The Differential Form Method for Finding Symmetries

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Abstract. This article reviews the use of differential forms and Lie derivatives to find symmetries of differential equations, as originally presented in Harrison and Estabrook 1971 [1]. An outline of the method is given, followed by examples and references to recent papers using the method.

Key words: symmetries; differential equations; differential forms

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1 Introduction

In 1969–70, Frank Estabrook and the present author found a method for finding symmetries of differential equations using differential forms and Cartan's formulation of differential equations [1]. (This will be called Paper I.) It was not something we were searching for; rather, we were simply trying to understand how the symmetries of Maxwell's equations could be found from the differential form version of those equations. Once we realized that the key to symmetries was the use of the Lie derivative, it became clear how to apply this to all differential equations. An outline of the method, with examples, will be given here. A few computer programs that use it will be mentioned, along with a number of published papers on symmetries which have used it.

The reader may wonder at the order of the names of the authors on our original paper. Since we had done roughly equal amounts of work on the research, the order was determined by the flip of a coin.

The method proceeds as follows. We consider a set of partial differential equations, defined on a differentiable manifold M of n independent variables and m dependent variables. (Ordinary differential equations constitute a special case; we will mention those later.) We define the partial derivatives of the dependent variables as new variables (prolongation) in sufficient number to write the equations as a set of first order equations, thus extending the manifold to a manifold M'. Then we can formulate those as differential forms. We speak of the set of forms, representing the equations, as an ideal I. It is to be closed.

We can recapture the original set of equations by two operations: we specialize to a submanifold by letting the dependent variables be functions of the independent variables ("sectioning") and then we set the pullback of the forms to zero ("annulling"). The resultant equations are the original set of first order partial differential equations, and the submanifold is a solution submanifold. (All calculations are local and we do not use any manifold structure except differentiability.)

Lie derivatives of geometrical objects, like tensors, are associated with symmetries of those objects. The Lie derivative of a geometrical object carries it along a path, determined by a vector $\mathbf{v}$, in its manifold. If the Lie derivative vanishes, then the vector $\mathbf{v}$ represents the direction of an infinitesimal symmetry transformation in the manifold. A differential form is a type of tensor (totally antisymmetric on the indices of the components), so it has a Lie derivative. We may construct the Lie derivative (symbolized by $\pounds$) of the forms in the ideal I. Setting the Lie derivative of these forms equal to zero should therefore represent symmetries — except for one thing. When we make an infinitesimal transformation away from the original variables, we require that the new form of the differential equations should vanish — but the old form must also vanish. Thus we want the original forms in I to vanish, but also their Lie derivative must vanish when that happens. In other words, we require that the Lie derivatives of the forms in I to be linear combinations of those forms themselves — and when they vanish, then the Lie derivatives also vanish. We can express this by writing $\pounds_{\mathbf{v}} I = 0 \pmod{I}$, or

$$\pounds_{\mathbf{v}} I \subset I. \tag{1}$$

This is satisfied by letting the Lie derivative of each differential form in I be a linear combination of the forms in I.

Equation (1) will contain a number of Lagrange multipliers, the coefficients of the forms in their linear combinations. Those are to be eliminated. Once they are eliminated, there remains a set of linear homogeneous first order equations for the components of $\mathbf{v}$ in M', which are the symmetry generators. The equations are simply the determining equations for the symmetries of the original set of differential equations, considered as point transformations in M'.

We note, in this set, that the derivatives of the components of $\mathbf{v}$ (the generators) will be taken with respect to both dependent variables (including prolonged ones) and independent variables. One often assumes that the generators for the independent variables (often denoted by ξ and η) are functions only of the independent variables (there are exceptions.) The determining equations will usually show that feature promptly.

Some examples of familiar equations will be presented to show how the method works. Some of this material was presented by the author at the second Kiev symmetry conference in 1997 and can be found in its Proceedings [2]. This will be denoted as paper II.

In paper I, we adopted the term "isovector" for the vector $\mathbf{v}$, describing a symmetry transformation in M', even though the term had been used elsewhere in the physics literature. We did not think there would be any confusion. A number of authors thus refer to this method as the "isovector" method.

2 Lie derivatives of differential forms

First we note some simple features of Lie derivatives of differential forms. (See paper I.)

(1) Lie differentiation preserves the rank of a form.

(2) The Lie derivative of a coordinate is simply the component of $\mathbf{v}$ in that direction:

$$\pounds_{\mathbf{v}} x^i = v^i.$$

(3) The Lie derivative of a function on M' (0-form) is simply its directional derivative:

$$\pounds_{\mathbf{v}} f = \mathbf{v}(f) = v^i f_{,i}.$$

(Commas represent partial derivatives. Sometimes they will omitted when the context is clear.)

