Scale-Dependent Functions, Stochastic Quantization and Renormalization

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Received November 25, 2005, in final form April 07, 2006; Published online April 24, 2006
Original article is available at http://www.emis.de/journals/SIGMA/2006/Paper046/

Abstract. We consider a possibility to unify the methods of regularization, such as the renormalization group method, stochastic quantization etc., by the extension of the standard field theory of the square-integrable functions $\phi(b) \in \mathrm{L}^2(\mathbb{R}^d)$ to the theory of functions that depend on coordinate b and resolution a. In the simplest case such field theory turns out to be a theory of fields $\phi_a(b,\cdot)$ defined on the affine group $G : x' = ax + b$, $a > 0, x, b \in \mathbb{R}^d$, which consists of dilations and translation of Euclidean space. The fields $\phi_a(b,\cdot)$ are constructed using the continuous wavelet transform. The parameters of the theory can explicitly depend on the resolution a. The proper choice of the scale dependence $g = g(a)$ makes such theory free of divergences by construction.

Key words: wavelets; quantum field theory; stochastic quantization; renormalization

2000 Mathematics Subject Classification: 37E20; 42C40; 81T16; 81T17

1 Introduction

In many problems of field theoretic description of infinite-dimensional systems the continuous description of fields and propagators faces the problem of infinities in loop integrals. The underlying physics is that field theoretic description includes integration over all microscopic scales smaller than the size of the system. In the simplest case small-scale fluctuations can be just averaged to get the classical equations for macroscopic fields. The best known example is the laminar flow hydrodynamics. In more complicated cases interaction strength (and even type) may depend on the scale. The examples are: turbulent fluid flow, critical phenomena, elementary particles interaction, etc.

For this reason we need a measure of integration that works better than Euclidean integration does. For some cases such measures are observed experimentally: these are fractal measure of energy dissipation in turbulent field [1, 2], fractal mass distribution of percolation clusters, etc. For high energies, when the fractal distribution of the field or order parameter can not be measured experimentally due to the uncertainty principle, the search for a better measure of integration remains a mathematical problem. Fortunately, all these phenomena share a common important feature – the self-similarity, or *scaling*. The percolation processes in nanoelectronics, the turbulent velocity field, and the nucleon scattering – all display the same picture if being observed at different scales. The self-similarity has initiated at least two important regularization methods for field theory models:

- The renormalization group (RG) method that makes use of substitution of initial fields

$\phi(x) \in \mathrm{L}^2(\mathbb{R}^d)$ by the scale-truncated fields

$$\phi_{(\frac{2\pi}{\Lambda})}(x) = \int_{|k|\le\Lambda} e^{-\imath kx}\tilde\phi(k)\frac{d^dk}{(2\pi)^d}, \qquad (1)$$

makes the coupling constants dependent on the cut-off momentum Λ, and requires that the final physical results should be independent of the introduced scale

$$\Lambda\partial_\Lambda(\text{Physical quantities}) = 0.$$

- Use of random processes that are self-similar by construction for the regularization of quantum field theory (QFT) models. This is known as *stochastic quantization*.

In present paper we summarize the both methods by introducing the fields $\phi_a(b)$ that explicitly depend on both position b and the scale (resolution) a. If $\phi(x)$ is considered as the wave function of a quantum particle, then the normalization $\int|\phi_a(b)|^2 d\mu(a,b) = 1$, where $d\mu(a,b)$ is the appropriate Haar measure on affine group, expresses the fact that the probability of locating the particle anywhere in space $-\infty < b < \infty$ changing the resolution $0 < a < \infty$ is exactly one. Thus, the RG symmetry related to the change of the scale a extends the symmetry of the theory by allowing the change of parameters with scale. Technically, construction of the scale-dependent fields $\phi_a(b)$ is performed by using continuous wavelet transform (CWT).

It should be noted that since field theoretic calculations are usually performed with the help of the Fourier transform the cut-off momentum Λ corresponds to the minimal coordinate scale $L = \frac{2\pi}{\Lambda}$. However, the Fourier transform is not localized in coordinate space. For this reason we denote the fields obtained from $\phi(x)$ by truncation of the Fourier harmonics with $|k| > \frac{2\pi}{L}$ (1) as $\phi_{(L)}(x)$; in contrast to it the localized view of a function ϕ at the position x and the scale a is denoted as $\phi_a(x)$. The mathematical meaning of the latter will be explained hereafter in terms of wavelet transform. The same thing concerns the parameters of field theory models – masses, coupling constants, etc. In the standard approach they are dependent on the cut-off momentum and it is tacitly understood that the experimental dependence of these parameters on the squared transferred momentum Q^2 is equivalent to the dependence on the cut-off. In our wavelet-based approach the $g = g(a)$ may be the coupling constant of the given field $\phi_a(b)$ of the given scale a, rather than the effective coupling constant for the harmonics of all scales up to a. For this reason different notations are used hereafter in the Fourier and the wavelet representations of field theories.

2 Wavelets and scale-dependent functions

2.1 What is wavelet transform

Wavelet transform entered mathematical physics from geophysical applications [3, 4] as a preferable alternative to the Fourier transform in the case when localization in both position and momentum is simultaneously required. Formally the Fourier transform and the wavelet transform are on the same footing: both are decomposition of a function with respect to the representations of a Lie group. However, the former uses Abelian group of translations, the latter uses non-Abelian group of affine transformations

$$x' = aR(\theta)x + b, \qquad R(\theta) \in SO_d, \quad x, x', b \in \mathbb{R}^d.$$

Let us remind the general formalism. For a locally compact Lie group G acting transitively on the Hilbert space $\mathcal{H}$ it is possible to decompose vectors $\phi \in \mathcal{H}$ with respect to the square-integrable representations $U(g)$ of the group G [5, 6]:

$$|\phi\rangle = C_\psi^{-1}\int_{g\in G} |U(g)\psi\rangle d\mu(g)\langle\psi U^*(g)|\phi\rangle, \qquad (2)$$

where $d\mu(g)$ is the left-invariant Haar measure on G. The normalization constant C_ψ is determined by the norm of the action of $U(g)$ on the fiducial vector $\psi \in \mathcal{H}$, i.e. any $\psi \in \mathcal{H}$ that satisfies the admissibility condition

$$C_\psi = \|\psi\|^{-2} \int_{g\in G} |\langle\psi|U(g)\psi\rangle|^2 d\mu(g) < \infty,$$

can be used as a basis of wavelet decomposition (2).

Hereafter we assume the basic wavelet ψ is invariant under SO_d rotations $\psi(\boldsymbol{x}) = \psi(|\boldsymbol{x}|)$ and drop the angular part of the measure for simplicity ($R(\theta) \equiv 1$). After this simplifying assumption, the left-invariant Haar measure on affine group is $d\mu(a, \boldsymbol{b}) = \frac{dad^d\boldsymbol{b}}{a^{d+1}}$. The representation $U(g)$ induced by a basic wavelet $\psi(x)$ is

$$g : \boldsymbol{x}' = a\boldsymbol{x} + \boldsymbol{b}, \qquad U(g)\psi(\boldsymbol{x}) = a^{-d/2}\psi\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right). \tag{3}$$

Therefore, in the Hilbert space of square-integrable functions $\mathrm{L}^2(\mathbb{R}^d)$, with the scalar product $(f, g) := \int \bar{f}(x)g(x)dx$, a function $\phi \in \mathrm{L}^2(\mathbb{R}^d)$ can be decomposed with respect to the representations of affine group (3):

$$\phi(x) = C_\psi^{-1} \int a^{-d/2}\psi\left(\frac{x-b}{a}\right)\phi_a(b)\frac{dad^db}{a^{d+1}}. \tag{4}$$

The coefficients of this decomposition are

$$\phi_a(b) = \int a^{-d/2}\bar{\psi}\left(\frac{x-b}{a}\right)\phi(x)d^dx. \tag{5}$$

The d-dimensional bold vector notations are dropped hereafter where it does not lead to a confusion, and the basic wavelet $\psi(x)$ is assumed to be isotropic. Here we use normalization for the rotationally invariant wavelets

$$C_\psi = \int \frac{|\tilde{\psi}(k)|^2}{S_d|k|^d} d^dk,$$

where the area of the unit sphere in d dimensions S_d has come from rotation symmetry.

