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q-Deformed KP Hierarchy and q-Deformed Constrained KP Hierarchy

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Abstract. Using the determinant representation of gauge transformation operator, we have shown that the general form of τ function of the q-KP hierarchy is a q-deformed generalized Wronskian, which includes the q-deformed Wronskian as a special case. On the basis of these, we study the q-deformed constrained KP (q-cKP) hierarchy, i.e. l-constraints of q-KP hierarchy. Similar to the ordinary constrained KP (cKP) hierarchy, a large class of solutions of q-cKP hierarchy can be represented by q-deformed Wronskian determinant of functions satisfying a set of linear q-partial differential equations with constant coefficients. We obtained additional conditions for these functions imposed by the constraints. In particular, the effects of q-deformation (q-effects) in single q-soliton from the simplest τ function of the q-KP hierarchy and in multi-q-soliton from one-component q-cKP hierarchy, and their dependence of x and q, were also presented. Finally, we observe that q-soliton tends to the usual soliton of the KP equation when $x \to 0$ and $q \to 1$, simultaneously.

Key words: q-deformation; τ function; Gauge transformation operator; q-KP hierarchy; q-cKP hierarchy

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1 Introduction

Study of the quantum calculus (or q-calculus) [1, 2] has a long history, which may go back to the beginning of the twentieth century. F.H. Jackson was the first mathematician who studied the q-integral and q-derivative in a systematic way starting about 1910 [3, 4][1]. Since 1980's, the quantum calculus was re-discovered in the research of quantum group inspired by the studies on quantum integrable model that used the quantum inverse scattering method [5] and on noncommutative geometry [6]. In particular, S. Majid derived the q-derivative from the braided differential calculus [7, 8].

The q-deformed integrable system (also called q-analogue or q-deformation of classical integrable system) is defined by means of q-derivative ∂_q instead of usual derivative ∂ with respect to x in a classical system. It reduces to a classical integrable system as $q \to 1$. Recently, the q-deformation of the following three stereotypes for integrable systems attracted more attention. The first type is q-deformed N-th KdV (q-NKdV or q-Gelfand–Dickey hierarchy) [9, 16], which is reduced to the N-th KdV (NKdV or Gelfand–Dickey) hierarchy when $q \to 1$. The N-th q-KdV hierarchy becomes q-KdV hierarchy for $N = 2$. The q-NKdV hierarchy inherited several integrable structures from classical N-th KdV hierarchy, such as infinite conservation laws [10],

[1] Detailed notes on the initial research of q-integral, q-derivative of Jackson and wide applications of q-series are easily available in the text.

bi-Hamiltonian structure [11, 12], τ function [13, 14], Bäcklund transformation [15]. The second type is the q-KP hierarchy [17, 22]. Its τ function, bi-Hamiltonian structure and additional symmetries have already been reported in [20, 21, 18, 22]. The third type is the q-AKNS-D hierarchy, and its bilinear identity and τ function were obtained in [23].

In order to get the Darboux–Bäcklund transformations, the two elementary types of gauge transformation operators, differential-type denoted by T (or T_D) and integral-type denoted by S (or T_I), for q-deformed N-th KdV hierarchy were introduced in [15]. Tu et al. obtained not only the q-deformed Wronskian-type but also binary-type representations of τ function of q-KdV hierarchy. On the basis of their results, He et al. [24] obtained the determinant representation of gauge transformation operators T_{n+k} ($n \geq k$) for q-Gelfand–Dickey hierarchy, which is a mixed iteration of n-steps of T_D and then k-steps of T_I. Then, they obtained a more general form of τ function for q-KdV hierarchy, i.e., generalized q-deformed Wronskian (q-Wronskian) IW^q_{n+k} [24]. It is important to note that for $k = 0$ IW^q_{n+k} reduces to q-deformed Wronskian and for $k = n$ to binary-type determinant [15]. On the other hand, Tu introduced the q-deformed constrained KP (q-cKP) hierarchy [22] by means of symmetry constraint of q-KP hierarchy, which is a q-analogue of constrained KP (cKP) hierarchy [25, 31].

The purpose of this paper is to construct the τ function of q-KP and q-cKP hierarchy, and then explore the q-effect in q-solitons. The main tool is the determinant representation of gauge transformation operators [32, 33, 34, 35]. The paper is organized as follows: In Section 2 we introduce some basic facts on the q-KP hierarchy, such as Lax operator, Z-S equations, the existence of τ function. On the basis of the [15], two kinds of elementary gauge transformation operators for q-KP hierarchy and changing rule of q-KP hierarchy under it are presented in Section 3. In Section 4, we establish the determinant representation of gauge transformation operator T_{n+k} for the q-KP hierarchy and then obtain the general form of τ function $\tau_q^{(n+k)} = IW^q_{n+k}$. In particular, by taking $n = 1$, $k = 0$ we will show q-effect of single q-soliton solution of q-KP hierarchy. A brief description of the sub-hierarchy of q-cKP hierarchy is presented in Section 5, from the viewpoint of the symmetry constraint. In Section 6, we show that the q-Wronskian is one kind of forms of τ function of q-cKP if the functions in the q-Wronskian satisfy some restrictions. In Section 7 we consider an example which illustrates the procedure reducing q-KP to q-cKP hierarchy. q-effects of the q-deformed multi-soliton are also discussed. The conclusions and discussions are given in Section 8. Our presentation is similar to the relevant papers of classical KP and cKP hierarchy [32, 34, 36, 37, 38].

At the end of this section, we shall collect some useful formulae for reader's convenience.

The q-derivative ∂_q is defined by

$$\partial_q(f(x)) = \frac{f(qx) - f(x)}{x(q-1)}$$

and the q-shift operator is given by

$$\theta(f(x)) = f(qx).$$

Let ∂_q^{-1} denote the formal inverse of ∂_q. We should note that θ does not commute with ∂_q,

$$(\partial_q \theta^k(f)) = q^k \theta^k(\partial_q f), \qquad k \in \mathbb{Z}.$$

In general, the following q-deformed Leibnitz rule holds:

$$\partial_q^n \circ f = \sum_{k \geq 0} \binom{n}{k}_q \theta^{n-k}(\partial_q^k f)\partial_q^{n-k}, \qquad n \in \mathbb{Z}, \tag{1.1}$$

where the q-number and the q-binomial are defined by

$$(n)_q = \frac{q^n-1}{q-1}, \qquad \binom{n}{k}_q = \frac{(n)_q(n-1)_q\cdots(n-k+1)_q}{(1)_q(2)_q\cdots(k)_q}, \qquad \binom{n}{0}_q = 1,$$

and "$\circ$" means composition of operators, defined by $\partial_q \circ f = (\partial_q \cdot f) + \theta(f)\partial_q$. In the remainder of the paper for any function f "$\cdot$" is defined by $\partial_q \cdot f = \partial_q(f) \triangleq (\partial_q f)$. For a q-pseudo-differential operator (q-PDO) of the form $P = \sum_{i=-\infty}^{n} p_i\partial_q^i$, we decompose P into the differential part $P_+ = \sum_{i\geq 0} p_i\partial_q^i$ and the integral part$P_- = \sum_{i\leq -1} p_i\partial_q^i$. The conjugate operation "$*$" for P is defined by $P^* = \sum_i (\partial_q^*)^i p_i$ with $\partial_q^* = -\partial_q\theta^{-1} = -\frac{1}{q}\partial_{\frac{1}{q}}$, $(\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta\partial_q^{-1}$. We can write out several explicit forms of (1.1) for q-derivative ∂_q, as

$$\partial_q \circ f = (\partial_q f) + \theta(f)\partial_q, \tag{1.2}$$
$$\partial_q^2 \circ f = (\partial_q^2 f) + (q+1)\theta(\partial_q f)\partial_q + \theta^2(f)\partial_q^2, \tag{1.3}$$
$$\partial_q^3 \circ f = (\partial_q^3 f) + (q^2+q+1)\theta(\partial_q^2 f)\partial_q + (q^2+q+1)\theta^2(\partial_q f)\partial_q^2 + \theta^3(f)\partial_q^3, \tag{1.4}$$

and ∂_q^{-1}

$$\begin{aligned}\partial_q^{-1} \circ f &= \theta^{-1}(f)\partial_q^{-1} - q^{-1}\theta^{-2}(\partial_q f)\partial_q^{-2} + q^{-3}\theta^{-3}(\partial_q^2 f)\partial_q^{-3} - q^{-6}\theta^{-4}(\partial_q^3 f)\partial_q^{-4} \\ &\quad + \frac{1}{q^{10}}\theta^{-5}(\partial_q^4 f)\partial_q^{-5} + \cdots + (-1)^k q^{-(1+2+3+\cdots+k)}\theta^{-k-1}(\partial_q^k f)\partial_q^{-k-1} + \cdots, \end{aligned} \tag{1.5}$$
$$\begin{aligned}\partial_q^{-2} \circ f &= \theta^{-2}(f)\partial_q^{-2} - \frac{1}{q^2}(2)_q\theta^{-3}(\partial_q f)\partial_q^{-3} + \frac{1}{q^{(2+3)}}(3)_q\theta^{-4}(\partial_q^2 f)\partial_q^{-4} \\ &\quad - \frac{1}{q^{(2+3+4)}}(4)_q\theta^{-5}(\partial_q^3 f)\partial_q^{-5} + \cdots \\ &\quad + \frac{(-1)^k}{q^{(2+3+\cdots+k+1)}}(k+1)_q\theta^{-2-k}(\partial_q^k f)\partial_q^{-2-k} + \cdots. \end{aligned} \tag{1.6}$$

More explicit expressions of $\partial_q^n \circ f$ are given in Appendix A. In particular, $\partial_q^{-1} \circ f$ has different forms,

$$\partial_q^{-1} \circ f = \theta^{-1}(f)\partial_q^{-1} + \partial_q^{-1} \circ (\partial_q^* f) \circ \partial_q^{-1},$$
$$\partial_q^{-1} \circ f \circ \partial_q^{-1} = (\partial_q^{-1} f)\partial_q^{-1} - \partial_q^{-1} \circ \theta(\partial_q^{-1} f),$$

which will be used in the following sections. The q-exponent $e_q(x)$ is defined as follows

$$e_q(x) = \sum_{i=0}^{\infty} \frac{x^n}{(n)_q!}, \qquad (n)_q! = (n)_q(n-1)_q(n-2)_q\cdots(1)_q.$$

Its equivalent expression is of the form

$$e_q(x) = \exp\left(\sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)}x^k\right). \tag{1.7}$$

The form (1.7) will play a crucial role in proving the existence [20] of τ function of q-KP hierarchy.

2 q-KP hierarchy

Similarly to the general way of describing the classical KP hierarchy [36, 37], we shall give a brief introduction of q-KP based on [20]. Let L be one q-PDO given by

$$L = \partial_q + u_0 + u_{-1}\partial_q^{-1} + u_{-2}\partial_q^{-2} + \cdots , \tag{2.1}$$

which is called Lax operator of q-KP hierarchy. There exist infinite q-partial differential equations relating to dynamical variables $\{u_i(x, t_1, t_2, t_3, \ldots), i = 0, -1, -2, -3, \ldots\}$, and they can be deduced from generalized Lax equation,

$$\frac{\partial L}{\partial t_n} = [B_n, L], \qquad n = 1, 2, 3, \ldots, \tag{2.2}$$

which are called q-KP hierarchy. Here $B_n = (L^n)_+ = \sum_{i=0}^{n} b_i\partial_q^i$ means the positive part of q-PDO, and we will use $L^n_- = L^n - L^n_+$ to denote the negative part. By means of the formulae given in (1.2)–(1.6) and in Appendices A and B, the first few flows in (2.2) for dynamical variables $\{u_0, u_{-1}, u_{-2}, u_{-3}\}$ can be written out as follows. The first flow is

$$\begin{aligned}
\partial_{t_1} u_0 &= \theta(u_{-1}) - u_{-1},\\
\partial_{t_1} u_{-1} &= (\partial_q u_{-1}) + \theta(u_{-2}) + u_0u_{-1} - u_{-2} - u_{-1}\theta^{-1}(u_0),\\
\partial_{t_1} u_{-2} &= (\partial_q u_{-2}) + \theta(u_{-3}) + u_0u_{-2} - u_{-3} - u_{-2}\theta^{-2}(u_0) + \frac{1}{q}u_{-1}\theta^{-2}(\partial_q u_0),\\
\partial_{t_1} u_{-3} &= (\partial_q u_{-3}) + \theta(u_{-4}) + u_0u_{-3} - u_{-4} - \frac{1}{q^3}u_{-1}\theta^{-3}(\partial_q^2 u_0)\\
&\quad + \frac{1}{q^2}(2)_q u_{-2}\theta^{-3}(\partial_q u_0) - u_{-3}\theta^{-3}(u_0).
\end{aligned}$$

The second flow is

$$\begin{aligned}
\partial_{t_2} u_0 &= \theta(\partial_q u_{-1}) + \theta^2(u_{-2}) + \theta(u_0)\theta(u_{-1}) + u_0\theta(u_{-1})\\
&\quad - \big((\partial_q u_{-1}) + u_{-1}u_0 + u_{-1}\theta^{-1}(u_0) + u_{-2}\big),\\
\partial_{t_2} u_{-1} &= q^{-1}u_{-1}\theta^{-2}(\partial_q u_0) + u_{-1}(\partial_q u_0) + (\partial_q^2 u_{-1}) + \big(\theta(u_0) + u_0\big)(\partial_q u_{-1})\\
&\quad + (q+1)\theta(\partial_q u_{-2}) + \theta(u_0)\theta(u_{-2}) + u_0\theta(u_{-2}) + \theta(u_{-1})u_{-1} + u_0^2u_{-1}\\
&\quad - u_{-1}\theta^{-1}(u_0^2) - u_{-1}\theta^{-1}(u_{-1}) - u_{-2}\theta^{-1}(u_0) - u_{-2}\theta^{-2}(u_0) + \theta^3(u_{-3}) - u_{-3},\\
\partial_{t_2} u_{-2} &= (\partial_q^2 u_{-2}) + (q+1)\theta(\partial_q u_{-3}) + (\partial_q u_{-2})v_1 + \theta^2(u_{-4}) + \theta(u_{-3})v_1 + u_{-2}v_0\\
&\quad - \big(q^{-3}u_{-1}\theta^{-3}(\partial_q^2 v_1) - q^{-1}u_{-1}\theta^{-2}(\partial_q v_0) - q^{-2}(2)_q u_{-2}\theta^{-3}(\partial_q v_1)\\
&\quad + u_{-2}\theta^{-2}(v_0) + u_{-3}\theta^{-3}(v_1) + u_{-4}\big),\\
\partial_{t_2} u_{-3} &= (\partial_q^2 u_{-3}) + (q+1)\theta(\partial_q u_{-4}) + (\partial_q u_{-3})v_1 + \theta^2(u_{-5}) + \theta(u_{-4})v_1 + u_{-3}v_0\\
&\quad - \big(- q^{-6}\theta^{-4}(\partial_q^3 v_1) + q^{-3}u_{-1}\theta^{-3}(\partial_q^2 v_0) + q^{-5}(3)_q u_{-2}\theta^{-4}(\partial_q^2 v_1)\\
&\quad - q^{-2}(2)_q u_{-2}\theta^{-3}(\partial_q v_0) - q^{-3}(3)_q u_{-3}\theta^{-4}(\partial_q v_1) + u_{-3}\theta^{-3}(v_0)\\
&\quad + u_{-4}\theta^{-4}(v_1) + u_{-5}\big).
\end{aligned}$$

The third flow is

$$\begin{aligned}
\partial_{t_3} u_0 &= (\partial_q^3 u_0) + (3)_q\theta(\partial_q^2 u_{-1}) + \tilde{s}_2(\partial_q^2 u_0) + (3)_q\theta^2(\partial_q u_{-2}) + (2)_q\theta(\partial_q u_{-1})\tilde{s}_2\\
&\quad + (\partial_q u_0)\tilde{s}_1 + \theta^3(u_{-3}) + \theta^2(u_{-2})\tilde{s}_2 + \theta(u_{-1})\tilde{s}_1 + u_0\tilde{s}_0\\
&\quad - \big(- q^{-1}\theta^{-2}(\partial_q\tilde{s}_2)u_{-1} + u_0\tilde{s}_0 + u_{-1}\theta^{-1}(\tilde{s}_1) + u_{-2}\theta^{-2}(\tilde{s}_2) + u_{-3} + (\partial_q\tilde{s}_0)\big),
\end{aligned}$$

$$
\begin{aligned}
\partial_{t_3} u_{-1} = {} & (\partial_q^3 u_{-1}) + (3)_q \theta(\partial_q^2 u_{-2}) + \tilde{s}_2(\partial_q^2 u_{-1}) + (3)_q \theta^2(\partial_q u_{-3}) + (2)_q \tilde{s}_2 \theta(\partial_q u_{-2}) \\
& + \tilde{s}_1(\partial_q u_{-1}) + \theta^3(u_{-4}) + \tilde{s}_2 \theta^3(u_{-3}) + \tilde{s}_1 \theta(u_{-2}) + \tilde{s}_0 u_{-1} \\
& - \big(q^{-3} u_{-1} \theta^{-3}(\partial_q^2 \tilde{s}_2) - q^{-1} u_{-1} \theta^{-2}(\partial_q \tilde{s}_1) - q^{-2}(2)_q u_{-2} \theta^{-3}(\partial_q \tilde{s}_2) \\
& + u_{-1} \theta^{-1}(\tilde{s}_0) + u_{-2} \theta^{-2}(\tilde{s}_1) + u_{-3} \theta^{-3}(\tilde{s}_2) + u_{-4}\big), \\
\partial_{t_3} u_{-2} = {} & (\partial_q^3 u_{-2}) + (3)_q \theta(\partial_q^2 u_{-3}) + \tilde{s}_2(\partial_q^2 u_{-2}) + (3)_q \theta^2(\partial_q u_{-4}) + (2)_q \tilde{s}_2 \theta(\partial_q u_{-3}) \\
& + \tilde{s}_1(\partial_q u_{-2}) + \theta^3(u_{-5}) + \tilde{s}_2 \theta^2(u_{-4}) + \tilde{s}_1 \theta(u_{-3}) + \tilde{s}_0 u_{-2} \\
& - \big(- q^{-6} u_{-1} \theta^{-4}(\partial_q^3 \tilde{s}_2) + q^{-3} u_{-1} \theta^{-3}(\partial_q^2 \tilde{s}_1) + q^{-5}(3)_q u_{-2} \theta^{-4}(\partial_q^2 \tilde{s}_2) \\
& - q^{-1} u_{-1} \theta^{-2}(\partial_q \tilde{s}_0) - q^{-2}(2)_q u_{-2} \theta^{-3}(\partial_q \tilde{s}_1) - q^{-3}(3)_q u_{-3} \theta^{-4}(\partial_q \tilde{s}_2) \\
& + u_{-2} \theta^{-2}(\tilde{s}_0) + u_{-3} \theta^{-3}(\tilde{s}_1) + u_{-4} \theta^{-4}(\tilde{s}_2) + u_{-5}\big), \\
\partial_{t_3} u_{-3} = {} & (\partial_q^2 u_{-3}) + (3)_q \theta(\partial_q^2 u_{-4}) + \tilde{s}_2(\partial_q^2 u_{-3}) + (3)_q \theta^2(\partial_q u_{-5}) + (2)_q \tilde{s}_2 \theta(\partial_q u_{-4}) \\
& + \tilde{s}_1(\partial_q u_{-3}) + \theta^3(u_{-6}) + \tilde{s}_2 \theta^2(u_{-5}) + \tilde{s}_1 \theta(u_{-4}) + \tilde{s}_0 u_{-3} \\
& - \big(q^{-10} u_{-1} \theta^{-5}(\partial_q^4 \tilde{s}_2) - q^{-6} u_{-1} \theta^{-4}(\partial_q^3 \tilde{s}_1) - q^{-9}(4)_q \theta^{-5}(\partial_q^3 \tilde{s}_2) \\
& + q^{-3} u_{-1} \theta^{-3}(\partial_q^2 \tilde{s}_0) + q^{-5}(3)_q u_{-2} \theta^{-4}(\partial_q^2 \tilde{s}_1) + q^{-7} \frac{(3)_q (4)_4}{(2)_q} u_{-3} \theta^{-5}(\partial_q^2 \tilde{s}_2) \\
& - q^{-2}(2)_q u_{-2} \theta^{-3}(\partial_q \tilde{s}_0) - q^{-3}(3)_q u_{-3} \theta^{-4}(\partial_q \tilde{s}_1) - q^{-4}(4)_q u_{-4} \theta^{-5}(\partial_q \tilde{s}_2) \\
& + u_{-3} \theta^{-3}(\tilde{s}_0) + u_{-4} \theta^{-4}(\tilde{s}_1) + u_{-5} \theta^{-5}(\tilde{s}_2) + u_{-6}\big).
\end{aligned}
$$