(4) The Lie derivative of a wedge product obeys the Leibniz rule (the subscript $\mathbf{v}$ may be suppressed where it is not necessary):

$$\pounds(\alpha \wedge \beta) = (\pounds\alpha) \wedge \beta + \alpha \wedge (\pounds\beta).$$

(5) The exterior derivative d and the Lie derivative $\pounds$ commute. In particular,

$$\pounds_{\mathbf{v}} dx^i = d(\pounds_{\mathbf{v}} x^i) = dv^i;$$

the Lie derivative of the differential of a variable equals the differential of the corresponding component of $\mathbf{v}$.

3 The one-dimensional heat equation

We write the one dimensional heat equation

$$u_{xx} = u_t \tag{2}$$

as a first order set of equations by defining a new variable w:

$$u_x = w, \qquad w_x = u_t. \tag{3}$$

The variables are x, t, u, and w. We construct two 2-forms by inspection:

$$\begin{aligned} \alpha &= du \wedge dt - w dx \wedge dt, \\ \beta &= dw \wedge dt + du \wedge dx \end{aligned} \tag{4}$$

(α is a contact form.) If we "section" these forms — specialize to a submanifold $u = u(x,t)$ and $w = w(x,t)$ — we get

$$\alpha = (u_x dx + u_t dt) \wedge dt - w dx \wedge dt = (u_x - w) dx \wedge dt$$

and

$$\beta = (w_x dx + w_t dt) \wedge dt + (u_x dx + u_t dt) \wedge dx = (w_x - u_t) dx \wedge dt$$

where we have used the antisymmetry of 1-forms. We now "annul" these forms — set them equal to zero — obtaining Eqs. (3), the original first-order set of equations. The forms α and β in Eqs. (4) now constitute the ideal I of forms representing the heat equation (2).

We note that I is not unique; we may as well represent the heat equation by defining $z = u_t$ and constructing an ideal I' with a 1-form

$$\gamma = -du + w dx + z dt,$$

its exterior derivative

$$d\gamma = dw \wedge dx + dz \wedge dt,$$

and the 2-form

$$\delta = dw \wedge dt - z dx \wedge dt.$$

We note that $\alpha = -\gamma \wedge dt$ and $\beta = \delta - \gamma \wedge dx$.

We work first in the ideal I. Write the Lie derivatives of α and β as linear combinations of themselves. Expand the Lie derivatives by the rules above. We also drop the wedge product $\wedge$ and the subscript $\mathbf{v}$ on $\pounds$ to save writing.

$$\begin{aligned} \pounds\alpha &= \pounds(du\,dt - w dx\,dt) \\ &= (\pounds du)dt + du(\pounds dt) - (\pounds w)dx\,dt - w(\pounds dx)dt - w dx(\pounds dt) \\ &= dv^u dt + du\,dv^t - v^w dx\,dt - w dv^x dt - w dx\,dv^t \\ &= \lambda_1(du\,dt - w dx\,dt) + \lambda_2(dw\,dt + du\,dx). \end{aligned}$$

The λ_i are 0-forms (functions). Expand the dv^i by the usual chain rule, since the v^i are functions in M', using all four variables. Since $dt\,dt = 0$, etc., by the antisymmetry of 1-forms, some terms drop out. We have

$$\begin{aligned}
&(v^u_{,u}du + v^u_{,x}dx + v^u_{,w}dw)\,dt + du\,(v^t_{,t}dt + v^t_{,x}dx + v^t_{,w}dw) \\
&\quad - v^w dx\,dt - w(v^x_{,x}dx + v^x_{,u}du + v^x_{,w}dw)\,dt \\
&\quad - wdx\,(v^t_{,t}dt + v^t_{,u}du + v^t_{,w}dw) \\
&\quad = \lambda_1(du\,dt - wdx\,dt) + \lambda_2(dw\,dt + du\,dx).
\end{aligned}$$

There will be $4!/2!2! = 6$ basis 2-forms ($dx\,dt$, $dx\,du$, $dx\,dw$, $dt\,du$, $dt\,dw$, and $du\,dw$.) We equate the coefficients of these 2-forms to get

$$\begin{aligned}
&v^u_{,x} - v^w - w(v^x_{,x} + v^t_{,t}) = -w\lambda_1, \\
&-v^t_{,x} - wv^t_{,u} = -\lambda_2, \\
&-wv^t_{,w} = 0, \\
&-v^u_{,u} - v^t_{,t} + wv^x_{,u} = -\lambda_1, \\
&-v^u_{,w} + wv^x_{,w} = -\lambda_2, \\
&v^t_{,w} = 0.
\end{aligned}$$

Eliminating the Lagrange multipliers λ_i gives us one half of the determining equations:

$$\begin{aligned}
&v^t_{,w} = 0, \\
&v^t_{,x} + wv^t_{,u} = v^u_{,w} - wv^x_{,w}, \\
&v^u_{,x} - v^w - wv^x_{,x} = -wv^u_{,u} + w^2 v^x_{,u}.
\end{aligned}$$

Expansion of $\pounds\beta$ gives us the other half. One quickly sees from them that v^t is a function of t only and that v^x is a function only of x and t. Further calculation gives the usual six generators plus addition of an arbitrary solution. Exponentiation of the transformation proceeds by setting $\mathbf{v}\cdot\gamma = 0$ (contraction of $\mathbf{v}$ and γ, symbolized by a dot) and solving, where w and z are replaced by their values as derivatives of u (the usual method).

