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Equivalence Groupoids in Group Classification Problems

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2*. Vaneeva O., Boyko V., Zhalij A. and Sophocleous C., Classification of reduction operators and exact solutions of variable coefficient Newell–Whitehead–Segel equations, *J. Math. Anal. Appl.* **474** (2019), no. 1, 264–275.

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Contents

Chapter 1

Equivalence Groupoids of Classes of Differential Equations

It is widely known that there is no general theory for integration of nonlinear partial differential equations (PDEs). Nevertheless, many special cases of complete integration or finding particular solutions are related to appropriate changes of variables. Nondegenerate point transformations that leave a differential equation invariant and form a connected Lie group are called Lie symmetries of this equation. Transformations of this kind are ones which are mostly used. In many cases, the algorithmic Lie reduction method, which uses known Lie symmetries, results in the construction of group-invariant solutions for a given PDE. This places the transformation methods among the most powerful analytic tools currently available in the study of nonlinear PDEs [273].

Another feature of Lie symmetries is that they reveal equations which are important for applications among wide set of admissible ones. Indeed, all basic equations of mathematical physics, e.g., the equations of Newton, Laplace, Euler–Lagrange, d'Alembert, Lamé, Hamilton–Jacobi, Maxwell, Schrödinger etc., have rich symmetry properties [89]. This property distinguishes these equations from other PDEs. Therefore, an important problem arises to single out from a given class of PDEs those admitting Lie symmetry algebra of the maximally possible dimension. This problem is called the *group classification problem* and is formulated as follows [131, 227]: given a class of PDEs, to classify all possible cases of extension of Lie invariance algebras of such equations with respect to the equivalence group of the class.

Another important task is to study transformational properties of classes of differential equations, i.e. to describe explicitly nondegenerate point transformations that link members of a given class. Indeed, if two differential equations are connected by such a transformation, then associated objects like exact solutions, local conservation laws, and different kinds of symmetries of these equations are also related by this transformation. Such equations are called equivalent (or *similar* in terms of [227]). Knowledge of an exact solution for one of two equivalent equations allows one to construct the corresponding exact solution for the other equation using the point transformation connecting them. A number of exact solutions for variable coefficient PDEs were constructed using the equivalence method, see, e.g., [175, 251, 300]. At the same time, nondegenerate point transformations appear to be a useful tool not only for finding exact solutions but also for exhaustive solving group classifications problems (see, e.g., [21, 175, 248] and reference therein), design of physical parameterization schemes [242], and study of integrability [45, 114, 172, 172, 304].

We firstly describe the basic notions on admissible transformations in classes of differential equations in Section 1.1. Then in Section 1.2 we relate existing notions and results from group analysis of differential equations to the groupoid-relevant terminology. The procedures for classifying admissible transformations and Lie symmetries for non-normalized classes of differential equations are presented in Section 1.3. The results of this chapter are published in [4*,13*,27*].

1.1. Preliminaries on Admissible Transformations

The systematic study of transformational properties of classes of nonlinear PDEs was initiated in 1991 by Kingston and Sophocleous [158, 159, 273]. These authors later named the transformations related two particular equations in a class of PDEs *form-preserving transformations* [160], because such transformations preserve the form of the equation in a class and change only its arbitrary elements. Only a year later in 1992 Gazeau and Winternitz started to investigate such transformations in classes of PDEs calling them *allowed transformations* [101,315]. Rigorous definitions and developed theory on the subject was proposed later by Popovych [239, 248]. As formalization of notion of form-preserving (allowed) transformations the term *admissible transformation* was suggested therein. In brief, an admissible transformation is a triple consisting of two fixed equations from a class and a transformation that links these equations. The set of admissible transformations considered with the standard operation of composition of transformations is also called the equivalence groupoid [242].

Equivalence transformations generate a subset in a set of admissible transformations. It is important that admissible transformations are not necessarily related to a group structure, but equivalence transformations always form a group. An equivalence transformation applied to any equation from the class always maps it to another equation from the same class. In other words, equivalence transformations preserve differential structure of the whole class. At the same time, an admissible transformation may exist only for a specific pair of equations from the class under consideration. For example, the point transformation $t'=e^{bt}/b$, $x'=x$, $u'=u-bt$ links equations $u_t=(e^u)_{xx}+ae^u+b$ and $u'_{t'}=(e^{u'})_{x'x'}+ae^{u'}$, where a and b are arbitrary constants with $b\neq 0$ [131]. Both these equations are members of the class $\mathcal{L}$: $u_t=(e^u)_{xx}+Q(u)$, where Q is a smooth function of u. Acting on other equation from this class, e.g., on $u_t=(e^u)_{xx}+e^{2u}+b$, this transformation maps it to the equation $u'_{t'}=(e^{u'})_{x'x'}+b\,t'e^{2u'}$, that is not constant coefficient one and does not belong to the class $\mathcal{L}$.

By Ovsiannikov, the equivalence group consists of the nondegenerate point transformations of the independent and dependent variables and of the arbitrary elements of the class, where transformations for independent and dependent variables are projectible on the space of these variables [227]. After appearance of other kinds of equivalence group the one used by Ovsiannikov is called now usual equivalence group. If the transformations for independent and dependent variables involve arbitrary elements, then the corresponding equivalence group is called the generalized equivalence group [194, 248]. If new arbitrary elements appear to depend on old ones in a nonlocal way (e.g., new arbitrary elements are expressed via integrals of old ones), then the corresponding equivalence group is called extended [137, 248]. Extended generalized equivalence group possesses both the aforementioned properties. A number of examples of usage of different kinds of equivalence groups are presented, e.g., in [138, 302].

If any admissible transformation in a given class is induced by a transformation from its equivalence group (usual / generalized / extended / extended generalized), then this class is called normalized in the corresponding sense.

In [292] we developed the groupoid theory in classes of differential equations, enhancing the approaches used in previous papers on admissible transformations in classes of differential equations and their group classification by the algebraic method, see e.g. [17,21,24,72,96,101,178,222,239,248]. Below we relate existing notions and results from group analysis of differential equations to the groupoid-relevant terminology and then present procedures for classifying admissible transformations and Lie symmetries for non-normalized classes of differential equations.

1.2. Equivalence Groupoids and Related Notions

Consider a class $\mathcal{L}|_{\mathcal{S}} = \{\mathcal{L}_\theta \mid \theta \in \mathcal{S}\}$ of systems of differential equations $\mathcal{L}_\theta$ for unknown functions $u = (u^1, \dots, u^m)$ of independent variables $x = (x_1, \dots, x_n)$ with the arbitrary-element tuple $\theta = (\theta^1, \dots, \theta^k)$ running through a set $\mathcal{S}$. Here $\mathcal{L}_\theta$ denotes a system of differential equations of the form $L(x, u_{(r)}, \theta(x, u_{(r)})) = 0$ with a fixed tuple L of rth order differential functions in u parameterized by θ. We use the short-hand notation $u_{(r)}$ for the tuple of derivatives of u with respect to x up to order r, including u as the zeroth order derivatives. The set $\mathcal{S}$ is the solution set of an auxiliary system of differential equations and inequalities in θ, where rth order jet variables $(x, u_{(r)})$ play the role of independent variables, $S(x, u_{(r)}, \theta_{(q)}) = 0$ and, e.g., $\Sigma(x, u_{(r)}, \theta_{(q)}) \neq 0$ with the tuple $\theta_{(q)}$ constituted by the derivatives of θ up to order q with respect to $(x, u_{(r)})$. Up to the gauge equivalence of systems from $\mathcal{L}|_{\mathcal{S}}$ [248], which is usually trivial, the correspondence $\theta \mapsto \mathcal{L}_\theta$ between $\mathcal{S}$ and $\mathcal{L}|_{\mathcal{S}}$ is bijective.

The *equivalence groupoid* $\mathcal{G}^\sim$ of the class $\mathcal{L}|_{\mathcal{S}}$ is the small category with $\mathcal{L}|_{\mathcal{S}}$ or, equivalently, with $\mathcal{S}$ as the set of objects and with the set of point transformations of (x, u), i.e., of (local) diffeomorphisms in the space with the coordinates (x, u), between pairs of systems from $\mathcal{L}|_{\mathcal{S}}$ as the set of arrows. Specifically,

$$\mathcal{G}^\sim = \big\{\mathcal{T} = (\theta, \Phi, \tilde\theta) \mid \theta, \tilde\theta \in \mathcal{S},\ \Phi \in \mathrm{Diff}^{\mathrm{loc}}_{(x,u)}\colon \Phi_*\mathcal{L}_\theta = \mathcal{L}_{\tilde\theta}\big\}.$$

Elements of $\mathcal{G}^\sim$ are called *admissible (point) transformations* within the class $\mathcal{L}|_{\mathcal{S}}$. The pushforward of θ by Φ is defined by $\Phi_*\theta = \tilde\theta$ if $\Phi_*\mathcal{L}_\theta = \mathcal{L}_{\tilde\theta}$.

The definitions of all notions related to groupoids are obvious. Thus, the *source* and *target* maps $\mathrm{s}, \mathrm{t}\colon \mathcal{G}^\sim \to \mathcal{S}$ are defined by $\mathrm{s}(\mathcal{T}) = \theta$ and $\mathrm{t}(\mathcal{T}) = \tilde\theta$ for any $\mathcal{T} = (\theta, \Phi, \tilde\theta) \in \mathcal{G}^\sim$, which gives rise to the groupoid notation $\mathcal{G}^\sim \rightrightarrows \mathcal{S}$, where the symbol "$\rightrightarrows$" denotes the pair of the source and target maps. Admissible transformations $\mathcal{T}$ and $\mathcal{T}' = (\theta', \Phi', \tilde\theta')$ are *composable* if $\tilde\theta = \theta'$, and then their *composition* is $\mathcal{T} \star \mathcal{T}' = (\theta, \Phi' \circ \Phi, \tilde\theta')$, which defines a natural partial multiplication on $\mathcal{G}^\sim$. For any $\theta \in \mathcal{S}$, the *unit at* θ is given by $\mathrm{id}_\theta := (\theta, \mathrm{id}_{(x,u)}, \theta)$, where $\mathrm{id}_{(x,u)}$ is the identity transformation of (x, u). This defines the *object inclusion map* $\mathcal{S} \ni \theta \mapsto \mathrm{id}_\theta \in \mathcal{G}^\sim$, i.e., the object set $\mathcal{S}$ can be regarded to coincide with the base groupoid $\mathcal{S} \rightrightarrows \mathcal{S} := \{\mathrm{id}_\theta \mid \theta \in \mathcal{S}\}$. The *inverse* of $\mathcal{T}$ is $\mathcal{T}^{-1} := (\tilde\theta, \Phi^{-1}, \theta)$, where Φ^{-1} is the inverse of Φ. All required properties like associativity of the partial multiplication, its consistency with the source and target maps, natural properties of units and inverses are obviously satisfied.

The s-*fibre over* $\theta\in\mathcal S$, $\mathrm s^{-1}(\theta)\subseteq\mathcal G^\sim$, is the set of possible admissible transformations within $\mathcal L|_{\mathcal S}$ with source at θ. Similarly, the t-*fibre over* $\theta\in\mathcal S$, $\mathrm t^{-1}(\theta)\subseteq\mathcal G^\sim$, is the set of possible admissible transformations within $\mathcal L|_{\mathcal S}$ with target at θ. The subset $\mathcal G(\theta,\tilde\theta):=\mathrm s^{-1}(\theta)\cap\mathrm t^{-1}(\tilde\theta)$ of $\mathcal G^\sim$ with $\theta,\tilde\theta\in\mathcal S$ corresponds to the set of point transformations mapping the system $\mathcal L_\theta$ to the system $\mathcal L_{\tilde\theta}$. The *vertex group* $\mathcal G_\theta:=\mathcal G(\theta,\theta)=\mathrm s^{-1}(\theta)\cap\mathrm t^{-1}(\theta)$ is associated with the point symmetry (pseudo)group G_θ of the system $\mathcal L_\theta$,

$$G_\theta=\big\{\Phi\in\mathrm{Diff}^{\rm loc}_{(x,u)}\mid(\theta,\Phi,\theta)\in\mathcal G_\theta\big\}.$$

The *orbit* $\mathcal O_\theta:=\mathrm t\big(\mathrm s^{-1}(\theta)\big)$ of θ is the subset of values of the arbitrary-element tuple such that the corresponding systems in the class $\mathcal L|_{\mathcal S}$ are similar to $\mathcal L_\theta$ with respect to point transformations.

Denote by ϖ and ϖ^r the projections from the space with the coordinates $(x,u_{(r)},\theta)$ to the spaces with the coordinates (x,u) and $(x,u_{(r)})$, respectively.

The *(usual) equivalence group* $G^\sim$ of the class $\mathcal L|_{\mathcal S}$ is the (pseudo)group of point transformations, $\mathcal T$, in the space with the coordinates $(x,u_{(r)},\theta)$ that are projectable to the spaces with the coordinates (x,u) and $(x,u_{(r)})$ with $\varpi^r_*\mathcal T$ being the standard prolongation of $\varpi_*\mathcal T$ to rth order jets $(x,u_{(r)})$ and that map the class $\mathcal L|_{\mathcal S}$ onto itself. The group $G^\sim$ can be considered to act in the space with the coordinates $(x,u_{(r')},\theta)$, where $r'<r$, if the arbitrary-element tuple depends only on $(x,u_{(r')})$. The notion of usual equivalence group can be generalized in several ways by weakening the specific restrictions on equivalence transformations, which are their projectability and their locality with respect to arbitrary elements. This gives the notions of generalized equivalence group and extended equivalence group, respectively, or the notions of extended generalized equivalence group if both restrictions are weakened simultaneously [138, 195, 222, 239, 248, 294, 300].

The *action groupoid* $\mathcal G^{G^\sim}$ of the equivalence group $G^\sim$ of the class $\mathcal L|_{\mathcal S}$,

$$\mathcal G^{G^\sim}:=\big\{(\theta,\varpi_*\mathcal T,\mathcal T_*\theta)\mid\theta\in\mathcal S,\ \mathcal T\in G^\sim\big\},$$

is a subgroupoid of the equivalence groupoid $\mathcal G^\sim$ of this class, $\mathcal G^{G^\sim}\subseteq\mathcal G^\sim$, with the same object set $\mathcal{S}$. We say that an admissible transformation $\mathcal{T}$ in the class $\mathcal{L}|_\mathcal{S}$ is generated by an equivalence transformation of this class if $\mathcal{T}\in\mathcal{G}^{G^\sim}$.

The *fundamental groupoid* $\mathcal{G}^{\rm f}$ of the class $\mathcal{L}|_\mathcal{S}$ is the disjoint union of the vertex groups $\mathcal{G}_\theta$, $\theta\in\mathcal{S}$, $\mathcal{G}^{\rm f}=\sqcup_{\theta\in\mathcal{S}}\mathcal{G}_\theta$. Since it has the same object set $\mathcal{S}$ and the same vertex groups as $\mathcal{G}^\sim$ and $\mathcal{T}^{-1}\mathcal{G}_{\tilde\theta}\mathcal{T}=\mathcal{G}_\theta$ for any $\mathcal{T}\in\mathcal{G}(\theta,\tilde\theta)$, it is a normal subgroupoid of the equivalence groupoid $\mathcal{G}^\sim$, which is also called the *fundamental subgroupoid* of $\mathcal{G}^\sim$. In other words, the groupoid $\mathcal{G}^{\rm f}$ is constituted by the admissible transformations generated by point symmetry transformations of systems from $\mathcal{L}|_\mathcal{S}$, $\mathcal{G}^{\rm f}:=\big\{(\theta,\Phi,\theta)\mid\theta\in\mathcal{S},\ \Phi\in G_\theta\big\}$.

The *kernel point symmetry group* $G^\cap:=\cap_{\theta\in\mathcal{S}}G_\theta$ of systems from the class $\mathcal{L}|_\mathcal{S}$, which consists of the common point symmetries of these systems, can be associated with the normal subgroup $\tilde G^\cap$ of $G^\sim$ whose elements are obtained from elements of $G^\cap$ by the standard prolongation to r'th order jets $(x,u_{(r')})$ and the trivial prolongation to the arbitrary-element tuple θ, $G^\cap=\varpi_*\tilde G^\cap$. Thus, $\tilde G^\cap$ is the unfaithful subgroup of $G^\sim$ under the action on $\mathcal{L}|_\mathcal{S}$.

The s-, the t- and the conjugation actions of $G^\sim$ on $\mathcal{G}^\sim$ respectively defined by $\mathcal{T}=(\theta,\Phi,\tilde\theta)\overset{\mathcal{T}}{\mapsto}(\mathcal{T}_*\theta,\Phi\circ(\varpi_*\mathcal{T})^{-1},\tilde\theta)$, $(\theta,(\varpi_*\mathcal{T})\circ\Phi,\mathcal{T}_*\tilde\theta)$, $(\mathcal{T}_*\theta,(\varpi_*\mathcal{T})\circ\Phi\circ(\varpi_*\mathcal{T})^{-1},\mathcal{T}_*\tilde\theta)$ for any $\mathcal{T}\in G^\sim$ and for any $\mathcal{T}=(\theta,\Phi,\tilde\theta)\in\mathcal{G}^\sim$, induce several equivalence relations on $\mathcal{G}^\sim$ (s-$G^\sim$-equivalence, t-$G^\sim$-equivalence, $G^\sim$-conjugation and $G^\sim$-equivalence).

Definition 1.1. Admissible transformations $\mathcal{T}^1=(\theta^1,\Phi^1,\tilde\theta^1)$ and $\mathcal{T}^2=(\theta^2,\Phi^2,\tilde\theta^2)$ in the class $\mathcal{L}|_\mathcal{S}$ are called *conjugate* with respect to the equivalence group $G^\sim$ of this class if there exists $\mathcal{T}\in G^\sim$ such that $\theta^2=\mathcal{T}_*\theta^1$, $\tilde\theta^2=\mathcal{T}_*\tilde\theta^1$ and $\Phi^2=(\varpi_*\mathcal{T})\circ\Phi^1\circ(\varpi_*\mathcal{T})^{-1}$. Admissible transformations $\mathcal{T}^1$ and $\mathcal{T}^2$ are called $G^\sim$*-equivalent* if there exist $\mathcal{T},\tilde{\mathcal{T}}\in G^\sim$ such that $\theta^2=\mathcal{T}_*\theta^1$, $\tilde\theta^2=\tilde{\mathcal{T}}_*\tilde\theta^1$ and $\Phi^2=(\varpi_*\tilde{\mathcal{T}})\circ\Phi^1\circ(\varpi_*\mathcal{T})^{-1}$. If additionally $\tilde{\mathcal{T}}={\rm id}_{(x,u_{(r)},\theta)}$ (resp. $\mathcal{T}={\rm id}_{(x,u_{(r)},\theta)}$), then the admissible transformations $\mathcal{T}^1$ and $\mathcal{T}^2$ are called s-$G^\sim$*-equivalent* (resp. t-$G^\sim$*-equivalent*).

A different terminology was used in [248], where the stronger equivalence relation of $G^\sim$-conjugation of admissible transformations was called $G^\sim$-equivalence, whereas in the present paper we use a weaker notion of $G^\sim$-equivalence of admissible transformations. An admissible transformation in $\mathcal{L}|_\mathcal{S}$ belongs to $\mathcal{G}^{G^\sim}$ if and only if this admissible transformation is $G^\sim$-equivalent in the above sense to the identity admissible transformation with the same source system.

Since the fundamental groupoid is a normal subgroupoid of $\mathcal{G}^\sim$, the Frobenius product $\mathcal{G}^\mathrm{f}\star\mathcal{G}^{G^\sim}=\big\{\mathcal{T}\star\mathcal{T}'\mid\mathcal{T}\in\mathcal{G}^\mathrm{f},\ \mathcal{T}'\in\mathcal{G}^{G^\sim},\ \mathrm{t}(\mathcal{T})=\mathrm{s}(\mathcal{T}')\big\}$ is a subgroupoid of $\mathcal{G}^\sim$, which coincides with the image of $\mathcal{G}^\mathrm{f}$ under the s-action (resp. the t-action) of $G^\sim$ on $\mathcal{G}^\sim$.

There are several kinds of classes of differential equations that are convenient for group classification by the algebraic method in different ways [21, 178, 239, 248].

Definition 1.2. The class $\mathcal{L}|_\mathcal{S}$ is called *normalized* if $\mathcal{G}^{G^\sim}=\mathcal{G}^\sim$. It is called *semi-normalized* if $\mathcal{G}^\mathrm{f}\star\mathcal{G}^{G^\sim}=\mathcal{G}^\sim$. Depending on the kind of the equivalence group $G^\sim$ (the usual, the generalized, the extended or the extended generalized equivalence group of $\mathcal{L}|_\mathcal{S}$), we distinguish the (semi-)normalization in the usual, the generalized, the extended or the extended generalized sense.

Definition 1.3. Let $\mathcal{G}^H$ be the action groupoid of a subgroup H of $G^\sim$. Suppose that a family $N_\mathcal{S}:=\{N_\theta<G_\theta\mid\theta\in\mathcal{S}\}$ of subgroups of the point symmetry groups G_θ with the associated subgroups $\mathcal{N}_\theta:=\{(\theta,\Phi,\theta)\mid\theta\in\mathcal{S},\ \Phi\in N_\theta\}$ of the vertex groups $\mathcal{G}_\theta$ satisfies the property $\mathcal{T}\mathcal{N}_\theta=\mathcal{N}_{\mathcal{T}\theta}\mathcal{T}$ for any $\theta\in\mathcal{S}$ and for any $\mathcal{T}\in\mathcal{G}^H$ with $\mathrm{s}(\mathcal{T})=\theta$. Then the Frobenius product

$$\mathcal{N}^\mathrm{f}\star\mathcal{G}^H=\big\{\mathcal{T}\star\mathcal{T}'\mid\mathcal{T}\in\mathcal{N}^\mathrm{f},\ \mathcal{T}'\in\mathcal{G}^H,\ \mathrm{t}(\mathcal{T})=\mathrm{s}(\mathcal{T}')\big\},$$

with $\mathcal{N}^\mathrm{f}:=\sqcup_{\theta\in\mathcal{S}}\mathcal{N}_\theta$ is a subgroupoid of $\mathcal{G}^\sim$, which coincides with the image of $\mathcal{N}^\mathrm{f}$ under the s-action (resp. the t-action) of H on $\mathcal{G}^\sim$. If $\mathcal{N}^\mathrm{f}\star\mathcal{G}^H=\mathcal{G}^\sim$, we call the class $\mathcal{L}|_\mathcal{S}$ *semi-normalized with respect to the subgroup H of $G^\sim$ and the family $N_\mathcal{S}$ of subgroups of the point symmetry groups*. If additionally

$\mathcal{G}^H \cap \mathcal{N}^{\mathrm{f}} = \mathcal{S} \rightrightarrows \mathcal{S}$, then the class $\mathcal{L}|_{\mathcal{S}}$ is called *disjointedly semi-normalized with respect to the subgroup H of $G^{\sim}$ and the family $N_{\mathcal{S}}$ of subgroups of the point symmetry groups.*

If $H = G^{\sim}$ and $N_\theta = \{\mathrm{id}_{(x,u)}\}$ for any $\theta \in \mathcal{S}$, then a class (disjointedly) semi-normalized with respect to the group H and the family $N_{\mathcal{S}}$ is literally normalized. If $H = G^{\sim}$ and $N_\theta = G_\theta$ for any $\theta \in \mathcal{S}$, then a class semi-normalized with respect to the group H and the family $N_{\mathcal{S}}$ is literally semi-normalized. It is obvious that a normalized class is semi-normalized.

1.3. Classification of Admissible Transformations and Group Classification Problems

The most powerful method for describing admissible transformations within a class of differential equations is still the *direct method*, which is based on the definition of admissible transformations. Applying this method to the class $\mathcal{L}|_{\mathcal{S}}$, we consider an arbitrary pair $(\theta, \tilde{\theta}) \in \mathcal{S} \times \mathcal{S}$ and a point transformation in the space with coordinates (x, u) of the most general form Φ: $\tilde{x} = X(x, u)$, $\tilde{u} = U(x, u)$ with nonzero Jacobian $|\partial(X, U)/\partial(x, u)|$ and assume that $\Phi_*\mathcal{L}_\theta = \mathcal{L}_{\tilde{\theta}}$. Expressing the required derivatives of $\tilde{u}$ with respect to $\tilde{x}$ in terms of derivatives of u with respect to x using the chain rule, we substitute the derived expressions into the system $\mathcal{L}_{\tilde{\theta}}$, obtaining the system $(\Phi^{-1})_*\mathcal{L}_{\tilde{\theta}}$, which should be identically satisfied by solutions of the system $\mathcal{L}_\theta$. To take into account the last condition, we fix a ranking of derivatives of u that is consistent with the structure of $\mathcal{L}_\theta$, substitute the expressions for the leading derivatives of u in view of the system $\mathcal{L}_\theta$ and its differential consequences into $(\Phi^{-1})_*\mathcal{L}_{\tilde{\theta}}$ and split the resulting system with respect to the involved parametric derivatives of u. As a result, we obtain a system that implies both the expression of $\tilde{\theta}$ via (θ, X, U) and the system DE of determining equations for components of Φ. The system DE involves only the arbitrary-element tuple θ (resp. $(\Phi^{-1})_*\tilde{\theta}$).

Assuming θ varying within $\mathcal{S}$ and splitting with respect to derivatives of θ in view of the auxiliary system defining the set $\mathcal{S}$,[1.1] we get the system DE$^\sim$ of determining equations for the (x,u)-components of usual equivalence transformations. After finding the (x,u)-components via the integration of DE$^\sim$, the θ-component of usual equivalence transformations is obtained from the above expression for $\tilde\theta$. As a result, we construct the usual equivalence group $G^\sim$ of the class $\mathcal{L}|_\mathcal{S}$.

If the solution sets of DE and DE$^\sim$ coincide, then $\mathcal{G}^\sim = \mathcal{G}^{G^\sim}$, i.e., the class $\mathcal{L}|_\mathcal{S}$ is normalized, which completes the description of the equivalence groupoid $\mathcal{G}^\sim$. The first example of such description in the literature was given for the normalized class of generalized Burgers equations of the form $u_t + uu_x + f(t,x)u_{xx} = 0$ in [159] although the normalization property was implicitly used there.

Otherwise, the class $\mathcal{L}|_\mathcal{S}$ is not normalized, and integrating DE, which can be carried out up to $G^\sim$-equivalence of admissible transformations, is a complicated problem. A number of various techniques can be used to simplify the solution of this problem. Below we present some of them.

Partition of classes. Let the set $\mathcal{S}$ be represented as a disjoint union of its subsets, $\mathcal{S} = \sqcup_{\gamma\in\Gamma}\mathcal{S}_\gamma$ with some index set Γ, where each of the subsets $\mathcal{S}_\gamma$ is singled out from $\mathcal{S}$ by additional constraints, which are differential equations or differential inequalities. The partition of $\mathcal{S}$ is equivalent to the partition of the class $\mathcal{L}|_\mathcal{S}$ into the subclasses $\mathcal{L}|_{\mathcal{S}_\gamma}$ with γ running through Γ, $\mathcal{L}|_\mathcal{S} = \sqcup_{\gamma\in\Gamma}\mathcal{L}|_{\mathcal{S}_\gamma}$. Denote by $G^\sim_\gamma$ and by $\mathcal{G}^\sim_\gamma$ the equivalence group and the equivalence groupoid of the subclass $\mathcal{L}|_{\mathcal{S}_\gamma}$, respectively.

If systems from different subclasses of the partition are not related by point transformations, then the partition of the class $\mathcal{L}|_\mathcal{S}$ induces the partition $\mathcal{G}^\sim = \sqcup_{\gamma\in\Gamma}\mathcal{G}^\sim_\gamma$ of its equivalence groupoid. In general, the structure

[1.1] This means that we set a ranking among the derivatives of θ that is consistent with structure of the auxiliary system, solve this system jointly with its differential consequences for the leading derivatives of θ, substitute the derived expressions into DE and split the obtained system with respect to the parametric derivatives of θ.

of the groupoid of a subclass may even be more complicated than that of the entire class. This is why a preliminary analysis of the system DE is needed for an appropriate partition of the class $\mathcal{L}|_{\mathcal{S}}$, where for any $\gamma\in\Gamma$ the structure of $\mathcal{G}^\sim_\gamma$ is simpler than the structure of $\mathcal{G}^\sim$. Then we can find the subgroupoids $\mathcal{G}^\sim_\gamma$ separately and unite them. The best kind of partitions is given by partitions into normalized subclasses, for which $\mathcal{G}^\sim_\gamma=\mathcal{G}^{G^\sim_\gamma}$ and thus $\mathcal{G}^\sim=\sqcup_{\gamma\in\Gamma}\mathcal{G}^{G^\sim_\gamma}$ [239, 248].

There are several generalizations of the partition technique.

Disjoint subclasses may be related by point transformations, and thus the partition of the class $\mathcal{L}|_{\mathcal{S}}$ into the subclasses $\mathcal{L}|_{\mathcal{S}_\gamma}$ does not induce the partition of the equivalence groupoid $\mathcal{G}^\sim$ into the equivalence groupoids $\mathcal{G}^\sim_\gamma$. Consider a simple situation, where we have a partition of $\mathcal{L}|_{\mathcal{S}}$ into normalized subclasses $\mathcal{L}|_{\mathcal{S}_\gamma}$, $\gamma\in\Gamma$, and for some fixed $\gamma_0\in\Gamma$ and for each $\gamma\in\Gamma$ there exists a point transformation Φ_γ that maps $\mathcal{L}|_{\mathcal{S}_\gamma}$ onto $\mathcal{L}|_{\mathcal{S}_{\gamma_0}}$. We can assume that $\Phi_{\gamma_0}=\mathrm{id}_{(x,u)}$. In fact, for any $\gamma\in\Gamma$ the normalization of $\mathcal{L}|_{\mathcal{S}_\gamma}$ follows from the normalization of $\mathcal{L}|_{\mathcal{S}_{\gamma_0}}$ and the existence of Φ_γ. Then

$$\begin{aligned}\mathcal{G}^\sim=\Big\{&\big((\Phi_\gamma^{-1})_*\theta,\Phi_{\gamma'}^{-1}\circ(\varpi\mathcal{T})\circ\Phi_\gamma,(\Phi_{\gamma'})_*(\mathcal{T}_*\theta)\big)\;\Big|\\ &\qquad\theta\in\mathcal{S}_{\gamma_0},\ \mathcal{T}\in G^\sim_{\gamma_0},\gamma,\gamma'\in\Gamma\Big\}.\end{aligned}\tag{1.1}$$

This structure is admitted by the groupoid of the class (4.46), where the parameter σ plays the role of γ, see Remark 4.22 below.

The condition that the class $\mathcal{L}|_{\mathcal{S}}$ is a disjoint union of appropriate subclasses can be weakened by allowing a proper intersection of subclasses in the union. Thus, in [294] a class of variable-coefficient reaction–diffusion equations with power nonlinearities was represented as a non-disjoint union of normalized subclasses, and its groupoid was proved to be constituted by the admissible transformations for the action groupoids of the subclasses and the compositions of such composable admissible transformations from the action groupoids of different subclasses with nonempty intersections.

Construction of generalized/extended/extended generalized equivalence *group.* If the class $\mathcal{L}|_\mathcal{S}$ is not normalized in the usual sense, one can try to describe its equivalence groupoid $\mathcal{G}^\sim$ via finding a generalized counterpart of the usual equivalence group $G^\sim$, with respect to which the class $\mathcal{L}|_\mathcal{S}$ is normalized in the corresponding sense [222, 248].

Mappings between classes. Suppose that there are a class $\mathcal{L}'|_{\mathcal{S}'}$ of (systems of) differential equations with the same independent and dependent variables x and u as systems from the class $\mathcal{L}|_\mathcal{S}$ and a family of point transformations $\mathcal{F} = \{\Psi^\theta \mid \theta \in \mathcal{S}\}$ such that $\Psi^\theta_*\mathcal{L}_\theta \in \mathcal{L}'|_{\mathcal{S}'}$ for any $\theta \in \mathcal{S}$, and for any $\theta' \in \mathcal{S}'$ there exists $\theta \in \mathcal{S}$ with $\Psi^\theta_*\mathcal{L}_\theta = \mathcal{L}'_{\theta'}$. Then we say that the family $\mathcal{F}$ generates the mapping $\mathcal{F}_*$ from the class $\mathcal{L}|_\mathcal{S}$ onto the class $\mathcal{L}'|_{\mathcal{S}'}$, where $\mathcal{F}_*\mathcal{L}_\theta := \Psi^\theta_*\mathcal{L}_\theta$, or, equivalently, the mapping $\mathcal{F}_*\colon \mathcal{S} \to \mathcal{S}'$ with $\mathcal{F}_*\theta = \theta'$ if $\Psi^\theta_*\mathcal{L}_\theta = \mathcal{L}'_{\theta'}$; see [248, 300] for the first explicit discussions of mappings between the classes. Via $\mathcal{F}_*$, the family $\mathcal{F}$ also induces the mapping from the equivalence groupoid $\mathcal{G}^\sim$ of the class $\mathcal{L}|_\mathcal{S}$ to the equivalence groupoid $\mathcal{G}^{\sim\prime}$ of the class $\mathcal{L}'|_{\mathcal{S}'}$ that is defined by

$$\mathcal{G}^\sim \ni \mathcal{T} = (\theta, \Phi, \tilde\theta) \overset{\mathcal{F}_*}{\mapsto} \big((\Psi^\theta)_*\theta, \Psi^{\tilde\theta} \circ \Phi \circ (\Psi^\theta)^{-1}, (\Psi^{\tilde\theta})_*\tilde\theta\big) \in \mathcal{G}^{\sim\prime}.$$

We will denote this mapping by the same symbol as the corresponding mapping between classes. The mapping $\mathcal{F}_*\colon \mathcal{G}^\sim \to \mathcal{G}^{\sim\prime}$ is in fact a groupoid homomorphism since

$$\mathcal{F}_*(\mathcal{T}_1 \star \mathcal{T}_2) = (\mathcal{F}_*\mathcal{T}_1) \star (\mathcal{F}_*\mathcal{T}_2), \quad \mathcal{F}_*(\mathrm{id}_\theta) = \mathrm{id}_{\mathcal{F}_*\theta}, \quad \mathcal{F}_*(\mathcal{T}^{-1}) = (\mathcal{F}_*\mathcal{T})^{-1}$$

for any $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{G}^\sim$, any $\theta \in \mathcal{S}$ and any $\mathcal{T} \in \mathcal{G}^\sim$. Moreover, this homomorphism is surjective. Indeed, take any $\mathcal{T}' = (\theta', \Phi', \tilde\theta') \in \mathcal{G}^{\sim\prime}$. By the choice of the family $\mathcal{F}$, there exist $\theta, \tilde\theta \in \mathcal{S}$ such that $\mathcal{F}_*(\theta) = \theta'$ and $\mathcal{F}_*(\tilde\theta) = \tilde\theta'$. The triple $\mathcal{T} = (\theta, \Phi, \tilde\theta)$ with $\Phi = (\Psi^{\tilde\theta})^{-1} \circ \Phi' \circ \Psi^\theta$ belongs to $\mathcal{G}^\sim$, and $\mathcal{F}_*\mathcal{T} = \mathcal{T}'$.

Under an appropriate choice of $\mathcal{F}$, the structure of $\mathcal{G}^{\sim\prime}$ is simpler than the structure of $\mathcal{G}^\sim$. Then after the study of $\mathcal{G}^{\sim\prime}$, we can pull back obtained results with respect to $\mathcal{F}$ and thus get results on $\mathcal{G}^\sim$. For example, an appropriate partition of a class into its subclasses can become evident only after a mapping of this class to another class [223]. The complete group classification of $\mathcal{L}|_{\mathcal{S}}$ up to $\mathcal{G}^{\sim}$-equivalence can easily be derived from the analogous classification of $\mathcal{L}|_{\mathcal{S}'}$ using the pullback by $\mathcal{F}$. In this way, the complete group classifications of the classes of (1+1)-dimensional Kolmogorov and Fokker–Planck equations modulo the general point equivalence were obtained from the classical group classification of the class of linear heat equations with potentials, see Corollaries 7 and 17 in [249]. Using mappings between classes for deriving complete group classifications up to $G^{\sim}$-equivalence needs a more delicate consideration [300].

An important particular case of mappings between classes is given by mappings of classes to their subclasses that are generated by subgroups of the corresponding equivalence groups. Let $\mathcal{S}'$ be a subset of $\mathcal{S}$ that is singled out from $\mathcal{S}$ by additional auxiliary differential equations or inequalities with respect to the arbitrary-element tuple θ. Thus, $\mathcal{L}|_{\mathcal{S}'}$ is a subclass of the class $\mathcal{L}|_{\mathcal{S}}$, and its equivalence groupoid $\mathcal{G}^{\sim\prime}$ is a subgroupoid of the equivalence groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$. Suppose that for some subgroup H of $G^{\sim}$ each orbit of the action groupoid $\mathcal{G}^{H}$ intersects $\mathcal{S}'$ by a single θ'. Denote by Ψ^{θ} the point transformation $\varpi_*\mathcal{T}$ with $\mathcal{T}\in H$ such that $\mathcal{T}_*\theta\in\mathcal{S}'$. Then the family $\mathcal{F}=\{\Psi^{\theta}\mid\theta\in\mathcal{S}\}$ satisfies the required conditions to generate the corresponding mapping $\mathcal{F}_*\colon\mathcal{L}|_{\mathcal{S}}\to\mathcal{L}|_{\mathcal{S}'}$ and the corresponding surjective homomorphism $\mathcal{F}_*\colon\mathcal{G}^{\sim}\to\mathcal{G}^{\sim\prime}$. In practice, such mappings are realized via gauging arbitrary elements by equivalence transformations.

Conditional equivalence groups. The equivalence group of a subclass of the class $\mathcal{L}|_{\mathcal{S}}$ is called a conditional equivalence group of this class. The conditional equivalence group $G^{\sim\prime}$ of $\mathcal{L}|_{\mathcal{S}}$ associated with the subclass $\mathcal{L}|_{\mathcal{S}'}$ is called maximal if for any subclass of $\mathcal{L}|_{\mathcal{S}}$ properly containing $\mathcal{L}|_{\mathcal{S}'}$, its equivalence group does not contain $G^{\sim\prime}$. The equivalence group $G^{\sim}$ of the entire class $\mathcal{L}|_{\mathcal{S}}$ acts on subclasses of $\mathcal{L}|_{\mathcal{S}}$ simultaneously with their equivalence groups, and the set of maximal conditional equivalence groups of $\mathcal{L}|_{\mathcal{S}}$ is closed under this action. Hence maximal conditional equivalence groups of $\mathcal{L}|_{\mathcal{S}}$ can be classified modulo $G^{\sim}$-equivalence. This classification can be a step in the description of $\mathcal{G}^{\sim}$ [248]. For some classes, this lone step gives the complete description of the corresponding equivalence groupoids [221, 222]. The classification of maximal conditional equivalence groups of the class $\mathcal{L}|_{\mathcal{S}}$ can be combined with a partition of $\mathcal{L}|_{\mathcal{S}}$ into subclasses that is consistent with the structure of the set of such groups. Generalized versions of conditional equivalence groups also can be considered [221, 306].

Generating set of admissible transformations. A set $\mathcal{B} = \{\mathcal{T}_\gamma \in \mathcal{G}^{\sim} \mid \gamma \in \Gamma\}$, where Γ is an index set, is called a generating set of admissible transformations for the class $\mathcal{L}|_{\mathcal{S}}$ up to $G^{\sim}$-equivalence if any admissible transformation of this class can be represented as the composition of a finite number of elements of the set $\mathcal{B} \cup \hat{\mathcal{B}} \cup \mathcal{G}^{G^{\sim}}$, where $\hat{\mathcal{B}}$ is the set of inverses of admissible transformations from $\mathcal{B}$, $\hat{\mathcal{B}} := \{\mathcal{T}^{-1} \mid \mathcal{T} \in \mathcal{B}\}$. To make the set $\mathcal{B}$ as small as possible, it is natural to choose $\mathcal{B}$ as a subset of $\mathcal{G}^{\sim} \setminus \mathcal{G}^{G^{\sim}}$. Moreover, if a canonical representative in a coset of $G^{\sim}$-equivalent admissible transformations can be assigned, only this representative should be selected from the coset for including in $\mathcal{B}$.

We call admissible transformations $\mathcal{T}_1$ and $\mathcal{T}_2$ for the class $\mathcal{L}|_{\mathcal{S}}$ *composable up to $G^{\sim}$-equivalence* if an admissible transformation that is $G^{\sim}$-equivalent to $\mathcal{T}_1$ is composable with $\mathcal{T}_2$ or, equivalently, if $\mathcal{T}_1$ is composable with an admissible transformation that is $G^{\sim}$-equivalent to $\mathcal{T}_2$. It happens if and only if there exists $\mathfrak{T} \in G^{\sim}$ such that $\mathfrak{T}_*\big(\mathrm{t}(\mathcal{T}_1)\big) = \mathrm{s}(\mathcal{T}_2)$. We call a subset $\mathcal{B}$ of $\mathcal{G}^{\sim}$ *self-consistent with respect to $G^{\sim}$-equivalence* if the composability of elements of $\mathcal{B} \cup \hat{\mathcal{B}}$ up to $G^{\sim}$-equivalence implies their usual composability. (The converse implication always holds.) A necessary condition for the self-consistency of $\mathcal{B}$ is the equality

$$\big(\mathrm{s}(\mathcal{B}) \times \mathrm{t}(\mathcal{B})\big) \cap (\mathrm{s} \times \mathrm{t})\big(\mathcal{G}^{G^{\sim}} \setminus \mathcal{G}^{\mathrm{f}}\big) = \varnothing$$

meaning that there is no element of the action groupoid $\mathcal{G}^{G^{\sim}}$ with different source and target in $\mathrm{s}(\mathcal{B}) \times \mathrm{t}(\mathcal{B})$. If $\mathcal{B}$ is a self-consistent generating set of $\mathcal{G}^{\sim}$

with respect to $G^\sim$-equivalence, then any element of $\mathcal{G}^\sim$ is $G^\sim$-equivalent to the composition of a finite number of elements of $\mathcal{B}\cup\hat{\mathcal{B}}$. More specifically, then for any $\mathcal{T}\in\mathcal{G}^\sim$ there exist $n\in\mathbb{N}\cup\{0\}$, $\mathcal{T}_0,\mathcal{T}_{n+1}\in\mathcal{G}^{G^\sim}$ and $\mathcal{T}_1,\dots,\mathcal{T}_n\in\mathcal{B}\cup\hat{\mathcal{B}}$ such that $\mathcal{T}=\mathcal{T}_0\star\mathcal{T}_1\star\dots\star\mathcal{T}_n\star\mathcal{T}_{n+1}$. If additionally $\mathcal{B}$ is minimal up to $G^\sim$-equivalence, then this representation for $\mathcal{T}$ is unique up to the transformations

$$\tilde{\mathcal{T}}_0=\mathcal{T}_0\star\breve{\mathcal{T}},\quad \tilde{\mathcal{T}}_{n+1}=\mathcal{T}_n^{-1}\star\dots\star\mathcal{T}_1^{-1}\star\breve{\mathcal{T}}^{-1}\star\mathcal{T}_1\star\dots\star\mathcal{T}_n\star\mathcal{T}_{n+1}$$

with an arbitrary $\breve{\mathcal{T}}\in\mathcal{G}_{\mathrm{t}(\mathcal{T}_0)}$ if $n>0$ and under setting $\mathcal{T}_{n+1}=\mathrm{id}_{\mathrm{t}(\mathcal{T}_0)}$ if $n=0$.

Furcate splitting. This technique was suggested in [209] as a refinement of the direct method of group classification. Its essence is a special way of handling the system of determining equations for Lie symmetries of systems from the class under study depending on the possible number of independent constraints on values of θ that are induced by this system. This is why it can be extended to descriptions of other objects that are related to systems from classes of differential equations and are computed via solving certain systems of determining equations, including conservation laws [25], conditional equivalence groups [222] and generating sets of admissible transformations (see footnote 4.2 below). The method of furcate splitting can further be enhanced by involving algebraic techniques [23, 25].

(Bijective) functors between groupoids. Suppose that we construct an isomorphism between $\mathcal{G}^\sim$ and the equivalence groupoid $\tilde{\mathcal{G}}^\sim$ of a class $\tilde{\mathcal{L}}|_{\tilde{\mathcal{S}}}$, and the description of $\tilde{\mathcal{G}}^\sim$ has been known or it is easier or more convenient to describe the groupoid $\tilde{\mathcal{G}}^\sim$ than the original groupoid $\mathcal{G}^\sim$. For the latter option, for example, some computation techniques that are relevant for $\tilde{\mathcal{G}}^\sim$ might be inapplicable to $\mathcal{G}^\sim$. Here it is not necessary for the classes $\mathcal{L}|_{\mathcal{S}}$ and $\tilde{\mathcal{L}}|_{\tilde{\mathcal{S}}}$ to be related by a family of point transformations. Then the description of $\tilde{\mathcal{G}}^\sim$ implies the description of $\mathcal{G}^\sim$.

The technique involving bijective functors is effectively applied to the study of equivalence groupoids of classes of differential equations for the first time in [292].

The infinitesimal counterparts of the (pseudo)groups G^θ, $G^\cap$ and $G^\sim$ are the Lie algebras $\mathfrak{g}^\theta$, $\mathfrak{g}^\cap$ and $\mathfrak{g}^\sim$ that are constituted by the generators of local one-parameter subgroups of the corresponding groups and which are called the *maximal Lie invariance algebras* of the systems $\mathcal{L}_\theta$, the *kernel invariance algebra* of systems from the class $\mathcal{L}|_\mathcal{S}$ and the *equivalence algebra* of the class $\mathcal{L}|_\mathcal{S}$, respectively. Note that $\mathfrak{g}^\cap = \cap_{\theta\in\mathcal{S}}\mathfrak{g}_\theta$.

The *(complete) group classification problem* for the class $\mathcal{L}|_\mathcal{S}$ up to $G^\sim$-equivalence (resp. up to $\mathcal{G}^\sim$-equivalence) is to find $\mathfrak{g}^\cap$ and an exhaustive list of $G^\sim$-inequivalent (resp. $\mathcal{G}^\sim$-inequivalent) values of θ jointly with the corresponding algebras $\mathfrak{g}_\theta$ for which $\mathfrak{g}_\theta \neq \mathfrak{g}^\cap$. An admissible transformation from $\mathcal{G}^\sim \setminus \mathcal{G}^{G^\sim}$ between systems from the final group classification list modulo $G^\sim$-equivalence is called an *additional equivalence transformation*. Supplementing the group classification up to $G^\sim$-equivalence with the complete set of additional equivalence transformations results in the group classification up to $\mathcal{G}^\sim$-equivalence.

Any version of the algebraic method of group classification in fact reduces to the classification, modulo $G^\sim$-equivalence, of certain subalgebras contained by the span $\mathfrak{g}_{\langle\rangle} := \langle\mathfrak{g}_\theta, \theta\in\mathcal{S}\rangle$. The efficiency of using the algebraic method depends on additional conditions satisfied by the class $\mathcal{L}|_\mathcal{S}$, in particular, how consistent the span $\mathfrak{g}_{\langle\rangle}$ is with $G^\sim$-equivalence [21, Section 12].

Normalized classes are the most convenient for group classification by the algebraic method. If the class $\mathcal{L}|_\mathcal{S}$ is normalized, then $\mathfrak{g}_{\langle\rangle} \subseteq \varpi_*\mathfrak{g}^\sim$, and the solution of the complete group classification problem for this class reduces to the classification of appropriate subalgebras of $\mathfrak{g}^\sim$ whose pushforwards by ϖ can be qualified as the maximal Lie invariance algebras of systems from $\mathcal{L}|_\mathcal{S}$. Since then $G^\sim$-equivalence coincides with $\mathcal{G}^\sim$-equivalence, it is obvious that there are no additional equivalence transformations between Lie-symmetry extensions classified modulo $G^\sim$-equivalence. Moreover, it is inessential which of the two equivalences is used in the course of the classification. The above is also true if the class $\mathcal{L}|_{\mathcal{S}}$ is semi-normalized with respect to a subgroup H of $G^{\sim}$ and a family $N_{\mathcal{S}}$ of subgroups of the point symmetry groups, and additionally the subgroup H and the family $N_{\mathcal{S}}$ are known. In this case, we call the class $\mathcal{L}|_{\mathcal{S}}$ *definitely semi-normalized*, and looking for $G^{\sim}$-inequivalent subalgebras of $\mathfrak{g}^{\sim}$ is substituted in the algebraic method by looking for H-inequivalent subalgebras of the infinitesimal counterpart $\mathfrak{h}$ of H [178]. The pure semi-normalization of $\mathcal{L}|_{\mathcal{S}}$ at least guarantees that the group classification $\mathcal{L}|_{\mathcal{S}}$ up to $G^{\sim}$-equivalence coincides with that up to $\mathcal{G}^{\sim}$-equivalence.

If the class $\mathcal{L}|_{\mathcal{S}}$ is not normalized, then some Lie-symmetry extensions within this class are not related to subalgebras of its equivalence algebra $\mathfrak{g}^{\sim}$.

Definition 1.4. We call the maximal Lie invariance algebra $\mathfrak{g}_\theta$ of a system $\mathcal{L}_\theta$ from the class $\mathcal{L}|_{\mathcal{S}}$ *regular* in this class if there exists a subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\sim}$ such that $\mathfrak{g}_\theta = \varpi_*\mathfrak{s}$, and *singular* in $\mathcal{L}|_{\mathcal{S}}$ otherwise.

If $\mathfrak{g}_\theta = \mathfrak{g}^{\cap}$, then the maximal Lie invariance algebra $\mathfrak{g}_\theta$ is regular in $\mathcal{L}|_{\mathcal{S}}$ since $\mathfrak{g}^{\cap} \subseteq \varpi_*\mathfrak{g}^{\sim}$.[1.2] If $\mathfrak{g}_\theta \neq \mathfrak{g}^{\cap}$ and $\mathfrak{g}_\theta$ is a regular (resp. singular) maximal Lie invariance algebra in the class $\mathcal{L}|_{\mathcal{S}}$, then we say that the pair constituted by the value of the arbitrary-element tuple and the algebra $\mathfrak{g}_\theta$ presents a regular (resp. singular) Lie-symmetry extension of $\mathfrak{g}^{\cap}$ in this class.

It is obvious that the sets of regular and singular Lie-symmetry extensions are separately invariant with respect to the action of $G^{\sim}$ but in general this is not the case for the action of $\mathcal{G}^{\sim}$. In other words, regular Lie-symmetry extensions are $G^{\sim}$-inequivalent to singular ones but may be $\mathcal{G}^{\sim}$-equivalent to them, see Remark 4.14 as an example on this claim. This also means that the Lie-symmetry extension for a system $\mathcal{L}_\theta$ with θ satisfying $\mathrm{s}^{-1}(\theta) \neq \mathrm{s}^{-1}(\theta) \cap \mathcal{G}^{G^{\sim}}$ or, equivalently, $\mathrm{t}^{-1}(\theta) \neq \mathrm{t}^{-1}(\theta) \cap \mathcal{G}^{G^{\sim}}$ (i.e., for a system being the source or the target of an admissible transformation that is not generated by an equivalence transformation in $\mathcal{L}|_{\mathcal{S}}$) may also be regular. This is definitely

[1.2]More specifically, the kernel invariance algebra $\mathfrak{g}^{\cap}$ is naturally embedded into $\mathfrak{g}^{\sim}$ via the standard prolongation of its elements to $u_{(r)}$ in view of the contact structure and the trivial prolongation to the arbitrary elements θ [72].

the case if the quotient of the set $\mathrm{s}^{-1}(\theta)$ with respect to t-$G^{\sim}$-equivalence of admissible transformations is discrete, as it appears for the regular Cases 14d and 19d of Table 4.6 below.

Using Definition 1.4, we suggest the following procedure of group classification for a non-normalized class $\mathcal{L}|_{\mathcal{S}}$ of differential equations within the framework of the algebraic method.

1. Describe the equivalence groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$ up to $G^{\sim}$-equivalence, e.g., via constructing a generating set $\mathcal{B}$ of admissible transformations. The further consideration simplifies if the set $\mathcal{B}$ is minimal and self-consistent with respect to $G^{\sim}$-equivalence.

2. Classify, modulo $\mathcal{G}^{\sim}$-equivalence, Lie symmetries of systems $\mathcal{L}_\theta$ with θ satisfying the condition $\mathrm{s}^{-1}(\theta) \neq \mathrm{s}^{-1}(\theta) \cap \mathcal{G}^{G^{\sim}}$ or, equivalently, $\mathrm{t}^{-1}(\theta) \neq \mathrm{t}^{-1}(\theta) \cap \mathcal{G}^{G^{\sim}}$. This leads to the complete list of $\mathcal{G}^{\sim}$-inequivalent Lie-symmetry extensions within the class $\mathcal{L}|_{\mathcal{S}}$ that are singular or regular but related to other Lie-symmetry extensions with elements from $\mathcal{G}^{\sim} \setminus \mathcal{G}^{G^{\sim}}$. Here both the direct and the algebraic methods of group classification might be applicable.

3. Carry out the (complete) preliminary group classification of the class $\mathcal{L}|_{\mathcal{S}}$. The optimized version of such classification includes the classification of candidates for appropriate subalgebras of the equivalence algebra $\mathfrak{g}^{\sim}$ up to $G^{\sim}$-equivalence and, whenever it is possible, the construction of systems from the class $\mathcal{L}|_{\mathcal{S}}$ that admit the projections of the above candidates by ϖ_* as their Lie invariance algebras. For each obtained system, we select the candidate that is maximal by inclusion; such a candidate always exists.

4. Merge the lists obtained in steps 2 and 3 and exclude repetitions up to $\mathcal{G}^{\sim}$-equivalence, which leads to the complete list of Lie-symmetry extensions within the class $\mathcal{L}|_{\mathcal{S}}$ up to $\mathcal{G}^{\sim}$-equivalence.

5. Extend the part of the list from step 4 that is related to the cases of step 2 by compositions of admissible transformations from the set $\mathcal{B}$ modulo $G^{\sim}$-equivalence. This gives the complete list of Lie-symmetry extensions within the class $\mathcal{L}|_{\mathcal{S}}$ up to $G^{\sim}$-equivalence. All possible additional equivalence transformations between cases in this list are generated by elements of $\mathcal{B}$ modulo $G^{\sim}$-equivalence.

The order of steps or even single operations may vary depending on the class of differential equations to be studied.

We demonstrate the application of this technique in Section 4.2.

Chapter 2

Equivalence Groupoids in Group Analysis of Second-Order Evolution Equations

A number of mathematical models in physics and biology are represented by (1+1)-dimensional second-order nonlinear evolution equations. Such models are used in such diverse fields as quantum field theory [78, pp. 294–341], physics of nanowire semiconductor devices [63], and population genetics [203]. Many nonlinear partial differential equations (PDEs) that are important for applications are parameterized by arbitrary elements (constants or functions) and constitute classes of PDEs. An important task is to study transformation properties of such classes. If two PDEs are connected by such a transformation, then associated objects like exact solutions, local conservation laws, and various kinds of symmetries of these equations are also related by the respective transformation. Such connected equations are called equivalent or similar [227]. In particular, the equivalence method allows one to construct exact solutions for variable coefficient PDEs using known exact solutions for their constant coefficient counterparts, see, e.g., [289, 290, 293]. At the same time, nondegenerate point transformations appear to be a useful tool not only for finding exact solutions but also for exhaustive solving group classifications problems (see, e.g., [178, 224, 248, 289, 290, 293]), design of physical parameterization schemes [242], and study of integrability [45, 148, 172, 304].

The core problem of group analysis is the classification of the reduction operators of differential equations. A reduction operator of a (1+1)-dimensional partial differential equation (PDE) with independent variables

t and x and the dependent variable u is a differential operator of the form $Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$, $(\tau,\xi) \neq (0,0)$, such that the corresponding invariant surface condition $Q[u] := \tau u_t + \xi u_x - \eta = 0$ leads to the construction of ansatz that reduces the number of independent variables of the respective equation by one. Thus, such operators allows one to reduce a (1+1)-dimensional PDE to an ordinary differential equation.

The reduction method is an efficient tool for seeking exact solutions of nonlinear PDEs as the general theory of integration of such equations does not exist. Among the most known reduction techniques are the prominent Lie reduction method that originates from works by S. Lie and the nonclassical reduction method suggested by G.W. Bluman in [28] (see also [35]). The criterion of "nonclassical" invariance was firstly formulated in [93] and the rigorous theory of the nonclassical reduction method, *theory of reduction modules*, was recently developed in [42]. The nonclassical reduction operators are also called *nonclassical symmetries* [218], *conditional symmetries* [185] and *Q-conditional symmetries* [91] (see the related discussion in [174] and some more research papers of interest [87,88,108,212,219,323]).

There is also a direct reduction method based on substitution of ansatz into a PDE in question [60,86]. A rigorous definition of reduction of PDEs was presented in [335]. It was proved therein that the direct approach of reduction, taken in its full generality, is equivalent to the non-classical (conditional symmetry) approach. The enhanced proof can be found in [42].

Therefore an important problem arises: to classify reduction operators for those classes of PDEs that are of interest for applications. Classification of Lie reduction operators is known as *group classification problem* and appears to be the central problem of the group analysis. The main benefit of Lie method is that the determining system for finding coefficients of Lie reduction operators consists of linear PDEs. That is why the construction of Lie symmetry operators for a fixed PDE is a routine task usually which can be performed using the packages of symbolic computations. See, for example, the Maple-based GEM package [57,58]. Unfortunately the group classification problems can be solved automatically using symbolic computations only for certain classes having simple structures. The majority of cases requires usage of the modern techniques of the group analysis such as mapping between classes of PDEs, gauging of arbitrary elements of the class, application of various types of equivalence groups, etc. (see, e.g., [245,294,300]).