Obviously, $\partial_{t_1} = \partial$ and equations of flows here are reduced to usual KP flows (4.10) and (4.11) in [39] when $q \to 1$ and $u_0 = 0$. If we only consider the first three flows, i.e. flows of (t_1, t_2, t_3), then $u_{-1} = u_{-1}(t_1, t_2, t_3)$ is a q-deformation of the solution of KP equation [39]

$$
\frac{\partial}{\partial t_1}\left(4\frac{\partial u}{\partial t_3} - 12u\frac{\partial u}{\partial t_1} - \frac{\partial^3 u}{\partial t_1^3}\right) - 3\frac{\partial^2 u}{\partial t_2^2} = 0.
$$

In other words, $u_{-1} = u(t_1, t_2, t_3)$ in the above equation when $q \to 1$, and hence u_{-1} is called a q-soliton if $u(t_1, t_2, t_3) = \lim_{q\to 1} u_{-1}$ is a soliton solution of KP equation.

On the other hand, L in (2.1) can be generated by dressing operator $S = 1 + \sum_{k=1}^{\infty} s_k \partial_q^{-k}$ in the following way

$$
L = S \circ \partial_q \circ S^{-1}. \tag{2.3}
$$

Further, the dressing operator S satisfies the Sato equation

$$
\frac{\partial S}{\partial t_n} = -(L^n)_- S, \qquad n = 1, 2, 3, \dots. \tag{2.4}
$$

The q-wave function $w_q(x, \bar{t})$ and q-adjoint wave function $w_q^*(x, \bar{t})$ for q-KP hierarchy are defined by

$$
w_q(x, \bar{t}; z) = \left(S e_q(xz) \exp\left(\sum_{i=1}^{\infty} t_i z^i\right)\right) \tag{2.5}
$$

and

$$
w^*(x, \bar{t}; z) = \left((S^*)^{-1}|_{x/q} e_{1/q}(-xz) \exp\left(-\sum_{i=1}^{\infty} t_i z^i\right)\right), \tag{2.6}
$$

where $\bar{t} = (t_1, t_2, t_3, \ldots)$. Here, for a q-PDO $P = \sum_i p_i(x)\partial_q^i$, the notation

$$P|_{x/t} = \sum_i p_i(x/t)t^i\partial_q^i$$

is used in (2.6). Note that $w_q(x,\bar{t})$ and $w_q^*(x,\bar{t})$ satisfy following linear q-differential equations,

$$\begin{aligned} &(Lw_q) = zw_q, && \frac{\partial w_q}{\partial t_n} = (B_n w_q), \\ &(L^*|_{x/q} w_q^*) = zw_q^*, && \frac{\partial w_q^*}{\partial t_n} = -((B_n|_{x/q})^* w_q^*). \end{aligned} \tag{2.7}$$

Furthermore, $w_q(x,\bar{t})$ and $w_q^*(x,\bar{t})$ can be expressed by sole function $\tau_q(x,\bar{t})$ as

$$\omega_q = \frac{\tau_q(x;\bar{t} - [z^{-1}])}{\tau_q(x;\bar{t})} e_q(xz)\exp\left(\sum_{i=1}^{\infty} t_i z^i\right), \tag{2.8}$$

$$\omega_q^* = \frac{\tau_q(x;\bar{t} + [z^{-1}])}{\tau_q(x;\bar{t})} e_{1/q}(-xz)\exp\left(-\sum_{i=1}^{\infty} t_i z^i\right),$$

where

$$[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots\right).$$

From comparison of (2.5) and (2.8), the dressing operator S has the form of

$$S = 1 - \left(\frac{1}{\tau_q}\frac{\partial}{\partial t_1}\tau_q\right)\partial_q^{-1} + \left[\frac{1}{2\tau_q}\left(\frac{\partial^2}{\partial t_1^2} - \frac{\partial}{\partial t_2}\right)\tau_q\right]\partial_q^{-2} + \cdots. \tag{2.9}$$

Using (2.9) in (2.3), and then comparing with Lax operator in (2.1), we can show that all dynamical variables u_i $(i = 0, -1, -2, -3, \ldots)$ can be expressed by $\tau_q(x,\bar{t})$, and the first two are

$$\begin{aligned} &u_0 = s_1 - \theta(s_1) = -x(q-1)\partial_q s_1 = x(q-1)\partial_q\partial_{t_1}\ln\tau_q, \\ &u_{-1} = -\partial_q s_1 + s_2 - \theta(s_2) + \theta(s_1)s_1 - s_1^2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{2.10}$$

We can see $u_0 = 0$, and $u_{-1} = (\partial_x^2 \log\tau)$ as classical KP hierarchy when $q \to 1$, where $\tau = \tau_q(x,\bar{t})|_{q\to 1}$. By considering u_{-1} depending only on (q, x, t_1, t_2, t_3), we can regard u_{-1} as q-deformation of solution of classical KP equation. We shall show the q-effect of this solution for q-KP hierarchy after we get τ_q in next section. In order to guarantee that $e_q(x)$ is convergent, we require the parameter $0 < q < 1$ and parameter x to be bounded.

Beside existence of the Lax operator, q-wave function, τ_q for q-KP hierarchy, another important property is the q-deformed Z-S equation and associated linear q-differential equation. In other words, q-KP hierarchy also has an alternative expression, i.e.,

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0, \qquad m, n = 1, 2, 3, \ldots. \tag{2.11}$$

The "eigenfunction" ϕ and "adjoint eigenfunction" ψ of q-KP hierarchy associated to (2.11) are defined by

$$\frac{\partial\phi}{\partial t_n} = (B_n\phi), \tag{2.12}$$

$$\frac{\partial\psi}{\partial t_n} = -(B_n^*\psi), \tag{2.13}$$

where $\phi = \phi(\lambda; x, \bar{t})$ and $\psi = \psi(\mu; x, \bar{t})$. Here (2.13) is different from the second equation in (2.7). $\phi_i \equiv \phi(\lambda_i; x, \bar{t})$ and $\psi_i \equiv \psi(\mu_i; x, \bar{t})$ will be generating functions of gauge transformations.

3 Gauge transformations of q-KP hierarchy

The authors in [15] reported two types of elementary gauge transformation operator only for q-Gelfand–Dickey hierarchy. We extended the elementary gauge transformations given in [15], for the q-KP hierarchy. At the same time, we shall add some vital operator identity concerning to the q-differential operator and its inverse. Here we shall prove two transforming rules of τ function, "eigenfunction" and "adjoint eigenfunction" of the q-KP hierarchy under these transformations. Majority of the proofs are similar to the classical case given by [32, 33] and [35], so we will omit part of the proofs.

Suppose T is a pseudo-differential operator, and

$$L^{(1)} = T \circ L \circ T^{-1}, \qquad B_n^{(1)} \equiv \big(L^{(1)}\big)^n_+,$$

so that

$$\frac{\partial}{\partial t_n} L^{(1)} = \big[B_n^{(1)}, L^{(1)}\big]$$

still holds for the transformed Lax operator $L^{(1)}$; then T is called a gauge transformation operator of the q-KP hierarchy.

Lemma 1. *The operator T is a gauge transformation operator, if*

$$\big(T \circ B_n \circ T^{-1}\big)_+ = T \circ B_n \circ T^{-1} + \frac{\partial T}{\partial t_n} \circ T^{-1}, \tag{3.1}$$

or

$$\big(T \circ B_n \circ T^{-1}\big)_- = -\frac{\partial T}{\partial t_n} \circ T^{-1}. \tag{3.2}$$

If the initial Lax operator of q-KP is a "free" operator $L = \partial_q$, then the gauge transformation operator is also a dressing operator for new q-KP hierarchy whose Lax operator $L^{(1)} = T \circ \partial_q \circ T^{-1}$, because of (3.2) becomes

$$T_{t_n} = -\big(T \circ B_n \circ T^{-1}\big)_- \circ T = -\big(T \circ \partial_q^n \circ T^{-1}\big)_- \circ T = -\big(L^{(1)}\big)^n_- \circ T, \tag{3.3}$$

which is the Sato equation (2.4). In order to prove existence of two types of the gauge transformation operator, the following operator identities are necessary.

Lemma 2. *Let f be a suitable function, and A be a q-deformed pseudo-differential operator, then*

$$\begin{aligned}
(1)\quad & \big(\theta(f) \circ \partial_q \circ f^{-1} \circ A \circ f \circ \partial_q^{-1} \circ (\theta(f))^{-1}\big)_+ \\
&\quad = \theta(f) \circ \partial_q \circ f^{-1} \circ A_+ \circ f \circ \partial_q^{-1} \circ (\theta(f))^{-1} \\
&\qquad - \theta(f) \circ \big[\partial_q\big(f^{-1} \cdot (A_+ \cdot f)\big)\big] \circ \partial_q^{-1} \circ (\theta(f))^{-1},
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
(2)\quad & \big(\theta^{-1}(f^{-1}) \circ \partial_q^{-1} \circ f \circ A \circ f^{-1} \circ \partial_q \circ \theta^{-1}(f)\big)_- \\
&\quad = \theta^{-1}(f^{-1}) \circ \partial_q^{-1} \circ f \circ A_- \circ f^{-1} \circ \partial_q \circ \theta^{-1}(f) \\
&\qquad - \theta^{-1}(f^{-1}) \circ \partial_q^{-1} \circ \theta^{-1}(f) \circ \partial_q\big(\theta^{-1}[f^{-1} \cdot \big(A_+^* \cdot f\big)]\big).
\end{aligned} \tag{3.5}$$

Remark 1. This lemma is a q-analogue of corresponding identities of pseudo-differential operators given by [33].

Theorem 1. *There exist two kinds of gauge transformation operator for the q-KP hierarchy, namely*

$$\text{Type I}: \quad T_D(\phi_1) = \theta(\phi_1) \circ \partial_q \circ \phi_1^{-1}, \tag{3.6}$$
$$\text{Type II}: \quad T_I(\psi_1) = (\theta^{-1}(\psi_1))^{-1} \circ \partial_q^{-1} \circ \psi_1. \tag{3.7}$$

Here ϕ_1 and ψ_1 are defined by (2.12) *and* (2.13) *that are called the generating functions of gauge transformation.*

Proof. First of all, for the Type I case (see (3.6)),

$$\begin{aligned}
B_n^{(1)} &\equiv \big(L^{(1)}\big)^n_+ = \big(T_D \circ (L)^n \circ T_D^{-1}\big)_+ \\
&= T_D \circ B_n \circ T_D^{-1} - \theta(\phi_1) \cdot \partial_q\big(\phi_1^{-1} \cdot (B_n \cdot \phi_1)\big) \circ \partial_q^{-1} \circ (\theta(\phi_1))^{-1} \\
&= T_D \circ B_n^{(0)} \circ T_D^{-1} - \Big(\theta(\phi_1) \circ \partial_q \circ \frac{(\phi_1)_{t_n}}{\phi_1} \circ \partial_q^{-1} \circ (\theta(\phi_1))^{-1} \\
&\quad - \theta(\phi_1) \circ \theta\Big(\frac{(\phi_1)_{t_n}}{\phi_1}\Big) \circ \partial_q \circ \partial_q^{-1} \circ (\theta(\phi_1))^{-1}\Big) \\
&= T_D \circ B_n \circ T_D^{-1} + \theta\Big(\frac{(\phi_1)_{t_n}}{\phi_1}\Big) - \theta(\phi_1) \circ \partial_q \circ \frac{(\phi_1)_{t_n}}{\phi_1} \circ \partial_q^{-1} \circ (\theta(\phi_1))^{-1}.
\end{aligned}$$

Here the operator identity (3.4), $B_n = (L)^n_+$, $(\phi_1)_{t_n} = (B_n \cdot \phi_1)$ and (1.2) were used. On the other hand,

$$\begin{aligned}
\frac{\partial T_D}{\partial t_n} \circ T_D^{-1} &= \big(\theta(\phi_1) \circ \partial_q \circ \phi_1^{-1}\big)_{t_n} \circ T_D^{-1} = \theta((\phi_1)_{t_n}) \circ \partial_q \circ \phi_1^{-1} \circ \phi_1 \circ \partial_q^{-1} \circ (\theta(\phi_1))^{-1} \\
&\quad - \theta(\phi_1) \circ \partial_q \circ \frac{(\phi_1)_{t_n}}{\phi_1^2} \circ \phi_1 \circ \partial_q^{-1} \circ (\theta(\phi_1))^{-1} \\
&= \theta\Big(\frac{(\phi_1)_{t_n}}{\phi_1}\Big) - \theta(\phi_1) \circ \partial_q \circ \frac{(\phi_1)_{t_n}}{\phi_1} \circ \partial_q^{-1} \circ (\theta(\phi_1))^{-1}.
\end{aligned}$$

Taking this expression back into $B_n^{(1)}$, we get

$$B_n^{(1)} \equiv \big(L^{(1)}\big)^n_+ = T_D \circ B_n \circ T_D^{-1} + \frac{\partial T_D}{\partial t_n} \circ T_D^{-1},$$

and that indicates that $\mathrm{T}_D(\phi_1)$ is indeed a gauge transformation operator via Lemma 1. Second, we want to prove that the equation (3.2) holds for Type II case (see (3.7)). By direct calculation the left hand side of (3.2) is in the form of

$$\begin{aligned}
\big(T_I \circ B_n \circ T_I^{-1}\big)_- &= \big((\theta^{-1}(\psi_1))^{-1} \circ \partial_q^{-1} \circ \psi_1 \circ B_n \circ \psi_1^{-1} \circ \partial_q \circ \theta^{-1}(\psi_1)\big)_- \\
&= (\theta^{-1}(\psi_1))^{-1} \circ \partial_q^{-1} \circ \psi_1 \circ (B_n)_- \circ \psi_1^{-1} \circ \partial_q \circ \theta^{-1}(\psi_1) \\
&\quad - (\theta^{-1}(\psi_1))^{-1} \circ \partial_q^{-1} \circ \theta^{-1}(\psi_1) \circ \left(\partial_q \theta^{-1}\left(\frac{(B_n^* \cdot \psi_1)}{\psi_1}\right)\right) \\
&= (\theta^{-1}(\psi_1))^{-1} \circ \partial_q^{-1} \circ \theta^{-1}(\psi_1) \circ \left(\partial_q \theta^{-1}\left(\frac{(\psi_1)_{t_n}}{\psi_1}\right)\right).
\end{aligned}$$

In the above calculation, the operator identity (3.5), $(B_n)_- = 0$, $(\psi_1)_{t_n} = -(B_n^* \cdot \psi_1)$ were used. Moreover, with the help of (1.2), we have

$$-\frac{\partial T_I}{\partial t_n} \circ T_I^{-1} = -\frac{\partial}{\partial t_n}\big((\theta^{-1}(\psi_1))^{-1} \circ \partial_q^{-1} \circ \psi_1\big) \circ \psi_1^{-1} \circ \partial_q \circ \theta^{-1}(\psi_1)$$

$$
\begin{aligned}
&= \frac{\theta^{-1}((\psi_1)_{t_n})}{(\theta^{-1}(\psi_1))^2} \circ \partial_q^{-1} \circ \psi_1 \circ \psi_1^{-1} \circ \partial_q \circ \theta^{-1}(\psi_1) \\
&\quad - (\theta^{-1}(\psi_1))^{-1} \circ \partial_q^{-1} \circ (\psi_1)_{t_n} \circ \psi_1^{-1} \circ \partial_q \circ \theta^{-1}(\psi_1) \\
&= \theta^{-1}\left(\frac{(\psi_1)_{t_n}}{\psi_1}\right) - \frac{1}{\theta^{-1}(\psi_1)} \circ \partial_q^{-1} \circ \left[\partial_q \circ \theta^{-1}\left(\frac{(\psi_1)_{t_n}}{\psi_1}\right)\right. \\
&\quad \left. - \left(\partial_q \cdot \theta^{-1}\left(\frac{(\psi_1)_{t_n}}{\psi_1}\right)\right)\right] \circ \theta^{-1}(\psi_1) = \theta^{-1}\left(\frac{(\psi_1)_{t_n}}{\psi_1}\right) \\
&\quad - \theta^{-1}\left(\frac{(\psi_1)_{t_n}}{\psi_1}\right) + \frac{1}{\theta^{-1}(\psi_1)} \circ \partial_q^{-1} \circ \left(\partial_q \cdot \theta^{-1}\left(\frac{(\psi_1)_{t_n}}{\psi_1}\right)\right) \circ \theta^{-1}(\psi_1) \\
&= \frac{1}{\theta^{-1}(\psi_1)} \circ \partial_q^{-1} \circ \left(\partial_q \cdot \theta^{-1}\left(\frac{(\psi_1)_{t_n}}{\psi_1}\right)\right) \circ \theta^{-1}(\psi_1).
\end{aligned}
$$

