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Symmetry Properties of Autonomous Integrating Factors

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Abstract. We study the symmetry properties of autonomous integrating factors from an algebraic point of view. The symmetries are delineated for the resulting integrals treated as equations and symmetries of the integrals treated as functions or configurational invariants. The succession of terms (pattern) is noted. The general pattern for the solution symmetries for equations in the simplest form of maximal order is given and the properties of the associated integrals resulting from this analysis are given.

Key words: autonomous integrating factors; maximal symmetry

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1 Introduction

It is well-known that, when a symmetry is used to determine a first integral for a differential equation, the symmetry provides an integrating factor for the equation and remains as a symmetry of the first integral. For first-order ordinary differential equations the direct determination of the integrating factor is known [1] and algorithms for finding integrating factors for equations of higher order have been developed. In 1999 Cheb-Terrab and Roche [2] presented a systematic algorithm for the construction of integrating factors for second-order ordinary differential equations and claimed that their algorithm gave integrating factors for equations which did not possess Lie point symmetries. In 2002 Leach and Bouquet [3] showed that for all equations except one of which Cheb-Terrab and Roche [2] had found integrating factors had symmetries which were not necessarily point symmetries but generalised or nonlocal. In the same year, 2002, Abraham-Shrauner [4] also wrote a paper to demonstrate the reduction of order of nonlinear ordinary differential equations by a combination of first integrals and Lie group symmetries. The latter and former motivated us hereby to investigate the underlying properties of autonomous integrating factors and the associated integrals treated as equations and as functions. Observations are made and inferred in general for any nth-order ordinary differential equation of maximal symmetry. These will also be extended to include other types of equations in a separate contribution.

Program `LIE` [5] is used to compute the symmetries for the different cases considered. The knowledge of the symmetries of first integrals of the equation does give rise to some interesting properties of the equation itself. For example, the Ermakov–Pinney equation [6, 7] which in its simplest form is

$$w'' + \frac{K}{w^3} = 0, \tag{1}$$

where K is a constant. In theoretical discussions the sign of the constant K is immaterial and in fact it is often rescaled to unity.

The general form of (1), *videlicet*

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3}$$

occurs in the study of the time-dependent linear oscillator, be it the classical or the quantal problem, as the differential equation which determines the time-dependent rescaling of the space variable and the definition of 'new time'. Some of the references for this are [8, 9].

Another origin of (1) — of particular interest in this work — is as an integral of the third-order equation of maximal symmetry which in its elemental form is $y''' = 0$.

2 Equations of maximal symmetry

Definition 1. We define a first integral I for an equation of maximal symmetry, $E = y^{(n)} = 0$, as $I = f\big(y, y', y', \dots, y^{(n-1)}\big)$, where

$$\left.\frac{\mathrm{d}I}{\mathrm{d}x}\right|_{E=0} = 0 \qquad \Longleftrightarrow \qquad \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{E=0} = 0.$$

This means that, if $g\big(x, y, y', y'', \dots, y^{(n-1)}\big)$ is an integrating factor, then

$$\left.\frac{\mathrm{d}I}{\mathrm{d}x}\right|_{E=0} = gE\left(x, y, y', \dots, y^{(n)}\right)\Big|_{E=0} = 0.$$

We start by considering the well-known third-order ordinary differential equation of maximal symmetry

$$y''' = 0 \tag{2}$$

which has seven Lie point symmetries. These are

$$\begin{aligned} &G_1 = \partial_y, \qquad G_2 = x\partial_y, \qquad G_3 = x^2\partial_y, \qquad G_4 = y\partial_y, \\ &G_5 = \partial_x, \qquad G_6 = x\partial_x + y\partial_y, \qquad G_7 = x^2\partial_x + 2xy\partial_y. \end{aligned} \tag{3}$$

The algebra is $\{A_1 \oplus_s sl(2,\mathbb{R})\} \oplus_s 3A_1$. The autonomous integrating factors for (2) are y'' and y. We list the symmetries and algebra when each of the integrals is treated as an equation and as a function.

