\mathcal{D}_j$ is according to (1.5) and ∂_j is the usual partial derivative with respect to x_j.

Lemma 2. *Let $m_0, m_1, \ldots, m_k \in \mathbb{N}$, and let $A : (0,\infty)^{k+1} \to \mathbb{R}$ be $m := m_0 + m_1 + \cdots + m_k$ times differentiable. Then, for $x_0, \ldots, x_k \neq 0$,*

$$\begin{aligned}\mathcal{D}_0^{m_0} \cdots \mathcal{D}_k^{m_k} [A(x_0^2, \ldots, x_k^2)] &= \sum_{j_0=0}^{m_0} \cdots \sum_{j_k=0}^{m_k} 2^{m-2(j_0+\cdots+j_k)} \\ &\quad \times \left(\partial_0^{m_0-j_0} \cdots \partial_k^{m_k-j_k} A\right)(x_0^2, \ldots, x_k^2) \frac{\mathcal{D}_0^{2j_0} \cdots \mathcal{D}_k^{2j_k}}{j_0! \cdots j_k!} x_0^{m_0} \cdots x_k^{m_k}.\end{aligned} \tag{5.4}$$

Warning: On the left-hand side of this formula the operators $\mathcal{D}_j$ are applied to the function $f(x_0, \ldots, x_k) := A(x_0^2, \ldots, x_k^2)$, whereas on the right-hand side the partial derivatives ∂_j are applied directly to A.

Proof. Let B be the Taylor polynomial of A of order m at a given point $(z_0, \ldots, z_k)$ with $z_j > 0$. Let $x_j := z_j^{1/2}$. Then (5.4) is true with $A - B$ in place of A at the point $x_0, \ldots, x_k$ (both sides of the equation are zero.) Therefore, it is sufficient to prove (5.4) for polynomials A, and so for monomials

$$A(z_0, \ldots, z_k) = z_0^{\ell_0} \cdots z_k^{\ell_k}.$$

In this case, we obtain (5.4) by applying Lemma 1 to each function $z_j^{\ell_j}$ and multiplying. ■

If $f(x_0, \ldots, x_k)$ is a polynomial, we will use the operator $f(\mathcal{D}_0, \ldots, \mathcal{D}_k)$. It is well-defined because the operators $\mathcal{D}_j$ commute. We use

$$r := (x_0^2 + \cdots + x_k^2)^{1/2}.$$

Theorem 6. *Let $f_m(x_0,\dots,x_k)$ be a homogeneous polynomial of degree m, and let $B:(0,\infty)\to\mathbb{R}$ be m times differentiable. Then, for all nonzero $(x_0,x_1,\dots,x_k)$,*

$$f_m(\mathcal{D}_0,\dots,\mathcal{D}_k)[B(r^2)]=\sum_{j=0}^{m}2^{m-2j}B^{(m-j)}(r^2)\frac{1}{j!}\Delta_{\boldsymbol{\alpha}}^j f_m(x_0,\dots,x_k). \tag{5.5}$$

Proof. It is sufficient to prove (5.5) for monomials

$$f_m(x_0,\dots,x_k)=x_0^{m_0}\cdots x_k^{m_k},\qquad m=m_0+\cdots+m_k.$$

In this case (5.5) follows from Lemma 2 with $A(z_0,\dots,z_k)=B(z_0+\cdots+z_k)$ by using

$$\partial_0^{m_0-j_0}\cdots\partial_k^{m_k-j_k}A(z_0,\dots,z_k)=B^{(m-j)}(z_0+\cdots+z_k)$$

with $j=j_0+\cdots+j_k$ and the multinomial formula

$$\Delta_{\boldsymbol{\alpha}}^j=(\mathcal{D}_0^2+\cdots+\mathcal{D}_k^2)^j=\sum_{j_0+\cdots+j_k=j}\binom{j}{j_0\cdots j_k}\mathcal{D}_0^{2j_0}\cdots\mathcal{D}_k^{2j_k}. \qquad \blacksquare$$

When we apply Theorem 6 to $B(z):=z^\gamma$ with $\gamma\in\mathbb{R}$, we obtain the following corollary.