The nonclassical reduction operators can be of regular and singular types. The problem of finding singular reduction operators reduces to solving an initial PDE, therefore this case is called the “no-go” case and often omitted in consideration (see more about “no-go” case in [42, 92, 173, 240, 241, 333]). But even in the case of regular nonclassical reduction operators the problem of their classification for classes of PDEs is difficult. This is due to the fact that finding coefficients of nonclassical reduction operators one requires to solve a system of nonlinear PDEs. That is why this method more often results in the complete solution when applied to a fixed PDE rather than to a class of PDEs. Indeed, there are quite few examples of successful classification of nonclassical reduction operators (even regular ones) in the literature. At the best of our knowledge, such classifications are performed for the class of semilinear diffusion equations with a source $u_t = u_{xx} + f(u)$ [13,61,90], the class of nonlinear reaction–diffusion equations $u_t = (D(u)u_x)_x + f(u)$ for the cases of exponential and power low diffusivity [12], the class of nonlinear filtration equations $u_t = f(u_x)u_{xx}$ [252], the class of variable coefficient Huxley equations $u_t = u_{xx} + k(x)u^2(1-u)$ [44,143], and the class of generalized Burgers equations $u_t = uu_x + f(t,x)u_{xx}$ [233].

In Section 2.1 we study transformation properties of the general class of (1+1)-dimensional second-order evolution equations $u_t = H(t,x,u,u_x,u_{xx})$, $H_{u_{xx}} \neq 0$ and construct a chain of its nested normalized subclasses. A special attention is paid to a class of variable coefficient equations of reaction-diffusion–convection type. For all the considered classes of nonlinear evolution equations, we construct their equivalence groups, which can further be used for group analysis of these classes.

The group classification of a class of variable coefficient reaction–diffusion equations with exponential nonlinearities $f(x)u_t = (g(x)e^{nu}u_x)_x + h(x)e^{mu}$ is carried out in Section 2.2. The equivalence groupoid of this class is exhaustively described via finding the complete family of maximal normalized subclasses and the associated conditional equivalence groups. Limit processes between variable coefficient reaction–diffusion equations with power nonlinearities and those with exponential nonlinearities are simultaneously studied with limit processes between objects related to these equations, including Lie symmetries, exact solutions and conservation laws.

In Section 2.3, we classify conservation laws and potential symmetries of diffusion equations in a porous medium $u_t = ((u^n)_x + f(x)u^m)_x$, $n \neq 0$.

The class of generalized Fisher equations with time-dependent coefficients, $u_t = b(t)u_{xx} + a(t)u(1-u)$, $ab \neq 0$ is studied from Lie-symmetry point of view in Section 2.4. We find the equivalence groupoid of this class and perform its exhaustive group classification. Exact solutions of equations from this class are constructed using the equivalence method and the method of mapping between classes.

A class of the Newell–Whitehead–Segel equations $u_t = a^2(t)u_{xx} + b(t)u - c(t)u^3$, $ac \neq 0$, is studied with in Section 2.5. We describe its equivalence groupoid and classify Lie reduction operators and regular nonclassical reduction operators of equations from this class. The criterion of reducibility of variable coefficient Newell–Whitehead–Segel equations to their constant-coefficient counterparts is derived. Wide families of exact solutions for variable coefficient equations from this class are constructed.

In Section 2.6, we study Lie symmetries of generalized Burgers equations from two classes. At first we carry out the group classification of a class of generalized Burgers equations with time-dependent viscosity $u_t + a(u^n)_x = g(t)u_{xx}$, $agn \neq 0$. Using computed Lie symmetries, we solve an associated boundary-value problem. Then the group classification of the generalized Burgers equations with linear damping $u_t + u^n u_x + h(t)u = g(t)u_{xx}$, $ng \neq 0$, is derived using the equivalence method suggested in [289].

In Section 2.7 the complete group classification problem for a class of (2+1)-dimensional nonlinear Kolmogorov equations of the general form $u_t = f(t)u_{yy} - g(t)[K(u)]_x$, $fgK_{uu} \neq 0$, is solved via gauging the arbitrary elements of the class by a family of equivalence transformations parameterized by the arbitrary elements, which reduces their number. Two possible gaugings are discussed in order to show how equivalence groups serve in making the optimal choice of gaugings.

The results presented in this chapter are based on publications [1*–3*,6*,7*,10*,18*,22*,29*].

2.1. Transformation Properties of Nonlinear Evolution Equations in 1+1 Dimensions

In this section we study transformation properties of the general class of nonlinear second-order evolution equations of the form

$$u_t = H(t, x, u, u_x, u_{xx}), \quad H_{u_{xx}} \neq 0, \tag{2.1}$$

to construct a chain of its nested normalized subclasses and to find their equivalence groupoids.

2.1.1. Admissible Transformations of Evolution Equations. Any nondegenerate point transformation $\mathcal{T}$ relating two fixed equations $u_t = H$ and $\tilde u_{\tilde t} = \tilde H$ from the class (3.67) has the form $\tilde t = T(t)$, $\tilde x = X(t, x, u)$, $\tilde u = U(t, x, u)$ with $T_t(X_xU_u - X_uU_x) \neq 0$ [158,160]. The partial derivatives are transformed as follows:

$$\tilde u_{\tilde t} = \frac{D_tU\,D_xX - D_xU\,D_tX}{T_tD_xX}, \quad \tilde u_{\tilde x} = \frac{D_xU}{D_xX}, \quad \tilde u_{\tilde x\tilde x} = \frac{1}{D_xX}D_x\left(\frac{D_xU}{D_xX}\right),$$

where $D_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \dots$ and $D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \dots$ are operators of the total differentiation with respect to t and x. Moreover, it was proved in [250] that class (3.67) is normalized. For further consideration we use the following statement.

Theorem 2.1 ([250]). *Class (3.67) is normalized in the usual sense. Its equivalence group is formed by the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = X(t,x,u), \quad \tilde{u} = U(t,x,u), \tag{2.2}$$

$$\tilde{H} = \frac{X_xU_u - X_uU_x}{T_tD_xX}H + \frac{U_tD_xX - X_tD_xU}{T_tD_xX}, \tag{2.3}$$

where $T_t(X_xU_u - X_uU_x) \neq 0$,

Formula (2.3) implies that the subclass of class (3.67) singled out by the condition $H_{u_{xx}u_{xx}} = 0$ has the same equivalence transformation components for variables. The following statement is true.

Theorem 2.2. *The class of quasilinear second-order evolution equations,*

$$u_t = G(t,x,u,u_x)u_{xx} + F(t,x,u,u_x), \quad G \neq 0, \tag{2.4}$$

is normalized in the usual sense. Its equivalence group is formed by the transformation components for variables (2.2) and the transformations for arbitrary elements

$$\tilde{G} = \frac{(D_xX)^2}{T_t}G, \quad \tilde{F} = \frac{X_xU_u - X_uU_x}{T_tD_xX}F + \frac{U_tD_xX - X_tD_xU}{T_tD_xX}$$
$$+ \frac{(X_{xx} + 2X_{xu}u_x + X_{uu}u_x^2)D_xU - (U_{xx} + 2U_{xu}u_x + U_{uu}u_x^2)D_xX}{T_tD_xX}G.$$

The transformation component for G implies that, if G does not depend on u_x, then $X_u = 0$. We formulate more generally Lemma 1 from [138].

Theorem 2.3. *The class*

$$u_t = G(t,x,u)u_{xx} + F(t,x,u,u_x), \quad G \neq 0, \tag{2.5}$$

is normalized in the usual sense. Its equivalence group comprises the transformations

$$\tilde t = T(t), \quad \tilde x = X(t,x), \quad \tilde u = U(t,x,u), \quad \tilde G = \frac{X_x^2}{T_t}G, \tag{2.6}$$
$$\tilde F = \frac{U_u}{T_t}F + \frac{U_tX_x - X_tD_xU}{T_tX_x} + \frac{X_{xx}D_xU - (U_{xx} + 2U_{xu}u_x + U_{uu}u_x^2)X_x}{T_tX_x}G,$$

where $T_tX_xU_u \neq 0$.

Particular important subclass of class (2.5) is one, where the function F is polynomial in u_x and especially when it is quadratic or linear in u_x. We formulate separate statements for each of these cases.

Theorem 2.4. *The class*

$$u_t = G(t,x,u)u_{xx} + \sum_{k=0}^{n} F^k(t,x,u)u_x^k, \;\; n \geq 2, \quad G \neq 0, \tag{2.7}$$

is normalized in the usual sense. Its equivalence group consists of the transformations (2.6) *and the transformation components for the arbitrary elements* F^k, $k = 0, \dots, n$, *are found as solutions of the algebraic system resulting from the splitting of the following equation with respect to different powers of* u_x

$$\sum_{k=0}^{n} \tilde F^k \left(\frac{U_u}{X_x}u_x + \frac{U_x}{X_x}\right)^k = \frac{1}{T_tX_x}\big[X_xU_u \sum_{k=0}^{n} F^k u_x^k + U_tX_x - X_tD_xU$$
$$+ \big(X_{xx}D_xU - (U_{xx} + 2U_{xu}u_x + U_{uu}u_x^2)X_x\big)\, G\big].$$

Theorem 2.5. *The class*

$$u_t = G(t,x,u)u_{xx} + F^2(t,x,u)u_x^2 + F^1(t,x,u)u_x + F^0(t,x,u), \quad G \neq 0, \tag{2.8}$$

is normalized in the usual sense. Its equivalence group is formed by the transformations

$$\tilde t = T(t), \quad \tilde x = X(t,x), \quad \tilde u = U(t,x,u),$$
$$\tilde G = \frac{X_x^2}{T_t}G, \quad \tilde F^2 = \frac{X_x^2}{T_tU_u^2}\left(U_uF^2 - U_{uu}G\right), \quad \tilde F^1 = \frac{1}{T_tU_u}\Big[X_xU_uF^1$$

$$+\,2\frac{X_xU_x}{U_u}(U_{uu}G-U_uF^2)-X_tU_u+(X_{xx}U_u-2U_{xu}X_x)G\Bigg],$$
$$\tilde F^0=\frac{1}{T_t}\left[\frac{U_x^2}{U_u}F^2-U_xF^1+U_uF^0+U_t+\left(2\frac{U_x}{U_u}U_{xu}-U_{xx}-\frac{U_x^2}{U_u^2}U_{uu}\right)G\right],$$

where $T_tX_xU_u\neq 0$.

If we consider the subclass of class (2.8) singled out by the condition $F^2=0$, then its equivalence group is a proper subgroup of the equivalence group of class (2.8). The constraints for the transformations are derived by setting $\tilde F^2=0$ and $F^2=0$ in Theorem 5. This implies that $U_{uu}=0$. The following statement is true.

Theorem 2.6. *The class*

$$u_t=G(t,x,u)u_{xx}+F^1(t,x,u)u_x+F^0(t,x,u),\quad G\neq 0,\tag{2.9}$$

is normalized in the usual sense. Its equivalence group comprises the transformations

$$\tilde t=T(t),\;\;\tilde x=X(t,x),\;\;\tilde u=U^1(t,x)u+U^0(t,x),\;\;T_tX_xU^1\neq 0,\tag{2.10}$$
$$\tilde G=\frac{X_x^2}{T_t}G,\quad \tilde F^1=\frac{1}{T_tU^1}\left(X_xU^1F^1-X_tU^1+(X_{xx}U^1-2U^1_xX_x)G\right),$$
$$\tilde F^0=\frac{1}{T_t}\Bigg[U^1F^0-(U^1_xu+U^0_x)F^1+U^1_tu+U^0_t$$
$$+\left(2\frac{U^1_x}{U^1}(U^1_xu+U^0_x)-U^1_{xx}u-U^0_{xx}\right)G\Bigg].$$

Consider one more subclass of class (2.8) for which the condition $U_{uu}=0$ holds for admissible transformations. This is the subclass singled out by the condition $F^2=G_u$,

$$u_t=(G(t,x,u)u_x)_x+K(t,x,u)u_x+P(t,x,u),\quad G\neq 0.\tag{2.11}$$

This class can be written in the form

$$u_t=Gu_{xx}+G_uu_x^2+(G_x+K)u_x+P,$$

where connections between arbitrary elements of the latter class and class (2.8) are given by the formulas $F^2 = G_u$, $F^1 = G_x + K$, $F^0 = P$. It contains derivatives of G and they naturally appear in transformation components for arbitrary elements of the equivalence group. This group can be considered as usual one if we extend the tuple of arbitrary elements by new elements G_u and G_x. Then the transformations of these variables take the form

$$\tilde G_{\tilde x} = \frac{X_x}{T_t} G_x + 2\frac{X_{xx}}{T_t} G, \quad \tilde G_{\tilde u} = \frac{X_x^2}{T_t U^1} G_u.$$

It is easy to see that the group is really usual one, representing the above class in the form

$$u_t = G u_{xx} + G^1 u_x^2 + (G^2 + K) u_x + P$$

with additional arbitrary elements $G^1 = G_u$, and $G^2 = G_x$.

Theorem 2.7. *Reparameterized class* (2.11) *is normalized in the usual sense. Its equivalence group is formed by the transformations*

$$\begin{aligned}
&\tilde t = T(t), \quad \tilde x = X(t,x), \quad \tilde u = U^1(t,x)u + U^0(t,x), \quad T_t X_x U^1 \neq 0,\\
&\tilde G = \frac{X_x^2}{T_t} G, \quad \tilde K = \frac{X_x}{T_t}\left[K - \left(\frac{X_{xx}}{X_x} + 2\frac{U^1_x}{U^1}\right) G - 2(U^1_x u + U^0_x)\frac{G_u}{U^1} - \frac{X_t}{X_x}\right],\\
&\tilde P = \frac{1}{T_t}\left[U^1 P + \frac{(U^1_x u + U^0_x)^2}{U^1} G_u - (U^1_x u + U^0_x)(G_x + K) + U^1_t u + U^0_t\right.\\
&\qquad\left. + \left(2\frac{U^1_x}{U^1}(U^1_x u + U^0_x) - U^1_{xx} u - U^0_{xx}\right) G\right].
\end{aligned}$$

The subclass of class (2.11) singled out by the condition $K = 0$,

$$u_t = (G(t,x,u)u_x)_x + P(t,x,u), \quad G \neq 0, \tag{2.12}$$

is not normalized anymore in contrast to its covering classes considered above.

Constraints for its admissible transformations are derived setting $K = 0$ and $\tilde K = 0$ in the transformations adduced in the previous theorem, which results in the equation

$$2\left(\frac{U^1_x}{U^1}u+\frac{U^0_x}{U^1}\right)G_u+\left(\frac{X_{xx}}{X_x}+2\frac{U^1_x}{U^1}\right)G+\frac{X_t}{X_x}=0.$$

The further constraints on forms of X, U^1 and U^0 depend on values of the function G. If G does not satisfy the equation of the form $(au+b)G_u+cG+d=0$, where a, b, c and d are functions of t and x, then the point transformations between equations from this class necessarily satisfy the conditions $X_t=X_{xx}=U^1_x=U^0_x=0$, and such a subclass of class (2.12) will be normalized in the usual sense. The whole class (2.12) is not normalized. We adduce its equivalence group in the following statement.

Theorem 2.8. *The equivalence group of class* (2.12) *is comprised of the transformations*

$$\tilde t=T(t),\quad \tilde x=\delta_1x+\delta_2,\quad \tilde u=U^1(t)u+U^0(t),\quad T_tU^1\delta_1\neq0,$$
$$\tilde G=\frac{\delta_1^2}{T_t}G,\quad \tilde P=\frac{1}{T_t}\left(U^1P+U^1_tu+U^0_t\right).$$

The description of the entire equivalence groupoid of class (2.12) needs additional study.

Classes of evolution equations with variable coefficients of u_t often appear in applications. That is why we additionally consider the generalization of equations (2.11) of the form

$$S(t,x)u_t=(G(t,x,u)u_x)_x+K(t,x,u)u_x+P(t,x,u),\quad SG\neq0. \tag{2.13}$$

In particular, the classes of variable-coefficient diffusion–reaction equations $f(x)u_t=(g(x)A(u)u_x)_x+h(x)B(u)$ and diffusion–convection equations $f(x)u_t=(g(x)A(u)u_x)_x+h(x)B(u)u_x$ ($fgA\neq0$) are subclasses of this class. Though the coefficient $S(t,x)$ can be gauged to one by the family of point transformation

$$\tilde t=t,\quad \tilde x=\textstyle\int_{x_0}^x S(t,y)\,\mathrm{d}y,\quad \tilde u=u,$$

we will consider class (2.13) separately since its transformational properties become more complicated in comparison with those of class (2.11). The following statement is true.

Theorem 2.9. *Any point transformation between two fixed equations from class* (2.13) *has the form* (2.10)*. Then the respective values of the arbitrary elements are related via the formulas*

$$\frac{\tilde K+\tilde G_{\tilde x}}{\tilde S}=\frac{X_x}{T_tS}\left[K+G_x+\left(\frac{X_{xx}}{X_x}-2\frac{U^1_x}{U^1}\right)G-2(U^1_xu+U^0_x)\frac{G_u}{U^1}-\frac{X_t}{X_x}S\right],$$
$$\frac{\tilde G}{\tilde S}=\frac{X_x^2}{T_t}\frac{G}{S},\quad \frac{\tilde P}{\tilde S}=\frac{1}{T_tS}\Bigg[U^1P+\frac{(U^1_xu+U^0_x)^2}{U^1}G_u+(U^1_tu+U^0_t)S$$
$$+\left(2\frac{U^1_x}{U^1}(U^1_xu+U^0_x)-U^1_{xx}u-U^0_{xx}\right)G-(U^1_xu+U^0_x)(K+G_x)\Bigg].$$

It is obvious that transformation properties of class (2.13) become more complicated in comparison with those of class (2.11). Transformations are defined only for fractions of arbitrary elements. It is explained by the fact that this class admits peculiar gauge equivalence transformation (an equivalence transformation for which independent and dependent variables do not transform but only arbitrary elements). This is the transformation

$$\tilde S=Z(t,x,S),\quad \tilde G=\frac{G}{S}Z,\quad \tilde K=\frac{K}{S}Z-G\left(\frac{Z}{S}\right)_x,\quad \tilde P=\frac{P}{S}Z,$$

where Z is an arbitrary smooth function of its variables with $Z_S\neq 0$.

Theorem 2.10. *The equivalence group of class* (2.13) *comprises the transformations*

$$\tilde t=T(t),\quad \tilde x=X(t,x),\quad \tilde u=U^1(t,x)u+U^0(t,x),\quad T_tX_xU^1\neq 0,$$
$$\tilde S=Z(t,x,S),\quad \tilde G=\frac{X_x^2}{T_t}\frac{G}{S}Z,$$
$$\tilde K=\frac{X_xZ}{T_tS}\left[K-\left(\frac{X_{xx}}{X_x}+2\frac{U^1_x}{U^1}\right)G-2\mathcal{U}\frac{G_u}{U^1}-\frac{X_t}{X_x}S\right]-\frac{X_x}{T_t}G\left(\frac{Z}{S}\right)_x,$$
$$\tilde P=\frac{Z}{T_tS}\Bigg[U^1P+\frac{\mathcal{U}^2}{U^1}G_u-\mathcal{U}(K+G_x)+(U^1_tu+U^0_t)S$$

$$+\left(2\frac{U^1_x}{U^1}\mathcal{U}-U^1_{xx}u-U^0_{xx}\right)G\Big],$$

where $\mathcal{U}=U^1_xu+U^0_x$.

Class (2.13) can be regarded as normalized in the usual sense since we can present it in the form $Su_t=Gu_{xx}+G^1u_x^2+G^2u_x+P$ with additional arbitrary elements $G^1=G_u$, and $G^2=K+G_x$.

We note that the subclass of class (2.13) singled out by the condition $K=0$, i.e. the class

$$S(t,x)u_t=(G(t,x,u)u_x)_x+P(t,x,u),\quad SG\neq 0, \tag{2.14}$$

is not normalized. In contrast to the case of class (2.13) the coefficient S is essential for class (2.14). The gauge equivalence transformations are quite simple in this case, namely, each coefficient can be multiplied by a nonvanishing smooth function of t. The equivalence group of class (2.14) is wider than the equivalence group of its subclass with $S=1$ (cf. Theorem 8). The following statement is true.

Theorem 2.11. *The equivalence group of class* (2.14) *consists of the transformations*

$$\tilde t=T(t),\quad \tilde x=X(x),\quad \tilde u=U^1(t)u+U^0(t),\quad T_tX_xU^1\neq 0,$$
$$\tilde S=\psi(t)\frac{T_t}{X_x}S,\quad \tilde G=\psi(t)X_xG,\quad \tilde P=\frac{\psi(t)}{X_x}\left(U^1P+(U^1_tu+U^0_t)S\right),$$

where $\psi(t)$ *is a nonvanishing smooth function of its variable.*

Concluding Remarks. The chain of nested subclasses of the general class of (1+1)-dimensional second-order nonlinear evolution equations is constructed. For those subclasses that are proved to be normalized the found equivalence groups give the exhaustive description of the respective equivalence groupoids of these classes. So, we firstly proved that classes (2.4), (2.5), (2.7)–(2.9), (2.11) and (2.13) are normalized and then looked for their equivalence groups, which lead to the complete description of the equivalence groupoids of these classes.

Finding the equivalence groupoids for the non-normalized classes is a difficult task since the determining equations for components of admissible transformations are nonlinear ones. That is why for such classes other techniques are needed like partition of the class into normalized subclasses and the method of furcate splitting [292]. In future work we plan to use these methods to find the entire equivalence groupoids of classes (2.12) and (2.14), that are of special interest for further applications. There are many models with application in physics and biology which are members of these classes, e.g. variable coefficient Fisher and Newell–Whitehead–Segel equations, which are also studied within this chapter. We note that the group classification for the general class of (1+1)-dimensional second-order quasilinear evolution equations $u_t = F(t,x,u,u_x)u_{xx} + G(t,x,u,u_x)$, $F \neq 0$, that contains classes considred in this chapter as subclasses was performed in [17]. Nevertheless those results obtained up to a very wide equivalence group seem to be inconvenient to derive group classification for its specific subclasses.

2.2. Extended Group Analysis of a Class of Reaction–Diffusion Equations with Exponential Nonlinearities

The problem of extended group analysis of variable coefficient reaction–diffusion equations of the general form

$$f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u), \tag{2.15}$$

where $fgA \neq 0$, was initiated in [294, 300]. The case of A and B being power functions, i.e., the class of equations having the form

$$f(x)u_t = (g(x)u^n u_x)_x + h(x)u^m, \tag{2.16}$$

where $fg \neq 0$ and $(n, mh) \neq (0, 0)$, was successfully investigated therein. Lie symmetry classifications of certain subclasses of this class were carried out in [70,83,225,275]. The most important studies on reduction operators (called more often nonclassical or Q-conditional symmetries) are presented in [12, 13, 61, 90, 241, 299, 301]. For many reasons, the natural continuation of the study in [294, 300] is to consider equations of the form (2.15) with A and B being exponential functions,

$$f(x)u_t = (g(x)e^{nu}u_x)_x + h(x)e^{mu}. \tag{2.17}$$

Here $f = f(x)$, $g = g(x)$ and $h = h(x)$ are arbitrary smooth functions of the variable x, $fg \neq 0$ and n and m are arbitrary constants. The linear case, which is singled out by the condition $n = m = 0$, is excluded from consideration as it is well investigated. The semilinear equations of the form (2.17), which correspond to the constraints $n = 0$ and $m \neq 0$, were already considered in [288, 301]. Moreover, equations of the form (2.17) with $n \neq 0$ are not related to linear and semilinear equations of the same form via point transformations. This is why in the present section we study only the class of equations of the form (2.17) with $fgn \neq 0$, which we briefly call class (2.17). Note that the parameter n can be gauged to 1 by a simple scaling of variables from the very beginning but we will not use this gauge and will deal with the general form (2.17).

2.2.1. Equivalence Transformations. We make a preliminary study of admissible transformations for class (2.17) using the direct method [160]. The obtained results are summarized in Theorems 2.12 and 2.13, where we use the notation $\Psi = e^{-\frac{n}{\delta_3}\psi}$ in order to simplify formulas.

Theorem 2.12. *The generalized extended equivalence group* $\hat{G}^{\sim}$ *of class* (2.17) *is formed by the transformations*

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \delta_3 u + \psi(x),$$
$$\tilde{f} = \frac{\delta_0\delta_1}{\varphi_x}\Psi f, \quad \tilde{g} = \delta_0\varphi_x\Psi^2 g, \quad \tilde{h} = \frac{\delta_0\delta_3}{\varphi_x}e^{-\frac{m}{\delta_3}\psi}\Psi h, \quad \tilde{n} = \frac{n}{\delta_3}, \quad \tilde{m} = \frac{m}{\delta_3},$$

where φ is an arbitrary smooth function of x, $\varphi_x \neq 0$; ψ and Ψ are determined by the formulas

$$\psi(x) = -\frac{\delta_3}{n}\ln|\Psi(x)|\,, \quad \Psi(x) = \delta_4\int\frac{dx}{g(x)} + \delta_5.$$

δ_j, $j = 0,\dots,5$, *are arbitrary constants,* $\delta_0\delta_1\delta_3 \neq 0$ *and* $(\delta_4, \delta_5) \neq (0,0)$.

Thus, elements of $\hat{G}^{\sim}$ are parameterized by five arbitrary constants and a single arbitrary smooth function of x. The usual equivalence group $G^{\sim}$ of class (2.17) is the subgroup of the generalized extended equivalence group $\hat{G}^{\sim}$, which is singled out with the condition $\delta_4 = 0$.

Theorem 2.13. *The generalized extended equivalence group of the class*

$$f(x)u_t = (g(x)e^{nu}u_x)_x + h(x)e^{nu} \quad \textit{with} \quad nfg \neq 0, \tag{2.18}$$

coincides with the usual equivalence group $G^{\sim}_{m=n}$ of this class and consists of the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \delta_3 u + \psi(x),$$

$$\tilde{f} = \frac{\delta_0\delta_1}{\varphi_x}\Psi f, \quad \tilde{g} = \delta_0\varphi_x\Psi^2 g, \quad \tilde{h} = \frac{\delta_0\delta_3}{n\varphi_x}\left[nh\Psi + (g\Psi_x)_x\right]\Psi, \quad \tilde{n} = \frac{n}{\delta_3},$$

where δ_j, $j = 0,1,2,3$, are arbitrary constants, $\delta_0\delta_1\delta_3 \neq 0$, φ and ψ are arbitrary smooth functions of x with $\varphi_x \neq 0$, $\Psi(x) = e^{-\frac{n\psi(x)}{\delta_3}}$.

Elements of $G^{\sim}_{m=n}$ are parameterized by four arbitrary constants and two arbitrary smooth functions of x. Therefore, the group $G^{\sim}_{m=n}$ is really a nontrivial conditional equivalence group of class (2.17).

In view of Theorem 2.12, the family of equivalence transformations

$$\tilde{t} = t, \quad \tilde{x} = \int_{x_0}^{x}\frac{dy}{g(y)}, \quad \tilde{u} = u, \tag{2.19}$$

parameterized by the arbitrary element g maps class (2.17) onto its subclass consisting of equations of the form $\tilde{f}\tilde{u}_{\tilde{t}} = (e^{\tilde{n}\tilde{u}}\tilde{u}_{\tilde{x}})_{\tilde{x}} + \tilde{h}e^{\tilde{m}\tilde{u}}$, with $\tilde{g} = 1$. The new arbitrary elements are expressed via the old ones in the following way:

$\tilde{f} = fg$, $\tilde{h} = gh$, $\tilde{m} = m$, $\tilde{n} = n$. Hence, within the framework of symmetry analysis it suffices, without loss of generality, to investigate the equations

$$f(x)u_t = (e^{nu}u_x)_x + h(x)e^{mu} \quad \text{with} \quad nf \neq 0 \tag{2.20}$$

instead of class (2.17) because all results on symmetries, solutions and conservation laws of equations from subclass (2.20) can be extended to the entire class (2.17) with transformations (2.19). In other words, up to $\hat{G}^{\sim}$-equivalence we can assign the gauge $g = 1$ for the arbitrary element g. Instead of $g = 1$ we can set $f = 1$, but the gauge $g = 1$ is more convenient because it results in simpler group classification of class (2.17). Simultaneously, we can assign the value 1 to arbitrary element n, but this gauge is not essential in the course of group classification and hence it will be used only in the presentation of the final classification list.

The description of the generalized equivalence group of class (2.20) and the generalized conditional equivalence group of the same class which is associated with the condition $m = n$ is deduced from Theorem 2.12 and Theorem 2.13 by setting $\tilde{g} = g = 1$.

Theorem 2.14. *The generalized equivalence group $\hat{G}_1^{\sim}$ of class (2.20) is formed by the transformations*

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \frac{\delta_6 x + \delta_7}{\delta_4 x + \delta_5}, \quad \tilde{u} = \delta_3 u - \frac{\delta_3}{n} \ln|\delta_4 x + \delta_5|,$$
$$\tilde{f} = \frac{\delta_1}{\Delta^2}|\delta_4 x + \delta_5|^3 f, \quad \tilde{h} = \frac{\delta_3}{\Delta^2}|\delta_4 x + \delta_5|^{\frac{m}{n}+3} h, \quad \tilde{n} = \frac{n}{\delta_3}, \quad \tilde{m} = \frac{m}{\delta_3},$$

where δ_j, $j = 1, \ldots, 7$, are arbitrary constants such that $\delta_1\delta_3 \neq 0$, $\Delta = \delta_5\delta_6 - \delta_4\delta_7 \neq 0$ and the tuple $(\delta_4, \delta_5, \delta_6, \delta_7)$ is defined up to a nonzero multiplier; e.g., we can set $\Delta = \pm 1$.

Theorem 2.15. *The class of equations*

$$f(x)u_t = (e^{nu}u_x)_x + h(x)e^{nu} \quad \textit{with} \quad nf \neq 0 \tag{2.21}$$

admits the generalized equivalence group $\hat{G}^{\sim}_{1,m=n}$ *consisting of the transformations*

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \delta_3 u + \frac{\delta_3}{2n} \ln |\delta_0^2 \varphi_x|, \qquad (2.22)$$
$$\tilde{f} = \delta_0\delta_1 |\varphi_x|^{-\frac{3}{2}} f, \quad \tilde{h} = \delta_3 \varphi_x^{-2} h + \frac{\delta_3}{n} |\varphi_x|^{-\frac{3}{2}} (|\varphi_x|^{-\frac{1}{2}})_{xx}, \quad \tilde{n} = \frac{n}{\delta_3},$$

where δ_j, $j = 0, \dots, 3$, *are arbitrary constants with* $\delta_0\delta_1\delta_3 \neq 0$ *and* $\varphi = \varphi(x)$ *is an arbitrary smooth function with* $\varphi_x \neq 0$.

Class (2.21) can be mapped onto a proper subclass with only one arbitrary element depending on x using an appropriate family of point transformations from the group $\hat{G}^{\sim}_{1,m=n}$. The most convenient gauges for arbitrary elements are the gauges $f = 1$ and $h = 0$. The first gauge can be realized by the transformation

$$\tilde{t} = \operatorname{sign}(f) t, \quad \tilde{x} = \int_{x_0}^{x} f(y)^{\frac{2}{3}} dy, \quad \tilde{u} = u + \frac{1}{3n} \ln |f|, \qquad (2.23)$$

which maps an equation of the form (2.21) to the equation $\tilde{u}_{\tilde{t}} = (e^{\tilde{n}\tilde{u}} \tilde{u}_{\tilde{x}})_{\tilde{x}} + \tilde{h} e^{\tilde{n}\tilde{u}}$, i.e., $\tilde{f} = 1$. The other arbitrary elements h and n are transformed to $\tilde{h} = f^{-1} \left(f^{-\frac{1}{3}} h + n^{-1} (f^{-\frac{1}{3}})_{xx} \right)$ and $\tilde{n} = n$, respectively.

Theorem 2.16. *The generalized equivalence group* $\hat{G}^{\sim}_{f=g=1,m=n}$ *of the class of equations*

$$u_t = (e^{nu} u_x)_x + h(x) e^{nu} \quad \textit{with} \quad n \neq 0, \qquad (2.24)$$

consists of the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_4 x + \delta_5, \quad \tilde{u} = \delta_3 u + \frac{\delta_3}{n} \ln \frac{\delta_4^2}{\delta_1}, \quad \tilde{h} = \frac{\delta_3}{\delta_4^2} h, \quad \tilde{n} = \frac{n}{\delta_3},$$

where δ_j, $j = 1, \dots, 5$ *are arbitrary constants,* $\delta_1\delta_3\delta_4 \neq 0$ *with* $\delta_1 > 0$.

The gauge $h = 0$ for class (2.21) can be realized by the family of transformations (2.22), where $\delta_1 = \delta_3 = 1$, $\delta_2 = 0$ and the function φ satisfies the ODE $(|\varphi_x|^{-\frac{1}{2}})_{xx} + nh |\varphi_x|^{-\frac{1}{2}} = 0$.

Theorem 2.17. *The generalized equivalence group* $\hat{G}^{\sim}_{1,h=0}$ *of the class of equations*

$$f(x)u_t = (e^{nu}u_x)_x \quad \textit{with} \quad nf \neq 0, \tag{2.25}$$

is projection of the group $\hat{G}^{\sim}_1$ *on the space* (t, x, u, f, n).

It is possible to carry out the complete group classification of equations from class (2.21) using either the gauge $f = 1$ or the gauge $h = 0$. We use the first gauge to perform the complete group classification of (2.21).

2.2.2. Lie Symmetries. The group classification of class (2.17) is carried out within the framework of the classical Lie approach [217, 227]. At the same time, we additionally apply a number of modern tools of symmetry analysis. Thus, gauging the arbitrary element g to 1 by equivalence transformations, we in fact classify subclass (2.20) instead of the entire class. The main equivalence relation involved in the consideration is generated by the generalized equivalence group $\hat{G}^{\sim}_1$ of this subclass, which contains the usual equivalence group of the same subclass as a proper subgroup. In other words, we use $\hat{G}^{\sim}_1$-equivalence, which is stronger than $G^{\sim}_1$-equivalence prescribed by the classical Lie approach. Moreover, for group classification of equations from subclass (2.20) with $m = n$ we involve the equivalence relation which is generated by the conditional generalized equivalence group $\hat{G}^{\sim}_{1,m=n}$ and is even stronger than $\hat{G}^{\sim}_1$-equivalence. In total, this leads to the reduction of classification cases and lowering the number of additional equivalence transformations to be constructed. Lie symmetry extensions are separated using the method of furcate splitting [138, 209].

The following statement is true.

Proposition 2.18. *The kernel algebra, i.e., the intersection of the maximal Lie invariance algebras of equations from class* (2.20) *(resp. class* (2.17)*) is the one-dimensional algebra* $A^{\cap} = \langle \partial_t \rangle$.

The following exclusive cases appear during classification:

1) $m \neq 0, n$; 2) $m = 0$; 3) $m = n$.

The case $h = 0$ is special as the value of m is undefined in this case but it can be included to the case $m = n$. This is why in the first two cases we assume that $h \neq 0$.

The results of group classification for class (2.17) are collected in Table 2.1. There exist additional equivalence transformations between classification cases presented in Table 2.1. Thus, the point transformation

$$t' = \frac{1}{\varepsilon n} e^{\varepsilon n t}, \quad x' = x, \quad u' = u - \varepsilon t \tag{2.26}$$

links the equations $f(x)u_t = (g(x)e^{nu}u_x)_x + \varepsilon f(x)$ and $f(x')u'_{t'} = (g(x')e^{nu'}u'_{x'})_{x'}$. This transformation belongs to no equivalence group found in previous section and reduces Cases 4–6 of Table 2.1 to the set of cases '$m = n$ or $h = 0$'. For the reduction to be precisely to Cases 7–10 of Table 2.1, for Case 5 transformation (2.26) should be composed with an appropriate transformation of the form (2.23). Such compositions map subcases of Case 5 to subcases of Case 8 and 9. The transformations described exhaust additional equivalence transformations within the classification list from Table 2.1. It is proved in the next section within the framework of admissible transformations. The transformation (2.26) can also be included in the framework of conditional equivalence but the corresponding conditional equivalence group is too complicated. The following statement is true.

Theorem 2.19. *Up to point transformations, a complete list of Lie symmetry extensions for equations from class* (2.17) *is exhausted by Cases 1–3 and 7–10 of Table 2.1.*

Corollary 2.20. *If an equation from class (2.17) is invariant with respect to a four-dimensional Lie algebra then it is reduced using point transformations to the equation* $u_t = (e^u u_x)_x$.

Table 2.1: Results of group classification of class (2.17) under the gauge $g = 1$.

no.	$f(x)$	$h(x)$	Basis of $A^{\max}$
General case of m			
1	$\forall$	$\forall$	∂_t
2	$f_1(x)$	$h_1(x)$	∂_t, $(d+2b-pn)t\partial_t + (nax^2+bx+c)\partial_x + (ax+p)\partial_u$
3	1	ε	∂_t, ∂_x, $2mt\partial_t + (m-n)x\partial_x - 2\partial_u$
$m=0,\quad h\neq 0,\quad (h/f)_x = 0$			
4	$\forall$	$\varepsilon f(x)$	∂_t, $e^{-\varepsilon nt}(\partial_t + \varepsilon\partial_u)$
5	$f_1(x)$	$\varepsilon f_1(x)$	∂_t, $e^{-\varepsilon nt}(\partial_t + \varepsilon\partial_u)$, $n(nax^2+bx+c)\partial_x + (nax+2b+d)\partial_u$
6	1	ε	∂_t, $e^{-\varepsilon nt}(\partial_t + \varepsilon\partial_u)$, ∂_x, $nx\partial_x + 2\partial_u$
$m=n\quad$ or $\quad h=0$			
7	$\forall$	$\forall$	∂_t, $nt\partial_t - \partial_u$
8	1	αx^{-2}	∂_t, $nt\partial_t - \partial_u$, $nx\partial_x + 2\partial_u$
9	1	ε	∂_t, $nt\partial_t - \partial_u$, ∂_x
10	1	0	∂_t, $nt\partial_t - \partial_u$, ∂_x, $nx\partial_x + 2\partial_u$

Here n, α and ε are nonzero constants, $n = 1 \bmod \hat{G}_1^{\sim}$, $\varepsilon = \pm 1 \bmod \hat{G}_1^{\sim}$,

$$f_1(x) = \exp\left(\int \tfrac{-3nax+d}{nax^2+bx+c}\,\mathrm{d}x\right), \quad h_1(x) = \varepsilon\exp\left(-\int \tfrac{(3n+m)ax+2b+(m-n)p}{nax^2+bx+c}\,\mathrm{d}x\right),$$

and up to $\hat{G}_1^{\sim}$-equivalence the parameter tuple (a,b,c,d,p) can be assumed to belong to the set

$$\{(0,1,0,\bar{d},(\bar{q}+2)/(n-m)),\ (0,0,1,1,\check{p}),\ (0,0,1,0,1),\ (1/n,0,1,\hat{d},\hat{p})\},$$

where $(\bar{d},\bar{q}) \neq (0,0), (-3,-3-m/n)$ and modulo $\hat{G}_1^{\sim}$ we can also set $\bar{d} \geqslant -3/2$ and, if $\bar{d} = -3/2$, $\bar{q} \geqslant -3/2 - m/(2n)$; $\hat{d} \geqslant 0$ and, if $\hat{d} = 0$, $\hat{p} \geqslant 0$. In Case 5 the parameter p should be neglected. In Case 7 the arbitrary element f (resp. h) can be additionally gauged by transformations from $\hat{G}^{\sim}_{1,m=n}$. For example, we can set $f = 1$.

Corollary 2.21. *If an equation from class (2.17) with $m \neq 0, n$ possesses a three-dimensional Lie invariance algebra then it is mapped by a point transformation to the equation $u_t = (e^u u_x)_x \pm e^{mu}$.*

2.2.3. Equivalence Groupoid. To complete the description of the equivalence groupoid of the class (2.17), that is not normalized, we need to derive a conditional equivalence group for one more its subclass.

Theorem 2.22. *The generalized equivalence group $G^{\sim}_{m=0,(h/f)_x=0}$ of the subclass of class* (2.17)*, which is singled out by the conditions $m = 0$ and $(h/f)_x = 0$, consists of the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \delta_3 u + \psi(t,x),$$
$$\tilde{f} = \frac{\delta_0}{\varphi_x}\Psi f, \quad \tilde{g} = \delta_0\varphi_x\Psi^2 g, \quad \tilde{n} = \frac{n}{\delta_3},$$

where the smooth function $T = T(t)$ with $T_t \neq 0$ is defined by the formulas:

$$\mu\tilde{\mu}\neq 0\colon\ \ \frac{e^{\tilde{n}\tilde{\mu}T}-1}{\tilde{n}\tilde{\mu}} = \delta_1\frac{e^{n\mu t}-1}{n\mu}+\delta_2, \quad \mu=0,\ \tilde{\mu}\neq 0\colon\ \ \frac{e^{\tilde{n}\tilde{\mu}T}-1}{\tilde{n}\tilde{\mu}} = \delta_1 t+\delta_2,$$
$$\mu\neq 0,\ \tilde{\mu}=0\colon\ \ T=\delta_1\frac{e^{n\mu t}-1}{n\mu}+\delta_2, \quad \mu=\tilde{\mu}=0\colon\ \ T=\delta_1 t+\delta_2,$$

where $\mu = h/f$ and $\tilde{\mu} = \tilde{h}/\tilde{f}$ are constants, φ is an arbitrary smooth function of x with $\varphi_x \neq 0$, $\psi(t,x) = -\frac{\delta_3}{n}\ln|T_t(t)\Psi(x)|$, $\Psi(x) = \delta_4\int\frac{dx}{g(x)} + \delta_5$, δ_j, $j = 0,\dots,5$, are arbitrary constants, $\delta_0\delta_1\delta_3 \neq 0$ and $(\delta_4,\delta_5)\neq(0,0)$.

The group $G^{\sim}_{m=0,(h/f)_x=0}$ is a generalized equivalence group even if n is fixed as it contains transformations with respect to t, which depend on the (constant) ratio of the arbitrary elements h and f. In contrast to $G^{\sim}_{m=n}$, we do not use $G^{\sim}_{m=0,(h/f)_x=0}$ in the course of group classification of class (2.20) because the application of this conditional equivalence group does not have a crucial influence on classification, and the corresponding system $m = 0$, $(h/f)_x = 0$ for arbitrary elements is less obvious. At the same time, transformations from $G^{\sim}_{m=0,(h/f)_x=0}$ play the role of additional equivalence transformations after completing the classification (see the previous section).

We summarize the investigation of admissible transformations in class (2.17) in the following assertion.

Theorem 2.23. *Let the equations*

$$f(x)u_t = (g(x)e^{nu}u_x)_x + h(x)e^{mu} \ \textit{and}\ \tilde f(\tilde x)u_t = (\tilde g(\tilde x)e^{\tilde n\tilde u}u_x)_x + \tilde h(\tilde x)e^{\tilde m\tilde u}$$

be connected via a point transformation $\mathcal{T}$ *in the variables* t, x *and* u*. Then*

$$\textit{either}\quad \frac{\tilde m}{\tilde n} = \frac{m}{n} \quad \textit{or} \quad (m, \tilde m) = (0, \tilde n) \quad \textit{or} \quad (m, \tilde m) = (n, 0).$$

The transformation $\mathcal{T}$ *is induced by a transformation from*

a) $G^{\sim}$ *if either* $m \neq 0, n$ *or* $m = 0$, $(h/f)_x \neq 0$*;*

b) $G^{\sim}_{m=n}$ *if* $m = n$ *and* $\tilde m \neq 0$*, then also* $\tilde m = \tilde n$*;*

c) $G^{\sim}_{m=0,(h/f)_x=0}$ *if* $m = \tilde m = 0$, $(h/f)_x = 0$*, then also* $(\tilde h/\tilde f)_x = 0$*.*

If $m = 0$ *and* $\tilde m = \tilde n$ *then* $(h/f)_x = 0$ *and the transformation* $\mathcal{T}$ *is the composition of two transformations, from* $G^{\sim}_{m=0,(h/f)_x=0}$ *and* $G^{\sim}_{m=n}$*, with the intermediate equation having* $h = 0$*.*

The case with $m = n$ *and* $\tilde m = 0$ *is similar to the previous one.*

Theorem 2.24. *Class (2.17) is represented as the union of its three maximal normalized subclasses separated by the conditions* $(h \neq 0,\ m \neq 0, n)$ *or* $(m = 0,\ (h/f)_x \neq 0)$*;* $m = 0,\ (h/f)_x = 0$*;* $m = n$*. Only the latter two subclasses have a non-empty intersection, and this intersection is the normalized subclass '*$h = 0$*'.*

2.2.4. Contractions. Examples of nontrivial limits between equations admitting Lie symmetry extensions are known for a long time. For instance, in [39] equations with exponential nonlinearities were excluded from the group classification list of nonlinear diffusion equations as a separate case and were just considered as a *limiting case* of equations with power nonlinearities. At the same time, it looks more convenient to include such cases to classification lists and then indicate connections between different classification cases via limiting processes. Using the analogy with theory of Lie algebras such connections are called *contractions*. A theoretical background on contractions of differential equations, their Lie symmetry algebras and solutions was first discussed in [139].

In this section we relate, via contractions, the group classification lists obtained for class (2.17) with the classification list for class (2.16) [294, Table 1]. Then contractions are used to construct exact solutions of equations from class (2.17) using known solutions of equations from class (2.16). We also demonstrate a similar consideration for conservation laws.

Contractions of Equations and of Lie Invariance Algebras. At first we apply the transformation

$$\tilde{t} = \delta t, \quad \tilde{x} = \sqrt{\delta}x, \quad \tilde{u} = \delta(u-1), \quad \tilde{n} = \frac{n}{\delta}, \quad \tilde{m} = \frac{m}{\delta} \tag{2.27}$$

parameterized by a positive constant parameter δ to the equation from class (2.16) with the values arbitrary elements $g = 1$ and f and h presented in Case 2 of Table 1 of [294]. The constant parameters a, b, c, d and p are transformed in the following way

$$\tilde{a} = a, \quad \tilde{b} = \frac{b}{\sqrt{\delta}}, \quad \tilde{c} = c, \quad \tilde{d} = \frac{d}{\sqrt{\delta}}, \quad \tilde{p} = \sqrt{\delta}p, \quad \tilde{\alpha} = \delta\alpha \tag{2.28}$$

wherever this is relevant, i.e., we change parameters if and only if they appear in the values of arbitrary elements of the initial equation. Then, we take the imaged equation and proceed to the limit $\delta \to +\infty$. This results in the equation from class (2.20) with the values of the arbitrary elements f and h presented in Case 2 of Table 2.1. The same procedure establishes a contraction between the associated Lie algebras of vector fields. The corresponding notation will be 2.2 $\to$ 1.2, where II in the first numbers indicate the Table 1 of [294] and I in the first numbers stands for Table 2.1 and the second numbers indicate the numbers of cases within these tables. We present the complete list of contractions which replace power nonlinearities by exponential ones and, therefore, connect cases of Lie symmetry extensions for classes (2.17) and (2.16):

$$\text{II.1} \to \text{I.1}, \quad \text{II.2} \to \text{I.2}, \quad \text{II.3} \to \text{I.3}, \quad \text{II.4} \to \text{I.4}, \quad \text{II.5} \to \text{I.5},$$

$$\text{II.6} \to \text{I.6}, \quad \text{II.8} \to \text{I.7}, \quad \text{II.9} \to \text{I.8}, \quad \text{II.10} \to \text{I.9}, \quad \text{II.11} \to \text{I.10}.$$

Contractions of Lie Reductions and Exact Solutions. In [294] we carried out Lie reductions and constructed Lie exact solution for equations from class (2.16) with the values of arbitrary elements presented in Cases 9 and 12 of Table 1 in [294], which admit three-dimensional Lie symmetry algebras. It is shown in the previous subsection that there is a contraction of Case 9 of Table 1 in [294] to Case 8 of Table 2.1. The corresponding equations from classes (2.17) and (2.16) are

$$\tilde{u}_{\tilde{t}} = \left(e^{\tilde{n}\tilde{u}}\tilde{u}_{\tilde{x}}\right)_{\tilde{x}} + \tilde{\alpha}\tilde{x}^{-2}e^{\tilde{n}\tilde{u}} \tag{2.29}$$

$$u_t = (u^n u_x)_x + \alpha x^{-2}u^{n+1}, \tag{2.30}$$

whose maximal Lie invariance algebras $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ are generated by the vector fields $\tilde{X}_1 = \partial_{\tilde{t}}$, $\tilde{X}_2 = n\tilde{t}\partial_{\tilde{t}} - \partial_{\tilde{u}}$, $\tilde{X}_3 = n\tilde{x}\partial_{\tilde{x}} + 2\partial_{\tilde{u}}$ and $X_1 = \partial_t$, $X_2 = nt\partial_t - u\partial_u$, $X_3 = nx\partial_x + 2u\partial_u$, respectively. The contraction II.9 $\to$ I.8 can be realized using the simpler transformation

$$\tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = \delta(u-1), \quad \tilde{n} = \frac{n}{\delta}, \quad \tilde{\alpha} = \delta\alpha \tag{2.31}$$

than transformation (2.27). In the course of this contraction the algebra $\mathfrak{g}$ is contracted to the algebra $\tilde{\mathfrak{g}}$ as a Lie algebra of vector fields in the space of (t,x,u). Namely, $X_1 \to \tilde{X}_1$, $X_2 \to \tilde{X}_2$ and $X_3 \to \tilde{X}_3$. Let us study the related contractions of Lie reductions of equation (2.30) to ones of equation (2.29). Inequivalent Lie reductions of equation (2.29) with respect to one-dimensional subalgebras of the corresponding maximal Lie invariance algebras are exhausted by those presented in Table 2.2. For convenience we omit tildes in Table 2.2. The transformations of the invariant independent and dependent variables, which are induced by transformation (2.31), take the form $\tilde{\varphi} = \delta(\varphi - 1)$ and $\tilde{\omega} = \omega$ in all Cases 9.1–9.4 of Table 2 in [294].

Consider Case 9.1 of [294, Table 2] in detail. The transformed version and the corresponding limit of the ansatz $u = |t|^{-\frac{1+2\mu}{n}}\varphi(\omega)$ are

$$\left(1+\frac{\tilde{u}}{\delta}\right)^{\delta\tilde{n}} = |\tilde{t}|^{-(1+2\mu)}\left(1+\frac{\tilde{\varphi}}{\delta}\right)^{\delta\tilde{n}} \to e^{\tilde{n}\tilde{u}} = |\tilde{t}|^{-(1+2\mu)}e^{\tilde{n}\tilde{\varphi}} \text{ at } \delta \to +\infty.$$

Table 2.2: Lie reductions for Case 8 of Table 2.1.

no.	X	ω	$u=$	Reduced ODE
1	$X_2-\mu X_3$	$x\|t\|^\mu$	$\varphi(\omega)-\frac{1+2\mu}{n}\ln\|t\|$	$(e^{n\varphi})_{\omega\omega}-\mu n\varepsilon\omega\varphi_\omega+(1+2\mu)\varepsilon+\alpha n\omega^{-2}e^{n\varphi}=0$, $\varepsilon=\operatorname{sign}t$
2	X_3	t	$\varphi(\omega)+\frac{2}{n}\ln\|x\|$	$n\varphi_\omega-(\alpha n+2)e^{n\varphi}=0$
3	$X_3\pm X_1$	$xe^{\mp t}$	$\varphi(\omega)\pm\frac{2}{n}t$	$(e^{n\varphi})_{\omega\omega}\pm n\omega\varphi_\omega\mp 2+\alpha n\omega^{-2}e^{n\varphi}=0$
4	X_1	x	$\varphi(\omega)$	$(e^{n\varphi})_{\omega\omega}+\alpha n\omega^{-2}e^{n\varphi}=0$

Therefore, the contracted ansatz is $\tilde u=\tilde\varphi(\tilde\omega)-\frac{1+2\mu}{\tilde n}\ln|t|$. The reduced equation from Case 9.1 of [294, Table 2] is mapped by transformation (2.31) to the equation

$$\frac{\delta\tilde n}{\delta\tilde n+1}\left[\left(1+\frac{\tilde\varphi}{\delta}\right)^{\delta\tilde n+1}\right]_{\tilde\omega\tilde\omega}-\mu\tilde n\varepsilon\tilde\omega\tilde\varphi_{\tilde\omega}$$
$$+(1+2\mu)\varepsilon\left(1+\frac{\tilde\varphi}{\delta}\right)+\frac{\tilde\alpha\tilde n}{\tilde\omega^2}\left(1+\frac{\tilde\varphi}{\delta}\right)^{\delta\tilde n+1}=0.$$

Then the limit process at $\delta\to+\infty$ leads to the equation

$$\left(e^{\tilde n\tilde\varphi}\right)_{\tilde\omega\tilde\omega}-\mu\tilde n\varepsilon\tilde\omega\tilde\varphi_{\tilde\omega}+(1+2\mu)\varepsilon+\tilde\alpha\tilde n\tilde\omega^{-2}e^{\tilde n\tilde\varphi}=0$$

which is also obtained from equation (2.29) by the reduction with respect to the contracted ansatz and presented by Case 1 of Table 2.2. Analogously we obtain contractions of the other reductions.

For Cases 9.2 and 9.4 of [294, Table 2] exact solutions of reduced equations were found in [294]. The substitution of these solutions to the respective anzatze results in the following exact solutions of equation (2.30):

$$u=\left|\frac{x^2}{C-(\alpha n+2+4n^{-1})t}\right|^{\frac{1}{n}},\tag{2.32}$$

$$u = \begin{cases} \left|C_1\sqrt{x}\ln x + C_2\sqrt{x}\right|^{\frac{1}{n+1}}, & \text{if} \quad \alpha' = 0, \\ \left|C_1 x^{\varkappa_1} + C_2 x^{\varkappa_2}\right|^{\frac{1}{n+1}}, & \text{if} \quad \alpha' > 0, \\ \left|C_1\sqrt{x}\sin(\sigma\ln x) + C_2\sqrt{x}\cos(\sigma\ln x)\right|^{\frac{1}{n+1}}, & \text{if} \quad \alpha' < 0, \end{cases} \tag{2.33}$$

where $\alpha' = 1 - 4\alpha(n+1)$, $\varkappa_{1,2} = \dfrac{1 \pm \sqrt{\alpha'}}{2}$, $\sigma = \dfrac{\sqrt{-\alpha'}}{2}$. Here and in what follows C, C_1 and C_2 are arbitrary constants. Applying transformation (2.31) to solution (2.32) and proceeding with the limit $\delta \to +\infty$, we obtain

$$\left(1 + \frac{\tilde{u}}{\delta}\right)^{\delta\tilde{n}} = \tilde{x}^2\left(C - \left(\tilde{\alpha}\tilde{n} + 2 + \frac{4}{\tilde{n}\delta}\right)\tilde{t}\right)^{-1} \to$$
$$e^{\tilde{n}\tilde{u}} = \tilde{x}^2\left(C - (\tilde{\alpha}\tilde{n} + 2)\,\tilde{t}\right)^{-1}.$$

As a result, we construct the exact solution

$$\tilde{u} = \frac{1}{\tilde{n}}\ln\left|\frac{\tilde{x}^2}{C - (\tilde{\alpha}\tilde{n} + 2)\,\tilde{t}}\right|$$

for equation (2.29). Applying the same technique to solutions (3.54) leads to the steady-state solutions of (2.29):

$$\tilde{u} = \begin{cases} \frac{1}{\tilde{n}}\ln\left|C_1\sqrt{\tilde{x}}\ln\tilde{x} + C_2\sqrt{\tilde{x}}\right|, & \text{if} \quad \tilde{\alpha}' = 0, \\ \frac{1}{\tilde{n}}\ln\left|C_1\tilde{x}^{\varkappa_1} + C_2\tilde{x}^{\varkappa_2}\right|, & \text{if} \quad \tilde{\alpha}' > 0, \\ \frac{1}{\tilde{n}}\ln\left|C_1\sqrt{\tilde{x}}\sin(\sigma\ln\tilde{x}) + C_2\sqrt{\tilde{x}}\cos(\sigma\ln\tilde{x})\right|, & \text{if} \quad \tilde{\alpha}' < 0, \end{cases}$$

where $\tilde{\alpha}' = 1 - 4\tilde{\alpha}\tilde{n}$, $\varkappa_{1,2} = \dfrac{1 \pm \sqrt{\tilde{\alpha}'}}{2}$, $\sigma = \dfrac{\sqrt{-\tilde{\alpha}'}}{2}$. Another way for finding this solution is to integrate the reduced equation of Case 4 from Table 2.2. By the obvious transformation $\hat{\varphi} = e^{n\varphi}$ the reduced equation is mapped to the Euler equation $\omega^2\hat{\varphi}_{\omega\omega} + \alpha n\hat{\varphi} = 0$.

Contractions of Conservation Laws. We use contractions in order to construct conservation laws of equations from class (2.17) with $g = 1$ using results obtained in [294] for equations from class (2.16) with the same gauge of g. Note that the consideration can be easily extended to the entire classes (2.17) and (2.16) using transformations from the corresponding equivalence groups.

Roughly speaking, a conservation law of a system $\mathcal{L}$ of differential equations is a divergence expression that vanishes on solutions of this system. Thus, in the case of two independent variables t and x and one unknown function u the general form of conservation laws is $D_tF(t,x,u_{(r)})+D_xG(t,x,u_{(r)})=0$ whenever u is a solution of $\mathcal{L}$. Here D_t and D_x are the operators of total differentiation with respect to t and x, respectively, and $u_{(r)}$ denotes the set of all the derivatives of the functions u with respect to t and x of order not greater than r, including u as the derivative of the zero order. The components F and G of the conserved vector (F,G) are called the *density* and the *flux* of the conservation law. Two conserved vectors (F,G) and (F',G') are equivalent if there exist such functions $\hat{F}$, $\hat{G}$ and H of t, x and derivatives of u that $\hat{F}$ and $\hat{G}$ vanish for all solutions of $\mathcal{L}$ and $F'=F+\hat{F}+D_xH$, $G'=G+\hat{G}-D_tH$. A conserved vector is called trivial if it is equivalent to the zero conserved vector.

It is found in [294] that there are three subclasses of equations of the form (2.16) which admit nontrivial conserved vectors. Thus, assuming the gauge $g=1$, each equation from class (2.16) with 1. $m=n+1$, 2. $m=1$ and $h=\mu f$, and 3. $m=0$, admits two linearly independent conservation laws. They contract to the cases 1. $m=n$, 2. $h=0$, and 3. $m=0$ of class (2.17), respectively.

In order to contract equations from class (2.16) to equations from class (2.17), we should vary the arbitrary element n. This is why only the case of general n is appropriate for contractions. There are three different ways in order to perform contractions of conservation laws. We will contract the equations and the conserved vectors of their conservation laws.