When we multiply $y''' = 0$ by the integrating factor y'', we obtain $y''y''' = 0$. Integration of this expression gives $\frac{1}{2}\left(y''\right)^2 = k$, where k is a constant of integration. This gives rise to three cases which we list as follows:

$$
\begin{array}{ll}
\qquad y''=0 & \qquad\qquad y''=k \\
G_1=\partial_y, & G_1=\partial_y, \\
G_2=x\partial_y, & G_2=x\partial_y, \\
G_3=y\partial_y, & G_3=\left(\frac{1}{2}x^2k-y\right)\partial_y, \\
G_4=\partial_x, & G_4=\partial_x+2xk\partial_y, \\
G_5=x\partial_x, & G_5=x\partial_x+x^2k\partial_y, \\
G_6=x^2\partial_x+xy\partial_y, & G_6=x^2\partial_x+\left(xy+\frac{1}{2}x^3k\right)\partial_y, \\
G_7=y\partial_x, & G_7=\left(y-\frac{3}{2}x^2k\right)\partial_x-x^3k^2\partial_y, \\
G_8=xy\partial_x+y^2\partial_y, & G_8=\left(xy-\frac{1}{2}x^3k\right)\partial_x+\left(y^2-\frac{1}{4}x^4k^2\right)\partial_y,
\end{array}
$$

and, when $y''=k$ is treated as a function, we have

$$G_1=\partial_y,\qquad G_2=x\partial_y,\qquad G_3=\partial_x,\qquad G_4=x\partial_x+2y\partial_y.$$

Remark 1. When $y''=k$ is treated as an equation, we have two cases, that is, $y''=0$ and $y''=k$ for which the algebra is $sl(3,\mathbb{R}):2A_1\oplus_s\{sl(2,\mathbb{R})\oplus A_1\}\oplus 2A_1$ [11, 12, 13]. If $y''=k$ is treated as a function, the algebra is $A^1_{4,9}:A_2\oplus_s 2A_1$ [11, 12, 13, 14].

If y is used an the integrating factor, we obtain $yy'''=0$. Integration of this equation gives $yy''-\frac{1}{2}y'^2=k$ which can be written as $\left(y^{1/2}\right)''=k/\left(y^{1/2}\right)^3$ and is the simplest form of the Ermakov–Pinney equation [6, 7]. As before we write down the point symmetries corresponding to the three cases of the differential equation $u''=k/u^3$, where $u=y^{1/2}$. Program `LIE` [5] gives the following:

$$
\begin{array}{lll}
\qquad u''=0 & \qquad u''=k/u^3 & \qquad u''=k/u^3 \\
G_1=\partial_u, & G_1=\partial_x, & G_1=\partial_x, \\
G_2=x\partial_u, & G_2=2x\partial_x+u\partial_u, & G_2=2x\partial_x+u\partial_u, \\
G_3=u\partial_u, & G_3=x^2\partial_x+xu\partial_u, & G_3=x^2\partial_x+xu\partial_u, \\
G_4=\partial_x, & & \\
G_5=x\partial_x, & & \\
G_6=x^2\partial_x+xu\partial_u, & & \\
G_7=u\partial_x, & & \\
G_8=xu\partial_x+u^2\partial_u. & &
\end{array}
$$

The transformation of $yy''-\frac{1}{2}y'^2=k$ to $u''=k/u^3$ does not make a difference in terms of the symmetries as we just have a point transformation in this case. The other obvious integrating factors for (2) are 1, x and $\frac{1}{2}x^2$ which give

$$
\begin{aligned}
1\cdot y'''=0 &\quad\longrightarrow\quad I_3=y'', \\
x\cdot y'''=0 &\quad\longrightarrow\quad I_2=xy''-y', \\
\frac{1}{2}x^2\cdot y'''=0 &\quad\longrightarrow\quad I_1=\frac{1}{2}x^2y''-xy'+y.
\end{aligned}
$$

(Note that the numbering of the fundamental first integrals follows the convention given in Flessas *et al* [15, 16].)

- The integration of equation (2), which is a feature of the calculation of the symmetries of all linear ordinary differential equations of maximal symmetry [10], by means of an integrating factor gives a variety of results depending upon the integrating factor used.
- The characteristic feature of the Ermakov–Pinney equation is that it possesses the three-element algebra of Lie point symmetries, $sl(2,\mathbb{R})$, which in itself is characteristic of all scalar ordinary differential equations of maximal symmetry.