Corollary 1. *Let $f_m(x_0,\dots,x_k)$ be a homogeneous polynomial of degree m. Then, for $\gamma\in\mathbb{R}$,*

$$f_m(\mathcal{D}_0,\dots,\mathcal{D}_k)[r^{2\gamma}]=\sum_{j=0}^{m}2^{m-2j}(-1)^{m-j}(-\gamma)_{m-j}r^{2(\gamma-m+j)}\frac{1}{j!}\Delta_{\boldsymbol{\alpha}}^j f_m(x_0,\dots,x_k). \tag{5.6}$$

Corollary 2. *Let f_m be a generalized spherical harmonic of degree m. Then, for $\gamma\in\mathbb{R}$,*

$$r^{2(m-\gamma)}f_m(\mathcal{D}_0,\dots,\mathcal{D}_k)[r^{2\gamma}]=2^m(-1)^m(-\gamma)_m f_m(x_0,\dots,x_k). \tag{5.7}$$

If we set

$$\gamma=\frac{1-k}{2}-|\boldsymbol{\alpha}|, \tag{5.8}$$

then a simple calculation shows that $\Delta_{\boldsymbol{\alpha}}r^{2\gamma}=0$. So $r^{2\gamma}$ plays the role of a fundamental solution of $\Delta_{\boldsymbol{\alpha}}u=0$ generalizing the solution $1/r$ of the Laplace equation $\Delta u=0$ in $\mathbb{R}^3$ with $\alpha=(0,0,0)$. Note that the number γ defined by (5.8) is always less than 1. It can be zero (for example for the Laplacian in the plane). In this case, $\ln r$ plays the role of a fundamental solution. The fundamental solution $r^{2\gamma}$ and an associated formula producing harmonic polynomials appeared in Xu [22].

Corollary 3. *If $f_m(x_0,\dots,x_k)$ is a homogeneous polynomial of degree m and γ is defined by* (5.8)*, then the right-hand side of equation* (5.6) *is a generalized spherical harmonic of degree m.*

Proof. Since $\Delta_{\boldsymbol{\alpha}}r^{2\gamma}=0$, this follows by applying $\Delta_{\boldsymbol{\alpha}}$ to both sides of (5.6). $\blacksquare$

6 Niven's formula

In this section we prove a generalization of Niven's formula expressing ellipsoidal harmonics in terms of sphero-conal harmonics. We follow the method of Hobson [6, p. 483].

Let $E = E_{\boldsymbol{n},\boldsymbol{p}}$ be a Stieltjes quasi-polynomial, and let $F = F_{\boldsymbol{n},\boldsymbol{p}}$, $G = G_{\boldsymbol{n},\boldsymbol{p}}$ be the corresponding ellipsoidal and sphero-conal harmonics written in the forms (3.10) and (4.8), respectively. It will be convenient to introduce the auxiliary polynomial

$$H(\boldsymbol{x}) := c_{\boldsymbol{n},\boldsymbol{p}}\boldsymbol{x}^{\boldsymbol{p}} \prod_{\ell=1}^{|\boldsymbol{n}|} \left(\sum_{j=0}^{k} \frac{x_j^2}{\theta_\ell - a_j} - \sum_{j=0}^{k} \frac{x_j^2}{t - a_j} \right),$$

where t is a fixed number greater than a_k. We define positive constants d_j by $d_j^2 = t - a_j$ for $j = 0, \dots, k$. Let γ be the constant defined by (5.8). We assume that $\gamma \neq 0$. The identity

$$\sum_{j=0}^{k} \frac{x_j^2}{\theta - a_j} - \sum_{j=0}^{k} \frac{x_j^2}{t - a_j} = (t - \theta) \sum_{j=0}^{k} \frac{x_j^2}{(\theta - a_j)(t - a_j)},$$

implies

$$H(d_0x_0, \dots, d_kx_k) = E(t)G(x_0, \dots, x_k). \tag{6.1}$$

By Corollary 2, we have

$$2^m(-1)^m(-\gamma)_m G(x_0, \dots, x_k) = r^{2(m-\gamma)} G(\mathcal{D}_0, \dots, \mathcal{D}_k)[r^{2\gamma}], \tag{6.2}$$

where $m := 2|\boldsymbol{n}| + |\boldsymbol{p}|$. Since

$$r^2 + (t - \theta) \sum_{j=0}^{k} \frac{x_j^2}{\theta - a_j} = \sum_{j=0}^{k} \frac{(t - a_j)x_j^2}{\theta - a_j},$$

we conclude that

$$G(d_0x_0, \dots, d_kx_k) = E(t)G(x_0, \dots, x_k) + r^2 P(x_0, \dots, x_k),$$

where P is a polynomial. It follows that

$$G(d_0\mathcal{D}_0, \dots, d_k\mathcal{D}_k) = E(t)G(\mathcal{D}_0, \dots, \mathcal{D}_k) + P(\mathcal{D}_0, \dots, \mathcal{D}_k)\Delta_{\boldsymbol{\alpha}}.$$