2.2 Scale-dependent functions

Up to this point we considered only a projection of a $\phi \in \mathrm{L}^2(\mathbb{R}^d)$ functions onto the basis constructed of shifts and dilations of the basic wavelet $\psi(x)$. If we substitute the inverse wavelet transform (4) into any physical theory we have started with we will reproduce it identically. However, the wavelet coefficients $\phi_a(b)$ may have their own physical meaning. The convolution (5) can be considered as a "microscope" that scrutinizes a field or a signal $\phi(x)$ at a point b at different resolutions. In this sense the function ψ can be considered as an apparatus function of the measuring device.

The most known case of such theory is the measurement of the turbulent velocity field: the mean velocity and the PDF of fluctuations of different scales should not be the same [1, 7]. Experimentally averaging of the field in a physical volume L^d is usually described as

$$v_L = L^{-d} \int_{L^d} K(x-y)v(y)d^dy,$$

where K is some averaging kernel. But the "no-scale" field $v(y)$ may not exist at all: consider a mean velocity of fluctuations at a scale L less than a mean free path of the particle.

That is why the wavelet coefficients $\phi_a(b)$ may have their own operational meaning, even if the "no-scale" function $\phi(x)$ does not exist. In this case a sum of all fluctuations with scales equal or greater than a given scale L can be defined [8]:

$$\phi_{(L)}(x) = \frac{2}{C_\psi} \int_L^\infty a^{-d/2} \psi\left(\frac{x-b}{a}\right) \phi_a(b) \frac{dad^db}{a^{d+1}}.$$

This is a close analogy to the Wilson's RG approach to critical phenomena [9, 10, 11], where a magnetization (or velocity, or other order parameter) of scale L is taken in the form (1).

2.3 Wavelets and random processes

If a function to be analyzed by continuous wavelet is a random function, its wavelet transform is also a random function. So, instead of the usual space of the random functions $f(x,\cdot) \in (\Omega, \mathcal{A}, P)$, where $f(x,\omega) \in \mathrm{L}^2(\mathbb{R}^d)$ for each given realization ω of the random process, we can go to the multiscale representation provided by the continuous wavelet transform (5):

$$W_\psi(a, \boldsymbol{b}, \cdot) = \int |a|^{-\frac{d}{2}} \overline{\psi\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)} f(\boldsymbol{x}, \cdot) d^d x.$$

The inverse wavelet transform

$$f(\boldsymbol{x}, \cdot) = C_\psi^{-1} \int |a|^{-\frac{d}{2}} \psi\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right) W_\psi(a, \boldsymbol{b}, \cdot) \frac{da d\boldsymbol{b}}{a^{d+1}}$$

reconstructs the common random process as a sum of its scale components, i.e. projections onto different resolution spaces.

The use of the scale components instead of the original stochastic process provides extra analytical flexibility of the method: there exist more than one set of random functions $W(a, \boldsymbol{b}, \cdot)$ the images of which have coinciding correlation functions in the space of $f(x, \cdot)$. It is easy to check that the random process generated by wavelet coefficients having in (a, k) space the correlation function

$$\langle \widetilde{W}(a_1, k_1) \widetilde{W}(a_2, k_2) \rangle = C_\psi (2\pi)^d \delta^d(k_1 + k_2) a_1^{d+1} \delta(a_1 - a_2) D_0$$

has the same correlation function as white noise has:

$$\begin{aligned}
&\langle f(x_1) f(x_2) \rangle = D_0 \delta^d(x_1 - x_2), \\
&\langle \tilde{f}(k_1) \tilde{f}(k_2) \rangle = (2\pi)^d D_0 \delta^d(k_1 + k_2), \\
&\langle \widetilde{W}_\psi(a_1, k_1) \widetilde{W}_\psi(a_2, k_2) \rangle = (2\pi)^d D_0 \delta^d(k_1 + k_2) (a_1 a_2)^{d/2} \overline{\tilde{\psi}(a_1 k_1) \tilde{\psi}(a_2 k_2)}.
\end{aligned}$$

Therefore, the space of scale-dependent random functions is just richer than the ordinary space of random functions: since we have an extra scale argument a here, we can play with the PDF and correlations of random functions $\phi_a(b, \cdot)$ to achieve required properties in ordinary space, holding at the same time some other limitations, say, on singular behavior.

3 Stochastic quantization

3.1 Parisi–Wu stochastic quantization

Quantum field theory, as is understood in technical sense, is a tool to calculate vacuum expectation values (v.e.v.) of the field operator products $\langle \phi(x_1) \dots \phi(x_n) \rangle$ – the Green functions.

Usually it is done in the functional integral formalism: if the action of the field ϕ $S[\phi]$ is known, the v.e.v. can be derived by taking functional derivatives of the generating functional

$$W[J(x)] = \int \mathcal{D}\phi e^{\imath S[\phi]+\imath\int J(x)\phi(x)dx} \tag{6}$$

with respect to the formal source $J(x)$. Unfortunately, infinite perturbation series obtained by such differentiation often give infinite results and require regularization.

However, there are alternatives to this method. Since any quantum system interacts with environment, it can not be in a pure state, and averaging over all field configurations should be performed taking into account averaging over the states of environment. This can be done by considering the system environment as a thermostat, described by a Gaussian random noise; the desired v.e.v. state will be a result of relaxation to the thermodynamic limit. Thus, instead of the Feynman path integral (6), one can operate with stochastic differential equations. Because of equivalence between Euclidean QFT and the stochastic problems, the stochastic methods have found wide applications in both numeric and analytic solution of QFT problems, especially in regularization. Among those, the method of stochastic quantization of gauge fields, proposed by G. Parisi and Y. Wu [12], have been attracting attention for more than 20 years. The idea of the method is as follows. Let $S_E[\phi]$ be the action Euclidean field theory in $\mathbb{R}^d$. Instead of direct calculation of the Green functions from the generation functional of the field theory, it is possible to introduce a *fictitious time* τ, make the quantum fields into stochastic fields $\phi(x) \to \phi(x,\tau)$, $x \in \mathbb{R}^d$, $\tau \in \mathbb{R}$ and evaluate the moments $\langle \phi(x_1,\tau_1)\ldots\phi(x_m,\tau_m)\rangle$ by averaging over a random process $\phi(x,\tau,\cdot)$ governed by the Langevin equation

$$\frac{\partial\phi(x,\tau)}{\partial\tau} + \frac{\delta S}{\delta\phi(x,\tau)} = \eta(x,\tau). \tag{7}$$

The Gaussian random force is δ-correlated in both the $\mathbb{R}^d$ coordinate and the fictitious time τ:

$$\langle \eta(x,\tau)\eta(x',\tau')\rangle = 2D_0\delta(x-x')\delta(\tau-\tau'), \qquad \langle \eta(x,\tau)\rangle = 0. \tag{8}$$

The physical Green functions are obtained by taking the steady state limit:

$$G(x_1,\ldots,x_m) = \lim_{\tau\to\infty} \langle \phi(x_1,\tau)\ldots\phi(x_m,\tau)\rangle.$$

The stochastic quantization method is much preferable to ordinary methods for it does not have problems with gauge fixing and does not require incorporation of higher derivatives as continuous regularization methods do.