Another way to proceed, which removes the need for the multipliers, is to use $\alpha = 0$ to replace $du\,dt$, anywhere that combination occurs in the expansion of the Lie derivatives of α and β, by $wdx\,dt$, and to use $\beta = 0$ to replace $dw\,dt$ by $-du\,dx$. This may save considerable work in complicated cases, especially in cases where not all forms in the ideal are of the same rank. In those cases, some of the Lagrange multipliers may need to be forms (of rank greater than zero) themselves in order for the right hand sides to be of the same rank as the left hand sides, and that means that there may be very many coefficients to be eliminated. If one can avoid that, labor may be saved.

One can also use the ideal I' for the heat equation. (This is the technique used in paper I.) There are now five variables in M': x, t, u, w, and z. In I', there is only one 1-form γ, and so its Lie derivative equation is simple:

$$\pounds\gamma = \lambda\gamma, \tag{5}$$

where λ is a multiplier. One can expand the Lie derivative by an identity for any form ω, using the contraction operator:

$$\pounds_{\mathbf{v}}\omega = d(\mathbf{v}\cdot\omega) + \mathbf{v}\cdot d\omega. \tag{6}$$

Write $F = \mathbf{v} \cdot \gamma$, which is a function, and expand Eq. (5) using Eq. (6) (with $\omega = \gamma$) and identities for contraction (e.g., $\mathbf{v} \cdot (dx\, dy) = v^x dy - v^y dx$) (see paper I). We get

$$F = -v^u + wv^x + zv^t \tag{7}$$

and

$$\begin{aligned} \pounds_{\mathbf{v}}\gamma &= dF + v^w dx - v^x dw + v^z dt - v^t dz \\ &= \lambda\gamma = \lambda(-du + wdx + zdt). \end{aligned}$$

Expand dF with the chain rule, equate coefficients, eliminate λ and use Eq. (7), and we get all generators v^i in terms of F and its derivatives (subscripts on F are derivatives):

$$\begin{aligned} &v^x = F_w, \qquad v^t = F_z, \qquad v^u = -F + wF_w + zF_z, \\ &v^w = -F_x - wF_u, \qquad v^z = -F_t - zF_u. \end{aligned}$$

The exterior derivative of Eq. (5) is

$$d(\pounds\gamma) = \pounds(d\gamma) = d\lambda \wedge \gamma + \lambda d\gamma$$

so that the Lie derivative of $d\gamma$ is also in the ideal. There is now only one equation left:

$$\pounds\delta = \lambda_1\delta + \lambda_2 d\gamma + \tau \wedge \gamma,$$

where the λ_i are 0-forms and τ is an arbitrary 1-form with five terms. The term in du in τ can be eliminated by substituting from γ, and that drops out. This procedure also gives the standard determining equations.

4 Computer programs

There are a few computer programs which use this technique. Some of these were written, in REDUCE, by D.G.B. Edelen [3]. The programs are probably still available from Lehigh University. Other programs were written by Gragert, Kersten, and Martini, also in REDUCE. They published several works which developed and used this software, including a program for symbolic integration of overdetermined systems [4, 5, 6, 7, 8, 9]. Problems treated in references [10, 11, 12] are studies of a nonlinear diffusion equation

$$\triangle(u^{p+1}) + ku^q = u_t,$$

(where $\triangle$ is the Laplacian), the massive Thirring model, and the Federbush model. The present author used one of the programs with E.D. Fackerell to explore a relativity problem, in unpublished work, and one of Fackerell's students, Ben Langton, used it for his Ph.D. dissertation on certain solutions of the Einstein equations [13].

Another useful computer program is **liesymm**, a program found in MAPLE, based on a paper by Carminati et al. [14]. It works quite well, and there is an additional program called **autosimp** in MAPLE which does some integration of the determining equations, although the integration may not be complete. These programs are discussed briefly in Refs. [15, 16]. A student of the author's, David Neilsen, did a master's thesis with **liesymm** on Einstein's equations [17].

An extensive review of symbolic software was done by Hereman [18] in 1997. A nice table of programs is provided. No specific distinction is made in that paper between the traditional method and the differential form method.

5 Nonlinear Boltzmann equation

This example is actually listed as an example in **liesymm** in Maple 9.5, although it is not worked out. The equation is:

$$u_{xt} + u_x + u^2 = 0.$$

Possible ideals are I' with five variables, x, t, u, $p = u_x$, $q = u_t$:

$$\begin{aligned} &\alpha = du - pdx - qdt, \\ &d\alpha = -dp\,dx - dq\,dt, \\ &\beta = -dt\,dq + (p + u^2)dx\,dt \end{aligned}$$

or I with four variables, x, t, u, p:

$$\begin{aligned} &\gamma = du\,dt - pdx\,dt, \\ &\delta = dp\,dx + (p + u^2)dt\,dx. \end{aligned}$$

The calculation is quite similar to that for the heat equation. There are four generators.