Using that $\Delta_{\boldsymbol{\alpha}} r^{2\gamma} = 0$, we obtain

$$G(d_0\mathcal{D}_0, \dots, d_k\mathcal{D}_k)\big[r^{2\gamma}\big] = E(t)G(\mathcal{D}_0, \dots, \mathcal{D}_k)\big[r^{2\gamma}\big]. \tag{6.3}$$

We now combine equations (6.1), (6.2), (6.3) and obtain

$$2^m(-1)^m(-\gamma)_m H(d_0x_0, \dots, d_kx_k) = r^{2(m-\gamma)} G(d_0\mathcal{D}_0, \dots, d_k\mathcal{D}_k)\big[r^{2\gamma}\big]. \tag{6.4}$$

Now (6.4) and Corollary 1 yield

$$H(d_0x_0, \dots, d_kx_k) = \sum_{i=0}^{m} \frac{r^{2i}}{2^{2i} i! (\gamma - m + 1)_i} \Delta_{\boldsymbol{\alpha}}^i [G(d_0x_0, \dots, d_kx_k)].$$

We replace the variables x_j by y_j/d_j and rename y_j as x_j again. This gives

$$H(\boldsymbol{x}) = \sum_{i=0}^{m} \frac{R^{2i}}{2^{2i} i! (\gamma - m + 1)_i} \big(d_0^2\mathcal{D}_0^2 + \dots + d_k^2\mathcal{D}_k^2\big)^i G(\boldsymbol{x}), \tag{6.5}$$

where

$$R^2 = \sum_{j=0}^{k} \frac{x_j^2}{d_j^2} = \sum_{j=0}^{k} \frac{x_j^2}{t - a_j}.$$

Since G satisfies $\Delta_{\boldsymbol{\alpha}} G = 0$, we see that the right-hand side of (6.5) does not change if we replace $d_j^2\mathcal{D}_j^2$ by $-a_j\mathcal{D}_j^2$. If positive numbers $x_0, \dots, x_k$ are given, we can choose t as the ellipsoidal coordinate $t = t_0$. Then $R = 1$ and $H(\boldsymbol{x}) = F(\boldsymbol{x})$. We have proved the following theorem.

Theorem 7. *Let $F_{\boldsymbol{n},\boldsymbol{p}}, G_{\boldsymbol{n},\boldsymbol{p}}$ be the generalized ellipsoidal and sphero-conal harmonics of degree $m = 2|\boldsymbol{n}| + |\boldsymbol{p}|$ defined by* (3.10), (4.8), *respectively. Assume that γ defined by* (5.8) *is nonzero. Then*

$$F_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{x}) = \sum_{i=0}^{m} \frac{(-1)^i}{2^{2i} i! (\gamma + 1 - m)_i} \big(a_0 \mathcal{D}_0^2 + \cdots + a_k \mathcal{D}_k^2\big)^i G_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{x}). \tag{6.6}$$

In the classical case $k = 2$ and $\alpha_0 = \alpha_1 = \alpha_2 = 0$ this is Niven's formula; see [6, p. 489].

7 A Dirichlet problem for ellipsoids

In this section we apply generalized ellipsoidal harmonics to solve the Dirichlet boundary value problem for the Dunkl equation on ellipsoids.

We consider the solid ellipsoid

$$\mathcal{E} := \left\{ \boldsymbol{x} \in \mathbb{R}^{k+1} : \sum_{j=0}^{k} \frac{x_j^2}{b_j^2} < 1 \right\},$$

with semi-axes $b_0 > b_1 > \cdots > b_k > 0$. Let $\partial\mathcal{E}$ be the boundary of $\mathcal{E}$. Given a function $f : \partial\mathcal{E} \to \mathbb{R}$ we want to find a solution u of Dunkl's equation (1.4) on $\mathcal{E}$ that assumes the given boundary values f on $\partial\mathcal{E}$ in the sense explained below.