Concerning the renormalization of stochastically quantized theory, the matter practically comes out to the renormalization of the Langevin equation (7) in $(d+1)$ dimensions instead of renormalization of the original theory with the action functional $S[\phi]$ in d dimensions. The renormalization of the Langevin equation is usually done by constructing the characteristic functional

$$Z[J] = \left\langle \exp\left(\int d^dx d\tau J(x,\tau)\phi(x,\tau)\right)\right\rangle_{\text{solutions}},$$

where the statistical averaging $\langle\cdots\rangle$ is taken over all solutions of the Langevin equation (7). The summation over all solutions in functional integral formalism is achieved by expressing the functional δ-function as a functional integral over an imaginary auxiliary field $\hat\phi$:

$$\delta\left(\frac{\partial\phi(x,\tau)}{\partial\tau} + \frac{\delta S}{\delta\phi(x,\tau)} - \eta(x,\tau)\right) \sim \int \mathcal{D}\hat\phi \exp\hat\phi\left(\frac{\partial\phi(x,\tau)}{\partial\tau} + \frac{\delta S}{\delta\phi(x,\tau)} - \eta(x,\tau)\right). \tag{9}$$

This is so-called Martin–Siggia–Rose field-doubling formalism [13]. The resulting theory has the form of a field theory with constraints in $(d+1)$ dimensions. Under the change of the cut-off scale all fields, coordinates and parameters in such a theory are transformed according to their canonical dimensions.

The renormalization of such a theory has much in common with the renormalization of gauge theories: for the constraint (9) is similar to gauge fixing, and the functional determinant arising from the this term under the change of variables results in ghost fields and BRST symmetry. The details of renormalization of stochastically quantized theories can be found e.g. in [14, 15].

3.2 Wavelet-based stochastic quantization

Taking into account the above mentioned flexibility of stochastic processes defined in the space of wavelet coefficients, it appears to be attractive to use wavelet-defined noise for stochastic quantization. Applying the wavelet transform (in spatial coordinates, but not in fictitious time)

$$\phi_a(b,\cdot) = \int a^{-d}\overline{\psi\left(\frac{x-b}{a}\right)}\phi(x,\cdot)d^dx$$

to the fields and the random force in the Langevin equation (7), we get the possibility to change the white noise (8) into a scale-dependent random force

$$\langle \tilde\eta_{a_1}(k_1,\tau_1)\tilde\eta_{a_2}(k_2,\tau_2)\rangle = C_\psi(2\pi)^d\delta^d(k_1+k_2)\delta(\tau_1-\tau_2)a_1\delta(a_1-a_2)D(a_1,k_1). \tag{10}$$

(Here and after we use CWT in L^1 norm instead of L^2.)

In the case when the spectral density of the random force is a constant $D(a_1,k_1)=D_0$, the inverse wavelet transform

$$\phi(x) = \frac{2}{C_\psi}\int_0^\infty \frac{da}{a}\int \frac{d^dk}{(2\pi)^d}\frac{d\omega}{2\pi}\exp(\imath(kx-\omega\tau))\tilde\psi(ak)\phi_a(k,\omega) \tag{11}$$

drives the process (10) into the white noise (8).

In the case of arbitrary functions $\phi_a(x,\cdot)$ we have more possibilities. In particular, we can define a narrow band forcing that acts at a single scale

$$D(a,k) = a_0\delta(a-a_0)D_0. \tag{12}$$

The contribution of the scales with the wave vectors apart from the the typical scale a_0^{-1} is suppressed by rapidly vanishing wings of the compactly supported wavelet $\tilde\psi(k)$.

Here we present two examples of the divergence free stochastic perturbation expansion: (i) the scalar field theory ϕ^3, (ii) the non-Abelian gauge field theory.

3.3 Scalar field theory

Let us turn to the stochastic quantization of the ϕ^3-theory with the scale-dependent noise [16]. The Euclidean action of the ϕ^3-theory is:

$$S_E[\phi(x)] = \int d^dx\left[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{3!}\phi^3\right].$$

The corresponding Langevin equation is written as

$$\frac{\partial\phi(x,\tau)}{\partial\tau} + \left(-\Delta\phi + m^2\phi + \frac{\lambda}{2!}\phi^2\right) = \eta(x,\tau). \tag{13}$$

Substituting the scale components in representation (11) into the Langevin equation (13) we get the integral equation for the stochastic fields:

$$(-\imath\omega + k^2 + m^2)\phi_a(k,\omega) = \eta_a(k,\omega) - \frac{\lambda}{2}\overline{\tilde{\psi}(ak)}\left(\frac{2}{C_\psi}\right)^2 \int \frac{d^dk_1}{(2\pi)^d}\frac{d\omega_1}{2\pi}\frac{da_1}{a_1}\frac{da_2}{a_2},$$
$$\tilde{\psi}(a_1k_1)\tilde{\psi}(a_2(k-k_1))\phi_{a_1}(k_1,\omega_1)\phi_{a_2}(k-k_1,\omega-\omega_1). \qquad (14)$$

Starting from the zero-th order approximation $\phi_0 = G_0\eta$, with the bare Green function $G_0(k,\omega) = 1/(-\imath\omega + k^2 + m^2)$, and iterating the integral equation (14), we get the one-loop correction to the stochastic Green function:

$$G(k,\omega) = G_0(k,\omega)$$
$$+ \lambda^2 G_0^2(k,\omega)\int \frac{d^dq}{(2\pi)^d}\frac{d\Omega}{2\pi} 2\Delta(q)|G_0(q,\Omega)|^2 G_0(k-q,\omega-\Omega) + \cdots, \qquad (15)$$

where $\Delta(k)$ is the scale-averaged effective force correlator

$$\Delta(k) \equiv \frac{2}{C_\psi}\int_0^\infty \frac{da}{a}|\tilde{\psi}(ak)|^2 D(a,k).$$

In the same way other stochastic moments are evaluated. Thus, the common stochastic diagram technique is reproduced, but with the scale-dependent random force (10) instead of the standard one (8). The diagrams corresponding to the stochastic Green function decomposition (15) are shown in Fig. 1.

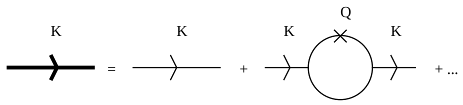

Figure 1. Diagram expansion of the stochastic Green function in ϕ^3-model.