6 Vacuum Maxwell equations

From paper I we write the usual 3-forms that represent the vacuum Maxwell equations in rectangular coordinates. Subscripts represent components.

$$\begin{aligned} &\alpha = dE_x\,dx\,dt + dE_y\,dy\,dt + dE_z\,dz\,dt + dB_x\,dy\,dz + dB_y\,dz\,dx + dB_z\,dx\,dy, \\ &\beta = dB_x\,dx\,dt + dB_y\,dy\,dt + dB_z\,dz\,dt - dE_x\,dy\,dz - dE_y\,dz\,dt - dE_z\,dx\,dy. \end{aligned}$$

We simplify these forms by defining $\gamma = \alpha + i\beta$ and $(A, B, C) = (\text{cyclic } E_k + iB_k) = \mathbf{h}$. Then (paper I) we can write

$$\gamma = d\mathbf{h} \cdot (d\mathbf{r}\,dt - (1/2)id\mathbf{r} \times d\mathbf{r})$$

or

$$\gamma = dA\,(dx\,dt - idy\,dz) + dB\,(dy\,dt - idz\,dx) + dC\,(dz\,dt - idx\,dy). \tag{8}$$

The forms in the ideal will then be γ and $\gamma*$, where the star represents complex conjugate. The variables will be t, x, y, z, A, B, C, A^*, B^*, C^*. The generators for the coordinates t, x, y, z will be real. The equations for the Lie derivatives are then

$$\pounds\gamma = \lambda\gamma + \mu\gamma*$$

and its complex conjugate.

In paper I the determining equations were worked out by using a vector-dyadic formalism. Here we use Eq. (8) for γ, which is a little clearer. We work with 3-forms in ten variables, so that there are $10!/7!3! = 120$ different basis 3-forms. There are two equations, for $\pounds\gamma$ and $\pounds\gamma*$, so that we apparently have 240 equations. However, 120 of them are simply the complex conjugates of the others. So we just look at the $\pounds\gamma$ equation. We see immediately by inspection that 3-forms with all terms being d(field variable) do not appear. Terms of the form $d(\text{field}) \wedge d(\text{field*}) \wedge d(\text{coordinate})$ yield only equations for the derivatives of the coordinate generators with respect to the complex conjugate fields (which are zero). Thus the coordinate generators do not depend on the complex conjugate fields, nor (by complex conjugation) on the fields. Thus they depend only on the coordinates themselves. It is also easy to show that the field generators depend only on the fields and not on their complex conjugates.

This reduces the number of equations to 22–18 with one field 1-form and two coordinate forms and four with three coordinate forms. From the first set we get the conformal Killing equations plus some expressions for the derivatives of the field generators. From the second we get equations for the field generators that are the Maxwell equations themselves. Solution of the determining equations gives the 17-generator set given in paper I (15 conformal Killing vectors, a scale change on the fields, and a duality change on the fields, plus the addition of an arbitrary solution). Steeb [19] presents other symmetries besides these, which depend on the derivatives of the fields. (Steeb and collaborators also treat various versions of the Dirac equation [20].)

7 Nonlinear Poisson equation

We consider the equation:

$$u_{xx} + u_{yy} + u_{zz} = f(u),$$

where $f(u)$ is an undetermined function. Subscripts represent derivatives. We define $r = u_x$, $s = u_y$, $t = u_z$. Then

$$r_x + s_y + t_z = f(u).$$

The ideal I consists of these forms:

$$\alpha = -du + rdx + sdy + tdz,$$
$$d\alpha = dr\,dx + ds\,dy + dt\,dz,$$
$$\beta = dr\,dy\,dz + ds\,dz\,dx + dt\,dx\,dy - f(u)\,dx\,dy\,dz.$$

There are seven variables.

We may approach this problem by defining a function $H = \mathbf{v} \cdot \alpha$, as we did with the heat equation. The Lie derivative of α gives all the generators in terms of H and its derivatives, as before. Then the only equation we have left is that for $\pounds\beta$, a 3-form. But equating it to a linear combination of α, $d\alpha$, and β is messy. The multiplier of $d\alpha$, a 2-form, must itself be a 1-form, which will have six coefficients (we do not include a term in du, because that can be replaced by α, and $d\alpha \wedge \alpha$ can be included in the α term.) The multiplier of α must be a 2-form — and again we can eliminate du terms because they can be replaced by α, and $\alpha \wedge \alpha = 0$. But that still leaves 15 coefficients. The multiplier of β will be a single coefficient. That totals 22 coefficients that must be eliminated.