The fourth-order ordinary differential equation $y^{\rm iv}=0$ has autonomous integrating factors y' and y'''. If we use y' as an integrating factor in the original equation and integrate, we obtain

$$y'y''' - \frac{1}{2}(y'')^2 = k. \tag{4}$$

Equation (4) is a generalised Kummer–Schwartz equation for $k=0$ and for $k\neq 0$ a variation on the Ermakov–Pinney equation as it can be written in the form

$$\left((y')^{1/2}\right)'' = \left(k\Big/\left((y')^{3/2}\right)\right).$$

The three cases for the integral in (4) treated as an equation and as a function give the following results:

$$\begin{array}{lll}
y'y''' - \frac{1}{2}(y'')^2 = 0 & y'y''' - \frac{1}{2}(y'')^2 = k & y'y''' - \frac{1}{2}(y'')^2 = k \\
G_1 = \partial_x, & G_1 = \partial_x, & G_1 = \partial_x, \\
G_2 = x\partial_x, & G_2 = \partial_y, & G_2 = \partial_y, \\
G_3 = y\partial_y, & G_3 = x\partial_x + 2y\partial_y, & G_3 = x\partial_x + 2y\partial_y, \\
G_4 = \partial_y. & &
\end{array}$$

The use of y''' as an integrating factor gives $y'''=k$. If $k=0$, then we just have seven point symmetries as those of equation (2). The two remaining cases give

$$\begin{array}{ll}
\qquad y''' = k & \qquad y''' = k \\
G_1 = \partial_x, & G_1 = \partial_x, \\
G_2 = x\partial_y, & G_2 = \partial_y, \\
G_3 = \frac{1}{2}x^2\partial_y, & G_3 = x\partial_y, \\
G_4 = \partial_y, & G_4 = x^2\partial_y, \\
G_5 = x\partial_x + \frac{1}{2}x^3k\partial_y & G_5 = x\partial_x + 3y\partial_y, \\
G_6 = \left(y - \frac{1}{6}x^3k\right)\partial_y, & \\
G_7 = x^2\partial_x + \left(2xy + \frac{1}{6}x^4k\right)\partial_y. &
\end{array}$$

Consider the fifth-order equation of maximal symmetry given by

$$y^{\rm v} = 0 \tag{5}$$

with autonomous integrating factors y, y'' and $y^{\rm iv}$. If we multiply (5) by the first integrating factor and integrate, we obtain the integral

$$yy^{\rm iv} - y'y''' + \frac{1}{2}(y'')^2 = k. \tag{6}$$

We consider the three cases for (6) treated as an equation with $k = 0$, $k \neq 0$ and as a function.

$$\begin{array}{lll} yy^{\mathrm{iv}} - y'y''' + \frac{1}{2}(y'')^2 = 0 & yy^{\mathrm{iv}} - y'y''' + \frac{1}{2}(y'')^2 = k & yy^{\mathrm{iv}} - y'y''' + \frac{1}{2}(y'')^2 = k \\ G_1 = \partial_x, & G_1 = \partial_x, & G_1 = \partial_x, \\ G_2 = x\partial_x, & G_2 = x\partial_x + 2y\partial_y, & G_2 = x\partial_x + 2y\partial_y, \\ G_3 = y\partial_y, & G_3 = x^2\partial_x + 4xy\partial_y, & G_3 = x^2\partial_x + 4xy\partial_y, \\ G_4 = x^2\partial_x + 4xy\partial_y. & & \end{array}$$

Remark 2. For easier closure of the algebra in the first case $x\partial x$ can be written as $x\partial_x + 2y\partial_y$.

- We also observe that there is no difference when the integral is treated as a function and as an equation. It is important to note that, if y is an integrating factor of $y^{(n)} = 0$, then the integral obtained using this integrating factor always has the $sl(2, \mathbb{R})$ subalgebra.
- We further observe that for the peculiar value of the constant, that is, $k = 0$, there is the splitting of the self-similarity symmetry into two homogeneity symmetries.

The integrating factor y'' with (5) gives the following results

$$\begin{array}{lll} y''y^{\mathrm{iv}} - \frac{1}{2}(y''')^2 = 0 & y''y^{\mathrm{iv}} - \frac{1}{2}(y''')^2 = k & y''y^{\mathrm{iv}} - \frac{1}{2}(y''')^2 = k \\ G_1 = \partial_x, & G_1 = \partial_x, & G_1 = \partial_x, \\ G_2 = x\partial_x, & G_2 = \partial_y, & G_2 = \partial_y, \\ G_3 = \partial_y, & G_3 = x\partial_y, & G_3 = x\partial_y, \\ G_4 = x\partial_y, & G_4 = x\partial_x + 3y\partial_y, & G_4 = x\partial_x + 3y\partial_y, \\ G_5 = y\partial_y. & & \end{array}$$