It can be seen that for a single-band forcing (12) and a suitably chosen wavelet the loop divergences are suppressed. For instance, use of the Mexican hat wavelet

$$\tilde{\psi}(k) = (2\pi)^{d/2}(-\imath k)^2 \exp(-k^2/2), \qquad C_\psi = (2\pi)^d,$$

and the single-band random force (12) gives the effective force correlator

$$\Delta(q) = (a_0q)^4 e^{-(a_0q)^2} D_0. \qquad (16)$$

The loop integrals, taken with this effective force (16), can be easily seen to be free of ultraviolet divergences at $q \to \infty$:

$$G_2(k,\omega) = G_0^2(k,\omega)\int \frac{d^dq}{(2\pi)^d} 2\Delta(q)\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\frac{1}{\Omega^2 + (q^2+m^2)^2}\frac{1}{-\imath(\omega-\Omega) + (k-q)^2 + m^2}.$$

3.4 Non-Abelian gauge theory

The Euclidean action of a non-Abelian gauge field is given by

$$S[A] = \frac{1}{4}\int d^dx F^a_{\mu\nu}(x)F^a_{\mu\nu}(x),$$
$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + gf^{abc}A^b_\mu(x)A^c_\nu(x), \qquad (17)$$

where f^{abs} are the structure constants of the gauge group, g is the coupling constant. The Langevin equation for the stochastic quantization of gauge theory (17) can be written as

$$\frac{\partial A^a_\mu(x,\tau)}{\partial\tau} + \left(-\delta_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu\right)A^a_\nu(x,\tau) = \eta^a_\mu(x,\tau) + U^a_\mu(x,\tau), \tag{18}$$

where $\eta^a_\mu(x,\tau)$ is the random force and $U^a_\mu(x,\tau)$ is the nonlinear interaction term

$$U[A] = \frac{g}{2}V^0(A,A) + \frac{g^2}{6}W^0(A,A,A).$$

The two terms standing in the free field Green function correspond to the transversal and the longtitudal mode propagation:

$$G^{ab}_{\mu\nu}(k) = \frac{T_{\mu\nu}(k)\delta_{ab}}{-\imath\omega + k^2} + \frac{L_{\mu\nu}(k)\delta_{ab}}{-\imath\omega}, \qquad T_{\mu\nu}(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \qquad L_{\mu\nu}(k) = \frac{k_\mu k_\nu}{k^2}.$$

Similarly to the scalar field theory, we can use the scale-dependent forcing (19) in the Langevin equation (18). Since there is no dynamic evolution for the longtitudal modes in the Langevin equation (18), it is natural to use the transversal scale-dependent random force

$$\langle \eta^a_{a_1\mu}(k_1,\tau_1)\eta^b_{\nu a_2}(k_2,\tau_2)\rangle = (2\pi)^d\delta^d(k_1+k_2)\delta(\tau_1-\tau_2)T_{\mu\nu}(k_1)C_\psi a_1\delta(a_1-a_2)D(a_1,k_1). \tag{19}$$

Let us consider a gluon loop with two cubic vertices. Summing up over the gauge group indices $\left(\frac{\imath}{2}g\right)^2 f^{abc}\delta_{bd}f^{der}\delta_{cr} = \frac{g^2}{4}\delta_{ae}C_2$, with $C_2 = N$ for SU_N gauge groups, we can write the gluon loop as a sum of two diagrams – those with the transversal and the longtitudal stochastic Green functions:

$$G^{ab}_{2\mu\nu}(k,\omega) = g^2\delta_{ab}C_2|G_0(k,\omega)|^2 \sum_{I=T,L}\int \frac{d\Omega}{2\pi}\frac{d^dq}{(2\pi)^d}N^I(k,\omega,q,\Omega)l^I_{\mu\nu}(k,q)2\Delta(q),$$

where

$$N(k,q) = \left|\frac{1}{-\imath\Omega+q^2}\right|^2 \begin{pmatrix} \frac{1}{-\imath(\omega-\Omega)+(k-q)^2} \\ \frac{1}{-\imath(\omega-\Omega)} \end{pmatrix},$$

$$l_{\mu\nu}(k,q) = V_{\mu\kappa\lambda}(k,k-q,q)T_{\lambda\gamma}(q)V_{\sigma\nu\gamma}(k-q,k,-q)\begin{pmatrix} T_{\kappa\sigma}(k-q) \\ L_{\kappa\sigma}(k-q)\end{pmatrix}.$$

As it can be observed after explicit evaluation of the tensor structures $l^T_{\mu\nu}$ and $l^L_{\mu\nu}$ using the force correlator (19), and integration over $d\Omega$, the wavelet factor in the effective force correlator $\Delta(q)$ suppresses the divergences in the case of a narrow-band forcing (12). The power factor k^n of the basic wavelet ψ that provides $\tilde\psi(0) = 0$ and makes the IR behavior softer. In this respect the wavelet regularization is different from the continuous regularization $\int d^dyR_\Lambda(\partial^2)\eta(y,\tau)$, which makes UV behavior softer by the factor $e^{-\frac{k^2}{\Lambda^2}}$, but do not affect the IR behavior, see e.g. [17].

4 Wavelet-based Euclidean field theory

4.1 QFT on a Lie group

Let us consider a Euclidean field theory determined by a characteristic functional

$$W_E[J] = \mathcal{N}\int \mathcal{D}\phi \exp\left[-S_E[\phi(x)] + \int d^dxJ(x)\phi(x)\right]$$

where $S_E[\phi(x)]$ is Euclidean action, $\mathcal{N}$ is formal normalization constant. The (connected) Green functions (m-point cumulative moments) are evaluated as functional derivatives of the logarithm of generating functional $W_E[J] = e^{-Z_E[J]}$:

$$G_m(x_1,\ldots,x_m) \equiv \langle\phi(x_1)\ldots\phi(x_m)\rangle = -\left.\frac{\delta^m}{\delta J(x_1)\cdots\delta J(x_m)}\right|_{J=0} \ln W_E[J].$$

Applying a formal partition of a unity (2) with respect to a Lie group G we yield a theory with the generating functional

$$W_g[J(g)] = \mathcal{N}\int \mathcal{D}\phi(g)\exp\left[-S_g[\phi(g)] + \int_G d\mu(g)J(g)\phi(g)\right], \qquad g\in G$$

and appropriately defined Green functions.

4.2 Wavelet-based action

Let us consider the theory of a massive scalar field with polynomial interaction

$$S_E[\phi] = \int_{\mathbb{R}^d} d^dx\left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2 + \lambda V(\phi)\right], \tag{20}$$

that can be alternatively interpreted as a theory of classical fluctuating field with the Wiener probability measure $\mathcal{D}P = e^{-S_E[\phi]}\mathcal{D}\phi$. In this case $m^2 = |T - T_c|$ is the deviation from critical temperature and λ is the fluctuation interaction strength.