The two equations obtained above show that $\mathrm{T}_I(\psi_1)$ satisfies (3.2), so $T_I(\psi_1)$ is also a gauge transformation operator of the q-KP hierarchy according to Lemma 1. ■

Remark 2. There are two convenient expressions for T_D and T_I,

$$T_D = \partial_q - \alpha_1, \qquad T_D^{-1} = \partial_q^{-1} + \theta^{-1}(\alpha_1)\partial_q^{-2} + \cdots, \qquad \alpha_1 = \frac{\partial_q \phi_1}{\phi_1}, \tag{3.8}$$

$$T_I = (\partial_q + \beta_1)^{-1} = \partial_q^{-1} - \theta^{-1}(\beta_1)\partial_q^{-2} + \cdots, \qquad \beta_1 = \frac{\partial_q \theta^{-1}(\psi_1)}{\psi_1}. \tag{3.9}$$

In order to get a new solution of q-KP hierarchy from the input solution, we should know the transformed expressions of $u_i^{(1)}$, $\tau_q^{(1)}$, $\phi_i^{(1)}$, $\psi_i^{(1)}$. The following two theorems are related to this. Before we start to discuss explicit forms of them, we would like to define the generalized q-Wronskian for a set of functions $\{\psi_k, \psi_{k-1}, \dots, \psi_1; \phi_1, \phi_2, \dots, \phi_n\}$ as

$$
IW_{k,n}^q(\psi_k, \dots, \psi_1; \phi_1, \dots, \phi_n) = \begin{vmatrix}
\partial_q^{-1}\psi_k\phi_1 & \partial_q^{-1}\psi_k\phi_2 & \cdots & \partial_q^{-1}\psi_k\phi_n \\
\vdots & \vdots & \cdots & \vdots \\
\partial_q^{-1}\psi_1\phi_1 & \partial_q^{-1}\psi_1\phi_2 & \cdots & \partial_q^{-1}\psi_1\phi_n \\
\phi_1 & \phi_2 & \cdots & \phi_n \\
\partial_q\phi_1 & \partial_q\phi_2 & \cdots & \partial_q\phi_n \\
\vdots & \vdots & \cdots & \vdots \\
\partial_q^{n-k-1}\phi_1 & \partial_q^{n-k-1}\phi_2 & \cdots & \partial_q^{n-k-1}\phi_n
\end{vmatrix},
$$

which reduce to the q-Wronskian when $k = 0$,

$$
W_n^q(\phi_1, \cdots, \phi_n) = \begin{vmatrix}
\phi_1 & \phi_2 & \cdots & \phi_n \\
\partial_q\phi_1 & \partial_q\phi_2 & \cdots & \partial_q\phi_n \\
\vdots & \vdots & \cdots & \vdots \\
\partial_q^{n-1}\phi_1 & \partial_q^{n-1}\phi_2 & \cdots & \partial_q^{n-1}\phi_n
\end{vmatrix}.
$$

Suppose $\phi_1(\lambda_1; x, \bar{t})$ is a known "eigenfunction" of q-KP with the initial function τ_q, which generates gauge transformation operator $T_D(\phi_1)$. Then we have

Theorem 2. *Under the gauge transformation $L^{(1)} = T_D(\phi_1) \circ L \circ (T_D(\phi_1))^{-1}$, new "eigenfunction", "adjoint eigenfunction" and τ function of the transformed q-KP hierarchy are*

$$\phi \longrightarrow \phi^{(1)}(\lambda; x, \bar{t}) = (T_D(\phi_1) \cdot \phi) = \frac{W_2^q(\phi_1, \phi)}{\phi_1},$$

$$\psi \longrightarrow \psi^{(1)}(\lambda; x, \bar{t}) = \big(\big(T_D(\phi_1)^{-1}\big)^* \cdot \psi\big) = \frac{\theta(\partial_q^{-1}\phi_1\psi)}{\theta(\phi_1)},$$
$$\tau_q \longrightarrow \tau_q^{(1)} = \phi_1\tau_q.$$

$\phi_k^{(1)} = \phi^{(1)}(\lambda = \lambda_k; x, \bar{t})$, $\psi_k^{(1)} = \psi^{(1)}(\lambda = \lambda_k; x, \bar{t})$. Note $\phi_1^{(1)} = 0$.

Proof. (1) By direct calculations, we have

$$\begin{aligned}\big(\partial_{t_n}\phi^{(1)}\big) &= (\partial_{t_n}(T_D\cdot\phi)) = (\partial_{t_n}T_D)\cdot\phi + (T_D\cdot\partial_{t_n}\phi)\\ &= \big(\partial_{t_n}T_D\circ T_D^{-1}\big)\cdot(T_D\phi) + T_D\cdot(B_n\phi) = \big(\partial_{t_n}T_D\circ T_D^{-1} + T_D\circ B_n\circ T_D^{-1}\big)\cdot(T_D\phi)\\ &= \big(B_n^{(1)}\cdot\phi^{(1)}\big),\end{aligned}$$

in which (2.12) and (3.1) were used.

(2) Similarly, with the help of $(B_n^{(1)})^* = (T_D^{-1})^*\circ\partial_{t_n}T_D^* + (T_D^{-1})^*\circ B_n^*\circ T_D^*$ and (2.13), we can obtain

$$\begin{aligned}\partial_{t_n}\psi^{(1)} &= \big((T_D^{-1})^*\cdot\psi\big)_{t_n} = \big(-(T_D^*)^{-1}\circ\partial_{t_n}T_D^*\circ(T_D^*)^{-1}\big)\cdot\psi + (T_D^*)^{-1}\cdot\partial_{t_n}\psi\\ &= -\big((T_D^{-1})^*\circ\partial_{t_n}T_D^* + (T_D^{-1})^*\circ B_n^*\circ T_D^*\big)\cdot\big((T_D^{-1})^*\cdot\psi\big) = -\big(B_n^{(1)}\big)^*\cdot\psi^{(1)}.\end{aligned}$$

(3) According to the definition of T_D in (3.6) and with the help of (3.8), $L^{(1)}$ can be expressed as

$$L_q^{(1)} = \partial_q + u_0^{(1)} + u_{-1}^{(1)}\partial_q^{-1} + \cdots, \qquad u_0^{(1)} = x(q-1)\partial_q\alpha_1 + \theta(u_0).$$

On the other hand, $(\phi_1)_{t_1} = ((L)_+\phi_1)$ implies $\alpha_1 = \partial_{t_1}\ln\phi_1 - u_0$, then $u_0^{(1)}$ becomes

$$\begin{aligned}u_0^{(1)} &= x(q-1)\partial_q\partial_{t_1}\ln\phi_1 + u_0 = x(q-1)\partial_q\partial_{t_1}\ln\phi_1 + x(q-1)\partial_q\partial_{t_1}\ln\tau_q\\ &= x(q-1)\partial_q\partial_{t_1}\ln\phi_1\tau_q.\end{aligned}$$

Then

$$\tau_q^{(1)} = \phi_1\tau_q.$$

This completes the proof of the theorem. ■

For the gauge transformation operator of Type II, there exist similar results. Let $\psi_1(\mu_1; x, \bar{t})$ be a known "adjoint eigenfunction" of q-KP with the initial function τ_q, which generates the gauge transformation operator $T_I(\psi_1)$. Then we have

Theorem 3. *Under the gauge transformation $L^{(1)} = T_I(\psi_1)\circ L\circ(T_I(\psi_1))^{-1}$, new "eigenfunction", "adjoint eigenfunction" and τ function of the transformed q-KP hierarchy are*

$$\phi \longrightarrow \phi^{(1)}(\lambda; x, \bar{t}) = (T_I(\psi_1)\cdot\phi) = \frac{(\partial_q^{-1}\psi_1\phi)}{\theta^{-1}(\psi_1)},$$
$$\psi \longrightarrow \psi^{(1)}(\lambda; x, \bar{t}) = \big(\big(T_I(\psi_1)^{-1}\big)^*\cdot\psi\big) = \frac{\widetilde{W}_2^q(\psi_1, \psi)}{\psi_1},$$
$$\tau_q \longrightarrow \tau_q^{(1)} = \theta^{-1}(\psi_1)\tau_q.$$

$\phi_k^{(1)} = \phi^{(1)}(\lambda = \lambda_k; x, \bar{t})$, $\psi_k^{(1)} = \psi^{(1)}(\lambda = \lambda_k; x, \bar{t})$. Note $\psi_1^{(1)} = 0$. $\widetilde{W}_n^q$ is obtained from W_n^q by replacing ∂_q with ∂_q^*.

Proof. The proof is analogous to the proof of the previous theorem. So it is omitted. ■

4 Successive applications of gauge transformations

We now discuss successive applications of the two types of gauge transformation operators in a general way, which is similar to the classical case [32, 34, 35]. For example, consider the chain of gauge transformation operators,

$$\begin{aligned}
&L \xrightarrow{T_D^{(1)}\left(\phi_1\right)} L^{(1)} \xrightarrow{T_D^{(2)}\left(\phi_2^{(1)}\right)} L^{(2)} \xrightarrow{T_D^{(3)}\left(\phi_3^{(2)}\right)} L^{(3)} \to \cdots \to L^{(n-1)} \xrightarrow{T_D^{(n)}\left(\phi_n^{(n-1)}\right)} L^{(n)} \\
&\xrightarrow{T_I^{(n+1)}\left(\psi_1\right)} L^{(n+1)} \xrightarrow{T_I^{(n+2)}\left(\psi_2^{(n+1)}\right)} L^{(n+2)} \to \cdots \to L^{(n+k-1)} \xrightarrow{T_I^{(n+k)}\left(\psi_k^{(n+k-1)}\right)} L^{(n+k)}. \quad (4.1)
\end{aligned}$$

Here the index "i" in a gauge transformation operator means the i-th gauge transformation, and $\phi_i^{(j)}$ (or $\psi_i^{(j)}$) is transformed by j-steps gauge transformations from ϕ_i (or ψ_i), $L^{(i)}$ is transformed by j-step gauge transformations from the initial Lax operator L. Successive applications of gauge transformation operator in (4.1) can be represented by

$$\begin{aligned}
T_{n+k} = {} & T_I^{(n+k)}\left(\psi_k^{(n+k-1)}\right)\cdots T_I^{(n+2)}\left(\psi_2^{(n+1)}\right)\circ T_I^{(n+1)}\left(\psi_1^{(n)}\right) \\
& \circ T_D^{(n)}\left(\phi_n^{(n-1)}\right)\cdots T_D^{(2)}\left(\phi_2^{(1)}\right)\circ T_D^{(1)}(\phi_1).
\end{aligned}$$

Our goal is to obtain $\phi^{(n+k)}$, $\psi^{(n+k)}$, $\tau_q^{(n+k)}$ of $L^{(n+k)}$ transformed from L by the T_{n+k} in the above chain. All of these are based on the determinant representation of gauge transformation operator T_{n+k}. As the proof of the determinant representation of T_{n+k} is similar extremely to the case of classical KP hierarchy [34], we will omit it.

Lemma 3. *The gauge transformation operator T_{n+k} has the following determinant representation ($n > k$):*

$$T_{n+k} = \frac{1}{IW_{k,n}^q(\psi_k,\ldots,\psi_1;\phi_1,\ldots,\phi_n)}
\begin{vmatrix}
\partial_q^{-1}\psi_k\phi_1 & \cdots & \partial_q^{-1}\psi_k\phi_n & \partial_q^{-1}\circ\psi_k \\
\vdots & \cdots & \vdots & \vdots \\
\partial_q^{-1}\psi_1\phi_1 & \cdots & \partial_q^{-1}\psi_1\phi_n & \partial_q^{-1}\circ\psi_1 \\
\phi_1 & \cdots & \phi_n & 1 \\
\partial_q\phi_1 & \cdots & \partial_q\phi_n & \partial_q \\
\vdots & \cdots & \vdots & \vdots \\
\partial_q^{n-k}\phi_1 & \cdots & \partial_q^{n-k}\phi_n & \partial_q^{n-k}
\end{vmatrix}$$

and

$$\begin{aligned}
T_{n+k}^{-1} = {} & \begin{vmatrix}
\phi_1\circ\partial_q^{-1} & \theta(\partial_q^{-1}\psi_k\phi_1) & \cdots & \theta(\partial_q^{-1}\psi_1\phi_1) & \theta(\phi_1) & \cdots & \theta(\partial_q^{n-k-2}\phi_1) \\
\phi_2\circ\partial_q^{-1} & \theta(\partial_q^{-1}\psi_k\phi_2) & \cdots & \theta(\partial_q^{-1}\psi_1\phi_2) & \theta(\phi_2) & \cdots & \theta(\partial_q^{n-k-2}\phi_2) \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\phi_n\circ\partial_q^{-1} & \theta(\partial_q^{-1}\psi_k\phi_n) & \cdots & \theta(\partial_q^{-1}\psi_1\phi_n) & \theta(\phi_n) & \cdots & \theta(\partial_q^{n-k-2}\phi_n)
\end{vmatrix} \\
& \times \frac{(-1)^{n-1}}{\theta(IW_{k,n}^q(\psi_k,\ldots,\psi_1;\phi_1,\ldots,\phi_n)}.
\end{aligned}$$

Lemma 4. *Under the case of $n = k$, T_{n+k} is given by*

$$T_{n+n} = \frac{1}{IW_{n,n}^q(\psi_n,\ldots,\psi_1;\phi_1,\ldots,\phi_n)}
\begin{vmatrix}
\partial_q^{-1}\psi_n\phi_1 & \cdots & \partial_q^{-1}\psi_n\phi_n & \partial_q^{-1}\circ\psi_n \\
\vdots & \cdots & \vdots & \vdots \\
\partial_q^{-1}\psi_1\phi_1 & \cdots & \partial_q^{-1}\psi_1\phi_n & \partial_q^{-1}\circ\psi_1 \\
\phi_1 & \cdots & \phi_n & 1
\end{vmatrix}$$

but T^{-1}_{n+n} *becomes*

$$T^{-1}_{n+n} = \begin{vmatrix} -1 & \psi_n & \cdots & \psi_1 \\ \phi_1 \circ \partial_q^{-1} & \theta(\partial_q^{-1}\psi_n\phi_1) & \cdots & \theta(\partial_q^{-1}\psi_1\phi_1) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_n \circ \partial_q^{-1} & \theta(\partial_q^{-1}\psi_n\phi_n) & \cdots & \theta(\partial_q^{-1}\psi_1\phi_n) \end{vmatrix} \frac{(-1)}{\theta(IW^q_{n,n}(\psi_n,\ldots,\psi_1;\phi_1,\ldots,\phi_n))}.$$

In the above lemmas, T_{n+k} are expanded with respect to the last column collecting all sub-determinants on the left of the symbols ∂_q^i $(i=-1,0,1,2,\ldots,n-k)$; T^{-1}_{n+k} are expanded with respect to the first column by means of collection of all minors on the right of $\phi_i\partial_q^{-1}$. Basing on the determinant representation, first of all, we would like to consider the case of $k=0$ in (4.1), i.e.

$$L \xrightarrow{T_D^{(1)}(\phi_1)} L^{(1)} \xrightarrow{T_D^{(2)}(\phi_2^{(1)})} L^{(2)} \xrightarrow{T_D^{(3)}(\phi_3^{(2)})} L^{(3)} \to \cdots \to L^{(n-1)} \xrightarrow{T_D^{(n)}(\phi_n^{(n-1)})} L^{(n)},$$

whose corresponding equivalent gauge transformation operator is

$$T_n = T_D^{(n)}(\phi_n^{(n-1)})\cdots T_D^{(2)}(\phi_2^{(1)})\circ T_D^{(1)}(\phi_1). \tag{4.2}$$

Theorem 4. *Under the gauge transformation* T_n $(n\geq 1)$,

$$\phi^{(n)}(\lambda;x,\bar{t}) = (T_n\cdot\phi) = \frac{W^q_{n+1}(\phi_1,\ldots,\phi_n,\phi)}{W^q_n(\phi_1,\ldots,\phi_n)}, \tag{4.3}$$

$$\psi^{(n)}(\mu;x,\bar{t}) = \left((T_n^{-1})^*\cdot\psi\right) = (-1)^n\theta\left(\frac{IW^q_{1,n}(\psi;\phi_1,\ldots,\phi_n)}{W^q_n(\phi_1,\ldots,\phi_n)}\right), \tag{4.4}$$

$$\tau_q^{(n)} = W_n^q(\phi_1,\ldots,\phi_n)\cdot\tau_q.$$

Furthermore, $\phi_i^{(n)} = \phi^{(n)}(\lambda=\lambda_i;x,\bar{t})$, $\psi_i^{(n)} = \psi^{(n)}(\mu=\mu_i;x,\bar{t})$. *Note* $\phi_i^{(n)}=0$ *if* $i\in\{1,2,\ldots,n\}$.

Proof. (1) Successive application of Theorem 2 implies

$$\begin{aligned}\phi^{(n)} &= T_D^{(n)}(\phi_n^{(n-1)})\phi^{(n-1)} = T_D^{(n)}(\phi_n^{(n-1)})T_D^{(n-1)}(\phi_{n-1}^{(n-2)})\phi^{(n-2)} = \cdots\\ &= T_D^{(n)}(\phi_n^{(n-1)})\cdots T_D^{(2)}(\phi_2^{(1)})\circ T_D^{(1)}(\phi_1)\phi = (T_n\cdot\phi).\end{aligned}$$

Using the determinant representation of T_n in it leads to $\phi^{(n)}$. Here $T_D^{(1)}(\phi_1) = T_D(\phi_1)$.

(2) Similarly, according to Theorem 2 we have

$$\begin{aligned}\psi^{(n)} &= \big(T_D^{(n)^{-1}}\big)^*\psi^{(n-1)} = \big(T_D^{(n)^{-1}}\big)^*\big(T_D^{(n-1)^{-1}}\big)^*\psi^{(n-2)} = \cdots\\ &= \Big(\big(T_D^{(n)^{-1}}\big)^*\big(T_D^{(n-1)^{-1}}\big)^*\cdots\big(T_D^{(3)^{-1}}\big)^*\big(T_D^{(2)^{-1}}\big)^*\big(T_D^{-1}\big)^*\Big)\cdot\psi = \big((T_n^{-1})^*\cdot\psi\big).\end{aligned}$$

Then $\psi^{(n)}$ can be deduced by using the determinant representation of T_n^{-1} in the Lemma 3 with $k=0$. Here we omit the generating functions in $T_D^{(i)}$ $(i=1,2,\ldots,n)$, which are the same as (1).