If we use y^{iv} as the integrating factor of (5) and integrate, we obtain

$$y^{\mathrm{iv}} = k.$$

We delineate the three cases below:

$$\begin{array}{lll} y^{\mathrm{iv}} = 0 & y^{\mathrm{iv}} = k & y^{\mathrm{iv}} = k \\ G_1 = \partial_y, & G_1 = \partial_y & G_1 = \partial_y, \\ G_2 = x\partial_y, & G_2 = x\partial_y & G_2 = x\partial_y, \\ G_3 = x^2\partial_y, & G_3 = \frac{1}{2}x^2\partial_y & G_3 = x^2\partial_y, \\ G_4 = x^3\partial_y, & G_4 = \frac{1}{6}x^3\partial_y & G_4 = x^3\partial_y, \\ G_5 = y\partial_y, & G_5 = \partial_x & G_5 = \partial_x, \\ G_6 = \partial_x, & G_6 = 6x\partial_x + x^4k\partial_y & G_6 = x\partial_x + 4y\partial_y, \\ G_7 = x\partial_x, & G_7 = \left(24y - x^3k\right)\partial_y, & \\ G_8 = x^2\partial_x + 3xy\partial_y, & G_8 = 24x^2\partial_x + \left(72xy + x^5k\right)\partial_y. & \end{array}$$

The differential equation

$$y^{\mathrm{vi}} = 0 \tag{7}$$

has integrating factors y', y''' and y^{v}. If we use y' as the integrating factor, we obtain

$$y'y^{\mathrm{v}} - y''y^{\mathrm{iv}} + \frac{1}{2}(y''')^2 = k$$

which leads to the cases below.

$$\begin{array}{lll}
y'y^{\rm v}-y''y^{\rm iv}+\frac{1}{2}(y''')^2=0 & y'y^{\rm v}-y''y^{\rm iv}+\frac{1}{2}(y''')^2=k & y'y^{\rm v}-y''y^{\rm iv}+\frac{1}{2}(y''')^2=k\\
G_1=\partial_y, & G_1=\partial_y, & G_1=\partial_y,\\
G_2=y\partial_y, & G_2=\partial_x, & G_2=\partial_x,\\
G_3=\partial_x, & G_3=x\partial_x+3y\partial_y, & G_3=x\partial_x+3y\partial_y,\\
G_4=x\partial_x. & &
\end{array}$$

The use of y''' as the integrating factor for (7) leads to

$$y'''y^{\rm v}-\frac{1}{2}\left(y^{\rm iv}\right)^2=k.$$

The three cases give the following results:

$$\begin{array}{lll}
y'''y^{\rm v}-\frac{1}{2}\left(y^{\rm iv}\right)^2=0 & y'''y^{\rm v}-\frac{1}{2}\left(y^{\rm iv}\right)^2=k & y'''y^{\rm v}-\frac{1}{2}\left(y^{\rm iv}\right)^2=k\\
G_1=\partial_y, & G_1=\partial_y, & G_1=\partial_y,\\
G_2=\partial_x, & G_2=\partial_x, & G_2=\partial_x,\\
G_3=x\partial_y, & G_3=x\partial_y, & G_3=x\partial_y,\\
G_4=x^2\partial_y, & G_4=x^2\partial_y, & G_4=x^2\partial_y,\\
G_5=y\partial_y, & G_5=x\partial_x+4y\partial_y, & G_5=x\partial_x+4y\partial_y,\\
G_6=x\partial_x. & &
\end{array}$$

If we use $y^{\rm v}$ as an integrating factor, we obtain

$$y^{\rm v}=k.$$

We also have the three cases as mentioned above to obtain

$$\begin{array}{lll}
y^{\rm v}=0 & y^{\rm v}=k & y^{\rm v}=k\\
G_1=\partial_y, & G_1=\partial_y, & G_1=\partial_y,\\
G_2=y\partial_y, & G_2=\partial_x, & G_2=\partial_x,\\
G_3=\partial_x, & G_3=x\partial_y, & G_3=x\partial_y,\\
G_4=x\partial_y, & G_4=x^2\partial_y, & G_4=x^2\partial_y,\\
G_5=x^2\partial_y, & G_5=x^3\partial_y, & G_5=x^3\partial_y,\\
G_6=x^3\partial_y, & G_6=x^4\partial_y, & G_6=x^4\partial_y,\\
G_7=x^4\partial_y, & G_7=x\partial_x+\frac{1}{24}kx^5\partial_y, & G_7=x\partial_x+5y\partial_y,\\
G_8=x\partial_x, & G_8=\left(y-\frac{1}{120}kx^5\right)\partial_y, & \\
G_9=x^2\partial_x+4xy\partial_y, & G_9=x^2\partial_x+\left(4xy+\frac{1}{120}kx^6\right)\partial_y. &
\end{array}$$