To introduce the scale-dependent fields $\phi_a(b)$ into the action we can just substitute the fields ϕ into (20) using wavelet transform (4). For convenience and recording purposes, we rewrite continuous wavelet transform (5), (4) using the L^1 norm

$$\begin{aligned}
\phi_a(b) &= \int \overline{\frac{1}{a^d}\psi\left(\frac{x-b}{a}\right)}\phi(x)d^dx,\\
\phi(x) &= \frac{2}{C_\psi}\int_0^\infty \frac{da}{a}\int_{\mathbb{R}^d} d^db\,\phi_a(b)\frac{1}{a^d}\psi\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right),\\
C_\psi &= S_d^{-1}\int\frac{|\tilde\psi(\boldsymbol{k})|^2}{|k|^d}d^dk = 2\int_0^\infty \frac{|\tilde\psi(a\boldsymbol{k})|^2}{a}da < \infty.
\end{aligned} \tag{21}$$

This provides the fields $\phi(x) = \langle x|\phi\rangle$ and $\phi_a(x) = \langle a,x;\psi|\phi\rangle$ with the same physical dimension. The generating functional for the scale-dependent functions $\phi_a(b)$ is

$$W[J_a(b)] = \int \mathcal{D}\phi_a(b)\left(-S_{EW}[\phi_a(b)] + \int J_a(b)\phi_a(b)\frac{da\,d^db}{a}\right),$$

where $J_a(b)$ is a formal source ("external force"), which corresponds to the fluctuations of given scale a localized near a given point b. The corresponding Green functions a given by

$$G(a_1,x_1;\ldots;a_n,x_n) = -\left.\frac{\delta\ln W[J_a(x)]}{\delta J_{a_1}(x_1)\delta J_{a_n}(x_n)}\right|_{J=0} = \langle\phi_{a_1}(x_1)\ldots\phi_{a_n}(x_n)\rangle.$$

Let us consider a theory with polynomial interaction $V(\phi) = \frac{g}{n!}\phi^n$

$$\begin{aligned}
S_{EW} = \int &-\frac{1}{2}\phi_{a_1}(b_1)D(a_1,a_2;b_1-b_2)\phi_{a_2}(b_2) + \frac{m^2}{2}|\phi_a(b)|^2\\
&+\frac{g}{n!}V^{a_1\ldots a_n}_{b_1\ldots b_n}\phi_{a_1}(b_1)\cdots\phi_{a_n}(b_n),
\end{aligned} \tag{22}$$

$$V^{a_1\ldots a_n}_{b_1\ldots b_n} = \int d^d x V(x_1,\ldots,x_n) \prod_{i=1}^n \frac{1}{a_i^d}\psi\left(\frac{x-b_i}{a_i}\right), \qquad (23)$$

where integration over all pairs of matching indices $\frac{da_i d^d b_i}{C_\psi a_i}$ is assumed. The inverse propagator is the wavelet image of $(-\partial^2+m^2)$, that is

$$D(a_1,a_2,k) = \overline{\tilde\psi(a_1k)}(k^2+m^2)\tilde\psi(a_2k).$$

As it is seen from (23), the local interaction term ϕ^n becomes nonlocal after the application of wavelet transform (21). Of course, since the wavelet transform provides the partition of a unity, the integration over all scale arguments in the theory (22) drives it back to the original field theory with polynomial interactions, which is UV divergent. However, using the Wilson's RG ideas we shall show how the nonlocal wavelet theory (22) can be made into a local one, which has UV finite behavior.

Let us suppose that we manage to make the wavelet theory with nonlocal interaction (23) into a local one with the coupling constant explicitly dependent on scale $g = g(a)$. The simplest case of the fourth power interaction of this type is

$$V_{int}[\phi] = \frac{1}{C_\psi}\int \frac{g(a)}{4!}\phi_a^4(b)\frac{dadb}{a}, \qquad g(a)\sim a^\nu. \qquad (24)$$

The one-loop order contribution to the Green function G_2 in the theory with local interaction (24) can be easily evaluated [18] by integration over a scalar variable $z = ak$:

$$\int \frac{a^\nu|\tilde\psi(ak)|^2}{k^2+m^2}\frac{d^dk}{(2\pi)^d}\frac{da}{a} = C_\psi^{(\nu)}\int \frac{d^dk}{(2\pi)^d}\frac{k^{-\nu}}{k^2+m^2},$$

where

$$C_\psi^{(\nu)} = C_\psi^{-1}\int |\tilde\psi(z)|^2 z^{\nu-1}dz.$$

Therefore, there are no UV divergences for $\nu > d-2$. However, the positive values of ν mean that the interaction strengths at large scales and diminishes at small. (This is a kind of asymptotically free theory that is hardly appropriate say to magnetic systems.)

We have to admit that the idea of using wavelets for regularization and blocking of degrees of freedom has been considered by G. Battle [19] and P. Federbush [20] in lattice settings. The idea was to sum up the discrete wavelet expansion of a function starting from a large infra-red scale (set to unity) up to the infinitesimally small scale instead of integrating in Fourier space. The expansion was taken in the form

$$\phi(x) = \sum_k \alpha_k u_k(x), \qquad u_k(x) = \frac{1}{\sqrt{-\Delta+m^2}}\Psi_k(x), \qquad (25)$$

where $k = (n,m,s)$ is a multiindex that incorporates translations and binary dilations on the lattice as well as internal degrees of freedom of the field ϕ, $\Psi_k(x)$ forms an orthogonal basis in $\mathrm{L}^2(\mathbb{R}^d)$. It have been understood then that different subgrids involved in the wavelet expansion (25) represent different independent degrees of freedom [21], rather than being just different approximation of the same field like that in Kadanoff block expansion. The corresponding diagram technique, called "wavelet diagrams" and originally proposed by Federbush, can be found in Battle's book [19].

For a particular type of wavelet basis, the Lemarié wavelets, it was shown that due to the big number of vanishing momenta of the basic wavelets Ψ_k the correlations between different block variables $\langle \alpha_k \alpha_{k'} \rangle$ decrease as a high inverse power of the separation between blocks. As a consequence of this a field theory $S[\phi(x)]$ becomes UV-finite without any renormalization. However the decoupling of interaction between different scales seems to have quite general nature and that is the reason we exploit this idea in present paper in terms of *continuous* wavelet transform without using any lattices.

Now we will try to show how the local interaction of type (24) can emerge. Following K. Wilson [10, 11] we consider a system of spins with spacing L_0. The idea of RG applied to a spin system is that the interaction of spins separated by a distance L_0, can be substituted by interaction of blocks (of 2^d spins in each) of size $2L_0$ with the high-frequency details absorbed into the interaction constants. The same procedure is then applied to the blocks of $2L_0$ size, blocks of $4L_0$ size etc. – this is so-called Kadanoff blocking procedure. On each step of the blocking procedure the transformation from the scale L to $2L$ is done by integrating over the high frequency degrees of freedom $K > \frac{2\pi}{L}$ with appropriate adjustment of the coupling constants.

In continuous limit, i.e. in Landau theory of ferromagnetism, when spins are averaged into the mean magnetization $\phi(x)$, the energy of the spin system is given by the Ginzburg–Landau free energy functional

$$F_E[\phi] = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{R}{2}\phi^2 + \frac{U}{4!}\phi^4 \right], \tag{26}$$

where integration is performed over the domain occupied by the spin system. The field theory (26) gives divergences in correlation functions. To cope with it we remind that $\phi(x)$ is the mean magnetization of a macroscopically big block, i.e. one that contains enough spins to make statistical averaging valid. This means that the field ϕ should be substituted by one containing only the fluctuations of size larger than a typical scale L:

$$\phi(x) \to \phi_{(L)}(x) = \int_{|k|<\frac{2\pi}{L}} e^{-\imath kx}\tilde\psi(k)\frac{d^d k}{(2\pi)^d}. \tag{27}$$

So, the original Ginzburg–Landau functional is made into effective action for the large-scale fields $\phi_{(L)}$

$$F_L[\phi_{(L)}] = \int d^d x \left[\frac{1}{2}(\partial\phi_{(L)})^2 + \frac{R_L}{2}\phi_{(L)}^2 + \frac{U_L}{4!}\phi_{(L)}^4 \right], \tag{28}$$

in which the effects of small-scale fluctuations with $|k| \ge \frac{2\pi}{L}$ are absorbed into the coupling constants R_L and U_L. It should be noted that the projection (27), performed by filtering out the Fourier harmonics with $|k| \ge \frac{2\pi}{L}$, is not the only possible type of smoothing: an exponential or proper-time cutoff can be used as well [22]. Similar projection can be obtained if we apply the wavelet transform with certain given kernel $\phi(x) \to \phi_a(b)$.