(3) Meanwhile, we can get $\tau^{(n)}$ by repeated iteration according to the rule in Theorem 2,

$$\begin{aligned}\tau_q^{(n)} &= \phi_n^{(n-1)}\tau_q^{(n-1)} = \phi_n^{(n-1)}\phi_{n-1}^{(n-2)}\tau_q^{(n-2)} = \phi_n^{(n-1)}\phi_{n-1}^{(n-2)}\phi_{n-2}^{(n-3)}\tau_q^{(n-3)} = \cdots\\ &= \phi_n^{(n-1)}\phi_{n-1}^{(n-2)}\phi_{n-2}^{(n-3)}\cdots\phi_4^{(3)}\phi_3^{(2)}\phi_2^{(1)}\phi_1\tau_q\\ &= \frac{W_n^q(\phi_1,\phi_2,\phi_3,\ldots,\phi_n)}{W_{n-1}^q(\phi_1,\phi_2,\phi_3,\ldots,\phi_{n-1})}\frac{W_{n-1}^q(\phi_1,\phi_2,\phi_3,\ldots,\phi_{n-1})}{W_{n-2}^q(\phi_1,\phi_2,\phi_3,\ldots,\phi_{n-2})}\frac{W_{n-2}^q(\phi_1,\phi_2,\phi_3,\ldots,\phi_{n-2})}{W_{n-3}^q(\phi_1,\phi_2,\phi_3,\ldots,\phi_{n-3})}\\ &\quad\cdots\frac{W_4^q(\phi_1,\phi_2,\phi_3,\phi_4)}{W_3^q(\phi_1,\phi_2,\phi_3)}\frac{W_3^q(\phi_1,\phi_2,\phi_3)}{W_2^q(\phi_1,\phi_2)}\frac{W_2^q(\phi_1,\phi_2)}{W_1^q(\phi_1)}\phi_1\tau_q = W_n^q(\phi_1,\phi_2,\ldots,\phi_n)\tau_q.\end{aligned}$$

with the help of the determinant representation of Lemma 3 with $k=0$. Here $W_1^q(\phi_1)=\phi_1$. ∎

It should be noted that there is a θ action in (4.4), which is the main difference between the q-KP and classical KP beside different elements in determinant. Furthermore, for more complicated chain of gauge transformation operators in (4.1), $\phi^{(n+k)}$, $\psi^{(n+k)}$, $\tau_q^{(n+k)}$ of $L^{(n+k)}$ can be expressed by the generalized q-Wronskian.

Theorem 5. *Under the gauge transformation T_{n+k} $(n > k > 0)$,*

$$\phi^{(n+k)}(\lambda; x, \bar{t}) = (T_{n+k} \cdot \phi) = \frac{IW^q_{k,n+1}(\psi_k, \ldots, \psi_1; \phi_1, \ldots, \phi_n, \phi)}{IW^q_{k,n}(\psi_k, \ldots, \psi_1; \phi_1, \ldots, \phi_n)},$$

$$\psi^{(n+k)}(\mu; x, \bar{t}) = \big((T^{-1}_{n+k})^* \cdot \psi\big) = (-1)^n \frac{IW^q_{k+1,n}(\psi, \psi_k, \psi_{k-1}, \ldots, \psi_1; \phi_1, \ldots, \phi_n)}{IW^q_{k,n}(\psi_k, \ldots, \psi_1; \phi_1, \ldots, \phi_n)},$$

$$\tau_q^{(n+k)} = IW^q_{k,n}(\psi_k, \ldots, \psi_1; \phi_1, \cdots, \phi_n) \cdot \tau_q.$$

Furthermore, $\phi_i^{(n+k)} = \phi^{(n+k)}(\lambda = \lambda_i; x, \bar{t})$; $\psi_i^{(n+k)} = \psi^{(n+k)}(\mu = \mu_i; x, \bar{t})$. Note $\phi_i^{(n+k)} = 0$ if $i \in \{1, 2, \ldots, n\}$, $\psi_i^{(n+k)} = 0$ if $i \in \{1, 2, \ldots, k\}$.

Proof. (1) The repeated iteration of Theorems 2 and 3 according to the ordering of T_I and T_D deduces

$$\begin{aligned}
\phi^{(n+k)} &= T_I^{(n+k)}\big(\psi_{n+k}^{(n+k-1)}\big) \cdot \phi^{(n+k-1)} \\
&= T_I^{(n+k)}\big(\psi_{n+k}^{(n+k-1)}\big)T_I^{(n+k-1)}\big(\psi_{n+k-1}^{(n+k-2)}\big) \cdot \phi^{(n+k-2)} = \cdots \\
&= T_I^{(n+k)}\big(\psi_{n+k}^{(n+k-1)}\big)T_I^{(n+k-1)}\big(\psi_{n+k-1}^{(n+k-2)}\big) \cdots T_I^{(n+2)}\big(\psi_{n+2}^{(n+1)}\big)T_I^{(n+1)}\big(\psi_{n+1}^{(n)}\big) \cdot \phi^{(n)}.
\end{aligned}$$

Then taking in it $\phi^{(n)} = (T_n \cdot \phi)$ from (4.3) , we get

$$\begin{aligned}
\phi^{(n+k)} &= \big(T_I^{(n+k)}\big(\psi_{n+k}^{(n+k-1)}\big)T_I^{(n+k-1)}\big(\psi_{n+k-1}^{(n+k-2)}\big) \cdots T_I^{(n+2)}\big(\psi_{n+2}^{(n+1)}\big)T_I^{(n+1)}\big(\psi_{n+1}^{(n)}\big)T_n\big) \cdot \phi \\
&= (T_{n+k} \cdot \phi).
\end{aligned}$$

Therefore the determinant form of $\phi^{(n+k)}$ is given by Lemma 3.

(2) Using Theorems 2 and 3 iteratively according to the chain in (4.1), similarly to the step (1), we can get

$$\begin{aligned}
\psi^{(n+k)} &= \big(T_I^{(n+k)^{-1}}\big)^* \cdot \psi^{(n+k-1)} = \big(T_I^{(n+k)^{-1}}\big)^*\big(T_I^{(n+k-1)^{-1}}\big)^* \cdot \psi^{(n+k-2)} = \cdots \\
&= \big(T_I^{(n+k)^{-1}}\big)^*\big(T_I^{(n+k-1)^{-1}}\big)^* \cdots \big(T_I^{(n+2)^{-1}}\big)^*\big(T_I^{(n+1)^{-1}}\big)^* \cdot \psi^{(n)}.
\end{aligned}$$

Noting that $\psi^{(n)}$ is given by (4.4), we get $\psi^{(n+k)} = ((T^{-1}_{n+k})^* \cdot \psi)$. The explicit form of $\psi^{(n+k)}$ is given from the determinant representation of T^{-1}_{n+k}.

(3) According to the changing rule under gauge transformation in Theorems 2 and 3, the new τ function of q-KP hierarchy $\tau_q^{(n+k)}$ produced by chain of gauge transformations in (4.1) is

$$\begin{aligned}
\tau_q^{(n+k)} &= \theta^{-1}\big(\psi_k^{(n+k-1)}\big)\tau_q^{(n+k-1)} = \theta^{-1}\big(\psi_k^{(n+k-1)}\big)\theta^{-1}\big(\psi_{k-1}^{(n+k-2)}\big)\tau_q^{(n+k-2)} \\
&= \theta^{-1}\big(\psi_k^{(n+k-1)}\big)\theta^{-1}\big(\psi_{k-1}^{(n+k-2)}\big) \cdots \theta^{-1}\big(\psi_2^{(n+1)}\big)\theta^{-1}\big(\psi_1^{(n)}\big)\tau_q^{(n)}.
\end{aligned}$$

So the explicit form of $\psi_i^{(n+i-1)}$ $(i = 1, 2, \ldots, k)$ and $\tau_q^{(n)}$ implies

$$\tau_q^{(n+k)} = (-1)^n \frac{IW^q_{k,n}(\psi_k, \psi_{k-1}, \ldots, \psi_1; \phi_1, \phi_2, \ldots, \phi_n)}{IW^q_{k-1,n}(\psi_{k-1}, \ldots, \psi_1; \phi_1, \phi_2, \ldots, \phi_n)}$$

$$
\begin{aligned}
&\times(-1)^n\frac{IW^q_{k-1,n}(\psi_{k-1},\psi_{k-1},\dots,\psi_1;\phi_1,\phi_2,\dots,\phi_n)}{IW^q_{k-2,n}(\psi_{k-2},\dots,\psi_1;\phi_1,\phi_2,\dots,\phi_n)}\\
&\cdots(-1)^n\frac{IW^q_{2,n}(\psi_2,\psi_1;\phi_1,\phi_2,\dots,\phi_n)}{IW^q_{1,n}(\psi_1;\phi_1,\phi_2,\dots,\phi_n)}\\
&\times(-1)^n\frac{IW^q_{1,n}(\psi_1;\phi_1,\phi_2,\dots,\phi_n)}{W^q_n(\phi_1,\phi_2,\dots,\phi_n)}W^q_n(\phi_1,\phi_2,\dots,\phi_n)\tau_q\\
\approx\ & IW^q_{k,n}(\psi_k,\psi_{k-1},\dots,\psi_1;\phi_1,\phi_2,\dots,\phi_n)\tau_q.
\end{aligned}
$$

We omitted the trivial factor $(-1)^n$ in the last step, because it will not affect u_i in the q-KP hierarchy. ■

Remark 3. There exists another complicated chain of gauge transformation operators for q-KP hierarchy (that may be regarded as motivated by the classical KP hierarchy)

$$
\begin{aligned}
&L\xrightarrow{T_I^{(1)}\left(\psi_1\right)}L^{(1)}\xrightarrow{T_I^{(2)}\left(\psi_2^{(1)}\right)}L^{(2)}\xrightarrow{T_I^{(3)}\left(\psi_3^{(2)}\right)}L^{(3)}\to\cdots\to L^{(n-1)}\xrightarrow{T_I^{(n)}\left(\psi_n^{(n-1)}\right)}L^{(n)}\\
&\xrightarrow{T_D^{(n+1)}\left(\phi_1^{(n)}\right)}L^{(n+1)}\xrightarrow{T_D^{(n+2)}\left(\phi_2^{(n+1)}\right)}L^{(n+2)}\to\cdots\to L^{(n+k-1)}\xrightarrow{T_D^{(n+k)}\left(\phi_k^{(n+k-1)}\right)}L^{(n+k)},
\end{aligned}
$$

that can lead to another form of $\tau_q^{(n+k)}$. This is parallel to the classical case of [32].

If the initial q-KP is a "free" operator, then $L=\partial_q$ that means the initial τ function is 1. We can write down the explicit form of q-KP hierarchy generated by T_{n+k}. Under this situation, (2.12) and (2.13) become

$$
\frac{\partial\phi}{\partial t_n}=(\partial_q^n\phi),\qquad \frac{\partial\psi}{\partial t_n}=-(\partial_q^{n*}\psi), \tag{4.5}
$$

that possess set of solution $\{\phi_i,\psi_i\}$ as follows

$$
\phi_i(x;\bar t)=e_q(\lambda_{i_1}x)e^{\sum\limits_{j=1}^{\infty}t_j\lambda_{i_1}^j}+a_ie_q(\mu_{i_1}x)e^{\sum\limits_{j=1}^{\infty}t_j\mu_{i_1}^j}, \tag{4.6}
$$

$$
\psi_i(x;\bar t)=e_{1/q}(-\lambda_{i_2}qx)e^{-\sum\limits_{j=1}^{\infty}t_j\lambda_{i_2}^j}+b_ie_{1/q}(-\mu_{i_2}qx)e^{-\sum\limits_{j=1}^{\infty}t_j\mu_{i_2}^j}. \tag{4.7}
$$

After the $(n+k)$-th step gauge transformation T_{n+k}, the final form of τ_q can be given in following corollary, which can be deduced directly from Theorems 4 and 5.

Corollary 1. *The gauge transformation can generate the following two forms of τ function of the q-KP hierarchy,*

$$
\tau_q^{(n+k)}=IW^q_{k,n}(\psi_k,\dots,\psi_1;\phi_1,\dots,\phi_n)=\begin{vmatrix}
\partial_q^{-1}\psi_k\phi_1 & \partial_q^{-1}\psi_k\phi_2 & \cdots & \partial_q^{-1}\psi_k\phi_n\\
\vdots & \vdots & \cdots & \vdots\\
\partial_q^{-1}\psi_1\phi_1 & \partial_q^{-1}\psi_1\phi_2 & \cdots & \partial_q^{-1}\psi_1\phi_n\\
\phi_1 & \phi_2 & \cdots & \phi_n\\
\partial_q\phi_1 & \partial_q\phi_2 & \cdots & \partial_q\phi_n\\
\vdots & \vdots & \cdots & \vdots\\
\partial_q^{n-k-1}\phi_1 & \partial_q^{n-k-1}\phi_2 & \cdots & \partial_q^{n-k-1}\phi_n
\end{vmatrix},
$$

$$
\tau_q^{(n)}=W^q_n(\phi_1,\dots,\phi_n)=\begin{vmatrix}
\phi_1 & \phi_2 & \cdots & \phi_n\\
\partial_q\phi_1 & \partial_q\phi_2 & \cdots & \partial_q\phi_n\\
\vdots & \vdots & \cdots & \vdots\\
\partial_q^{n-1}\phi_1 & \partial_q^{n-1}\phi_2 & \cdots & \partial_q^{n-1}\phi_n
\end{vmatrix}. \tag{4.8}
$$

Here $\{\phi_i,\psi_i\}$ are defined by (4.6) *and* (4.7).

On the other hand, we know from (3.3) that T_n defined by (4.2) is a dressing operator if its generating functions are given by (4.6). Therefore we can define one q-wave function

$$\omega_q = T_n\partial_q^{-n} e_q(xz)e^{\sum\limits_{i=1}^{\infty} z^i t_i} = \frac{1}{W_n^q}\begin{vmatrix} \phi_1 & \cdots & \phi_n & z^{-n} \\ \partial_q\phi_1 & \cdots & \partial_q\phi_n & z^{-n+1} \\ \vdots & \cdots & \vdots & \vdots \\ \partial_q^n\phi_1 & \cdots & \partial_q^n\phi_n & 1 \end{vmatrix} e_q(xz)e^{\sum\limits_{i=1}^{\infty} z^i t_i}. \tag{4.9}$$

Corollary 2. *The relationship in* (2.8) *between the q-wave function and* τ_q *is satisfied by* $\tau_q^{(n)}$ *in* (4.8) *and q-wave function in* (4.9)*, i.e.,*

$$\omega_q = \frac{\tau_q^{(n)}(x;\bar{t}-[z^{-1}])}{\tau_q^{(n)}(x;\bar{t})} e_q(xz)\exp\left(\sum_{i=1}^{\infty} t_i z^i\right). \tag{4.10}$$

Proof. We follow the Dickey's method on page 100 of [37] to prove the corollary. By direct computations,

$$\begin{aligned} \phi_k(x;t-[z^{-1}]) &= e_q(\lambda_k x)e^{\sum\limits_{i=1}^{\infty} z^i t_i - \left(\frac{\lambda_k}{z}+\frac{\lambda_k^2}{2z^2}+\cdots\right)} + a_k e_q(\mu_k x)e^{\sum\limits_{i=1}^{\infty} z^i t_i - \left(\frac{\mu_k}{z}+\frac{\mu_k^2}{2z^2}+\cdots\right)} \\ &= \phi_k - \frac{1}{z}\partial_q\phi_k \end{aligned}$$

whence

$$\frac{\tau_q^{(n)}(x;\bar{t}-[z^{-1}])}{\tau_q^{(n)}(x;\bar{t})} = \frac{1}{W_q}\begin{vmatrix} \phi_1-\frac{1}{z}\partial_q\phi_1 & \phi_2-\frac{1}{z}\partial_q\phi_2 & \cdots & \phi_n-\frac{1}{z}\partial_q\phi_n \\ \partial_q\phi_1-\frac{1}{z}\partial_q^2\phi_1 & \partial_q\phi_2-\frac{1}{z}\partial_q^2\phi_2 & \cdots & \partial_q\phi_n-\frac{1}{z}\partial_q^2\phi_n \\ \vdots & \vdots & \cdots & \vdots \\ \partial_q^{n-1}\phi_1-\frac{1}{z}\partial_q^n\phi_1 & \partial_q^{n-1}\phi_2-\frac{1}{z}\partial_q^n\phi_2 & \cdots & \partial_q^n\phi_n-\frac{1}{z}\partial_q^n\phi_n \end{vmatrix}.$$

Comparing the fraction above of the determinant term with (4.9), we can see that they are similar, although the form of the determinant in the numerator is different. The determinant in the numerator of (4.9) can be reduced to the same form of (4.10) if the second row, divided by z, is subtracted from the first one, the third from the second etc. ■