For the differential equation $y^{\rm vii}=0$ we have the integrating factors y, y'', $y^{\rm iv}$ and $y^{\rm vi}$. The integrals corresponding to these integrating factors respectively are

$$\begin{aligned}
&yy^{\rm vi}-y'y^{\rm v}+y''y^{\rm iv}-\frac{1}{2}(y''')^2=k,\\
&y''y^{\rm vi}-y'''y^{v}+\frac{1}{2}\left(y^{\rm iv}\right)^2=k,\\
&y^{\rm iv}y^{\rm vi}-\frac{1}{2}\left(y^{\rm v}\right)^2=k,\\
&y^{\rm vi}=k.
\end{aligned}\tag{8}$$

If y is used as the integrating factor, we have the integral $yy^{\rm vi} - y'y^{\rm v} + y''y^{\rm iv} - \frac{1}{2}(y''')^2 = k$ which is treated as an equation for $k = 0$, $k \neq 0$ and as a function. This gives the following results:

$$\begin{array}{lll}
G_1 = \partial_x, & G_1 = \partial_x, & G_1 = \partial_y, \\
G_2 = x\partial_x, & G_2 = x\partial_x + 3y\partial_y, & G_2 = x\partial_x + 3y\partial_y, \\
G_3 = y\partial_y, & G_3 = x^2\partial_x + 6xy\partial_y, & G_3 = x^2\partial_x + 6xy\partial_y, \\
G_4 = x^2\partial_x + 6xy\partial_y. & &
\end{array}$$

The integral corresponding to the integrating factor y'' leads to the following cases:

$$\begin{array}{lll}
y''y^{\rm vi} - y'''y^{\rm v} + \frac{1}{2}\left(y^{\rm iv}\right)^2 = 0 & y''y^{\rm vi} - y'''y^{\rm v} + \frac{1}{2}\left(y^{\rm iv}\right)^2 = k & y''y^{\rm vi} - y'''y^{\rm v} + \frac{1}{2}\left(y^{\rm iv}\right)^2 = k \\
G_1 = \partial_y, & G_1 = \partial_y, & G_1 = \partial_y, \\
G_2 = \partial_x, & G_2 = \partial_x, & G_2 = \partial_x, \\
G_3 = x\partial_y, & G_3 = x\partial_y, & G_3 = x\partial_y, \\
G_4 = x\partial_x, & G_4 = x\partial_x + 4y\partial_y, & G_4 = x\partial_x + 4y\partial_y, \\
G_5 = y\partial_y. & &
\end{array}$$

For the integrating factor $y^{\rm iv}$ we have the cases:

$$\begin{array}{lll}
y^{\rm iv}y^{\rm vi} - \frac{1}{2}\left(y^{\rm v}\right)^2 = 0 & y^{\rm iv}y^{\rm vi} - \frac{1}{2}\left(y^{\rm v}\right)^2 = k & y^{\rm iv}y^{\rm vi} - \frac{1}{2}\left(y^{\rm v}\right)^2 = k \\
G_1 = \partial_y, & G_1 = \partial_y, & G_1 = \partial_y, \\
G_2 = \partial_x, & G_2 = \partial_x, & G_2 = \partial_x, \\
G_3 = y\partial_y, & G_3 = x\partial_y, & G_3 = x\partial_y, \\
G_4 = x\partial_y, & G_4 = x^2\partial_y, & G_4 = x^2\partial_y, \\
G_5 = x^2\partial_y, & G_5 = x^3\partial_y, & G_5 = x^3\partial_y, \\
G_6 = x^3\partial_y, & G_6 = x\partial_x + 5y\partial_y, & G_6 = x\partial_x + 5y\partial_y, \\
G_7 = x\partial_x. & &
\end{array}$$