Wilson suggested an elegant way to determine the dependence of R_L and U_L on the cutoff scale L by averaging over fluctuations in the shell $[L, L+\delta L)$ [10, 9, 11]. Since we understand the fluctuations of a given scale $\phi_a(b)$ – now in wavelet notation – as physical fields measured at scale a, we will reproduce the Wilson's derivation integrating the scale-dependent free-energy $F[\phi_a(b)]$ in appropriate limits over the logarithm of scale $a^{-1}da$. The larger is the scale L, the more fluctuations of smaller scales contribute to free energy.

Let L_0 be the smallest size of the system – the distance between spins in the theory of ferromagnetism. Then, the Kadanoff blocking procedure is a mapping

$$\begin{aligned}
&\{\psi_k^{(0)}\} \to \{\psi_k^{(1)}\} \to \{\psi_k^{(2)}\} \to \cdots, \\
&H^{(0)}(R_0, U_0) \to H^{(1)}(R_1, U_1) \to H^{(2)}(R_2, U_2) \to \cdots,
\end{aligned}$$

where $\{\psi_k^{(0)}\}$ is the basis for the finest resolution Hilbert space, with $H^{(0)}(R_0, U_0)$ being the Hamiltonian acting on this space, $\{\psi_k^{(1)}\}$ is the basis of the next coarse-grained space of the resolution $2L_0$, etc. At each step of the coarse graining process some details are lost, so any function $\psi_k^{(1)}$ of the basis $\{\psi_k^{(1)}\}$ can be expanded in the basis $\{\psi_k^{(0)}\}$; and similarly for all next scales:

$$\psi_i^{(j)} = \sum_k c_{ik}^{j+1} \psi_k^{(j)},$$

since $\{\psi_k^{(j)}\}$ is more detailed than $\{\psi_k^{(j+1)}\}$.

Expanding the more coarse-grained basic functions in terms of the less coarse-grained one we obtain the expansion of the parameters of the coarser Hamiltonian $H^{(j+1)}$ in terms of the finer Hamiltonian $H^{(j)}$. Wilson suggested a qualitative way of doing that. Suppose we know the free energy functional $F_L[\phi]$ of scale L and need to calculate the next coarser functional $F_{L+\delta L}[\phi]$ of scale $L+\delta L$. Then, we have to add fluctuations of all scales within the range $[L, L+\delta L)$ to the original theory and integrate over those fluctuations. If $\psi(x)$ is a normalized basic function of scale L, e.g. a wave packet with $k \approx \frac{2\pi}{L}$, such that

$$\int d^d x |\psi(x)|^2 = 1, \qquad \int d^d x |\partial\psi(x)|^2 \simeq L^{-2}, \tag{29}$$

we just make $\phi(x) \to \phi(x) + c\psi(x)$ and integrate over the scalar amplitude c:

$$e^{-F_{L+\delta L}[\phi]} = \int_{-\infty}^{\infty} dm e^{-F_L[\phi+m\psi]}. \tag{30}$$

Substituting the free energy (28) into the condition (30) and taking into account the conditions (29) we get

$$e^{-F_{L+\delta L}[\phi]} = e^{-F_L[\phi]} \int_{-\infty}^{\infty} dm e^{-\frac{m^2}{2}\left(\frac{1}{L^2}+R_L+\frac{1}{2}U_L\phi^2\right)}. \tag{31}$$

The derivation is not very rigorous, and referring the reader to the original paper [11] for the details, we just have to state that using the formal properties of Gaussian integration in (31), up to the numerical factors, we get a logarithmic contribution to the free energy

$$F_{L+\delta L}[\phi] = F_L[\phi] + \frac{1}{2}\ln\left(\frac{1}{L^2} + R_L + \frac{1}{2}U_L\phi^2\right). \tag{32}$$

To derive the dependence of R_L and U_L on scale L two more tricks were used by Wilson: (i) the logarithm in (32) is decomposed into a Taylor series; (ii) integration over the phase space volume corresponding to the d.o.f. related to the wavepacket ψ is changed into integration over the coordinate volume δV using the uncertainty principle.

The logarithm is decomposed into the power series only up to the second order – to get the forth order in the fields. Up to the terms that do not depend on ϕ we get

$$\frac{1}{2}\ln\left(\frac{1}{L^2} + R_L + \frac{1}{2}U_L\phi^2\right) = \frac{1}{4}\left(U_L L^2\phi^2 - \frac{1}{4}L^4\phi^4 - R_L U_L L^4\phi^2\right) + \cdots. \tag{33}$$

The next trick is to count the degrees of freedom corresponding to the localized wavepacket ψ. Since the scales are in the range $[L, L+\delta L)$, the volume in momentum space is $\frac{\delta p}{2\pi} \sim \frac{\delta L}{L^2}$. The phase space volume, since $p \approx 2\pi/L$, is

$$\delta\Gamma \sim V p^{d-1}\delta p \sim V L^{-1-d}\delta L,$$

and, hence, the coordinate volume per one degree of freedom is

$$\delta V \sim \frac{L^{1+d}}{\delta L}.$$

Thus the equation (33) can be formally multiplied by $1 = \delta V L^{-1-d} \delta L$ and integrated over the volume δV. This results in equations

$$\begin{aligned} R_{L+\delta L} &= R_L + \frac{1}{2}\left(L^{1-d}U_L - L^{3-d}R_L U_L\right)\delta L, \\ U_{L+\delta L} &= U_L - \frac{3}{2}L^{3-d}U_L^2 \delta L. \end{aligned}$$

See [23] for general theory of renormalization in field theory and critical phenomena.

The above Wilson's consideration was presented only to show how integration in the thin shell below the cutoff renormalizes the parameters of the action. More rigorous formalism that accounts for the change of the action parameters by means of integration over the infinitesimally thin shell $1 - \delta l < |q| < 1$ in momentum space, followed by rescaling, consists in constructing differential equations for the variation of the action functional. It is known as exact renormalization group (ERG). The ERG equations are rather complicated and can be found elsewhere [24, 10, 25]. If solved numerically, the ERG equations could provide exact scale dependence of the coupling constants on cutoff scale in terms of the considered model. The problems are, however, that (i) it is often impossible to solve these equations, (ii) the cutoff scale is not the same as the scale of observation, and hence it is not obvious how to interpret the ERG results; (iii) and the last, but not the least, that it is not clear how to separate the modes to be integrated over from those to be renormalized by this integration using the space of functions that depend only on momentum $\phi(k)$ ([10], Fig. 11.2). Here we propose an alternative solution of the mode separation problem: we just extend the space of functions adding the scale argument explicitly $\phi(\cdot) \to \phi_a(\cdot)$.

Having the information on the dependence on scale of the effective mass R_L and coupling constant U_L we can address the question, how the effective theory (28) that comprises nonlocal interactions of all modes larger than L can be transformed into a theory with the local interactions of scales of the type (24). To do this we approximate the original Ginzburg–Landau action F, by a new action

$$F_E[\phi] = \int \frac{da d^d b}{a}\left[-\frac{1}{2}\phi_{a_1}(b_1) D \phi_{a_2}(b_2) + \frac{m^2(a)}{2}|\phi_a(b)|^2 + \frac{g(a)}{4!}|\phi_a(b)|^4\right]. \quad (34)$$

That meets physical requirements by construction: for a free field ($g(a) = 0$) it coincides with the original free action; for the interacting fields, it just satisfies the Wilson's assumption that *only the fluctuations of the close scale interact*: in the case of the action (34), certainly the equal scales only.