At the end of this section, we would like to discuss q-effects in the solution of q-KP hierarchy. By direct calculation, we get that the first flow of q-KP is

$$\begin{aligned} &\partial_{t_1}u_0 = x(q-1)(\partial_q u_1), \\ &\partial_{t_1}u_{-1} = (\partial_q u_{-1}) + u_0u_{-1} + \theta(u_{-2}) - u_{-2} - u_{-1}\theta^{-1}(u_0)), \\ &\partial_{t_1}u_{-2} = (\partial_q u_{-2}) + u_0u_{-2} + \theta(u_{-3}) + \big[-u_{-3} + q^{-1}u_{-1}\theta^{-2}(\partial_q u_0) - u_{-2}\theta^{-2}(u_0)\big], \\ &\partial_{t_1}u_{-3} = (\partial_q u_{-3}) + u_0u_{-3} + \theta(u_{-4}) + \big[-u_{-4} - q^{-3}u_{-1}\theta^{-3}(\partial_q^2 u_0) \\ &\qquad\quad + (q^{-1}+q^{-2})u_{-2}\theta^{-3}(\partial_q u_0) - u_{-3}\theta^{-3}(u_0)\big], \\ &\partial_{t_1}u_{-i} = (\partial_q u_{-i}) + u_0u_{-i} + \theta(u_{-i-1}) + \big[-u_{-i-1} + (\cdots) - u_{-i}\theta^{-i}(u_0)\big], \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

in which $(\cdots) = \sum\limits_{k=1}^{i-1} a_{-k}u_{-k}\theta^{-i}(\partial_q^{i-k}u_0)$ $(i = 2,3,\ldots)$, and a_{-k} depends on q only. We can see that

$$\partial_{t_1}u_0 = 0, \qquad \partial_{t_1}u_{-i} = \partial_q u_{-i} = \partial_x u_{-i}, \qquad i \geq 1.$$

when $q \to 1$. This result shows that the variable t_1 in q-KP hierarchy is corresponding to the variable x in KP hierarchy. So we have two global parameters in q-KP hierarchy, namely x and q. In order to show q-effect, we will write out the concrete form of single q-soliton of q-KP equation, namely, we let u_{-1} depend on three variable (t_1, t_2, t_3) beside two parameters (x, q). We consider $L^{(1)}$ generated by one step of $T_D(\phi_1)$ from $L = \partial_q$, and the generating function is given by

$$\phi_1 = e_q(\lambda_1 x)e^{\xi_1} + B_1 e_q(\lambda_2 x)e^{\xi_2} \tag{4.11}$$

from (4.6), then the Corollary 1 shows that the τ function of $L^{(1)}$ is $\tau_q^{(1)} = \phi_1$ in (4.11). Here $\xi_k = \lambda_k t_1 + \lambda_k^2 t_2 + \lambda_k^3 t_3$ $(k = 1, 2)$, B_1 is real constant. Taking this $\tau_q^{(1)}$ back into (2.9), then (2.10), we get q-soliton of q-KP as

$$\begin{aligned} u_{-1} = &\left[1 + x(q-1)\left(\frac{\lambda_1 e_q(\lambda_1 x)e^{\xi_1} + \lambda_2 B_1 e_q(\lambda_2 x)e^{\xi_2}}{e_q(\lambda_1 x)e^{\xi_1} + B_1 e_q(\lambda_2 x)e^{\xi_2}}\right)\right] \\ &\times \left\{ \frac{(\lambda_1^2 e_q(\lambda_1 x)e^{\xi_1} + B_1\lambda_2^2 e_q(\lambda_2 x)e^{\xi_2})(e_q(\lambda_1 qx)e^{\xi_1} + B_1 e_q(\lambda_2 qx)e^{\xi_2})}{(e_q(\lambda_1 qx)e^{\xi_1} + B_1 e_q(\lambda_2 qx)e^{\xi_2})(e_q(\lambda_1 x)e^{\xi_1} + B_1 e_q(\lambda_2 x)e^{\xi_2})} \right. \\ &\left. - \frac{(\lambda_1 e_q(\lambda_1 x)e^{\xi_1} + B_1\lambda_2 e_q(\lambda_2 x)e^{\xi_2})(\lambda_1 e_q(\lambda_1 qx)e^{\xi_1} + B_1\lambda_2 e_q(\lambda_2 qx)e^{\xi_2})}{(e_q(\lambda_1 qx)e^{\xi_1} + B_1 e_q(\lambda_2 qx)e^{\xi_2})(e_q(\lambda_1 x)e^{\xi_1} + B_1 e_q(\lambda_2 x)e^{\xi_2})} \right\}. \end{aligned}$$

In particular, if $q \to 1$, we have

$$u_{-1} = \frac{B_1(\lambda_1 - \lambda_2)^2}{e^{\hat{\xi}_1 - \hat{\xi}_2} + B_1^2 e^{\hat{\xi}_2 - \hat{\xi}_1} + 2B_1},$$

which is a single soliton of the classical KP when $x \to 0$. Here $\hat{\xi}_k = \lambda_k x + \xi_k$ $(k = 1, 2)$. In order to plot a figure for u_{-1}, we fix $\lambda_1 = 2$, $\lambda_2 = -1.5$ and $B_1 = 1$, so $u_{-1} = u_{-1}(x, t_1, t_2, t_3, q)$. The single q-soliton $u_{-1}(0.001, t_1, t_2, t_3, 0.999)$ is plotted in Fig. 1, which is close to classical soliton of KP equation as we analysed above. From Figs. 2–5[2] we can see the varying trends of $\triangle u_{-1} = u_{-1}(0.5, t_1, t_2, 0, 0.999) - u_{-1}(0.5, t_1, t_2, 0, q) \triangleq u_{-1}(q = 0.999) - u_{-1}(q)$ for certain values of q, where $q = 0.7, 0.5, 0.3, 0.1$ respectively. Furthermore, in order to see the q-effects more clearly, we further fixed $t_2 = -5$ in $\triangle u_{-1}$, which are plotted in Figs. 6–9. Dependence of $\triangle u_{-1} = u_{-1}(x, t_1, -5, 0, 0.999) - u_{-1}(x, t_1, -5, 0, 0.1) \triangleq u_{-1}(x, q = 0.999) - u_{-1}(x, q = 0.1)$ on x is shown in Figs. 10–14, and $x = 0.3, 0.4, 0.52, 0.54, 0.55$ respectively. It is obvious from figures that $\triangle u_{-1}$ goes to zero when $q \to 1$ and $x \to 0$, q-soliton (u_{-1}) of q-KP goes to a usual soliton of KP, which reproduces the process of q-deformation. On the other hand, Figs. 10–14[3] show parameter x amplifies q-effects. In other word, for a given $\triangle q$, $\triangle u_{-1}$ will increase along x. However, x is bounded so that $e_q(\lambda_k x)$ and $e_q(\lambda_k qx)$ $(k = 1, 2)$ are convergent. This is the reason for plotting u_{-1} with $x \le 0.55$. Obviously, the convergent interval depends on q and λ_k. We would like to emphasize that from Figs. 6–14[4] the q-deformation does not destroy the profile of soliton; it just similar to an "impulse" to soliton.

5 Symmetry constraint of q-KP: q-cKP hierarchy

We know that there exists a constrained version of KP hierarchy, i.e. the constrained KP hierarchy (cKP) [25, 31], introduced by means of the symmetry constraint from KP hierarchy.

[2]For Figs. 2–5, q-effect $\mathrm{D}u_{-1} \equiv \triangle u_{-1} \triangleq u_{-1}(q = 0.999) - u_{-1}(q = i)$ with $x = 0.5$ and $t_3 = 0$, where $i = 0.7, 0.5, 0.3, 0.1$. Figs. 6–9, are projection of Figs. 2–5, by fixing $t_2 = -5$.

[3]For Figs. 10–14, the variable x, varies as follows: $0.3, 0.4, 0.52, 0.54, 0.55$, while $q = i = 0.1$ in $\mathrm{D}u_{-1}$ is fixed.

[4]For Figs. 6–14, $\mathrm{D}u_{-1}$, $u_{-1}(q = 0.999)$, are represented by continuous line and dashed line (long), respectively, while dashed line (short) represent $u_{-1}(q = i)$, $i = 0.7, 0.5, 0.3, 0.1$ for Figs. 6–9.

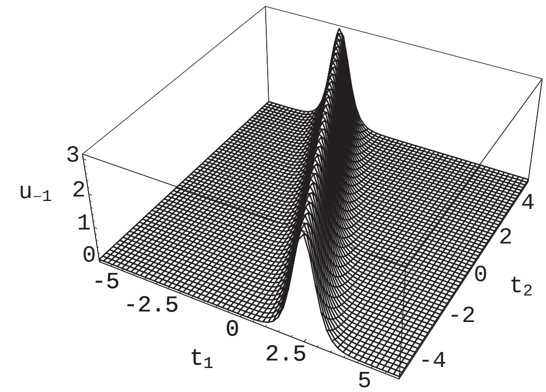

Figure 1. $u_{-1}(x = 0.001, q = 0.999)$ with $t_3 = 0$.

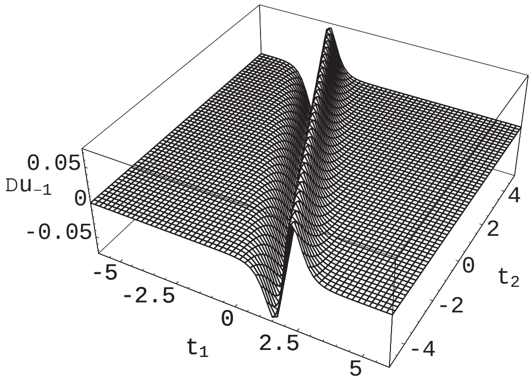

Figure 2.

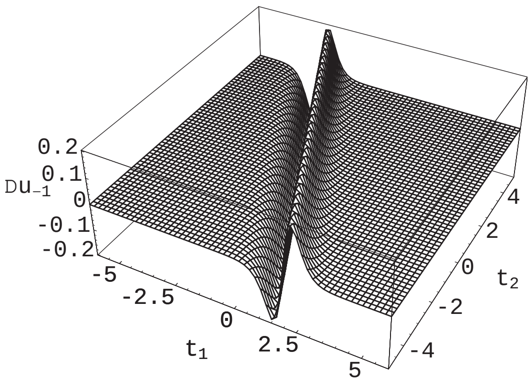

Figure 3.

With inspiration from it, the symmetry of q-KP was established in [22]. In the same article the authors defined one kind of constrained q-KP (q-cKP) hierarchy by using the linear combination of generators of additional symmetry. In this section, we shall briefly introduce the symmetry and q-cKP hierarchy [22].

The linearization of (2.2) is given by

$$\partial_{t_m}(\delta L) = [\delta B_m, L] + [B_m, \delta L], \tag{5.1}$$

where

$$\delta B_m = \left(\sum_{r=1}^{m} L^{m-r}\delta L L^{r-1}\right)_+.$$

We call $\delta L = \delta u_0 + \delta u_1 \partial_q^{-1} + \cdots$ the symmetry of the q-KP hierarchy, if it satisfies (5.1). Let L be a "dressed" operator from ∂_q, we find

$$\delta L = \delta S \partial_q S^{-1} - S\partial_q S^{-1}\delta S S^{-1} = [\delta S S^{-1}, L] = [K, L], \tag{5.2}$$

where $\delta S = \delta s_1 \partial_q^{-1} + \delta s_2 \partial_q^{-2} + \cdots$, and $K = \delta S S^{-1}$. Therefore

$$\delta B_m = [K, L^m]_+ = [K, B_m]_+,$$

the last identity is resulted by $K = K_-$ and $[K, L_-^m]_+ = 0$. Then the linearized equation (5.1) is equivalent to

$$\partial_{t_m} K = [B_m, K]_-, \qquad \delta S = KS. \tag{5.3}$$

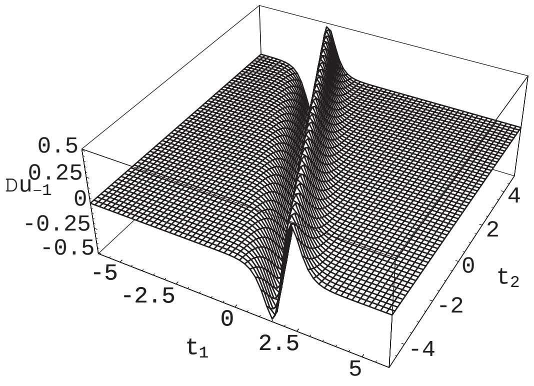

Figure 4.

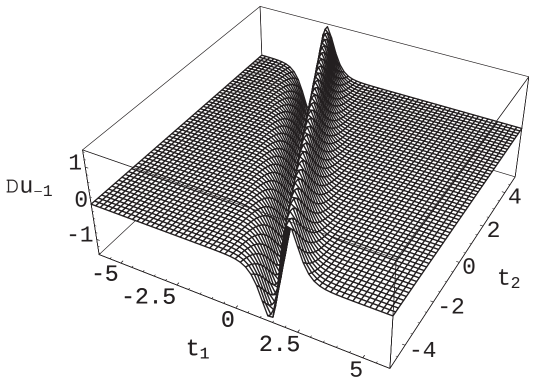

Figure 5.

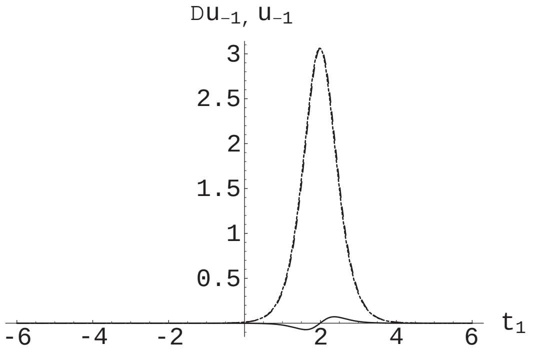

Figure 6.

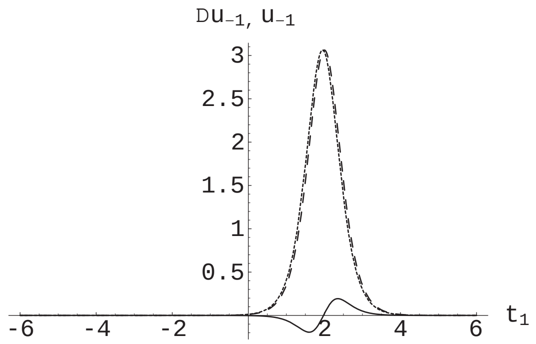

Figure 7.

Let $K_n = -(L^n)_-$ $(n = 1, 2, \ldots)$, then it can easily be checked that K_n satisfies (5.3). For each K_n, δL is given by $\delta L = -[(L^n)_-, L] = [B_n, L]$ from (5.2). So the q-KP hierarchy admits a reduction defined by $(L^n)_- = 0$, which is called q-deformed n-th KdV hierarchy. For example, $n = 2$, it leads to q-KdV hierarchy, whose q-Lax operator is

$$L_{q\mathrm{KdV}} = L^2 = L_+^2 = \partial_q^2 + x(q-1)u\partial_q + u.$$

There is also another symmetry called additional symmetry, which is $K = (M^m L^l)_-$ [22], and it also satisfies (5.3). Here the operator M is defined by

$$\partial_{t_k} M = [L_k^+, M], \qquad M = S\Gamma_q S^{-1},$$

and Γ_q is defined as

$$\Gamma_q = \sum_{i=1}^{\infty} \left[i t_i + \frac{(1-q)^i}{1-q^i} x^i \right] \partial_q^{i-1}.$$

The more general generators of additional symmetry are in form of

$$Y_q(\mu, \lambda) = \sum_{m=0}^{\infty} \frac{(\mu-\lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l-1} \big(M^m L^{m+l}\big)_-,$$

which are constructed by combination of $K = (M^m L^l)_-$. The operator $Y_q(\mu, \lambda)$ can be expressed as

$$Y_q(\mu, \lambda) = \omega_q(x, \bar{t}; \mu) \circ \partial_q^{-1} \circ \theta(\omega_q^*(x, \bar{t}; \lambda)).$$

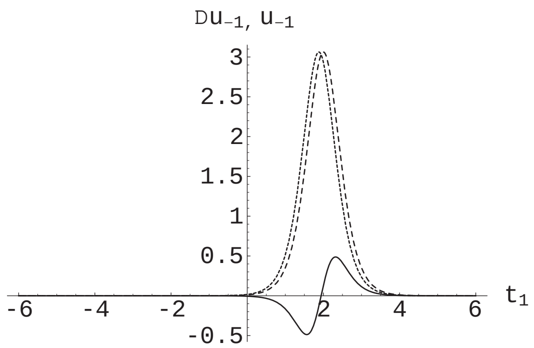

Figure 8.

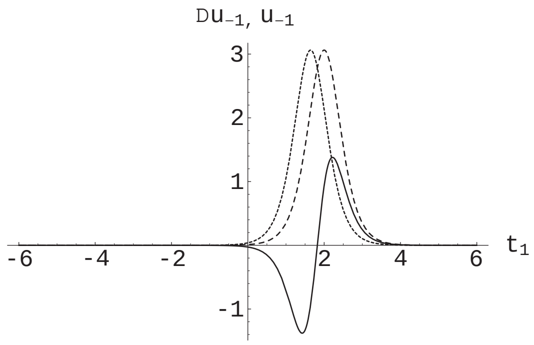

Figure 9.

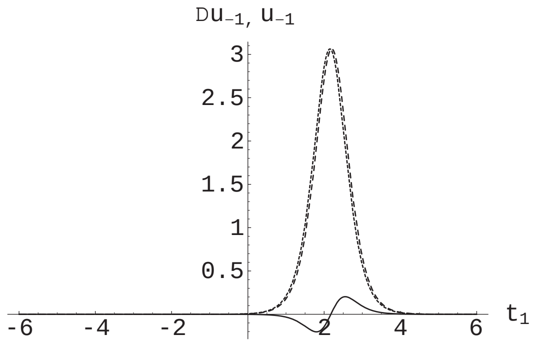

Figure 10.

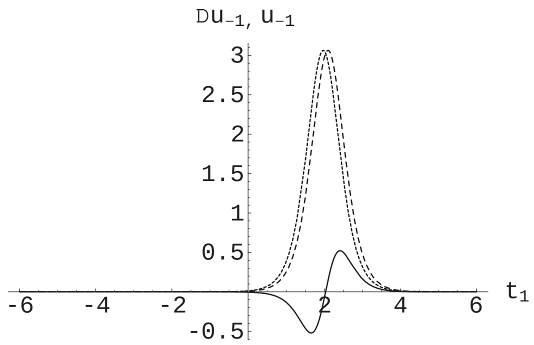

Figure 11.

In order to define the q-analogue of the constrained KP hierarchy, we need to establish one special generator of symmetry $Y(t) = \phi(t) \circ \partial_q^{-1} \circ \psi(t)$ based on $Y_q(\mu, \lambda)$, where

$$\phi(t) = \int \rho(\mu)\omega_q(x, \bar{t}; \mu)\mathrm{d}\mu, \qquad \psi(t) = \int \chi(\lambda)\theta(\omega_q^*(x, \bar{t}; \lambda))\mathrm{d}\lambda,$$

further $\phi(t)$ and $\psi(t)$ satisfy (2.12) and (2.13). In other words, we get a new symmetry of q-KP hierarchy,

$$K = \phi(\lambda; x, \bar{t}) \circ \partial_q^{-1} \circ \psi(\mu; x, \bar{t}), \tag{5.4}$$

where $\phi(\lambda; x, \bar{t})$ and $\psi(\mu; x, \bar{t})$ is an "eigenfunction" and an "adjoint eigenfunction", respectively. We can regard from the process above that $K = \phi(\lambda; x, \bar{t}) \circ \partial_q^{-1} \circ \psi(\mu; x, \bar{t})$ is a special linear combination of the additional symmetry generator $(M^m L^l)_-$. It is obvious that generator K in (5.4) satisfies (5.3), because of the following two operator identities,

$$(A \circ a \circ \partial_q^{-1} \circ b)_- = (A \cdot a) \circ \partial_q^{-1} \circ b, (a \circ \partial_q^{-1} \circ b \circ A)_- = a \circ \partial_q^{-1} \circ (A^* \cdot b). \tag{5.5}$$

Here A is a q-PDO, and a and b are two functions. Naturally, q-KP hierarchy also has a multi-component symmetry, i.e.

$$K = \sum_i^n \phi_i \circ \partial_q^{-1} \circ \psi_i.$$

It is well known that the integrable KP hierarchy is compatible with generalized l-constraints of this type $(L^l)_- = \sum_i q_i \circ \partial_x^{-1} \circ r_i$. Similarly, the l-constraints of q-KP hierarchy

$$(L^l)_- = K = \sum_{i=1}^m \phi_i \circ \partial_q^{-1} \circ \psi_i$$

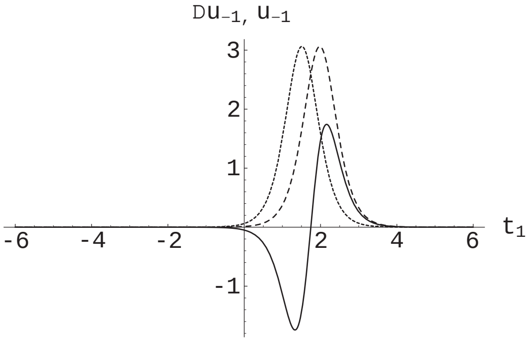

Figure 12.