So we consider an easier way. We define a new ideal I', made up of four 3-forms: β, $\alpha\,dy\,dz = (-du + rdx)\,dy\,dz$, $\alpha\,dz\,dx$, and $\alpha\,dx\,dy$. The latter three forms are equivalent to α alone. The Lie derivative of each 3-form must be a linear combination of all four, thus giving four multipliers in each equation to be eliminated. The equations are much simpler; it is easy to eliminate four multiplier coefficients in each equation than 22, even though there are now still $4 \times 4 = 16$ multipliers. The easiest procedure is to write out the equation for the Lie derivative of $\alpha\,dy\,dz$, eliminate the multipliers to get a set of determining equations and then to permute x, y, z (and s, t) cyclically. One quickly gets the result that the generators for x, y, z, and u are functions only of x, y, z, u, and the generators for r, s, and t are given in terms of a function which is precisely the H defined above, $H = \mathbf{v} \cdot \alpha = -v^u + rv^x + sv^y + tv^z$.

The equation for $\pounds\beta$ now has four multipliers, which are easily eliminated. We find quickly that v^x, v^y, v^z depend only on x, y and z and that they obey the Killing equations. The generators for r, s, and t are written out easily, and one ends up with a single equation involving $f(u)$ and $f'(u)$. Solution of that equation for the given f then leads to the final result.

There is a small technical point. We did not include $d\alpha$ in I', even though we did represent α as three 3-forms. Should we have done so? The answer is no. If we take the exterior derivative of the 3-forms $\alpha\, dy\, dz$, etc. we get terms like $dr\, dx\, dy\, dz$ — in other words, just $d\alpha\, dy\, dz$. We get three of those equations, which are equivalent to the equation for $\pounds d\alpha$. The determining equations for I and I' give the same result.

A similar treatment is used by Satir [21], who writes a set of two-dimensional bosonic membrane equations as eight 3-forms. He then uses a REDUCE program together with the EXCALC differential geometry package to find a 12 parameter group. He remarks that the use of differential forms enabled the calculation to go more quickly that the conventional method.

8 Nonlinear diffusion equation

This an equation treated in paper II, originally due to Fushchych — a nonlinear diffusion equation with an additional condition. We can write a 1-form as was done above, its exterior derivative, and a 4-form for the main field equation. In paper II, it was assumed *a priori* that the generators for the coordinates depend only on the coordinates and that those for the derivatives of the field were linear in those derivatives. It then turns out that much of the analysis of the Lie derivative of the 4-form can be done by inspection.

9 One-dimensional compressible fluid dynamics

The equations considered here are (see paper I):

$$\rho_t + (\rho u)_x = 0,$$
$$\rho u_t + \rho u u_x + c^2 \rho_x = 0,$$

where isentropic flow is considered so that the pressure is only a function of the density, $P = P(\rho)$, and $c^2 = dP/d\rho$. The generators for x and t include a ρ- and u-dependent case, which turns out to give the hodograph transformation.

10 Nonclassical symmetries

One can generalize the ideal I by including contractions of $\mathbf{v}$ with some of the differential forms. An example of this was provided in paper I for the heat equation, in which it was shown that one gets the equations for "nonclassical symmetries" of that equation, the same equations found by Bluman and Cole in 1969 [22]. While this technique has not been explored in detail by this author, Webb has studied this set of equations — referred to as a coupled nonlinear Burgers-heat equations system — with differential forms and has searched for Bäcklund transformations for the set [23].

11 Ordinary differential equations

We consider an example:

$$y'' = f(x, y, y'),$$

where the prime indicates differentiation with respect to x. We put $z = y'$ and write two 1-forms, $\alpha = dy - zdx$ and $\beta = dz - f(x,y,z)dx$. There are three variables. The Lie derivative equation for α is

$$\pounds\alpha = dv^y - v^z dx - z dv^x = \lambda_1(dy - zdx) + \lambda_2(dz - fdx).$$

There are three equations, for the coefficients of dx, dy, and dz; elimination of the multipliers gives a single equation, which is an expression for v^z. The Lie derivative equation for β gives another single equation. We assume that the generators v^x and v^y (usually written as ξ and η, respectively) are functions only of x and y. In that case, v^z becomes the usual extended generator for $z = y'$ and the remaining equation is the usual determining equation for ξ and η, as given, e.g., in Stephani [24].

12 Advantages of using differential forms

These have been treated in paper II, but are reviewed here. The method is easy to apply. One simply writes all equations as first order equations; the differential forms can be written by inspection. Calculations may be long because of the necessity of introducing the Lagrange multipliers; however, one can choose the ideal to minimize this. One can make use of symmetries of form (e.g., cyclic symmetry of coordinates), or one can use the forms to substitute for certain terms in the Lie derivative expansion, thus removing the need for multipliers. Independent variable generators may easily be considered as functions of the independent variables only (just assume that and that simplifies the expansion of the differentials of those generators).