It will be convenient to parameterize $\partial\mathcal{E}$ by the unit sphere $\boldsymbol{S}^k$ employing the map

$$T : \boldsymbol{S}^k \to \partial\mathcal{E} \tag{7.1}$$

defined by

$$T(y_0, y_1, \dots, y_k) := (b_0 y_0, b_1 y_1, \dots, b_k y_k).$$

We suppose that the given boundary value function $f : \partial\mathcal{E} \to \mathbb{R}$ has the property that the function $f \circ T$ is in $L^2_w(\boldsymbol{S}^k)$, where the weight function w is defined in (4.11). A solution of the Dirichlet boundary value problem for the Dunkl equation with given boundary value function f is a function $u \in C^2(\mathcal{E})$ which satisfies (1.4) in $\mathcal{E}$ and assumes the boundary value f in the following sense. For sufficiently small $\delta > 0$, form the confocal ellipsoids

$$\left\{ \boldsymbol{x} \in \mathbb{R}^{k+1} : \sum_{j=0}^{k} \frac{x_j^2}{b_j^2 - \delta} = 1 \right\}, \tag{7.2}$$

and let T_δ be defined as T but with respect to the ellipsoid (7.2) in place of $\partial\mathcal{E}$. Then we require that

$$u \circ T_\delta \to f \circ T \quad \text{in} \quad L^2_w(\boldsymbol{S}^k) \quad \text{as} \quad \delta \to 0. \tag{7.3}$$

We now show how to construct a solution of this Dirichlet problem. We choose any real number ω (we can take $\omega = 0$ if we wish), and define numbers $a_0 < a_1 < \cdots < a_k$ by

$$a_j := \omega - b_j^2.$$

Corresponding to these numbers a_j we introduce sphero-conal coordinates $(r, s_1, \dots, s_k)$ for cartesian coordinates $\boldsymbol{y} = (y_0, y_1, \dots, y_k)$ and ellipsoidal coordinates $(t_0, t_1, \dots, t_k)$ for cartesian coordinates $\boldsymbol{x} = (x_0, x_1, \dots, x_k)$. Note that if $\boldsymbol{x} = T\boldsymbol{y}$ and $r = 1$ then $s_j = t_j$ for each $j = 1, 2, \dots, k$.

Since the function $f \circ T$ lies in $L^2_w(\boldsymbol{S}^k)$, we can expand $f \circ T$ in the orthonormal basis of Theorem 4:

$$f \circ T = \sum_{\boldsymbol{n},\boldsymbol{p}} f_{\boldsymbol{n},\boldsymbol{p}} e_{\boldsymbol{n},\boldsymbol{p}} G_{n,p}, \tag{7.4}$$

where the factors $e_{\boldsymbol{n},\boldsymbol{p}}$ are determined by

$$e_{\boldsymbol{n},\boldsymbol{p}} \|G_{\boldsymbol{n},\boldsymbol{p}}\|_w = 1,$$

and

$$f_{\boldsymbol{n},\boldsymbol{p}} := \langle f \circ T, e_{\boldsymbol{n},\boldsymbol{p}} G_{\boldsymbol{n},\boldsymbol{p}} \rangle_w. \tag{7.5}$$

Then

$$\sum_{\boldsymbol{n},\boldsymbol{p}} |f_{\boldsymbol{n},\boldsymbol{p}}|^2 = \|f \circ T\|_w^2 < \infty. \tag{7.6}$$

The expansion (7.4) converges in $L^2_w(\boldsymbol{S}^k)$. We are going to prove that

$$u(\boldsymbol{x}) := \sum_{\boldsymbol{n},\boldsymbol{p}} \frac{f_{\boldsymbol{n},\boldsymbol{p}} e_{\boldsymbol{n},\boldsymbol{p}}}{E_{\boldsymbol{n},\boldsymbol{p}}(\omega)} F_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{x}) \tag{7.7}$$

is the desired solution of our Dirichlet problem.

Theorem 8. *The function u defined by* (7.7) *is infinitely many times differentiable and solves Dunkl's equation* (1.4) *on the open ellipsoid $\mathcal{E}$, and it assumes the given boundary value f in the sense of* (7.3).