We illustrate this in detail using equations

$$f(x)u_t=(u^nu_x)_x+\alpha f(x)u \tag{2.34}$$

with $n\neq-1$, whose conserved vectors are given by

$$n\neq-1\colon\quad \begin{array}{l}\big(\,xe^{-\mu t}fu,\ e^{-\mu t}\big(-xu^nu_x+\frac{u^{n+1}}{n+1}\big)\,\big),\quad \lambda^1=xe^{-\mu t},\\ \big(\,e^{-\mu t}fu,\ -e^{-\mu t}u^nu_x\,\big),\quad \lambda^2=e^{-\mu t}.\end{array} \tag{2.35}$$

We consider as an example the first conserved vector. At first we apply equivalence transformation

$$\tilde t=t,\quad \tilde x=x,\quad \tilde u=\delta(u-1),\quad \tilde n=\frac{n}{\delta},\quad \tilde\mu=\delta^2\mu \tag{2.36}$$

to equation (2.34) and proceed to the limit $\delta\to+\infty$. As a result, we obtain the class of equations (tildes are omitted)

$$f(x)u_t=(e^{nu}u_x)_x, \tag{2.37}$$

i.e., equations from class (2.20) with $h=0$.

For the image $\tilde\lambda^1$ of the characteristic $\lambda^1=xe^{-\mu t}$ with respect to transformation (2.31) we have that $\tilde\lambda^1\to x$ if $\delta\to+\infty$. Now we are able to construct the corresponding conservation law of (2.37) using the characteristic obtained as an integrating factor. After the multiplication by x the equation (2.37) can be written in divergence form as

$$D_t\,(xfu)+D_x\left(-xe^{nu}u_x+\frac{1}{n}e^{nu}\right)=0. \tag{2.38}$$

Therefore, we constructed conservation law (2.38) of equation (2.37) via carrying out a limiting process of characteristics. Another way is to directly deal with divergence expressions. Thus the conservation law

$$D_t\left(xe^{-\mu t}fu\right)+D_x\left(-e^{-\mu t}xu^nu_x+e^{-\mu t}\frac{u^{n+1}}{n+1}\right)=0$$

of (2.34) with $n\neq-1$ is transformed by (2.31) to

$$D_{\tilde t}\left(\tilde xe^{-\frac{\tilde\mu}{\delta^2}\tilde t}f\left(\frac{\tilde u}{\delta}+1\right)\right)\\ +D_{\tilde x}\left(-e^{-\frac{\tilde\mu}{\delta^2}\tilde t}\tilde x\left(\frac{\tilde u}{\delta}+1\right)^{\tilde n\delta}\frac{\tilde u_{\tilde x}}{\delta}+e^{-\frac{\tilde\mu}{\delta^2}\tilde t}\frac{1}{\tilde n\delta+1}\left(\frac{\tilde u}{\delta}+1\right)^{\tilde n\delta+1}\right)=0.$$

Multiplying the obtained expression by δ and adding the term $-\tilde{x}f\delta$ under $D_{\tilde{t}}$, we proceed to the limit $\delta \to +\infty$. (As the term $-\tilde{x}f\delta$ depends only on $\tilde{x}$, it is negligible in view of the differentiation with respect to $\tilde{t}$.) Omitting tildes, we exactly obtain the conservation law (2.38).

Conserved vectors can be contracted in the same way using the fact that we can add expression which depends on $\tilde{x}$ only to the density component up to equivalence of conserved vectors.

Contracting the conservation laws obtained in [294] jointly with the corresponding equations, we obtain the following assertion.

Theorem 2.25. *A complete list of equations from class* (2.20) *possessing nontrivial conservation laws is exhausted by the following ones.*

1. $m = n$: $\quad \big(\varphi^i fu,\ -\varphi^i e^{nu}u_x + \frac{1}{n}\varphi^i_x e^{nu}\big),\ \varphi^i,\ i = 1, 2.$

2. $m = 0$: $\quad \big(xfu,\ -xe^{nu}u_x + \frac{1}{n}e^{nu} - \int xh\,\mathrm{d}x\big),\ x;$

$\quad \big(fu,\ -e^{nu}u_x - \int h\,\mathrm{d}x\big),\ 1.$

Here the functions $\varphi^i = \varphi^i(x)$, $i = 1, 2$, form a fundamental set of solutions of the second-order linear ordinary differential equation $\varphi_{xx} + nh\varphi = 0$.

Remark 2.26. The case $h = 0$:

$$\big(xfu,\ -xe^{nu}u_x + \frac{1}{n}e^{nu}\big),\ x; \quad \big(fu,\ -e^{nu}u_x\big),\ 1,$$

appears as a particular subcase of cases 1 and 2 adduced in Theorem 2.25.

Simultaneously with constraints on the arbitrary elements we also present conserved vectors and characteristics of the basis elements of the corresponding space of conservation laws.

2.3. Potential Symmetries of a Class of Porous Medium Equations

Bluman et al. [38, 39] introduced a method for finding a new class of symmetries (non-Lie ones) for a system of PDEs $\Delta(x, u)$, in the case that this system has at least one conservation law. If we introduce potential variables v for the equations written in conserved forms as further unknown functions, we obtain a system $Z(x,u,v)$. Any Lie symmetry for $Z(x,u,v)$ induces a symmetry for $\Delta(x,u)$. When at least one of the infinitesimals which correspond to the variables x and u depends explicitly on potentials, then the local symmetry of $Z(x,u,v)$ induces a nonlocal symmetry of $\Delta(x,u)$. These nonlocal symmetries are called *potential symmetries*. More details about potential symmetries and their applications can be found in [30, 31, 38]. Potential symmetries were investigated for quite general classes of differential equations. The problem of finding criteria for the existence of potential symmetries for classes of differential equations was posed in [253]. Some useful criteria were derived for PDEs in two independent variables. Nonclassical potential symmetries of such equations were discussed in [261].

In [98] the construction of hidden potential symmetries for some classes of diffusion equations is claimed. Here we show that these symmetries are usual potential symmetries that can be derived using the conventional method by Bluman and collaborators.

A complete classification of potential symmetries can be achieved by considering all potential systems that correspond to the conservation laws. It is known [245] that the equivalence group for a class of systems of differential equations or the symmetry group for a single system can be prolonged to potential variables. It is natural to use these prolonged equivalence groups for classification of possible potential symmetries. In view of this statement we will classify potential symmetries of diffusion equations up to the (trivial) prolongation of their equivalence groups to the corresponding potentials.

The first step in the investigation of potential symmetries is to calculate the conservation laws. The conventional symmetry approach for this is based on Noether's theorem but it cannot be directly used for evolution equations. There exists no Lagrangian for which an evolution equation is an Euler–Lagrange equation. Hence the application of Noether's theorem in this case is possible only for particular equations and after special technical tricks. At the same time, the definition of conservation laws itself gives rise to a method of finding conservation laws, which is called *direct* and can be applied to any system of differential equations with no restriction on its structure. The technique of calculations used within the framework of this method is similar to the classical Lie method yielding symmetries of differential equations. Four versions of it are distinguished in the literature depending on the way of taking into account systems under consideration and the usage either the definition of conserved vectors or the characteristic form of conservation laws. See, e.g., [10,11,247] on details of the calculation technique. The necessary theoretical background is given in [217]. In the present work we employ the most direct version [247] based on immediate solving of determining equations for conserved vectors of conservations laws on the solution manifolds of investigated systems and additionally combined with techniques involving symmetry or equivalence transformations.

In [98] the porous medium equations of the form

$$u_t = ((u^n)_x + f(x)u^m)_x\,, \tag{2.39}$$

with $n \neq 0$ was given without considering its potential symmetries. It was stated that the complete classification of potential symmetries was carried out in [97]. There are three remarks on this statement.

1) In [97] the case $m = 0$ was omitted from the consideration since another representation for which the value $m = 0$ is singular was used for equations from class (2.39). At the same time, the corresponding subclass of class (2.39) contains well-known equations, e.g., the linearizable equation $u_t = (u^{-2}u_x)_x + 1$. Moreover, some equations from the cases $m = 0$ and $m \neq 0$ are connected within both the point and potential frame.

2) The description of potential symmetries in [97] was not a classifica-

tion since no equivalence relations of equations or symmetries were used.

3) Only simplest potential symmetries arising under the study of the corresponding "natural" potential systems were found. The problem on the construction of the other simplest and, moreover, general potential symmetries of equations from class (2.39) was still open.

We employ class (2.39) in order to give the basic steps for the exhaustive classification of the simplest potential symmetries. Moreover, in the next section we completely describe potential symmetries of some equations from class (2.39).

The equivalence group $G^{\sim}$ of class (2.39) is formed by the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x + \delta_4, \quad \tilde{u} = {\delta_1}^{-\frac{1}{n-1}} {\delta_3}^{\frac{2}{n-1}} u,$$
$$\tilde{f} = {\delta_1}^{\frac{m-n}{n-1}} {\delta_3}^{\frac{n-2m+1}{n-1}} f, \quad \tilde{n} = n, \quad \tilde{m} = m,$$

where δ_i, $i = 1, \ldots, 4$, are arbitrary constants, $\delta_1\delta_3 \neq 0$. Additionally, the subclass singled out from (2.39) by the condition $m = n$ can be mapped to the subclass consisting of the equations

$$e^{-\frac{n+1}{n}\int f(x)dx} \tilde{u}_{\tilde{t}} = (\tilde{u}^n)_{\tilde{x}\tilde{x}}$$

by the transformation

$$\tilde{t} = t, \quad \tilde{x} = \int e^{\int f(x)dx} dx, \quad \tilde{u} = e^{\frac{1}{n}\int f(x)dx} u. \tag{2.40}$$

We present the conservation laws for (2.39) and the subsequent potential systems.

Theorem 2.27. *Any equation from class* (2.39) *has the conservation law of form* $D_tF + D_xG = 0$, *whose density and flux are, respectively,*

$$1.\ F = u, \quad G = -nu^{n-1}u_x - fu^m. \tag{2.41}$$

A complete list of $G^{\sim}$*-inequivalent equations* (2.39) *having additional (i.e. linear independent with* (2.41)*) conservation laws is exhausted by the following ones*

2. $m=n\neq 1:\ F=u\int e^{\int f dx}dx,\ G=-\int e^{\int f dx}dx\,(nu^{n-1}u_x{+}fu^n){+}e^{\int f dx}u^n$,
3. $n\neq 1,\ m=0:\ F=xu,\ G=-x(nu^{n-1}u_x+f)+u^n+\int f dx$,
4. $n\neq 1,\ m=1,\ f=1:\ F=(t+x)u,\ G=-(t+x)(nu^{n-1}u_x+u)+u^n$,
5. $n\neq 1,\ m=1,\ f=\varepsilon x:\ F=e^{\varepsilon t}xu,\ G=-e^{\varepsilon t}x(nu^{n-1}u_x+xu)+e^{\varepsilon t}u^n$,
6. $n=1,\ m=0:\ F=\alpha u,\ G=-\alpha(u_x+f)+\alpha_x u+\int \alpha_x f dx$,
7. $n=1,\ m=1:\ F=\beta u,\ G=-\beta(u_x{+}fu){+}\beta_x u$, *where* $\varepsilon=\pm 1 \bmod G^{\sim}$, $\alpha=\alpha(t,x)$ *and* $\beta=\beta(t,x)$ *are arbitrary solutions of the linear equations* $\alpha_t{+}\alpha_{xx}=0$ *and* $\beta_t{+}\beta_{xx}{-}f\beta_x=0$, *respectively. (Together with restrictions on values* f, n *and* m *we also adduce densities and fluxes of additional conservation laws.)*

These conservation laws can be used for the construction of potential systems that lead to potential symmetries for the equation (2.39). The associated characteristics are equal to the coefficients of u in the presented expressions for F. Here we consider only simplest potential systems (i.e., potential systems with one potential variable, constructed with usage of single conserved vectors of basis conservation laws) of equations from class (2.39), having the form $v_x=F$, $v_t=-G$. Cases 6 and 7 of Theorem 2.27 can be excluded from the investigation since they concern linear equations studied in [249]. Then, the equations of Case 2 are reducible to diffusion equations of form (A.52) by transformation (2.40). The equations of Cases 4 and 5 are reducible to the constant coefficient diffusion equation $\tilde u_{\tilde t}=(\tilde u^n)_{\tilde x\tilde x}$ by means of the Galilei transformation

$$\tilde t=t,\quad \tilde x=x+t,\quad \tilde u=u$$

and the transformation

$$\tilde t=\begin{cases}\frac{1}{\varepsilon(n+1)}e^{\varepsilon(n+1)t}, & n\neq -1,\\ t, & n=-1,\end{cases}\qquad \tilde x=e^{\varepsilon t}x,\quad \tilde u=e^{-\varepsilon t}u,$$

respectively.

Therefore, we have to investigate only two potential systems

$$v_x = u, \quad v_t = nu^{n-1}u_x + fu^m, \tag{2.42}$$

$$v^*_x = xu, \quad v^*_t = x(nu^{n-1}u_x + f) - u^n - \int f dx \tag{2.43}$$

corresponding to cases 1 and 3 of Theorem 2.27. (To distinguish the potential introduced, we denote the second potential by v^*.)

Lie point symmetries of the potential system (2.42) give nontrivial potential symmetries of equation (2.39) in the following cases:

1. $f = x^{-2}$, $n = -1$, $m = -2$:
   $\langle \partial_t,\ \partial_v,\ x\partial_x - u\partial_u,$
   $12t\partial_t + (3\ln x - v)x\partial_x + (3 - 3\ln x + xu + v)u\partial_u + 2(3v - t)\partial_v\rangle;$
2. $f = \varepsilon x \ln|x|$, $\varepsilon = \pm 1$, $n = -1$, $m = 1$:
   $\langle \partial_t,\ \partial_v,\ e^{-\varepsilon t}(x\partial_x - u\partial_u),\ e^{-\varepsilon t}(-\varepsilon xv\partial_x + \varepsilon(xu^2 + uv)\partial_u + 2\partial_v)\rangle;$
3. $f = x$, $n = -1$, $m = 0$:
   $\langle \partial_t,\ \partial_v,\ \partial_x + t\partial_v,\ 2t\partial_t - x\partial_x + 2u\partial_u + v\partial_v,$
   $t^2\partial_t + (v - tx)\partial_x + (2t - u)u\partial_u + tv\partial_v\rangle;$
4. $f = 0$, $n = -1$, $m = 0$:
   $\langle \partial_t,\ \partial_v,\ x\partial_x - u\partial_u,\ 2t\partial_t + u\partial_u + v\partial_v,\ vx\partial_x - (v + xu)u\partial_u + 2t\partial_v,$
   $4t^2\partial_t + (v^2 - 2t)x\partial_x - (v^2 + 2vxu - 6t)u\partial_u + 4tv\partial_v,\ \beta\partial_x - \beta_v u^2\partial_u\rangle;$
5. $f = 1$, $n = -1$, $m = 1$:
   $\langle \partial_t,\ \partial_v,\ (x + t)\partial_x - u\partial_u,\ 2t\partial_t + 2x\partial_x - u\partial_u + v\partial_v,\ \beta\partial_x - \beta_v u^2\partial_u,$
   $v(x + t)\partial_x - [(x + t)u + v]u\partial_u + 2t\partial_v,$
   $4t^2\partial_t + [(x + t)v^2 - 2tx - 6t^2]\partial_x - [v^2 - 6t + 2(x + t)vu]u\partial_u + 4tv\partial_v\rangle;$
6. $f = \varepsilon x$, $\varepsilon = \pm 1$, $n = -1$, $m = 1$:
   $\langle \partial_t,\ \partial_v,\ x\partial_x - u\partial_u,\ 2t\partial_t - 2\varepsilon tx\partial_x + (1 + 2\varepsilon t)u\partial_u + v\partial_v,$
   $4t^2\partial_t + \left(v^2 - 2t - 4\varepsilon t^2\right)x\partial_x - \left(v^2 - 6t + 2xvu - 4\varepsilon t^2\right)u\partial_u + 4tv\partial_v,$
   $vx\partial_x - (xu + v)u\partial_u + 2t\partial_v,\ e^{-\varepsilon t}(\beta\partial_x - \beta_v u^2\partial_u)\rangle;$
7. $n = -1$, $m = -1$:

$\langle \partial_t,\ \partial_v,\ \psi\partial_x - (1-f\psi)u\partial_u,\ 2t\partial_t + u\partial_u + v\partial_v,$
$4t^2\partial_t + (v^2-2t)\psi\partial_x - [2\psi vu + (v^2-2t)(1-f\psi) - 4t]u\partial_u + 4tv\partial_v,$
$\psi v\partial_x - ((1-f\psi)v + \psi u)u\partial_u + 2t\partial_v,\ \varphi(\beta\partial_x - (\beta_v u^2 - f\beta u)\partial_u)\rangle;$

8. $f=1,\ n=1,\ m=2$:
$\langle \partial_t,\ \partial_x,\ \partial_v,\ 2t\partial_t + x\partial_x - u\partial_u,\ 4t^2\partial_t + 4tx\partial_x - 2(2tu+x)\partial_u - (2t+x^2)\partial_v,$
$2t\partial_x - \partial_u - x\partial_v,\ e^{-v}[(\alpha u - \alpha_x)\partial_u - \alpha\partial_v]\rangle.$

Here $\alpha = \alpha(t,x)$ and $\beta = \beta(t,v)$ run through the solution sets of the linear heat equation $\alpha_t - \alpha_{xx} = 0$ and backward linear heat equation $\beta_t + \beta_{vv} = 0$, respectively, $\varphi(x) = e^{-\int f dx}$, $\psi(x) = e^{-\int f dx}\int e^{\int f dx} dx$.

Note 2.28. *Equations from class (2.39) with $n=-1$, $m=0$ and $f\in\{0,1\}$ are just different representations of the same equation. Potential systems corresponding to these two cases are connected via the transformation $\tilde v = v + t$ of potential variable v. This transformation maps case 4 to the case*

$f=1,\ n=-1,\ m=0$:
$\langle \partial_t,\ \partial_{\tilde v},\ x\partial_x - u\partial_u,\ 2t\partial_t + u\partial_u + (t+\tilde v)\partial_{\tilde v},$
$4t^2\partial_t + [(\tilde v - t)^2 - 2t]x\partial_x - [(\tilde v - t)^2 + 2(\tilde v - t)xu - 6t]u\partial_u + 4t\tilde v\partial_{\tilde v},$
$(\tilde v - t)x\partial_x - (\tilde v - t + xu)u\partial_u + 2t\partial_{\tilde v},\ e^{\frac{t}{4}-\frac{\tilde v}{2}}[\tilde\beta\partial_x - (\tilde\beta_{\tilde v} - \frac12\tilde\beta)u^2\partial_u]\rangle,$

where the function $\tilde\beta = \tilde\beta(t,\tilde v)$ runs through the solution set of the backward linear heat equation $\tilde\beta_t + \tilde\beta_{\tilde v\tilde v} = 0$.

Note 2.29. *Cases 5, 6 and 7 are reduced to case 4 by the point transformations $\{\tilde x = x + t,\ \tilde u = u\}$, $\{\tilde x = e^{\varepsilon t}x,\ \tilde u = e^{-\varepsilon t}u\}$ and (2.40), respectively. The variables t and v are identically transformed.*

As mentioned, Lie symmetries of potential system (2.42) were investigated in [97] only for $m \neq 0$ and hence cases 3 and 4 were omitted there. It is explained by choice of another representation of equation (2.39). For all values of m, potential symmetries of equation (2.39) associated with potential system (2.42) are first classified above.

There exists only one inequivalent case of potential system (2.43) that gives nontrivial potential symmetries for equation (2.39), namely, $f = x$, $n = -1$ and $m = 0$. Lie algebra of potential symmetries in this case is

9. $f = x$, $n = -1$, $m = 0$:

$\langle \partial_t,\ \partial_v,\ 2t\partial_t - x\partial_x + 2u\partial_u,\ e^{-v^*/2}(2x\partial_x + (x^2u - 2)u\partial_u - 4\partial_{v^*})\rangle.$

Here we give only one example on potential symmetries of equations from class (2.39) with $f \neq 0$, which involve two potentials. They are easily constructed via the point transformation (2.40) from potential symmetries presented in Section A.4. Namely, the equation $x^{-6}u_t = (u^{-2/3}u_x)_x$ from class (A.52) (case 3.4, $f = x^{-6}$, $n = -2/3$) is mapped by the transformation $\tilde{t} = \frac{3}{4}t$, $\tilde{x} = |x|^{-1/2}$, $\tilde{u} = |x|^{-9/2}u$ to the equation

$$\tilde{u}_{\tilde{t}} = ((\tilde{u}^{1/3})_{\tilde{x}} - 3\tilde{x}^{-1}\tilde{u}^{1/3})_{\tilde{x}} \tag{2.44}$$

from class (2.39), where $\tilde{n} = \tilde{m} = 1/3$ and $\tilde{f} = -3\tilde{x}^{-1}$. For coefficients to be simpler, we have additionally combined the corresponding transformation of form (2.40) with a scaling. The second order potential system for equation (2.44), which is constructed from (A.54) via the transformation prolonged to the potentials as $\tilde{v} = -v^1/2$ and $\tilde{w} = w/4$, is

$$\tilde{v}_{\tilde{x}} = \tilde{u}, \quad \tilde{w}_{\tilde{x}} = \tilde{x}^{-3}\tilde{v}, \quad \tilde{w}_{\tilde{t}} = \tilde{x}^{-3}\tilde{u}^{1/3}.$$

The associated potential symmetry algebra of (2.44) is (we omit tildes for convenience)

$$\begin{aligned}
&\langle \partial_t,\ \partial_w,\ 2\partial_v - x^2\partial_w,\ 2t\partial_t + 3u\partial_u + 3v\partial_v + 3w\partial_w,\\
&x\partial_x - 3u\partial_u - 2v\partial_v - 4w\partial_w,\\
&x^{-1}\partial_x + 3x^{-2}u\partial_u + 2(2w + x^{-2}v)\partial_v - 2x^{-2}w\partial_w,\\
&xw\partial_x - 3(x^{-2}v + w)u\partial_u - (x^{-2}v + 2w)v\partial_v - 2w^2\partial_w\rangle.
\end{aligned}$$

2.4. Group Classification of the Fisher equation with Time Dependent Coefficients

The classical Fisher equation

$$u_t = bu_{xx} + au(1-u), \quad ab \neq 0, \tag{2.45}$$

first appeared seventy-five years ago in the seminal paper [80]. This equation was originally derived to model the propagation of a gene in a population. More precisely, the dependent variable, u, stands for the frequency of the mutant gene in a population distributed in a linear habitat, such as a shore line, with uniform density. Theorems on existence and uniqueness of bounded solutions for equations of the more general form $u_t = u_{xx} + F(t,x,u)$ were proved in [161]. Traveling-wave solutions of (2.45) were constructed in [5] (see also [52, 66, 168, 208]). In [117, 213] it was proposed to consider generalized Fisher equations with a time dependent diffusion coefficient, b, and a time dependent favorability coefficient, a, namely the class of equations

$$u_t = b(t)u_{xx} + a(t)u(1-u), \tag{2.46}$$

where $a(t)$ and $b(t)$ are smooth nonvanishing functions. In practice these coefficients could represent long term changes in climate or short term seasonality [117]. Solutions for equations from this class were constructed in [117, 213]. In this section we study symmetry properties of equations from class (2.46) and find their exact solutions [291, 303].

2.4.1. Admissible Transformations. We search the admissible transformations using the direct method [160]. The following statement is true.

Theorem 2.30. *The usual equivalence group $G^{\sim}$ of class (2.46) comprises the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \varepsilon u + \frac{1-\varepsilon}{2}, \quad \tilde{a} = \frac{a}{T_t \varepsilon}, \quad \tilde{b} = \frac{\delta_1^2}{T_t} b,$$

where $T(t)$ is an arbitrary smooth function with $T_t \neq 0$, δ_1 and δ_2 are arbitrary constants with $\delta_1 \neq 0$ and $\varepsilon = \pm 1$.

Remark 2.31. Up to composing to each other and to continuous equivalence transformations, the equivalence group $G^{\sim}$ contains three independent discrete transformations

$$\begin{aligned}
\mathcal{T}_1 &\colon\ (t,x,u,a,b) \mapsto (-t,x,u,-a,-b),\\
\mathcal{T}_2 &\colon\ (t,x,u,a,b) \mapsto (t,-x,u,a,b),\\
\mathcal{T}_3 &\colon\ (t,x,u,a,b) \mapsto (t,x,1-u,-a,b).
\end{aligned}$$

Theorem 2.32. *Class (2.46) is normalized with respect to its generalized extended equivalence group $\hat{G}^{\sim}$ formed by the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \omega(t)u + \theta(t),$$
$$\tilde{a} = \frac{a}{T_t\omega}, \quad \tilde{b} = \frac{{\delta_1}^2}{T_t} b,$$

where $T(t)$ is an arbitrary smooth function with $T_t \neq 0$, δ_1 and δ_2 are arbitrary constants with $\delta_1 \neq 0$,

$$\omega = \frac{(\alpha e^{\int a\,dt} + \beta)(\gamma e^{\int a\,dt} + \delta)}{(\alpha\delta - \beta\gamma)e^{\int a\,dt}}, \quad \theta = -\gamma\frac{\alpha e^{\int a\,dt} + \beta}{\alpha\delta - \beta\gamma},$$

the constant pairs (α,β) and (γ,δ) are defined up to nonvanishing multipliers and $\alpha\delta - \beta\gamma \neq 0$. In other words, the equivalence groupoid of class (2.46) is generated by the generalized extended equivalence group $\hat{G}^{\sim}$ of this class.

The proof can be found in [303]. It is obvious that there are equations in class (2.46) that are $\hat{G}^{\sim}$-equivalent but not $G^{\sim}$-equivalent. Therefore the usage of the group $\hat{G}^{\sim}$ strongly simplifies the group analysis of class (2.46).

Corollary 2.33. *Equation (2.46) reduces to the classical Fisher equation*

$$u_t = u_{xx} + u(1-u) \tag{2.47}$$

by a point transformation if and only if for some positive constant λ the coefficients a and b satisfy the condition

$$\lambda b^2 - 2\frac{b_{tt}}{b} + 3\frac{b_t^2}{b^2} = a^2 - 2\frac{a_{tt}}{a} + 3\frac{a_t^2}{a^2}. \tag{2.48}$$

Remark 2.34. After the exclusion of the constant λ by differentiation with respect to t, the condition (2.48) reduces to the condition

$$\frac{b_{ttt}}{b} + 6\frac{b_t^3}{b^3} - 6\frac{b_{tt}b_t}{b^2} = \frac{a_{ttt}}{a} - 4\frac{a_{tt}a_t}{a^2} + 3\frac{a_t^3}{a^3} - aa_t - 2\frac{a_{tt}}{a}\frac{b_t}{b} + 3\frac{a_t^2}{a^2}\frac{b_t}{b} + a^2\frac{b_t}{b},$$

which is more convenient for checking using a computer algebra package.

Remark 2.35. The condition (2.48) is satisfied if and only if the function b is expressed in terms of a as

$$b = \frac{\lambda(\alpha\delta - \beta\gamma)ae^{\int a\,dt}}{(\alpha e^{\int a\,dt} + \beta)(\gamma e^{\int a\,dt} + \delta)},$$

where λ is a positive constant, the constant pairs (α, β) and (γ, δ) are defined up to nonvanishing multipliers and $\alpha\delta - \beta\gamma \neq 0$.

2.4.2. Gauging of Arbitrary Elements. Equivalence transformations allow us to simplify the group classification problem by gauging arbitrary elements. For example, there is one arbitrary parameter-function $T(t)$ in the equivalence groups $G^{\sim}$ and $\hat{G}^{\sim}$ of class (2.46). It means that we can gauge an arbitrary element, either a or b, to a simple constant value, e.g., to 1. Thus the equivalence transformation $\tilde{t} = \int b\,dt$, $\tilde{x} = x$, $\tilde{u} = u$ belonging to the group $G^{\sim}$ maps class (2.46) onto its subclass singled out by the constraint $b = 1$. The arbitrary element $\tilde{a}$ of the mapped class equals a/b. The gauge $a = 1$ is realized by the similar point transformation from $G^{\sim}$ $\tilde{t} = \int a\,dt$, $\tilde{x} = x$, $\tilde{u} = u$. In the corresponding mapped class we have $\tilde{a} = 1$ and $\tilde{b} = b/a$.

Since class (2.46) is normalized in the generalized sense, it is easy to find the equivalence group of its subclass with $b = 1$ (resp. $a = 1$) by setting $\tilde{b} = b = 1$ (resp. $\tilde{a} = a = 1$) in the transformations from $\hat{G}^{\sim}$. We obtain the following corollaries of Theorem 2.32.

Corollary 2.36. *The class of equations of the general form*

$$u_t = u_{xx} + a(t)u(1-u), \tag{2.49}$$

where a runs through the set of nonvanishing smooth functions of t, is normalized in the generalized extended sense. The generalized extended equivalence group $\hat{G}^{\sim}_a$ of this class consists of the transformations

$$\tilde{t} = \delta_1^2 t + \delta_0, \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \omega(t)u + \theta(t), \quad \tilde{a} = \frac{a}{\delta_1^2\omega},$$

where δ_i, $i = 0, 1, 2$, are arbitrary constants with $\delta_1 \neq 0$,

$$\omega = \frac{(\alpha e^{\int a\,dt} + \beta)(\gamma e^{\int a\,dt} + \delta)}{(\alpha\delta - \beta\gamma)e^{\int a\,dt}}, \quad \theta = -\gamma\frac{\alpha e^{\int a\,dt} + \beta}{\alpha\delta - \beta\gamma},$$

the constant pairs (α, β) and (γ, δ) are defined up to nonvanishing multipliers and $\alpha\delta - \beta\gamma \neq 0$.

Corollary 2.37. *The class of equations of the general form*

$$u_t = b(t)u_{xx} + u(1-u), \tag{2.50}$$

where b runs through the set of nonvanishing smooth functions of t, is normalized in the usual sense. The usual equivalence group $G^{\sim}_b$ of this class consists of the transformations

$$\tilde{t} = \ln\frac{\alpha e^t + \beta}{\gamma e^t + \delta}, \quad \tilde{x} = \delta_1 x + \delta_2,$$
$$\tilde{u} = \frac{(\alpha e^t + \beta)(\gamma e^t + \delta)}{(\alpha\delta - \beta\gamma)e^t}u - \gamma\frac{\alpha e^t + \beta}{\alpha\delta - \beta\gamma}, \quad \tilde{b} = \frac{\delta_1^2(\alpha e^t + \beta)(\gamma e^t + \delta)}{(\alpha\delta - \beta\gamma)e^t}b,$$

where δ_j, $j = 1, 2$, are arbitrary constants with $\delta_1 \neq 0$, the constant quadruple $(\alpha, \beta, \gamma, \delta)$ is defined up to a nonzero multiplier and $\alpha\delta - \beta\gamma \neq 0$.

Remark 2.38. The group $G^{\sim}_b$ contains two discrete equivalence transformations

$$\mathcal{T}' : (t, x, u, b) \mapsto (t, -x, u, b), \quad \mathcal{T}'' : (t, x, u, b) \mapsto (-t, x, 1-u, -b).$$

An interesting question is which of the above two gauges is preferable for further consideration. Class (2.49) is still normalized only in the generalized extended sense. At the same time class (2.50) is normalized with respect to its usual equivalence group. This is why we can expect that it is easier to perform the group classification in class (2.50) rather than in class (2.49).

Corollary 2.39. *Eq. (2.50) reduces to the classical Fisher equation (2.47) by a point transformation if and only if the coefficient b has the form*

$$b(t) = \frac{\lambda(\alpha\delta - \beta\gamma)e^t}{(\alpha e^t + \beta)(\gamma e^t + \delta)},$$

where λ is a positive constant, the constant quadruple $(\alpha, \beta, \gamma, \delta)$ is defined up to a nonzero multiplier and $\alpha\delta - \beta\gamma \neq 0$.

2.4.3. Lie Symmetries. We study the Lie symmetries of equations from class (2.50) using the classical approach [227]. We fix an equation, $\mathcal{L}$, from class (2.50) and search for vector fields of the form

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$$

that generate one-parameter point symmetry groups of $\mathcal{L}$. The determining equations derived using the the infinitesimal invariance criterion imply $\tau = \tau(t)$ and $\xi = \xi(t,x)$. This completely agrees with the general results on point transformations between evolution equations (see [160] and Section 1.1). Then the remaining determining equations take the form

$$\begin{gathered}\eta_{uu} = 0, \quad 2b\,\xi_x = (b\tau)_t, \quad 2b\,\eta_{xu} = b\xi_{xx} - \xi_t,\\ \eta - \eta_t + b\eta_{xx} + (\tau_t - 2\eta - \eta_u)u + (\eta_u - \tau_t)u^2 = 0.\end{gathered}$$

The integration of the first two equations of this system results in

$$\xi = \frac{(b\tau)_t}{2b}x + \zeta(t) \quad \text{and} \quad \eta = \eta^1(t,x)u + \eta^0(t,x),$$

where ζ, η^1 and η^0 are arbitrary functions of their arguments. Then the third equation becomes

$$\left(\frac{1}{2b}(b\tau)_t\right)_t x + \zeta_t + 2b\eta^1_x = 0. \tag{2.51}$$

After substituting the above expression for η into the fourth equation and splitting this equation with respect to u, we get

$$\eta^1 = -\tau_t, \quad \eta^0_t - b\eta^0_{xx} - \eta^0 = 0, \quad 2\eta^0 = \tau_t + b\eta^1_{xx} - \eta^1_t.$$

The last system implies that η^1 and η^0 do not depend upon x and are expressed via derivatives of the function τ as follows

$$\eta^1 = -\tau_t, \quad \eta^0 = \frac{1}{2}(\tau_t + \tau_{tt}).$$

The function τ satisfies the equation $\tau_{ttt} - \tau_t = 0$, i.e.,

$$\tau = c_1 e^t + c_2 e^{-t} + c_3$$

for some constants c_1, c_2 and c_3. We take into account all the constraints derived and split Eq. (2.51) with respect to x. As a result we immediately obtain $\zeta = c_0 = \mathrm{const}$ and the classifying equation $((b\tau)_t/b)_t = 0$ which essentially includes both the residuary uncertainties in the coefficients of the vector field Q and the arbitrary element b. We integrate the classifying equation once to obtain

$$(c_1 e^t + c_2 e^{-t} + c_3)b_t = (-c_1 e^t + c_2 e^{-t} + c_4)b \tag{2.52}$$

with one more constant, c_4. To find the common part of the maximal Lie invariance algebras (the kernel algebra) of equations from class (2.50), we split equation (2.52) with respect to b and b_t, which gives $c_i = 0$, $i = 1, 2, 3, 4$. The only nonzero constant c_0 corresponds to the operator ∂_x (Case 0 of Table 2.3).

The set V_b of coefficient tuples of equations of the form (2.52) satisfied by a fixed value of the parameter-function b is a linear space. The dimension, k_b, of this space coincides with the dimension of Lie symmetry extension for the same value of b. It is easy to prove that $k_b < 3$ and, if $k_b = 2$, any element of V_b satisfies the equation $c_4^2 = c_3^2 - 4c_1c_2$.

An at least one-dimensional extension of Lie symmetry exists only for values of b satisfying (2.52) with $(c_1, c_2, c_3) \neq (0, 0, 0)$, i.e., if

$$b(t) = c_5 \exp\left(\int \frac{-c_1e^t + c_2e^{-t} + c_4}{c_1e^t + c_2e^{-t} + c_3}\mathrm{dt}\right) \tag{2.53}$$

for some nonzero constant c_5. Then an extension operator is of the form

$$X_2 = (c_1e^t + c_2e^{-t} + c_3)\partial_t + \frac{c_4}{2}x\partial_x + \left[(c_2e^{-t} - c_1e^t)u + c_1e^t\right]\partial_u.$$

The integration in (2.53) gives the following values of b:

$$b = \frac{c_5}{c_1e^t + c_2e^{-t} + c_3}\left|\frac{2c_1e^t + c_3 - \nu}{2c_1e^t + c_3 + \nu}\right|^{\frac{c_4}{\nu}} \quad \text{if} \quad D > 0,\ c_1 \neq 0,$$

$$b = \frac{c_5e^{\frac{c_4}{c_3}t}}{(c_2e^{-t} + c_3)^{1-\frac{c_4}{c_3}}} \quad \text{if} \quad D > 0,\ c_1 = 0,$$

$$b = \frac{c_5}{c_1e^t + c_2e^{-t} + c_3}\exp\left(-\frac{2c_4}{2c_1e^t + c_3}\right) \quad \text{if} \quad D = 0,\ c_1 \neq 0,$$

$$b = c_5\exp\left(t + \frac{c_4}{c_2}e^t\right) \quad \text{if} \quad D = 0,\ c_1 = 0,$$

$$b = \frac{c_5}{c_1e^t + c_2e^{-t} + c_3}\exp\left(\frac{2c_4}{\nu}\arctan\frac{2c_1e^t + c_3}{\nu}\right) \quad \text{if} \quad D < 0.$$

Here $D = c_3^2 - 4c_1c_2$ and $\nu = \sqrt{|D|}$. When one uses the scaling transformation with respect to x, the constant c_5 can be set to $\operatorname{sign} c_5$, i.e., $c_5 = \pm 1 \bmod G_b^{\sim}$.

In fact the above expressions for b can be simplified more by transformations from the group $G_b^{\sim}$. Up to $G_b^{\sim}$-equivalence the parameter quadruple (c_1, c_2, c_3, c_4) can be assumed to belong to the set

$$\{(0, 0, 1, \sigma),\ (0, 1, 0, \kappa),\ (1, 1, 0, \rho) \mid \sigma \geqslant 0,\ \kappa = \pm 1,\ \rho \in \mathbb{R}\}.$$

Indeed, combined with multiplication by a nonzero constant, each transformation from the equivalence group $G_b^\sim$ is extended to the coefficient quadruple of equation (2.52) in the following way:

$$\tilde{c}_1 = c_1\delta^2 - c_3\gamma\delta + c_2\gamma^2, \qquad \tilde{c}_2 = c_1\beta^2 - c_3\alpha\beta + c_2\alpha^2,$$
$$\tilde{c}_3 = -2c_1\beta\delta + c_3(\alpha\delta + \beta\gamma) - 2c_2\alpha\gamma, \qquad \tilde{c}_4 = (\alpha\delta - \beta\gamma)c_4.$$

There are three $G_b^\sim$-inequivalent reduced forms of the triple (c_1, c_2, c_3) depending upon the sign of D,

$$(0,0,1) \quad \text{if} \quad D > 0, \quad (0,1,0) \quad \text{if} \quad D = 0, \quad (1,1,0) \quad \text{if} \quad D < 0.$$

So, up to $G_b^\sim$-equivalence, which coincides with the general point equivalence, there are three types of equations from class (2.46) the maximal Lie symmetry algebras of which are two-dimensional. They are represented by Cases 1–3 of Table 2.3. In view of the constraint $c_4^2 = c_3^2 - 4c_1c_2$ any case of extension of the kernel algebra by two linearly independent operators reduces by equivalence transformations to Case 4 of Table 2.3, where $b = \pm e^t$. Since class (2.50) is normalized, there are no additional equivalence transformations between the cases listed in Table 2.3.

As a result we have proven the following theorem.

Theorem 2.40. *The kernel algebra of class (2.50) is $A^\cap = \langle \partial_x \rangle$. $G_b^\sim$-inequivalent Lie symmetry extensions for class (2.50) are exhausted by those presented in Table 2.3.*

The classification list adduced in Table 2.3 represents by itself the result of group classification problem for class (2.46) up to $\hat{G}^\sim$-equivalence.

2.4.4. Exact Solutions. Criterion (2.48) shows that the subclass of equations (2.46) of the form

$$u_t = \frac{\lambda \Delta a(t) h(t)}{(\alpha h(t) + \beta)(\gamma h(t) + \delta)} u_{xx} + a(t)u(1-u), \tag{2.54}$$

Table 2.3: The group classification of class (2.50).

no.	b	Basis elements of $A^{\max}$
0	$\forall$	∂_x
1	$\varepsilon e^{\sigma t}$	$\partial_x,\quad \partial_t + \frac{\sigma}{2}x\partial_x$
2	$\varepsilon \exp(t+\kappa e^t)$	$\partial_x,\quad e^{-t}\partial_t + \frac{\kappa}{2}x\partial_x + e^{-t}u\partial_u$
3	$\frac{\varepsilon}{\cosh t}\exp\left(\rho \arctan e^t\right)$	$\partial_x,\quad 2\cosh t\,\partial_t + \frac{\rho}{2}x\partial_x + \left(e^t - 2\sinh t\, u\right)\partial_u$
4	εe^t	$\partial_x,\quad \partial_t + \frac{1}{2}x\partial_x,\quad e^{-t}(\partial_t + u\partial_u)$

Here ε, σ, κ, ρ are constants, $\varepsilon = \pm 1 \bmod G^{\sim}_b$, $\sigma \geqslant 0 \bmod G^{\sim}_b$, $\sigma \neq 1$ and $\kappa = \pm 1 \bmod G^{\sim}_b$.

where $h(t) = e^{\int a(t)\mathrm{d}t}$, and $\Delta = \alpha\delta - \beta\gamma \neq 0$, reduces by point transformations to classical Fisher equation (2.47). Using theorem 2.32 we find the family of transformations that maps equation (2.54) into equation (2.47), that is of the form

$$\tilde{t} = \ln\frac{\alpha h(t)+\beta}{\gamma h(t)+\delta} + c_1, \quad \tilde{x} = \frac{x}{\sqrt{\lambda}} + c_2, \tag{2.55}$$
$$\tilde{u} = \frac{(\alpha h(t)+\beta)(\gamma h(t)+\delta)}{h(t)\Delta}u - \gamma\frac{\alpha h(t)+\beta}{\Delta},$$

where c_1 and c_2 are arbitrary constants. We apply the found transformations to known solutions of the classical Fisher equation (2.54) and get the following family of exact solutions of the equation (2.54):

$$u = \frac{h\Delta\exp\left(\frac{5}{3}\tilde{t} + \frac{\sqrt{6}}{3}\tilde{x}\right)\wp\left(\exp\left(\frac{5}{6}\tilde{t} + \frac{\sqrt{6}}{6}\tilde{x}\right) + \tilde{C}, 0, \hat{C}\right)}{(\alpha h+\beta)(\gamma h+\delta)} + \frac{\gamma h}{\gamma h+\delta},$$
$$u = \frac{h\Delta}{(\alpha h+\beta)(\gamma h+\delta)}\frac{1}{\left(C\exp\left(\frac{\sqrt{6}}{6}\tilde{x} - \frac{5}{6}\tilde{t}\right) \pm 1\right)^2} + \frac{\gamma h}{\gamma h+\delta},$$

Here $\tilde{t}$ and $\tilde{x}$ are defined in (2.55), $\wp(z, k_1, k_2)$ is Weierstrass elliptic function, c_1, c_2, C, $\tilde{C}$, $\hat{C}$ are arbitrary constants with $C \neq 0$.

As Fisher equation admits the discrete symmetry transformation $x \mapsto -x$, the above solutions with opposite signs of x satisfy respective equation (2.54).

The family of point transformations parameterized by the arbitrary element $a(t)$ of the class (2.46),

$$\tilde{t}=\int a(t)\mathrm{e}^{\int a(t)\mathrm{d}t}\mathrm{d}t,\quad \tilde{x}=x,\quad \tilde{u}=-\mathrm{e}^{-\int a(t)\mathrm{d}t}u, \tag{2.56}$$

maps the class (2.46) into the class of quasilinear reaction–diffusion equations with quadratic nonlinearity and and an arbitrary element depending on variable t,

$$\tilde{u}_{\tilde{t}}=\tilde{b}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}}+\tilde{u}^2,\quad \tilde{b}\neq 0, \tag{2.57}$$

Arbitrary elements of the classes (2.46) and (2.57) are related via the formula

$$\tilde{b}=\frac{b(t)}{a(t)}\mathrm{e}^{-\int a(t)\mathrm{d}t}.$$

For the equation $u_t=u_{xx}+u^2$ several exact solutions are known (see [16, 208] and [237, p. 157]). Using the transformation (2.56), we find new exact solutions for variable coefficient Fisher equations

$$u_t=a(t)\mathrm{e}^{\int a(t)\mathrm{d}t}u_{xx}+a(t)u(1-u):$$

$$u=\frac{12\big(4\pm\sqrt{6}\big)x(x+c_1)+120(12\pm 5\sqrt{6})\Theta(t)+12\big(2\pm\sqrt{6}\big)c_2+6c_1^2}{\mathrm{e}^{-\int a(t)\mathrm{d}t}\big(x^2+c_1x+10(3\pm\sqrt{6})\Theta(t)+c_2\big)^2},$$

$$u=\mathrm{e}^{\int a(t)\mathrm{d}t}\wp\left(\frac{x}{\sqrt{6}},0,\hat{C}\right),$$

where $\Theta(t)=\int a(t)\mathrm{e}^{\int a(t)\mathrm{d}t}\mathrm{d}t$, c_1, c_2, $\hat{C}$ are arbitrary constants.

Concluding Remarks. It turns out that class (2.46) has a number of interesting properties. In particular it possesses a nontrivial generalized extended equivalence group and it is normalized with respect to this group. It is also mapped by proper gauging of arbitrary elements to its subclass (2.50) that is normalized in the usual sense and the equivalence algebra of which is finite dimensional.

When one analyzes results of group classification problems for various classes of variable-coefficient PDEs (see, e.g., [138, 245, 248, 288, 300]), one can observe that constant coefficient equations usually admit the widest Lie symmetry groups within such classes. In other words they represent the most symmetric cases. We unexpectedly discovered that the Lie symmetry group of the classical Fisher equation (2.45) is not widest within class (2.46). This fact can easily be interpreted in terms of mappings between classes of differential equations [248, 288, 300]. Namely class (2.46) is mapped by a family of point transformations to a class of similar but simpler structure. The mapped class consists of equations $u_t = f(t)u_{xx} + g(t)u^2$, where the arbitrary elements f and g run though the set of smooth nonvanishing functions of t, and this class is more convenient for group classification than the initial class (2.46). Under the above mapping the classical Fisher equation is transformed to a variable-coefficient equation and the equation presenting the Lie symmetry extension of highest dimension in class (2.46) (Case 4 of Table 2.3) is transformed to a constant-coefficient equation. Using this fact we have constructed some exact solutions for variable coefficient Fisher equations.

2.5. Classification of Reduction Operators of Variable Coefficient Newell–Whitehead–Segel Equations

In this section we aim to perform exhaustive classifications of Lie and regular nonclassical reduction operators for the class of equations of the form

$$u_t = a^2(t)u_{xx} + b(t)u - c(t)u^3, \tag{2.58}$$

where $a(t)$, $b(t)$ and $c(t)$ are arbitrary smooth functions, $a(t)$ and $c(t)$ are nonvanishing. This is a class of variable coefficient Newell–Whitehead–

Segel equations called also in the literature generalized Fisher equations and Newell–Whitehead equations.

The classical Newell–Whitehead–Segel equation, $u_t = u_{xx} + u - u^3$, was derived in [206, 265] and it is particular case of generalized Fisher equations

$$u_t = \big(u^m u_x\big)_x + u^p\big(1 - u^q\big), \tag{2.59}$$

which appear as insect and animal dispersal and invasion models in the mathematical biology (cf. equation (13.40) in [203]). Here t and x are time and spatial coordinates, respectively, u is a population density, p, q and m are positive parameters. The equations (2.58) with $b(t) = c(t) = 1$ were studied in [213] using the truncated Painlevé expansion method in order to construct their exact solutions. Having the same goal the whole class (2.58) was considered recently in [285]. It appears that all the found in [285] "solutions" are stationary ones and moreover do not satisfy the respective equations due to wrong signs of constants appearing therein. We aim to construct exact solutions for equations (2.58) and also to present the complete classifications of not only Lie reduction operators but also regular nonclassical ones.

2.5.1. Equivalence Groupoid. Using the direct method we deduce that the equivalence groupoid of class (2.58) is generated by the usual *equivalence transformations* from this class. The following statement is true.

Theorem 2.41. *The equivalence group $G^{\sim}$ of class (2.58) is formed by the transformations*

$$\begin{gathered}\tilde t = \theta(t), \quad \tilde x = \delta_1 x + \delta_2, \quad \tilde u = \varphi(t)u,\\ \tilde a^2(\tilde t) = \frac{{\delta_1}^2}{\theta_t}a^2(t), \quad \tilde b(\tilde t) = \frac{1}{\varphi\theta_t}(\varphi b(t) + \varphi_t), \quad \tilde c(\tilde t) = \frac{1}{\varphi^2\theta_t}c(t),\end{gathered} \tag{2.60}$$

where δ_1 and δ_2 are arbitrary constants with $\delta_1 \neq 0$, and the functions $\theta(t)$ and $\varphi(t)$ are arbitrary smooth functions with $\theta_t\varphi \neq 0$.

These transformations generate the equivalence groupoid of class (2.58).

The proof can be found in [293]. Using Theorem 2.41, we can find the conditions for arbitrary elements a, b, and c, for which variable coefficient Newell–Whitehead–Segel equations are reducible to constant coefficient equations from the same class. To derive such a condition we set $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ to be constants in the formulas (2.60) and find compatibility condition for the obtained system. This results in the statement.

Theorem 2.42. *A variable-coefficient equation from class* (2.58) *is reduced to a constant-coefficient equation from the same class by a point transformation if and only if for some constant* λ *the corresponding coefficients* $a(t)$*,* $b(t)$ *and* $c(t)$ *satisfy the condition*

$$\frac{b}{a^2}+\frac{1}{2}\frac{\left(c/a^2\right)_t}{c}=\lambda. \tag{2.61}$$

The criterion (2.61) is rather useful for checking whether a given Newell–Whithead–Segel equation with time-dependent coefficients is similar to a constant coefficient equation from the same class. In [285] "solutions" were found for equations (2.58) with $b(t)=c_1k^2a^2(t)$ and $c(t)=c_2k^2a^2(t)$, where c_1, c_2 and k are constants. It is easy to see that for such values of $b(t)$ and $c(t)$ the condition (2.61) is satisfied. In Section 2.5.4 we show how to get wide families of non-stationary solutions for the subclass of equations, whose coefficients satisfy (2.61) using the equivalence method.

Equivalence transformations allow us to simplify the initial class essentially. The arbitrary element $b(t)$ can be set to zero whereas $a(t)$ to a nonzero constant, for example, to one. Indeed, the transformation

$$\tilde{t}=\textstyle\int a^2(t)\,\mathrm{d}t,\quad \tilde{x}=x,\quad \tilde{u}=\mathrm{e}^{-\int b(t)\mathrm{d}t}u \tag{2.62}$$

maps class (2.58) to its subclass (the tildes in it are omitted)

$$u_t=u_{xx}-c(t)u^3. \tag{2.63}$$

Admissible transformations of the class (2.63) can be easily derived from Theorem 2.41, where we set $\tilde{a}^2=a^2=1$ and $\tilde{b}=b=0$. It guarantees the complete result since superclass (2.58) of class (2.63), is normalized. The result is summarized in the following statement.

Theorem 2.43. *Class* (2.63) *is normalized. The equivalence groupoid of* (2.63) *is generated by transformations which form its usual equivalence group* $G_1^\sim$*:*

$$\tilde{t} = {\delta_1}^2 t + \delta_0, \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u, \quad \tilde{c}(\tilde{t}) = \frac{1}{{\delta_1}^2 {\delta_3}^2} c(t),$$

where δ_i*,* $i = 0, 1, 2, 3$*, are arbitrary constants with* $\delta_1\delta_3 \neq 0$*.*

Therefore, we reduce the problem of classification of reduction operators for class (2.58) up to the $G^\sim$-equivalence to the similar problem for class (2.63), that contains only one arbitrary element $c(t)$, up to the $G_1^\sim$-equivalence.

2.5.2. Lie Symmetries. The group classification for the class (2.63) is performed using the standard technique [217, 227]. We omit the details and summarise the results of the classification in the following theorem.

Theorem 2.44. *The kernel of maximal Lie symmetry algebras of equations from the class* (2.63) *is the one-dimensional algebra* $\langle \partial_x \rangle$*. A complete list of* $G^\sim$*-inequivalent Lie symmetry extensions in class* (2.63) *is exhausted by the cases* 1–3 *given in Table* 2.4*.*

Table 2.4 represents also the group classification results for class (2.58) up to the $G^\sim$-equivalence. We recall that $a^2(t) = 1 \bmod G^\sim$, $b(t) = 0 \bmod G^\sim$ for all the cases of Lie symmetry extension.

For the practical use of the group classification results it is convenient to have also the list of Lie symmetry extensions which is not simplified by equivalence transformations. To get such a list we use the algorithm described in [289]. Firstly we write down the most general forms of the function $c(t)$ that correspond to equations from class (2.63) with Lie symmetry extensions. These are the cases:

Table 2.4: The group classification of class (2.63) up to the $G_1^{\sim}$-equivalence.

no.	$c(t)$	Basis of $A^{\max}$
0	$\forall$	∂_x
1	εt^ρ	$\partial_x,\ 2t\partial_t + x\partial_x - (\rho+1)u\partial_u$
2	$\varepsilon \mathrm{e}^{\pm t}$	$\partial_x,\ 2\partial_t \mp u\partial_u$
3	ε	$\partial_x,\ \partial_t,\ 2t\partial_t + x\partial_x - u\partial_u$

Here ρ is an arbitrary nonzero constant, $\varepsilon = \pm 1 \bmod G_1^{\sim}$.

Table 2.5: The group classification of class (2.58) without usage of the equivalence group.

no.	$c(t)$	Basis of $A^{\max}$
0	$\forall$	∂_x
1	$\mu a^2 \mathrm{e}^{-2\int b\,\mathrm{d}t}(\gamma T+\delta)^\rho$	$\partial_x,\ \ \frac{2}{a^2}(\gamma T+\delta)\partial_t + \gamma x\partial_x + \big(\frac{2}{a^2}(\gamma T+\delta)b - (\rho+1)\gamma\big)u\partial_u$
2	$\mu a^2 \mathrm{e}^{\sigma T - 2\int b\,\mathrm{d}t}$	$\partial_x,\ \ \frac{2}{a^2}\partial_t + \big(\frac{2b}{a^2} - \sigma\big)u\partial_u$
3	$\mu a^2 \mathrm{e}^{-2\int b\,\mathrm{d}t}$	$\partial_x,\ \ \frac{1}{a^2}\left(\partial_t + bu\partial_u\right),\ \ \frac{2}{a^2}\partial_t + x\partial_x + \big(\frac{2b}{a^2} - 1\big)u\partial_u$

Here $a = a(t)$ and $b = b(t)$ are arbitrary nonvanishing smooth functions, $T = \int a^2(t)\,\mathrm{d}t$; μ, σ, δ and ρ are arbitrary constants with $\mu\sigma\rho \neq 0$.

1) $c(t) = \mu(\gamma t+\delta)^\rho$: $\quad A^{\max} = \langle \partial_x,\, 2(\gamma t+\delta)\partial_t + \gamma x\partial_x - \gamma(\rho+1)u\partial_u\rangle$;

2) $c(t) = \mu \mathrm{e}^{\sigma t}$: $\quad A^{\max} = \langle \partial_x,\, 2\partial_t - \sigma u\partial_u\rangle$;

3) $c(t) = \mu$: $\quad A^{\max} = \langle \partial_x,\, \partial_t,\, 2t\partial_t + x\partial_x - u\partial_u\rangle$.

Here μ, γ, ρ and σ are arbitrary nonzero constants and δ is an arbitrary constant. Using the transformation (2.62) and the latter classification list it's easy to obtain the classification list for class (2.58) where arbitrary elements are not gauged by the equivalence transformations. The results are summarized in Table 2.5. The latter list reveals the Newell–Whitehead–Segel equations which are of more interest for applications and for which the classical Lie reduction method can be utilized. It is also necessary for the study of nonclassical reduction operators that we perform in the next section to get truly nontrivial ones, i.e., those which are not equivalent to Lie reduction operators.

2.5.3. Nonclassical Method. Given a (1+1)-dimensional evolution equation with the independent variables t and x and the dependent variable u, its reduction operators have the general form

$$X = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u \tag{2.64}$$

with $(\tau, \xi) \neq (0,0)$. The reduction operators (2.64) with nonvanishing coefficients of ∂_t are regular, and the other its reduction operators are singular [173]; see also [42]. The singular case $\tau = 0$ was exhaustively investigated for general evolution equation in [173, 333].

Consider the case $\tau \neq 0$. We can assume $\tau = 1$ up to the usual equivalence of reduction operators. This equivalence relation means that reduction operators X and $\tilde{X}$ are equivalent if $\tilde{X} = \Lambda(t,x,u)X$, where $\Lambda(t,x,u)$ is a nonvanishing smooth function of its arguments. Then the nonclassical invariance criterion implies the following determining equations for the coefficients ξ and η, and also for the arbitrary element $c(t)$:

$$\begin{gathered} \xi_{uu} = 0, \quad \eta_{uu} = 2(\xi_{xu} - \xi\xi_u), \\ \eta_t - \eta_{xx} + 2\xi_x\eta + (2\xi_x - \eta_u)\, cu^3 + 3\eta cu^2 + c_t u^3 = 0, \\ \xi_t - \xi_{xx} + 2\xi\xi_x - 2\xi_u\eta + 2\eta_{xu} - 3\xi_u cu^3 = 0. \end{gathered} \tag{2.65}$$

Integration of the first two equations of system (2.65) gives us the following expressions for the coefficients ξ and η

$$\xi = fu + g, \quad \eta = -\tfrac{1}{3}f^2u^3 + (f_x - fg)u^2 + hu + k,$$

where $f = f(t,x)$, $g = g(t,x)$, $h = h(t,x)$ and $k = k(t,x)$. We further substitute the derived forms of ξ and η into the rest two equations of system (2.65) and split the resulting equations with respect to variable u. This leads to a system of nine determining equations involving operator coefficients f, g, h, and k as well as the arbitrary element $c(t)$ of class (2.65). One of the equations is $f(9c-2f^2)=0$. The further consideration splits into two cases $f\neq 0$ and $f=0$.