The last of the four integrating factors $y^{\rm vi}$ leads to $y^{\rm vi} = k$. We have for the three cases the following results:

$$\begin{array}{lll}
y^{\rm vi} = 0 & y^{\rm vi} = k & y^{\rm vi} = k \\
G_1 = \partial_y, & G_1 = \partial_y, & G_1 = \partial_y, \\
G_2 = \partial_x, & G_2 = \partial_x, & G_2 = \partial_x, \\
G_3 = y\partial_y, & G_3 = x\partial_y, & G_3 = x\partial_y, \\
G_4 = x\partial_y, & G_4 = \frac{1}{2}x^2\partial_y, & G_4 = x^2\partial_y, \\
G_5 = x^2\partial_y, & G_5 = \frac{1}{6}x^3\partial_y, & G_5 = x^3\partial_y, \\
G_6 = x^3\partial_y, & G_6 = \frac{1}{24}x^4\partial_y, & G_6 = x^4\partial_y, \\
G_7 = x^4\partial_y, & G_7 = \frac{1}{120}x^5\partial_y, & G_7 = x^5\partial_y, \\
G_8 = x^5\partial_y, & G_8 = x\partial_x + \frac{1}{120}kx^6\partial_y, & G_8 = x\partial_x + 6y\partial_y, \\
G_9 = x\partial_x, & G_9 = \left(y - \frac{1}{720}kx^6\right)\partial_y, & \\
G_{10} = x^2 + 5xy\partial_y, & G_{10} = x^2\partial_x + \left(5xy + \frac{1}{3600}k5x^7\right)\partial_y. &
\end{array}$$

The differential equation

$$y^{\mathrm{viii}} = 0 \tag{9}$$

has integrating factors y', y''', y^{v} and y^{vii}. If we use y' in (9) and integrate the resulting equation, we obtain the integral

$$y'y^{\mathrm{vii}} - y''y^{\mathrm{vi}} + y'''y^{\mathrm{v}} - \frac{1}{2}\left(y^{\mathrm{iv}}\right)^2 = k. \tag{10}$$

The three cases of the integral in (10) being treated as an equation with $k = 0$ and $k \neq 0$ and as a function are given respectively below:

$$\begin{array}{lll}
G_1 = \partial_y & G_1 = \partial_y & G_1 = \partial_y, \\
G_2 = y\partial_y, & G_2 = \partial_x, & G_2 = \partial_x, \\
G_3 = \partial_x, & G_3 = x\partial_x + 4y\partial_y, & G_3 = x\partial_x + 4y\partial_y, \\
G_4 = x\partial_x. & &
\end{array}$$

If y''' is used as an integrating factor, we obtain

$$y'''y^{\mathrm{vii}} - y^{\mathrm{iv}}y^{\mathrm{vi}} + \frac{1}{2}\left(y^{\mathrm{v}}\right)^2 = k$$

with the following respective cases:

$$\begin{array}{lll}
G_1 = \partial_y, & G_1 = \partial_y, & G_1 = \partial_y, \\
G_2 = y\partial_y, & G_2 = \partial_x, & G_2 = \partial_x, \\
G_3 = \partial_x, & G_3 = x\partial_y, & G_3 = x\partial_y, \\
G_4 = x\partial_y, & G_4 = x^2\partial_y, & G_4 = x^2\partial_y, \\
G_5 = x^2\partial_y. & &
\end{array}$$

The use of y^{v} as an integrating factor gives

$$y^{v}y^{\mathrm{vii}} - \frac{1}{2}\left(y^{\mathrm{vi}}\right)^2 = k. \tag{11}$$

Equation (11) is of the Ermakov–Pinney type. The three cases can be delineated as follows:

$$\begin{array}{lll}
y^{\mathrm{v}}y^{\mathrm{vii}} - \frac{1}{2}\left(y^{\mathrm{vi}}\right)^2 = 0 & y^{\mathrm{v}}y^{\mathrm{vii}} - \frac{1}{2}\left(y^{\mathrm{vi}}\right)^2 = k & y^{\mathrm{v}}y^{\mathrm{vii}} - \frac{1}{2}\left(y^{\mathrm{vi}}\right)^2 = k \\
G_1 = \partial_y, & G_1 = \partial_y, & G_1 = \partial_y, \\
G_2 = y\partial_y, & G_2 = \partial_x, & G_2 = \partial_x, \\
G_3 = \partial_x, & G_3 = x\partial_y, & G_3 = x\partial_y, \\
G_4 = x\partial_y, & G_4 = x^2\partial_y, & G_4 = x^2\partial_y, \\
G_5 = x^2\partial_y, & G_5 = x^3\partial_y, & G_5 = x^3\partial_y, \\
G_6 = x^3\partial_y, & G_6 = x^4\partial_y, & G_6 = x^4\partial_y, \\
G_7 = x^4\partial_y, & G_7 = x\partial_x + 6y\partial_y, & G_7 = x\partial_x + 6y\partial_y, \\
G_8 = x\partial_x. & &
\end{array}$$