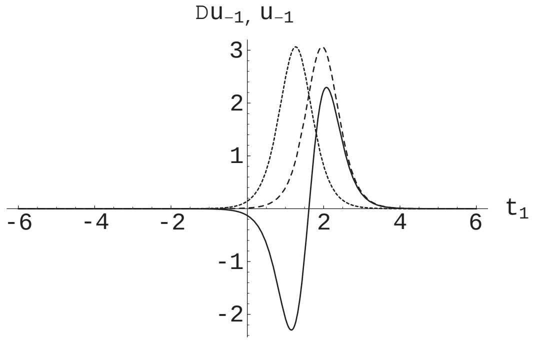

Figure 13.

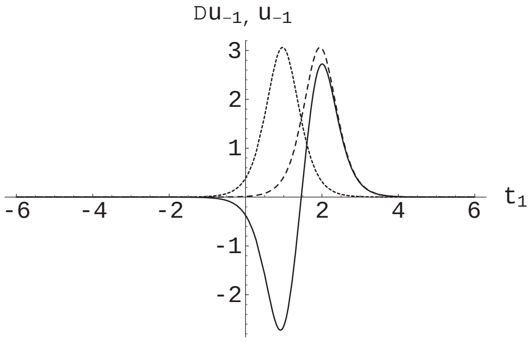

Figure 14.

also lead to q-cKP hierarchy. The flow equations of this q-cKP hierarchy

$$\partial_{t_k} L^l = [L_+^k, L^l], \qquad L^l = (L^l)_+ + \sum_{i=1}^{m} \phi_i \circ \partial_q^{-1} \circ \psi_i \tag{5.6}$$

are compatible with

$$(\phi_i)_{t_k} = ((L^k)_+ \phi_i), \qquad (\psi_i)_{t_k} = -((L^{*k})_+ \psi_i).$$

It can be obtained directly by using the operator identities in (5.5). An important fact is that there exist two m-th order q-differential operators

$$A = \partial_q^m + a_{m-1}\partial_q^{m-1} + \cdots + a_0, \qquad B = \partial_q^m + b_{m-1}\partial_q^{m-1} + \cdots + b_0,$$

such that AL^l and L^lB are differential operators. From $(AL^l)_- = 0$ and $(L^lB)_- = 0$, we get that A and B annihilate the functions ϕ_i and ψ_i, i.e., $A(\phi_1) = \cdots = A(\phi_m) = 0$, $B^*(\psi_1) = \cdots = B^*(\psi_m) = 0$, that implies $\phi_i \in \operatorname{Ker}(A)$. It should be noted that $\operatorname{Ker}(A)$ has dimension m. We will use this fact to reduce the number of components of the q-cKP hierarchy in the next section.

6 q-Wronskian solutions of q-cKP hierarchy

We know from Corollary 1 that q-Wronskian

$$\tau_q^{(N)} = W_N^q(\phi_1, \ldots, \phi_N) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_N \\ \partial_q\phi_1 & \partial_q\phi_2 & \cdots & \partial_q\phi_N \\ \vdots & \vdots & \cdots & \vdots \\ \partial_q^{N-1}\phi_1 & \partial_q^{N-1}\phi_2 & \cdots & \partial_q^{N-1}\phi_N \end{vmatrix}, \tag{6.1}$$

is a τ function of q-KP hierarchy. Here ϕ_i $(i = 1, 2, \ldots, N)$ satisfy linear q-partial differential equations,

$$\frac{\partial \phi_i}{\partial t_n} = (\partial_q^n \phi_i), \qquad n = 1, 2, 3, \ldots. \tag{6.2}$$

In this section, we will reduce $\tau_q^{(N)}$ in (6.1) to a τ function of q-cKP hierarchy. To this end, we will find the additional conditions satisfied by ϕ_i except the linear q-differential equation (6.2).

Corollary 1 also shows that the q-KP hierarchy with Lax operator $L^{(N)} = T_N \circ \partial_q \circ T_N^{-1}$ is generated from the "free" Lax operator $L = \partial_q$, which has the τ function $\tau_q^{(N)}$ in (6.1). In order to get the explicit form of such Lax operator $L^{(N)}$, the following lemma is necessary.

Lemma 5.

$$T_N = \frac{1}{W_N^q(\phi_1, \ldots, \phi_N)} \begin{vmatrix} \phi_1 & \cdots & \phi_N & 1 \\ \partial_q \phi_1 & \cdots & \partial_q \phi_N & \partial_q \\ \vdots & \cdots & \vdots & \vdots \\ \partial_q^N \phi_1 & \cdots & \partial_q^N \phi_N & \partial_q^N \end{vmatrix}$$

and

$$T_N^{-1} = \begin{vmatrix} \phi_1 \circ \partial_q^{-1} & \theta(\phi_1) & \cdots & \theta(\partial_q^{N-2}\phi_1) \\ \phi_2 \circ \partial_q^{-1} & \theta(\phi_2) & \cdots & \theta(\partial_q^{N-2}\phi_2) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_N \circ \partial_q^{-1} & \theta(\phi_N) & \cdots & \theta(\partial_q^{N-2}\phi_N) \end{vmatrix} \cdot \frac{(-1)^{N-1}}{\theta(W_N^q(\phi_1, \ldots, \phi_N))} = \sum_{i=1}^{N} \phi_i \circ \partial_q^{-1} \circ g_i$$

with

$$g_i = (-1)^{N-i} \theta\Big(\frac{W_N^q(\phi_1, \ldots, \phi_{i-1}, \hat{i}, \phi_{i+1}, \ldots, \phi_N)}{W_N^q(\phi_1, \ldots, \phi_{i-1}, \phi_i, \phi_{i+1}, \ldots, \phi_N)}\Big). \tag{6.3}$$

Here $\hat{i}$ means that the column containing ϕ_i is deleted from $W_N^q(\phi_1, \ldots, \phi_{i-1}, \phi_i, \phi_{i+1}, \ldots, \phi_N)$, and the last row is also deleted.

Proof. The proof is a direct consequence of Lemma 3 and Theorem 4 from the initial "free" Lax operator $L = \partial_q$. The generating functions $\{\phi_i, i = 1, 2, \ldots, N\}$ of T_N satisfies equations (6.2), which is obtained from definition of "eigenfunction" (2.12) of the KP hierarchy under $B_n = \partial_q^n$. ■

In particular, $(T_N \cdot \phi_1) = (T_N \cdot \phi_2) = \cdots = (T_N \cdot \phi_N) = 0$.

Now we can give one theorem reducing the q-Wronskian τ function $\tau_q^{(N)}$ in (6.1) of q-KP hierarchy to the q-cKP hierarchy defined by (5.6).

Theorem 6. *$\tau_q^{(N)}$ is also a τ function of the q-cKP hierarchy whose Lax operator $L^l = (L^l)_+ + \sum_{i=1}^{M} q_i \circ \partial_q^{-1} \circ r_i$ with some suitable functions $\{q_i, i = 1, 2, \ldots, M\}$ and $\{r_i, i = 1, 2, \ldots, M\}$ if and only if*

$$W_{N+M+1}^q(\phi_1, \ldots, \phi_N, \partial_q^l \phi_{i_1}, \ldots, \partial_q^l \phi_{i_{M+1}}) = 0 \tag{6.4}$$

for any choice of $(M+1)$-indices $(i_1, i_2, \ldots, i_{M+1})$ $1 \leqslant i_1 < \cdots < i_{M+1} \leq N$, which can be expressed equivalently as

$$W_{M+1}^q\left(\frac{W_{N+1}^q(\phi_1, \ldots, \phi_N, \partial_q^l \phi_{i_1})}{W_N^q(\phi_1, \ldots, \phi_N)}, \frac{W_{N+1}^q(\phi_1, \ldots, \phi_N, \partial_q^l \phi_{i_2})}{W_N^q(\phi_1, \ldots, \phi_N)}, \ldots,\right.$$

$$\left.\frac{W^q_{N+1}(\phi_1,\dots,\phi_N,\partial_q^l\phi_{i_{M+1}})}{W^q_N(\phi_1,\dots,\phi_N)}\right) = 0 \tag{6.5}$$

for all indices. Here $\{\phi_i,\, i=1,2,\dots,N\}$ satisfy (6.2)*.*

Remark 4. This theorem is a q-analogue of the classical theorem on cKP hierarchy given by [38].

Proof. The q-Wronskian identity proven in Appendix C

$$W^q_{M+1}\left(\frac{W^q_{N+1}(\phi_1,\dots,\phi_N,f_1)}{W^q_N(\phi_1,\dots,\phi_N)},\dots,\frac{W^q_{N+1}(\phi_1,\dots,\phi_N,f_{M+1})}{W^q_N(\phi_1,\dots,\phi_N)}\right)$$
$$= \frac{W^q_{N+M+1}(\phi_1,\dots,\phi_N,f_1,\dots,f_{M+1})}{W^q_N(\phi_1,\dots,\phi_N)}$$

implies equivalence between (6.4) and (6.5). Using T_N and T_N^{-1} in Lemma 5 and the operator identity in (5.5) we have

$$(L^l)_- = (T_N\circ\partial_q^l\circ T_N^{-1})_- = \sum_{i=1}^N (T_N(\partial_q^l\phi_i))\circ\partial_q^{-1}\circ g_i, \tag{6.6}$$

where g_i is given by (6.3) and T_N acting on $(\partial_q^l\phi_i)$ is $T_N(\partial_q^l\phi_i) = \dfrac{W^q_{N+1}(\phi_1,\phi_2,\dots,\phi_N,\partial_q^l\phi_i)}{W^q_N(\phi_1,\phi_2,\dots,\phi_N)}$. So $\tau_q^{(N)}$ is automatically a tau function of N-component q-cKP hierarchy with the form (6.6). Next, we can reduce the N-component to the M-component ($M<N$) by a suitable constraint of ϕ_i.

Suppose that the M-component ($M<N$) q-cKP hierarchy is obtained by constraint of qKP hierarchy generated by T_N, i.e., there exist suitable functions $\{q_i,r_i\}$ such that

$$(L^l)_- = \sum_{i=1}^M q_i\circ\partial_q^{-1}\circ r_i = \sum_{i=1}^N (T_N(\partial_q^l\phi_i))\circ\partial_q^{-1}\circ g_i.$$

As we pointed out in previous section, for a Lax operator whose negative part is in the form of $(L^l)_- = \sum\limits_{i=1}^M q_i\circ\partial_q^{-1}\circ r_i$, there exists an M-th order q-differential operator A such that AL^l is a q-differential operator, then we have

$$0 = AL^l(T_N(\phi_i)) = AT_N\partial_q^l(\phi_i) = A(T_N(\partial_q^l\phi_i))$$

from $T_N(\phi_i)=0$that implies $T_N(\partial_q^l\phi_i)\in\operatorname{Ker}(A)$. Therefore, at most M of these functions $T_N(\partial_q^l\phi_i)$ can be linearly independent because the Kernel of A has dimension M. So (6.5) is deduced.

Conversely, suppose (6.5) is true, we will show that there exists one M-component q-ckP ($M<N$) constrained from (6.6). The equation (6.5) implies that at most M of functions $T_N(\partial_q^l\phi_i)$ ($i=1,2,\dots,N$) are linearly independent. Then we can find suitable M functions $\{q_1,q_2,\dots,q_M\}$, which are linearly independent, to express functions $T_N(\partial_q^l\phi_i)$ as

$$T_N(\partial_q^l\phi_i) = \frac{W^q_{N+1}(\phi_1,\phi_2\cdots,\phi_N,\partial_q^l\phi_i)}{W^q_N(\phi_1,\phi_2,\dots,\phi_N)} = \sum_{j=1}^M c_{ij}q_j, \qquad i=1,\cdots,N$$

with some constants c_{ij}. Taking this back into (6.6), it becomes

$$(L^l)_- = \sum_{i=1}^{N}\left(\sum_{j=1}^{M} c_{ij}q_j\right)\circ\partial_q^{-1}\circ g_i = \sum_{j=1}^{M} q_j\circ\partial_q^{-1}\circ\left(\sum_{i=1}^{N} c_{ij}g_i\right) = \sum_{j=1}^{M} q_j\circ\partial_q^{-1}\circ r_j,$$

which is an M-component q-cKP hierarchy as we expected. ■

7 Example reducing q-KP to q-cKP hierarchy

To illustrate the method in Theorem 6 reducing the q-KP to multi-component a q-cKP hierarchy, we discuss the q-KP generated by $T_N|_{N=2}$. In order to obtain the concrete solution, we only consider the three variables (t_1, t_2, t_3) in $\bar{t}$. Furthermore, the q_1, r_1 and u_{-1} are constructed in this section.

According to Theorem 6, the q-KP hierarchy generated by $T_N|_{N=2}$ possesses a tau function

$$\begin{aligned}\tau_q^{(2)} &= W_2^q(\phi_1,\phi_2) = \phi_1(\partial_q\phi_2) - \phi_2(\partial_q\phi_1)\\ &= (\lambda_2-\lambda_1)e_q(\lambda_1 x)e_q(\lambda_2 x)e^{\xi_1+\xi_2} + (\lambda_3-\lambda_1)e_q(\lambda_1 x)e_q(\lambda_3 x)e^{\xi_1+\xi_3}\\ &\quad + (\lambda_3-\mu)e_q(\mu x)e_q(\lambda_3 x)e^{\xi+\xi_3} + (\lambda_2-\mu)e_q(\mu x)e_q(\lambda_2 x)e^{\xi+\xi_2}\end{aligned}\tag{7.1}$$

with

$$\phi_1 = e_q(\lambda_1 x)e^{\xi_1} + e_q(\mu x)e^{\xi}, \qquad \phi_2 = e_q(\lambda_2 x)e^{\xi_2} + e_q(\lambda_3 x)e^{\xi_3}.$$

Here $\xi_i = c_i + \lambda_i t_1 + \lambda_i^2 t_2 + \lambda_i^3 t_3$ $(i = 1,2,3)$, and $\xi = d + \mu t_1 + \mu^2 t_2 + \mu^3 t_3$, c_i and d are arbitrary constants. These functions satisfy the linear equations

$$\frac{\partial\phi_i}{\partial t_n} = \partial_q^n\phi_i, \qquad n = 1,2,3, \quad i = 1,2,$$

as a special case of (6.2). From (6.6), the q-KP hierarchy generated by $T_N|_{N=2}$ is in the form of

$$L^l = (L^l)_+ + (T_2(\partial_q^l\phi_1))\circ\partial_q^{-1}\circ g_1 + (T_2(\partial_q^l\phi_2))\circ\partial_q^{-1}\circ g_2, \tag{7.2}$$

$$\overset{\text{constraint}}{=\!=\!=\!=\!=} (L^l)_+ + q_1\circ\partial_q\circ r_1. \tag{7.3}$$

Here q_1 and r_1 are undetermined, which can be expressed by ϕ_1 and ϕ_2 as follows.