I. If $f\neq 0$, then $9c-2f^2=0$, which means $f_x=0$ and f is a function of t only. Then the rest of the determining equations imply $g=k=0$, $h=\alpha$, $f=\beta\mathrm{e}^{2\alpha t}$, and $c=\frac{2}{9}\beta^2\mathrm{e}^{4\alpha t}$, where α and $\beta\neq 0$ are constants. Therefore, the equation

$$u_t=u_{xx}-\frac{2}{9}\beta^2\mathrm{e}^{4\alpha t}u^3 \tag{2.66}$$

admits the nonclassical reduction operator

$$X_1=\partial_t+\beta\mathrm{e}^{2\alpha t}u\partial_x+\left(\alpha-\tfrac{1}{3}\beta^2\mathrm{e}^{4\alpha t}u^2\right)u\partial_u.$$

The constants α and β can be additionally gauged by equivalence transformations, see Case 1 of Table 2.6.

II. If $f=0$, then $k=0$, $h=-g_x-\frac{1}{2}\frac{\dot c}{c}$ and the rest of the determining equations are

$$g_t+2gg_x-3g_{xx}=0,\quad g_{tx}+2g_x^2-g_{xxx}+\frac{\dot c}{c}g_x+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot c}{c}\right)=0.$$

This system of two partial differential equations for the function $g(t,x)$, one of which involves arbitrary element $c(t)$ of the class. The investigation of compatibility of this system implies that $c(t)$ can be only a power, exponential or constant function, otherwise the system is inconsistent. Truly non-Lie reduction operators arise only if $c(t)$ is either an exponential function or a constant. The list of the equations admitting nontrivial nonclassical reduction operators with $\xi_u=0$ is the following:

$$\begin{aligned}
&u_t=u_{xx}-\mu u^3\colon\\
&X_2=\partial_t-\frac{3}{x}\partial_x-\frac{3}{x^2}u\partial_u.\\
&u_t=u_{xx}-\mu\mathrm{e}^{\sigma t}u^3\colon\\
&X_3=\partial_t-\tfrac{3}{2}\sqrt{\sigma}\tanh\left(\tfrac{\sqrt{\sigma}}{2}x\right)\partial_x-\tfrac{3}{4}\sigma\left(\tanh^2\left(\tfrac{\sqrt{\sigma}}{2}x\right)-\tfrac{1}{3}\right)u\partial_u,\quad\sigma>0;
\end{aligned}\tag{2.67}$$

Table 2.6: Nonclassical reduction operators of equations (2.63).

no.	$c(t)$	Reduction operators
$1_{a,b}$	$\mathrm{e}^{\pm t}$	$\partial_t+\frac{3\sqrt{2}}{2}\mathrm{e}^{\pm\frac{1}{2}t}u\partial_x-\frac{1}{2}\left(3\mathrm{e}^{\pm t}u^2\mp\frac{1}{2}\right)u\partial_u$
1_a	$\varepsilon\mathrm{e}^{t}$	$\partial_t-\frac{3}{2}\tanh\left(\frac{1}{2}x\right)\partial_x-\frac{3}{4}\left(\tanh^2\left(\frac{1}{2}x\right)-\frac{1}{3}\right)u\partial_u$
		$\partial_t-\frac{3}{2}\coth\left(\frac{1}{2}x\right)\partial_x-\frac{3}{4}\left(\coth^2\left(\frac{1}{2}x\right)-\frac{1}{3}\right)u\partial_u$
1_b	$\varepsilon\mathrm{e}^{-t}$	$\partial_t+\frac{3}{2}\tan\left(\frac{1}{2}x\right)\partial_x-\frac{3}{4}\left(\tan^2\left(\frac{1}{2}x\right)+\frac{1}{3}\right)u\partial_u$
2	ε	$\partial_t-\frac{3}{x}\partial_x-\frac{3}{x^2}u\partial_u$

$$X_4=\partial_t-\tfrac{3}{2}\sqrt{\sigma}\coth\left(\tfrac{\sqrt{\sigma}}{2}x\right)\partial_x-\tfrac{3}{4}\sigma\left(\coth^2\left(\tfrac{\sqrt{\sigma}}{2}x\right)-\tfrac{1}{3}\right)u\partial_u,\quad \sigma>0;$$
$$X_5=\partial_t+\tfrac{3}{2}\sqrt{-\sigma}\tan\left(\tfrac{\sqrt{-\sigma}}{2}x\right)\partial_x+\tfrac{3}{4}\sigma\left(\tan^2\left(\tfrac{\sqrt{-\sigma}}{2}x\right)+\tfrac{1}{3}\right)u\partial_u,\quad \sigma<0.$$

Here μ and σ are arbitrary nonzero constants. Both of them can be gauged by the equivalence transformations to be equal to 1 or -1 depending on their signs, namely $\mu\mapsto\operatorname{sign}\mu$, and $\sigma\mapsto\operatorname{sign}\sigma$.

We summarize the results on classification of nonclassical reduction operators of equations (2.63) up to the $G_1^\sim$-equivalence in Table 2.6. In all the cases of Table 2.6 $\varepsilon=\pm1$. The same table represents the results on classification of nonclassical reduction operators of equations (2.58) up to the $G^\sim$-equivalence ($a(t)=1 \bmod G^\sim$ and $b(t)=0 \bmod G^\sim$ in this case).

Theorem 2.42 implies that equations (2.66) and (2.67) are reducible to constant coefficient Newell–Whitehead–Segel equations (2.58) by equivalence transformations from the group $G^\sim$. Indeed, the transformation $\tilde t=t$, $\tilde x=x$, $\tilde u=\mathrm{e}^{\frac{\sigma}{2}t}u$ maps equation (2.67) to the equation $\tilde u_{\tilde t}=\tilde u_{\tilde x\tilde x}+\frac{\sigma}{2}\tilde u-\mu\tilde u^3$. The latter observation means that direct reduction of equations (2.66) and (2.67) using the nonclassical symmetry operators is not the optimal way for finding their exact solutions. More convenient way is the reduction of their constant coefficient counterparts (or immediate usage of exact solutions of constant coefficient equations, if such solutions are known) and then derivation of exact solutions by the equivalence method, see the related discussion in [251]. The next section is devoted to construction of exact solutions for equations from class (2.58) using the equivalence transformations.

2.5.4. Exact Solutions. Theorem 2.42 implies that equations of the form

$$u_t = a^2(t)u_{xx} + \left(\lambda a^2(t) + \frac{\dot a(t)}{a(t)} - \frac{1}{2}\frac{\dot c(t)}{c(t)}\right)u - c(t)u^3, \tag{2.68}$$

where $a(t)$ and $c(t)$ are nonvanishing smooth functions and λ is a nonzero constant, are similar to the constant-coefficient equation

$$u_t = u_{xx} + \varepsilon u - u^3 \tag{2.69}$$

with $\varepsilon = \operatorname{sign}\lambda$. The latter equation is well studied by various techniques and a number of its exact solutions are known, see, e.g., [237, p. 177] and [300], and references therein. The similarity is established by the transformation

$$\tilde t = |\lambda| \textstyle\int a^2(t)\mathrm{d}t, \quad \tilde x = \sqrt{|\lambda|}\,x, \quad \tilde u = \frac{\sqrt{c(t)}}{a(t)\sqrt{|\lambda|}}\,u. \tag{2.70}$$

for the case $\lambda \neq 0$ and by the transformation

$$\tilde t = \textstyle\int a^2(t)\mathrm{d}t, \quad \tilde x = x, \quad \tilde u = \frac{\sqrt{c(t)}}{a(t)}\,u, \tag{2.71}$$

otherwise. There are obvious restrictions for this transformations to connect two real valued exact solutions for the physical case $t > 0$. It works fine for all functions $c(t) > 0$ when $t > 0$, for example for power coefficient $c(t)$ that is used most frequently in applications.

We illustrate the possibility of generation of solutions for equations (2.68) by the following example. The transformation (2.70) maps the known traveling wave solution $u = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\sqrt{2}}{4}x - \frac{3}{4}t\right)$ of the constant-coefficient equation (2.69) with $\varepsilon = 1$ [313] to new exact solution

$$u = \frac{1}{2}a(t)\sqrt{\frac{\lambda}{c(t)}}\left(1 - \tanh\left(\frac{\sqrt{2\lambda}}{4}x - \frac{3}{4}\lambda\int a^2(t)\mathrm{d}t\right)\right)$$

of variable-coefficient equation (2.68) with $\lambda > 0$ and $c(t) > 0$ for $t > 0$.

A number of other exact solutions of the equation (2.69) are collected in [208, 237, 300]. We consider the exact solutions of the equation (2.69) collected in [300] and apply to them either transformation (2.70) in the case $\lambda \neq 0$ or transformation (2.71), otherwise. As a result we obtain wide families of exact solutions of variable coefficient Newell–Whitehead–Segel equations (2.68).

Hereafter $T = |\lambda|\int a^2(t)\mathrm{d}t$; the functions $\mathrm{cn}(z,k)$, $\mathrm{sn}(z,k)$, and $\mathrm{ds}(z,k)$ are Jacobian elliptic functions [316].

$\lambda > 0$:

$$u = a(t)\sqrt{\frac{\lambda}{c(t)}}\frac{C_1\exp\left(\frac{\sqrt{2\lambda}}{2}x\right) - C_1'\exp\left(-\frac{\sqrt{2\lambda}}{2}x\right)}{C_2\mathrm{e}^{-\frac{3}{2}T} + C_1\exp\left(\frac{\sqrt{2\lambda}}{2}x\right) + C_1'\exp\left(-\frac{\sqrt{2\lambda}}{2}x\right)},$$

$$u = \frac{C_1\sqrt{\lambda}a(t)}{\sqrt{c(t)}}\mathrm{e}^{\frac{3}{2}T}\sinh\left(\frac{\sqrt{2\lambda}}{2}x\right)\mathrm{ds}\left(C_1\mathrm{e}^{\frac{3}{2}T}\cosh\left(\frac{\sqrt{2\lambda}}{2}x\right) + C_2, \frac{\sqrt{2}}{2}\right),$$

$$u = \frac{C_1\sqrt{\lambda}a(t)}{\sqrt{c(t)}}\mathrm{e}^{\frac{3}{2}T}\cosh\left(\frac{\sqrt{2\lambda}}{2}x\right)\mathrm{ds}\left(C_1\mathrm{e}^{\frac{3}{2}T}\sinh\left(\frac{\sqrt{2\lambda}}{2}x\right) + C_2, \frac{\sqrt{2}}{2}\right),$$

$$u = \frac{C_1\sqrt{\lambda}a(t)}{2\sqrt{c(t)}}\mathrm{e}^{\frac{3}{2}T}\sinh\left(\frac{\sqrt{2\lambda}}{2}x\right)\frac{1 + \mathrm{cn}\left(C_1\mathrm{e}^{\frac{3}{2}T}\cosh\left(\frac{\sqrt{2\lambda}}{2}x\right) + C_2, \frac{\sqrt{2}}{2}\right)}{\mathrm{sn}\left(C_1\mathrm{e}^{\frac{3}{2}T}\cosh\left(\frac{\sqrt{2\lambda}}{2}x\right) + C_2, \frac{\sqrt{2}}{2}\right)},$$

$$u = \frac{C_1\sqrt{\lambda}a(t)}{2\sqrt{c(t)}}\mathrm{e}^{\frac{3}{2}T}\cosh\left(\frac{\sqrt{2\lambda}}{2}x\right)\frac{1 + \mathrm{cn}\left(C_1\mathrm{e}^{\frac{3}{2}T}\sinh\left(\frac{\sqrt{2\lambda}}{2}x\right) + C_2, \frac{\sqrt{2}}{2}\right)}{\mathrm{sn}\left(C_1\mathrm{e}^{\frac{3}{2}T}\sinh\left(\frac{\sqrt{2\lambda}}{2}x\right) + C_2, \frac{\sqrt{2}}{2}\right)}.$$

$\lambda < 0$:

$$u = a(t)\sqrt{\frac{-\lambda}{c(t)}}\frac{\sin\left(\frac{\sqrt{-2\lambda}}{2}x\right)}{C_2\mathrm{e}^{\frac{3}{2}T} + \cos\left(\frac{\sqrt{-2\lambda}}{2}x\right)},$$

$$u = a(t)\sqrt{\frac{-\lambda}{c(t)}}C_1\mathrm{e}^{-\frac{3}{2}T}\sin\left(\frac{\sqrt{-2\lambda}}{2}x\right)\mathrm{ds}\left(C_1\mathrm{e}^{-\frac{3}{2}T}\cos\left(\frac{\sqrt{-2\lambda}}{2}x\right) + C_2, \frac{\sqrt{2}}{2}\right),$$

$$u = \frac{C_1 a(t)\mathrm{e}^{-\frac{3}{2}T}}{2}\sqrt{\frac{-\lambda}{c(t)}}\cos\left(\frac{\sqrt{-2\lambda}}{2}x\right)\frac{1+\mathrm{cn}\left(C_1\mathrm{e}^{-\frac{3}{2}T}\sin\left(\frac{\sqrt{-2\lambda}}{2}x\right)+C_2, \frac{\sqrt{2}}{2}\right)}{\mathrm{sn}\left(C_1\mathrm{e}^{-\frac{3}{2}T}\sin\left(\frac{\sqrt{-2\lambda}x}{2}\right)+C_2, \frac{\sqrt{2}}{2}\right)}.$$

$\lambda = 0$:

$$u = 2\sqrt{2}\,x\,\frac{a(t)}{\sqrt{c(t)}}\,\mathrm{ds}\left(x^2 + 6\int a^2(t)\mathrm{d}t, \frac{\sqrt{2}}{2}\right),$$

$$u = \sqrt{2}\,x\,\frac{a(t)}{\sqrt{c(t)}}\,\frac{1 + \mathrm{cn}\left(x^2 + 6\int a^2(t)\mathrm{d}t, \frac{\sqrt{2}}{2}\right)}{\mathrm{sn}\left(x^2 + 6\int a^2(t)\mathrm{d}t, \frac{\sqrt{2}}{2}\right)},$$

$$u = \frac{a(t)}{\sqrt{c(t)}}\frac{2\sqrt{2}\,x}{x^2 + 6\int a^2(t)\mathrm{d}t}, \quad u = \frac{a(t)}{\sqrt{c(t)}}\frac{\sqrt{2}}{x},$$

$$u = \sqrt{2}\,\frac{a(t)}{\sqrt{c(t)}}\,\mathrm{ds}\left(x, \frac{\sqrt{2}}{2}\right), \quad u = \frac{\sqrt{2}}{2}\,\frac{a(t)}{\sqrt{c(t)}}\,\frac{1 + \mathrm{cn}\left(x, \frac{\sqrt{2}}{2}\right)}{\mathrm{sn}\left(x, \frac{\sqrt{2}}{2}\right)}.$$

As equation (2.63) admits the equivalence transformation of the alternating sign $u \mapsto -u$ all the above presented solutions can also have the forms with the opposite sign.

2.6. Lie Symmetries of Generalized Burgers Equations and their Application to Solving Boundary Value Problems

Lie symmetry methods play an important role in solving nonlinear PDEs providing us with the algorithmic method of Lie reduction. There exist several approaches exploiting Lie symmetries in reduction of boundary-value problems (BVPs) for PDEs to those for ODEs. The classical technique is to require that both equation and boundary conditions are left invariant under the action of a one-parameter Lie group of transformations. Of course the infinitesimal approach is usually applied, i.e., a basis of operators of Lie invariance algebra is used instead of finite transformations from the corresponding Lie symmetry group (see, e.g., [32, Section 4.4]). The first works in this direction appeared in the late sixties (see, e.g., [29, 35, 201, 202]). Bluman used the approach [29, 35], that generally can be termed as the "direct" one, namely firstly the symmetries of a PDE were derived and then the boundary conditions were checked to determine whether they are also invariant under the action of the generators of symmetry found. In the case of a positive answer the BVP for the PDE was reduced to a BVP for an ODE. Using this technique a number of boundary-value problems were solved (see, e.g., [48, 214, 277, 278]).

The method suggested by Moran and Gaggioli in [202] uses specific one-parameter Lie groups of transformations of the independent and dependent variables of the PDE system as well as of all arbitrary elements which appear in the equations under study and in initial and boundary conditions. Namely, only the groups of scalings and translations are considered which can lead to self-similar or travelling-wave solutions only. After the admitted Lie group of scalings and/or translations is specified, the complete set of absolute invariants has to be found. Then a boundary-value problem for the PDE system is reduced to similar but simpler problem for the ODE system. Such an approach was applied to a number of engineering problems (see, e.g., [1] and references therein).

There is also the approach in which the group classification of a PDE system and associated boundary conditions is performed simultaneously (see, e.g., [164, 165]). Lie symmetries can also be used for certain cases when boundary conditions themselves are not invariant with respect to the corresponding Lie group of transformations [102].

2.6.1. A Class of Generalized Burgers Equations. In this section we demonstrate that the "direct" approach is much easier than that one suggested in [202] and used, e.g., in [1]. To illustrate this we use an example of a generalized Burgers equation of the form

$$u_t + a(u^n)_x = g(t)u_{xx}, \tag{2.72}$$

where a is a nonzero constant, g is an arbitrary smooth nonvanishing function of t and $n \neq 0, 1$. If $n = 2$, $a = 1/2$, and $g = -\nu$, where ν is a nonzero constant, equation (2.72) becomes the prominent Burgers equation, $u_t + uu_x + \nu u_{xx} = 0$, that is one of the simplest nonlinear $(1+1)$-dimensional evolution equations that is exactly solvable. It has a long history as it was already known to Forsyth [82] and discussed by Bateman not many years later [19]. However, it was a serious contribution made by Burgers which led to its present name [49]. Burgers equation has been used to describe many processes in fluid mechanics and a variety of other fields which seem to be rather disparate. Its remarkable feature is that it can be transformed to the standard heat equation by means of the Hopf–Cole transformation [62, 121]. Therefore it is C-integrable [50].

The generalized Burgers equations (2.72) with $n = 2$ and a nonconstant function g were derived in [116] and describe the propagation of weakly nonlinear acoustic waves under the influence of geometrical spreading and thermoviscous diffusion. Lie symmetries of such equations were studied in [73, 310]. This and other generalizations of the Burgers equations are discussed, e.g., in [262, 263].

We solve the group classification problem for class (2.72) in the framework of modern group analysis. Then we consider the class of BVPs and solve successfully a specific BVP satisfying a requirement of invariance with respect to Lie symmetries obtained. In contrast to the work performed in [1] we use the "direct" approach [32, 35] to solve the equation with associated boundary conditions and show that it is easier to implement and more transparent.

Equivalence Groupoid. We study all admissible transformations in class (2.72). They appear to be exhausted by equivalence once. The results of the study are summarized in the following statements.

Theorem 2.45. *The usual equivalence group $G^{\sim}$ of class (2.72) comprises the transformations*

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x + \delta_4, \quad \tilde{u} = \delta_5 u,$$
$$\tilde{a} = \frac{\delta_3}{\delta_1}\delta_5^{1-n} a, \quad \tilde{g} = \frac{{\delta_3}^2}{\delta_1} g, \quad \tilde{n} = n,$$

where δ_j, $j = 1, \ldots, 5$, are arbitrary constants with $\delta_1\delta_3\delta_5 \neq 0$.

If $n = 2$, class (2.72) admits a nontrivial conditional equivalence group which is wider than $G^{\sim}$.

Theorem 2.46. *The generalized equivalence group $\hat{G}_2^{\sim}$ of the class,*

$$u_t + a(u^2)_x = g(t)u_{xx}, \tag{2.73}$$

consists of the transformations

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{\kappa x + \mu_1 t + \mu_0}{\gamma t + \delta}, \quad \tilde{a} = \frac{a}{\sigma},$$
$$\tilde{u} = \frac{\sigma}{2a(\alpha\delta - \beta\gamma)}\left(2a\kappa(\gamma t + \delta)u - \kappa\gamma x + \mu_1\delta - \mu_0\gamma\right), \quad \tilde{g} = \frac{\kappa^2}{\alpha\delta - \beta\gamma} g,$$

where $\alpha, \beta, \gamma, \delta, \kappa, \mu_1, \mu_0, \sigma$ are constants defined up to a nonzero multiplier, $\alpha\delta - \beta\gamma \neq 0$ and $\kappa\sigma \neq 0$.

Theorem 2.47. *Let two equations from class (2.72), $u_t + a(u^n)_x = g(t)u_{xx}$ and $\tilde{u}_{\tilde{t}} + \tilde{a}(\tilde{u}^{\tilde{n}})_{\tilde{x}} = \tilde{g}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}}$, be connected by a point transformation $\mathcal{T}$ in the variables t, x and u. Then the transformation $\mathcal{T}$ is the projection on the space (t, x, u) of a transformation from the group $G^{\sim}$ if $n \neq 2$, or from the group $\hat{G}_2^{\sim}$, if $n = 2$.*

Table 2.8: Group classification of the class $u_t + a(u^n)_x = g(t)u_{xx}$, $n \neq 0, 1$.

no.	n	g	Basis operators of $A^{\max}$
1	$\neq 2$	$\forall$	∂_x
2	$\neq 2$	εt^ρ	$\partial_x,\quad 2t\partial_t + (\rho+1)x\partial_x + \dfrac{\rho-1}{n-1}u\partial_u$
3	$\neq 2$	εe^t	$\partial_x,\quad 2\partial_t + x\partial_x + \frac{1}{n-1}u\partial_u$
4	$\neq 2$	1	$\partial_x,\quad \partial_t,\quad 2t\partial_t + x\partial_x - \frac{1}{n-1}u\partial_u$
5	2	$\forall$	$\partial_x,\quad t\partial_x + \partial_u$
6	2	εt^ρ	$\partial_x,\quad t\partial_x + \partial_u,\quad 2t\partial_t + (\rho+1)x\partial_x + (\rho-1)u\partial_u$
7	2	εe^t	$\partial_x,\quad t\partial_x + \partial_u,\quad 2\partial_t + x\partial_x + u\partial_u$
8	2	$\varepsilon e^{2\rho \arctan t}$	$\partial_x,\quad t\partial_x + \partial_u,\quad (t^2+1)\partial_t + (t+\rho)x\partial_x + (x + (\rho - t)u)\partial_u$
9	2	1	$\partial_x,\quad t\partial_x + \partial_u,\quad \partial_t,\quad 2t\partial_t + x\partial_x - u\partial_u,\quad t^2\partial_t + tx\partial_x + (x - tu)\partial_u$

Here $\varepsilon = \pm 1 \bmod G^\sim$ and ρ is a nonzero constant. In all cases $a = 1/n \bmod G^\sim$. In Case 6 we can set, $\bmod \hat{G}_2^\sim$, either $\rho > 0$ or $\rho < 0$.

Lie Symmetries. We perform the group classification of class (2.72) within the framework of the classical Lie approach [217, 227]. It is convenient to perform the group classification for class (2.72) up to $G^\sim$-equivalence and for its subclass (2.73) up to $\hat{G}_2^\sim$-equivalence.

Theorem 2.48. *The kernel of the maximal Lie invariance algebras of equations from class (2.72) with $n \neq 2$ coincides with the one-dimensional algebra $\langle \partial_x \rangle$. All possible $G^\sim$-non-equivalent cases of extension of the maximal Lie invariance algebras are exhausted by the cases 2–4 of Table 2.8.*

Theorem 2.49. *The kernel of the maximal Lie invariance algebras of equations from class (2.73) coincides with the two-dimensional Abelian algebra $\langle \partial_x,\ 2at\partial_x + \partial_u \rangle$. All possible $\hat{G}_2^\sim$-non-equivalent cases of extension of the maximal Lie invariance algebras are exhausted by the cases 6–9 of Table 2.8.*

Solution of a Boundary-Value Problem Using Lie Symmetries.

We consider the class of BVPs

$$
\begin{aligned}
&u_t + a(u^n)_x = g(t)u_{xx}, \quad x \in [0, +\infty), \; t > 0, \\
&\lim_{t\to+\infty} u(t,x) = 0, \quad x \in (0, +\infty), \\
&u(t,0) = q(t), \quad t > 0, \\
&\lim_{x\to+\infty} u(t,x) = 0, \quad t > 0,
\end{aligned}
\tag{2.74}
$$

where a is a nonzero constant, g and q are arbitrary smooth nonvanishing functions and $n \neq 0, 1$ and search those for which the "direct" approach suggested by Bluman [32, 35] is applicable.

We have derived the Lie symmetries for the variable coefficient equation (2.72) and now we examine which of these symmetries leave the initial and boundary conditions of the problem (2.74) invariant. The procedure starts by assuming a general symmetry of the form

$$
X = \sum_{i=1}^{m} \alpha_i X_i, \tag{2.75}
$$

where m is the number of basis operators of maximal Lie symmetry algebra of a given PDE and α_i, $i = 1, \ldots, m$, are constants to be determined.

Lie symmetries for equation (2.72) appear in Table 2.8. In Case 2, for which $g(t) = \varepsilon t^\rho$, the generator (B.13) takes the form

$$
X = \alpha_1 \partial_x + \alpha_2 \Big(2t\partial_t + (\rho+1)x\partial_x + \frac{\rho-1}{n-1} u\partial_u \Big).
$$

Application of X to the first boundary condition which is written as $x = 0$ and $u(t,0) = q(t)$ gives

$$
\alpha_1 = 0 \quad \text{and} \quad \alpha_2 \left(-2t\frac{dq}{dt} + \frac{\rho-1}{n-1} q \right) = 0.
$$

For nonzero α_2 we have

$$
q(t) = \gamma t^{\frac{\rho-1}{2n-2}},
$$

where $\gamma > 0$ is a constant. It can be shown that the symmetry X with $\alpha_1 = 0$ leaves invariant the other boundary conditions. Hence the admitted Lie symmetry can be used to reduce BVP (2.74) to a problem with the governing equation being an ordinary differential equation. In fact the Lie symmetry $2t\partial_t + (\rho+1)x\partial_x + ((\rho-1)/(n-1))\,u\partial_u$ produces the transformation

$$u = t^{\frac{\rho-1}{2n-2}}\phi(\eta), \quad \text{where} \quad \eta = xt^{-\frac{\rho+1}{2}}, \tag{2.76}$$

that reduces (2.74) into the BVP for ODE

$$2\varepsilon\phi'' + (\rho+1)\eta\phi' - 2a(\phi^n)' - \frac{\rho-1}{n-1}\phi = 0, \quad \eta \in [0, +\infty), \tag{2.77}$$

$$\phi(0) = \gamma, \tag{2.78}$$

$$\lim_{\eta\to+\infty}\phi(\eta) = 0. \tag{2.79}$$

Let $\rho = (2-n)/n$. Then (2.77) takes the form $\varepsilon\phi'' + (\eta\phi' + \phi)/n - a(\phi^n)' = 0$ and can be integrated once to give $\varepsilon\phi' + \eta\phi/n - a\phi^n + c = 0$, where c is an integration constant. When we set $c = 0$, this equation becomes the Bernoulli equation that is linearizable by the substitution $\phi^{1-n} = z$ to the form

$$\frac{\varepsilon}{1-n}z' + \frac{1}{n}\eta z - a = 0.$$

The general solution of this equation is

$$z = e^{-\frac{1-n}{2n\varepsilon}\eta^2}\left(C + \frac{a(1-n)}{\varepsilon}\int_0^\eta e^{\frac{1-n}{2n\varepsilon}\theta^2}\mathrm{d}\theta\right),$$

where C is an arbitrary constant. If $\varepsilon n(n-1) > 0$, the solution can be written in terms of the error function as

$$z = e^{\frac{\eta^2}{\sigma^2}}\left(C + \frac{a(1-n)\sqrt{\pi}}{2\varepsilon\sigma}\operatorname{erf}(\sigma\eta)\right),$$

where $\sigma = \sqrt{\frac{n-1}{2\varepsilon n}}$, $\operatorname{erf}(\theta) = \frac{2}{\sqrt{\pi}}\int_0^\theta e^{-s^2}\mathrm{d}s$. Therefore a particular solution of the second-order ODE on the function ϕ is

$$\phi = \begin{cases} e^{-\frac{1}{2\varepsilon n}\eta^2}\left(C + \frac{a(1-n)}{\varepsilon}\int_0^\eta e^{\frac{1-n}{2n\varepsilon}\theta^2}\mathrm{d}\theta\right)^{\frac{1}{1-n}}, & \text{if} \quad \varepsilon n(n-1) < 0, \\ e^{-\frac{1}{2\varepsilon n}\eta^2}\left(C + \frac{a(1-n)\sqrt{\pi}}{2\varepsilon\sigma}\operatorname{erf}(\sigma\eta)\right)^{\frac{1}{1-n}}, & \text{if} \quad \varepsilon n(n-1) > 0, \end{cases} \tag{2.80}$$

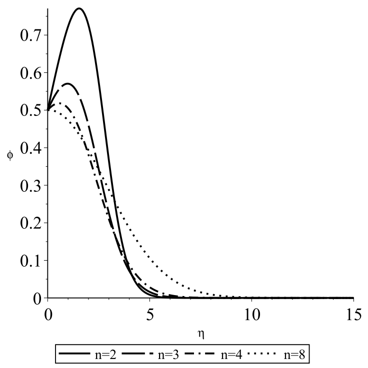

Figure 2.1: Solution (2.80) for $\varepsilon = 1$, $\gamma = 0.5$, $a = 1$ and various n.

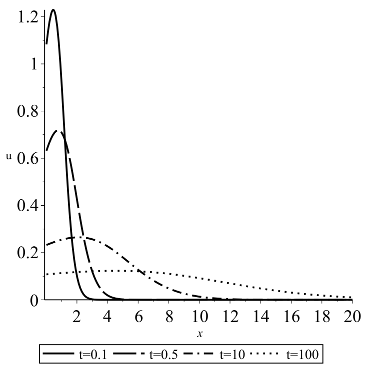

Figure 2.2: Solution (2.82) for $\varepsilon = 1$, $\gamma = 0.5$, $a = 1$ and $n = 3$ (evolution in time).

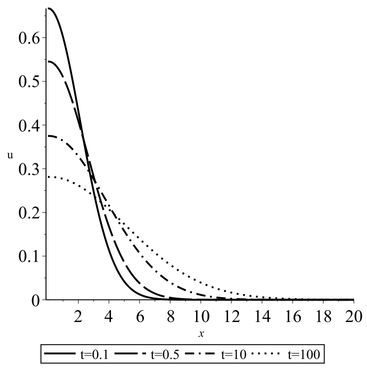

Figure 2.3: Solution (2.82) for $\varepsilon = 1$, $\gamma = 0.5$, $a = 1$ and $n = 8$ (evolution in time).

where $\sigma = \sqrt{(n-1)/(2\varepsilon n)}$. This is the solution of BVP (2.77)–(2.79) with $\rho = (2-n)/n$, when $C = \gamma^{1-n}$ and $\varepsilon n > 0$. Its typical behaviour is shown on Figure 2.1.

Now we use (2.76) and obtain the solution of the following BVP

$$\begin{aligned}
&u_t + a(u^n)_x = \varepsilon t^{\frac{2-n}{n}} u_{xx}, \quad x \in [0,+\infty),\ t > 0,\\
&\lim_{t\to+\infty} u(t,x) = 0, \quad x \in (0,+\infty),\\
&u(t,0) = \gamma t^{-\frac{1}{n}}, \quad t > 0,\\
&\lim_{x\to+\infty} u(t,x) = 0, \quad t > 0.
\end{aligned} \tag{2.81}$$

For $\varepsilon > 0$ and $n > 1$ the solution has the form

$$u = t^{-\frac{1}{n}} \exp\left[-\frac{1}{2\varepsilon n} x^2 t^{-\frac{2}{n}}\right] \left(\gamma^{1-n} + \frac{a(1-n)\sqrt{\pi}}{2\varepsilon\sigma} \operatorname{erf}(\sigma x t^{-\frac{1}{n}})\right)^{\frac{1}{1-n}}, \tag{2.82}$$

where $\sigma = \sqrt{\frac{n-1}{2n\varepsilon}}$. Note that the solution satisfies BVP (2.81) for all values of positive γ if $a < 0$. If $a > 0$, then the parameters have to satisfy the inequality $\gamma^{1-n} > a(n-1)\sqrt{\pi}/(2\varepsilon\sigma)$. The typical behaviour of the latter solution is shown on Figures 2.2 and 2.3 for the values $n = 3$ and $n = 8$, respectively.

The above procedure can be applied to the remaining cases that appear in Table 2.8. If we omit Case 9 which is the well-known Burgers equation, and constant coefficient Case 4, only in Case 6 there exists a Lie symmetry that leaves the boundary and initial conditions invariant. However, the results for this case can be obtained from the above by setting $n = 2$ and BVP (2.81) reduces to one with a constant coefficient governing equation.

Concluding Remarks. One performing research in the fields of engineering or physical sciences often encounters the problem of solving boundary-value problems (BVPs) for nonlinear partial differential equations. It is important to choose the method for solution which is easier to implement and which leads to more general results than others. Some of the analytical methods are based on the usage of Lie symmetry groups. We have applied the classical "direct" technique involving Lie symmetries of PDEs [32] to the class of BVPs for generalized Burgers equation with time-dependent viscosity coefficient and have solved the particular subcase. The used approach is more straightforward than the one suggested in [202]. One more disadvantage of the latter technique is that it uses only scalings and translations. Lie symmetry groups of some BVPs are wider and are not exhausted by scalings and translations only (see, e.g., [165]). So, the "direct" approach is also more general. However, as we have seen, the present method is applicable only for specific forms of $g(t)$ and $q(t)$ in (14). In other words, the method has its limitations, but nevertheless is applied to certain nonlinear problems. Some recent examples of its successful usage can be found in [175, 298].

The list of Lie symmetries derived for the class (2.72) comes to complete the existing results that appear in the literature [73, 310] and presents all non-equivalent cases for which the algorithmic Lie reduction method can be applied.

2.6.2. A Class of Generalized Burgers Equations With Linear Damping. In this section we shortly adduce the classification results derived in [234] for another class of variable coefficient generalized Burgers equations with linear damping of the form

$$u_t + u^n u_x + h(t)u = g(t)u_{xx}, \quad ng \neq 0. \tag{2.83}$$

Here $h(t)$ and $g(t)$ are arbitrary smooth functions with $g \neq 0$, and n is an arbitrary nonzero constant.

Theorem 2.50. *The generalized equivalence group $\hat{G}^{\sim}$ of the class (2.83) consists of the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_0, \quad \tilde{u} = \left(\frac{\delta_1}{T_t}\right)^{\frac{1}{n}} u,$$
$$\tilde{h} = \frac{1}{T_t}h + \frac{T_{tt}}{n{T_t}^2}, \quad \tilde{g} = \frac{{\delta_1}^2}{T_t}g, \quad \tilde{n} = n,$$

where δ_1 and δ_0 are arbitrary constants and $T = T(t)$ is an arbitrary smooth function with $\delta_1 T_t > 0$. The equivalence groupoid of the subclass of the class (2.83) singled out by the condition $n \neq 1$ is generated by elements of $\hat{G}^{\sim}$, i.e., this subclass is normalized in the generalized sense.

Corollary 2.51. *The generalized equivalence group $\hat{G}^{\sim}_{h=\mathrm{const}}$ of the class (2.83) with $h = \mathrm{const}$ consists of the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_0, \quad \tilde{u} = \left(\frac{\delta_1}{\alpha}\right)^{\frac{1}{n}} e^{ht-\tilde{h}T}u,$$
$$\tilde{g} = \frac{{\delta_1}^2}{\alpha}e^{nht-n\tilde{h}T}g, \quad \tilde{n} = n,$$

where the function $T = T(t)$ depends on h and $\tilde{h}$ and is defined by the formulae

$$h\tilde{h} \neq 0\colon \quad \frac{e^{-n\tilde{h}T}-1}{-n\tilde{h}} = \alpha\frac{e^{-nht}-1}{-nh} + \beta,$$
$$h \neq 0,\ \tilde{h} = 0\colon \quad T = \alpha\frac{e^{-nht}-1}{-nh} + \beta,$$
$$h = 0,\ \tilde{h} \neq 0\colon \quad \frac{e^{-n\tilde{h}T}-1}{-n\tilde{h}} = \alpha t + \beta,$$
$$h = \tilde{h} = 0\colon \quad T = \alpha t + \beta.$$

Here α, β, δ_0 and δ_1 are arbitrary constants with $\alpha\delta_1 > 0$.

Theorem 2.52. *The generalized extended equivalence group* $\hat{G}^{\sim}_{n=1}$ *of the class*

$$u_t + uu_x + h(t)u = g(t)u_{xx} \tag{2.84}$$

consists of the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = (x+\delta_1)X^1 + \delta_0, \quad \tilde{u} = \frac{X^1}{T_t}\left(u + (x+\delta_1)\frac{X^1_t}{X^1}\right),$$
$$\tilde{h} = \frac{1}{T_t}\left(h + \frac{T_{tt}}{T_t} - 2\frac{X^1_t}{X^1}\right), \quad \tilde{g} = \frac{\left(X^1\right)^2}{T_t}g.$$

Here δ_0, δ_1 *are arbitrary constants,* $T = T(t)$ *is an arbitrary smooth function with* $T_t \neq 0$, *and* $X^1 = \left(\gamma \int e^{-\int h(t)\,\mathrm{d}t}\,\mathrm{d}t + \delta\right)^{-1}$.

Class (2.84) *is normalized in the generalized extended sense.*

Therefore, the admissible transformations in the class (2.83) are exhaustively described. The following statement is true.

Theorem 2.53. *The class* (2.83), *where the exponent* n *varies, is not normalized. It can be partitioned into the normalized subclasses each of which is singled out by fixing a value of* n, *and these classes are not connected by point transformations. Each subclass of* (2.83) *with a specified value of* n, $n \neq 1$, *is normalized in the usual sense, whereas the subclass* (2.84) *corresponding to the value* $n = 1$ *is normalized in the generalized extended sense only. Every union of any subclasses of* (2.83) *with* $n \neq 1$ *is normalized in the generalized sense.*

The transformations from $\hat{G}^{\sim}$ are parameterized by an arbitrary function $T = T(t)$. This allows us to gauge one of the arbitrary elements, g or h, to a simple constant value. For example, we can set g to one or h to zero. The gauging $h = 0$ looks more convenient since in this case the class (2.83) reduces to another one, for which the group classification problem has been solved in the previous section.[2.1] This gauging can be

[2.1] Group classification problems for certain subclasses of (2.83) with $h = 0$ were considered in [73, 286, 287, 310] and the complete group classification of this class was achieved in [307].

realized with the transformation

$$\mathcal{T}\colon\quad \hat t=\textstyle\int e^{-n\int h(t)\,\mathrm{d}t}\,\mathrm{d}t,\quad \hat x=x,\quad \hat u=e^{\int h(t)\,\mathrm{d}t}u \tag{2.85}$$

from the group $\hat G^{\sim}$, which links the class (2.83) with the class $\hat u_{\hat t}+\hat u^n\hat u_{\hat x}=\hat g(\hat t)\hat u_{\hat x\hat x}$, where the new arbitrary element $\hat g$ depends on h and g as

$$\hat g=e^{n\int h(t)\,\mathrm{d}t}g. \tag{2.86}$$

Of course the transformation $\mathcal{T}$ for this gauging is not unique. If $n\neq 1$, then the most general transformation is

$$\hat t=\alpha\textstyle\int e^{-n\int h(t)\,\mathrm{d}t}\,\mathrm{d}t+\beta,\quad \hat x=\delta_1x+\delta_0,\quad \hat u=\left(\dfrac{\delta_1}{\alpha}\right)^{\frac1n}e^{\int h(t)\,\mathrm{d}t}u,$$

where α, β, δ_1 and δ_0 are arbitrary constants with $\alpha\delta_1\neq 0$. In the case $n=1$ the most general transformation takes the form

$$\begin{aligned}&\tilde t=\frac{\alpha\hat t+\beta}{\gamma\hat t+\delta},\quad \tilde x=\frac{x+\mu_1\hat t+\mu_0}{\gamma\hat t+\delta},\\ &\tilde u=\frac{(\gamma\hat t+\delta)e^{\int h(t)\,\mathrm{d}t}u-\gamma x+\mu_1\delta-\mu_0\gamma}{\alpha\delta-\beta\gamma},\end{aligned}$$

where α, β, γ, δ, μ_0, and μ_1 are arbitrary constants with $\alpha\delta-\beta\gamma\neq 0$, and $\hat t=\int e^{-n\int h(t)\,\mathrm{d}t}\,\mathrm{d}t$.

If h is a nonzero constant, then the transformation $\mathcal{T}$ gauging h to zero has the form

$$\hat t=-\frac{1}{nh}e^{-nht},\quad \hat x=x,\quad \hat u=e^{ht}u.$$

Remark 2.54. The alternative gauge $g=1$ can be set using a parameterized family of point transformations that are projections of transformations from $\hat G^{\sim}$ to the space of independent and dependent variables,

$$\hat t=\textstyle\int g(t)\,\mathrm{d}t,\quad \hat x=x\operatorname{sgn}g(t),\quad \hat u=|g(t)|^{-\frac1n}u.$$

This family of transformations maps the class (2.83) onto the class $\hat u_{\hat t}+\hat u^n\hat u_{\hat x}+\hat h(\hat t)\hat u=\hat u_{\hat x\hat x}$, where the new arbitrary element $\hat h$ depends on h and g as $\hat h=\frac{h}{g}+\frac{g_t}{ng^2}$.

Table 2.10: The complete list of Lie symmetry extensions for the class (2.83).

no.	g	Basis operators of $A^{\max}$
		$n\neq 1$
1	$\forall$	∂_x
2	$\lambda T_t(\alpha T+\beta)^\rho$	$\partial_x,\ 2n(\alpha T+\beta)T_t^{-1}\partial_t+\alpha n(\rho{+}1)x\partial_x$ $+\left(\alpha(\rho{-}1)-2nh(t)(\alpha T+\beta)T_t^{-1}\right)u\partial_u$
3	$\lambda T_t e^{\alpha T}$	$\partial_x,\ 2nT_t^{-1}\partial_t+\alpha nx\partial_x+\left(\alpha-2nh(t)T_t^{-1}\right)u\partial_u$
4	λT_t	$\partial_x,\ T_t^{-1}\partial_t-h(t)T_t^{-1}u\partial_u,$ $2nTT_t^{-1}\partial_t+nx\partial_x-\left(2nh(t)TT_t^{-1}+1\right)u\partial_u$
		$n=1$
5	$\forall$	$\partial_x,\ T\partial_x+T_t\partial_u$
6	$\lambda T_t\left(\frac{\alpha T+\beta}{\gamma T+\delta}\right)^\rho$	$\partial_x,\ T\partial_x+T_t\partial_u,\ (\alpha T+\beta)(\gamma T+\delta)T_t^{-1}\partial_t$ $+\left(\frac12(\rho-1)\Delta+\alpha(\gamma T+\delta)\right)x\partial_x+\Big(\alpha\gamma T_tx$ $+\left[-\alpha(\gamma T+\delta)-h(t)(\alpha T+\beta)(\gamma T+\delta)T_t^{-1}+\frac12(\rho+1)\Delta\right]u\Big)\partial_u$
7	$\lambda T_t e^{\frac{\alpha T+\beta}{\gamma T+\delta}}$	$\partial_x,\ T\partial_x+T_t\partial_u,$ $(\gamma T+\delta)^2\,T_t^{-1}\partial_t+\left(\gamma(\gamma T+\delta)+\frac12\Delta\right)x\partial_x$ $+\left(\left[-\gamma(\gamma T+\delta)-h(t)(\gamma T+\delta)^2T_t^{-1}+\frac12\Delta\right]u+\gamma^2T_tx\right)\partial_u$
8	$\lambda T_t e^{2\rho\arctan(\alpha T+\beta)}$	$\partial_x,\ T\partial_x+T_t\partial_u,$ $\left((\alpha T+\beta)^2+1\right)T_t^{-1}\partial_t+\alpha\left(\alpha T+\rho+\beta\right)x\partial_x$ $+\left(\left[\alpha(-\alpha T+\rho-\beta)-h(t)\left((\alpha T+\beta)^2+1\right)T_t^{-1}\right]u+\alpha^2T_tx\right)\partial_u$
9	λT_t	$\partial_x,\ T\partial_x+T_t\partial_u,\ 2TT_t^{-1}\partial_t+x\partial_x-\left(2h(t)TT_t^{-1}+1\right)u\partial_u,$ $T_t^{-1}\partial_t-h(t)T_t^{-1}u\partial_u,$ $T^2T_t^{-1}\partial_t+Tx\partial_x+\left(T_tx-\left(h(t)T^2T_t^{-1}+T\right)u\right)\partial_u$

Here $T=T(t)=\int e^{-n\int h(t)\,\mathrm{d}t}\,\mathrm{d}t$, and the function $h(t)$ is arbitrary in all cases. Here λ and ρ are nonzero constants. $\alpha=\pm1$ in Case 2 and $\alpha\neq0$ in Cases 3 and 8. In Cases 6 and 7 α, β, γ, and δ are arbitrary constants defined up to a nonzero multiplier (with additional possibility of scaling in Case 6) such that $\Delta=\alpha\delta-\beta\gamma\neq0$. In Case 7 we can set $(\alpha,\beta,\gamma,\delta)\in\{(\alpha',0,0,1),(0,\beta',1,\delta')\}$.

Theorems 2.50, 2.52, and 2.53 exhaustively describe the equivalence groupoid of the class (2.83).

The complete list of Lie symmetry extensions for equations (2.83) are presented in Table 2.10. We do not present the list of point-inequivalent cases of Lie symmetry extensions since it in fact coincides with the cases presented in Table 2.8. Note that the cases $n \neq 2$ and $n = 2$ for equations (2.72) correspond to the cases $n \neq 1$ and $n = 1$ for equations (2.83), respectively. The classification list adduced in Table 2.10 derived using the equivalence based approach suggested in [289].

In [307] we also derived solutions of equations from the initial class (2.83) with arbitrary values of $h(t)$ and specific values of $g(t)$,

$$(i) \qquad u_t + u^n u_x + h(t)u = \varepsilon T^{\frac{1-n}{1+n}} e^{-n\int h(t)\,\mathrm{d}t} u_{xx}:$$

$$u = \frac{T^{-\frac{1}{n+1}} \exp\left(-\frac{\mu}{n} x^2 T^{-\frac{2}{n+1}}\right) e^{-\int h(t)\,\mathrm{d}t}}{\left(c_1 - 2\mu T^{-\frac{1}{n+1}} \int e^{-\mu x^2 T^{-\frac{2}{n+1}}}\,\mathrm{d}x\right)^{\frac{1}{n}}},$$

$$(ii) \quad u_t + u^n u_x + h(t)u = e^{-n\int h(t)\,\mathrm{d}t} u_{xx}:$$

$$u = \left(\frac{a(n+1)}{1 + c_1 e^{an(x-aT)}}\right)^{\frac{1}{n}} e^{-\int h(t)\,\mathrm{d}t},$$

$$u = \left(\frac{n+1}{c_1 - nx}\right)^{\frac{1}{n}} e^{-\int h(t)\,\mathrm{d}t},$$

$$(iii) \quad u_t + uu_x + h(t)u = g(t)u_{xx} \quad \forall g:$$

$$u = \frac{x + c_0}{\int e^{-\int h(t)\,\mathrm{d}t}\,\mathrm{d}t + a} e^{-\int h(t)\,\mathrm{d}t}.$$

Here a and c are arbitrary constants, $\varepsilon = \pm 1$, $\mu = \frac{n}{2\varepsilon(n+1)}$, and the function $T = T(t)$ is defined as $T(t) = \int e^{-n\int h(t)\,\mathrm{d}t}\,\mathrm{d}t$.

The equation (ii) can be rewritten as

$$u_t + u^n u_x - \frac{1}{n}\frac{k_t}{k} u = k u_{xx}, \tag{2.87}$$

where the functions $k(t)$ and $h(t)$ are related via the formula $k = e^{-n\int h(t)\,\mathrm{d}t}$. For $n = 1$ the equation (2.87) coincides with equation (3.262) in [262, p. 90], which includes the Burgers model for turbulence but with variable diffusivity (applicable to modeling of acoustic waves in the atmosphere). We have derived two families of exact solutions for the equation (ii). The behavior of the solution

$$u = \left(\frac{a(n+1)}{1 + c_1 e^{an(x - aT)}} \right)^{\frac{1}{n}} e^{-\int h(t)\,\mathrm{d}t} \tag{2.88}$$

for two different values of n and forms of variable diffusivity coefficients h is illustrated at Fig. 2.4.

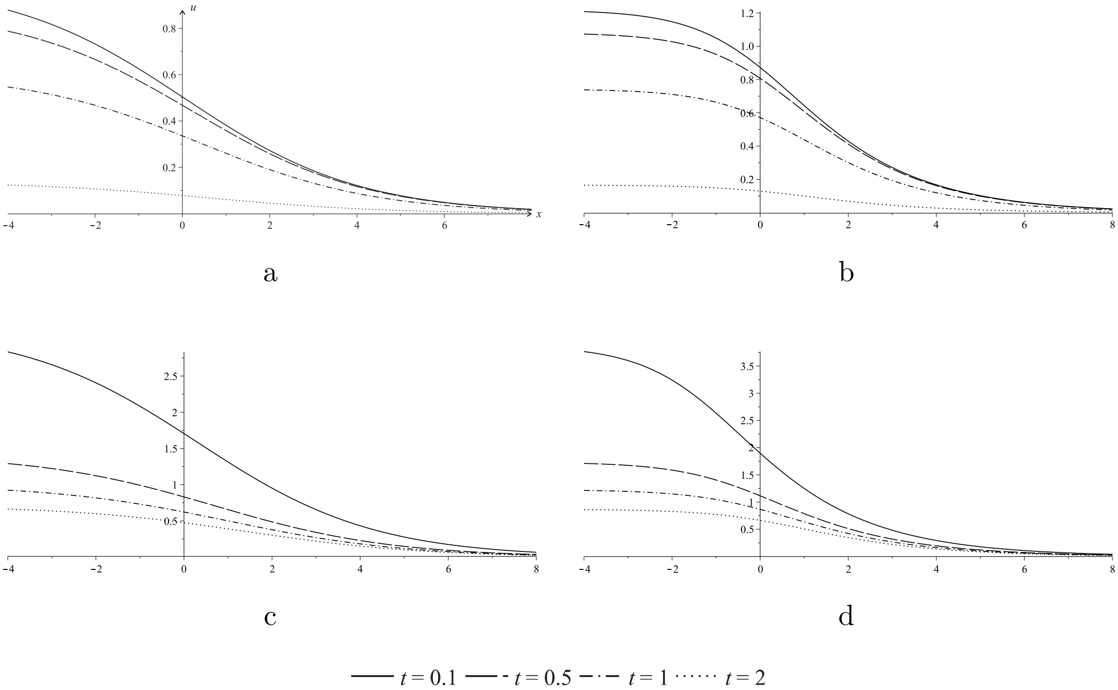

Figure 2.4: The behavior of the solution (2.88) for $c_1 = 1$, $a = 0.5$. Here $n = 1$, and $h = t$ at Fig. a, $n = 2$, and $h = t$ at Fig. b; $n = 1$, and $h = (2t)^{-1}$ at Fig. c, $n = 2$, and $h = (2t)^{-1}$ at Fig. d.

We also found that all equations from the class (2.83) that are linearized to the heat equation

$$\hat{v}_{\hat{t}} = \lambda \hat{v}_{\hat{x}\hat{x}} \tag{2.89}$$

have the form

$$u_t + uu_x + h(t)u = \lambda e^{-\int h(t)\,\mathrm{d}t} u_{xx}. \tag{2.90}$$

The corresponding transformation

$$\hat{t} = \int e^{-\int h(t)\,\mathrm{d}t}\,\mathrm{d}t, \quad \hat{x} = x, \quad -2\lambda \frac{\hat{v}_{\hat{x}}}{\hat{v}} = e^{\int h(t)\,\mathrm{d}t} u$$

gives the formula for generating solutions of the equation (2.90) from solutions of the heat equation

$$u(t,x) = -2\lambda e^{-\int h(t)\,\mathrm{d}t}\frac{\hat{v}_x(\hat{t},x)}{\hat{v}(\hat{t},x)}, \quad \text{where} \quad \hat{t} = \int e^{-\int h(t)\,\mathrm{d}t}\,\mathrm{d}t.$$

Consider, for example, the exact solution $\hat{v} = ce^{a\hat{x}+a^2\lambda\hat{t}}(\hat{x}+2a\lambda\hat{t})$ of the heat equation (2.89). Here c and a are arbitrary nonzero constants. Using the last transformation, we get an exact solution of (2.90),

$$u(t,x) = -2\lambda\frac{ax + 2a^2\lambda\int e^{-\int h(t)\,\mathrm{d}t}\,\mathrm{d}t + 1}{x + 2a\lambda\int e^{-\int h(t)\,\mathrm{d}t}\,\mathrm{d}t}e^{-\int h(t)\,\mathrm{d}t}.$$

2.7. Group Classification of Variable Coefficient Nonlinear Kolmogorov Equations in 2+1 Dimensions

Second-order partial differential equations of the form

$$u_t = Du_{yy} + \nu\,[K(u)]_x\,, \tag{2.91}$$

where D and ν are nonzero constants, and K is a smooth nonlinear function of the dependent variable u, appear in various applications. In particular, they describe diffusion–convection processes [77], model an interaction of particles with two kinds of particles on a lattice [8], arise in mathematical finance, when studying agents' decisions under risk [59, 229]. Equations (2.91) are called in the literature diffusion–advection equations, nonlinear ultraparabolic equations and nonlinear Kolmogorov equations.

Lie symmetries of equations (2.91) and the corresponding group invariant solutions were classified by Demetriou et al [67]. There are also studies on Lie symmetries of linear Kolmogorov equations [166, 279] and of constant coefficient nonlinear Kolmogorov equations of the form $u_t - u_{yy} - uu_x = f(u)$ [268].

An attempt of group classification of a class of nonlinear Kolmogorov equations more general than (2.91), namely, such equations with time dependent coefficients,

$$u_t = f(t)u_{yy} - g(t)[K(u)]_x\,, \quad fgK_{uu} \neq 0, \tag{2.92}$$

was recently made [171]. Here f and g are smooth nonvanishing functions of the variable t, and K is a smooth nonlinear function of u. Nevertheless the complete classification of Lie symmetries of class (2.92) was not achieved in [171], in particular, the case $K = u\ln u$ was missed and dimensions of maximal Lie symmetry algebras as well as some of their basis elements for the other cases of extensions were presented incorrectly. The case $K = u^2$ that is important for applications was not studied with Lie symmetry point of view at all.

In this section we perform the complete group classification of equations (2.92). As class (2.92) is parameterized by three arbitrary elements, $K(u)$, $f(t)$ and $g(t)$, the group classification problem appears to be too complicated to be solved completely without modern approaches based on the usage of point equivalence transformations (see also [296]). One of such tools is the gauging of arbitrary elements by equivalence transformations (i.e., reducing of a class to a subclass with fewer number of arbitrary elements). In section 2.7.3 we discuss how to choose an optimal gauging among possible ones. To illustrate that the chosen gauging is optimal, we also present results on group classification of class (2.92) carried out for an alternative gauging.

2.7.1. Admissible Transformations. To find the admissible transformations we use the direct method [160]. The details of calculations are skipped for brevity. As it is more convenient for the study of Lie symmetries to consider the equivalent form of the above class,

$$u_t = f(t)u_{yy} - g(t)k(u)u_x, \quad fgk_u \neq 0, \tag{2.93}$$

we present transformations for both K and $k = K_u$ in the theorems below.

Theorem 2.55. *The generalized extended equivalence group* $\hat{G}^{\sim}$ *of class (2.92) (resp. (2.93)) is formed by the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2 \int g(t)\,\mathrm{d}t + \delta_3, \quad \tilde{y} = \delta_4 y + \delta_5,$$
$$\tilde{u} = \delta_6 u + \delta_7, \quad \tilde{f}(\tilde{t}) = \frac{{\delta_4}^2}{T_t} f(t), \quad \tilde{g}(\tilde{t}) = \frac{\varepsilon_1}{T_t} g(t),$$
$$\tilde{K}(\tilde{u}) = \frac{\delta_6}{\varepsilon_1}\left(\delta_1 K(u) + \delta_2 u + \varepsilon_2\right), \quad \left(\textit{resp. } \tilde{k}(\tilde{u}) = \frac{1}{\varepsilon_1}(\delta_1 k(u) + \delta_2),\right)$$

where δ_i, $i = 1, \dots, 7$, ε_1 *and* ε_2 *are arbitrary constants with* $\delta_1\delta_4\delta_6\varepsilon_1 \neq 0$, $T(t)$ *is an arbitrary smooth function with* $T_t \neq 0$.

The usual equivalence group of class (2.92) (resp. (2.93)) consists of the above transformations with $\delta_2 = 0$.

The group $\hat{G}^{\sim}$ contains a subgroup of gauge equivalence transformations, i.e. the transformations that change only arbitrary elements while the independent and dependent variables remain unchanged [248]. This subgroup is formed by the transformations $\tilde{t} = t$, $\tilde{x} = x$, $\tilde{y} = y$, $\tilde{u} = u$, $\tilde{f} = f$, $\tilde{g} = \varepsilon_1 g$, $\tilde{K} = (K + \varepsilon_2)/\varepsilon_1$ (resp. $\tilde{k} = k/\varepsilon_1$). It is more convenient to consider class (2.93) than class (2.92) as in this case the dimension of the gauge equivalence subgroup reduces.

It appears that the subclass of equations (2.92) with K quadratic in u (resp. (2.93) with k linear in u) admits a wider equivalence group. Up to the $\hat{G}^{\sim}$-equivalence we can consider the case $K = u^2$ (resp. $k = u$).

Theorem 2.56. *The generalized extended equivalence group* $\hat{G}_1^{\sim}$ *of the class*

$$u_t = f(t)u_{yy} - g(t)uu_x, \quad fg \neq 0, \tag{2.94}$$

comprises the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = X(t)x + \delta_3 \int g(t)X(t)^2 \mathrm{d}t + \delta_4, \quad \tilde{y} = \delta_1 y + \delta_2,$$
$$\tilde{u} = \delta_5\left(\frac{u}{X(t)} - \delta_6 x + \delta_3\right), \quad \tilde{f}(\tilde{t}) = \frac{\delta_1^2}{\delta_5 T_t} f(t), \quad \tilde{g}(\tilde{t}) = \frac{X(t)^2}{\delta_5 T_t} g(t),$$

where $X(t) = \left(\delta_6 \int g(t)\,\mathrm{d}t + \delta_7\right)^{-1}$, δ_i, $i = 1, \ldots, 7$, *are arbitrary constants with* $\delta_1\delta_5(\delta_6^2+\delta_7^2) \neq 0$, *and* $T(t)$ *is an arbitrary smooth function with* $T_t \neq 0$.

