On extremal problem for area functional

B. A. Klishchuk, R. R. Salimov

Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

kban1988@gmail.com, ruslan623@yandex.ru

Let Γ be a family of curves γ in the complex plane $\mathbb{C}$. A Borel function $\varrho\colon \mathbb{C}\to[0,\infty]$ is called *admissible* for Γ, abbr. $\varrho\in\operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma}\varrho(z)\,|dz|\ \geqslant\ 1$$

for all $\gamma\in\Gamma$. The *p–modulus* of Γ is the quantity defined by

$$\mathcal{M}_p(\Gamma)\ =\ \inf\limits_{\varrho\in\operatorname{adm}\Gamma}\int\limits_{\mathbb{C}}\varrho^p(z)\,dx\,dy\,,\qquad p\geqslant 1.$$

Let E, F and G be arbitrary sets in $\mathbb{C}$. Denote by $\Delta(E,F,G)$ a family of all continuous curves $\gamma\colon[a,b]\to\mathbb{C}$ joining E and F in G, i.e. $\gamma(a)\in E$, $\gamma(b)\in F$ and $\gamma(t)\in G$ for $a<t<b$. Given a domain $D\subset\mathbb{C}$ and $z_0\in D$, denote by $\mathbb{A}(z_0,r_1,r_2)=\{z\in\mathbb{C}\colon r_1\leqslant|z-z_0|\leqslant r_2\}<d_0$, where $d_0=\operatorname{dist}(z_0,\partial D)$. Now let $Q\colon D\to[0,\infty]$ be a (Lebesgue) measurable function. A homeomorphism $f\colon D\to\mathbb{C}$ is called *a ring Q-homeomorphism with respect to p-modulus* at $z_0\in D$, if

$$\mathcal{M}_p(\Delta(fS_1,fS_2,fD))\ \leqslant\ \int\limits_{\mathbb{A}}Q(z)\,\eta^p(|z-z_0|)\,dx\,dy$$

for every ring $\mathbb{A}=\mathbb{A}(z_0,r_1,r_2)$, $0<r_1<r_2<d_0$, and every measurable function $\eta:(r_1,r_2)\to[0,\infty]$ such that $\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1$. Let $\mathbb{B}=\{z\in\mathbb{C}\colon\ |z|\leqslant 1\}$ and $Q\colon\mathbb{B}\to[0,\infty]$ be a (Lebesgue) measurable function. For $p>2$ denote by $\mathcal{H}$ a set of all ring Q-homepmorphisms $f\colon\mathbb{B}\to\mathbb{C}$ with respect to p-modulus at the origin satisfying

$$q(t)=\frac{1}{2\pi t}\int\limits_{S_t}Q(z)\,|dz|\leqslant q_0\,t^{-\alpha},\ q_0\in(0,\infty)\,,\ \alpha\in[0,\infty)\,,$$

for almost all $t\in(0,1)$. Here $S_t=\{z\in\mathbb{C}\colon|z|=t\}$. Let $\mathbf{S}_r(f)=|fB_r|$ be an area functional over the class $\mathcal{H}$ where $B_r=\{z\in\mathbb{C}\colon|z|\leqslant r\}$. The following statement provides an extremal bound for the functional $\mathbf{S}_r(f)$.

Theorem 1. *For all* $r\in[0,1]$

$$\min\limits_{f\in\mathcal{H}}\mathbf{S}_r(f)=\pi\,\left(\frac{p-2}{\alpha+p-2}\right)^{\frac{2(p-1)}{p-2}}\ q_0^{\frac{2}{2-p}}\,r^{\frac{2(\alpha+p-2)}{p-2}}\,.$$