Proof. We first show that the infinite series in (7.7) converges. We know from [14, Lemma 2.2] that there is a sequence K_m of polynomial growth such that

$$|e_{\boldsymbol{n},\boldsymbol{p}} G_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{y})| \le K_m \qquad \text{for} \quad m = 2|\boldsymbol{n}| + |\boldsymbol{p}|, \quad \boldsymbol{y} \in \boldsymbol{S}^k. \tag{7.8}$$

For given $t \in (a_k, \omega)$ we consider the solid ellipsoid

$$\mathcal{E}_t := \left\{ \boldsymbol{x} \in \mathbb{R}^{k+1} : \sum_{j=0}^{k} \frac{x_j^2}{t - a_j} \le 1 \right\}$$

which is a subset of $\mathcal{E}$. By comparing (3.8) and (4.6), we get from (7.8)

$$|e_{\boldsymbol{n},\boldsymbol{p}} F_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{x})| \le E_{\boldsymbol{n},\boldsymbol{p}}(t) K_m \qquad \text{for} \quad \boldsymbol{x} \in \partial\mathcal{E}_t. \tag{7.9}$$

The Stieltjes quasi-polynomial $E_{\boldsymbol{n},\boldsymbol{p}}$ has degree $m/2$ and all of its zeros lie in the interval $[a_0, a_k]$. Hence we have the inequality

$$0 < E_{\boldsymbol{n},\boldsymbol{p}}(t) \le E_{\boldsymbol{n},\boldsymbol{p}}(\omega) \left(\frac{t - a_0}{\omega - a_0} \right)^{m/2} \qquad \text{for} \quad a_k < t \le \omega. \tag{7.10}$$

Now we obtain from (7.9), (7.10)

$$|e_{\boldsymbol{n},\boldsymbol{p}} F_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{x})| \le \left(\frac{t - a_0}{\omega - a_0} \right)^{m/2} E_{\boldsymbol{n},\boldsymbol{p}}(\omega) K_m \qquad \text{for} \quad x \in \mathcal{E}_t. \tag{7.11}$$

Since the set of numbers $f_{\boldsymbol{n},\boldsymbol{p}}$ is bounded by (7.6), K_m grows only polynomially with m and $\left(\frac{t-a_0}{\omega-a_0}\right)^{m/2}$ goes to 0 exponentially as $m \to \infty$, we see that the series in (7.7) converges uniformly in $\mathcal{E}_t$ and thus in every compact subset of $\mathcal{E}$.

The next step is to show that u solves equation (1.4). This follows if we can justify interchanging the operator $\Delta_{\boldsymbol{\alpha}}$ with the sum in (7.7). Consider the series

$$\sum_{\boldsymbol{n},\boldsymbol{p}} \frac{f_{\boldsymbol{n},\boldsymbol{p}} e_{\boldsymbol{n},\boldsymbol{p}}}{E_{\boldsymbol{n},\boldsymbol{p}}(\omega)} \frac{\partial}{\partial x_j} F_{\boldsymbol{n},\boldsymbol{p}}(x_0, x_1, \dots, x_k) \tag{7.12}$$

that we obtain from (7.7) by differentiating each term with respect to x_j. In order to show uniform convergence of this series on $\mathcal{E}_t$ we need a bound for the partial derivatives of $F_{\boldsymbol{n},\boldsymbol{p}}$. We obtain such a bound from the following result due to Kellog [11]. If $P(y_0, y_1, \dots, y_k)$ is a polynomial of total degree N then

$$\max\{\|\mathrm{grad}\, P(\boldsymbol{y})\| : \|y\| \le 1\} \le N^2 \max\{|P(\boldsymbol{y})| : \|y\| \le 1\}, \tag{7.13}$$

where $\|\cdot\|$ denotes euclidian norm in $\mathbb{R}^{k+1}$. If we use a mapping like (7.1) to transform the ellipsoid to the unit ball we find that

$$\left|\frac{\partial}{\partial x_j} F_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{x})\right| \le \frac{m^2}{(t-a_k)^{1/2}} \max\{|F_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{z})| : \boldsymbol{z} \in \mathcal{E}_t\} \qquad \text{for} \quad \boldsymbol{x} \in \mathcal{E}_t. \tag{7.14}$$

Using this estimate we show as before that (7.12) converges uniformly on $\mathcal{E}_t$. In a similar way we argue for the second term in the generalized partial derivative (1.5). Since we can repeat the procedure we see that u is infinitely many times differentiable on $\mathcal{E}$ and it solves Dunkl's equation.