The usual equivalence group of class (2.94) consists of the above transformations with $\delta_3 = \delta_6 = 0$.

As there is one arbitrary function, $T(t)$, in the transformations from the group $\hat{G}^{\sim}$, we can set one of the arbitrary elements f or g of the initial class equals to a nonzero constant value. We choose to perform the gauging $g = 1$ by using the transformation $\tilde{t} = \int g(t)\,\mathrm{d}t$, $\tilde{x} = x$, $\tilde{u} = u$. Then, any equation from class (2.92) (resp. (2.93)) is mapped to an equation from its subclass that is singled out by the condition $g = 1$. Without loss of generality, we can restrict ourselves to the study of class (2.92) with $g = 1$ or, what is more convenient, its equivalent form

$$u_t = f(t)u_{yy} - k(u)u_x, \quad f k_u \neq 0. \tag{2.95}$$

The generalized extended equivalence groups of class (2.95) and its subclass with $k = u$ coincide with their usual equivalence groups.

Theorem 2.57. *The usual equivalence group* $G^{\sim}$ *of class (2.95) consists of the transformations*

$$\tilde{t} = \varepsilon_1 t + \varepsilon_0, \quad \tilde{x} = \delta_1 x + \delta_2 t + \delta_3, \quad \tilde{y} = \delta_4 y + \delta_5, \quad \tilde{u} = \delta_6 u + \delta_7,$$
$$\tilde{f}(\tilde{t}) = \frac{{\delta_4}^2}{\varepsilon_1} f(t), \quad \tilde{k}(\tilde{u}) = \frac{1}{\varepsilon_1}(\delta_1 k(u) + \delta_2),$$

where δ_i, $i = 1, \ldots, 7$, ε_1 *and* ε_0 *are arbitrary constants with* $\delta_1\delta_4\delta_6\varepsilon_1 \neq 0$.

Theorem 2.58. *The usual equivalence group* $G_1^{\sim}$ *of the class*

$$u_t = f(t)u_{yy} - uu_x, \quad f \neq 0, \tag{2.96}$$

is formed by the transformations

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{\kappa x + \mu t + \nu}{\gamma t + \delta}, \quad \tilde{y} = \lambda y + \varepsilon,$$
$$\tilde{u} = \frac{1}{\Delta}\left(\kappa(\gamma t + \delta)u - \kappa\gamma x + \delta\mu - \gamma\nu\right), \quad \tilde{f}(\tilde{t}) = \frac{\lambda^2}{\Delta}(\gamma t + \delta)^2 f(t),$$

where α, β, γ, δ, κ, μ, *and* ν *are arbitrary constants defined up to a nonzero multiplier with* $\Delta=\alpha\delta-\beta\gamma\neq 0$, $\kappa\neq 0$*;* λ *and* ε *are arbitrary constants,* $\lambda\neq 0$*.*

Theorem 2.58 implies that any equation (2.96) with $f=a(t+b)^{-2}$, where $a\neq 0$ and b are constants, is mapped by a point transformation to a constant-coefficient equation from the same class.

We also present equivalence transformations for the subclass of class (2.93) singled out by the condition $f=1$, which we will use for the comparison of the cases $f=1$ and $g=1$ in Section 2.7.3.

Theorem 2.59. *The generalized extended equivalence group* $\hat{G}_2^{\sim}$ *of the class*

$$u_t=u_{yy}-g(t)k(u)u_x,\quad gk_u\neq 0, \tag{2.97}$$

comprises the transformations

$$\begin{gathered}\tilde t=\delta_4^2t+\delta_0,\quad \tilde x=\delta_1x+\delta_2\textstyle\int g(t)\,\mathrm{d}t+\delta_3,\quad \tilde y=\delta_4y+\delta_5,\\ \tilde u=\delta_6u+\delta_7,\quad \tilde g(\tilde t)=\dfrac{\varepsilon_1}{\delta_4^2}g(t),\quad \tilde k(\tilde u)=\dfrac{1}{\varepsilon_1}\left(\delta_1k(u)+\delta_2\right),\end{gathered}$$

where δ_i, $i=0,1,\dots,7$, *and* ε_1 *are arbitrary constants with* $\delta_1\delta_4\delta_6\varepsilon_1\neq 0$*.*

Theorem 2.60. *The generalized extended equivalence group* $\hat{G}_3^{\sim}$ *of the class*

$$u_t=u_{yy}-g(t)uu_x,\quad g\neq 0, \tag{2.98}$$

consists of the transformations

$$\begin{gathered}\tilde t=\delta_1^2t+\delta_2,\quad \tilde x=\frac{1}{\theta(t)}(x+\delta_4)+\delta_5,\quad \tilde y=\delta_1y+\delta_3,\\ \tilde u=\delta_6\left(\theta(t)u-\gamma_1(x+\delta_4)\right),\quad \tilde g(\tilde t)=\frac{g(t)}{\delta_1^2\delta_6\theta(t)^2},\end{gathered}$$

where δ_i, $i=1,\dots,6$, $\theta(t)=\gamma_1\int g(t)\mathrm{d}t+\gamma_2$, γ_1 *and* γ_2 *are arbitrary constants with* $\delta_1\delta_6(\gamma_1^2+\gamma_2^2)\neq 0$*.*

2.7.2. Lie Symmetries. The group classification problem for class (2.93) up to $\hat{G}^{\sim}$-equivalence reduces to the similar problem for class (2.95) up to $G^{\sim}$-equivalence (resp. the group classification problem for class (2.94) up to $\hat{G}_1^{\sim}$-equivalence reduces to such a problem for class (2.96) up to $G_1^{\sim}$-equivalence). To solve the group classification problem for class (2.95) we use the classical approach based on integration of determining equations implied by the infinitesimal invariance criterion [227]. We search for symmetry operators of the form $Q = \tau(t,x,y,u)\partial_t + \xi(t,x,y,u)\partial_x + \eta(t,x,y,u)\partial_y + \theta(t,x,y,u)\partial_u$ generating one-parameter Lie groups of transformations that leave equations (2.95) invariant [217, 227]. It is required that the action of the second prolongation $Q^{(2)}$ of the operator Q on (2.95) vanishes identically modulo equation (2.95),

$$Q^{(2)}\{u_t - f(t)u_{yy} + k(u)u_x\}|_{u_t=f(t)u_{yy}-k(u)u_x} = 0. \tag{2.99}$$

The infinitesimal invariance criterion (2.99) implies the determining equations, simplest of which result in

$$\tau = \tau(t), \quad \xi = \xi(t,x), \quad \eta = \eta^1(t)y + \eta^0(t),$$
$$\theta = \varphi(t,x,y)u + \psi(t,x,y),$$

where τ, ξ, η^1, η^0, φ and ψ are arbitrary smooth functions of their variables. Then the rest of the determining equations are

$$\tau f_t = (2\eta^1 - \tau_t)f, \quad 2f\varphi_y = -\eta^1_t y - \eta^0_t, \tag{2.100}$$
$$(\varphi u + \psi)k_u + (\tau_t - \xi_x)k = \xi_t, \tag{2.101}$$
$$(\varphi_x u + \psi_x)k + (\varphi_t - f\varphi_{yy})u + \psi_t - f\psi_{yy} = 0. \tag{2.102}$$

Firstly we integrate equations (2.101) and (2.102) for k up to the $G^{\sim}$-equivalence taking into account that $k_u \neq 0$. The method of furcate splitting [138, 209] is further used. For any operator $Q \in A^{\max}$ equation (2.101) gives equations on k of the general form

$$(au + b)k_u + ck = d, \tag{2.103}$$

where a, b, c, and d are constants. The number s of such independent equations is not greater than two, otherwise they form incompatible system for k. If $s=0$, then (2.103) is not an equation on k but an identity, this corresponds to the case of arbitrary k. If $s=1$, then the integration of (2.103) up to the $G^{\sim}$-equivalence gives three different cases: (i) $k=u^n$, $n\neq 0,1$; (ii) $k=e^u$; (iii) $k=\ln u$. If $s=2$, then the function k is linear in u, $k=u \bmod G^{\sim}$.

The determining equation (2.102) implies that there exist two essentially different cases of classification: I. $k_{uu}\neq 0$, and II. $k_{uu}=0$ ($k=u \bmod G^{\sim}$). Consider firstly the case of arbitrary function k. In this case equations (2.101) and (2.102) should be split with respect to k and k_u. The splitting results in the equations $\varphi=\psi=\xi_t=\tau_t-\xi_x=0$. Therefore $\tau=c_1t+c_2$, $\xi=c_1x+c_3$. As $\varphi=0$, the second equation of (2.100) implies $\eta^1_t=\eta^0_t=0$, i.e. $\eta^1=c_4$, and $\eta^0=c_5$. Here c_i, $i=1,\dots,5$, are arbitrary constants. Then the general form of the infinitesimal generator is $Q=(c_1t+c_2)\partial_t+(c_1x+c_3)\partial_x+(c_4y+c_5)\partial_y$ and the first equation of (2.100) takes the form

$$(c_1t+c_2)f_t=(2c_4-c_1)f. \tag{2.104}$$

This is the classifying equation for f. If f is an arbitrary nonvanishing smooth function, then the latter equation should be split with respect to f and its derivative, which results in $c_1=c_2=c_4=0$. Therefore, the kernel $A^{\cap}$ of the maximal Lie invariance algebras of equations from class (2.95) is $A^{\cap}=\langle\partial_x,\,\partial_y\rangle$ (Case 1 of Table 2.11). To perform the further classification we integrate equation (2.104) up to the $G^{\sim}$-equivalence. All $G^{\sim}$-inequivalent values of f that provide Lie symmetry extensions for equations from class (2.95) with arbitrary k are exhausted by the following values: $f=t^\rho$, $\rho\neq 0$; $f=e^t$; $f=1$. The corresponding bases of maximal Lie invariance algebras are presented by Cases 2–4 of Table 2.11.

If $k=u^n$, $n\neq 0,1$, then splitting equations (2.101) and (2.102) with respect to different powers of u leads to the system $\xi_t=\psi=\varphi_x=0$,

Table 2.11: The group classification of class (2.95) up to the $G^{\sim}$-equivalence.

no.	$f(t)$	Basis of $A^{\max}$
		Arbitrary k
1	$\forall$	$\partial_x,\ \partial_y$
2	t^ρ	$\partial_x,\ \partial_y,\ 2t\partial_t+2x\partial_x+(\rho+1)y\partial_y$
3	e^t	$\partial_x,\ \partial_y,\ 2\partial_t+y\partial_y$
4	1	$\partial_x,\ \partial_y,\ \partial_t,\ 2t\partial_t+2x\partial_x+y\partial_y$
		$k=u^n,\ n\neq 0,1$
5	$\forall$	$\partial_x,\ \partial_y,\ nx\partial_x+u\partial_u$
6	t^ρ	$\partial_x,\ \partial_y,\ nx\partial_x+u\partial_u,\ 2t\partial_t+2x\partial_x+(\rho+1)y\partial_y$
7	e^t	$\partial_x,\ \partial_y,\ nx\partial_x+u\partial_u,\ 2\partial_t+y\partial_y$
8	1	$\partial_x,\ \partial_y,\ nx\partial_x+u\partial_u,\ \partial_t,\ 2t\partial_t+2x\partial_x+y\partial_y$
		$k=e^u$
9	$\forall$	$\partial_x,\ \partial_y,\ x\partial_x+\partial_u$
10	t^ρ	$\partial_x,\ \partial_y,\ x\partial_x+\partial_u,\ 2t\partial_t+2x\partial_x+(\rho+1)y\partial_y$
11	e^t	$\partial_x,\ \partial_y,\ x\partial_x+\partial_u,\ 2\partial_t+y\partial_y$
12	1	$\partial_x,\ \partial_y,\ x\partial_x+\partial_u,\ \partial_t,\ 2t\partial_t+2x\partial_x+y\partial_y$
		$k=\ln u$
13	$\forall$	$\partial_x,\ \partial_y,\ t\partial_x+u\partial_u$
14	t^ρ	$\partial_x,\ \partial_y,\ t\partial_x+u\partial_u,\ 2t\partial_t+2x\partial_x+(\rho+1)y\partial_y$
15	e^t	$\partial_x,\ \partial_y,\ t\partial_x+u\partial_u,\ 2\partial_t+y\partial_y$
16	1	$\partial_x,\ \partial_y,\ t\partial_x+u\partial_u,\ \partial_t,\ 2t\partial_t+2x\partial_x+y\partial_y$

Here n and ρ are arbitrary nonzero constants, and $n\neq 1$.

$\varphi_t=f\varphi_{yy}$, $n\varphi+\tau_t-\xi_x=0$. These equations together with (2.100) imply $\tau=c_1t+c_2$, $\xi=(c_1+nc_6)x+c_3$, $\eta=c_4y+c_5$, $\varphi=c_6$, where c_i, $i=1,\dots,6$, are arbitrary constants. The classifying equation for f takes the form (2.104). Therefore, the cases of Lie symmetry extensions are given by the same forms of f as in previous case, namely, arbitrary, power, exponential and constant. See Cases 5–8 of Table 2.11. The dimensions of the respective Lie symmetry algebras increase by one in comparing with the case of arbitrary k. The highest dimension is five, not six as it was

Table 2.12: The group classification of class (2.96) up to the $G_1^{\sim}$-equivalence.

no.	$f(t)$	Basis of $A^{\max}$
1	$\forall$	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ t\partial_x+\partial_u$
2	$\dfrac{e^{\sigma\arctan t}}{t^2+1}$	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ t\partial_x+\partial_u,\ (t^2+1)\partial_t+tx\partial_x+\frac{1}{2}\sigma y\partial_y+(x-tu)\partial_u$
3	t^ρ	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ t\partial_x+\partial_u,\ 2t\partial_t+(\rho+1)y\partial_y-2u\partial_u$
4	e^t	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ t\partial_x+\partial_u,\ 2\partial_t+y\partial_y$
5	1	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ t\partial_x+\partial_u,\ \partial_t,\ 2t\partial_t+y\partial_y-2u\partial_u$

Here ρ and σ are arbitrary constants with $\rho\neq 0,-2$. Moreover $\rho\leq -1 \bmod G_1^{\sim}$.

stated in [171].

The consideration of the cases $k=e^u$ and $k=\ln u$ is rather similar to the case of $k=u^n$ with $n\neq 0,1$, therefore, we omit the details of calculations. The classification results are presented in Cases 9–16 of Table 2.11.

Consider the case of linear k, then up to the equivalence we can assume $k=u$. We substitute $k=u$ to equations (2.101) and (2.102) and further split them with respect to different powers of u. This leads to the system $\psi=\xi_t$, $\tau_t-\xi_x+\varphi=0$, $\varphi_x=0$, $\psi_x+\varphi_t-f\varphi_{yy}=0$, and $\psi_t-f\psi_{yy}=0$. We differentiate the first and the second equation of this system with respect to the variable y and get the additional conditions $\varphi_y=\psi_y=0$. Then also $\psi_t=\psi_{xx}=\varphi_{tt}=0$ and the second equation of (2.100) gives $\eta^1_t=\eta^0_t=0$. The general form of the infinitesimal operator Q is $Q=(c_2t^2+c_1t+c_0)\partial_t+((c_2t+c_4)x+c_3t+c_5)\partial_x+(c_6y+c_7)\partial_y+((c_4-c_1-c_2t)u+c_2x+c_3)\partial_u$, where c_i, $i=0,\dots,7$, are arbitrary constants. The classifying equation is

$$(c_2t^2+c_1t+c_0)f_t=(2c_6-c_1-2c_2t)f. \tag{2.105}$$

If this is not an equation on f but an identity, then $c_0=c_1=c_2=c_6=0$. Therefore, the constants c_3, c_4, c_5, c_7 appearing in the infinitesimal generator Q are arbitrary and the maximal Lie invariance algebra of the equations (2.96) with arbitrary f is the four-dimensional algebra $\langle\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ t\partial_x+\partial_u\rangle$ (Case 1 of Table 2.12).

The further group classification of equations (2.95) with $k=u$, i.e.

equations (2.96), is equivalent to the integration of the equation on f:

$$(at^2 + bt + c)f_t = (d - 2at)f, \tag{2.106}$$

where a, b, c and d are arbitrary constants with $(a, b, c) \neq (0, 0, 0)$. Up to $G_1^\sim$-equivalence the parameter quadruple (a, b, c, d) can be assumed to belong to the set $\{(1, 0, 1, \sigma),\ (0, 1, 0, \rho),\ (0, 0, 1, 1),\ (0, 0, 1, 0)\}$, where σ, ρ are nonzero constants, $\rho \leq -1$. The proof is similar to ones presented in [302, 307]. It is based on the fact that transformations from the equivalence group $G_1^\sim$ can be extended to the coefficients a, b, c and d as follows

$$\tilde{a} = \mu(a\delta^2 - b\gamma\delta + c\gamma^2), \quad \tilde{b} = \mu(-2a\beta\delta + b(\alpha\delta + \beta\gamma) - 2c\alpha\gamma),$$
$$\tilde{c} = \mu(a\beta^2 - b\alpha\beta + c\alpha^2), \quad \tilde{d} = \mu(d\Delta + 2a\beta\delta - 2b\beta\gamma + 2c\alpha\gamma),$$

where $\Delta = \alpha\delta - \beta\gamma \neq 0$ and μ is an arbitrary nonzero constant.

Integration of the equation (2.106) for four inequivalent cases of the quadruple (a, b, c, d) gives respectively $f = \dfrac{e^{\sigma \arctan t}}{t^2 + 1}$, $f = t^\rho$, $\rho \neq 0$, $f = e^t$ and $f = 1$. We further substitute the obtained inequivalent values of f into equation (2.105) and find the corresponding values of constants c_i and, therefore, the general forms of the infinitesimal generators. The results of the group classification of class (2.96) are presented in Table 2.12.

The classification lists presented in Tables 2.11 and 2.12 give the exhaustive group classification of the class of variable coefficient nonlinear Kolmogorov equations (2.93) with nonlinear k and of the class of equations (2.94) up to the $\hat{G}^\sim$- and $\hat{G}_1^\sim$-equivalences, respectively.

2.7.3. Discussion on the Choice of the Optimal Gauging. Appropriate choice of gauging of the arbitrary elements is a crucial step in solving group classification problems. The gauging $f = 1$ could seem more convenient if one look for the determining equations for finding Lie symmetries. For class (2.97) they have the form

$$2\eta_y = \tau_t, \quad \eta_{yy} - \eta_t = 2\varphi_y, \quad (\varphi u + \psi)gk_u + [\tau g_t + (\tau_t - \xi_x)g]k = \xi_t,$$
$$(\varphi_x u + \psi_x)gk + (\varphi_t - \varphi_{yy})u + \psi_t - \psi_{yy} = 0.$$

For the case $k \neq u$ the difference in classification is not so crucial (cf. Table 2.11 with Table 2.13). Though one can see that for $k = \ln u$ the operator $t\partial_x + u\partial_u$ appearing in Cases 13–16 of Table 2.11 transforms to various forms in the respective cases of Table 2.13. For the case $k = u$ the difficulty of group classification of the class (2.93) with $f = 1$ increases essentially in comparison with the gauging $g = 1$. Solving the determining equations results in the following form of the infinitesimal generator

$$Q = (c_1 t + c_0)\partial_t + [(c_2 x + c_3)\textstyle\int g(t)\mathrm{d}t + c_4 x + c_5]\partial_x +$$
$$(\tfrac{1}{2}c_1 y + c_6)\partial_y + [(c_7 - c_2 \textstyle\int g(t)\mathrm{d}t)u + c_2 x + c_3]\partial_u,$$

where c_i, $i = 0, \dots, 7$, are arbitrary constants. The classifying equation for g is the integro-differential equation

$$(c_1 t + c_0)g_t + \left(c_1 - c_4 + c_7 - 2c_2 \textstyle\int g(t)\,\mathrm{d}t\right) g = 0$$

(cf. with the classifying equation (2.105) for f that is much simpler). The results of group classification for class (2.98) are presented in Table 2.14. Comparing Tables 2.12 and 2.14 one can conclude that forms of the basis operators of the maximal Lie invariance algebras are more cumbersome in Table 2.14.

The links between equations of the form (2.98) are also more tricky than those between equations from class (2.96). For example, the equation

$$u_t = u_{yy} - \frac{1}{t\cosh^2(\nu \ln t)} u u_x,$$

where the variable coefficient can be rewritten as $\dfrac{4}{t(t^\nu + t^{-\nu})^2}$, admits the five-dimensional maximal Lie invariance algebra with the basis operators ∂_x, ∂_y, $\tanh(\nu\ln t)\partial_x + \nu\partial_u$, $x\partial_x + u\partial_u$, and $t\partial_t - \nu x\tanh(\nu\ln t)\partial_x + \frac{1}{2}y\partial_y - \nu(\nu x - \tanh(\nu\ln t)u)\partial_u$. The equivalence of this equation and the equation

$$\tilde u_{\tilde t} = \tilde u_{\tilde y\tilde y} - \tilde t^{\,2\nu-1}\tilde u\tilde u_{\tilde x}$$

Table 2.13: The group classification of class (2.97) up to the $\hat{G}_2^{\sim}$-equivalence.

no.	$g(t)$	Basis of $A^{\max}$
Arbitrary k		
1	$\forall$	$\partial_x,\ \partial_y$
2	t^ρ	$\partial_x,\ \partial_y,\ 2t\partial_t+2(\rho+1)x\partial_x+y\partial_y$
3	e^t	$\partial_x,\ \partial_y,\ \partial_t+x\partial_x$
4	1	$\partial_x,\ \partial_y,\ \partial_t,\ 2t\partial_t+2x\partial_x+y\partial_y$
$k=u^n,\ n\neq 0,1$		
5	$\forall$	$\partial_x,\ \partial_y,\ nx\partial_x+u\partial_u$
6	t^ρ	$\partial_x,\ \partial_y,\ nx\partial_x+u\partial_u,\ 2t\partial_t+2(\rho+1)x\partial_x+y\partial_y$
7	e^t	$\partial_x,\ \partial_y,\ nx\partial_x+u\partial_u,\ \partial_t+x\partial_x$
8	1	$\partial_x,\ \partial_y,\ nx\partial_x+u\partial_u,\ \partial_t,\ 2t\partial_t+2x\partial_x+y\partial_y$
$k=e^u$		
9	$\forall$	$\partial_x,\ \partial_y,\ x\partial_x+\partial_u$
10	t^ρ	$\partial_x,\ \partial_y,\ x\partial_x+\partial_u,\ 2t\partial_t+2(\rho+1)x\partial_x+y\partial_y$
11	e^t	$\partial_x,\ \partial_y,\ x\partial_x+\partial_u,\ \partial_t+x\partial_x$
12	1	$\partial_x,\ \partial_y,\ x\partial_x+\partial_u,\ \partial_t,\ 2t\partial_t+2x\partial_x+y\partial_y$
$k=\ln u$		
13	$\forall$	$\partial_x,\ \partial_y,\ \int g(t)\,\mathrm{d}t\,\partial_x+u\partial_u$
14_a	$t^\rho,\ \rho\neq -1$	$\partial_x,\ \partial_y,\ t^{\rho+1}\partial_x+(\rho+1)u\partial_u,\ 2t\partial_t+2(\rho+1)x\partial_x+y\partial_y$
14_b	t^{-1}	$\partial_x,\ \partial_y,\ \ln t\,\partial_x+u\partial_u,\ 2t\partial_t+y\partial_y$
15	e^t	$\partial_x,\ \partial_y,\ e^t\partial_x+u\partial_u,\ \partial_t+x\partial_x$
16	1	$\partial_x,\ \partial_y,\ t\partial_x+u\partial_u,\ 2t\partial_t+2x\partial_x+y\partial_y,\ \partial_t$

Here n and ρ are arbitrary nonzero constants.

from the same class does not seem obvious. Nevertheless, there exists the transformation from the equivalence group $\hat{G}_3^{\sim}$,

$$\tilde{t}=t,\quad \tilde{x}=\frac{1}{4}x(t^{2\nu}+1),\quad \tilde{y}=y,\quad \tilde{u}=\frac{u}{t^{2\nu}+1}+\frac{\nu}{2}x,$$

that establishes a link between these equations. This shows that the distinguishing inequivalent cases of Lie symmetry extensions for class (2.98) is also a more difficult task than for class (2.96).

Table 2.14: The group classification of class (2.98) up to the $\hat{G}_3^{\sim}$-equivalence.

no.	$g(t)$	Basis of $A^{\max}$
1	$\forall$	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ \int g(t)\,\mathrm{d}t\,\partial_x+\partial_u$
2	$\dfrac{1}{t\cos^2(\nu\ln t)}$	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ \tan(\nu\ln t)\partial_x+\nu\partial_u,$ $t\partial_t+\nu x\tan(\nu\ln t)\partial_x+\frac{1}{2}y\partial_y+\nu(\nu x-\tan(\nu\ln t)u)\partial_u$
3	$\dfrac{1}{\cos^2 t}$	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ \tan t\partial_x+\partial_u,\ \partial_t+x\tan t\partial_x+(x-u\tan t)\partial_u$
4_a	t^ρ	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ t^{\rho+1}\partial_x+(\rho+1)\partial_u,\ 2t\partial_t+2(\rho+1)x\partial_x+y\partial_y$
4_b	t^{-1}	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ \ln t\,\partial_x+\partial_u,\ 2t\partial_t+y\partial_y$
5	e^t	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ e^t\partial_x+\partial_u,\ \partial_t+x\partial_x$
6	1	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ t\partial_x+\partial_u,\ \partial_t,\ 2t\partial_t+2x\partial_x+y\partial_y$

Here ρ and ν are arbitrary constants with $\nu\neq 0$, $\rho\neq -2,-1,0$. Moreover $\rho<-1 \bmod \hat{G}_3^{\sim}$.

Therefore, the gauging $g=1$ is without a doubt the right choice to perform a group classification for the class (2.93) and especially for its subclass (2.94). So, is there a regular way that can help one to choose a preferable gauging among several possible ones? Equivalence groups appear to be indicators showing the right choice of gauging. Indeed, the comparison of the equivalence groups presented in Theorems 3 and 4 with those given in Theorems 5 and 6 shows that the equivalence groups of class (2.95) and its subclass (2.96) are usual whereas the equivalence groups of class (2.97) and its subclass (2.98) remain to be generalized extended as the equivalence group of the initial class. Transformations from the generalized extended groups become point only after fixing arbitrary elements and integrals of g then naturally appear in the forms of Lie symmetry generators and even in the classifying equation. This of course makes the calculations more difficult. Therefore, the widest possible equivalence group should be necessarily found even before applying the Lie invariance criterion to equations under study in order to choose the optimal gauging and to optimize the entire process of group classification.

The values of arbitrary element K of class (2.92) can be derived using the values of arbitrary element k of class (2.93) adduced in Tables 1 and 3, namely: $k = u^n$, $n \neq 0, -1$, $\leftrightarrow K = u^{n+1}$; $k = u^{-1} \leftrightarrow K = \ln u$; $k = e^u$ $\leftrightarrow K = e^u$; $k = \ln u \leftrightarrow K = u \ln u$.

Application of the widest possible (generalized extended) equivalence groups allowed us to write down classification lists in the explicit and concise form. We have also shown that the equivalence group is that indicator that helps one to choose the optimal gauging among several possible ones.

Chapter 3

Equivalence Groupoids in the Study of KdV-Like Equations and Related Models

The classical Korteweg-de Vries (KdV) equation and its generalizations model various physical systems, including gravity waves, plasma waves and waves in lattices [144]. In particular, the KdV equation arises in the modeling of one-dimensional plane waves in cold quasi-neutral collision-free plasma propagating along the x-direction under the presence of a uniform magnetic field [149].

The KdV equation is widely recognised as a paradigm for the description of weakly nonlinear long waves in many branches of physics and engineering. It describes how waves evolve under the competing but comparable effects of weak nonlinearity and weak dispersion.

In the last decades there is a great interest to variable coefficient models that in many cases describe the real world phenomena with more accuracy. Classifications of Lie symmetries are usual tasks in studies of such models. This is due to the fact that Lie symmetries allows one not only to reduce a model PDE to a PDE with fewer number of independent variables or to an ODE but also to derive cases that are potentially more interesting for applications [89].

In Section 3.1 we completely describe the equivalence groupoid of the class $\mathcal{L}$ of variable coefficient KdV equations $u_t + f(t,x)uu_x + g(t,x)u_{xxx} = 0$, $fg \neq 0$. The class $\mathcal{L}$ is partitioned into six disjoint subclasses, which induces a partition of the equivalence groupoid of the class $\mathcal{L}$ since equations from different subclasses of the partition are not related by point transformations. It is proved that only one of the subclasses is normalized in the usual sense, whereas other subclasses are normalized in the generalized extended sense. The usual and generalized extended equivalence groups are computed for each of the subclasses, which leads, in view of their normalization, to the description of their equivalence groupoids and, therefore, to the description of the equivalence groupoid of the entire class $\mathcal{L}$. Ways for improvement of transformational properties of the subclasses that are not normalized in the usual sense are considered. These ways involve gaugings of arbitrary elements by families of equivalence transformations and mappings between classes generated by families of equivalence transformations. The group classification of one of the subclasses is carried out as an illustrative example.

In Section 3.2 we construct a hierarchy of normalized classes of third-order $(1 + 1)$-dimensional evolution equations, which is related to the Korteweg–de Vries equation and their generalizations. This gives the complete description of the equivalence groupoids of the classes from the hierarchy. For two wide classes of the variable-coefficient KdV and mKdV equations, we derive the necessary and sufficient conditions of similarity of such equations to the standard KdV and mKdV equations, respectively. We also carry out exhaustive group analysis of a normalized class of variable-coefficient KdV equations $u_t + f(t)uu_x + g(t)u_{xxx} = 0$, $fg \neq 0$.

The exhaustive group classification of variable coefficient generalized Kawahara equations $u_t + \alpha(t)u^n u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0$ is carried out in Section 3.3. Lie reductions of these equations to ordinary differential equations are classified. We also present some examples on the construction of exact and numerical solutions of these equations, in particular, the solution of a related boundary-value problem modelling the propagation of long nonlinear waves in the water covered by ice whose thickness grows in time.

In Section 3.4 we investigate Benjamin–Bona–Mahony (BBM) equations that is similarly to KdV equation model the unidirectional propagation of moderately long waves with small finite amplitude in systems that manifest nonlinear and dispersive effects. Group classification of BBM equations with time dependent coefficients $u_t + f(t)u_x + g(t)uu_x + h(t)u_{xxt} = 0$, $gh \neq 0$, is carried out using the method of mapping between classes. As by-product of this approach the complete group classification of a class of variable-coefficient BBM equations with forcing term is derived.

In Section 3.5 we discuss how point transformations can be used for the study of integrability, in particular, for deriving classes of integrable variable coefficient differential equations. The procedure of computing equivalence groupoids is specified for the case of nth-order evolution equations. A class of fifth-order variable-coefficient KdV-like equations is considered within the framework suggested.

The results of this chapter are published in [4*,7*,9*,13*,17*,19*,24*,28*].

3.1. Equivalence Groupoid of a Class of Variable Coefficient Korteweg–de Vries Equations

Notably that one of the first classes whose transformational properties were investigated was the class of remarkable variable-coefficient Korteweg–de Vries equations

$$\mathcal{L}: \quad u_t + f(t,x)uu_x + g(t,x)u_{xxx} = 0, \quad fg \neq 0. \tag{3.1}$$

The classical Korteweg–de Vries equation and its generalizations model various physical systems, including classical water waves, gravity waves, plasma waves and waves in lattices [?, 53, 105, 144]. There are a number of works, where equations from the class $\mathcal{L}$ were investigated from various points of view. We mention only the works most related to our study. In papers [120, 172] the Painlevé analysis of equations (3.1) was performed.

It was shown that such equations pass Painlevé test only if the coefficients f and g satisfy the conditions $f_x = g_x = 0$, and $(g/f)_t/f = \text{const}$. These conditions coincide with those of reducibility of variable coefficient equations (3.1) to the standard Korteweg–de Vries equation, which is foreseeable.

Lie symmetries and allowed transformations of equations from the class $\mathcal{L}$ were studied in [101, 110, 315]. We aim to present enhanced results on transformational properties of these equations, which will be investigated with modern point of view. It was deduced correctly in [101, 315] that there are six different subclasses of the class (3.1) which differ by their transformational properties. There are several reasons to reconsider the problem of classification of admissible transformations in this class:

- The consideration in [101,315] was restricted to fiber-preserving point transformations without a proof that transformations of such kind exhaust all possible admissible point transformations. We aim to show that not only point but even contact admissible transformations are exhausted by fiber-preserving ones.

- When six subclasses that differ by their transformational properties were derived in [101,315] the arbitrary elements were gauged whenever possible from the very beginning. The admissible transformations were indicated for the simplified (gauged) forms of equations. Our strategy is firstly to consider the subclasses without simplification and only then the gauged ones.

- Only transformation components for independent and dependent variables were adduced in [101,315] without rules for changing arbitrary elements. We formulate results in terms of equivalence groups of appropriate kinds (usual or generalized extended), where transformations for both independent and dependent variables t, x, u and arbitrary elements f, g are presented.

- Some admissible transformations were found in [101,315] in an implicit form, namely they included functions that satisfy certain ordinary differential equations. We aim to present all the results explicitly.

The concept of normalized classes allows us to explain why subclasses with different transformational properties arise in the course of the investigation. In fact they are maximal normalized subclasses in the class under study. Therefore, all the consideration within the modern approach becomes complete and clear.

3.1.1. Classification of Admissible Transformations. Though the study of transformational properties in classes of PDEs are more often restricted to point admissible transformations it is possible also to consider contact admissible transformations and the respective contact equivalence groupoid. It was shown in [304] that for the normalized class of evolution equations of the form $\bar{\mathcal{L}}$: $u_t = F(t, x, u, u_x)u_n + G(t, x, u, u_1, \ldots, u_{n-1})$ for $n \geq 3$ its contact equivalence groupoid coincides with the respective point one, i.e., any contact admissible transformation between two equations from the class $\bar{\mathcal{L}}$ is the first prolongation of a point transformation between these equations. Therefore, for subclass (3.1) of the class $\bar{\mathcal{L}}$ it suffices to investigate only point admissible transformations.

It is well known that any point transformation $\mathcal{T}$ relating two fixed evolution equations in 1+1 dimensions has the form $\tilde{t} = T(t)$, $\tilde{x} = X(t, x, u)$, $\tilde{u} = U(t, x, u)$ with $T_t(X_xU_u - X_uU_x) \neq 0$. The partial derivatives involved in equations (3.1) are transformed as follows:

$$\tilde{u}_{\tilde{t}} = \frac{1}{T_t}\left(D_tU - \frac{D_tX}{D_xX}D_xU\right), \ \tilde{u}_{\tilde{x}} = \frac{D_xU}{D_xX},$$
$$\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = \frac{1}{D_xX}D_x\left(\frac{1}{D_xX}D_x\left(\frac{D_xU}{D_xX}\right)\right),$$

where $D_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \ldots$ and $D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \ldots$ are operators of the total differentiation with respect to t and x, restectively. Here and below subscripts of functions indicate partial derivatives with respect to the corresponding variables.

We substitute these expressions into the equation $\tilde{u}_{\tilde{t}} + \tilde{f}(\tilde{t},\tilde{x})\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{g}(\tilde{t},\tilde{x})\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$ and obtain an equation in terms of the untilded variables. In order to confine this equation to the manifold defined by (3.1) in the third-order jet space with the independent variables (t,x) and the dependent variable u we further substitute into it the expression $u_t = -f(t,x)uu_x - g(t,x)u_{xxx}$. Splitting the obtained identity with respect to the derivatives of u leads to the determining equations on the functions T, X, and U. In particular, the coefficient of u_{xxx} is $(\tilde{g}T_t - g(X_x + X_u u_x)^3)(X_xU_u - X_uU_x)$. Setting it to zero and taking into account the nondegeneracy condition $T_t(X_xU_u - X_uU_x) \neq 0$ we get $\tilde{g}T_t - g(X_x + X_u u_x)^3 = 0$. Further splitting with respect to u_x leads to the conditions $X_u = 0$ and $\tilde{g}T_t = gX_x^3$. Therefore, $X = X(t,x)$ and it means that there are no other admissible transformations except fiber-preserving ones. This agrees with the results of papers [160, 304], derived for more general classes of evolution equations. The condition $X_u = 0$ leads to essential simplification of the determining equations, thus we get additional constraints for the coefficients X and U: $U_{uu} = U_{xu} = X_{xx} = 0$. Therefore, the transformation components for independent and dependent variables of admissible transformations have the form

$$\tilde{t} = \theta(t), \quad \tilde{x} = \alpha(t)x + \beta(t), \quad \tilde{u} = \varphi(t)u + \psi(t,x), \tag{3.2}$$

where θ, α, β, φ and ψ are arbitrary smooth functions of their variables with $\dot{\theta}\alpha\varphi \neq 0$. The formulas

$$\tilde{f} = \frac{\alpha}{\dot{\theta}\varphi}f, \quad \tilde{g} = \frac{\alpha^3}{\dot{\theta}}g \tag{3.3}$$

establish the connection between values of arbitrary elements of the initial equation and the target equation. We denote by overdots ordinary derivatives with respect to the variable t.

The classifying equations involving θ, α, β, φ and ψ and the arbitrary functions f and g are of the form

$$f\psi_x + \dot{\varphi} = 0, \tag{3.4}$$

$$f\alpha\psi = \varphi(\dot\alpha x + \dot\beta), \tag{3.5}$$

$$\psi_t + g\psi_{xxx} = 0. \tag{3.6}$$

To deduce the equivalence group of the class (3.1) equations (3.4)–(3.6) should be split with respect to arbitrary elements f and g. This results in the equations $\dot\alpha = \dot\beta = \dot\varphi = \psi = 0$. The following statement is true.

Theorem 3.1. *The class $\mathcal{L}$ of equations (3.1) admits the usual equivalence group $G^{\sim}$ consisting of the transformations*

$$\tilde t = \theta(t), \quad \tilde x = \delta_1 x + \delta_2, \quad \tilde u = \delta_3 u,$$
$$\tilde f(\tilde t, \tilde x) = \frac{\delta_1}{\delta_3 \dot\theta(t)} f(t,x), \quad \tilde g(\tilde t, \tilde x) = \frac{{\delta_1}^3}{\dot\theta(t)} g(t,x),$$

where θ is an arbitrary function of t with $\dot\theta \neq 0$, δ_j $(j=1,2,3)$ are arbitrary constants with $\delta_1\delta_3 \neq 0$.

The class $\mathcal{L}$ is not normalized in any sense, there exist subclasses of this class singled out by setting restrictions on values of arbitrary elements f and g which possess wider equivalence groups than the group $G^{\sim}$. These subclasses and the corresponding *maximal conditional equivalence groups* [248, Definition 7] can be found investigating the system (3.4)–(3.6).

Equation (3.4) results in $\psi = \dfrac{1}{f}\dfrac{\varphi}{\alpha}\left(\dot\alpha x + \dot\beta\right)$, then equation (3.5) takes the form

$$\left(\frac{\dot\alpha}{\alpha} x + \frac{\dot\beta}{\alpha}\right) f_x = \left(\frac{\dot\varphi}{\varphi} + \frac{\dot\alpha}{\alpha}\right) f.$$

There are only three possibilities for the nonvanishing function $f(t,x)$ to satisfy this equation: I. $f = f(t) \neq 0$; II. $f = p(t)e^{q(t)x}$, $pq \neq 0$, and III. $f = p(t)(x+q(t))^{r(t)}$, $pr \neq 0$. Further we consider each case separately.

I. $f = f(t) \neq 0$. We substitute this form of f into the equations (3.4)–(3.6). This results in the conditions $\psi_t = \psi_{xx} = 0$ and therefore $\psi = c_1 x + c_2$, where c_1 and c_2 are arbitrary constants. Equation (3.6) becomes an identity, hence there is no restriction on the form of the function g and it remains arbitrary. Equations (3.4), (3.5) imply the system of equations for α, β an φ of the form: $\dot{\varphi} = -c_1 f$, $\varphi\dot{\alpha} = c_1 f\alpha$, and $\varphi\dot{\beta} = c_2 f\alpha$. The general solution of this system and formulas (3.3) give the full description of admissible transformations of the subclass of class (3.1) with $f_x = 0$. After redenotion of constants $c_1 = -\delta_3$, $c_2 = -\delta_2\delta_3/\delta_1$ we get the following assertion.

Theorem 3.2. *The generalized extended equivalence group $\hat{G}_1^{\sim}$ of the class*

$$\mathcal{L}_1: \quad u_t + f(t)uu_x + g(t,x)u_{xxx} = 0, \tag{3.7}$$

consists of the transformations

$$\tilde{t} = \theta(t), \quad \tilde{x} = \frac{\delta_1 x + \delta_2}{\delta_3 \int f(t)\,\mathrm{d}t + \delta_4} + \delta_5,$$

$$\tilde{u} = \left(\delta_3 \int f(t)\,\mathrm{d}t + \delta_4\right) u - \delta_3 x - \delta_2\delta_3\delta_1^{-1},$$

$$\tilde{f}(\tilde{t}) = \frac{\delta_1 f(t)}{\dot{\theta}(t)(\delta_3 \int f(t)\,\mathrm{d}t + \delta_4)^2}, \quad \tilde{g}(\tilde{t},\tilde{x}) = \frac{{\delta_1}^3 g(t,x)}{\dot{\theta}(t)(\delta_3 \int f(t)\,\mathrm{d}t + \delta_4)^3},$$

where θ is an arbitrary function of t with $\dot{\theta} \neq 0$, and δ_j $(j = 1,2,3,4,5)$ are arbitrary constants with $\delta_1({\delta_3}^2 + {\delta_4}^2) \neq 0$.

The usual equivalence group of the class (3.7) coincides with the equivalence group $G^{\sim}$ of its superclass (3.1).

II. $f = p(t)e^{q(t)x}$, where $pq \neq 0$. In this case we have in fact reparametrization of the class, thus instead of arbitrary element $f(t,x)$ there are two arbitrary elements $p(t)$ and $q(t)$. The equivalence relations for them are given by $\tilde{q} = \dfrac{q}{\alpha}$, and $\tilde{p} = \dfrac{p\alpha}{\varphi\dot{\theta}}\exp(-q\beta/\alpha)$. Equations (3.4)–(3.6) lead to the conditions $\psi = \dfrac{\dot{\varphi}}{pq}e^{-qx}$, $\dot{\alpha} = 0$, $\dot{\beta} = \dfrac{\dot{\varphi}\alpha}{\varphi q}$. The function g has the form $g = \dfrac{s(t) - \dot{q}x}{q^3}$ with the additional constraint $s = \dfrac{\ddot{\varphi}}{\dot{\varphi}} - \dfrac{\dot{p}}{p} - \dfrac{\dot{q}}{q}$. The function g is also reparameterized now and the function s can be regarded as another arbitrary element. The admissible transformations in the deduced subclass are described by the following statement.

Theorem 3.3. *The class of equations*

$$\mathcal{L}_2: \quad u_t + p(t)e^{q(t)x}uu_x + \frac{s(t)-\dot{q}(t)x}{q(t)^3}u_{xxx} = 0 \tag{3.8}$$

admits the generalized extended equivalence group $\hat{G}_2^{\sim}$ *consisting of the transformations*

$$\tilde{t} = \theta(t), \quad \tilde{x} = \delta_1 x + \beta(t), \quad \tilde{u} = \varphi(t)u + \frac{\dot{\varphi}(t)}{p(t)q(t)}e^{-q(t)x},$$
$$\tilde{p}(\tilde{t}) = \frac{\delta_1 p(t)}{\dot{\theta}(t)\varphi(t)}e^{-\frac{1}{\delta_1}\beta(t)q(t)}, \quad \tilde{q}(\tilde{t}) = \frac{q(t)}{\delta_1}, \quad \tilde{s}(\tilde{t}) = \frac{\delta_1 s(t) + \dot{q}(t)\beta(t)}{\delta_1\dot{\theta}(t)},$$

where θ *is an arbitrary function of* t *with* $\dot{\theta} \neq 0$,

$$\beta(t) = \delta_1\delta_3\int\frac{p(t)e^{\int s(t)\,\mathrm{d}t}}{\varphi(t)}\,\mathrm{d}t + \delta_2, \quad \varphi(t) = \delta_3\int p(t)q(t)e^{\int s(t)\,\mathrm{d}t}\mathrm{d}t + \delta_4.$$

Here δ_j $(j=1,2,3,4)$ *are arbitrary constants with* $\delta_1({\delta_3}^2 + {\delta_4}^2) \neq 0$.

III. $f = p(t)(x+q(t))^{r(t)}$ with $pr \neq 0$. In this case equation (3.5) implies $\psi = \dfrac{\varphi(\dot{\alpha}x+\dot{\beta})}{\alpha p(x+q)^r}$, then (3.4) leads to the conditions $\dfrac{\dot{\varphi}}{\varphi} = (r-1)\dfrac{\dot{\alpha}}{\alpha}$ and $\dot{\beta} = q\dot{\alpha}$. In view of the latter condition the function ψ can be rewritten as $\psi = \dfrac{\varphi}{p}\dfrac{\dot{\alpha}}{\alpha}(x+q)^{1-r}$. Then the equation (3.6) containing g takes the form $r(r^2-1)g = (x+q)^3\left(s - \dot{r}\ln(x+q) + \dfrac{(1-r)\dot{q}}{x+q}\right)$, where the function s, that can be regarded as new arbitrary element, satisfies the equation $s\dfrac{\varphi}{p}\dfrac{\dot{\alpha}}{\alpha} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\varphi}{p}\dfrac{\dot{\alpha}}{\alpha}\right)$. It is easy to see that in two special cases, namely, when $r=1$ or $r=-1$, the function g remains arbitrary. We consider the cases $r \neq \pm 1$, $r=1$ and $r=-1$ separately. The general solution of the determining equations gives in each of these three cases full description of admissible transformations in the corresponding subclass of the class (3.1). The results are collected in Theorems 3.4, 3.5 and 3.6. We note that in the case $r=-1$ the nontrivial equivalence group which is wider than $G^{\sim}$ exists only if the functions $q(t)=c$, where c is an arbitrary constant. In this case we use one more reparameterization introducing the new function $k = 1/p$, so f takes the form $f = k(t)/(x + c)$.

Theorem 3.4. *The generalized extended equivalence group $\hat{G}_3^{\sim}$ of the class*

$$\begin{aligned}\mathcal{L}_3\colon \quad & u_t + p(t)(x+q(t))^{r(t)}uu_x + \\ & \frac{(x+q(t))^3}{r(t)(r(t)^2-1)}\left(s(t) - \dot{r}(t)\ln(x+q(t)) + \frac{(1-r(t))\dot{q}(t)}{x+q(t)}\right)u_{xxx} = 0,\end{aligned}$$

$r \neq 0, \pm 1$, *consists of the transformations*

$$\begin{aligned}&\tilde{t} = \theta(t), \quad \tilde{x} = \alpha(t)x + \beta(t), \quad \tilde{u} = \varphi(t)u + \delta_1 e^{\int s(t)\,\mathrm{d}t}(x+q(t))^{1-r(t)},\\ &\tilde{p}(\tilde{t}) = \frac{\alpha(t)^{1-r(t)}}{\phi(t)\dot{\theta}(t)}p(t), \quad \tilde{q}(\tilde{t}) = q(t)\alpha(t) - \beta(t),\\ &\tilde{s}(\tilde{t}) = \frac{1}{\dot{\theta}(t)}\left(s(t) + \dot{r}(t)\ln\alpha(t)\right), \quad \tilde{r}(\tilde{t}) = r(t),\end{aligned}$$

where θ is an arbitrary nonvanishing function of t, $\dot{\theta} \neq 0$,

$$\begin{aligned}&\varphi(t) = \delta_1 \textstyle\int p(t)(r(t)-1)\, e^{\int s(t)\,\mathrm{d}t}\mathrm{d}t + \delta_2,\\ &\alpha(t) = \delta_3 \exp\left[\delta_1 \displaystyle\int \frac{p(t)}{\varphi(t)}\, e^{\int s(t)\,\mathrm{d}t}\mathrm{d}t\right], \quad \beta(t) = \textstyle\int q(t)\dot{\alpha}(t)\mathrm{d}t + \delta_4.\end{aligned}$$

Here δ_j $(j = 1, 2, 3, 4)$ are arbitrary constants with $({\delta_1}^2 + {\delta_2}^2)\delta_3 \neq 0$.

Note, that if $r \neq 0, \pm 1$ is a constant, then the coefficient α can be simplified as follows $\alpha(t) = \delta_3\varphi(t)^{\frac{1}{r-1}}$.

Theorem 3.5. *The class of equations of the form*

$$\mathcal{L}_4: \quad u_t + (p(t)x + q(t))uu_x + g(t,x)u_{xxx} = 0 \tag{3.9}$$

admits the generalized extended equivalence group $\hat{G}_4^{\sim}$ consisting of the transformations

$$\begin{aligned}&\tilde{t} = \theta(t), \quad \tilde{x} = \alpha(t)x + \beta(t), \quad \tilde{u} = \delta_1(u + \delta_2), \quad \tilde{p}(\tilde{t}) = \frac{p(t)}{\delta_1\dot{\theta}(t)},\\ &\tilde{q}(\tilde{t}) = \frac{1}{\delta_1\dot{\theta}(t)}(\alpha(t)q(t) - \beta(t)p(t)), \quad \tilde{g}(\tilde{t},\tilde{x}) = \frac{\alpha(t)^3}{\dot{\theta}(t)}g(t,x),\end{aligned}$$

where θ is an arbitrary function of t with $\dot{\theta} \neq 0$,

$$\alpha(t) = \delta_3 e^{\delta_2 \int p(t)\,\mathrm{d}t}, \quad \beta(t) = \delta_2\delta_3 \int q(t) e^{\delta_2 \int p(t)\,\mathrm{d}t}\,\mathrm{d}t + \delta_4.$$

Here δ_j $(j = 1, 2, 3, 4)$ are arbitrary constants with $\delta_1\delta_3 \neq 0$.

Theorem 3.6. *The generalized extended equivalence group $\hat{G}_5^{\sim}$ of the class*

$$\mathcal{L}_5: \quad u_t + \frac{k(t)}{x+c}\,uu_x + g(t,x)u_{xxx} = 0 \tag{3.10}$$

comprises the transformations

$$\tilde{t} = \theta(t), \quad \tilde{x} = \alpha(t)(x+c) + \delta_2, \quad \tilde{u} = \frac{\delta_1}{\alpha(t)^2}\,u - \frac{1}{2}\delta_1\delta_3(x+c)^2,$$
$$\tilde{k}(\tilde{t}) = \frac{\alpha(t)^4}{\delta_1\dot{\theta}(t)}k(t), \quad \tilde{c} = -\delta_2, \quad \tilde{g}(\tilde{t},\tilde{x}) = \frac{\alpha(t)^3}{\dot{\theta}(t)}g(t,x),$$

where θ is an arbitrary function of t with $\dot{\theta} \neq 0$,

$$\alpha(t) = \pm\left(\delta_3 \int k(t)\,\mathrm{d}t + \delta_4\right)^{-\frac{1}{2}},$$

and δ_j $(j = 1, 2, 3, 4)$ are arbitrary constants with $\delta_1({\delta_3}^2 + {\delta_4}^2) \neq 0$.

Theorem 3.7. *Classes $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, and $\mathcal{L}_5$ are normalized in the generalized extended sense, their equivalence groupoids are induced by transformations from the corresponding generalized extended equivalence groups $\hat{G}_i^{\sim}$, $i = 1, \ldots, 5$.*

Theorem 3.8. *The subclass $\mathcal{L}_0 = \mathcal{L} \setminus \bigcup_{i=1}^{5} \mathcal{L}_i$ that is the complement of the union of the subclasses $\mathcal{L}_i$, $i = 1, \ldots, 5$, in the class $\mathcal{L}$ is normalized in the usual sense. Its equivalence group coincides with the usual equivalence group $G^{\sim}$ of the whole class $\mathcal{L}$.*

There are no point transformations between any two equations from different subclasses $\mathcal{L}_i$, $i = 0, \ldots, 5$, of the class (3.1).

3.1.2. Equivalence Groups of the Gauged Subclasses $\mathcal{L}_1, \dots, \mathcal{L}_5$. Using equivalence transformations we can perform the gauging of the arbitrary elements depending on t in each subclass $\mathcal{L}_1, \dots, \mathcal{L}_5$ of the class $\mathcal{L}$ and therefore, to reduce the number of their arbitrary elements. We consider each subclass separately.

$\boldsymbol{\mathcal{L}_1}$. The equivalence transformations with $\tilde{t} = \int f(t)\mathrm{d}t$, $\tilde{x} = x$, and $\tilde{u} = u$, which belong the group $\hat{G}_1^{\sim}$, reduces each equation from class (3.7) to the equation from the same class with $\tilde{f} = 1$. Therefore, without loss of generality the gauged subclass

$$\check{\mathcal{L}}_1: \quad u_t + uu_x + g(t,x)u_{xxx} = 0$$

can be investigated instead of $\mathcal{L}_1$. As $\mathcal{L}_1$ is normalized it is easy to deduce the equivalence group of its subclass $\check{\mathcal{L}}_1$. We put $\tilde{f} = f = 1$ in transformations from the group $\hat{G}_1^{\sim}$ and get the equation on θ, $\dot{\theta} = \delta_1/(\delta_3 t + \delta_4)^2$. In order to write the obtained transformations in the unified form, which include both cases $\delta_3 = 0$ and $\delta_3 \neq 0$, we redenote the constants involved in transformations and get the following assertion (the corollary of Theorem 3.2).

Corollary 3.9. *The generalized extended equivalence group of the class $\check{\mathcal{L}}_1$ is trivial. It coincides with the usual equivalence group of this class, which is comprised of transformations*

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{\kappa x + \mu_1 t + \mu_0}{\gamma t + \delta},$$
$$\tilde{u} = \frac{\kappa(\gamma t + \delta) - \kappa\gamma x + \mu_1\delta - \mu_0\gamma}{\alpha\delta - \beta\gamma}, \quad \tilde{g}(\tilde{t},\tilde{x}) = \frac{\kappa^3}{\alpha\delta - \beta\gamma}\frac{g(t,x)}{\gamma t + \delta},$$

$\alpha, \beta, \gamma, \delta, \kappa, \mu_1, \mu_0$ are constants defined up to a nonzero multiplier, $\kappa(\alpha\delta - \beta\gamma) \neq 0$, and without loss of generality we can assume that $\alpha\delta - \beta\gamma = \pm 1$.

This group is in fact coincides with the equivalence group of subclass of class $\check{\mathcal{L}}_1$, singled out by the condition $g_x = 0$, which was derived in [251].

$\boldsymbol{\mathcal{L}_2}$–$\boldsymbol{\mathcal{L}_4}$. Each equation from the classes $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$, is reducible to an equation from the respective class with $\tilde{p} = 1$ by the equivalence transformation with $\tilde t=\int p(t)\,\mathrm{d}t$, $\tilde x=x$, and $\tilde u=u$. The equivalence groups of the gauged classes directly follow from Theorems 3.3, 3.4 and 3.5 if we put $\tilde p=1$ and $p=1$ in the transformations therein. The respective Corollaries 3.10, 3.11 and 3.12 are adduced in the following assertions.

Corollary 3.10. *The generalized extended equivalence group of the class*

$$\check{\mathcal{L}}_2:\quad u_t+e^{q(t)x}uu_x+\frac{s(t)-\dot q(t)x}{q(t)^3}u_{xxx}=0$$

consists of the transformations

$$\tilde t=\delta_1\int\frac{1}{\varphi(t)}e^{-\frac{1}{\delta_1}\beta(t)q(t)}\mathrm{d}t+\delta_0,\quad \tilde x=\delta_1x+\beta(t),$$
$$\tilde u=\varphi(t)u+\delta_3e^{-q(t)x+\int s(t)\,\mathrm{d}t},$$
$$\tilde q(\tilde t)=\frac{q(t)}{\delta_1},\quad \tilde s(\tilde t)=\frac{1}{{\delta_1}^2}\varphi(t)e^{\frac{1}{\delta_1}\beta(t)q(t)}\left(\delta_1s(t)+\dot q(t)\beta(t)\right),$$

where δ_j $(j=0,1,2,3,4)$ are arbitrary constants, $\delta_1({\delta_3}^2+{\delta_4}^2)\neq0$,

$$\beta(t)=\delta_1\delta_3\int\frac{e^{\int s(t)\,\mathrm{d}t}}{\varphi(t)}\mathrm{d}t+\delta_2,\quad \varphi(t)=\delta_3\int q(t)e^{\int s(t)\,\mathrm{d}t}\mathrm{d}t+\delta_4.$$

Corollary 3.11. *The generalized extended equivalence group of the class*

$$\check{\mathcal{L}}_3:\quad u_t+(x+q(t))^{r(t)}uu_x+\frac{(x+q(t))^3}{r(t)(r(t)^2-1)}\left(s(t)-\dot r(t)\ln(x+q(t))+\frac{(1-r(t))\dot q(t)}{x+q(t)}\right)u_{xxx}=0,$$

consists of the transformations

$$\tilde t=\theta(t),\quad \tilde x=\alpha(t)x+\beta(t),\quad \tilde u=\varphi(t)u+\delta_1e^{\int s(t)\,\mathrm{d}t}(x+q(t))^{1-r(t)},$$
$$\tilde q(\tilde t)=q(t)\alpha(t)-\beta(t),\quad \tilde s(\tilde t)=\alpha(t)^{r(t)-1}\varphi(t)\left(s(t)+\dot r(t)\ln\alpha(t)\right),$$
$$\tilde r(\tilde t)=r(t),$$

where θ is an arbitrary nonvanishing function of t, $\dot\theta\neq0$,

$$\varphi(t)=\delta_1\textstyle\int(r(t)-1)\,e^{\int s(t)\mathrm{d}t}\mathrm{d}t+\delta_2,$$
$$\alpha(t)=\delta_3\exp\left[\delta_1\int\frac{e^{\int s(t)\mathrm{d}t}}{\varphi(t)}\,\mathrm{d}t\right],\quad \beta(t)=\textstyle\int q(t)\dot\alpha(t)\mathrm{d}t+\delta_4.$$

Here δ_j $(j=1,2,3,4)$ are arbitrary constants with $({\delta_1}^2+{\delta_2}^2)\delta_3\neq0$.