According to (6.4), the restriction for ϕ_1 and ϕ_2 to reduce (7.2) to (7.3) is given by

$$\begin{aligned}0 &= W_2^q(\phi_1,\phi_2,\partial_q^l\phi_1,\partial_q^l\phi_2) = (\mu^l-\lambda_1^l)(\lambda_2^l-\lambda_3^l)V(\lambda_1,\lambda_2,\lambda_3,\mu)e^{c_1+c_2+c_3+d}e^{(\lambda_1+\lambda_2+\lambda_3+\mu)t_1}\\ &\quad\times e^{(\lambda_1^2+\lambda_2^2+\lambda_3^2+\mu^2)t_2}e^{(\lambda_1^3+\lambda_2^3+\lambda_3^3+\mu^3)t_3}e_q(\lambda_1 x)e_q(\lambda_2 x)e_q(\lambda_3 x)e_q(\mu x)\end{aligned}\tag{7.4}$$

with

$$V(\lambda_1,\lambda_2,\lambda_3,\mu) = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3\\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3\\ 1 & \lambda_3 & \lambda_3^2 & \lambda_3^3\\ 1 & \mu & \mu^2 & \mu^3\end{vmatrix}.$$

Obviously, we can let $\mu = \lambda_2$ and $d = c_2$ such that (7.4) holds for ϕ_1 and ϕ_2. Then the τ function of a single component q-cKP defined by (7.3) is

$$\tau_{q\text{cKP}} = (\lambda_2-\lambda_1)e_q(\lambda_1 x)e_q(\lambda_2 x)e^{\xi_1+\xi_2} + (\lambda_3-\lambda_1)e_q(\lambda_1 x)e_q(\lambda_3 x)e^{\xi_1+\xi_3}$$

$$+ (\lambda_3 - \lambda_2)e_q(\lambda_2 x)e_q(\lambda_3 x)e^{\xi_2+\xi_3},$$

which is deduced from (7.1). That means we indeed reduce the τ function $\tau_q^{(2)}$ in (7.1) of the q-KP hierarchy generated by $T_N|_{N=2}$ to the τ function $\tau_{q\text{cKP}}$ of the one-component q-cKP hierarchy. Furthermore, we would like to get the explicit expression of (q_1, r_1) of q-cKP in (7.3). Using the determinant representation of $T_N|_{N=2}$ and $T_N^{-1}|_{N=2}$, we have

$$f_1 \triangleq (T_2(\partial_q^l \phi_1)) = \frac{(\lambda_1^l - \lambda_2^l)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)e_q(\lambda_1 x)e_q(\lambda_2 x)e_q(\lambda_3 x)e^{\xi_1+\xi_2+\xi_3}}{\tau_{q\text{cKP}}},$$

$$f_2 \triangleq (T_2(\partial_q^l \phi_2) = \frac{(\lambda_3^l - \lambda_2^l)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)e_q(\lambda_1 x)e_q(\lambda_2 x)e_q(\lambda_3 x)e^{\xi_1+\xi_2+\xi_3}}{\tau_{q\text{cKP}}},$$

$$g_1 = -\theta\left(\frac{\phi_2}{\tau_{q\text{cKP}}}\right), \qquad g_2 = \theta\left(\frac{\phi_1}{\tau_{q\text{cKP}}}\right),$$

under the restriction $\mu = \lambda_2$ and $d = c_2$. One can find that f_1 and f_2 are linearly dependent, and $(\lambda_3^l - \lambda_2^l)f_1 = (\lambda_1^l - \lambda_2^l)f_2$. So (7.2) and (7.3) reduce to

$$\begin{aligned} L_-^l &= f_1 \circ \partial_q^{-1} \circ g_1 + f_2 \circ \partial_q^{-1} \circ g_2 \\ &= (\lambda_3^l - \lambda_2^l)f_1 \circ \partial_q^{-1} \circ \frac{g_1}{(\lambda_3^l - \lambda_2^l)} + (\lambda_1^l - \lambda_2^l)f_2 \circ \partial_q^{-1} \circ \frac{g_2}{(\lambda_1^l - \lambda_2^l)} = q_1 \circ \partial_q^{-1} \circ r_1, \end{aligned}$$

in which

$$\begin{aligned} q_1 &\triangleq (\lambda_3^l - \lambda_2^l)f_1 = (\lambda_1^l - \lambda_2^l)f_2 \\ &= \frac{(\lambda_1^l - \lambda_2^l)\lambda_3^l - \lambda_2^l)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)e_q(\lambda_1 x)e_q(\lambda_2 x)e_q(\lambda_3 x)e^{\xi_1+\xi_2+\xi_3}}{\tau_{q\text{cKP}}}, \\ r_1 &\triangleq \left(\frac{g_1}{(\lambda_3^l - \lambda_2^l)} + \frac{g_2}{(\lambda_1^l - \lambda_2^l)}\right) = \frac{1}{(\lambda_1^l - \lambda_2^l)(\lambda_3^l - \lambda_2^l)} \\ &\times \theta\left(\frac{e^{-(\xi_1+\xi_2+\xi_3)}\big((\lambda_3^l - \lambda_2^l)e_q(\lambda_1 x)e^{\xi_1} + (\lambda_3^l - \lambda_1^l)e_q(\lambda_2 x)e^{\xi_2} + (\lambda_2^l - \lambda_1^l)e_q(\lambda_3 x)e^{\xi_3}\big)}{\tau_{q\text{cKP}}}\right). \end{aligned}$$

In particular, we can let $\lambda_1 = \lambda$, $\lambda_2 = 0$, $\lambda_3 = -\lambda$, $c_1 = c$, $c_2 = -0$, $c_3 = -c$, then

$$q_1 = \frac{(-1)^l \lambda^{2l+2} e_q(\lambda x)e_q(-\lambda x)}{e_q(\lambda x)e_q(-\lambda x) + \frac{e_q(\lambda x)e^{\eta} + e_q(-\lambda x)e^{-\eta}}{2} e^{-\lambda^2 t_2}}$$

and

$$r_1 = \begin{cases} -\dfrac{1}{\lambda^{l+1}}\theta\left[\dfrac{e^{-\lambda^2 t_2} + \frac{e_q(\lambda x)e^{\eta} + e_q(-\lambda x)e^{-\eta}}{2}}{e^{\lambda^2 t_2}e_q(\lambda x)e_q(-\lambda x) + \frac{e_q(\lambda x)e^{\eta} + e_q(-\lambda x)e^{-\eta}}{2}}\right] & \text{if } l \text{ is odd}, \\ -\dfrac{1}{\lambda^{l+1}}\theta\left[\dfrac{\frac{e_q(\lambda x)e^{\eta} - e_q(-\lambda x)e^{-\eta}}{2}}{e^{\lambda^2 t_2}e_q(\lambda x)e_q(-\lambda x) + \frac{e_q(\lambda x)e^{\eta} + e_q(-\lambda x)e^{-\eta}}{2}}\right] & \text{if } l \text{ is even}, \end{cases}$$

where $\eta = c + \lambda t_1 + \lambda^3 t_3$.

In general, the l-constrained one-component q-KP hierarchy has the Lax operator $L = \partial_q + u_0 + q_1 \circ \partial_q^{-1} \circ r_1$ when $l = 1$. On the other hand, its Lax operator can also be expressed as $L = \partial_q + u_0 + u_{-1}\partial_q^{-1} + u_{-2}\partial_q^{-2} + \cdots$. So all of the dynamical variables $\{u_{-i},\ i = 1, 2, 3, \ldots\}$ of q-KP hierarchy are given by

$$u_{-i-1} = (-1)^i q^{-i(i+1)/2} q_1 \theta^{-i-1}(\partial_q^i r_1), \qquad i \geq 0.$$

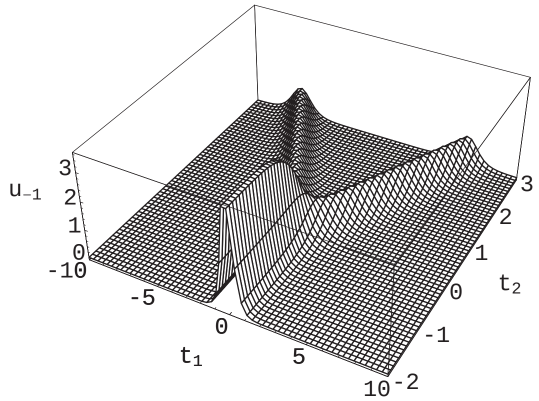

Figure 15. $u_{-1}(x = 0.001, q = 0.999)$ from q-cKP and $t_3 = 0$.

For the present situation, $u_{-1} = u_{-1}(t_1, t_2, t_3) = q_1\theta^{-1}(r_1)$ represents the q-deformed solution of the classical KP eqution, which is constructed from the components of q-cKP hierarchy, and is of the form

$$q_1 = \frac{-\lambda^4 e_q(\lambda x)e_q(-\lambda x)}{e_q(\lambda x)e_q(-\lambda x) + \frac{e_q(\lambda x)e^{\eta}+e_q(-\lambda x)e^{-\eta}}{2}e^{-\lambda^2 t_2}}, \tag{7.5}$$

$$r_1 = -\frac{1}{\lambda^2}\theta\left[\frac{e^{-\lambda^2 t_2} + \frac{e_q(\lambda x)e^{\eta}+e_q(-\lambda x)e^{-\eta}}{2}}{e^{\lambda^2 t_2}e_q(\lambda x)e_q(-\lambda x) + \frac{e_q(\lambda x)e^{\eta}+e_q(-\lambda x)e^{-\eta}}{2}}\right], \tag{7.6}$$

$$\begin{aligned} u_{-1} = {} & \frac{\lambda^2 e_q(\lambda x)e_q(-\lambda x)}{e_q(\lambda x)e_q(-\lambda x) + \frac{e_q(\lambda x)e^{\eta}+e_q(-\lambda x)e^{-\eta}}{2}e^{-\lambda^2 t_2}} \\ & \times \frac{e^{-\lambda^2 t_2} + \frac{e_q(\lambda x)e^{\eta}+e_q(-\lambda x)e^{-\eta}}{2}}{e^{\lambda^2 t_2}e_q(\lambda x)e_q(-\lambda x) + \frac{e_q(\lambda x)e^{\eta}+e_q(-\lambda x)e^{-\eta}}{2}}. \end{aligned} \tag{7.7}$$

Obviously, they will approach to the classical results on the cKP hierarchy in [38] when $x \to 0$ and $q \to 1$. We will fix $\lambda = 2$, $t_3 = 0$ and $c = 0$ to plot their figures, then get $q_1 = q_1(x, t_1, t_2, q)$, $r_1 = r_1(x, t_1, t_2, q)$ and $u_{-1} = u_{-1}(x, t_1, t_2, q)$ from (7.5)–(7.7). To save space, we plot the figures for u_{-1} and q_1 in (t_1, t_2, t_3) dimension spaces. It can be seen that Fig. 15 of $u_{-1}(0.001, t_1, t_2, 0.999)$ and Fig. 20 of $q_1(0.001, t_1, t_2, 0.999)$ match with the profile of u_1 and q in [38] with the same parameters. So we define q-effects quantity $\triangle u_{-1} = u_{-1}(0.5, t_1, t_2, 0.999) - u_{-1}(0.5, t_1, t_2, q) = u_{-1}(q = 0.999) - u_{-1}(q)$, $\triangle q_1 = q_1(0.5, t_1, t_2, 0.999) - q_1(.5, , t_1, t_2, q) = q_1(q = 0.999) - q_1(q)$, to show their dependence on q. Figs. 16–19[5] and Figs. 21–24[6] are plotted for $\triangle u_{-1}$ and $\triangle q_1$, respectively, where $q = 0.7, 0.5, 0.3, 0.1$. Obviously, they are decreasing to almost zero when q goes from 0.1 to 1 with fixed $x = 0.5$. Furthermore, Figs. 25–29[7] show that the dependence of the q-effects $\triangle u_{-1} = u_{-1}(x, t_1, t_2, 0.999) - u_{-1}(x, t_1, t_2, 0.1) = u_{-1}(x, q = 0.999) - u_{-1}(x, q = 0.1)$ on x,where $x = 0.2, 0.4, 0.51, 0.53, 0.55$ in order. These figures give us again an opportunity to observe the process of q-deformation in q-soliton solution of q-KP equation. They also demonstrate that q-deformation keep the profile of the soliton, although there exists deformation in some degree. On the other hand, in fact, (q_1, r_1) can be regarded as

[5] For Figs. 16–19, q-effect $\mathrm{Du}_{-1} \equiv \triangle u_{-1} \triangleq u_{-1}(q = 0.999) - u_{-1}(q = i)$, where $i = 0.7, 0.5, 0.3, 0.1$, from q-cKP with $x = 0.5$ and $t_3 = 0$.

[6] For Figs. 21–24, q-effect $\mathrm{Dq}_1 \equiv \triangle q_1 \triangleq q_1(q = 0.999) - q_1(q = i)$, where $i = 0.7, 0.5, 0.3, 0.1$, from q-cKP with $x = 0.5$ and $t_3 = 0$.

[7] For Figs. 25–29, q-effect $\mathrm{Du}_{-1} \equiv \triangle u_{-1} \triangleq u_{-1}(x = i, q = 0.999) - u_{-1}(x = i, q = 0.1)$ from q-cKP with $t_3 = 0$, where $i = 0.2, 0.4, 0.51, 0.53, 0.55$.

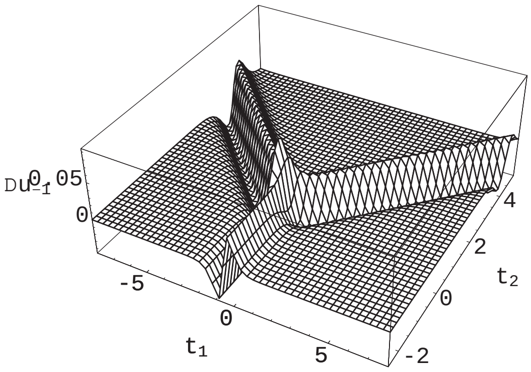

Figure 16.

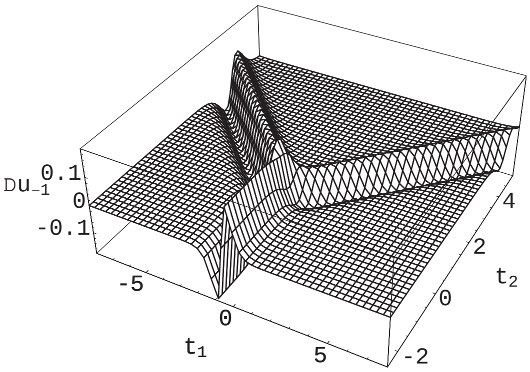

Figure 17.

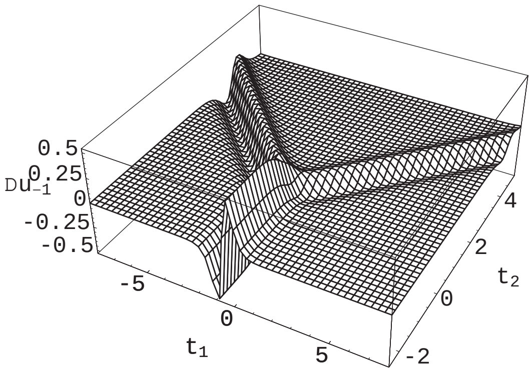

Figure 18.

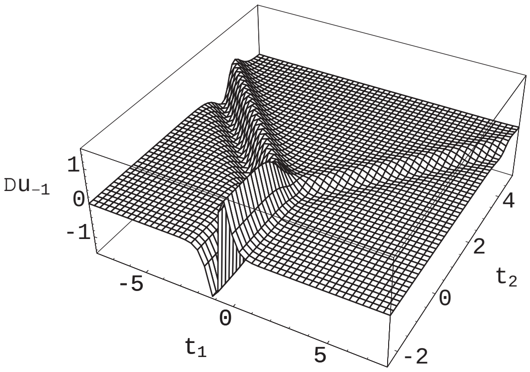

Figure 19.

a q-deformation of dynamical variables (q, r) of AKNS hierarchy, because cKP possessing Lax operator $L = \partial + q \circ \partial^{-1} \circ r$ is equivalent to the AKNS hierarchy.

8 Conclusions and discussions

In this paper, we have shown in Theorem 1 that there exist two types of elementary gauge transformation operators for the q-KP hierarchy. The changing rules of q-KP under the gauge transformation are given in Theorems 2 and 3. We mention that these two types of elementary gauge transformation operators are introduced first by Tu et al. [15] for q-NKdV hierarchy. Considering successive application of gauge transformation, we established the determinant representation of the gauge transformation operator of the q-KP hierarchy in Lemma 3 and the corresponding results on the transformed new q-KP are given in Theorem 5. For the q-KP hierarchy generated by T_{n+k} from the "free" Lax operator $L = \partial_q$ (i.e. the Lax operator is $L^{(n+k)} = T_{n+k} \circ \partial_q \circ T_{n+k}^{-1}$), Corollary 1 shows that the generalized q-Wronskian $IW_{k,n}^q$ of functions $\{\phi_i, \psi_j\}$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, k)$ is a general τ function of it, and q-Wronskian W_n^q of functions $\phi_i (i = 1, 2, \ldots, n)$ is also a special one. Here $\{\phi_i\}$ and $\{\psi_j\}$ satisfy special linear q-partial differential equations (4.5).

The symmetry and symmetry constraint of q-KP (q-cKP) hierarchy are discussed in Section 5. On the basis of the representation of T_N in Lemma 5, the q-KP hierarchy whose Lax operator $L^l = T_N \circ \partial_q^l \circ T_N^{-1}$ is generated from the "free" Lax operator $L = \partial_q$. The explicit form of its negative part L_-^l is given in (6.6), which is called l-constraint of the q-KP hierarchy. Further we found necessary and sufficient conditions that are given in Theorem 6, reducing a q-Wronskian solution in (6.1) of the q-KP hierarchy to solutions of the multi-component q-cKP hierarchy. One example is given in Section 7 to illustrate the method, i.e., the q-KP generated by $T_N|_{N=2}$

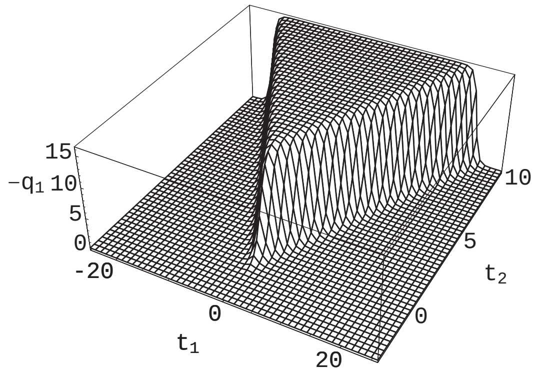

Figure 20. $q_1(x = 0.001, q = 0.999)$ and $t_3 = 0$.

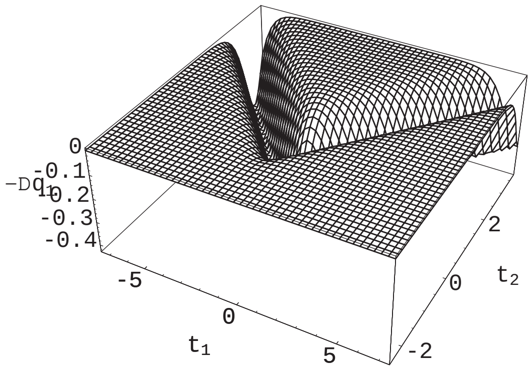

Figure 21.

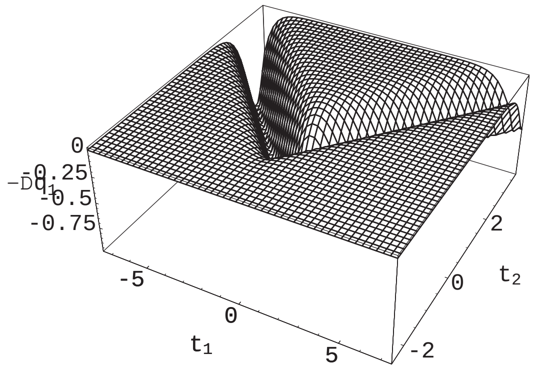

Figure 22.

is reduced to one-component q-cKP hierarchy. By taking finite variables (t_1, t_2, t_3) in $\bar{t}$, the component q_1 and r_1 are written out. Our results can be reduced to the classical results in [38].

As we pointed out in Section 2, u_{-1} is the q-analogue of the solution of classical KP equation if we only consider three variables (t_1, t_2, t_3) in $\bar{t}$. Therefore, the solution u_{-1} is called q-soliton of the q-KP equation, although we do not write out the q-KP equation on u_{-1}. One can find that the equations of dynamical variables $\{u_0, u_{-i}\}$ in q-KP hierarchy are coupled with each other and can not get one q-partial differential equation associated only with one dynamical variable, like classical KP equation has one dynamical variable u_{-1}. The reason is that the q-Leibnitz rule contains q-differential operation and θ operation, however, the Leibnitz rule of the standard calculus only contains one differential operation. We get a single q-soliton u_{-1} by means of the simplest τ function $\tau_q = W_1^q(\phi_1) = \phi_1$ in Section 4. Meanwhile, the multi-q-soliton u_{-1} is obtained from one-component q-cKP hierarchy in Section 7. Figures of q-effect $\triangle u_{-1}$ show that q-soliton u_{-1} indeed goes to classical soliton of KP equation when $x \to 0$ and $q \to 1$ and q-deformation does not destroy the rough profile of the q-soliton. In other worlds, the figure of q-soliton is similar to the classical soliton of KP equation. We also show the trends of the q-effect $\triangle u_{-1}$ depends on x and q; x plays a role of the amplifier of q-effects. In conclusion, the figures of q-effects $\triangle u_{-1}$ let us know the process of q-deformation in integrable systems for the first time. Of course, it is a long way to explore the physical meaning of q from the soliton theory.