If y^{vii} is used as an integrating factor in (9), we obtain $y^{\text{vii}} = k$ with the following symmetries for each of the three cases:

$y^{\text{vii}} = 0$	$y^{\text{vii}} = k$	$y^{\text{vii}} = k$
$G_1 = \partial_y,$	$G_1 = \partial_y,$	$G_1 = \partial_y,$
$G_2 = y\partial_y,$	$G_2 = \partial_x,$	$G_2 = \partial_x,$
$G_3 = \partial_x,$	$G_3 = x\partial_y,$	$G_3 = x\partial_y,$
$G_4 = x\partial_y,$	$G_4 = \frac{1}{2}x^2\partial_y,$	$G_4 = x^2\partial_y,$
$G_5 = x^2\partial_y,$	$G_5 = \frac{1}{6}x^3\partial_y,$	$G_5 = x^3\partial_y,$
$G_6 = x^3\partial_y,$	$G_6 = \frac{1}{24}x^4\partial_y,$	$G_6 = x^4\partial_y,$
$G_7 = x^4\partial_y,$	$G_7 = \frac{1}{120}x^5\partial_y,$	$G_7 = x^5\partial_y,$
$G_8 = x^5\partial_y,$	$G_8 = \frac{1}{720}x^6\partial_y,$	$G_8 = x^6\partial_y,$
$G_9 = x^6\partial_y,$	$G_9 = x\partial_x + \frac{1}{720}kx^7\partial_y,$	$G_9 = x\partial_x + 7y\partial_y,$
$G_{10} = x\partial_x,$	$G_{10} = \left(y - \frac{1}{5040}kx^7\right)\partial_y,$	
$G_{11} = x^2\partial_x + 6xy\partial_y,$	$G_{11} = x^2\partial_x + \left(6xy + \frac{x^8}{5040}k\right)\partial_y.$	

3 Relationship between fundamental integrals and integrals obtained from integrating factors

Consider the example of the third-order ordinary differential equation $y''' = 0$ with the three fundamental integrals together with the appropriate associated point symmetries from the subalgebra $sl(2,\mathbb{R})$:

$$\begin{aligned}
&G_7 = x^2\partial_x + 2xy\partial_y, && I_1 = \frac{1}{2}x^2y'' - xy' + y,\\
&G_6 = x\partial_x + y\partial_y, && I_2 = xy'' - y',\\
&G_5 = \partial_x, && I_3 = y''.
\end{aligned}$$

The numbering of the symmetries follows that of the listing of Lie point symmetries in (3) and the ordering of the integrals is in terms of their solution symmetries. Then the autonomous integral associated with the integrating factor y comes from the combination

$$J = I_1I_3 - \frac{1}{2}I_2^2 = yy'' - \frac{1}{2}y'^2.$$

Proposition 1. *All the integrals obtained using y as an integrating factor always have the $sl(2,\mathbb{R})$ subalgebra whereas the fundamental integrals only have one of the $sl(2,\mathbb{R})$ elements.*

Proof. To prove the first proposition we consider the $sl(2,\mathbb{R})$ subalgebra $\Lambda_1 = \partial_x$, $\Lambda_2 = x\partial_x + y\partial_y$ and $\Lambda_3 = x^2\partial_x + 2xy\partial_y$ and the fundamental integrals I_1, I_2 and I_3 respectively. Then we have the following:

$$\begin{aligned}
&\Lambda_1I_1 = I_2, && \Lambda_2I_1 = I_1, && \Lambda_3I_1 = 0,\\
&\Lambda_1I_2 = I_3, && \Lambda_2I_2 = 0, && \Lambda_3I_2 = -2I_1,\\
&\Lambda_1I_3 = 0, && \Lambda_2I_3 = -I_3, && \Lambda_3I_3 = -2I_2.
\end{aligned}$$