Corollary 3.12. *The generalized extended equivalence group* $\check{G}_4^{\sim}$ *of the class*

$$\check{\mathcal{L}}_4: \quad u_t + (x+q(t))uu_x + g(t,x)u_{xxx} = 0 \tag{3.11}$$

is formed by of the transformations

$$\tilde t = \tfrac{1}{\delta_1}t + \delta_0, \quad \tilde x = \delta_3 e^{\delta_2 t}x + \delta_2\delta_3 \textstyle\int q(t)e^{\delta_2 t}\,\mathrm{d}t + \delta_4, \quad \tilde u = \delta_1(u+\delta_2),$$
$$\tilde q(\tilde t) = \delta_3 e^{\delta_2 t}q(t) - \delta_2\delta_3 \textstyle\int q(t)e^{\delta_2 t}\,\mathrm{d}t - \delta_4, \quad \tilde g(\tilde t,\tilde x) = \delta_1{\delta_3}^3 e^{3\delta_2 t}g(t,x),$$

where δ_j $(j=0,1,2,3,4)$ *are arbitrary constants with* $\delta_1\delta_3 \neq 0$*.*

$\boldsymbol{\mathcal{L}_5}$. In the case of subclass $\mathcal{L}_5$ of class (3.1) two arbitrary elements can be gauged: the function $k(t)$ can be set to one, and the constant c to zero. This gauge is realized by the equivalence transformation with $\tilde t = \int k(t)\,\mathrm{d}t$, $\tilde x = x + c$, and $\tilde u = u$. The equivalence group of the gauged class

$$\check{\mathcal{L}}_5: \quad u_t + \frac{1}{x}uu_x + g(t,x)u_{xxx} = 0$$

is presented in the following statement.

Corollary 3.13. *The generalized extended equivalence group of the class* $\check{\mathcal{L}}_5$ *is trivial. It coincides with the usual equivalence group of the class comprised of the transformations*

$$\tilde t = \frac{\alpha t+\beta}{\gamma t+\delta}, \quad \tilde x = (\gamma t+\delta)^{-\frac12}x, \quad \tilde u = \frac{1}{\alpha\delta-\beta\gamma}\left((\gamma t+\delta)u - \frac12\gamma x^2\right),$$
$$\tilde g(\tilde t,\tilde x) = \frac{(\gamma t+\delta)^{\frac12}}{\alpha\delta-\beta\gamma}g(t,x),$$

where α, β, γ, *and* δ *are constants defined up to a nonzero multiplier with* $\alpha\delta-\beta\gamma\neq 0$*, without loss of generality we can assume that* $\alpha\delta-\beta\gamma = \pm 1$*.*

All the gauged subclasses remain normalized, their equivalence groupoids are generated by the respective equivalence groups presented in Corollaries 3.9–3.13. The following statement is true.

Theorem 3.14. *The classes* $\check{\mathcal{L}}_1$ *and* $\check{\mathcal{L}}_5$ *are normalized in the usual sense. The classes* $\check{\mathcal{L}}_2$*,* $\check{\mathcal{L}}_3$*, and* $\check{\mathcal{L}}_4$ *are normalized in the generalized extended sense.*

This theorem implies that the transformational properties of classes $\check{\mathcal{L}}_1$ and $\check{\mathcal{L}}_5$ are nicer than such properties of their superclasses $\mathcal{L}_1$ and $\mathcal{L}_5$. These classes are quite convenient already to investigate them with Lie symmetry or other points of view. The classes $\check{\mathcal{L}}_2$, $\check{\mathcal{L}}_3$, and $\check{\mathcal{L}}_4$ are still normalized only in the generalized extended sense, therefore, the links between equations in each of them are quite complicated and this courses difficulties in their investigation. In the next section we propose a way of improvement of the transformational properties of the class $\check{\mathcal{L}}_4$.

3.1.3. Possible Improvement of Transformational Properties via Mappings Between Classes. Classes of differential equations that possess generalized extended equivalence groups are more complicated for investigation than those whose equivalence groups are usual ones. To deal with such classes the method based on mappings between classes was proposed in [300]. This method appears to be very efficient, in particular, for solving the group classification problems. We will illustrate the method using the example of the class $\check{\mathcal{L}}_4$. This class can be mapped to the related class of KdV-like equations

$$u_t + xuu_x + h(t)u_x + g(t,x)u_{xxx} = 0 \tag{3.12}$$

by the family of point transformation

$$\hat{t} = t, \quad \hat{x} = x + q(t), \quad \hat{u} = u, \tag{3.13}$$

parameterized by the arbitrary element q of the class. The element h in the imaged class is connected with the arbitrary element q in the initial class via the formula $h(t) = \dot{q}(t)$. Therefore, the case $q = \mathrm{const}$ corresponds to the case $h = 0$. In contrast to the class $\check{\mathcal{L}}_4$ that is normalized in the

Table 3.1: The group classification of the class $u_t+xuu_x+h(t)u_x+g(t,x)u_{xxx}=0,\ hg\neq 0.$

no.	Equation	Constraints	Basis of $A^{\max}$
1	$u_t+xuu_x+t^{m-1}u_x+t^{3m-1}\Phi(x/t^m)u_{xxx}=0$	$\Phi(z)\neq\lambda z^2$ if $m=1$	$t\partial_t+mx\partial_x-u\partial_u$
2	$u_t+xuu_x+e^{\frac{\varepsilon}{2}t^2}u_x+e^{\frac{3\varepsilon}{2}t^2}\Phi\left(xe^{-\frac{\varepsilon}{2}t^2}\right)u_{xxx}=0$		$\partial_t+\varepsilon tx\partial_x+\varepsilon\partial_u$
3	$u_t+xuu_x+u_x+\Phi(x)u_{xxx}=0$	$\Phi(x)\neq\lambda x^2$	∂_t
4	$u_t+xuu_x+u_x+\lambda x^2u_{xxx}=0$		$\partial_t,\ t\partial_t+x\partial_x-u\partial_u$

Here $\varepsilon=\pm1 \bmod G^\sim_4$, λ and m are arbitrary constants, $\lambda\neq 0$.

generalized extended sense, class (3.12) is normalized in the usual sense. The equivalence groupoid of class (3.12) is induced by the transformations from its usual equivalence group $G^\sim_4$:

$$\tilde t=\delta_1t+\delta_2,\quad \tilde x=\delta_3e^{\delta_4t}x,\quad \tilde u=\frac{1}{\delta_1}(u+\delta_4),$$
$$\tilde h(\tilde t)=\frac{\delta_3}{\delta_1}e^{\delta_4t}h(t),\quad \tilde g(\tilde t,\tilde x)=\frac{{\delta_3}^3}{\delta_1}e^{3\delta_4t}g(t,x),$$

where δ_j $(j=1,2,3,4)$ are arbitrary constants, $\delta_1\delta_3\neq 0$.

The group classification of class (3.12) can be performed using the standard method [227] based on integration of the determining equations up to the $G^\sim_4$-equivalence. The results of the group classification of equations (3.12) with $h\neq 0$ (resp. $h=0$) are presented in Table 3.1 (resp. Table 3.2). The group classification list for equations (3.11) with $q=\mathrm{const}$ coincides with such a list for equations (3.12) with $h=0$ and, therefore, it is presented by the list given in Table 3.2. The group classification of equations (3.11) with $q\neq\mathrm{const}$ can be recovered from the classification results obtained for equations (3.12) with $h\neq 0$ by using the transformation (3.13). The results are presented in Table 3.3.

Note that in [109] the group classification for more general class of KdV-like equations that includes classes (3.11), (3.12) was carried out. However those results obtained up to very wide equivalence group seem to

Table 3.2: The group classification of the class $u_t + xuu_x + g(t,x)u_{xxx} = 0$, $g \neq 0$.

no.	Equation	Constraints	Basis of $A^{\max}$
1	$u_t + xuu_x + \Phi(x)u_{xxx} = 0$	$\Phi(x) \neq \lambda x^m$	∂_t
2	$u_t + xuu_x + t^{3m-1}\Phi(x/t^m)u_{xxx} = 0$	$\Phi(z) \neq \lambda z^3, \lambda z^{\frac{3m-1}{m}}$	$t\partial_t + mx\partial_x - u\partial_u$
3	$u_t + xuu_x + e^{\frac{3\varepsilon}{2}t^2}\Phi\left(xe^{-\frac{\varepsilon}{2}t^2}\right)u_{xxx} = 0$	$\Phi(z) \neq \lambda z^3$	$\partial_t + \varepsilon tx\partial_x + \varepsilon\partial_u$
4	$u_t + xuu_x + x^m u_{xxx} = 0$	$m \neq 3$	$\partial_t,\ t\partial_t + \frac{1}{3-m}x\partial_x - u\partial_u$
5	$u_t + xuu_x + \Phi(t)x^3u_{xxx} = 0$	$\Phi(t) \neq \lambda, \dfrac{1}{\lambda t + \nu}$	$x\partial_x,\ tx\partial_x + \partial_u$
6	$u_t + xuu_x + x^3u_{xxx} = 0$		$\partial_t,\ x\partial_x,\ tx\partial_x + \partial_u$
7	$u_t + xuu_x + \lambda\dfrac{x^3}{t}u_{xxx} = 0$		$x\partial_x,\ t\partial_t - u\partial_u,$ $tx\partial_x + \partial_u$

Here $\varepsilon = \pm 1 \bmod G_4^{\sim}$, m, λ and ν are arbitrary constants, $\lambda \neq 0$.

Table 3.3: The group classification of the class $u_t+(x + q(t))uu_x+g(t,x)u_{xxx} = 0$, $\dot{q}g \neq 0$.

no.	Equation	Basis of $A^{\max}$
1	$u_t + (x + t^m)\,uu_x + t^{3m-1}\Phi\left(\frac{x}{t^m}\right)u_{xxx} = 0$	$t\partial_t + mx\partial_x - u\partial_u$
2	$u_t + (x + \ln t)\,uu_x + t^{-1}\Phi\,(x + \ln t)\,u_{xxx} = 0$	$t\partial_t - \partial_x - u\partial_u$
3	$u_t + \omega uu_x + e^{\frac{3\varepsilon}{2}t^2}\Phi\left(\omega e^{-\frac{\varepsilon}{2}t^2}\right)u_{xxx} = 0$	$\partial_t + \left(\varepsilon t\omega - e^{\frac{\varepsilon}{2}t^2}\right)\partial_x + \varepsilon\partial_u$
4	$u_t + (x + t)uu_x + \Phi(x + t)u_{xxx} = 0$	$\partial_t - \partial_x$
5	$u_t + (x + t)uu_x + \lambda(x + t)^2u_{xxx} = 0$	$\partial_t - \partial_x,\quad t\partial_t + x\partial_x - u\partial_u$

Here $\omega = x + \int e^{\frac{\varepsilon}{2}t^2}\mathrm{d}t$, $\varepsilon = \pm 1 \bmod \check{G}_4^{\sim}$, λ and m are arbitrary nonzero constants. In Case 1 $\Phi(z) \neq \lambda(z + 1)^2$ if $m = 1$. In Case 4 $\Phi(z) \neq \lambda z^2$.

be inconvenient to derive group classifications for classes (3.11) and (3.12).

Concluding Remarks. We have shown that the class (3.1) is not normalized. Its equivalence groupoid has a complicated structure, namely, the class (3.1) can be partitioned into the six disjoint normalized subclasses:

- the subclass $\mathcal{L}_0 = \mathcal{L} \setminus \bigcup_{i=1}^{5} \mathcal{L}_i$ is normalized in the usual sense, the set of its admissible transformations is generated by point transformations from the usual equivalence group $G^{\sim}$ of the entire class (3.1) adduced in Theorem 3.1;

- the subclasses $\mathcal{L}_i$, $i = 1, \ldots, 5$, which are normalized in the generalized extended sense, the sets of their admissible transformations are generated by transformations from the corresponding generalized extended groups $\hat{G}_i^{\sim}$ presented in Theorems 3.2–3.6.

3.2. Group Analysis of Korteweg–de Vries Equations with Time Dependent Coefficients

3.2.1. Admissible Point Transformations in Classes of Generalized KdV Equations. Following [248], we start from the general class of third order evolution equations and construct a hierarchy of nested normalized subclasses of this class, which consist of different generalizations of the Korteweg–de Vries (KdV) and Korteweg–de Vries (mKdV) equations. In this way we describe the entire sets of admissible point transformations (equivalence groupoids) of these subclasses.

Consider the general class $\mathcal{E}^3$ of third order evolution equations. They have the form

$$u_t = H(t, x, u, u_x, u_{xx}, u_{xxx}),$$

where $H_{u_{xxx}} \neq 0$. To find the set of admissible transformations of the class $\mathcal{E}^3$ and its complete equivalence group including both discrete and continuous equivalence transformations, we apply the direct method. The calculations are simplified with taking into account the well-known fact that the expression for t in any point (and, even, contact) transformation connecting two $(1+1)$-dimensional evolution equations depends only on t [160, 192]. Thus, a point transformation maps an equation from $\mathcal{E}^3$ to an equation from the same class if and only if it has the form

$$\tilde{t} = T(t), \quad \tilde{x} = X(t, x, u), \quad \tilde{u} = U(t, x, u). \tag{3.14}$$

The functions T, X and U have to satisfy the nondegeneracy assumption $T_t\Delta\neq 0$, where $\Delta=X_xU_u-X_uU_x$. The equivalence group $G_0^\sim$ of $\mathcal{E}^3$ consists of the transformations

$$\begin{gathered}\tilde t=T(t),\quad \tilde x=X(t,x,u),\quad \tilde u=U(t,x,u),\\ \tilde H=\frac{\Delta}{T_tD_xX}H+\frac{U_tD_xX-X_tD_xU}{T_tD_xX},\end{gathered}\tag{3.15}$$

where T, X and U run through the corresponding sets of smooth functions satisfying the above nondegeneracy assumption and D_x denotes the operator of total differentiation with respect to the variable x, $D_x=\partial_x+u_x\partial_u+u_{tx}\partial_{u_t}+u_{xx}\partial_{u_x}+\cdots$. Therefore, the class $\mathcal{E}^3$ is normalized.

The subclass $\mathcal{E}^3_{0.1}$ singled out from $\mathcal{E}^3$ by the constraint $H_{u_{xxx}u_{xxx}}=0$ has the same equivalence group $G_0^\sim$ and, therefore, is normalized. The same claim is true for the subclass $\mathcal{E}^3_{0.2}$ of $\mathcal{E}^3_{0.1}$, associated with the additional constraint $H_{u_{xxx}u_{xx}}=0$.

Setting one more constraint $H_{u_{xxx}u_x}=0$ leads to the normalized subclass $\mathcal{E}^3_{1.1}$ of $\mathcal{E}^3_{0.2}$ whose equivalence group $G_1^\sim$ is a proper subgroup of $G_0^\sim$ consisting of the transformations (3.15) with $X_u=0$. The normalized subclass $\mathcal{E}^3_{1.2}$ singled out from $\mathcal{E}^3_{1.1}$ by the constraint $H_{u_{xxx}u}=0$ has the same equivalence group $G_1^\sim$.

The equivalence group $G_2^\sim$ of the normalized subclass $\mathcal{E}^3_2$ nested in $\mathcal{E}^3_{1.2}$ and associated with the additional constraint $H_{u_{xxx}x}=0$ is properly contained in $G_1^\sim$. Its elements additionally satisfy the condition $X_{xx}=0$.

Another possibility is to impose the constraint $H_{u_{xx}}=0$ within the subclass $\mathcal{E}^3_{0.1}$. The corresponding subclass $\mathcal{E}^3_3$ is normalized and possesses the equivalence group $G_3^\sim$ formed by the transformations (3.15) for which $X_u=0$, $U_{uu}=0$ and $2U_{ux}X_x=U_uX_{xx}$.

Then the subclass $\mathcal{E}^3_4=\mathcal{E}^3_2\cap\mathcal{E}^3_3$ is normalized and its equivalence group is $G_4^\sim=G_2^\sim\cap G_3^\sim$. Integrating the total set of the constraints imposed on the arbitrary element H and the conditions obtained for elements of $G_4^\sim$

gives that the equations from the class $\mathcal{E}_4^3$ have the form

$$u_t + g(t)u_{xxx} = F(t, x, u, u_x),$$

where $g \neq 0$. The equivalence group $G_4^\sim$ consists of the transformations (we present only components corresponding to the equation variables)

$$\tilde{t} = \alpha(t), \quad \tilde{x} = \beta(t)x + \gamma(t), \quad \tilde{u} = \theta(t)u + \Phi(t, x), \tag{3.16}$$

where α, β, γ, θ and Φ are arbitrary smooth functions of their arguments, $\alpha_t\beta\theta \neq 0$.

Further we consider two more special subclasses of $\mathcal{E}_4^3$. The first subclass is formed by the variable-coefficient KdV equations

$$\begin{aligned} &u_t + f(t)uu_x + g(t)u_{xxx} \\ &\qquad + h(t)u + (p(t) + q(t)x)u_x + k(t)x + l(t) = 0, \end{aligned} \tag{3.17}$$

where all the parameters are arbitrary smooth functions of t, $fg \neq 0$. This subclass is normalized. Its equivalence group (in terms of the arbitrary element H) is singled out from $G_4^\sim$ by the condition $\Phi_{xx} = 0$. Hence the components of transformations corresponding to the equation variables have the simple general form

$$\tilde{t} = \alpha(t), \quad \tilde{x} = \beta(t)x + \gamma(t), \quad \tilde{u} = \theta(t)u + \varphi(t)x + \psi(t), \tag{3.18}$$

where α, β, γ, θ, φ and ψ run through the set of smooth functions of t, $\alpha_t\beta\theta \neq 0$. The arbitrary elements of (3.17) are transformed as follows

$$\begin{aligned}
&\tilde{f} = \frac{\beta}{\alpha_t\theta}f, \quad \tilde{g} = \frac{\beta^3}{\alpha_t}g, \quad \tilde{h} = \frac{1}{\alpha_t}\left(h - \frac{\varphi}{\theta}f - \frac{\theta_t}{\theta}\right), \\
&\tilde{q} = \frac{1}{\alpha_t}\left(q - \frac{\varphi}{\theta}f + \frac{\beta_t}{\beta}\right), \quad \tilde{p} = \frac{1}{\alpha_t}\left(\beta p - \gamma q + \frac{\gamma\varphi - \beta\psi}{\theta}f + \gamma_t - \gamma\frac{\beta_t}{\beta}\right), \\
&\tilde{k} = \frac{1}{\alpha_t\beta}\left(\theta k - \varphi(h+q) + \frac{\varphi^2}{\theta}f - \varphi_t + \varphi\frac{\theta_t}{\theta}\right), \quad \tilde{l} = -\frac{\varphi p + \psi_t}{\alpha_t} \\
&+ \frac{1}{\alpha_t}\left(\theta l - \frac{\gamma}{\beta}\left(\theta k - \varphi(h{+}q){+}\frac{\varphi^2}{\theta}f - \varphi_t{+}\varphi\frac{\theta_t}{\theta}\right) - \psi\left(h - \frac{\varphi}{\theta}f - \frac{\theta_t}{\theta}\right)\right).
\end{aligned}$$

Any equation from class (3.17) can be reduced by point transformations to the form (3.23) with $g = 1$ and forms (3.25) and (3.28). The necessary and sufficient condition of similarity of such equations to the standard KdV equation is

$$s_t = 2gs^2 - 3qs + \frac{f}{g}k, \quad \text{where} \quad s := \frac{2q - h}{g} + \frac{f_t g - f g_t}{f g^2}. \tag{3.19}$$

The second subclass consists of the variable-coefficient mKdV equations

$$\begin{aligned} &u_t + f(t)u^2u_x + g(t)u_{xxx} + h(t)u \\ &\quad + (p(t) + q(t)x)u_x + k(t)uu_x + l(t) = 0, \end{aligned} \tag{3.20}$$

where all the parameters are arbitrary smooth functions of t, $fg \neq 0$. This subclass is also normalized. Its equivalence group (in terms of the arbitrary element H) is singled out from $G_4^\sim$ by the condition $\Phi_x = 0$, i.e., the components of transformations corresponding to the equation variables are of the form (3.18) with $\varphi = 0$, where α, β, γ, θ and ψ run through the set of smooth functions of t, $\alpha\beta\theta \neq 0$. The arbitrary elements of (3.20) are transformed by the formulas

$$\begin{aligned} &\tilde{f} = \frac{\beta}{\alpha_t\theta^2}f, \quad \tilde{g} = \frac{\beta^3}{\alpha_t}g, \quad \tilde{h} = \frac{1}{\alpha_t}\left(h - \frac{\theta_t}{\theta}\right), \quad \tilde{q} = \frac{1}{\alpha_t}\left(q + \frac{\beta_t}{\beta}\right), \\ &\tilde{p} = \frac{1}{\alpha_t}\left(\beta p - \gamma q + \beta\frac{\psi^2}{\theta^2}f - \beta\frac{\psi}{\theta}k + \gamma_t - \gamma\frac{\beta_t}{\beta}\right), \\ &\tilde{k} = \frac{\beta}{\alpha_t\theta}\left(k - 2\frac{\psi}{\theta}f\right), \quad \tilde{l} = \frac{1}{\alpha_t}\left(\theta l - \psi h - \psi_t + \psi\frac{\theta_t}{\theta}\right). \end{aligned} \tag{3.21}$$

Five of the arbitrary elements can be gauged to simple constant values. For example, it is possible to set $g = 1$ and $h = p = q = l = 0$.

Proposition 3.15. *An equation of form (3.20) is similar to the standard (constant coefficient) mKdV equation if and only if its coefficients satisfy the conditions*

$$2h - 2q = \frac{f_t}{f} - \frac{g_t}{g}, \quad 2lf = k_t + kh - k\frac{f_t}{f}. \tag{3.22}$$

In the next section we present examples on similarity of equations from classes (3.17) and (3.20) to the standard KdV and mKdV equations. We also solve several classification problems for classes of KdV-like equations.

3.2.2. Group Analysis of a Class of KdV Equations. Firstly we study a class of variable-coefficient KdV equations

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0, \tag{3.23}$$

where f and g are arbitrary (smooth) functions of t, $fg \neq 0$.

As shown in Section 3.2.1, for the general values of the parameter-functions (arbitrary elements) f and g equation (3.23) is not equivalent to the standard KdV equation up to point transformations but at least one of the parameters (f or g) can be set equal to 1 using a point transformation. (The equations of the form (3.23) with $g = 1$ are called the transitional KdV equations [51].) Thus, after the transformation

$$\tilde{t} = \textstyle\int f(t)\,dt, \quad \tilde{x} = x, \quad \tilde{u} = u \tag{3.24}$$

equation (3.23) takes the same form with $\tilde{f}(\tilde{t}) = 1$ and $\tilde{g}(\tilde{t}) = g(t)/f(t)$. Therefore, without lost of generality we can consider the class of equations

$$u_t + uu_x + g(t)u_{xxx} = 0, \tag{3.25}$$

where g is an arbitrary (smooth) nonvanishing function of t, since this class is the image of class (3.23) under the mapping generated by the family of point transformations (3.24). (See [248, 300] for related definitions.)

We carry out the exhaustive group classification of class (3.25).

The separate consideration of subclass (3.25) is justified by the fact that it has nicer transformational properties than the superclasses (3.23) and, especially, (3.1).

The equivalence group $G^{\sim}$ of class (3.25) consists of the transformations

$$\begin{aligned} &\tilde{t} = \frac{at+b}{ct+d}, \quad \tilde{x} = \frac{e_2x+e_1t+e_0}{ct+d}, \\ &\tilde{u} = \frac{e_2(ct+d)u - e_2cx - e_0c + e_1d}{\varepsilon}, \quad \tilde{g} = \frac{{e_2}^3}{ct+d}\frac{g}{\varepsilon}, \end{aligned} \tag{3.26}$$

where a, b, c, d, e_0, e_1 and e_2 are arbitrary constants with $\varepsilon = ad - bc \neq 0$ and $e_2 \neq 0$, the tuple $(a, b, c, d, e_0, e_1, e_2)$ is defined up to nonzero multiplier and hence without loss of generality we can assume that $\varepsilon = \pm 1$.

Equations from class (3.25) are similar only if they are $G^{\sim}$-equivalent. Moreover, all admissible transformations in this class are generated by transformations from $G^{\sim}$, i.e., the class (3.25) is normalized in the usual sense. (The initial class (3.23) is normalized only with respect to the extended generalized equivalence group and its superclass class (3.1) possesses no normalization properties.) This implies the following claim: *An equation of form* (3.25) *is similar to the KdV equation if and only if* $g_{tt} = 0$. *Any transformation realizing the similarity belongs to* $G^{\sim}$. Therefore, an equation of form (3.23) is reduced to the KdV equation by a point transformation if and only if

$$g(t) = f(t) \left(c_1 \int f(t)\, dt + c_0 \right), \tag{3.27}$$

where c_0 and c_1 are constants, $(c_0, c_1) \neq (0, 0)$ [105], that well agrees with Theorem 3 of [315] (cf. also equation (3.19)). Equation (3.27) coincides with the constraint on arbitrary elements of the equations from class (3.23) which have the Painlevé property [148].

The kernel $A^{\cap} = \cap_{g \neq 0} A^g$ of the maximal Lie invariance algebras of equations from class (3.25) is $A^{\cap} = \langle \partial_x,\, t\partial_x + \partial_u \rangle$. All $G^{\sim}$-inequivalent cases of Lie symmetry extension are exhausted by the Cases 1–4 of Table 3.4

The presented group classification gives all inequivalent values of g for which the classical method of Lie reduction can be effectively used.

The class (3.23) can be also mapped to the class

$$\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + h(\tilde{t})\tilde{u} = 0 \tag{3.28}$$

by the family of point transformations $\tilde{t} = \int g(t)\, dt$, $\tilde{x} = x$, $\tilde{u} = \dfrac{f}{g} u$. The arbitrary element h of the mapped class is expressed via the arbitrary

Table 3.4: The group classification of the class $u_t + uu_x + g\, u_{xxx} = 0$, $g \neq 0$.

no.	$g(t)$	Basis of $A^{\max}$
0	$\forall$	$\partial_x,\ t\partial_x + \partial_u$
1	t^n	$\partial_x,\ t\partial_x + \partial_u,\ 3t\partial_t + (n+1)x\partial_x + (n-2)u\partial_u$
2	e^t	$\partial_x,\ t\partial_x + \partial_u,\ 3\partial_t + x\partial_x + u\partial_u$
3	$e^{\delta \arctan t}\sqrt{t^2+1}$	$\partial_x,\ t\partial_x + \partial_u,\ 3(t^2+1)\partial_t + (3t+\delta)x\partial_x + ((-3t+\delta)u + 3x)\partial_u$
4	1	$\partial_x,\ t\partial_x + \partial_u,\ 3t\partial_t + x\partial_x - 2u\partial_u,\ \partial_t$

Here n, δ are arbitrary constants, $n \geq 1/2$, $n \neq 1$, $\delta \geq 0 \bmod G_0^{\sim}$.

elements f and g in the following way:

$$h(\tilde{t}) = \frac{f(t)g_t(t) - f_t(t)g(t)}{f(t)(g(t))^2}.$$

Thus, the equation (3.25) with $g = t$ is then mapped to the cylindrical KdV equation ($h = (2t)^{-1}$) whose similarity to the standard KdV equation is known for a long time [191]. Analogously, the value $g = e^t$ corresponds to the spherical KdV equation ($h = t^{-1}$) which is not integrable.

Below we adduce several examples on similarity of KdV equations.

Example 3.16. Some traveling wave solutions of the "compound/combined KdV–mKdV equation"

$$u_t + (\alpha + \beta u)uu_x + \gamma u_{xxx} = 0, \tag{3.29}$$

where α, β and γ are real constants, $\beta\gamma \neq 0$, were constructed in [76, 135, 271, 317, 325, 330, 332]. In fact, this equation is called the Gardner equation (α should be scaled to a standard value) and is obviously similar to the mKdV equation $\tilde{u}_{\tilde{t}} + \varepsilon\tilde{u}^2\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$, where $\varepsilon = \operatorname{sign}(\beta\gamma)$, with respect to the point transformation

$$\tilde{t} = \gamma t, \quad \tilde{x} = x + \frac{\alpha^2}{4\beta}t, \quad \tilde{u} = \sqrt{\left|\frac{\beta}{\gamma}\right|}\left(u + \frac{\alpha}{2\beta}\right)$$

which is well known for a long time [197]. Therefore, each solution of equation (3.29) is represented in the form

$$u(t,x) = \sqrt{\left|\frac{\gamma}{\beta}\right|}\,\tilde{u}\left(\gamma t, x + \frac{\alpha^2}{4\beta}t\right) - \frac{\alpha}{2\beta},$$

where $\tilde{u}$ is a solution of the mKdV equation, and for any solution $\tilde{u}$ of the mKdV equation this representation gives a solution of (3.29).

Other close class of the equations

$$u_t + u_x + \alpha u^2 u_x + u_{xxx} = 0, \tag{3.30}$$

where α runs through the set of real nonvanishing constants, was considered in [328]. Only specific traveling wave solutions were found using the so-called "Exp-function method". Any equation of form (3.30) is reduced by the trivial point transformation

$$\tilde{t} = t, \quad \tilde{x} = x - t, \quad \tilde{u} = \sqrt{|\alpha|}u$$

to the mKdV equation $\tilde{u}_{\tilde{t}} + \varepsilon\tilde{u}^2\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$, where $\varepsilon = \operatorname{sign}\alpha$. Equation (3.30) with $\alpha = 1$ was also investigated using "extended F-expansion method" in [190].

Example 3.17. The authors of [260] apply "generalized expansion method" to find exact solutions of generalized KdV equations with variable coefficients, which have the form

$$\begin{aligned} &u_t + g(t)(6uu_x + u_{xxx}) + 6f(t)g(t)u \\ &\qquad = x(f_t(t) + 12g(t)f^2(t)) + M(t) \end{aligned} \tag{3.31}$$

with $g \neq 0$. The whole class (3.31) is mapped to the KdV equation $\tilde{u}_{\tilde{t}} + 6\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$ by the family of point transformations

$$\tilde{t} = \textstyle\int g\gamma^3\,dt, \quad \tilde{x} = \gamma x - 6\int g\,\gamma^3\beta\,dt, \quad \tilde{u} = \dfrac{u - fx}{\gamma^2} - \beta,$$

where $\gamma = e^{-6\int fg\,dt}$ and $\beta = \int M\gamma^{-2}\,dt$. This means that the function $u = u(t,x)$ satisfies an equation of form (3.31) if and only if it is represented via a is a solution $\tilde{u}$ of the KdV equation by the expression

$$u = \gamma^2\, \tilde{u}\left(\textstyle\int g\gamma^3\, dt, \gamma x - 6\int g\, \gamma^3\beta\, dt\right) + fx + \gamma^2\beta.$$

The subclass of the equations (3.31) with $g = 1$ and $M = 0$ arose in [283], where symmetry properties of such equations were studied. It was also mentioned in [283] that obtained results can be extended to the equations of the more general form

$$u_t + 6uu_x + u_{xxx} + 6f(t)u = x(f_t(t){+}12f^2(t)) + h_t(t) + 12f(t)h(t).$$

The family of point transformations mapping the latter class to the KdV equation consists of the transformations

$$\tilde{t} = \textstyle\int \gamma^3\, dt, \quad \tilde{x} = \gamma x - 6\int h\gamma\, dt, \quad \tilde{u} = \dfrac{u - fx - h}{\gamma^2},$$

where $\gamma = e^{-6\int f\, dt}$.

The similarity can be applied not only for generating new solutions from known ones and simplifying calculations. The similarity approach is easily extended to different local objects and properties related to differential equations, e.g., Lie and point symmetries [227, 248, 300], conservation laws and potential symmetries [246, 247, 249], reduction operators (i.e., nonclassical symmetries) [252, 300], Bäcklund transformations, etc.

3.2.3. Group Analysis of a Class of KdV-Like Equations via Equivalence Method. We perform the group classification of a class of variable coefficient KdV equations using equivalence based approach. Namely, we investigate Lie symmetry properties and exact solutions of variable coefficient KdV equations of the form

$$u_t + uu_x + g(t)u_{xxx} + h(t)u = 0, \tag{3.32}$$

where g and h are arbitrary smooth functions of the variable t, $g \neq 0$. The group classification of class (3.32) with $h = 0$ is carried out in the previous section. So, using the known classification list and equivalence transformations we present group classification of the initial class (3.32) without direct calculations.

Class (3.32) is normalized, therefore, there are no additional equivalence transformations between cases of the classification list, which is constructed using the equivalence relations associated with the corresponding equivalence group. In other words, the same list represents the group classification result for the corresponding class up to the general equivalence with respect to point transformations. Recently the authors of [145] obtained a partial group classification of class (3.32) (the notation a and b was used there instead of h and g, respectively). The reason of failure was neglecting an opportunity to use equivalence transformations. This is why only some cases of Lie symmetry extensions were found, namely the cases with $h = \mathrm{const}$, $h = 1/t$ and $h = 2/t$.

In fact the group classification problem for class (3.32) up to its equivalence group is already solved since this class is reducible to class (3.32) with $h = 0$ (class (3.25)) whose group classification is carried out in [251]. Using the known classification list and equivalence transformations we present group classifications of class (3.32) without the simplification of both equations admitting extensions of Lie symmetry algebras and these algebras themselves by equivalence transformations. The extended classification list can be useful for applications and convenient to be compared with the results of [145].

Class (3.32) is a subclass of class (3.17) singled out by the conditions $f = 1$ and $p = q = k = l = 0$. Substituting these values of the functions f, p, q, k and l to (3.19) we obtain the following assertion.

Corollary 3.18. *An equation from class (3.32) is reduced to the standard KdV equation by a point transformation if and only if there exist a constant* c_0 *and* $\varepsilon \in \{0, 1\}$ *such that*

$$h = \frac{\varepsilon}{2} \frac{g}{\int g\, dt + c_0} - \frac{g_t}{g}. \tag{3.33}$$

Class (3.32) admits generalized extended equivalence group and it is normalized in generalized sense only. The following statement is true.

Theorem 3.19. *The generalized extended equivalence group $\hat{G}_1^{\sim}$ of class (3.32) consists of the transformations*

$$\tilde{t} = \alpha, \quad \tilde{x} = \beta x + \gamma, \quad \tilde{u} = \lambda(\beta u + \beta_t x + \gamma_t),$$
$$\tilde{h} = \lambda\, h - 2\lambda\frac{\beta_t}{\beta} - \lambda_t, \quad \tilde{g} = \beta^3\lambda\, g,$$

where $\beta = (\delta_1 \int e^{-\int hdt}dt + \delta_2)^{-1}$, $\gamma = \delta_3 \int \beta^2 e^{-\int hdt}dt + \delta_4$, $\delta_1, \ldots, \delta_4$ are arbitrary constants with $(\delta_1, \delta_2) \neq (0, 0)$; α is an arbitrary smooth function of t with $\alpha_t \neq 0$, and $\lambda = 1/\alpha_t$.

The usual equivalence group $G_1^{\sim}$ of class (3.32) is the subgroup of the generalized extended equivalence group $\hat{G}_1^{\sim}$, which is singled out with the condition $\delta_1 = \delta_3 = 0$.

The gauge $h = 0$ in class (3.32) can be made by the equivalence transformation

$$\hat{t} = \int e^{-\int h(t)\, dt}dt, \quad \hat{x} = x, \quad \hat{u} = e^{\int h(t)\, dt}u, \tag{3.34}$$

that connects equation (3.32) with the equation $\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} + \hat{g}(\hat{t})\hat{u}_{\hat{x}\hat{x}\hat{x}} = 0$. The new arbitrary element $\hat{g}$ is expressed via g and h in the following way: $\hat{g}(\hat{t}) = e^{\int h(t)\, dt}g(t)$.

For any equation from class (3.32) there exists an imaged equation in class (3.25) with respect to transformation (3.34). The equivalence group $G^{\sim}$ of class (3.25) given by (3.47) is induced by the equivalence group $\hat{G}_1^{\sim}$ of class (3.32) which, in turn, is induced by the equivalence group of their superclass (3.17). These guarantee that Table 3.4 presents also the group classification list for class (3.32) up to $\hat{G}_1^{\sim}$-equivalence (resp. for the class (3.17) up to its equivalence group). As all of the above classes are normalized, we can state that we obtain Lie symmetry classifications of these classes up to general point equivalence. This leads to the following assertion.

Proposition 3.20. *An equation from class (3.32) (resp. (3.17)) admits a four-dimensional Lie invariance algebra if and only if it is reduced by a point transformation to constant coefficient KdV equation, i.e., if and only if condition (3.33) (resp. (3.19)) holds.*

To derive the group classification of class (3.32) which is not simplified by equivalence transformations we use the algorithm for construction of the complete list of Lie symmetry extensions from the list of inequivalent ones suggested in [289]. We first apply transformations from the group $G^{\sim}$ to the classification list presented in Table 3.4 and obtain the following extended list:

0. arbitrary $\hat g$: $\langle \partial_{\hat x},\ \hat t\partial_{\hat x}+\partial_{\hat u}\rangle$;

1. $\hat g=c_0(a\hat t+b)^n(c\hat t+d)^{1-n}$, $n\neq 0,1$: $\langle \partial_{\hat x},\ \hat t\partial_{\hat x}+\partial_{\hat u},\ X_3\rangle$, where $X_3=3(a\hat t+b)(c\hat t+d)\partial_{\hat t}+\big(3ac\hat t+ad(n+1)+bc(2-n)\big)\,\hat x\partial_{\hat x}+\big[3ac\hat x-(3ac\hat t+ad(2-n)+bc(n+1))\hat u\big]\,\partial_{\hat u}$;

2. $\hat g=c_0(c\hat t+d)\exp\left(\dfrac{a\hat t+b}{c\hat t+d}\right)$: $\langle \partial_{\hat x},\ \hat t\partial_{\hat x}+\partial_{\hat u},\ X_3\rangle$, where $X_3=3(c\hat t+d)^2\partial_{\hat t}+\big(3c(c\hat t+d)+\varepsilon\big)\,\hat x\partial_{\hat x}+\big[3c^2\hat x+(\varepsilon-3c(c\hat t+d))\hat u\big]\,\partial_{\hat u}$;

3. $\hat g=c_0e^{\delta\arctan\left(\frac{a\hat t+b}{c\hat t+d}\right)}\sqrt{(a\hat t+b)^2+(c\hat t+d)^2}$: $\langle \partial_{\hat x},\ \hat t\partial_{\hat x}+\partial_{\hat u},\ X_3\rangle$, where $X_3=3\left((a\hat t+b)^2+(c\hat t+d)^2\right)\partial_{\hat t}+\big(3a(a\hat t+b)+3c(c\hat t+d)+\varepsilon\delta\big)\,\hat x\partial_{\hat x}+\big(3(a^2+c^2)\hat x-(3a(a\hat t+b)+3c(c\hat t+d)-\varepsilon\delta)\hat u\big)\,\partial_{\hat u}$;

4a. $\hat g=c_0$: $\langle \partial_{\hat x},\ \hat t\partial_{\hat x}+\partial_{\hat u},\ \partial_{\hat t},\ 3\hat t\partial_{\hat t}+\hat x\partial_{\hat x}-2\hat u\partial_{\hat u}\rangle$;

4b. $\hat g=c\hat t+d$, $c\neq 0$: $\langle \partial_{\hat x},\ \hat t\partial_{\hat x}+\partial_{\hat u},\ 3(c\hat t+d)\partial_{\hat t}+2c\hat x\partial_{\hat x}-c\hat u\partial_{\hat u},\ X_4\rangle$, where $X_4=(c\hat t+d)^2\partial_{\hat t}+c(c\hat t+d)\hat x\partial_{\hat x}+c(c\hat x-(c\hat t+d)\hat u)\partial_{\hat u}$. Here c_0, a, b, c, d and δ are arbitrary constants, $(a^2+b^2)(c^2+d^2)\neq 0$, $\varepsilon=ad-bc$, $c_0\neq 0$.

Then we find preimages of equations from the class $\hat u_{\hat t}+\hat u\hat u_{\hat x}+\hat g(\hat t)\hat u_{\hat x\hat x\hat x}=0$ with arbitrary elements collected in the above list with respect to transformation (3.34). The last step is to transform basis operators of the corresponding Lie symmetry algebras. The results are presented in Table 3.5.

Generation of Exact Solutions. The N-soliton solution of the KdV

Table 3.5: The group classification of the class $u_t + uu_x + gu_{xxx} + hu = 0$, $g \neq 0$.

no.	$g(t)$	Basis of $A^{\max}$
0	$\forall$	$\partial_x,\ T\partial_x + T_t\partial_u$
1	$c_0 T_t(aT+b)^n(cT+d)^{1-n}$	$\partial_x,\ T\partial_x + T_t\partial_u,\ 3T_t^{-1}(aT+b)(cT+d)\partial_t + \big[3acT + ad(n+1) + bc(2-n)\big]x\partial_x + \Big(3acxT_t - \big[3acT + 3hT_t^{-1}(aT+b)(cT+d) + bc(n+1) + ad(2-n)\big]u\Big)\partial_u$
2	$c_0 T_t(cT+d)\exp\left(\frac{aT+b}{cT+d}\right)$	$\partial_x,\ T\partial_x + T_t\partial_u,\ 3T_t^{-1}(cT+d)^2\partial_t + (3c(cT+d)+\varepsilon)x\partial_x + \left[3c^2xT_t + \left(\varepsilon - 3(cT+d)(c + h(cT+d)T_t^{-1})\right)u\right]\partial_u$
3	$c_0 T_t e^{\delta \arctan\left(\frac{aT+b}{cT+d}\right)} G(t)$	$\partial_x,\ T\partial_x + T_t\partial_u,\ 3T_t^{-1}G^2\partial_t + \big[3a(aT+b) + 3c(cT+d) + \varepsilon\delta\big]x\partial_x + \big[3(a^2+c^2)xT_t - \big(3a(aT+b) + 3c(cT+d) - \varepsilon\delta + 3hT_t^{-1}G^2\big)u\big]\partial_u$
4a	$c_0 T_t$	$\partial_x,\ T\partial_x + T_t\partial_u,\ T_t^{-1}(\partial_t - hu\partial_u)$, $3TT_t^{-1}\partial_t + x\partial_x - (2 + 3TT_t^{-1}h)u\partial_u$
4b	$(cT+d)T_t$	$\partial_x,\ T\partial_x + T_t\partial_u,\ T_t^{-1}(cT+d)^2\partial_t + c(cT+d)x\partial_x + [c^2xT_t - (cT+d)(c + T_t^{-1}(cT+d)h)u]\partial_u$, $3T_t^{-1}(cT+d)\partial_t + 2cx\partial_x - (c + 3T_t^{-1}(cT+d)h)u\partial_u$

Here $T = \int e^{-\int h(t)\,dt}dt$, $T_t = e^{-\int h(t)\,dt}$, $G = \sqrt{(aT+b)^2 + (cT+d)^2}$; n c_0, a, b, c, d and δ are arbitrary constants, $(a^2+b^2)(c^2+d^2) \neq 0$, $\varepsilon = ad - bc$, $c_0 \neq 0$, $n \neq 0, 1$. In the case (4b) $c \neq 0$. The function h ia arbitrary in all cases.

equation in the canonical form

$$U_t - 6UU_x + U_{xxx} = 0 \tag{3.35}$$

was constructed as early as in the seventies by Hirota [238]. The two-soliton solution of equation (3.35) has the form

$$U = -2\frac{\partial^2}{\partial x^2}\ln\left(1 + b_1 e^{\theta_1} + b_2 e^{\theta_2} + Ab_1 b_2 e^{\theta_1+\theta_2}\right), \tag{3.36}$$

where a_i, b_i are arbitrary constants, $\theta_i = a_i x - a_i^3 t$, $i = 1, 2$; $A = \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2$.

Combining the simple transformation $\hat{u} = -6U$ that connects (3.35) with the KdV equation of the form

$$\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} + \hat{u}_{\hat{x}\hat{x}\hat{x}} = 0 \tag{3.37}$$

and transformation (3.34), we obtain the formula

$$u = -6e^{-\int h(t)dt}\, U\left(\int e^{-\int h(t)\,dt}dt,\, x\right)$$

for generation of exact solutions for the equations of the general form

$$u_t + uu_x + e^{-\int h(t)\,dt}u_{xxx} + h(t)u = 0. \tag{3.38}$$

These equations are preimages of (3.37) with respect to transformation (3.34). Here h is an arbitrary nonvanishing smooth function of the variable t.

The two-soliton solution (3.36) leads to the following solution of (3.38)

$$u = 12e^{-\int h(t)dt}\frac{\partial^2}{\partial x^2}\ln\left(1 + b_1e^{\theta_1} + b_2e^{\theta_2} + Ab_1b_2e^{\theta_1+\theta_2}\right),$$

where a_i, b_i are arbitrary constants, $\theta_i = a_ix - a_i^3\int e^{-\int h(t)\,dt}dt$, $i = 1, 2$; $A = \left(\frac{a_1-a_2}{a_1+a_2}\right)^2$. In a similar way one can construct other types of solutions for equations from class (3.38) using known solutions of classical KdV equation.

3.3. Extended Group Analysis of Variable Coefficient Generalized Kawahara Equations

In this section we study generalized Kawahara equations with time-dependent coefficients

$$u_t + \alpha(t)u^nu_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \tag{3.39}$$

from the Lie symmetry point of view. Here n is an arbitrary nonzero integer, α, β and σ are smooth nonvanishing functions of the variable t.

If α, β and σ are not functions but constants, equations (3.39) become classical models appearing in the solitary waves theory. Here we present a brief overview on applications of Kawahara equations and the related results. In the usual sense, solitary waves are nonlinear waves of constant form which decay rapidly in their tail regions. The rate of this decay is usually exponential. However, under critical conditions in dispersive systems (e.g., the magneto-acoustic waves in plasmas, the waves with surface tension, etc.), unexpected rise of weakly nonlocal solitary waves occurs. These waves consist of a central core which is similar to that of classical solitary waves, but they are accompanied by copropagating oscillatory tails which extend indefinitely far from the core with a nonzero constant amplitude. In order to describe and clarify the properties of these waves Kawahara introduced generalized nonlinear dispersive equations which have a form of the KdV equation with an additional fifth order derivative term, namely,

$$u_t + \alpha u u_x + \beta u_{xxx} + \sigma u_{xxxxx} = 0,$$

where α, β and σ are nonzero constants [118,157]. This equation was heavily studied from different points of view. The exact solitary wave solution was presented in [320]. In [126] the existence of travelling wave solutions of the Kawahara equation being considered as a formal asymptotic approximation for water waves with surface tension was shown. In [41] various numerical computations of both infinite interval and spatially periodic solutions to a one-dimensional wave equation which models capillary-gravity waves were done. Using techniques of exponential asymptotics it was shown in [107] that solitary wave solutions of the Kawahara equation form a one-parameter family characterized by the phase shift of the trailing oscillations. An explicit asymptotic formula relating the oscillation amplitude to the phase shift was obtained therein. Solvability of the Cauchy problem (local and global existence) of the Kawahara equation was studied in [65,133]. Various studies on behavior of solutions of the Kawahara equations were presented, e.g., in [**?**, 15, 69, 74, 112, 321]. Generalized and formal symmetries as well as local conservation laws of the constant coefficient Kawahara equations with arbitrary nonlinearity $u_t + f(u)u_x + \beta u_{xxx} + \sigma u_{xxxxx} = 0$, $f_u\beta\sigma \neq 0$, were classified recently in [309].

Generalized constant coefficient models related to the Kawahara equation have appeared later. For example, long waves in a shallow liquid under ice cover in the presence of tension or compression were described by the equation $u_t + u_x + \alpha uu_x + \beta u_{xxx} + \sigma u_{xxxxx} = 0$ [193, 284]. This equation is similar to the classical Kawahara equation with respect to the simple changes of variables: $\tilde{x} = x - t$, where t and u are not transformed, or $\tilde{u} = 1 + \alpha u$, where t and x are not transformed. A stability of solitons, described by the modified Kawahara equations $u_t + \alpha u^n u_x + \beta u_{xxx} + \sigma u_{xxxxx} = 0$, where α, β and σ are nonzero constants, and $n \in \mathbb{N}$, was given in [153, 154]. It appears that solitons are stable for $n < 8$.

We note that neither the classical Kawahara equation nor its generalization adduced above are integrable by the inverse scattering transform method [196, 269].

Last time much attention is paid to variable coefficient models, like variable coefficient KdV, Burgers, and Schrödinger equations [251]. This is due to the fact that variable coefficient equations can model certain real-world phenomena with more accuracy than their constant coefficient counterparts. In the recent paper [156] Lie symmetries were applied for finding exact solutions of variable coefficient Kawahara and modified Kawahara equations, which are of the form (3.39) with $n = 1$ and $n = 2$, respectively. The presence of three arbitrary coefficients depending on t makes the task of finding Lie symmetries too difficult to get complete results without reducing the number of coefficients by equivalence transformations. That is why only few results on Lie symmetries were derived in [156]. In the present paper we show that the use of such transformations is a cornerstone in the complete solution of the problem.

3.3.1. Admissible Transformations. We search for admissible transformations in class (3.39) using the direct method [160]. The investigation results in the statements presented in Theorems 3.21 and 3.22.

Theorem 3.21. *The usual equivalence group $G^{\sim}$ of class* (3.39) *consists of the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u,$$

$$\tilde{\alpha}(\tilde{t}) = \frac{\delta_1}{{\delta_3}^n T_t}\alpha(t), \quad \tilde{\beta}(\tilde{t}) = \frac{\delta_1^3}{T_t}\beta(t), \quad \tilde{\sigma}(\tilde{t}) = \frac{\delta_1^5}{T_t}\sigma(t), \quad \tilde{n} = n,$$

where δ_j, $j = 1, 2, 3$, are arbitrary constants with $\delta_1\delta_3 \neq 0$, T is an arbitrary smooth function with $T_t \neq 0$.

Theorem 3.22. *The generalized extended equivalence group $\hat{G}^{\sim}_{n=1}$ of the class of equations*

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0 \tag{3.40}$$

is formed by the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = (x + \delta_1)X^1 + \delta_0, \quad \tilde{u} = \frac{\delta_2}{X^1}u - \delta_2\delta_3(x + \delta_1),$$

$$\tilde{\alpha}(\tilde{t}) = \frac{(X^1)^2}{\delta_2 T_t}\alpha(t), \quad \tilde{\beta}(\tilde{t}) = \frac{(X^1)^3}{T_t}\beta(t), \quad \tilde{\sigma}(\tilde{t}) = \frac{(X^1)^5}{T_t}\sigma(t),$$

where $X^1 = (\delta_3 \int \alpha(t)\mathrm{d}t + \delta_4)^{-1}$, δ_j, $j = 0, \ldots, 4$, are arbitrary constants with $\delta_2({\delta_3}^2 + {\delta_4}^2) \neq 0$; $T = T(t)$ is a smooth function with $T_t \neq 0$. The usual equivalence group $G^{\sim}_{n=1}$ of class (3.40) *comprises the above transformations with $\delta_1 = \delta_3 = 0$.*

Theorem 3.23. *A variable coefficient equation from class* (3.39) *is reducible to constant coefficient equation from the same class if and only if the coefficients α, β and σ satisfy the conditions*

$$\left(\frac{\beta}{\alpha}\right)_t = \left(\frac{\sigma}{\alpha}\right)_t = 0, \quad \textit{for} \quad n \neq 1, \tag{3.41}$$

$$\left(\frac{1}{\alpha}\left(\frac{\beta}{\alpha}\right)_t\right)_t = 0, \quad \left(\frac{\sigma\alpha^2}{\beta^3}\right)_t = 0, \quad \textit{for} \quad n = 1. \tag{3.42}$$

The presence of the arbitrary function $T(t)$ in the equivalence transformations adduced in Theorems 1 and 2 allows one to gauge one of the arbitrary functions α, β and σ to a simple constant value, e.g., to 1. An interesting question is which one of the three possible gauges is preferable for further consideration. Class (3.40) with $\beta = 1$ or $\sigma = 1$ is still normalized only in the generalized extended sense, since transformations of independent and dependent variables still involve $\int \alpha(t)\mathrm{d}t$. At the same time class (3.40) with $\alpha = 1$ is normalized with respect to its usual equivalence group, as X^1 appearing in Theorem 3.22 in this case takes the form $X^1 = (\delta_3 t + \delta_4)^{-1}$. This is why we can expect that in the case $n = 1$ it is easier to carry out the group classification under the gauge $\alpha = 1$ rather than under other possible gauges. If $n \neq 1$ all the three suggested gauges look equally convenient, and we choose the gauge $\alpha = 1$ just to present the group classification in the uniform way.

The gauge $\alpha = 1$ is realized by the point transformation

$$\hat{t} = \int \alpha(t)\,\mathrm{d}t, \quad \hat{x} = x, \quad \hat{u} = u. \tag{3.43}$$

Then class (3.39) is mapped to its subclass with $\hat{\alpha} = 1$, $\hat{\beta} = \beta/\alpha$ and $\hat{\sigma} = \sigma/\alpha$. Therefore, without loss of generality we can restrict ourselves to the study of the class

$$u_t + u^n u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \tag{3.44}$$

since all results on symmetries, conservation laws, classical solutions and other related objects for equations (3.39) can be found using the similar results derived for equations from class (3.44).

To derive the equivalence group for subclass of class (3.39) with $\alpha = 1$ we set $\tilde{\alpha} = \alpha = 1$ in the transformations presented in Theorems 3.21 and 3.22.

Corollary 3.24. *The generalized equivalence group* $\hat{G}^{\sim}_{\alpha=1}$ *of class* (3.44) *comprises the transformations*

$$\tilde{t} = \delta_1 {\delta_3}^{-n} t + \delta_0, \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u,$$
$$\tilde{\beta}(\tilde{t}) = {\delta_1}^2 {\delta_3}^n \beta(t), \quad \tilde{\sigma}(\tilde{t}) = {\delta_1}^4 {\delta_3}^n \sigma(t), \quad \tilde{n} = n, \tag{3.45}$$

where δ_j, $j = 0, 1, 2, 3$, are arbitrary constants with $\delta_1\delta_3 \neq 0$.

Remark 3.25. If we assume that the constant n varies in class (3.44), then the equivalence group $\hat{G}^{\sim}_{\alpha=1}$ is generalized since n is involved explicitly in the transformation of the variable t. From the other hand, n is invariant under the action of transformations from the equivalence group, so class (3.44) can be considered as the union of all its subclasses with fixed n. For each such subclass the group $\hat{G}^{\sim}_{\alpha=1}$ is usual equivalence group.

In the case $n = 1$ we put $\alpha = \tilde{\alpha} = 1$ in transformation from Theorem 3.22 and redenote the constants δ_j, $j = 0, \ldots, 4$, to write the transformations in a more compact form.

Corollary 3.26. *The usual equivalence group $G^{\sim}_{\alpha=n=1}$ of the class*

$$u_t + uu_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0 \tag{3.46}$$

consists of the transformations

$$\tilde{t} = \frac{at+b}{ct+d}, \quad \tilde{x} = \frac{e_2x + e_1t + e_0}{ct+d}, \quad \tilde{u} = \frac{e_2(ct+d)u - e_2cx - e_0c + e_1d}{\Delta},$$
$$\tilde{\beta} = \frac{{e_2}^3}{ct+d}\frac{\beta}{\Delta}, \quad \tilde{\sigma} = \frac{{e_2}^5}{(ct+d)^3}\frac{\sigma}{\Delta}, \tag{3.47}$$

where a, b, c, d, e_0, e_1 and e_2 are arbitrary constants with $\Delta = ad - bc \neq 0$ and $e_2 \neq 0$, the tuple $(a, b, c, d, e_0, e_1, e_2)$ is defined up to a nonzero multiplier and hence without loss of generality we can assume that $\Delta = \pm 1$.

3.3.2. Lie Symmetries. The group classification of equations of the form (3.44) with $n \neq 1$ up to $G^{\sim}_{\alpha=1}$-equivalence (resp. up to $G^{\sim}_{\alpha=n=1}$-equivalence if $n = 1$) coincides with the group classification of equations of the form (3.39) with $n \neq 1$ up to $G^{\sim}$-equivalence (resp. up to $\hat{G}^{\sim}$-equivalence if $n = 1$). We have proven the following assertions.

Table 3.6: The group classification of the class (3.39).

no.	$\beta(t)$	$\sigma(t)$	Basis of $A^{\max}$
			$n\neq 1$. This case is classified up to $G^{\sim}$-equivalence.
0	$\forall$	$\forall$	∂_x
1	λt^{ρ}	$\delta t^{\frac{5\rho+2}{3}}$	$\partial_x,\ 3nt\partial_t+(\rho+1)nx\partial_x+(\rho-2)u\partial_u$
2	λe^{t}	$\delta e^{\frac{5}{3}t}$	$\partial_x,\ 3n\partial_t+nx\partial_x+u\partial_u$
3	λ	δ	$\partial_x,\ \partial_t$
			$n=1$. This case is classified up to $\hat{G}^{\sim}$-equivalence.
$0'$	$\forall$	$\forall$	$\partial_x,\ t\partial_x+\partial_u$
$1'$	λt^{ρ}	$\delta t^{\frac{5\rho+2}{3}}$	$\partial_x,\ t\partial_x+\partial_u,\ 3t\partial_t+(\rho+1)x\partial_x+(\rho-2)u\partial_u$
$2'$	λe^{t}	$\delta e^{\frac{5}{3}t}$	$\partial_x,\ t\partial_x+\partial_u,\ 3\partial_t+x\partial_x+u\partial_u$
$3'$	λ	δ	$\partial_x,\ t\partial_x+\partial_u,\ \partial_t$
$4'$	$\lambda(t^2+1)^{\frac{1}{2}}e^{3\nu\arctan t}$	$\delta(t^2+1)^{\frac{3}{2}}e^{5\nu\arctan t}$	$\partial_x,\ t\partial_x+\partial_u,$ $(t^2+1)\partial_t+(t+\nu)x\partial_x+((\nu-t)u+x)\partial_u$

Here $\alpha=1 \bmod G^{\sim}$, ρ and ν are arbitrary constants, $\rho\geqslant 1/2$, $\nu\geqslant 0$; δ and λ are nonzero constants, $\delta=\pm 1 \bmod G^{\sim}$.