It remains to show that u satisfies the boundary condition (7.3). For $y \in \boldsymbol{S}^k$, we find

$$f \circ T(\boldsymbol{y}) - f \circ T_\delta(\boldsymbol{y}) = \sum_{\boldsymbol{n},\boldsymbol{p}} f_{\boldsymbol{n},\boldsymbol{p}} e_{\boldsymbol{n},\boldsymbol{p}} \left(1 - \frac{E_{\boldsymbol{n},\boldsymbol{p}}(\omega-\delta)}{E_{\boldsymbol{n},\boldsymbol{p}}(\omega)}\right) G_{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{y}).$$

Hence we obtain

$$\|f \circ T - f \circ T_\delta\|_w^2 = \sum_{\boldsymbol{n},\boldsymbol{p}} f_{\boldsymbol{n},\boldsymbol{p}}^2 \left(1 - \frac{E_{\boldsymbol{n},\boldsymbol{p}}(\omega-\delta)}{E_{\boldsymbol{n},\boldsymbol{p}}(\omega)}\right)^2$$

so (7.3) follows easily. ∎

8 Examples

Formulas in this paper have been checked with the software *Maple* for some Stieltjes polynomials represented in explicit form. For example, take $k = 2$,

$$a_0 = 0, \qquad a_1 = 3, \qquad a_2 = 5, \quad \alpha_0 = \frac{229}{54}, \qquad \alpha_1 = \frac{71}{54}, \qquad \alpha_2 = \frac{25}{6},$$

and

$$n_1 = 2, \qquad n_2 = 1, \qquad p_0 = p_1 = p_2 = 0.$$

Then the corresponding Stieltjes polynomial $E_{\boldsymbol{n},\boldsymbol{p}}$ is given by

$$E_{\boldsymbol{n},\boldsymbol{p}}(t) = (t-1)(t-2)(t-4).$$

Indeed, $E_{\boldsymbol{n},\boldsymbol{p}}$ satisfies equation (2.1) with

$$\lambda_0 = \frac{1120}{9}, \qquad \lambda_1 = -\frac{119}{3},$$

and it has two zeros between a_0 and a_1, and one zero between a_1 and a_2.

The simplest way to compute such examples is to use the fact that the zeros θ_ℓ of $E_{\boldsymbol{n},\boldsymbol{0}}$ are characterized by the system of equations

$$\sum_{\substack{q=1\\ q\neq\ell}}^{|\boldsymbol{n}|} \frac{2}{\theta_\ell - \theta_q} + \sum_{j=0}^{k} \frac{\alpha_j + \frac{1}{2}}{\theta_\ell - a_j} = 0, \qquad \ell = 1, 2, \ldots, |\boldsymbol{n}|; \tag{8.1}$$

see [18, (6.81.5)].

The corresponding ellipsoidal and sphero-conal harmonics are

$$F_{\boldsymbol{n},\boldsymbol{p}} = -192\left(x_0^2 - \tfrac{1}{2}x_1^2 - \tfrac{1}{4}x_2^2 - 1\right)\left(\tfrac{1}{2}x_0^2 - x_1^2 - \tfrac{1}{3}x_2^2 - 1\right)\left(\tfrac{1}{4}x_0^2 + x_1^2 - x_2^2 - 1\right),$$
$$G_{\boldsymbol{n},\boldsymbol{p}} = -192\left(x_0^2 - \tfrac{1}{2}x_1^2 - \tfrac{1}{4}x_2^2\right)\left(\tfrac{1}{2}x_0^2 - x_1^2 - \tfrac{1}{3}x_2^2\right)\left(\tfrac{1}{4}x_0^2 + x_1^2 - x_2^2\right).$$

One can check that these polynomials do satisfy equation (1.4). Also, applying formula (6.6) to $G_{\boldsymbol{n},\boldsymbol{p}}$ we obtain $F_{\boldsymbol{n},\boldsymbol{p}}$ as claimed.

We now take the same a_j but replace the parameters α_j by

$$\alpha_0 = \frac{229}{54} - 1 = \frac{175}{54}, \qquad \alpha_1 = \frac{71}{54} - 1 = \frac{17}{54}, \qquad \alpha_3 = \frac{25}{6}.$$

Moreover, let

$$n_1 = 2, \qquad n_2 = 1, \qquad p_0 = 1, \qquad p_1 = 1, \qquad p_2 = 0.$$

Then the Stieltjes quasi-polynomial is

$$E_{\boldsymbol{n},\boldsymbol{p}}(t) = \sqrt{|t|}\sqrt{|t-3|}(t-1)(t-2)(t-4).$$

It satisfies equation (2.1) with

$$\lambda_0 = \frac{2855}{18}, \qquad \lambda_1 = -\frac{440}{9}.$$

The corresponding ellipsoidal and sphero-conal harmonics are as before but with -192 replaced by $-192\sqrt{15}\sqrt{6}$ and the extra factor x_1x_2 added. Again it can be checked that these polynomials satisfy equation (1.4), and formula (6.6) holds.

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