We also observe that $\Lambda_i J = 0$ for $i = 1, 2, 3$. In fact it is easy to show that $\Lambda_i J = \epsilon_{ijk} I_j I_k$. This is shown below as follows:

$$\Lambda_1 J = I_2 I_3 - I_2 I_3 = 0,$$
$$\Lambda_2 J = I_1 I_3 - I_1 I_3 = 0,$$
$$\Lambda_3 J = -2I_2 I_1 + 2I_1 I_2 = 0. \qquad \blacksquare$$

In general we have

$$I_{ni} = \sum_{j=0}^{n-i-1} \frac{(-1)^j x^{(n-j-i-1)}}{(n-j-i-1)!} y^{(n-j-1)}, \qquad i = 0, 1, \dots, n-1, \tag{12}$$

so that for $n = 3$, $I_{30} = I_1$, $I_{31} = I_2$ and $I_{32} = I_3$. The symmetries Λ_1, Λ_2 and Λ_3 operating on the fundamental integrals then yield in general

$$\Lambda_1 I_{ni} = I_{n,i+1},$$
$$\Lambda_2 I_{ni} = (1-i) I_{ni},$$
$$\Lambda_3 I_{ni} = -(n+i-3)(n-i) I_{n,i-1},$$
$$I_{nn} = 0.$$

If we take for example $\Lambda_3 I_{3i} = -i(3-i) I_{3,i-1}$ with $n = 3$ and $i = 0, 1, 2$, we obtain

$$\Lambda_3 I_{30} = 0,$$
$$\Lambda_3 I_{31} = -2I_{30},$$
$$\Lambda_3 I_{32} = -2I_{31},$$

where as above $I_{30} = I_1$, $I_{31} = I_2$ and $I_{32} = I_3$.

Proposition 2 ([17]). *If we take the equation of maximal symmetry $y^{(n)} = 0$, the $sl(2, \mathbb{R})$ subalgebra maps back to itself and is preserved.*

Proposition 3. *For the fifth-order equation $y^{\mathrm{v}} = 0$ the autonomous integral emanating from the integrating factor y can be obtained from $J = I_0 I_4 - I_1 I_3 + \frac{1}{2} I_2^2$, where*

$$I_0 = \frac{1}{24} x^4 y^{\mathrm{iv}} - \frac{1}{6} x^3 y''' + \frac{1}{2} x^2 y'' - xy' + y,$$
$$I_1 = \frac{1}{6} x^3 y^{\mathrm{iv}} - \frac{1}{2} x^2 y''' + xy'' - y',$$
$$I_2 = \frac{1}{2} x^2 y^{\mathrm{iv}} - xy''' + y'',$$
$$I_3 = xy^{\mathrm{iv}} - y''',$$
$$I_4 = y^{\mathrm{iv}}.$$

Proposition 4. *The fourth-order equation also has an autonomous integral J defined as $J = I_1 I_3 - \frac{1}{2} I_2^2$, where*

$$I_1 = \frac{1}{2} x^2 y''' - xy'' + y',$$
$$I_2 = xy''' - y'',$$
$$I_3 = y'''.$$

Proposition 5. *It can be shown that the differential equation $y^{\mathrm{vi}} = 0$ also has the autonomous integral J which is defined as $J = I_0 I_6 - I_1 I_5 + I_2 I_4 - \frac{1}{2} I_3^2$ with the I_i $(i = 0, 6)$ being redefined appropriately.*

4 Conclusion

If $y^{(n)} = f\big(x, y, y', \ldots, y^{n-1}\big)$ is an nth-order ordinary differential equation and $g\big(x, y, y', \ldots, y^{n-1)}\big) = k$ is an integral, the integral obtained by multiplying the equation by the integrating factor and integrating once possesses certain symmetries when treated as a function, an equation for the general constant and a configurational invariant (k=0). It is important to note that, if y is an integrating factor of $y^{(n)} = 0$, then the integral obtained using this integrating factor always has the $sl(2, \mathbb{R})$ subalgebra whereas the fundamental integrals only have one of the $sl(2, \mathbb{R})$ elements. We further observe that for the peculiar value of the constant, $k = 0$, there is the splitting of the self-similarity symmetry into two homogeneity symmetries. The third-order ordinary differential equation is actually special and leads to the Ermakov–Pinney type equation. The fourth-order ordinary differential equation $y^{\rm iv} = 0$ has y' as one of its autonomous integrating factors which leads together with the the original equation upon integration to the generalised Kummer–Schwartz equation. An extension to other types of equations will be completed in a separate contribution. The question of what Lie point symmetries of an ordinary differential equation are also shared by all its first integrals will form the basis for the next contribution.

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