Theorem 3.27. *The kernel of the maximal Lie invariance algebras of equations from class* (3.44) (*resp.* (3.39)) *with* $n\neq 1$ *coincides with the one-dimensional algebra* $\langle\partial_x\rangle$. *All possible* $\hat{G}^{\sim}_{\alpha=1}$*-inequivalent* (*resp.* $G^{\sim}$*-inequivalent*) *cases of extension of the maximal Lie invariance algebras are exhausted by the cases 1–3 of Table 3.6.*

Theorem 3.28. *The kernel of the maximal Lie invariance algebras of equations from class* (3.46) (*resp.* (3.40)) *coincides with the two-dimensional algebra* $\langle\partial_x,\ t\partial_x+\partial_u\rangle$. *All possible* $G^{\sim}_{\alpha=n=1}$*-inequivalent* (*resp.* $\hat{G}^{\sim}_{n=1}$*-inequivalent*) *cases of extension of the maximal Lie invariance algebras are exhausted by the cases* $1'$–$4'$ *of Table 3.6.*

To derive the complete list of Lie symmetry extensions for the entire class (3.39), where arbitrary elements are not simplified by point trans-

Table 3.7: The group classification of the class (3.39) using no equivalence.

no.	$\beta(t)$	$\sigma(t)$	Basis of $A^{\max}$
		$n\neq 1$	
0	$\forall$	$\forall$	∂_x
1	$\lambda_1\alpha(T+l)^\rho$	$\lambda_2\alpha(T+l)^{\frac{5\rho+2}{3}}$	$\partial_x,\ 3n(T+l)\alpha^{-1}\partial_t$ $+n(\rho+1)x\partial_x+(\rho-2)u\partial_u$
2	$\lambda_1\alpha e^{mT}$	$\lambda_2\alpha e^{\frac{5}{3}mT}$	$\partial_x,\ 3n\alpha^{-1}\partial_t+nmx\partial_x+mu\partial_u$
3	$\lambda_1\alpha$	$\lambda_2\alpha$	$\partial_x,\ \alpha^{-1}\partial_t$
		$n=1$	
$0'$	$\forall$	$\forall$	$\partial_x,\ T\partial_x+\partial_u$
$1'$	$\lambda_1\alpha(aT+b)^\rho$ $\times\,(cT+d)^{1-\rho}$	$\lambda_2\alpha(aT+b)^{\frac{5\rho+2}{3}}$ $\times\,(cT+d)^{\frac{7-5\rho}{3}}$	$\partial_x,\ T\partial_x+\partial_u,\ 3(aT+b)(cT+d)\alpha^{-1}\partial_t$ $+(3acT+ad(\rho+1)+bc(2-\rho))\,x\partial_x$ $+(3acx-(3acT+ad(2-\rho)+bc(\rho+1))u)\partial_u$
$2'$	$\lambda_1\alpha(cT+d)\mathrm{e}^{\frac{aT+b}{cT+d}}$	$\lambda_2\alpha(cT+d)^3\mathrm{e}^{\frac{5}{3}\frac{aT+b}{cT+d}}$	$\partial_x,\ T\partial_x+\partial_u,\ 3(cT+d)^2\alpha^{-1}\partial_t$ $+\left(3c(cT+d)+\Delta\right)x\partial_x$ $+\left(3c^2x+(\Delta-3c(cT+d))u\right)\partial_u$
$3'$	$\lambda_1\alpha(cT+d)$	$\lambda_2\alpha(cT+d)^3$	$\partial_x,\ T\partial_x+\partial_u,\ (cT+d)^2\alpha^{-1}\partial_t+$ $c(cT+d)x\partial_x+c(cx-(cT+d)u)\partial_u$
$4'$	$\lambda_1\alpha e^{3\nu\arctan\frac{aT+b}{cT+d}}$ $\times\left((aT+b)^2+(cT+d)^2\right)^{\frac{1}{2}}$	$\lambda_2\alpha e^{5\nu\arctan\frac{aT+b}{cT+d}}$ $\times\left((aT+b)^2+(cT+d)^2\right)^{\frac{3}{2}}$	$\partial_x,\ T\partial_x+\partial_u,$ $\left((aT+b)^2+(cT+d)^2\right)\alpha^{-1}\partial_t$ $+\left(a(aT+b)+c(cT+d)+\Delta\nu\right)x\partial_x$ $+\big[-(a(aT+b)+c(cT+d)-\Delta\nu)u$ $+(a^2+c^2)x\big]\partial_u$

Here λ_1, λ_2, a, b, c, d, l, m, ρ and ν are arbitrary constants, $\lambda_1\lambda_2(c^2+d^2)\neq 0$, $\Delta=ad-bc\neq 0$, α is an arbitrary nonvanishing smooth function of t, $T=\int\alpha(t)\mathrm{d}t$.

formations, we use the equivalence-based approach [289]. The results are collected in Table 3.7.

The presented group classification reveals equations of the form (3.39) that may be of interest for applications and for which the classical Lie reduction method can be used.

3.3.3. Symmetry Reductions and Exact Solutions. The Lie symmetry operators derived as a result of solving the group classification problem can be applied to construction of exact solutions of the corresponding equations. The reduction method with respect to subalgebras of Lie invariance algebras is algorithmic and well-known; we refer to the classical textbooks on the subject [217, 227]. In order to get an optimal system of group-invariant solutions reductions should be performed with respect to subalgebras from the optimal system [217, Section 3.3].

Consider firstly the structure of the two and three-dimensional Lie algebras spanned by the generators presented in Table 3.6, using notations of [230]. In Cases 1–3 and Case $0'$ the maximal Lie-invariance algebras are two-dimensional. In Case $0'$, Case 1 with $\rho=-1$, and Case 3 they are Abelian ($2A_1$). The algebras adduced in Case 1 with $\rho\neq-1$ and Case 2 are non-Abelian (A_2). The algebras with basis operators presented in Cases $1'$–$4'$ are three-dimensional. In Case $1'$ with $\rho\neq-1,2$ the maximal Lie invariance algebra is of the type $A_{3.4}$ if $\rho=1/2$, $A^a_{3.5}$ with $a=\frac{\rho-2}{\rho+1}$ or $a=\frac{\rho+1}{\rho-2}$ if $\rho>1/2$ or $\rho<1/2$, respectively. If $\rho=-1$ or $\rho=2$, then $A^{\max}$ from Case $1'$ is $A_1\oplus A_2$. In other cases the maximal Lie invariance algebras are of the following types: Case $2'$ — $A_{3.2}$, Case $3'$ — the Weyl algebra $A_{3.1}$, Case $4'$ — $A^a_{3.7}$ with $a=|\nu|$.

If a one-dimensional invariance algebra is spanned by an operator $Q=\tau\partial_t+\xi\partial_x+\eta\partial_u$, then the associated ansatz reducing the corresponding PDE with two independent variables to an ODE is found as a solution of the invariant surface condition $Q[u]:=\tau u_t+\xi u_x-\eta=0$. In practice the related characteristic system $\frac{\mathrm{d}t}{\tau}=\frac{\mathrm{d}x}{\xi}=\frac{\mathrm{d}u}{\eta}$ has to be solved. Ansatzes and reduced equations obtained for equations from class (3.44) using one-dimensional subalgebras from Table 3.8 are collected in Table 3.9. Re-

Table 3.8: Optimal systems of one-dimensional subalgebras of $A^{\max}$ from Table 3.6.

no.	Optimal system
$1_{\rho\neq-1}$	$\mathfrak{g}_0=\langle\partial_x\rangle,\quad \mathfrak{g}_{1.1}=\langle 3nt\partial_t+(\rho+1)nx\partial_x+(\rho-2)u\partial_u\rangle$
$1_{\rho=-1}$	$\mathfrak{g}_0=\langle\partial_x\rangle,\quad \mathfrak{g}^a_{1.2}=\langle nt\partial_t+a\partial_x-u\partial_u\rangle$
2	$\mathfrak{g}_0=\langle\partial_x\rangle,\quad \mathfrak{g}_2=\langle 3n\partial_t+nx\partial_x+u\partial_u\rangle$
3	$\mathfrak{g}_0=\langle\partial_x\rangle,\quad \mathfrak{g}^a_3=\langle\partial_t+a\partial_x\rangle$
$0'$	$\mathfrak{g}_0=\langle\partial_x\rangle,\quad \mathfrak{g}^a_{0'}=\langle(t+a)\partial_x+\partial_u\rangle$
$1'_{\rho\neq-1,2}$	$\mathfrak{g}_0=\langle\partial_x\rangle,\quad \mathfrak{g}^\sigma_{0'}=\langle(t+\sigma)\partial_x+\partial_u\rangle\quad \mathfrak{g}_{1'.1}=\langle 3t\partial_t+(\rho+1)x\partial_x+(\rho-2)u\partial_u\rangle$
$1'_{\rho=-1}$	$\mathfrak{g}_0=\langle\partial_x\rangle,\quad \mathfrak{g}^\sigma_{0'}=\langle(t+\sigma)\partial_x+\partial_u\rangle,\quad \mathfrak{g}^a_{1'.2}=\langle t\partial_t+a\partial_x-u\partial_u\rangle,$
$1'_{\rho=2}$	$\mathfrak{g}_0=\langle\partial_x\rangle,\quad \mathfrak{g}^\sigma_{0'}=\langle(t+\sigma)\partial_x+\partial_u\rangle,\quad \mathfrak{g}^a_{1'.3}=\langle t\partial_t+(x+at)\,\partial_x+a\partial_u\rangle,$
$2'$	$\mathfrak{g}_0=\langle\partial_x\rangle,\quad \mathfrak{g}_{0'}=\langle t\partial_x+\partial_u\rangle,\quad \mathfrak{g}_{2'}=\langle 3\partial_t+x\partial_x+u\partial_u\rangle,$
$3'$	$\mathfrak{g}_0=\langle\partial_x\rangle,\quad \mathfrak{g}_{3'.1}=\langle\partial_t\rangle,\quad \mathfrak{g}^a_{3'.2}=\langle a\partial_t+2t\partial_x+2\partial_u\rangle$
$4'$	$\mathfrak{g}_0=\langle\partial_x\rangle,\quad \mathfrak{g}_{4'}=\langle(t^2+1)\partial_t+(t+\nu)x\partial_x+(x+(\nu-t)u)\partial_u\rangle$

In all cases $a\in\mathbb{R}$, $n\neq0$, $\sigma\in\{-1,0,1\}$.

ductions associated with the subalgebra $\mathfrak{g}_0$ are not considered since they lead to constant solutions only. We do not present reductions with respect to the subalgebras $\mathfrak{g}_{1'.1}$, $\mathfrak{g}^a_{1'.2}$ and $\mathfrak{g}_{2'}$ since these subalgebras are specifications of the subalgebras $\mathfrak{g}_{1.1}$, $\mathfrak{g}^a_{1.2}$ and $\mathfrak{g}_2$ for the case $n=1$. The reduction for the case $1'_{\rho=2}$ is not performed because this case is equivalent to $1'_{\rho=-1}$. Indeed, the equations $u_t+uu_x+\lambda t^2u_{xxx}+\delta t^4u_{xxxxx}=0$ and $u'_{t'}+u'u'_{x'}+\lambda/t'u'_{x'x'x'}+\delta/t'u'_{x'x'x'x'x'}=0$ are linked by the transformation $t'=1/t$, $x'=-x/t$, $u'=tu-x$.

The first-order reduced equation from Table 3.9, $(\omega+a)\varphi'+\varphi=0$, gives the "degenerate" solution of (3.44) for arbitrary values of $\beta(t)$ and $\sigma(t)$, $u=(x+c)/(t+a)$, where c and a are arbitrary constants. Using transformation (B.28) we get the "degenerate" exact solution of equation (3.40) in

Table 3.9: Similarity reductions of the equations (3.39) $\alpha = 1$ and $n\beta\sigma \neq 0$.

Case	$\mathfrak{g}$	ω	Ansatz, $u =$	Reduced ODE
Reductions for arbitrary nonzero n				
$1_{\rho\neq-1}$	$\mathfrak{g}_{1.1}$	$xt^{-\frac{\rho+1}{3}}$	$t^{\frac{\rho-2}{3n}}\varphi(\omega)$	$\delta\varphi''''' + \lambda\varphi''' + \left(\varphi^n - \frac{\rho+1}{3}\omega\right)\varphi' + \frac{\rho-2}{3n}\varphi = 0$
$1_{\rho=-1}$	$\mathfrak{g}^a_{1.2}$	$x - \frac{a}{n}\ln t$	$t^{-\frac{1}{n}}\varphi(\omega)$	$\delta\varphi''''' + \lambda\varphi''' + \left(\varphi^n - \frac{a}{n}\right)\varphi' - \frac{1}{n}\varphi = 0$
2	$\mathfrak{g}_2$	$xe^{-\frac{1}{3}t}$	$e^{\frac{1}{3n}t}\varphi(\omega)$	$\delta\varphi''''' + \lambda\varphi''' + \left(\varphi^n - \frac{1}{3}\omega\right)\varphi' + \frac{1}{3n}\varphi = 0$
3	$\mathfrak{g}^a_3$	$x - at$	$\varphi(\omega)$	$\delta\varphi''''' + \lambda\varphi''' + (\varphi^n - a)\varphi' = 0$
Specific reductions for $n = 1$				
$0'$	$\mathfrak{g}^a_{0'}$	t	$\varphi(\omega) + \dfrac{x}{t+a}$	$(\omega + a)\varphi' + \varphi = 0$
$3'$	$\mathfrak{g}^a_{3'.1}$	x	$\varphi(\omega)$	$\delta\varphi''''' + \lambda\varphi''' + \varphi\varphi' = 0$
$3'$	$\mathfrak{g}^a_{3'.2}$	$x - t^2/a$	$2t/a + \varphi(\omega),\ \ a \neq 0$	$\delta\varphi''''' + \lambda\varphi''' + \varphi\varphi' + 2/a = 0$
$4'$	$\mathfrak{g}_{4'}$	$\dfrac{xe^{-\nu\arctan t}}{\sqrt{t^2+1}}$	$\dfrac{e^{\nu\arctan t}}{\sqrt{t^2+1}}\varphi(\omega) + \dfrac{xt}{t^2+1}$	$\delta\varphi''''' + \lambda\varphi''' + (\varphi - \nu\omega)\varphi' + \nu\varphi + \omega = 0$

Here a is an arbitrary constant.

the form

$$u = \frac{x + c}{\int \alpha(t)\mathrm{d}t + a}. \tag{3.48}$$

Consider fifth-order reduced ODEs from Table 3.9. Cases 3 and $3'$ correspond to the constant-coefficient generalized Kawahara equations. The corresponding ODEs were heavily studied in the literature, see, e.g., [14, 68, 169, 228] and references therein. We concentrate our attention on variable coefficient cases.

3.3.4. Exact Solutions for Equations Reducible to Their Constant Coefficients Counterparts. In recent papers [156, 314] different techniques for finding exact solutions were applied to construct exact solutions of Kawahara equations with time-dependent coefficients. In both papers exact solutions were derived for equations whose coefficients obey additional constraints, namely, when all the coefficients are proportional to each other. Theorem 3.23 implies that such variable coefficient equations from class (3.39) are reducible to constant coefficient Kawahara equations.

In our opinion the optimal way to get exact solutions for equations from (3.39) that are reducible to the constant-coefficient equations from this class is to take known solutions for constant coefficient equations and then to make a corresponding change of variables. In such a way it is possible to construct exact solution not only for the case when the coefficients in (3.39) are proportional but also (if $n = 1$) for equations of the form (3.39) whose coefficients satisfy conditions (B.23).

We derive the corresponding changes of variables using Theorem 3.21 for the case $n \neq 1$ and Theorem 3.21 for the case $n = 1$. The following statement is true. The equations from class (3.39)

$$u_t + \alpha(t)u^n u_x + \tilde{\beta}\alpha(t)u_{xxx} + \tilde{\sigma}\alpha(t)u_{xxxxx} = 0, \quad \text{and,} \tag{3.49}$$

$$\begin{aligned} u_t + \alpha(t)uu_x &+ \tilde{\beta}\alpha(t)(\delta_3 \textstyle\int \alpha(t)\mathrm{d}t + \delta_4)u_{xxx} \\ &+ \tilde{\sigma}\alpha(t)(\delta_3 \textstyle\int \alpha(t)\mathrm{d}t + \delta_4)^3 u_{xxxxx} = 0, \end{aligned} \tag{3.50}$$

where $\alpha(t)$ is a smooth nonvanishing function, reduce to the constant coefficient Kawahara equations

$$\tilde{u}_{\tilde{t}} + \tilde{\alpha}\tilde{u}^n\tilde{u}_{\tilde{x}} + \tilde{\beta}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{\sigma}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0, \quad \text{and} \tag{3.51}$$

$$\tilde{u}_{\tilde{t}} + \tilde{\alpha}\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{\beta}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{\sigma}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0 \tag{3.52}$$

via the transformations

$$\tilde{t} = \textstyle\int \alpha(t)\mathrm{d}t, \quad \tilde{x} = x, \quad \tilde{u} = \tilde{\alpha}^{-\frac{1}{n}}u, \quad \text{and}$$

$$\begin{aligned} &\tilde{t} = \textstyle\int \alpha(t)(\delta_3 \int \alpha(t)\mathrm{d}t + \delta_4)^{-2}\mathrm{d}t, \\ &\tilde{x} = (x + \delta_1)(\delta_3 \textstyle\int \alpha(t)\mathrm{d}t + \delta_4)^{-1}, \\ &\tilde{u} = \big((\delta_3 \textstyle\int \alpha(t)\mathrm{d}t + \delta_4)u - (x + \delta_1)\delta_3\big)/\tilde{\alpha}, \end{aligned} \tag{3.53}$$

respectively. Here δ_i, $i = 1, 3, 4$, $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\sigma}$ are arbitrary constants with $\tilde{\alpha}\tilde{\beta}\tilde{\sigma}(\delta_3^2 + \delta_4^2) \neq 0$.

We take a family of solitary wave solutions of the Kawahara equation (3.52) of the form

$$\tilde{u} = -\frac{264992\tilde{\sigma}^2\kappa^5 - 7280\tilde{\beta}\tilde{\sigma}\kappa^3 - 31\tilde{\beta}^2 k + 507\tilde{\sigma}\mu}{507\tilde{\alpha}\tilde{\sigma}\kappa} - \frac{280\kappa^2(\tilde{\beta} - 104\tilde{\sigma}\kappa^2)}{13\tilde{\alpha}}\tanh^2(\kappa\tilde{x} + \mu\tilde{t} + \chi) - \frac{1680\tilde{\sigma}\kappa^4}{\tilde{\alpha}}\tanh^4(\kappa\tilde{x} + \mu\tilde{t} + \chi)$$

with κ given by

$$\kappa_{1,2} = \pm\frac{\sqrt{-13\tilde{\beta}\tilde{\sigma}}}{26\tilde{\sigma}}, \quad \kappa_{3,4} = \pm\frac{\sqrt{65\tilde{\beta}\tilde{\sigma}(31-3i\sqrt{31})}}{260\tilde{\sigma}}, \quad \kappa_{5,6} = \pm\frac{\sqrt{65\tilde{\beta}\tilde{\sigma}(31+3i\sqrt{31})}}{260\tilde{\sigma}},$$

μ and χ being arbitrary constants [169]. The corresponding exact solution of (3.50), derived with the usage of (3.53), is

$$u = \frac{1}{\delta_3\int\alpha(t)\mathrm{d}t+\delta_4}\Bigg(\delta_3(x+\delta_1) - \frac{280}{13}\kappa^2(\tilde{\beta}-104\tilde{\sigma}\kappa^2)\tanh^2(\kappa\tilde{x}+\mu\tilde{t}+\chi) - \frac{264992\,\tilde{\sigma}^2\kappa^5 - 7280\tilde{\beta}\tilde{\sigma}\kappa^3 - 31\tilde{\beta}^2\kappa + 507\tilde{\sigma}\mu}{507\tilde{\sigma}\kappa} - 1680\tilde{\sigma}\kappa^4\tanh^4(\kappa\tilde{x}+\mu\tilde{t}+\chi)\Bigg),$$

where $\tilde{t} = \int\alpha(t)(\delta_3\int\alpha(t)\mathrm{d}t + \delta_4)^{-2}\mathrm{d}t$, $\tilde{x} = (x+\delta_1)(\delta_3\int\alpha(t)\mathrm{d}t + \delta_4)^{-1}$, δ_1, μ and χ are arbitrary constants, κ takes the six values adduced above.

A family of solutions for equation (3.49) with $n = 2$ has the form

$$u = \frac{40k^2\tilde{\sigma} - \tilde{\beta}}{\sqrt{-10\tilde{\sigma}}} + 6k^2\sqrt{-10\tilde{\sigma}}\tanh^2\left(kx + \frac{k}{10\tilde{\sigma}}(240k^4\tilde{\sigma}^2 + \tilde{\beta}^2)\int\alpha(t)\mathrm{d}t + \chi\right), \tag{3.54}$$

where k and χ are arbitrary constants with $k \neq 0$. On Figs. 1–3 we present the graphs of solution (3.54) for certain values of parameters and different time inhomogeneities.

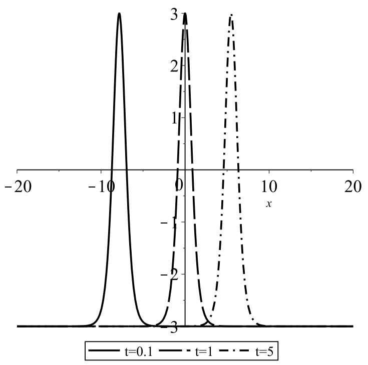

Figure 3.1: Solution (3.54) for $\alpha(t) = 1/t$, $\sigma = -0.1$, $\beta = -1$, $k = 1$, $\chi = 0$.

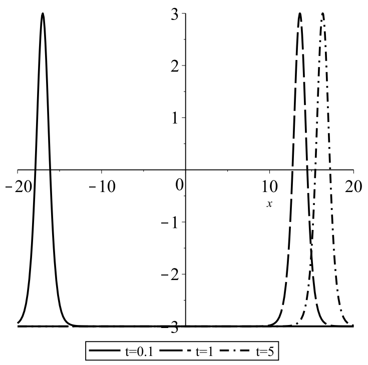

Figure 3.2: Solution (3.54) for $\alpha(t) = 1/t^2$, $\sigma = -0.1$, $\beta = -1$, $k = 1$, $\chi = -17$.

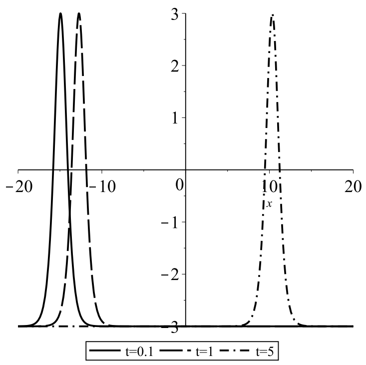

Figure 3.3: Solution (3.54) for $\alpha(t) = \sqrt{t}$, $\sigma = -0.1$, $\beta = -1$, $k = 1$, $\chi = 15$.

3.3.5. Numerical Solutions Using Lie Symmetries. Exact solutions of the fifth-order ODEs presented in Cases 1, 2 and 4′ of Table 3.9 are not known. At the same time behavior of solutions for variable coefficients models is what we are most interested in. The Lie reductions obtained can be useful in seeking solutions of equations (3.39) accompanied with boundary conditions that are invariant with respect to the corresponding Lie symmetry algebras [32]. Consider a class of boundary value problems (BVPs) for variable coefficient generalized Kawahara equations,

$$u_t + u^n u_x + \lambda t^\rho u_{xxx} + \delta t^{\frac{5\rho+2}{3}} u_{xxxxx} = 0, \ t > t_0, \ x > 0, \ n \in \mathbb{N}, \quad (3.55)$$

$$u(t,0) = \gamma_0 t^{\frac{\rho-2}{3n}}, \ \left.\frac{\partial^i u(t,x)}{\partial x^i}\right|_{x=0} = \gamma_i t^{\frac{\rho-2-n(\rho+1)i}{3n}}, \ t > t_0, \ i = 1,\dots,4, \quad (3.56)$$

where γ_i, $i = 0,\dots,4$, λ and δ are arbitrary constants with $\gamma_0\lambda\delta \neq 0$. Both equation and boundary conditions are invariant with respect to the scaling symmetry operator $Q = 3nt\partial_t + (\rho+1)nx\partial_x + (\rho-2)u\partial_u$ (Case 1 of Table 3.6). Using the corresponding ansatz (Case $1_{\rho\neq-1}$ of Table 3.9) this problem reduces to the initial value problem (IVP) for a fifth-order ODE,

$$\begin{aligned}
&\delta\varphi''''' + \lambda\varphi''' + \left(\varphi^n - \tfrac{\rho+1}{3}\omega\right)\varphi' + \tfrac{\rho-2}{3n}\varphi = 0, \\
&\varphi(0) = \gamma_0, \quad \left.\frac{\mathrm{d}^i\varphi(\omega)}{\mathrm{d}\omega^i}\right|_{\omega=0} = \gamma_i, \quad i = 1,\dots,4.
\end{aligned} \quad (3.57)$$

After the problem for the latter IVP is solved numerically, then the corresponding solution of BVP (3.55)–(3.56) can be recovered using the similarity transformation $u = t^{\frac{\rho-2}{3n}}\varphi(\omega)$ with $\omega = xt^{-\frac{\rho+1}{3}}$.

We illustrate the usage of Lie symmetries for the construction of numerical solutions for the Kawahara equations with time-dependent coefficients by the following example.

Example 3.29. Consider the equation

$$v_t + v_x + \tfrac{3}{2}\varepsilon v v_x + \tfrac{1}{2}\varkappa v_{xxx} + \tfrac{1}{2}\gamma v_{xxxxx} = 0$$

that arises as a model describing the propagation of long nonlinear waves in the water covered by ice [103, 134, 193, 284, 319]. Here

$$\varepsilon = \frac{a}{H}, \quad \varkappa = \frac{h}{\rho_\omega g \lambda^2}(\sigma_0 - \sigma_{xx}), \quad \gamma = \frac{Eh^3}{12(1-\nu^2)\rho_\omega g \lambda^4},$$

where v is the dimensionless amplitude of the oscillations of the under-ice surface of the fluid about the horizontal equilibrium position, a is the characteristic wave amplitude, H is the depth of the fluid, $2\pi\lambda$ is the characteristic wavelength, ρ_ω and ρ_i are the densities of the fluid and ice, respectively; h, E, and ν are the thickness, Young's modulus and Poisson's ratio of the ice, and σ_{xx} is a component of the ice sheet stress tensor, $\sigma_0 = gH[\rho_\omega H/(3h) + \rho_i]$. It is assumed that $\sigma_{xx} \approx 10^5\mathrm{N/m}^2$ is the result of external forces [134].

We suppose that the growth of ice thickness is described by the law $h = 0.04\sqrt{t}$, which for certain weather conditions is in well agreement with the data obtained for the sea of Azov for 10 days (240 hours) of observations of ice growth starting from h=0.1m [47]. Then for the values $\lambda \approx 100\mathrm{m}$, $H \approx 10\mathrm{m}$, $E \approx 3\cdot 10^9\mathrm{N/m}^2$, $a \approx 0.1\mathrm{m}$, $\rho_\omega \approx 1030\mathrm{kg/m}^3$, $\rho_\omega \approx 916\mathrm{kg/m}^3$ and $\sigma_0 \approx 1.2\cdot 10^6\mathrm{N/m}^2$ that is calculated for average ice thickness $h_a \approx 0.3\mathrm{m}$ we will have a model equation of the form

$$v_t + v_x + \alpha v v_x + \lambda t^{\frac{1}{2}} v_{xxx} + \delta t^{\frac{3}{2}} v_{xxxxx} = 0,$$

where $\alpha = 1.5 \cdot 10^{-2}$, $\beta \approx 2.20215 \cdot 10^{-5}$ and $\delta \approx 1.05566 \cdot 10^{-8}$ (after converting time in E, σ_0 and σ_{xx} in hours). To reduce this equation to the form (3.55) we make the change of the dependent variable $u = 1 + \alpha v$ and get the equation

$$u_t + uu_x + \lambda t^{\frac{1}{2}} u_{xxx} + \delta t^{\frac{3}{2}} u_{xxxxx} = 0 \tag{3.58}$$

where λ and δ remain the same. We consider the boundary conditions

$$u(t,0) = \gamma_0 t^{-\frac{1}{2}},\ u_x(t,0) = u_{xx}(t,0) = u_{xxx}(t,0) = u_{xxxx}(t,0) = 0, \tag{3.59}$$

that are invariant with respect to the operator of scaling symmetry $2t\partial_t + x\partial_x - u\partial_u$ of the latter equation. Such a BVP reduces to the following initial value problem

$$\begin{aligned} &\delta\varphi''''' + \lambda\varphi''' + \left(\varphi - \tfrac{1}{2}\omega\right)\varphi' - \tfrac{1}{2}\varphi = 0,\\ &\varphi(0) = \gamma_0, \quad \varphi'(0) = \varphi''(0) = \varphi'''(0) = \varphi''''(0) = 0. \end{aligned} \tag{3.60}$$

The numerical solution for this initial value problem is presented on Fig. 4. The corresponding numerical solution of equation (3.58) with the associated boundary conditions (3.56) is presented on Fig. 5.

Concluding Remarks. In this section the group classification problem for class (3.39) of variable coefficient generalized Kawahara equations was

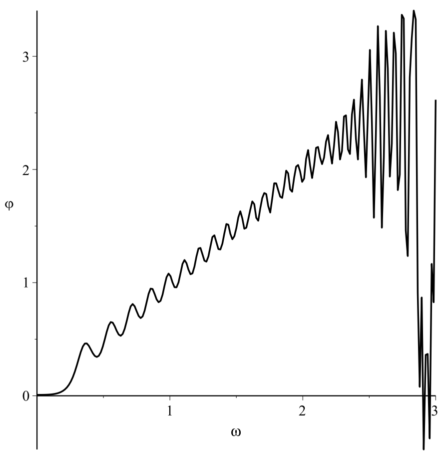

Figure 3.4: Solution of IVP (3.60), $\gamma_0 = 1/120$.

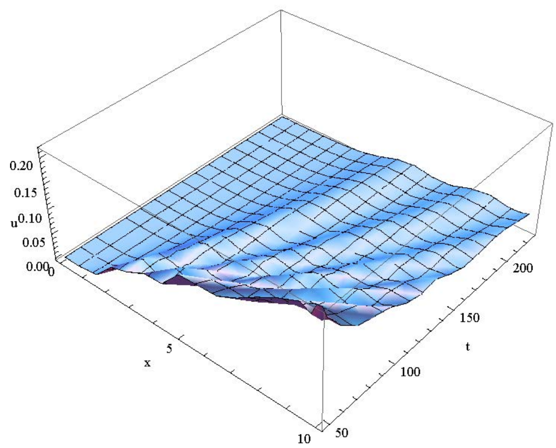

Figure 3.5: Solution of BVP (3.58)-(3.59), $\gamma_0 = 1/120$.

solved exhaustively. As a result, new variable coefficient nonlinear models admitting Lie symmetry extensions were derived. This became possible due to an appropriate gauge of arbitrary elements of the class. Namely, the gauge $\alpha = 1$ was utilized. The use of different equivalence groups for the cases $n \neq 1$ and $n = 1$, which were found in the course of the study of admissible transformations in class (3.39), allowed us to write down the classification list in a simple and concise form. We also construceted zero-order conservation laws of equations of the form (3.44), given by conserved vectors with characteristics 1 and u

$$\left(u, \tfrac{1}{n+1}\,\alpha(t)u^{n+1}+\beta(t)u_{xx}+\sigma(t)u_{xxxx}\right),$$
$$\left(\tfrac{1}{2}u^2, \tfrac{1}{n+2}\,\alpha(t)u^{n+2}+\beta(t)\left(uu_{xx}-\tfrac{1}{2}u_x^2\right)+\sigma(t)\left(uu_{xxxx}-u_xu_{xxx}+\tfrac{1}{2}u_{xx}^2\right)\right).$$

These are conservation laws of momentum and energy, respectively.

3.4. Lie Symmetries and Conservation Laws of Generalized Benjamin–Bona–Mahony Equations

The third-order nonlinear partial differential equation

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

named these days the Benjamin–Bona–Mahony (BBM) equation, appeared in [20, 231] as an alternative to the Korteweg–de Vries equation, $u_t + u_x + uu_x + u_{xxx} = 0$, model for the unidirectional propagation of moderately long waves with small but finite amplitude in systems that manifest nonlinear and dispersive effects. Numerical studies showed that the BBM equation admits soliton solutions whose interaction is inelastic though close to elastic [2,40]. It was proved in [75] for the equivalent form $u_t = uu_x + u_{xxt}$ of the BBM equation that it has no conserved quantity in addition to those found by Benjamin, Bona and Mahony: u (mass), $(u^2 + u_x^2)/2$ (energy), and $u^3/3$ (momentum). Lie symmetries and the corresponding reductions of the BBM equation in the above equivalent form were obtained in [163] (these results were also presented in [131, pp 194–196]). It was found that the maximal Lie symmetry algebra of this equation is a three-dimensional Lie algebra of the type $A_{2.1} \oplus A_1$ spanned by the vector fields ∂_t, $t\partial_t - u\partial_u$ and ∂_x.

There are several recent works (see [199] and references therein) devoted to the study of Lie symmetries of variable-coefficient BBM equations of the general form

$$u_t + f(t)u_x + g(t)uu_x + h(t)u_{xxt} = 0, \tag{3.61}$$

where f, g and h are arbitrary smooth functions of the variable t with $gh \neq 0$. However none of these works contains exhaustive and completely correct results. We aim to fill up this gap by presenting the exhaustive group classification of equations from class (3.61) and classifying local conservation laws of these equations.

Theorem 3.30. *The usual equivalence group $G_1^{\sim}$ of class (3.61) is comprised of the transformations*

$$\begin{aligned}
&\tilde t = T(t), \quad \tilde x = \delta_1 x + \delta_2, \quad \tilde u = \delta_3 u + \delta_4,\\
&\tilde f = \frac{\delta_1}{T_t\delta_3}(\delta_3 f - \delta_4 g), \quad \tilde g = \frac{\delta_1}{T_t\delta_3} g, \quad \tilde h = \delta_1^2 h,
\end{aligned}$$

where δ_j, $j = 1, 2, 3, 4$, are arbitrary constants with $\delta_1\delta_3 \neq 0$ and $T = T(t)$ is an arbitrary smooth function with $T_t \neq 0$. Class (3.61) is normalized in the usual sense.

Thus, each point transformation between equations from class (3.61) is induced by an element of the group $G_1^{\sim}$. In order to find which variable-coefficient equations of the form (3.61) admit constant-coefficient counterparts, we assume that the transformed arbitrary elements $\tilde f$, $\tilde g$ and $\tilde h$ are constants in equivalence transformations. This results in the following assertion:

Proposition 3.31. *A variable-coefficient equation from class (3.61) is reduced to a constant-coefficient equation from the same class by a point transformation if and only if the corresponding coefficients f, g and h satisfy the conditions*

$$\left(\frac{f}{g}\right)_t = h_t = 0,$$

i.e., h is a constant and f is proportional to g.

Equivalence transformations allow us to simplify the consideration by reducing the number of arbitrary elements. For example, we can set the gauge $g = 1$ using the family of point transformations

$$\tilde{t} = \textstyle\int g(t)\mathrm{d}t, \quad \tilde{x} = x, \quad \tilde{u} = u \tag{3.62}$$

parameterized by the arbitrary element g and related to equivalence transformations from the group $G_1^\sim$. Then the other arbitrary elements are changed as $\tilde{f}(\tilde{t}) = f(t)/g(t)$, and $\tilde{h}(\tilde{t}) = h(t)$. Here and below an integral with respect to t should be interpreted as a fixed antiderivative.

Therefore, without loss of generality we can restrict ourselves by the study of the class

$$u_t + f(t)u_x + uu_x + h(t)u_{xxt} = 0. \tag{3.63}$$

Since class (3.61) is normalized, the equivalence group of its subclass (3.63) can be easily found as the subgroup of the group $G_1^\sim$ whose elements preserve the gauge $g = 1$.

Corollary 3.32. *Class (3.63) is normalized in the usual sense. Its usual equivalence group $G_2^\sim$ is constituted by the transformations*

$$\tilde{t} = \frac{\delta_1}{\delta_3}t + \delta_0, \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u + \delta_4, \quad \tilde{f} = \delta_3 f - \delta_4, \quad \tilde{h} = \delta_1^2 h,$$

where δ_j, $j = 0, \ldots, 4$, are arbitrary constants with $\delta_1\delta_3 \neq 0$.

There are no truly variable-coefficient equations in class (3.63) that are reduced by point transformations to constant-coefficient equations from the same class.

3.4.1. Lie Symmetries. We use the method of mapping between classes, which was suggested in [300] for solving the group classification problem. This method has been successfully applied to several classes of nonlinear partial differential equations (see, e.g., [297]).

Class (3.61) can be mapped to a similar class of third-order partial differential equations of the form

$$u_t + uu_x + h(t)u_{xxt} = l(t), \quad h \neq 0. \tag{3.64}$$

The mapping is realized by the family of point transformations

$$\tilde{t} = \textstyle\int g(t)\mathrm{d}t, \quad \tilde{x} = x, \quad \tilde{u} = u + \dfrac{f(t)}{g(t)}, \tag{3.65}$$

parameterized by two arbitrary elements of class (3.61). The arbitrary elements in the imaged equations take values (tildes in (3.64) are omitted) $\tilde{h}(\tilde{t}) = h(t)$, $l(\tilde{t}) = \frac{1}{g(t)}\left(\frac{f(t)}{g(t)}\right)_t$. Following the method of mapping between classes, we first classify Lie symmetries of the imaged class (3.64) and then use the family of point transformations (3.65) to extend the result to the initial class (3.61).

In order to efficiently solve the group classification problem for class (3.64), we look for admissible transformations in this class using the direct method. It appears that such transformations are exhausted by transformations from the usual equivalence group of this class.

Theorem 3.33. *The usual equivalence group $G_3^\sim$ of class* (3.64) *consists of the transformations*

$$\tilde{t} = \frac{\delta_1}{\delta_3}t + \delta_0, \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u, \quad \tilde{h} = \delta_1^2 h, \quad \tilde{l} = \frac{\delta_3^2}{\delta_1} l,$$

where δ_j, $j = 0, 1, 2, 3$, are arbitrary constants with $\delta_1\delta_3 \neq 0$. Class (3.64) *is normalized in the usual sense.*

Using the classical Lie infinitesimal method, we get the complete group classification of equations from class (3.64). The results are summarized in the following assertion.

Table 3.10: The group classification of class (3.64) up to $G_3^{\sim}$-equivalence.

no.	$h(t)$	$l(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_x
1	εt^ρ	$\lambda t^{\frac{\rho-4}{2}}$	$\partial_x,\ 2t\partial_t+\rho x\partial_x+(\rho-2)u\partial_u$
2	εe^t	$\lambda e^{\frac{1}{2}t}$	$\partial_x,\ 2\partial_t+x\partial_x+u\partial_u$
3	ε	1	$\partial_x,\ \partial_t$
4	ε	0	$\partial_x,\ \partial_t,\ t\partial_t-u\partial_u$

Here ρ and λ are arbitrary constants, $\varepsilon=\pm1 \bmod G_3^{\sim}$, in Case 1 $(\rho,\lambda)\neq(0,0)$.

Theorem 3.34. *The kernel of the maximal Lie invariance algebras of equations from class* (3.64) *is the one-dimensional algebra* $\langle\partial_x\rangle$. *All possible* $G_3^{\sim}$*-inequivalent cases of extension of the maximal Lie invariance algebras are exhausted by Cases* 1–4 *of Table 3.10.*

Remark 3.35. The most general forms of the functions h and l that correspond to equations from class (3.64) with Lie symmetry extensions are

1. $h=\lambda_1(\varepsilon t+\kappa)^\rho,\quad l=\lambda_2(\varepsilon t+\kappa)^{\frac{\rho-4}{2}}$:
   $A^{\max}=\langle\partial_x,\ 2(\varepsilon t+\kappa)\partial_t+\varepsilon\rho x\partial_x+\varepsilon(\rho-2)u\partial_u\rangle$;
2. $h=\lambda_1 e^{\sigma t},\quad l=\lambda_2 e^{\frac{1}{2}\sigma t}$: $\quad A^{\max}=\langle\partial_x,\ 2\partial_t+\sigma x\partial_x+\sigma u\partial_u\rangle$;
3. $h=\lambda_1,\quad l=\lambda_2$: $\quad A^{\max}=\langle\partial_x,\ \partial_t\rangle$;
4. $h=\lambda_1,\quad l=0$: $\quad A^{\max}=\langle\partial_x,\ \partial_t,\ t\partial_t-u\partial_u\rangle$.

Here λ_1, λ_2, ε, κ and ρ are arbitrary constants with $\lambda_1\sigma\varepsilon\neq0$. Additionally, in Case 1 $(\rho,\lambda_2)\neq(0,0)$ and in Case 3 $\lambda_2\neq0$. Due to the presence of arbitrary constants λ_1 and λ_2, the constant ε can be assumed to take the values ±1 only.

The following example shows how to recover the group classification of class (3.61) using the results obtained for class (3.64).

Consider Case 1 of Table 3.10 extended by the equivalence transformations from $G_3^{\sim}$, i.e., the first case presented in Remark 3.35, were

$\tilde{h}=\lambda_1(\varepsilon\tilde{t}+\kappa)^{\rho}$, $l=\lambda_2(\varepsilon\tilde{t}+\kappa)^{\frac{\rho-4}{2}}$. We denote $\int g(t)\mathrm{d}t$ by T. As $\tilde{t}=T$ and $l(T)=(f/g)_t/g$, we get $(f/g)_t=\lambda_2 g(t)(\varepsilon T+\kappa)^{\frac{\rho-4}{2}}$. Finally,

$$f(t)=\begin{cases}\lambda_2 g(t)\left(\frac{2}{\varepsilon(\rho-2)}(\varepsilon T+\kappa)^{\frac{\rho-2}{2}}+\lambda_3\right), & \text{if} \quad \rho\neq 2,\\ \lambda_2 g(t)\left(\frac{1}{\varepsilon}\ln(\varepsilon T+\kappa)+\lambda_3\right), & \text{if} \quad \rho=2.\end{cases}$$

After re-denoting the constants λ_i, $i=1,2,3$, it is easy to see that we get Cases 1 and 2 of Table 3.12, respectively. To obtain the corresponding Lie symmetry operators one should make the change of variables $\tilde{t}=T$, $\tilde{x}=x$, $\tilde{u}=u+\mu_2(\varepsilon T+\kappa)^{\frac{\rho-2}{2}}+\mu_3$ (resp. $\tilde{u}=u+\mu_2\ln(\varepsilon T+\kappa)+\mu_3$ for the second case) in the vector fields $X_1=\partial_{\tilde{x}}$ and $X_2=2(\varepsilon\tilde{t}+\kappa)\partial_{\tilde{t}}+\varepsilon\rho\tilde{x}\partial_{\tilde{x}}+\varepsilon(\rho-2)\tilde{u}\partial_{\tilde{u}}$.

It is interesting to note that the images of two distinct inequivalent cases of Lie symmetry extensions in class (3.61) (Cases 1 and 2 of Table 3.11) belong to the same case of Lie symmetry extensions for class (3.64) (Case 1 of Table 3.10).

The other cases are easily treated in the same way.

Theorem 3.36. *The kernel of the maximal Lie invariance algebras of equations from class* (3.63) *is the one-dimensional algebra* $\langle\partial_x\rangle$*. All possible* $G_2^{\sim}$*-inequivalent cases of Lie symmetry extensions are exhausted by Cases 1–5 of Table 3.11.*

Corollary 3.37. *The group classification for class* (3.61) *up to* $G_1^{\sim}$*-equivalence results in the list presented in Table 3.10, where the arbitrary element* g *is assumed to be equal* 1*.*

In order to get the classification list for class (3.61), where forms of arbitrary elements are not simplified by equivalence transformations, we apply transformation (3.62) combined with transformations from the equivalence group $G_2^{\sim}$ to equations of the form (3.63) with f and h presented in Table 3.10. Basis elements of the corresponding maximal Lie invariance algebras are pushed forward by the same transformations. Then we re-denote the constants and collect the obtained results in Table 3.12. The detailed procedure of the *equivalence based approach* for deriving most general forms

Table 3.11: The group classification of class (3.63) up to $G_2^{\sim}$-equivalence.

no.	$h(t)$	$f(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_x
1	εt^ρ	$\lambda t^{\frac{\rho-2}{2}}$	$\partial_x,\ 2t\partial_t+\rho x\partial_x+(\rho-2)u\partial_u$
2	εt^2	$\ln t$	$\partial_x,\ t\partial_t+x\partial_x-\partial_u$
3	εe^t	$\lambda e^{\frac{1}{2}t}$	$\partial_x,\ 2\partial_t+x\partial_x+u\partial_u$
4	ε	t	$\partial_x,\ \partial_t-\partial_u$
5	ε	0	$\partial_x,\ \partial_t,\ t\partial_t-u\partial_u$

Here ρ and λ are arbitrary constants, $\varepsilon=\pm1 \bmod G_2^{\sim}$. In Case 1 $(\rho,\lambda)\neq(0,0)$.

Table 3.12: The group classification of class (3.61) using no equivalence.

no.	$h(t)$	$f(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_x
1	$\mu_1(\varepsilon T+\kappa)^\rho$	$\mu_2 g(\varepsilon T+\kappa)^{\frac{\rho-2}{2}}+\mu_3 g$	$\partial_x,\ \frac{2}{g}(\varepsilon T+\kappa)\partial_t+\varepsilon\rho x\partial_x+\varepsilon(\rho-2)(u+\mu_3)\partial_u$
2	$\mu_1(\varepsilon T+\kappa)^2$	$\mu_2 g\ln(\varepsilon T+\kappa)+\mu_3 g$	$\partial_x,\ \frac{1}{g}(\varepsilon T+\kappa)\partial_t+\varepsilon x\partial_x-\varepsilon\mu_2\partial_u$
3	$\mu_1\exp(\sigma T)$	$\mu_2 g\exp(\frac{1}{2}\sigma T)+\mu_3 g$	$\partial_x,\ \frac{2}{g}\partial_t+\sigma x\partial_x+\sigma(u+\mu_3)\partial_u$
4	μ_1	$\mu_2 gT+\mu_3 g$	$\partial_x,\ \frac{1}{g}\partial_t-\mu_2\partial_u$
5	μ_1	$\mu_3 g$	$\partial_x,\ \frac{1}{g}\partial_t,\ \frac{T}{g}\partial_t-(u+\mu_3)\partial_u$

Here g is an arbitrary nonvanishing smooth function, $T=\int g(t)\,\mathrm{d}t$; $\varepsilon=\pm1$; μ_1, μ_2, μ_3, ν and ρ are arbitrary constants satisfying the following constraints: $\mu_1\lambda\neq0$; in Case 1 $\rho\mu_2\neq0$; and in Case 4 $\mu_2\neq0$.

of arbitrary elements and basis elements of the corresponding maximal Lie invariance algebras can be found in [289].

Comparing the results of [199] with those collected in Table 3.11, we conclude that Lie symmetry extensions presented in [199] are particular specifications of Cases 1–5 from Table 3.11 for certain fixed values of the arbitrary element g. For example, there are two cases in the classification list derived in [199] with the three-dimensional maximal Lie symmetry algebras (Cases 4 and 10 of Table 1 therein). These cases are particular subcases of Case 5 of Table 3.11 for $g = g_0 = \text{const}$ and $g = g_0 e^{kt}$.

3.4.2. Conservation Laws. We classify (local) conservation laws of equations from class (3.61), applying the most direct method based on the definition of conservation laws [10, 11, 217].

The classification of local conservation laws of equations from the class (3.61) is as follows.

Case 0. Each equation from the class (3.61) admits the "natural" conservation law with the constant characteristic $\lambda^1 = 1$. The corresponding density and flux are

$$F^1 = u, \quad G^1 = f(t)u + \frac{1}{2}g(t)u^2 + h(t)u_{tx}.$$

For general admitted values of the arbitrary elements f, g and h the associated space of conservation laws is one-dimensional.

Case 1. If the arbitrary elements satisfy the equation $((1/h)_t/g)_t = 0$, and, therefore, $h(t) = \left(\rho_1 \int g(t)\mathrm{d}t + \rho_2\right)^{-1}$, where ρ_1 and ρ_2 are constants with $(\rho_1, \rho_2) \neq (0, 0)$, then the space of conservation laws of the corresponding equation of the form (3.61) is at least two-dimensional. The second basis conservation law can be chosen to have the following characteristic, density and flux:

$$\lambda^2 = \frac{u}{h} - \rho_1\left(x - \textstyle\int f(t)\mathrm{d}t\right), \quad F^2 = \frac{u^2}{2h(t)} - \frac{u_x^2}{2} - \rho_1\left(x - \textstyle\int f(t)\mathrm{d}t\right)u,$$
$$G^2 = \frac{g(t)}{3h(t)}u^3 + \frac{f(t)}{2h(t)}u^2 + uu_{tx} + \rho_1 h(t)u_t$$
$$- \rho_1\left(x - \textstyle\int f(t)\mathrm{d}t\right)\left(f(t)u + \frac{1}{2}g(t)u^2 + h(t)u_{tx}\right),$$

where $(1/h)_t/g = \rho_1 = \text{const}$.

Using the family of point transformations $\tilde t = \rho_1 \int g(t)\mathrm{d}t + \rho_2$, $\tilde x = x$, $\tilde u = u/\rho_1$ related to the group $G_1^\sim$, we can reduce any equation of this case with $\rho_1 \neq 0$ to the form $u_t + f(t)u_x + uu_x + t^{-1}u_{txx} = 0$ (tildes are omitted in the latter equation).

Case 2. One more case with at least two-dimensional spaces of conservation laws is given by the arbitrary elements satisfying the condition $((f/g)_t/g)_t = 0$, i.e., if $f(t) = \big(\sigma_1 \int g(t)\mathrm{d}t + \sigma_2\big)\, g(t)$, where σ_1 and σ_2 are arbitrary constants. The second basis conservation law can be chosen to have the following characteristic, density and flux:

$$\lambda^3 = W^2 - 2\sigma_1 x + 2\frac{h}{g}u_{tx}, \quad F^3 = \frac{1}{3}W^3 - 2\sigma_1 xu - \frac{1}{3}\frac{f(t)^3}{g(t)^3},$$

$$G^3 = \frac{h(t)}{g(t)}W_t^2 + \frac{h(t)^2}{g(t)}u_{tx}^2 + h(t)W^2u_{tx} - 2\sigma_1 h(t)xu_{tx}$$
$$- 2\sigma_1 f(t)xu - \sigma_1 g(t)xu^2 + \frac{g(t)}{4}W^4,$$

where $W = u + f(t)/g(t)$, $(f/g)_t/g = \sigma_1 = \text{const}$.

Using the family of point transformations $\tilde{t} = \sigma_1 \int g(t)\mathrm{d}t + \sigma_2$, $\tilde{x} = \sigma_1 x$, $\tilde{u} = u$ related to the group $G_1^{\sim}$, we can reduce any equation of this case with $\sigma_1 \neq 0$ to the form $u_t + tu_x + uu_x + h(t)u_{txx} = 0$ (tildes are omitted in the latter equation).

Case 3. The maximal dimension of the spaces of conservation laws for equations from class (3.61) equals three and is reached for the intersection of Cases 1 and 2, where arbitrary elements satisfy the both constraints, $((f/g)_t/g)_t = 0$ and $((1/h)_t/g)_t = 0$. Then for each of the spaces, a basis consists of conservation laws with the characteristics λ^1, λ^2 and λ^3 and the conserved currents (F^1, G^1), (F^2, G^2) and (F^3, G^3), respectively. The corresponding equation can be reduced to the form

$$u_t + (\sigma_1 t + \sigma_2)u_x + uu_x + (\rho_1 t + \rho_2)^{-1}u_{txx} = 0$$

by transformation (3.62). The further simplification is possible by transformations from the group $G_2^{\sim}$. For example, we can set one of the linear combinations $\sigma_1 t + \sigma_2$ or $\rho_1 t + \rho_2$ to t if $\sigma_1 \neq 0$ or $\rho_1 \neq 0$, respectively.

A well-studied subcase of Case 3 is constituted by constant-coefficient equations, for which $\rho_1 = \sigma_1 = 0$ [20, 75, 216]. Up to $G_1^{\sim}$-equivalence any constant-coefficient equation from class (3.61) can be mapped to the equation $u_t = uu_x + \varepsilon u_{xxt}$, where $\varepsilon = \operatorname{sgn} h = \pm 1$. Then the characteristics and the components of the conserved currents of the above basis conservation laws take the form (cf. [131, p. 195])

$$\begin{aligned}
&\lambda^1 = 1, \quad F^1 = u, \quad G^1 = -\frac{1}{2}u^2 - \varepsilon u_{tx},\\
&\lambda^2 = \varepsilon u, \quad F^2 = \frac{u^2}{2} + \varepsilon \frac{u_x^2}{2}, \quad G^2 = -\frac{1}{3}u^3 - \varepsilon u u_{tx},\\
&\lambda^3 = u^2 + 2\varepsilon u_{tx}, \quad F^3 = \frac{1}{3}u^3, \quad G^3 = \varepsilon u_t^2 - u_{tx}^2 - \varepsilon u^2 u_{tx} - \frac{1}{4}u^4.
\end{aligned}$$

The results on conservation laws are quite expectable and, at the same time, are not trivial. They naturally generalize well-known results of constant-coefficient BBM equations and need the completion of the most significant and tricky part of the proof, which is deriving an upper bound for order of conservation laws similarly to [75].

3.5. Equivalence Transformations in the Study of Integrability

Since late 1960s there is an unceasing interest to the study of exactly solvable (integrable) partial differential equations (PDEs) that model real-world phenomena. Thus, the inverse scattering transform method was introduced in [100] and was applied therein to the prominent Korteweg–de Vries (KdV) equation $u_t = u_{xxx} + 6uu_x$ [162] in order to find its soliton solutions. The notion of soliton had appeared earlier in [329]. It was shown in [198] that the KdV equation possesses an infinite set of conservation laws of arbitrarily high orders, and this property appeared to be typical for integrable equations. A new direct method (the Hirota bilinear method) for finding multisoliton solutions to integrable nonlinear evolution equations was suggested in [119]. In contrast to the inverse scattering transform method, the Hirota bilinear method is algebraic rather than analytical.

These and other methods were then applied to a wide range of integrable equations [269].

According to [50], integrable equations can be divided into those that are linearizable by an appropriate *Change of variables* (C-integrable equations) and equations integrable by inverse scattering transform method (*Spectral transform* technique) (S-integrable equations).

Among the C-integrable equations there are, e.g., the famous Burgers equation $u_t+uu_x = \nu u_{xx}$ [49] that can be linearized to the heat equation by the Hopf–Cole transformation [62, 121], the Sharma–Tasso–Olver equation $u_t + u_{xxx} + 3u^2u_x + 3u_x^2 + 3uu_{xx} = 0$ [215, 270], which is the second member of the Burgers hierarchy, the u^{-2}-diffusion equation (named also Fujita–Storm equation) $u_t = (u^{-2}u_x)_x + au$ [36, 280], the Fokas–Yortsos equation $u_t = (u^{-2}u_x)_x + au^{-2}u_x$ [81, 281]. Further examples in (1+1)-dimensions can be found in [50].

S-integrable equations in (1+1)-dimensions include the KdV and modified KdV equations, the Gardner equation (the combined KdV–mKdV equation) $u_t + uu_x + u^2u_x + u_{xxx} = 0$ [197], the cylindrical KdV equation $u_t = u_{xxx} + 6uu_x - \frac{1}{2t}u$ [191], the Dym equation $u_t = u^3u_{xxx}$ [167], the sine-Gordon equation $u_{tt} - u_{xx} + \sin u = 0$, etc. See other examples of integrable equations, e.g., in [269].

Most of integrable PDEs considered in the beginning of the development of integrability theory were constant-coefficient ones. At the same time, many model equations appearing in applications explicitly involve independent variables. For example, the generalized Burgers equations describing the propagation of weakly nonlinear acoustic waves under the influence of geometrical spreading and thermoviscous diffusion in non-dimensional variables are represented as $u_t + uu_x = g(t)u_{xx}$ with $g \neq 0$ [116] (these equations are not C-integrable for nonconstant values of g). The KdV and cubic Schrödinger equations with time-dependent coefficients,

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0 \text{ and } iu_t + f(t)u_{xx} + g(t)|u|^2u = 0, \quad (3.66)$$

respectively, also appear in different applications [105, 106]. Here f and g are nonvanishing smooth functions of t.

Many papers devoted to the study of variable-coefficient equations were published, especially, in recent years. Usual topics of these papers are the application of Painlevé test in order to single out subclasses of integrable equations within wider classes of variable-coefficient equations, the construction of conservation laws, Lax pairs and bilinear representations and finding exact soliton solutions by the Hirota bilinear method. Since sometimes variable-coefficient models are quite complicated and the number of variable coefficients varies from one to five or even to ten in some cases, packages of symbolic computations are widely used to complete these tasks. At the same time, the equivalence between equations in a class under study is neglected in many works, see a discussion in [251]. Though even in pioneering works on exactly solvable models it was shown that if an integrable PDE is related to another PDE by certain change of variables (point or non-point), then the latter PDE is also integrable. The classical examples are the connection between the KdV and mKdV equation via the Miura transformation, the reducibility of the Gardner equation to the mKdV [197] and of the cylindrical KdV equation to the classical KdV [147, 191]. Other examples are given in [148]. It was shown that the KdV and nonlinear Schrödinger equations with time-dependent coefficients (3.66) pass the Painlevé test if and only if the coefficients f and g satisfy the conditions $g(t) = f(t)(a_1\int^t f(s)\,\mathrm{d}s + a_0)$ (resp. $g(t) = f(t)/(a_1\int^t f(s)\,\mathrm{d}s + a_0)$), where a_1 and a_0 are constants with $a_1^2 + a_0^2 \neq 0$. These conditions coincide with those of reducibility of equations (3.66) to their constant-coefficient counterparts, which were obtained in [105, 106].

Another way for construction of variable-coefficient integrable models from constant-coefficient members of integrable hierarchies was presented in [84, Theorem 3.1]. Any “linear superposition”, with arbitrary time-dependent coefficients, of members of an integrable evolution hierarchy that correspond to mutually commuting flows proved to be again integrable.