13 Other examples

We mention here some research papers in which the differential form method is used. Papachristou generalized the method to vector-valued or Lie algebra-valued differential forms and treated the two-dimensional Dirac equation and the Yang–Mills free-field equations in Minkowski spacetime [25] (as part of a Ph.D. dissertation with the author.) This was later used to investigate self-dual Yang–Mills equations, which work showed connections between symmetry and integrability (in the form of Bäcklund transformations) of those equations [26, 27, 28]. Waller, in three similar papers, treats nonlinear diffusion equations (or reaction-diffusion equations) arising in plasma physics [29, 30, 31]. He uses the technique of writing a 1-form and contracting it with $\mathbf{v}$, as done in the second treatment of the heat equation above.

Edelen has developed the theory of the differential form method extensively. His computer programs have already been mentioned. At least two books [32, 33] and several papers [34, 35, 36, 37] explore the use in differential forms in physics, including the method discussed here. In papers [35, 36] he considers a method of characteristics in any number of dimensions, using isovector treatments. With this he can write parametric solutions of differential equations. One equation he considers is [36]

$$u_t\, u_x = 4u.$$

He gives a solution for the equation as an initial value problem: if $u(x,0) = \alpha(x)$, with $\alpha'(x) \neq 0$, then

$$x = z + \alpha(z)(\exp(4\tau) - 1)/\alpha'(z),$$
$$t = (1/4)\alpha'(z)(\exp(4\tau) - 1),$$
$$u = \alpha(z)\exp(8\tau),$$

where z is a parameter and τ is an arbitrary function. Instead, if one defines $r = u_x$, one can write a simple 1-form for the equation:

$$\alpha = -du + r dx + (4u/r)dt.$$

Then $\pounds\alpha = \lambda\alpha$ gives equations which yield most of the generators in terms of a function $F = \mathbf{v}\cdot\alpha$, which satisfies the linear first order equation (subscripts are derivatives)

$$F_t + 4F_r + (4u/r^2)F_x + (8u/r)F_u = 4F/r.$$

The special solution $F = u/r$ seems to give Edelen's solution, although not all details are worked out yet and it is a little uncertain. In Ref. [37] he considers "inverse" isovector methods.

Webb et al consider nonlinear Schrödinger equations for a type of MHD waves, using the differential form method [38]. He also analyzes a nonlinear magnetic potential equation, with conservation laws, with the Liouville equation as a special case [39]. Pakdemirli and others treat boundary layer equations for non-Newtonian fluids, including arbitrary shear stress, power law fluids, and other models [40, 41]. Şuhubi and others, in a number of papers, consider general approaches to equations of balance and other equations [42, 43, 44, 45, 46, 47, 48]. A number of these discuss equivalence groups, as a generalization of symmetry groups. One paper with Ozer [47] treats nonvacuum Maxwell equations with nonlinear constitutive relations. Another discusses steady boundary layer flow past a semi-infinite flat plate [44].

Bhutani and Bhattacharya study n-dimensional Klein–Gordon and Liouville equations with an interesting approach [49]. Various types of diffusion equations are treated in Refs. [50, 51, 52]. Viscoelastic-viscoplastic rods are studied in Ref. [53] and power law creep in Ref. [54]. Equations of meteorology, here meaning steady two-dimensional incompressible inviscid flow with a Coriolis term, are studied in Ref. [55]. Hu considers the principal chiral model [56], using differential forms and ideas from Ref. [27]. An interesting paper is that by Barco, who shows for a second-order hyperbolic or parabolic differential equation, with one dependent variable and two independent variables, that an isovector can be used to generate a similarity solution by using a particular Cauchy characteristic vector field [57].

Nonlinear thermoelasticity was treated by Kalpakides [58]. His work is related to that of Şuhubi [42, 43, 44, 45, 46, 47, 48]. Harnad and Winternitz considered a generalized nonlinear Schrödinger equation,

$$iz_t + z_{xx} = f(z, z*),$$

with attention to both symmetries and Bäcklund transformations [59].

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[1] Harrison B.K., Estabrook F.B., Geometric approach to invariance groups and solution of partial differential systems, *J. Math. Phys.*, 1971, V.12, 653–666 (Paper I).

[2] Harrison B.K., Differential form symmetry analysis of two equations cited by Fushchych, in Proceedings of Second International Conference "Symmetry in Nonlinear Mathematical Physics" (July 7–13, 1997, Kyiv), Editors M.I. Shkil, A.G. Nikitin and V.M. Boyko, V.1, 21–33 (Paper II).

[3] Edelen D.G.B., Programs for calculation of isovector fields in the REDUCE.2 environment, Center for the Application of Mathematics, Lehigh University, 1981.

[4] Gragert P.K.H., Symbolic computations in prolongation theory, Ph.D. Thesis, Twente University of Technology, Enschede, The Netherlands, 1981.