Since the action (34) explicitly involves integration in scale variable da/a, we can attribute the difference in the free energies (33) to interaction of the appropriate "renormalized" modes, governed by the equation (34), in the shell $[L, L + \delta L)$.

The $n > 2$ the polynomial interactions are nonlocal in wavelet space. To derive their interaction constant dependence on scale, we use the physical assumptions that only the fluctuations of close scales can directly interact to each other. So, in the self-interaction of the field ϕ

$$U_L \int_L \phi(x)^n d^d x = U_L \int_L V^{a_1 \ldots a_n}_{b_1 \ldots b_n} \phi_{a_1}(b_1) \cdots \phi_{a_n}(b_n) \prod_{i=1}^n \frac{da_i d^d b_i}{a_i}$$

the terms with $|\ln a_i - \ln a_j| \ll 1$ should dominate, from where we can postulate that

$$U_L \int_L V^{a_1\ldots a_n}_{b_1\ldots b_n} \phi_{a_1}(b_1)\cdots\phi_{a_n}(b_n) \prod_{i=1}^n \frac{da_i d^d b_i}{a_i} = \int_L \frac{da}{a} \int g(a)|\phi_a(b)|^n d^d b \tag{35}$$

for some $g(a)$.

It should be noted that similar idea of orthogonalization of fluctuations of different scales has been already applied to Ginzburg–Landau model by C.Best using discrete wavelet transform. However, the Best's paper [26], having started from continuous Ginzburg–Landau model, uses *orthogonal wavelets* and sufficiently strong assumptions that fluctuations of different scales are δ-correlated in both position and scale (equation (4) from [26]). This is definitely not the case for arbitrary (non-orthogonal) wavelets and standard assumption of Gaussian nature of fluctuations in ordinary continuous model (26). Nevertheless, the Best's idea is numerically convenient and it is interesting for what type of wavelets such behavior is really observed.

Let us perform the asymptotic estimation of the behavior of $g(a)$ at the known behavior of U_L. Let us assume that interaction takes place at the thin shell of scales $[L, L+\delta L)$ and set $a_1 = \cdots = a_n = L$. Since we need the behavior of the local interaction of the type (24), we set $b_1 = \cdots = b_n = 0$ in the r.h.s. of (35) to get the upper bound for $g(a)$. Doing so, we get

$$V^{a\ldots a}_{0\ldots 0} = \int d^d x \frac{1}{a^{nd}} \psi^n\left(\frac{x}{a}\right) = a^{(1-n)d} \int d^d y \psi^n(y).$$

In Wilson's theory ($n=4$, $d<4$)

$$U_L = \frac{1}{\frac{3}{2}\frac{L^{4-d}}{4-d} + \mathrm{const}} \stackrel{L\to\infty}{\sim} L^{d-4},$$

and hence

$$g(L) \sim L^{-3d} U_L \sim L^{-2d-4}. \tag{36}$$

The important point is that in the theory (34) the observable quantities are the correlators of the scale-dependent functions $\langle \phi_{a_1}(b_1)\phi_{a_2}(b_2)\rangle$. If we deal with "renormalized" action (34), where only equal scales interact polynomially, the loop integrals can be easily evaluated for a fixed value of scale. In the fourth power interaction theory the one-loop contribution to the Green function will be

$$\int_{a_{\min}}^{a_{\max}} \frac{da}{a} g(a) \int \frac{d^d k}{(2\pi)^d} \frac{|\tilde\psi(ak)|^2}{k^2+m^2} = \int_{a_{\min}}^{a_{\max}} \frac{da}{a} g(a) L(a). \tag{37}$$

The k-dependent integral in the last equation (37) can be explicitly evaluated. As an example, let us present a "bad" case of the Morlet wavelet $\tilde\psi(k) = \exp\left(\imath k z - \frac{k^2}{2}\right)$, which is not admissible in the sense $C_\psi = \infty$,

$$L(a) = S_d \int_0^\infty \frac{e^{-a^2 q^2} q^{d-1}}{q^2+m^2} \frac{d^d q}{(2\pi)^d} \stackrel{d=4,m=1}{\longrightarrow} \frac{-a^4 e^{a^2} Ei(1,a^2) + a^2}{a^2} \frac{a^{2-d}}{2}.$$

For admissible wavelets ($C_\psi < \infty$) the IR behavior is milder.

Certainly, the integral $\int_{a_{\min}}^{a_{\max}} \frac{da}{a} g(a) L(a)$ in any positive range of scales and with the coupling constant $g(a)$ taken from (36) is finite.

5 Conclusion

In this paper we attempt to construct a theory of scale-dependent fields $\phi_a(b)$ starting from an action functional that explicitly depends of these fields, rather than being a local functional $S[\phi]$ truncated at certain scale L by averaging over the short-wave fluctuations $|k| > \frac{2\pi}{L}$. The difference is that in our theory it is meaningful to consider the correlation functions $\langle\phi_{a_1}(b_1)\phi_{a_2}(b_2)\rangle$ between different points and different scales. This is important because any experimental measurement is performed not exactly in a point, but in a certain vicinity of a point, the size of which is constrained at least by the uncertainty principle: the higher momentum is used in the measurement, the smaller is the vicinity and the stronger is the perturbation caused by the measurement. Thus we have a strong need do construct a theory in terms of what is really measured in experiment – scale-dependent (wavelet) fields $\phi_a(b)$, rather than in terms of abstract "no-scale" functions $\phi(x)$.

Standard RG approach meets this need only partially: it enables to study the dependence of correlations on separation between points, $b = b_1 - b_2$ in our terms, but not on the typical scale of interaction. The latter is described in terms of the correlation length $\xi = \xi(T)$, which has a universal behavior near the critical temperature $\xi(T) \sim (T - T_c)^{-\nu}$. The Kadanoff hypothesis that blocks of spins interact exactly as spins themselves becomes strictly valid only in critical regime $\xi \to \infty$ at $T \to T_c$. In other cases there are no reasons to assume that fluctuations of different sizes do not interact to each other and $\langle\phi_{a_1}(b_1)\phi_{a_2}(b_2)\rangle = 0$ for $a_1 \neq a_2$, instead the inter-scale interaction is expressed by wavelet diagrams.

As it concerns the criticality, our approach may be close to the ideas of functional self-similarity (FS) [27]: at the critical point the blocks of spins can interact with each other in the same way as primitive spins themselves, but the coupling constant is scale-covariant, i.e. its transformation with changing of scale depends on the value of coupling constant, but not on the scale.