In comparison with the research of classical soliton theory [40], in particular, the KP hierarchy [36, 37], the cKP [25, 31] hierarchy and the AKNS [40] hierarchy, there exist at least several topics needed to be discussed in order to research the integrability property of nonlinear q-partial differential equations. For instance, the Hamiltonian structure the q-cKP hierarchy and its q-W-algebra; the gauge transformation of the q-cKP hierarchy; the q-Hirota equation as-

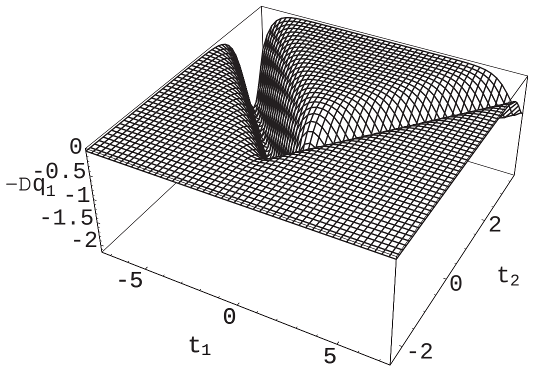

Figure 23.

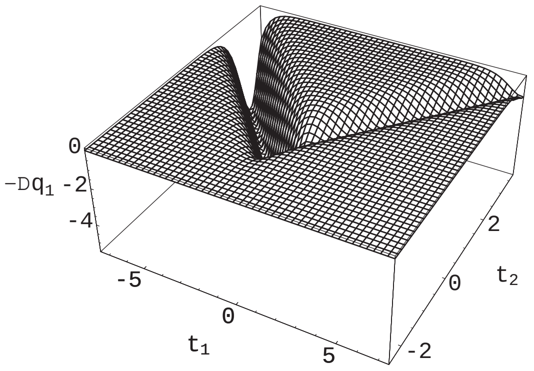

Figure 24.

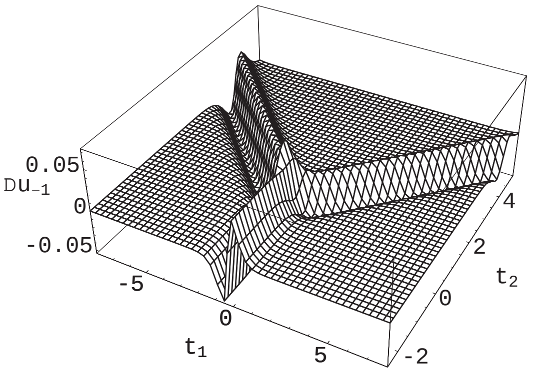

Figure 25.

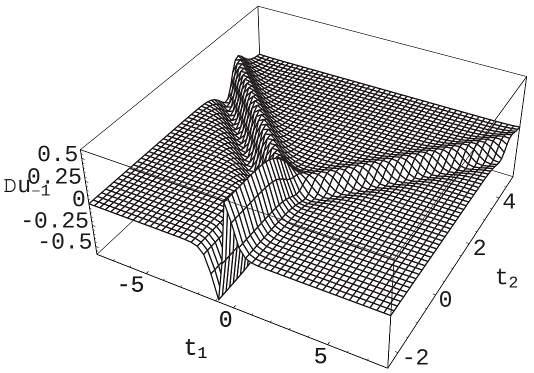

Figure 26.

sociated with the bilinear identity of the q-KP hierarchy; the symmetry analysis of q-differential equation and q-partial differential equations; the interaction of q-solitons; the q-AKNS hierarchy and its properties. Since the KP hierarchy has B-type and C-type sub-hierarchies, what are q-analogues of them? In particular, we showed in the previous sections that convergence of $e_q(x)$ affects the q-soliton, so the analytic property of $e_q(x)$ is a basis for research the interaction of q-solitons. We will try to investigate these questions in the future.

Acknowledgements

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A More explicit expressions of $\partial_q^n \circ f$

For $n \geq 1$, we have

$$\partial_q^4 \circ f = (\partial_q^4 f) + (4)_q \theta(\partial^3 f)\partial_q + \frac{(4)_q(3)_q}{(2)_q}\theta^2(\partial_q^2 f)\partial_q^2 + (4)_q\theta^3(\partial_q f)\partial_q^3 + \theta^4(f)\partial_q^4,$$

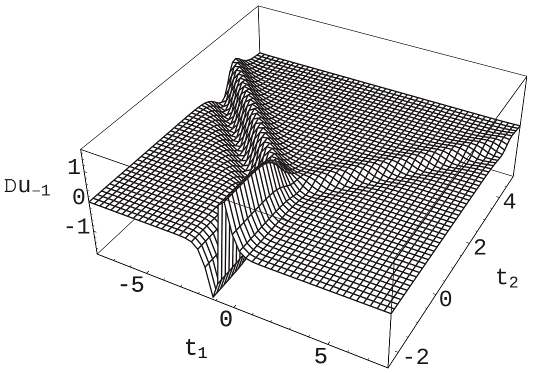

Figure 27.

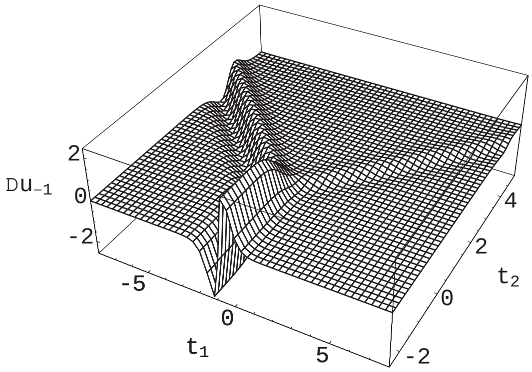

Figure 28.

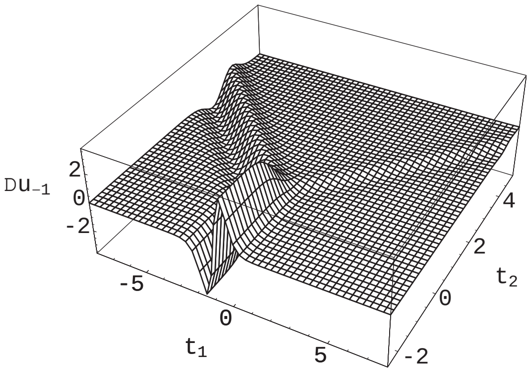

Figure 29.

$$\partial_q^5 \circ f = (\partial_q^5 f) + (5)_q \theta(\partial_q^4 f)\partial_q + \frac{(5)_q(4)_q}{(2)_q}\theta^2(\partial_q^3 f)\partial_q^2 + \frac{(5)_q(4)_q}{(2)_q}\theta^3(\partial_q^2 f)\partial_q^3 + (5)_q\theta^4(\partial_q f)\partial_q^4 + \theta^5(f)\partial_q^5.$$

On the other hand, several examples of an explicit expression for $\partial_q^{-n} \circ f$ $(n \geq 1)$ are

$$\partial_q^{-3} \circ f = \theta^{-3}(f)\partial_q^{-3} - \frac{(3)_q}{q^3}\theta^{-4}(\partial_q f)\partial_q^{-4} + \frac{(3)_q(4)_q}{(2)_q q^{3+4}}\theta^{-5}(\partial_q^2 f)\partial_q^{-5} - \frac{(4)_q(5)_q}{q^{3+4+5}(2)_q}\theta^{-6}(\partial_q^3 f)\partial_q^{-6} + \frac{(5)_q(6)_q}{q^{3+4+5+6}(2)_q}\theta^{-7}(\partial_q^4 f)\partial_q^{-7} + \cdots + \frac{(-1)^k(k+1)_q(k+2)_q}{q^{3+4+5+\cdots+(k+1)+(k+2)}(2)_q}\theta^{-3-k}(\partial_q^k f)\partial_q^{-3-k} + \cdots,$$

$$\partial_q^{-4} \circ f = \theta^{-4}(f)\partial_q^{-4} - \frac{(4)_q}{q^4}\theta^{-5}(\partial_q f)\partial_q^{-5} + \frac{(4)_q(5)_q}{q^{4+5}(2)_q}\theta^{-6}(\partial_q^2 f)\partial_q^{-6} - \frac{(4)_q(5)_q(6)_q}{q^{4+5+6}(2)_q(3)_q}\theta^{-7}(\partial_q^3 f)\partial_q^{-7} + \frac{(5)_q(6)_q(7)_q}{q^{4+5+6+7}(2)_q(3)_q}\theta^{-8}(\partial_q^4 f)\partial_q^{-8} + \cdots + \frac{(-1)^k(k+1)_q(k+2)_q(k+3)_q}{q^{4+5+6+\cdots+(k+2)+(k+3)}(2)_q(3)_q}\theta^{-4-k}(\partial_q^k f)\partial_q^{-4-k} + \cdots,$$

$$\partial_q^{-5} \circ f = \theta^{-5}(f)\partial_q^{-5} - \frac{(5)_q}{q^5}\theta^{-6}(\partial_q f)\partial_q^{-6} + \frac{(5)_q(6)_q}{q^{5+6}(2)_q}\theta^{-7}(\partial_q^2 f)\partial_q^{-7} - \frac{(5)_q(6)_q(7)_q}{q^{5+6+7}(2)_q(3)_q}\theta^{-8}(\partial_q^3 f)\partial_q^{-8} + \frac{(5)_q(6)_q(7)_q(8)_q}{q^{5+6+7+8}(2)_q(3)_q(4)_q}\theta^{-9}(\partial_q^4 f)\partial_q^{-9} + \cdots + \frac{(-1)^k(k+1)_q(k+2)_q(k+3)_q(k+4)_q}{q^{5+6+7+\cdots+(k+3)+(k+4)}(2)_q(3)_q(4)_q}\theta^{-5-k}(\partial_q^k f)\partial_q^{-5-k} + \cdots.$$

B Positive part of L^n $(n = 1, 2, 3)$

The first few of B_n are in the form of

$$B_1 = \partial_q + u_0, \qquad B_2 = \partial_q^2 + v_1\partial_q + v_0, \qquad B_3 = \partial_q^3 + \tilde{s}_2\partial_q^2 + \tilde{s}_1\partial_q + \tilde{s}_0,$$

where

$$v_1 = \theta(u_0) + u_0, \qquad v_0 = (\partial_q u_0) + \theta(u_{-1}) + u_0^2 + u_{-1},$$
$$v_{-1} = (\partial_q u_{-1}) + \theta(u_{-2}) + u_0 u_{-1} + u_{-1}\theta^{-1}(u_0) + u_{-2},$$

and

$$\tilde{s}_2 = \theta(v_1) + u_0, \qquad \tilde{s}_1 = (\partial_q v_1) + \theta(v_0) + u_0 v_1 + u_{-1},$$
$$\tilde{s}_0 = (\partial_q v_0) + \theta(v_{-1}) + u_0 v_0 + u_{-1}\theta^{-1}(v_1) + u_{-2}.$$

Note that v_{-1} comes from $L^2 = B_2 + v_{-1}\partial_q^{-1} + v_{-2}\partial_q^{-2} + \cdots$.

C Proof of the q-Wronskian identity

1) The first N steps. Consider the gauge transformation generated by the order of $\{\phi_i,\ i = 1, 2, \dots, N\}$

$$T_D^{(1)}(\phi_1) \longrightarrow T_D^{(2)}\big(\phi_2^{(1)}\big) \longrightarrow \cdots \longrightarrow T_D^{(i)}\big(\phi_i^{(i-1)}\big) \longrightarrow \cdots \longrightarrow T_D^{(N)}\big(\phi_N^{(N-1)}\big).$$

Assume there are l functions $\{\phi_{N+j}^{(N)},\ j = 1, 2, \dots, l\}$ expressed by

$$\phi_{N+j}^{(N)} = (T_N \cdot \phi_{N+j}) = \frac{W_{N+1}^q(\phi_1, \phi_2, \dots, \phi_N, \phi_{N+j})}{W_N^q(\phi_1, \phi_2, \dots, \phi_N)},$$

which are generated by T_N from $\{\phi_j\}$. Here ϕ_i $(i = 1, 2, \dots, N + l)$ are arbitrary functions such that the gauge transformations can be constructed.

2) The last $l - 1$ steps. Let $y_j = \phi_{N+j}^{(N)}$ $(j = 1, 2, \dots, l)$. Using y_j $(j = 1, 2, \dots, l - 1)$ as the generating functions in order of T_D, we can construct $(l - 1)$ steps of gauge transformation operators as

$$T_D^{(1)}(y_1) \longrightarrow T_D^{(2)}\big(y_2^{(1)}\big) \longrightarrow T_D^{(3)}\big(y_3^{(2)}\big) \longrightarrow \cdots \longrightarrow T_D^{(l-1)}\big(y_{l-1}^{(l-2)}\big).$$

According to the determinant of $T_N|_{N=j}$ $(j = 1, 2, \dots, l - 1)$, we have

$$y_i^{(j)} = (T_j \cdot y_i) \begin{cases} 0 & \text{if } j \geq i, \\ \dfrac{W_{j+1}^q(y_1, y_2, \dots, y_j, y_i)}{W_j^q(y_1, y_2, \dots, y_j)} & \text{if } j < i, \end{cases}$$

then

$$\begin{aligned} y_1 \cdot y_2^{(1)} \cdot y_3^{(2)} \cdots y_{l-1}^{(l-2)} y_l^{(l-1)} &= y_1 \frac{W_2^q(y_1, y_2)}{W_1^q(y_1)} \frac{W_3^q(y_1, y_2, y_3)}{W_2^q(y_1, y_2)} \cdots \\ &\frac{W_{l-1}^q(y_1, y_2, \dots, y_{l-2}, y_{l-1})}{W_{l-2}^q(y_1, y_2, \dots, y_{l-2})} \frac{W_l^q(y_1, y_2, \dots, y_{l-2}, y_{l-1}, y_l)}{W_{l-1}^q(y_1, y_2, \dots, y_{l-2}, y_{l-1})} \\ &= W_l^q(y_1, y_2, \dots, y_l) = W_l^q\big(\phi_{N+1}^{(N)}, \phi_{N+2}^{(N)}, \dots, \phi_{N+l}^{(N)}\big). \end{aligned} \tag{C.1}$$

3) Combine two chains of gauge transformations above. In fact, we can combine two chains into one,

$$\begin{aligned}
&T_D^{(1)}(\phi_1) \longrightarrow T_D^{(2)}\big(\phi_2^{(1)}\big) \longrightarrow \cdots \longrightarrow T_D^{(i)}\big(\phi_i^{(i-1)}\big) \longrightarrow \cdots \longrightarrow T_D^{(N)}\big(\phi_N^{(N-1)}\big),\\
&T_D^{(N+1)}\big(\phi_{N+1}^{(N)}\big) \longrightarrow T_D^{(N+2)}\big(\phi_{N+2}^{(N+1)}\big) \longrightarrow T_D^{(N+3)}\big(\phi_{N+3}^{(N+2)}\big) \longrightarrow \cdots \longrightarrow T_D^{(N+l-1)}\big(\phi_{N+l-1}^{(N+l-2)}\big).
\end{aligned}$$

The determinant representation of $T_N|_{N+j}$ implies ($1 < i,\ j < l$):

$$\phi_{N+i}^{(N+j)} = (T_{N+j}\cdot\phi_{N+i}) = \begin{cases} 0 & \text{if } j \ge i,\\ \dfrac{W_{N+j+1}^q(\phi_1,\phi_2\cdots,\phi_N,\phi_{N+1},\cdots,\phi_{N+j},\phi_{N+i})}{W_{N+j}^q(\phi_1,\phi_2\cdots,\phi_N,\phi_{N+1},\cdots,\phi_{N+j})} & \text{if } j < i.\end{cases}$$

So

$$\begin{aligned}
\phi_{N+1}^{(N)}\cdot\phi_{N+2}^{(N+1)}\cdot\phi_{N+3}^{(N+2)}\cdots\phi_{N+l-1}^{(N+l-2)}\phi_{N+l}^{(N+l-1)} &= \frac{W_{N+1}^q(\phi_1,\phi_2,\ldots,\phi_N,\phi_{N+1})}{W_N^q(\phi_1,\phi_2,\ldots,\phi_N)}\times\\
&\times\frac{W_{N+2}^q(\phi_1,\phi_2,\ldots,\phi_{N+1},\phi_{N+2})}{W_{N+1}^q(\phi_1,\phi_2,\ldots,\phi_{N+1})}\frac{W_{N+3}^q(\phi_1,\phi_2,\ldots,\phi_{N+2},\phi_{N+3})}{W_{N+2}^q(\phi_1,\phi_2,\ldots,\phi_{N+2})}\cdots\\
&\times\frac{W_{N+l-1}^q(\phi_1,\phi_2,\ldots,\phi_{N+l-2},\phi_{N+l-1})}{W_{N+l-2}^q(\phi_1,\phi_2,\ldots,\phi_{N+l-2})}\frac{W_{N+l}^q(\phi_1,\phi_2,\ldots,\phi_{N+l-1},\phi_{N+l})}{W_{N+l-1}^q(\phi_1,\phi_2,\ldots,\phi_{N+l-1})}\cdots\\
&=\frac{W_{N+l}^q(\phi_1,\phi_2,\ldots,\phi_{N+l-1},\phi_{N+l})}{W_N^q(\phi_1,\phi_2,\ldots,\phi_N)}.
\end{aligned}\tag{C.2}$$

The left hand side of (C.1) equals the left hand side of (C.2), which is followed by

$$\frac{W_{N+l}^q(\phi_1,\phi_2,\ldots,\phi_N,\phi_{N+1},\ldots,\phi_{N+l-1},\phi_{N+l})}{W_N^q(\phi_1,\phi_2,\ldots,\phi_N)} = W_l^q\big(\phi_{N+1}^{(N)},\phi_{N+2}^{(N)},\ldots,\phi_{N+l}^{(N)}\big).$$

It should be noted that the proof above is independent of the form of ϕ_k, so we can replace ϕ_{N+j} with $(\partial_q^l\phi_{N+j})$. This completes the proof of the q-Wronskian identity.

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