[5] Kersten P.H.M., Gragert P.K.H., Symbolic integration of overdetermined systems of partial differential equations, Memorandum 430, Twente University of Technology, Enschede, The Netherlands, 1983.

[6] Gragert P.K.H., Kersten P.H.M., Martini A., Symbolic computations in applied differential geometry, *Acta Appl. Math.*, 1983, V.1, 43–77.

[7] Kersten P.H.M., Infinitesimal symmetries: a computational approach, Ph.D. Thesis, Twente University of Technology, Enschede, The Netherlands, 1985.

[8] Kersten P.H.M., Software to compute infinitesimal symmetries of exterior differential systems, with applications, *Acta Appl. Math.*, 1989, V.16, 207–229.

[9] Gragert P.K.H., Kersten P.H.M., Differential geometric computations and computer algebra, *Math. Comp. Modelling*, 1997, V.25, 11–24.

[10] Kersten P.H.M., Gragert P.K.H., The Lie algebra of infinitesimal symmetries of nonlinear diffusion equation, *J. Phys. A: Math. Gen.*, 1983, V.16, L685–L688.

[11] Kersten P.H.M., Martini R., Lie–Bäcklund transformations for the massive Thirring model, *J. Math. Phys.*, 1985, V.26, 822–825.

[12] Kersten P.H.M., Creating and annihilating Lie–Bäcklund transformations of the Federbush model, *J. Math. Phys.*, 1986, V.27, 1139–1144.

[13] Langton B.T., Lie symmetry techniques for exact interior solutions of the Einstein field equations for axially symmetric, stationary, rigidly rotating perfect fluids, Ph.D. Thesis, University of Sydney, Australia, 1997.

[14] Carminati J., Devitt J.S., Fee G.J., Isogroups of differential-equations using algebraic computing, *J. Symbolic Comp.*, 1992, V.14, 103–120.

[15] Heck A., Introduction to Maple, Springer, New York, 1993, 413–417.

[16] Char B.W., Geddes K.O., Gonnet G.H., Leong B.L., Monagan M.B., Watt S.M., Maple V library reference manual, New York, Springer, 1991, 540–560.

[17] Neilsen D.W., A search for an interior solution in general relativity using Lie–Bäcklund symmetries, M.S. Thesis, Provo (Utah, USA), Brigham Young University, 1995.

[18] Hereman W., Review of symbolic software for Lie symmetry analysis, *Math. Comp. Modelling*, 1997, V.25, 115–132.

[19] Steeb W.H., Symmetries and vacuum Maxwell's equations, *J. Math. Phys.*, 1980, V.21, 1656–1680.

[20] Steeb W.H., Erig W., Strampp W., Symmetries and the Dirac equation, *J. Math. Phys.*, 1981, V.22, 970–973.

[21] Satir A., Duff–Inami–Pope–Sezgin–Stelle bosonic membrane equations as an involutory system, *Progr. Theoret. Phys.*, 1998, V.100, 1273–1280.

[22] Bluman G.W., Cole J.D., The general similarity solution of the heat equation, *J. Math. Mech.*, 1969, V.18, 1025–1042.

[23] Webb G.M., Lie symmetries of a coupled nonlinear Burgers heat-equation system, *J. Phys. A: Math. Gen.*, 1990, V.23, 3885–3894.

[24] Stephani H., Differential equations. Their solution using symmetries, Cambridge, Cambridge University Press, 1989, 28.

[25] Papachristou C.J., Harrison B.K., Isogroups of differential ideals of vector-valued differential forms: application to partial differential equations, *Acta Appl. Math.*, 1988, V.11, 155–175.

[26] Papachristou C.J., Harrison B.K., Symmetry groups of partial differential equations associated with vector-valued differential forms, in Proceedings of the XV International Colloquium in Group Theoretical Methods in Physics, Editor R. Gilmore, Singapore, World Scientific, 1987, 440–445.

[27] Papachristou C.J., Harrison B.K., Some aspects of the isogroup of the self-dual Yang–Mills system, *J. Math. Phys.*, 1987, V.28, 1261–1264.

[28] Papachristou C.J., Harrison B.K., Nonlocal symmetries and Bäcklund transformations for the self-dual Yang–Mills system, *J. Math. Phys.*, 1988, V.29, 238–243.

[29] Waller S.M., A three-parameter group similarity solution for a one dimensional nonlinear diffusion equation, *Phys. Scripta*, 1990, V.41, 193–196.

[30] Waller S.M., Isogroup and general similarity solution of a nonlinear diffusion equation, *J. Phys. A: Math. Gen.*, 1990, V.23, 1035–1040.

[31] Waller S.M., Invariant group similarity solution for a class of reaction-diffusion-equations, *Phys. Scripta*, 1990, V.42, 385–388.

[32] Edelen D.G.B., Isovector methods for equations of balance. With programs for computer assistance in operator calculations and an exposition of practical topics of the exterior calculus, *Monographs and Textbooks on Mechanics of Solids and Fluids, Mechanics: Analysis*, The Hague, Martinus Nijhoff Publishers, 1980.