For a quantum field theory models, which arise from the high energy physics problems, the Kadanoff block-averaging procedure can not be directly applied. Instead the universal behavior of the coupling constant $g = g(a)$ with respect to the scale transformations $a' = e^\lambda a$ should be considered as a fundamental symmetry of the physical system. Indeed, the coupling constant $g(a)$ is a charge of a given particle with respect to a given interaction measured at a given scale a. Since the experiments can be performed at different scales $a_0 < a_1 < a_2 < \cdots$, we should expect a universal relation between the values of charges at different scales, and it should be an *invariant charge*, such that $g_0 \equiv \zeta(a_0, e_0) = \zeta(a_2/a_0, e_2) = \zeta(a_1/a_0, e_1)$, or, since $e_1 = \zeta(a_2/a_1, e_2)$,

$$\zeta(a, g) = \zeta(a/t, \zeta(t, g)). \tag{38}$$

The transformation of the charges between different scales (38) has a structure of the Abelian group $T_{t_1 t_2} = T_{t_1} \circ T_{t_2}$. It describes the evolution of the coupling constant with the change of measurement parameter – the resolution a.

In differential form the self-similarity equation (38) is most naturally expressed in the logarithmic coordinate $l = \log a$:

$$G(l + \lambda, g) = G(l, G(\lambda, g)),$$

or

$$\left(\frac{\partial}{\partial l} - \beta(g)\frac{\partial}{\partial g}\right) G(l, g) = 0, \qquad \text{where} \quad \beta(g) = \left.\frac{\partial G(\lambda, g)}{\partial \lambda}\right|_{\lambda=0}.$$

The latter is referred to as the Lie–Ovsyannikov equation.

In general case a quantity $f(l,g)$, which is not an invariant charge, is not nullified by the generator of RG transform $e^{-\lambda\hat R}f(l,g) \neq f(l,g)$, but the generator

$$\hat R = \left(\frac{\partial}{\partial l} - \beta(g)\frac{\partial}{\partial g}\right)$$

determines the evolution of physical quantities under the scale transformations.

In our approach we extend the space of functions $\phi(x) \in \mathrm{L}^2(\mathbb{R}^d)$ to the space of multiscale (wavelet) fields $\phi_a(b)$ for which both the position b and the scale a are considered as *independent* variables. The RG transform therefore becomes just an Abelian subgroup of the whole symmetry group of the action $S[\phi_a(b)]$ described above in this paper. The difference between our approach and the exact renormalization group approach (ERG) consists in the fact that ERG generating functional

$$Z_\Lambda[J] = \int \prod_{|p|\le\Lambda} \mathcal{D}\phi(p) e^{-S[\phi;\Lambda]-\int dp J\phi}$$

is just a sum of all fluctuations truncated by a cutoff momentum Λ in their Fourier representation, and therefore having certain symmetries related to the rescaling of Λ – the cutoff *parameter*. In our approach the logarithm of scale $l = \log a$ becomes an independent variable, so the technique of Lie symmetry analysis can be applied with respect to l on the same footing as with respect to the position x. In this way the vector field that generates arbitrary transformations of the action $S[\phi_a(b)]$ due to the transformation of dependent and independent variables takes the form

$$\hat V = \lambda\left(\frac{\partial}{\partial l} - \beta(g)\frac{\partial}{\partial g}\right) + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial \phi}.$$

We hope that such symmetries could be used to find solutions of the wavelet renormalization equation (35).

Acknowledgement

The author is thankful to the referee for useful comments and references.

[1] Benzi R., Paladin G., Parizi G., Vulpiani A., On multifractal nature of fully developed turbulence and chaotic systems, *J. Phys. A: Math. Gen.*, 1984, V.17, 3521–3531.

[2] Muzy J.F., Bacry E., Arneodo A., Wavelets and multifractal formalism for singular signals: application to turbulence data, *Phys. Rev. Lett.*, 1991, V.67, 3515–3518.

[3] Zimin V.D., Hierarchic turbulence model, *Izv. Atmos. Ocean. Phys.*, 1981, V.17, 941–946.

[4] Goupillaud P., Grossmann A., Morlet J., Cycle-octave and related transforms in seismic signal analysis, *Geoexploration*, 1984/85, V.23, 85–102.

[5] Carey A.L., Square-integrable representations of non-unimodular groups, *Bull. Austr. Math. Soc.*, 1976, V.15, 1–12.

[6] Duflo M., Moore C.C., On regular representations of nonunimodular locally compact group, *J. Func. Anal.*, 1976, V.21, 209–243.

[7] Frish U., Turbulence, Cambridge University Press, 1995.

[8] Altaisky M.V., Scale-dependent function in statistical hydrodynamics: a functional analysis point of view, *Eur. J. Phys. B*, 1999, V.8, 613–617, chao-dyn/9707011.

[9] Wilson K.G., Renormalization group and critical phenomena. II. Phase-space cell analysis of critical behavior, *Phys. Rev. B*, V.4, 1971, 3184–3205.

[10] Wilson K.G., Kogut J., Renormalization group and ϵ-expansion, *Phys. Rep. C*, 1974, V.12, 77–200.

[11] Wilson K.G., The renormalization group and critical phenomena, *Rev. Modern Phys.*, V.55, 1983, 583–600.

[12] Parisi G., Wu Y.-S., Perturbation theory without gauge fixing, *Scientica Sinica*, 1981, V.24, 483–496.

[13] Martin P.C., Sigia E.D., Rose H.A., Statistical dynamics of classical systems, *Phys. Rev. A*, 1973, V.8, 423–437.

[14] Zinn-Justin J., Renormalization and stochastic quantization, *Nuclear Phys. B*, 1986, V.275, 135–159.

[15] Namiki M., Stochastic quantization, Springer, 1992.

[16] Altaisky M.V., Wavelet based regularization for Euclidean field theory, in Proc. of the 24th Int. Coll. "Group Theoretical Methods in Physics" (July 2002, Paris), Editors J.-P. Gazeau, R. Kerner, J.-P. Antoine, S. Metens and J.-Y. Thibon, *IOP Conference Series*, 2003, V.173, 893–897, hep-th/0305167.

[17] Halpern M.B., Universal continuum regularization of quantum field theory, *Progr. Theor. Phys. Suppl.*, 1993, V.111, 163–184.

[18] Altaisky M., ϕ^4-field theory on a Lie group, in Proc. 4th Int. Conf. "Frontiers of Fundamental Physics", (December 2000, Hyderabad), Editors B.G. Sidharth and M.V. Altaisky, NY, Kluwer Academic, 2001, 121–128, hep-th/0007180.

[19] Battle G., Wavelets and renormalization, World Scientific, 1989.

[20] Federbush P.G., A mass zero cluster expansion, *Comm. Math. Phys.*, 1981, V.81, 327–340.

[21] Federbush P.G., A phase cell approach to Yang–Mills theory, *Comm. Math. Phys.*, 1986, V.107, 319–329.

[22] Gaite J., Stochastic formulation of the renormalization group: supersymmetric structure and the topology of the space of couplings, *J. Phys. A: Math. Gen.*, 2004, V.37, 10409–10419, hep-th/0404212.

[23] Zinn-Justin J., Quantum field theory and critical phenomena, Oxford University Press, 1989.

[24] Wegner F.G., Houghton A., Renormalization group equations for critical phenomena, *Phys. Rev. A*, 1973, V.8, 401–412.

[25] Kubishin Yu., Exact renormalization group approach to scalar and fermionic theories, *Intern. J. Modern Phys. B*, 1998, V.12, 1321–1341.

[26] Best C., Wavelet-induced renormalization group for Landau–Ginzburg model, *Nucl. Phys. B Proc. Suppl.*, 2000, V.83–84, 848–850, hep-lat/9909151.

[27] Kovalev V.F., Shirkov D.V., The Bogoliuobov renormalization group and solution symmetry in mathematical physics, *Phys. Rep.*, 2001, V.352, 219–249, hep-th/0001210.