[33] Edelen D.G.B., Applied exterior calculus, New York, John Wiley and Sons, 1985, Chap. 6.

[34] Edelen D.G.B., Isovector fields for problems in the mechanics of solids and fluids, *Internat. J. Engrg. Sci.*, 1982, V.20, 803–815.

[35] Edelen D.G.B., On solving problems in the mechanics of solids and fluids by a generalized method of characteristics, *Internat. J. Engrg. Sci.*, 1988, V.26, 361–372.

[36] Edelen D.G.B., Order-independent method of characteristics, *Internat. J. Theoret. Phys.*, 1989, V.28, 303–333.

[37] Edelen D.G.B., Implicit similarities and inverse isovector methods, *Arch. Rat. Mech. and Anal.*, 1983, V.82, 181–189.

[38] Webb G.M., Brio M., Zank G.P., Symmetries of the triple degenerate DNLS equations for weakly nonlinear dispersive MHD waves, *J. Plasma Phys.*, 1995, V.54, 201–244.

[39] Webb G.M., Similarity considerations and conservation laws for magneto-static atmospheres, *Solar Phys.*, 1986, V.106, 287–313.

[40] Pakdemirli M., Yürüsoy M., Küçükbursa A., Symmetry groups of boundary layer equations of a class of non-Newtonian fluids, *Internat. J. Non-Linear Mech.*, 1996, V.31, 267–276.

[41] Pakdemirli M., Yürüsoy M., Equivalence transformations applied to exterior calculus approach for finding symmetries: an example of non-Newtonian fluid flow, *Internat. J. Engrg. Sci.*, 1999, V.37, 25–32.

[42] Şuhubi E.S., Chowdhury K.L., Isovectors and similarity solutions for nonlinear reaction-diffusion equations, *Internat. J. Engrg. Sci.*, 1988, V.26, 1027–1041.

[43] Şuhubi E.S., Bakkaloğlu A., Group properties and similarity solutions for a quasi-linear wave-equation in the plane, *Internat. J. Non-Linear Mech.*, 1991, V.26, 567–584.

[44] Şuhubi E.S., Isovector fields and similarity solutions for general balance-equations, *Internat. J. Engrg. Sci.*, 1991, V.29, 133–150.

[45] Pakdemirli M., Şuhubi E.S., Similarity solutions of boundary-layer equations for second-order fluids, *Internat. J. Engrg. Sci.*, 1992, V. 30, 611–629.

[46] Şuhubi E.S., Equivalence groups for second order balance equations, *Internat. J. Engrg. Sci.*, 1999, V.37, 1901–1925.

[47] Ozer S., Şuhubi E.S., Equivalence transformations for first order balance equations, *Internat. J. Engrg. Sci.*, 2004, V.42, 1305–1324.

[48] Şuhubi E.S., Equivalence groups for balance equations of arbitrary order, Part I, *Internat. J. Engrg. Sci.*, 2004, V.42, 1729–1751.

[49] Bhutani O.P., Bhattacharya L., Isogroups and exact-solutions for some Klein–Gordon and Liouville-type equations in n-dimensional Euclidean space, *J. Math. Phys.*, 1995, V.36, 3759–3770.

[50] Bhutani O.P., Vijayakumar K., On the isogroups of the generalized diffusion equation, *Internat. J. Engrg. Sci.*, 1990, V.28, 375–387.

[51] Bhutani O.P., Singh K., On certain exact solutions of a generalized K-dV-Burger type equation via isovector method-I, *Internat. J. Engrg. Sci.*, 2000, V.38, 1741–1753.

[52] Chowdhury K.L., On the isovectors of a class of nonlinear diffusion-equations, *Internat. J. Engrg. Sci.*, 1986, V.24, 1597–1605.

[53] Chowdhury K.L., A note on the isovector of rate-type materials, *Internat. J. Engrg. Sci.*, 1986, V.24, 819–826.

[54] Delph T.J., Isovector fields and self-similar solutions for power law creep, *Internat. J. Engrg. Sci.*, 1983, V.21, 1061–1067.

[55] Vijayakumar K., Isogroup classification of equations of meteorology, *Internat. J. Engrg. Sci.*, 1996, V.34, 1157–1164.

[56] Hu Z-J., On the isovectors of the principal chiral model, *J. Math. Phys.*, 1991, V.32, 2540–2542.

[57] Barco M.A., An application of solvable structures to classical and nonclassical similarity solutions, *J. Math. Phys.*, 2001, V.42, 3714–3734.

[58] Kalpakides V.K., Isovector fields and similarity solutions of nonlinear thermoelasticity, *Internat. J. Engrg. Sci.*, 1998, V.36, 1103–1126.

[59] Harnad J., Winternitz P., Pseudopotentials and Lie symmetries for the generalized nonlinear Schrödinger equation, *J. Math. Phys.*, 1982, V.23, 517–525.