The study of point transformations within a given class of variable-coefficient PDEs and the knowledge of reducibility conditions to constant-coefficient integrable equations allow one to obtain solutions, conservation laws, other objects and related information in an easier way than using the direct computations for variable-coefficient equations. This section is devoted to the discussion of this subject. The consideration is illustrated by variable-coefficient fifth-order KdV-like equations.

3.5.1. Admissible Transformations in Classes of Differential Equations. Consider the class of nth order (1+1)-dimensional evolution equations,

$$u_t = H(t, x, u_0, u_1, \ldots, u_n), \tag{3.67}$$

where $n \geqslant 2$, $u_j \equiv \partial^j u/\partial x^j$, $j = 1, 2, \ldots$, and $u_0 \equiv u$. We shall also employ, depending on convenience or necessity, the following notation for low-order derivatives: $u_x = u_1$, $u_{xx} = u_2$, $u_{xxx} = u_3$, etc. In general, a subscript of a function denotes the differentiation with respect to the corresponding variable, e.g., $u_t \equiv \partial u/\partial t$, $H_{u_i} \equiv \partial H/\partial u_i$. For the above class, the tuple of arbitrary elements θ consists of a single arbitrary smooth function H of its arguments. The auxiliary equations to H singling out evolution equations among all nth order two-dimensional partial differential equations form the system

$$H_{u_{it}} = 0,\ i = 0, \ldots, n-1, \quad H_{u_{itt}} = 0, \quad i = 0, \ldots, n-2, \quad \ldots,$$

meaning that the arbitrary element H does not depend on derivatives of u involving the differentiation with respect to t. The condition that the equation order equals n leads to the auxiliary inequality $H_{u_n} \neq 0$. For quasilinear evolution equations the arbitrary element H is linear in the highest-order derivative u_n, i.e., the subclass $\mathcal{E}_{\rm ql}$ of such equations is singled out from the entire class (3.67) by the additional auxiliary equation $H_{u_n u_n} =$

0. Representing H in the form $H = Fu_n + G$ and interpreting $F = F(t, x, u_0, u_1, \dots, u_{n-1})$ and $G = G(t, x, u_0, u_1, \dots, u_{n-1})$ as new arbitrary elements, we re-parameterize the subclass $\mathcal{E}_{\rm ql}$. In terms of F and G, the system of auxiliary equations and inequality for the subclass $\mathcal{E}_{\rm ql}$ is

$$
\begin{aligned}
&F_{u_{it}} = G_{u_{it}} = 0, \quad i = 0, \dots, n-1,\\
&F_{u_{itt}} = G_{u_{itt}} = 0, \quad i = 0, \dots, n-2, \quad \dots,\\
&F_{u_n} = G_{u_n} = 0, \quad F \neq 0.
\end{aligned}
\tag{3.68}
$$

Imposing additional auxiliary equations on H (resp. F and G), one can construct a tree of nested subclasses of evolution equations.

We consider a chain of nested normalized classes of evolution equations, which is of interest in view of the subject of the present paper. It is a well-known folklore assertion [192] that any contact transformation $\mathcal{T}$ relating two fixed equations $u_t = H$ and $\tilde{u}_{\tilde{t}} = \tilde{H}$ from the class (3.67) has the form $\tilde{t} = T(t)$, $\tilde{x} = X(t, x, u, u_x)$, $\tilde{u} = U(t, x, u, u_x)$. In comparison with the general contact transformation in the space of (t, x, u), the peculiarity is that the transformation component for t depends only on t and the transformation component for all the variables does not depend on u_t. The contact and nondegeneracy assumptions are reduced for $\mathcal{T}$ to the conditions $(U_x{+}U_u u_x)X_{u_x} = (X_x{+}X_u u_x)U_{u_x}$ and $T_t \neq 0$, $\operatorname{rank} \partial(X, U)/\partial(x, u, u_x) = 2$, respectively. The standard prolongation of $\mathcal{T}$ to the derivatives $u_1, \dots, u_n$ is carried out using the chain rule, which gives $\tilde{u}_{\tilde{x}} = V(t, x, u, u_x)$, where $V = (U_x{+}U_u u_x)/(X_x{+}X_u u_x)$ or $V = U_{u_x}/X_{u_x}$ if $X_x{+}X_u u_x \neq 0$ or $X_{u_x} \neq 0$, respectively, and $\tilde{u}_i \equiv \partial^i \tilde{u}/\partial \tilde{x}^i = ((1/D_x X)D_x)^{i-1}V$, $i = 2, \dots, n$. Here $D_x = \partial_x + u_x \partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \cdots$ is the operator of total differentiation with respect to the variable x. The possibility of simultaneous vanishing $X_x + X_u u_x$ and X_{u_x} is ruled out by the nondegeneracy assumption. Moreover, the contact and nondegeneracy assumptions jointly imply that $(X_u, U_u) \neq (0, 0)$. The transformed arbitrary element $\tilde{H}$ is equal to

$$
\tilde{H} = \frac{U_u - X_u V}{T_t} H + \frac{U_t - X_t V}{T_t}.
$$

Each of the above contact transformations maps the entire class (3.67) onto itself. Therefore, its prolongation to the arbitrary element H belongs to the contact equivalence group $G^{\sim}_{\rm c}$ of the class (3.67), and any element of $G^{\sim}_{\rm c}$ is obtained in this way. In other words, the equivalence group $G^{\sim}_{\rm c}$ generates the whole contact equivalence groupoid $\mathcal{G}^{\sim}_{\rm c}$ of the class (3.67), i.e., this class is contact-normalized, which obviously implies its normalization also with respect to point transformations. If $n\geqslant 3$, the subclass $\mathcal{E}_{\rm ql}$ of nth order quasilinear evolution equations has the same contact equivalence group and is also contact-normalized.

Each next subclass is singled out from the previous one by sequently adding more equations to the system (3.68) in the way preserving the property of usual normalization. The additional constraints $F_{u_2}=\dots=F_{u_{n-1}}=0$ (i.e., $F=F(t,x,u,u_x)$) lead to principally narrowing the equivalence groupoid of the corresponding subclass: Its contact equivalence group coincides with its point equivalence group. Therefore, in the course of the consideration of equations from this subclass it suffices to use only the point equivalence. Imposing additionally the constraint $F_{u_1}=0$, we obtain a subclass in which the x-component of any equivalence transformation does not involve u, $X_u=0$, i.e., all equivalence transformations are fiber preserving. The equivalence group of the subclass of equations with F depending only on t, $F=F(t)$, consists of transformations satisfying the equation $X_{xx}=0$. Finally, for the subclass of equations of the form

$$
\begin{aligned}
&u_t=F(t)u_n+G(t,x,u_0,u_1,\dots,u_{n-1}),\quad n\geqslant 2,\\
&F\neq 0,\quad G_{u_iu_{n-1}}=0,\ i=1,\dots,n-1,
\end{aligned}
\tag{3.69}
$$

any equivalence transformation is linear in u since $U_{uu}=0$. Collecting all determining equations for admissible transformations in the class (3.69), which are exhausted by the above equations $X_u=X_{xx}=U_{uu}=0$, we can claim that its usual point equivalence group consists of the transformations

$$
\tilde t=T(t),\quad \tilde x=X^1(t)x+X^0(t),\quad \tilde u=U^1(t,x)u+U^0(t,x),
\tag{3.70}
$$

$$\tilde{F} = \frac{(X^1)^n}{T_t} F, \quad \tilde{G} = \frac{1}{T_t} \Bigg[U^1 G - \left(\sum_{k=0}^{n-1} \binom{n}{k} U^1_{(n-k)} u_{(k)} + U^0_{(n)} \right) F$$
$$+ U^1_t u + U^0_t - \frac{X^1_t x + X^0_t}{X^1} \left((U^1 u)_x + U^0_x \right) \Bigg],$$

where $T_t X^1 U^1 \neq 0$. The class (3.69) is normalized with respect to this group.

3.5.2. Integrable Subclasses in a Class of Fifth-Order Variable Coefficient KdV Equations. Consider the class of variable-coefficient fifth-order KdV-like equations of the form

$$\begin{aligned} &u_t + a(t) u u_{xxx} + b(t) u_x u_{xx} + c(t) u^2 u_x + f(t) u u_x + g(t) u_{xxxxx} \\ &\quad + h(t) u_{xxx} + m(t) u + n(t) u_x + k(t) x u_x = 0, \end{aligned} \tag{3.71}$$

where the functions a, b, c, f, g, h, m, n, and k are arbitrary smooth functions of the time variable t with $g(a^2 + b^2 + c^2) \neq 0$. Recently certain subclasses of this class were studied, e.g., in [318, 326, 327].

Thus, in [318] the integrability of equations from the class (3.71) with $k = 0$ (and the re-denoted coefficients $f = d$, $g = l$ and $h = e$) was investigated using the Painlevé test. It was found that such equations are Painlevé integrable in the following three cases

I. $b = a, c = \mu_1 a e^{\int m\,\mathrm{dt}}, f = 2\mu_2 a, g = \frac{a}{5\mu_1} e^{-\int m\,\mathrm{dt}}, h = \frac{\mu_2}{\mu_1} a e^{-\int m\,\mathrm{dt}}$;

II. $b = 2a,\ c = \mu_1 a e^{\int m\,\mathrm{dt}},\ f = 2\mu_1 h e^{\int m\,\mathrm{dt}},\ g = \frac{3a}{10\mu_1} e^{-\int m\,\mathrm{dt}}$;

III. $b = \frac{5}{2} a,\ c = \mu_1 a e^{\int m\,\mathrm{dt}},\ f = 2\mu_2 a,\ g = \frac{a}{5\mu_1} e^{-\int m\,\mathrm{dt}},\ h = \frac{\mu_2}{\mu_1} a e^{-\int m\,\mathrm{dt}}$.

In all three cases μ_1 and μ_2 are arbitrary constants with $\mu_1 \neq 0$, the functions a, m and n are arbitrary. In Case II the function h is also arbitrary. Here and in what follows an integral with respect to t should be interpreted as a fixed antiderivative. N-soliton solutions were constructed for the first two cases whereas only one- and two-soliton solutions were presented in Case III.

The same subclass of equations with $k = 0$ was treated earlier in [326]. Although it was stated that both the Painlevé test and the mapping to the completely integrable constant-coefficient counterparts were applied for separating integrable cases, N-soliton solutions, a Bäcklund transformation and a Lax pair were constructed therein only for equations with additional constraints $a = b = 15g\nu e^{\int m\,\mathrm{d}t}$, $c = 45g\nu^2 e^{2\int m\,\mathrm{d}t}$, $f = h = 0$, where ν is a nonzero constant ($\nu = 1/\rho$ in the notation of [326]), which gives a particular subcase of Case I. The other integrable cases were missed.

In [327] such objects were constructed for equations of the form (3.71) with $f = h = 0$ (and the re-denoted coefficients $g = d$ and $k = l$) under the constraints $a = b = 15g\nu e^{\int (m-2k)\,\mathrm{d}t}$, $c = 45g\nu^2 e^{2\int (m-2k)\,\mathrm{d}t}$. It was also indicated that these constraints are derived both by Painlevé analysis and by mapping the corresponding variable coefficient models to their completely integrable constant-coefficient counterparts. In fact, this consideration just extend results of [326] to the case of nonzero k, although the parameter k is not essential and can be set to zero by a point transformation.

We show that all the mentioned cases of integrable equations from class (3.71) are reduced by point transformations to well-known fifth-order integrable evolution equations. To achieve this goal, we present a complete description of admissible transformations between equations from this class.

Theorem 3.38. *The generalized extended equivalence group $G^{\sim}$ of the class* (3.71) *consists of the transformations*

$$\begin{gathered}
\tilde t = \alpha(t), \quad \tilde x = \beta(t)x + \gamma(t), \quad \tilde u = \varphi(t)\left(u + \sigma e^{-\int m\,\mathrm{d}t}\right),\\
\tilde a = \frac{\beta^3}{\alpha_t\varphi}a, \quad \tilde b = \frac{\beta^3}{\alpha_t\varphi}b, \quad \tilde c = \frac{\beta}{\alpha_t\varphi^2}c, \quad \tilde f = \frac{\beta}{\alpha_t\varphi}\left(f - 2\sigma c e^{-\int m\,\mathrm{d}t}\right),\\
\tilde g = \frac{\beta^5}{\alpha_t}g, \quad \tilde h = \frac{\beta^3}{\alpha_t}\left(h - \sigma a e^{-\int m\,\mathrm{d}t}\right), \quad \tilde m = \frac{1}{\alpha_t}\left(m - \frac{\varphi_t}{\varphi}\right), \qquad (3.72)\\
\tilde n = \frac{\beta}{\alpha_t}\left(n + \left(\frac{\gamma}{\beta}\right)_t - k\frac{\gamma}{\beta} + \sigma^2 c e^{-2\int m\,\mathrm{d}t} - \sigma f e^{-\int m\,\mathrm{d}t}\right),\\
\tilde k = \frac{1}{\alpha_t}\left(k + \frac{\beta_t}{\beta}\right).
\end{gathered}$$

where α*,* β*,* γ*, and* φ *run through the set of smooth functions of* t *with* $\alpha_t\beta\varphi\neq 0$*, and* σ *is an arbitrary constant. This group generates the entire equivalence groupoid* $\mathcal{G}^{\sim}$ *of the class* (3.71)*, i.e., the class* (3.71) *is normalized in the generalized extended sense.*

The complete proof of this theorem can be found in [304].

Corollary 3.39. *The subclass of the class* (3.71) *with* $fh=0$ *and* $(a,c)\neq(0,0)$ *is normalized with respect to its usual equivalence group consisting of the transformations* (3.72) *with* $\sigma=0$*.*

Corollary 3.40. *The subclass of the class* (3.71) *with* $k=0$ *is normalized with respect to its generalized extended equivalence group that comprises the transformations* (3.72) *with* $\beta=\mathrm{const}$*.*

Corollary 3.41. *Any equation from the class* (3.71) *can be reduced by the point transformation*

$$\tilde t=\int ge^{-5\int k\,\mathrm{d}t}\,\mathrm{d}t,\quad \tilde x=e^{-\int k\,\mathrm{d}t}x-\int ne^{-\int k\,\mathrm{d}t}\,\mathrm{d}t,\quad \tilde u=e^{\int m\,\mathrm{d}t}u \tag{3.73}$$

to an equation from the same class with $g=1$ *and* $m=n=k=0$*. The subclass of the class* (3.71) *singled out by the constraints* $g=1$ *and* $m=n=k=0$ *is normalized with respect to its generalized extended equivalence group* $G_1^{\sim}$ *consisting of the transformations* (3.72) *with* $\beta_t=\varphi_t=0$*,* $\alpha_t=\beta^5$ *and* $\gamma_t=\sigma\beta f-\sigma^2\beta c$*.*

Transformations from the equivalence group $G^{\sim}$ have a nice particular structure. The principal property is that they are fiber-preserving (i.e., the transformation components corresponding to the independent variables t and x depend only on these variables) and, moreover, linear in u. An additional bonus is that the transformation component for t depends only on t and the transformation component for x is linear in x. Therefore, the entire study of equations from the class (3.71) within integrability theory can be implemented up to $G^{\sim}$-equivalence, which coincides for this class with general contact (resp. point) equivalence since the class (3.71) is normalized with respect to $G^\sim$ in both the contact-transformation and point-transformation frameworks.

Consider the class of constant-coefficient fifth-order KdV equations of the form

$$u_t + Auu_{xxx} + Bu_xu_{xx} + Cu^2u_x + u_{xxxxx} = 0, \tag{3.74}$$

where A, B and C are nonzero constants. Up to scale transformations, there exist three inequivalent triples (A, B, C) such that the corresponding equations of the form (3.74) are integrable. These are the triples $(10, 20, 30)$, $(15, 15, 45)$ and $(10, 25, 20)$ [196], which respectively give

- Lax's fifth-order KdV equation [184]

$$u_t + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x + u_{xxxxx} = 0; \tag{3.75}$$

- the Sawada–Kotera equation [264] (equivalent to the Caudrey–Dodd–Gibbon equation [54])

$$u_t + 15uu_{xxx} + 15u_xu_{xx} + 45u^2u_x + u_{xxxxx} = 0; \tag{3.76}$$

- the Kaup–Kupershmidt equation [155]

$$u_t + 10uu_{xxx} + 25u_xu_{xx} + 20u^2u_x + u_{xxxxx} = 0. \tag{3.77}$$

Corollary 3.42. *The usual equivalence group $G^\sim_{\rm const}$ of the class (3.74) consists of the transformations*

$$\tilde t = \beta^5 t + \delta, \quad \tilde x = \beta x + \gamma, \quad \tilde u = \frac{u}{\beta^2\lambda},$$
$$\tilde A = \lambda A, \quad \tilde B = \lambda B, \quad \tilde C = \lambda^2 C.$$

Here β, γ, δ, and λ are arbitrary constants with $\beta\lambda \neq 0$. This group generates the entire equivalence groupoid $\mathcal{G}^\sim_{\rm const}$ of the class (3.74), i.e., the class (3.74) is normalized in the usual sense.

In view of this assertion it is obvious, e.g., that the Caudrey–Dodd–Gibbon equation in which $(\tilde{A}, \tilde{B}, \tilde{C}) = (30, 30, 180)$ is similar to the Sawada–Kotera equation (3.76). The similarity between the equations is realized by the scale transformation $\tilde{t} = t$, $\tilde{x} = x$, $\tilde{u} = \frac{1}{2}u$.

Theorem 3.43. *An equation from the class* (3.71) *is similar to a constant-coefficient equation of the form* (3.74) *with $ABC \neq 0$ if and only if its coefficients satisfy the conditions*

$$\begin{aligned} &\left(\frac{b}{a}\right)_t = \left(\frac{b^2}{cg}\right)_t = \left(\frac{f}{c}e^{\int m\,\mathrm{dt}}\right)_t = 0, \\ &\left(\frac{b}{g}\right)_t = \frac{b}{g}(m-2k), \quad af = 2ch. \end{aligned} \tag{3.78}$$

The coefficients of all integrable equations considered in [318, 326, 327] (except the family of equations from [318] with coefficients presented in Case II) satisfy conditions (3.78). Therefore, these equations are similar to constant-coefficient ones. Namely, the equation [327]

$$\begin{aligned} &u_t + 15g\Upsilon uu_{xxx} + 15g\Upsilon u_xu_{xx} + 45g\Upsilon^2u^2u_x + gu_{xxxxx} \\ &\qquad + mu + nu_x + kxu_x = 0, \end{aligned} \tag{3.79}$$

where $\Upsilon = \nu e^{\int (m-2k)\,\mathrm{dt}}$ and ν is a nonzero constant, is mapped to the Sawada–Kotera equation (3.76) by the transformation that differs from (3.73) in the additional scaling of u by ν,

$$\tilde{t} = \textstyle\int ge^{-5\int k\,\mathrm{dt}}\,\mathrm{dt}, \quad \tilde{x} = e^{-\int k\,\mathrm{dt}}x - \textstyle\int ne^{-\int k\,\mathrm{dt}}\,\mathrm{dt}, \quad \tilde{u} = \nu e^{\int m\,\mathrm{dt}}u. \tag{3.80}$$

The same transformation maps the equation (3.71) with $f = h = 0$ and a, b, and c given by

$$\begin{aligned} &a = 10g\Upsilon, \quad b = 20g\Upsilon, \quad c = 30g\Upsilon^2 \qquad \text{or} \\ &a = 10g\Upsilon, \quad b = 25g\Upsilon, \quad c = 20g\Upsilon^2 \end{aligned}$$

to the constant-coefficient integrable equations (3.75) or (3.77), respectively. Therefore, these two integrable cases were missed in [327].

Another point transformation of the form

$$\tilde{t} = \frac{1}{5\mu_1}\int ae^{-\int m\,\mathrm{d}t}\,\mathrm{d}t, \quad \tilde{x} = x - \int\left(n - \frac{\mu_2^2}{\mu_1}ae^{-\int m\,\mathrm{d}t}\right)\mathrm{d}t,$$
$$\tilde{u} = \kappa\left(e^{\int m\,\mathrm{d}t}u + \frac{\mu_2}{\mu_1}\right)$$

with $\kappa = \mu_1/3$ (resp. $\kappa = \mu_1/2$) maps equations from the class (3.71) with $k = 0$ and the other coefficients satisfying conditions I (resp. III) to the Sawada–Kotera equation (3.76) (resp. the Kaup–Kupershmidt equation (3.77)).

The equation (3.71) with $k = 0$ and coefficients presented in Case II is reduced to the variable coefficient equation

$$\begin{aligned} &u_t + 10uu_{xxx} + 20u_xu_{xx} \\ &\quad + 30u^2u_x + u_{xxxxx} + \psi(t)(6uu_x + u_{xxx}) = 0, \end{aligned} \tag{3.81}$$

where $\psi(t) = \frac{10\mu_1}{3}\frac{h}{a}e^{\int m\,\mathrm{d}t}$, by the transformation

$$\tilde{t} = \frac{3}{10\mu_1}\int ae^{-\int m\,\mathrm{d}t}\,\mathrm{d}t, \quad \tilde{x} = x - \int n\,\mathrm{d}t, \quad \tilde{u} = \frac{\mu_1}{3}e^{\int m\,\mathrm{d}t}u.$$

The equation (3.81) is integrable since it is a "linear superposition", with time-dependent coefficients, of Lax's fifth-order KdV equation (3.75) and the classical KdV equation $u_t + 6uu_x + u_{xxx} = 0$, which are integrable and whose associated evolution vector fields commute [84, Theorem 3.1].

Using point transformations we have explained the appearance of all integrable cases found in [318, 326, 327] and have found that two integrable cases were missed in [326, 327]. Therefore, all these variable coefficient integrable equations could be found using equivalence transformations.

3.5.3. Applications of Point Transformations for Finding Lax Pairs. When the similarity of integrable equations from the class (3.71) to well-known integrable equations is established, the further consideration is needless as all objects related to general integrable equations from the class (3.71) and their properties can be easily derived from those of the classical similar equations using the similarity. We demonstrate this derivation only for Lax pairs, for which the special structure of transformations from $G^{\sim}$ is essential, since the same procedure, e.g., for symmetries, conservation laws and exact solutions is already absolutely conventional.

The Sawada–Kotera equation (3.76) admits the Lax pairs

$$\begin{aligned}
&L = {\partial_x}^3 + 3u\partial_x,\\
&P = 9{\partial_x}^5 + 45u{\partial_x}^3 + 45u_x{\partial_x}^2 + 15(2u_{xx} + 3u^2)\partial_x;
\end{aligned}$$

$$\begin{aligned}
&L = {\partial_x}^3 + 3u\partial_x + 3u_x,\\
&P = 9{\partial_x}^5 + 45u{\partial_x}^3 + 90u_x{\partial_x}^2 + 15(5u_{xx} + 3u^2)\partial_x + 30(u_{xxx} + 3uu_x).
\end{aligned}$$

Carrying out the transformation (3.80) in the associated spectral problems, $L\psi = \lambda\psi$, $\psi_t = P\psi$, we derive the corresponding Lax pairs for the variable coefficient equation (3.79),

$$\begin{aligned}
&L = e^{3\int k\,\mathrm{d}t}({\partial_x}^3 + 3\Upsilon u\partial_x),\\
&P = 9g{\partial_x}^5 + 45g\Upsilon u{\partial_x}^3 + 45g\Upsilon u_x{\partial_x}^2 + (30g\Upsilon u_{xx} + 45g\Upsilon^2u^2 - kx - n)\partial_x;
\end{aligned}$$

$$\begin{aligned}
&L = e^{3\int k\,\mathrm{d}t}({\partial_x}^3 + 3\Upsilon u\partial_x + 3\Upsilon u_x),\\
&P = 9g{\partial_x}^5 + 45g\Upsilon u{\partial_x}^3 + 90g\Upsilon u_x{\partial_x}^2\\
&\qquad + (75g\Upsilon u_{xx} + 45g\Upsilon^2u^2 - kx - n)\partial_x + 30g\Upsilon u_{xxx} + 90g\Upsilon^2uu_x,
\end{aligned}$$

respectively. Here and in what follows we again use the notation $\Upsilon = \nu e^{\int (m-2k)\,\mathrm{d}t}$ with a nonzero constant ν.

The Kaup–Kupershmidt equation (3.77) admits the Lax pair

$$\begin{aligned}
&L = {\partial_x}^3 + 2u\partial_x + u_x,\\
&P = 9{\partial_x}^5 + 30u{\partial_x}^3 + 45u_x{\partial_x}^2 + 5(7u_{xx} + 4u^2)\partial_x + 10(u_{xxx} + 2uu_x).
\end{aligned}$$

Using the transformation (3.80) it is possible to derive the corresponding Lax pair for the equation

$$u_t + 10g\Upsilon uu_{xxx} + 25g\Upsilon u_xu_{xx} + 20g\Upsilon^2u^2u_x + gu_{xxxxx}$$

$$+ mu + nu_x + kxu_x = 0,$$

which is of the form

$$L = e^{3\int k\,\mathrm{d}t}({\partial_x}^3 + 2\Upsilon u\partial_x + \Upsilon u_x),$$
$$P = 9g{\partial_x}^5 + 30g\Upsilon u{\partial_x}^3 + 45g\Upsilon u_x{\partial_x}^2$$
$$+ (35g\Upsilon u_{xx} + 20g\Upsilon^2u^2 - kx - n)\partial_x + 10g\Upsilon u_{xxx} + 20g\Upsilon^2uu_x.$$

Note that the above Lax pairs, which are constructed using the similarity to well-known constant-coefficient integrable equations, are still associated with isospectral problems in contrast to, e.g., Lax pairs constructed in [327] directly for variable coefficient equations.

Discussion. As we have shown, equivalence transformations fit well into the study of integrability of variable coefficient PDEs, where they can be used for several targets:

- to look for inessential arbitrary elements of the class of variable coefficient PDEs under consideration and to gauge these elements to chosen simple values from the very beginning;
- to establish the similarity of integrable variable coefficient PDEs, which are separated by another method (e.g., the Painlevé test) from the class under consideration, to well-known (usually, constant-coefficient) integrable equations; or, more generally, to select canonical representatives in the obtained list of integrable equations;
- to check listed integrable cases using the established similarity to previously known integrable equations;
- to derive all objects related to a singled out integrable equation and their properties from those of a similar well-studied integrable equation; such objects include, but are not exhausted by, local symmetries, cosymmetries, conservation laws, recursion operators, Bäcklund transformations, exact solutions, the Painlevé expansion, bilinear representations and Lax pairs.

Chapter 4

Algebraic Method of Group Classification and its Extensions

Group classification is concerned with finding an exhaustive list of inequivalent equations from a class of differential equations containing one or more arbitrary elements. It was originally motivated from theoretical physics, where traditionally those equations admitting the maximal number of symmetries among equations from a given class yield the most promising model describing real-world phenomena. Mathematically, group classification problems for classes of differential equations have been intensively investigated, starting with Sophus Lie's classifications of second-order ordinary differential equations [188] and of second-order linear partial differential equations with two independent variables [186]. Recently, a number of novel techniques of group classification have been introduced, which include various flavors of the algebraic method [21, 72, 178, 222, 248] and of the advanced modification of the direct method called the method of furcate splitting [23, 209, 224]. The algebraic method of group classification has proven so far to be the most powerful since it has been efficiently applied to classes of differential equations with arbitrary elements that are functions of several arguments.

Among the classes considered in the literature on group classification, the most prominent ones are classes of (1+1)-dimensional evolution equations, see e.g. [7, 17, 18, 24, 38, 71, 72, 101, 109, 124, 125, 131, 138, 192, 222–224, 227, 245, 289, 294, 297, 300] and references therein. It is thus also no coincidence that in the field of invariant discretization, which is concerned with deriving numerical schemes for differential equations possessing the same symmetries as the original, undiscretized equation, mostly evolutionary equations have been considered in the past, see e.g. [26]. What distinguishes evolutionary equations from the symmetry-perspective is the special role of the time variable, which is similar to the role of a parameter. Thus, the time component of any point or contact transformation between evolution equations only depends on the time variable [160, 192]. This considerably simplifies the classification procedure.

The complete group classification of the class of mKdV equations with time-dependent coefficients $u_t + u^2u_x + g(t)u_{xxx} + h(t)u = 0$, $g \neq 0$, is carried out in Section 4.1 using the standard algebraic method. Then the equivalence method is applied to the group classification of related classes of mKdV-like equations with variable coefficients. We prove that the classes under consideration are normalized. This allows us to formulate the classification results in three ways: up to two kinds of equivalence, which are respectively generated by the corresponding equivalence groups and by all admissible point transformations, and using no equivalence. Some exact solutions of mKdV-like equations are also constructed.

In Section 4.2 we extend the algebraic method of group classification to non-normalized classes of differential equations. Enhancing and essentially generalizing previous results on a class on (1+1)-dimensional nonlinear wave and elliptic equations, we exhaustively describe its equivalence groupoid. Then the complete group classification problem for the class under study is achieved up to both usual and general point equivalences. The solution includes the complete preliminary group classification of the class and the construction of singular Lie-symmetry extensions, which are not related to subalgebras of the equivalence algebra. The complete preliminary group classification is based on classifying appropriate subalgebras of the entire infinite-dimensional equivalence algebra whose projections are qualified as maximal extensions of the kernel invariance algebra.

A preliminary study of Lie symmetries in a class of nonlinear Dirac equations in two spatial dimensions is given in Section 4.3, using a version of the algebraic approach. A complete list of inequivalent nonlinearities for which such equations admit one-dimensional extensions of the kernel Lie invariance algebras is presented. Some solutions for the equations under study are constructed.

The results of this chapter are based on works [20*,21*,24*,27*].

4.1. Lie Symmetries and Exact solutions of Variable Coefficient mKdV Equations

In this section we investigate Lie symmetry properties and exact solutions of variable coefficient mKdV equations of the form

$$u_t + u^2 u_x + g(t) u_{xxx} + h(t) u = 0, \tag{4.1}$$

where g and h are arbitrary smooth functions of the variable t, $g \neq 0$. It is shown that using equivalence transformations the function h can be always set to the zero value and therefore the form of h does not affect results of group classification. So, at first we carry out the exhaustive group classification of the subclass of class (4.1) singled out by the condition $h = 0$. Then using the classification list obtained and equivalence transformations we present group classification of the initial class (4.1).

Moreover, equivalence transformations appear to be powerful enough to present the group classification for much wider class of variable coefficient mKdV equations (3.20), where all parameters are smooth functions of the variable t, $fg \neq 0$ and the parameters f, h, k and l satisfy the condition

$$2lf = k_t + kh - k\frac{f_t}{f}, \tag{4.2}$$

i.e. the equations

$$\begin{aligned} &u_t + f(t)u^2 u_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x \\ &\qquad + k(t)uu_x + \frac{1}{2f(t)}(\dot{k}(t) + k(t)h(t) - k(t)\dot{f}(t)/f(t)) = 0. \end{aligned} \tag{4.3}$$

This result can be easily obtained due to the fact that the group classification problem for class (3.20) can be reduced to the similar problem for class (4.1) with $h = 0$ if and only if condition (4.2) holds. Namely, equations (3.20) whose coefficients satisfy (4.2) are transformed to equations from class (4.1) with $h = 0$ by the point transformations (see Remark 1 for details). Equations from class (3.20) are important for applications and, in particular, describe atmospheric blocking phenomenon [282].

The above classes of differential equations is normalized, i.e., all admissible point transformations within these classes are generated by transformations from the corresponding equivalence groups. Therefore, there are no additional equivalence transformations between cases of the classification lists, which are constructed using the equivalence relations associated with the corresponding equivalence groups. In other words, the same lists represent the group classification results for the corresponding classes up to the general equivalence with respect to point transformations.

Recently the authors of [146] obtained a partial group classification of class (4.1) (the notation a and b was used there instead of h and g, respectively.) The reason of failure was neglecting an opportunity to use equivalence transformations. This is why only some cases of Lie symmetry extensions were found, namely the cases with $h = \text{const}$ and $h = 1/t$.

In this section we at first carry out the group classification problems for classes (4.1) and (4.3) up to the respective equivalence groups. Then using the obtained classification lists and equivalence transformations we present group classifications of these classes without the simplification of both equations admitting extensions of Lie symmetry algebras and these algebras themselves by equivalence transformations. The extended classification lists can be useful for applications and convenient to be compared with the results of [146].

Then we show how equivalence transformations can be used to construct exact solutions for those equations from class (4.3) and its subclass (4.1) which are reducible to the standard mKdV equation.

4.1.1. Equivalence Transformations. We find the equivalence group $G_1^\sim$ of class (4.1) using the results obtained in [251] for more general class of variable coefficient mKdV equations. Namely, in [251] a hierarchy of normalized subclasses of the general third-order evolution equations was constructed. The equivalence group for normalized class of variable coefficient mKdV equations (3.20) as well as criterion of reducibility of equations from this class to the standard mKdV equation were found therein.

The equivalence group $G^\sim$ of class (3.20) consists of the transformations

$$\tilde t = \alpha(t), \quad \tilde x = \beta(t)x + \gamma(t), \quad \tilde u = \theta(t)u + \psi(t), \tag{4.4}$$

where α, β, γ, θ and ψ run through the set of smooth functions of t, $\alpha_t\beta\theta \neq 0$. The arbitrary elements of (3.20) are transformed by the formulas (3.21) The criterion of reducibility to the standard mKdV equation obtained in [251] adduced in Proposition 3.15.

Class (4.1) is a subclass of class (3.20) singled out by the conditions $f = 1$ and $p = q = k = l = 0$. Substituting these values of the functions f, p, q, k and l to (3.22) we obtain the following assertion.

Corollary 4.1. *An equation from class (4.1) is reduced to the standard mKdV equation by a point transformation if and only if*

$$2h = -\frac{g_t}{g},$$

i.e. if and only if $g(t) = c_0 \exp(-2\int h(t)\mathrm{d}t)$, *where* c_0 *is an arbitrary nonzero constant.*

As class (3.20) is normalized [251], its equivalence group $G^\sim$ generates the entire set of admissible (form-preserving) transformations for this class. Therefore, to describe of the set of admissible transformations for class (4.1) we should set $\tilde f = f = 1$, $\tilde p = p = \tilde q = q = \tilde k = k = \tilde l = l = 0$ in (3.21) and solve the resulting equations with respect to transformation parameters.

It appears that projection of the obtained transformations on the space of the variables t, x and u can be applied to an arbitrary equation from class (4.1). It means that set of admissible transformations of class (4.1) is generated by transformations from its equivalence group and therefore this class is also normalized.

Summing up the above consideration, we formulate the following theorem.

Theorem 4.2. *Class* (4.1) *is normalized. The equivalence group* $G_1^{\sim}$ *of this class consists of the transformations*

$$\tilde{t} = \beta \int \frac{\mathrm{d}t}{\theta(t)^2}, \quad \tilde{x} = \beta x + \gamma, \quad \tilde{u} = \theta(t) u,$$
$$\tilde{h} = \frac{\theta}{\beta} \left(\theta h - \theta_t \right), \quad \tilde{g} = \beta^2 \theta^2 g,$$

where β *and* γ *are arbitrary constants,* $\beta \neq 0$ *and the function* θ *is an arbitrary nonvanishing smooth function of the variable* t.

The parameterization of transformations from the equivalence group $G_1^{\sim}$ by the arbitrary function $\theta(t)$ allows us to simplify the group classification problem for class (4.1) via reducing the number of arbitrary elements. For example, we can gauge arbitrary elements via setting either $h = 0$ or $g = 1$. Thus, the gauge $h = 0$ can be made by the equivalence transformation

$$\tilde{t} = \int e^{-2\int h(t)\,\mathrm{d}t} \mathrm{d}t, \quad \tilde{x} = x, \quad \tilde{u} = e^{\int h(t)\,\mathrm{d}t} u, \tag{4.5}$$

that connects equation (4.1) with the equation $\tilde{u}_{\tilde{t}} + \tilde{u}^2 \tilde{u}_{\tilde{x}} + \tilde{g}(\tilde{t}) \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$. The new arbitrary element $\tilde{g}$ is expressed via g and h in the following way:

$$\tilde{g}(\tilde{t}) = e^{2\int h(t)\,\mathrm{d}t} g(t).$$

This is why without loss of generality we can restrict the study to the class

$$u_t + u^2 u_x + g(t) u_{xxx} = 0, \tag{4.6}$$

since all results on symmetries and exact solutions for this class can be extended to class (4.1) with transformations of the form (4.5).

The equivalence group for class (4.6) can be obtained from Theorem 4.2 by setting $\tilde{h} = h = 0$. Note that class (4.6) is also normalized.

Theorem 4.3. *The equivalence group $G_0^\sim$ of class (4.6) is formed by the transformations*

$$\tilde{t} = \frac{\delta_2}{\delta_4^2}t + \delta_1, \quad \tilde{x} = \delta_2 x + \delta_3, \quad \tilde{u} = \delta_4 u, \quad \tilde{g} = \delta_2^2\delta_4^2 g,$$

where δ_j, $j = 1, \dots, 4$, are arbitrary constants, $\delta_2\delta_4 \neq 0$.

Corollary 4.4. *The equivalence algebra $\mathfrak{g}^\sim$ of class (4.6) is spanned by the operators ∂_t, ∂_x, $t\partial_t - \frac{1}{2}u\partial_u - g\partial_g$ and $t\partial_t + x\partial_x + 2g\partial_g$.*

Remark 4.5. An equation from class (3.20) is reducible to an equation from class (4.6) by a point transformation if and only if its coefficients f, h, k and l satisfy the second condition of (3.22), i.e., condition (4.2). The corresponding transformation from $G^\sim$ has the form

$$\begin{aligned} &\tilde{t} = \textstyle\int f e^{-\int (q+2h)\mathrm{d}t}\mathrm{d}t, \quad \tilde{x} = e^{-\int q\mathrm{d}t}x - \int \left(p - \frac{k^2}{4f}\right) e^{-\int q\mathrm{d}t}\mathrm{d}t, \\ &\tilde{u} = e^{\int h\mathrm{d}t}\left(u + \tfrac{k}{2f}\right), \quad \tilde{g} = \frac{g}{f}e^{2\int (h-q)\mathrm{d}t}. \end{aligned} \tag{4.7}$$

In particular, condition (4.2) implies that all equations from class (3.20) with $k = l = 0$ are reducible to equations from class (4.6).

4.1.2. Lie Symmetries. We at first carry out the group classification of class (4.6) up to $G_0^\sim$-equivalence. In this way we simultaneously solve the group classification problems for class (4.1) up to $G_1^\sim$-equivalence and for the class (4.3) up to $G^\sim$-equivalence (see explanations below). Then using the obtained classification lists and equivalence transformations we are able to present group classifications of classes (4.1) and (4.3) without the simplification of equations with wider Lie invariance algebras by equivalence transformations. These extended classification lists can be useful for applications and convenient to be compared with the results of [146].

Table 4.1: The group classification of the class (4.6).

no.	$g(t)$	Basis of $A^{\max}$
0	$\forall$	∂_x
1	δt^n, $n \neq 0$	∂_x, $6t\partial_t + 2(n+1)x\partial_x + (n-2)u\partial_u$
2	δe^t	∂_x, $6\partial_t + 2x\partial_x + u\partial_u$
3	δ	∂_x, ∂_t, $3t\partial_t + x\partial_x - u\partial_u$

Here $\delta = \pm 1 \bmod G_0^{\sim}$, n is an arbitrary nonzero constant.

Using the criterion of infinitesimal invariance we get the operators which generate one-parameter groups of point symmetry transformations of equations from class (4.6) have the form

$$Q = (c_1 t + c_2)\partial_t + (c_3 x + c_4)\partial_x + \frac{1}{2}(c_3 - c_1)u\partial_u,$$

and the *classifying* equation which includes arbitrary element g

$$(c_1 t + c_2)\, g_t = (3c_3 - c_1)g. \tag{4.8}$$

The study of the classifying equation leads to the following theorem.

The following statement is true.

Theorem 4.6. *The kernel $\mathfrak{g}^{\cap}$ of the maximal Lie invariance algebras of equations from class (4.6) coincides with the one-dimensional algebra $\langle\partial_x\rangle$. All possible $G_0^{\sim}$-inequivalent cases of extension of the maximal Lie invariance algebras are exhausted by Cases 1–3 of Table 4.1.*

Proof. As class (4.6) is normalized, it is also convenient to use a version of the algebraic method of group classification or combine this method with the direct investigation of the classifying equation [72]. The procedure which we use is the following. We consider the projection $\mathrm{P}\mathfrak{g}^{\sim}$ of the equivalence algebra $\mathfrak{g}^{\sim}$ of class (4.6) to the space of the variables (t, x, u). It is spanned by the operators ∂_t, ∂_x, $D^t = t\partial_t - \frac{1}{2}u\partial_u$ and $D^x = x\partial_x + \frac{1}{2}u\partial_u$. For any g the maximal Lie invariance algebra of the corresponding equation from class (4.6) is a subalgebra of $\mathrm{P}\mathfrak{g}^{\sim}$ in view of the normalization of this class and contains the kernel algebra $\mathfrak{g}^{\cap} = \langle \partial_x \rangle$. The algebra $\mathrm{P}\mathfrak{g}^{\sim}$ can be represented in the form $\mathrm{P}\mathfrak{g}^{\sim} = \mathfrak{g}^{\cap} \in \mathfrak{g}^{\mathrm{ext}}$, where $\mathfrak{g}^{\cap}$ and $\mathfrak{g}^{\mathrm{ext}} = \langle D^t, D^x, \partial_t \rangle$ is an ideal and a subalgebra of $\mathrm{P}\mathfrak{g}^{\sim}$, respectively. Therefore, each extension of the kernel algebra $\mathfrak{g}^{\cap}$ is associated with a subalgebra of $\mathfrak{g}^{\mathrm{ext}}$. In other words, to classify Lie symmetry extensions in class (4.6) up to $G_0^{\sim}$-equivalence it is sufficient to classify $G_0^{\sim}$-inequivalent subalgebras of $\mathfrak{g}^{\mathrm{ext}}$ and then check what subalgebras are agreed with the classifying equation and corresponds to a maximal extension. The complete list of $G_0^{\sim}$-inequivalent subalgebras of $\mathfrak{g}^{\mathrm{ext}}$ is exhausted by the following subalgebras:

$$
\begin{aligned}
&\mathfrak{g}_0 = \{0\}, \ \mathfrak{g}^a_{1.1} = \langle D^t + aD^x \rangle, \ \mathfrak{g}^b_{1.2} = \langle D^x + b\partial_t \rangle, \ \mathfrak{g}_{1.3} = \langle \partial_t \rangle, \\
&\mathfrak{g}_{2.1} = \langle D^t, D^x \rangle, \ \mathfrak{g}^a_{2.2} = \langle D^t + aD^x, \partial_t \rangle, \ \mathfrak{g}_{2.3} = \langle D^x, \partial_t \rangle, \\
&\mathfrak{g}_3 = \langle D^t, D^x, \partial_t \rangle,
\end{aligned}
$$

where the parameter b can be scaled to any appropriate value if it is nonzero. We fix a subalgebra from the above list and substitute the coefficients of each basis element of this subalgebra into the classifying equation (4.8). As a result, we obtain a system of ordinary differential equations with respect to the arbitrary element g. The systems associated with the subalgebras $\mathfrak{g}^0_{1.2}$, $\mathfrak{g}^a_{2.2}$, where $a \neq 1/3$, $\mathfrak{g}_{2.3}$ and $\mathfrak{g}_3$ are not consistent with the condition $g \neq 0$. The extensions given by the subalgebras $\mathfrak{g}_{1.3}$ and $\mathfrak{g}^{1/3}_{1.1}$ are not maximal since the maximal Lie invariance algebra in the case $g_t = 0$ coincides with $\mathfrak{g}_3$. The subalgebras $\mathfrak{g}_0$, $\mathfrak{g}^a_{1.1}$, $\mathfrak{g}^b_{1.2}$ and $\mathfrak{g}^{1/3}_{2.2}$, where $a \neq 1/3$ and $b \neq 0$, correspond to cases 0, 1, 2 and 3, respectively. The parameter n appearing in case 2 is connected with the parameter a via the formula $n = 3a - 1$, in case 3 the parameter b is scaled to the value $b = 3$. □

For any equation from class (4.1) there exists an imaged equation in class (4.6) with respect to transformation (4.5) (resp. in class (4.3) with respect to transformation (4.7)). The equivalence group $G_0^{\sim}$ of class (4.6) is induced by the equivalence group $G_1^{\sim}$ of class (**??**)♣which, in turn, is induced by the equivalence group $G^{\sim}$ of class (3.20). These guarantee that Table 1 presents also the group classification list for class (4.1) up to $G_1^{\sim}$-equivalence (resp. for the class (4.3) up to $G^{\sim}$-equivalence). As all of the above classes are normalized, we can state that we obtain Lie symmetry classifications of these classes up to general point equivalence. This leads to the following corollary of Theorem 3.

Corollary 4.7. *An equation from class* (4.1) *(resp. class* (3.20)*) admits a three-dimensional Lie invariance algebra if and only if it is reduced by a point transformation to constant coefficient mKdV equation, i.e., if and only if $g(t) = c_0 \exp(-2\int h(t)\mathrm{d}t)$, where c_0 is an arbitrary nonzero constant (resp. if and only if conditions* (3.22) *hold).*

To derive group classification of class (4.1) which are not simplified by equivalence transformations, we at first apply equivalence transformations from the group $G_0^{\sim}$ to the classification list presented in Table 4.1 and obtain the following extended list:

0. arbitrary $\tilde g$: $\langle\partial_{\tilde x}\rangle$;
1. $\tilde g = c_0(\tilde t + c_1)^n$: $\langle\partial_{\tilde x},\, 6(\tilde t + c_1)\partial_{\tilde t} + 2(n+1)\tilde x\partial_{\tilde x} + (n-2)\tilde u\partial_{\tilde u}\rangle$;
2. $\tilde g = c_0 e^{m\tilde t}$: $\langle\partial_{\tilde x},\, 6\partial_{\tilde t} + 2m\tilde x\partial_{\tilde x} + m\tilde u\partial_{\tilde u}\rangle$;
3. $\tilde g = c_0$: $\langle\partial_{\tilde x},\, \partial_{\tilde t},\, 3\tilde t\partial_{\tilde t} + \tilde x\partial_{\tilde x} - \tilde u\partial_{\tilde u}\rangle$.

Here c_0, c_1, m and n are arbitrary constants, $c_0 mn \neq 0$.

Then we find preimages of equations from class $\tilde u_{\tilde t} + \tilde u^2\tilde u_{\tilde x} + \tilde g(\tilde t)\tilde u_{\tilde x\tilde x\tilde x} = 0$ with arbitrary elements collected in the above list with respect to transformation (4.5). The last step is to transform basis operators of the corresponding Lie symmetry algebras. The results are presented in Table 4.3.

Now it is easy to see that Table 4.3 includes all cases presented in [146] as partial cases.

In a similar way, using transformations (4.7) we obtain group classification of class (4.3) without simplification by equivalence transformations. The corresponding results are collected in Table 4.5.

Table 4.3: The group classification of the class (4.1).

no.	$h(t)$	$g(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_x
1	$\forall$	$c_0\left(\int e^{-2\int h\,\mathrm{d}t}\mathrm{d}t+c_1\right)^n e^{-2\int h\mathrm{d}t}$	∂_x, $H\partial_t+2(n+1)x\partial_x+(n-2-hH)u\partial_u$
2	$\forall$	$c_0e^{\int\left(me^{-2\int h\,\mathrm{d}t}-2h\right)\mathrm{d}t}$	∂_x, $6e^{2\int h\mathrm{d}t}\partial_t+2mx\partial_x+\left(m-6he^{2\int h\mathrm{d}t}\right)u\partial_u$
3	$\forall$	$c_0e^{-2\int h\mathrm{d}t}$	∂_x, $e^{2\int h\mathrm{d}t}\left(\partial_t-hu\partial_u\right)$, $H\partial_t+2x\partial_x-(2+hH)u\partial_u$

Here c_0, c_1, m and n are arbitrary constants with $c_0mn\neq0$, and $H=6e^{2\int h\mathrm{d}t}\left(\int e^{-2\int h\mathrm{d}t}\mathrm{d}t+c_1\right)$. In case 3 $c_1=0$ in the formula for H.

4.1.3. Construction of Exact Solutions Using Equivalence Transformations. A number of recent papers concern the construction of exact solutions to different classes of KdV- or mKdV-like equations using e.g. such methods as "generalized (G'/G)-expansion method", "Exp-function method", "Jacobi elliptic function expansion method", etc. A number of references are presented in [251]. Nevertheless, almost in all cases exact solutions were constructed only for equations which are reducible to the standard KdV or mKdV equations by point transformations and usually these were only solutions similar to the well-known one-soliton solutions. In this section we show that the usage of equivalence transformations allows one to obtain more results in a simpler way.

The N-soliton solution of the mKdV equation in the canonical form

$$U_t+6U^2U_x+U_{xxx}=0 \tag{4.9}$$

were constructed as early as in the seventies using the Hirota's method [4]. The one- and two-soliton solutions of equation (4.9) have the form

$$U=a+\frac{k_0^2}{\sqrt{4a^2+k_0^2}\cosh(k_0x-k_0(6a^2+k_0^2)t+b)+2a}, \tag{4.10}$$

Table 4.5: The group classification of the class (4.3).

no.	$g(t)$	Basis of $A^{\max}$
0	$\forall$	$e^{\int q\mathrm{d}t}\partial_x$
1	$c_0 f e^{2\int(q-h)\mathrm{d}t}\left(\dfrac{H}{F}\right)^n$	$e^{\int q\mathrm{d}t}\partial_x,\ H\partial_t+\Big[(qH+2n+2)x+H\left(p-\frac{k^2}{4f}\right)$ $-2(n+1)Q\Big]\partial_x+\Big[(n-2-hH)u+\frac{k}{2f}(n-2)-lH\Big]\partial_u$
2	$c_0 f e^{\int\left(mfe^{-\int(q+2h)\,\mathrm{d}t}+2q-2h\right)\mathrm{d}t}$	$e^{\int q\mathrm{d}t}\partial_x,\ F\partial_t+\Big[(qF+2m)x+F\left(p-\frac{k^2}{4f}\right)-2mQ\Big]\partial_x$ $+\Big[(m-hF)u+\frac{m}{2}\frac{k}{f}-lF\Big]\partial_u$
3	$c_0 f e^{2\int(q-h)\mathrm{d}t}$	$e^{\int q\mathrm{d}t}\partial_x,\ F\Big[\partial_t+\left(qx+p-\frac{k^2}{4f}\right)\partial_x-(hu+l)\,\partial_u\Big],$ $H\partial_t+\Big[(qH+2)x+H\left(p-\frac{k^2}{4f}\right)-2Q\Big]\partial_x$ $-\Big[(2+hH)u+\frac{k}{f}+lH\Big]\partial_u$

The functions f, h, p, q and k are arbitrary functions of the variable t in all cases, $f\neq 0$. c_0, c_1, m and n are arbitrary constants, $c_0mn\neq 0$,

$$F=\frac{6}{f}\,e^{\int(q+2h)\mathrm{d}t},\quad H=F\left(\int fe^{-\int(q+2h)\,\mathrm{d}t}\mathrm{d}t+c_1\right),$$
$$Q=e^{\int q\mathrm{d}t}\int\left(p-\tfrac{k^2}{4f}\right)e^{-\int q\mathrm{d}t}\mathrm{d}t,\quad l=\tfrac{1}{2f}\left(k_t+kh-k\tfrac{f_t}{f}\right).$$

In case 3 $c_1=0$ in the formula for H.

$$U=\frac{e^{\theta_1}\left(1+\dfrac{A}{4a_2^2}\,e^{2\theta_2}\right)+e^{\theta_2}\left(1+\dfrac{A}{4a_1^2}\,e^{2\theta_1}\right)}{\left(\dfrac{1}{2a_1}\,e^{\theta_1}+\dfrac{1}{2a_2}\,e^{\theta_2}\right)^2+\left(1-\dfrac{A}{4a_1a_2}\,e^{\theta_1+\theta_2}\right)^2},\tag{4.11}$$

where k_0, a, b, a_i, b_i are arbitrary constants, $\theta_i=a_ix-a_i^3t+b_i$, $i=1,2$; $A=\left(\dfrac{a_1-a_2}{a_1+a_2}\right)^2$. Rational solutions which can be recovered by taking a long wave limit of soliton solutions are also known for a long time [3, 220].

Thus, the one- and two-soliton solutions give the rational solutions

$$U = a - \frac{4a}{4a^2z^2+1} \quad \text{and}$$
$$U = a - \frac{12a\left(z^4 + \dfrac{3}{2a^2}z^2 - \dfrac{3}{16a^4} - 24tz\right)}{4a^2\left(z^3+12t-\dfrac{3}{4a^2}\,z\right)^2 + 9\left(z^2+\dfrac{1}{4a^2}\right)^2}, \tag{4.12}$$

respectively, where $z = x - 6a^2t$ and a is an arbitrary constant. These solutions can be found also in [238]. Note that solution (4.11) and the second solution of (4.12) are presented in [238] with misprints.

Combining the simple transformation $\tilde{u} = \sqrt{6}U$ that connects the form (4.9) of the mKdV equation with the form

$$\tilde{u}_{\tilde{t}} + \tilde{u}^2\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0 \tag{4.13}$$

and transformation (4.5), we obtain the formula

$$u = \sqrt{6}e^{-\int h(t)\mathrm{d}t}\, U\left(\int e^{-2\int h(t)\,\mathrm{d}t}\mathrm{d}t,\, x\right).$$

Using this formula and solutions (4.10)–(4.12) we can easily construct exact solutions for the equations of the general form

$$u_t + u^2u_x + e^{-2\int h\,\mathrm{d}t}u_{xxx} + hu = 0, \tag{4.14}$$

which are preimages of (4.13) with respect to transformation (4.5). Here $h = h(t)$ is an arbitrary nonvanishing smooth function of the variable t.

For example, the two-soliton solution (4.11) leads to the following solution of (4.14)

$$u = \sqrt{6}e^{-\int h\,\mathrm{d}t}\frac{e^{\theta_1}\left(1+\dfrac{A}{4a_2^2}\,e^{2\theta_2}\right) + e^{\theta_2}\left(1+\dfrac{A}{4a_1^2}\,e^{2\theta_1}\right)}{\left(\dfrac{1}{2a_1}\,e^{\theta_1} + \dfrac{1}{2a_2}\,e^{\theta_2}\right)^2 + \left(1 - \dfrac{A}{4a_1a_2}\,e^{\theta_1+\theta_2}\right)^2},$$

where a_i, b_i are arbitrary constants, $\theta_i = a_ix - a_i^3\int e^{-2\int h\,\mathrm{d}t}\mathrm{d}t + b_i$, $i = 1, 2$; $A = \left(\frac{a_1-a_2}{a_1+a_2}\right)^2$. In a similar way one can easily construct one-soliton and rational solutions for equations from class (4.14).

More complicated transformation of the form

$$u=\sqrt{6}e^{-\int h\mathrm{d}t}\,U\Big(\int fe^{-\int(q+2h)\mathrm{d}t}\mathrm{d}t,\ e^{-\int q\mathrm{d}t}x-\int\big(p-\tfrac{k^2}{4f}\big)e^{-\int q\mathrm{d}t}\mathrm{d}t\Big)-\tfrac{k}{2f}.$$

allows us to use solutions (11)–(13) of equation (4.9) for construction of exact solutions of equations of the form

$$\begin{gathered}u_t+fu^2u_x+fe^{2\int(q-h)\mathrm{d}t}u_{xxx}+h\,u+(p+qx)u_x\\+k\,uu_x+\tfrac{1}{2f}\left(k_t+kh-k\tfrac{f_t}{f}\right)=0,\end{gathered}\tag{4.15}$$

which are preimages of equation (4.13) with respect to transformation (4.7). Here f,h,k,p and q are arbitrary smooth functions of the variable t, $f\neq 0$.

For example, the solution of (4.15) obtained from the one-soliton solution (11) has the form

$$u=\sqrt{6}e^{-\int h\mathrm{d}t}\left(a+\frac{k_0^2}{\sqrt{4a^2+k_0^2}\cosh z+2a}\right)-\frac{k}{2f},$$

$z=k_0e^{-\int q\mathrm{d}t}x-k_0\int\big(p-\tfrac{k^2}{4f}\big)e^{-\int q\mathrm{d}t}\mathrm{d}t-k_0(6a^2+k_0^2)\int fe^{-\int(q+2h)\mathrm{d}t}\mathrm{d}t+b,$ where k_0,a and b are arbitrary constants. In a similar way one can easily construct other types of solutions for equations from class (4.15).

4.2. Generalization of the Algebraic Method of Group Classification: Nonlinear Wave and Elliptic Equations

Wave and elliptic equations play an important role in physics and the mathematical sciences since wave equations model the transport of quantities at finite speeds whereas elliptic equations describe stationary processes. From the symmetry perspective, such equations are challenging since all the independent variables in them enter at equal footing. Lie symmetries of wave and elliptic equations with two independent variables have also been studied extensively, see e.g. [21, 33, 34, 37, 38, 99, 122, 123, 131, 132, 179, 181, 182] and references therein. Note that the first investigations of such equations within the framework of group analysis of differential equations, which are relevant for the subject of the present paper, were carried out by Sophus Lie in the course of his classification of second-order linear partial differential equations with two independent variables [186] and in the course of his study of contact transformations between nonlinear Klein–Gordon equations of the form $d^2z/dx\,dy = F(z)$ [187]. Exact solutions constructed for nonlinear wave and elliptic equations using group-theoretical and related methods are collected, e.g., in [131, 235–237, 336].

In this section we exhaustively solve the group classification problem for the class $\mathcal{W}$ of nonlinear wave and elliptic equations of the form

$$u_{tt} = f(x,u)u_{xx} + g(x,u). \tag{4.16}$$

We need to explicitly impose two auxiliary inequalities on the arbitrary-element tuple $\theta = (f,g)$ in order to precisely describe the class $\mathcal{W}$, which is also referred to as the class (4.16) in the paper. The auxiliary inequality $f \neq 0$ is natural since equations of the form (4.16) with $f = 0$ are not true partial differential equations.[4.1] We denote by $\mathcal{W}_{\rm gen}$ the superclass of equations of the form (4.16) with $f \neq 0$. In order to guarantee nonlinearity of equations from the class $\mathcal{W}$, the definition of this class should also include the auxiliary inequality $(f_u, g_{uu}) \neq (0,0)$. The subclass $\mathcal{W}_{\rm lin}$ of linear equations in $\mathcal{W}_{\rm gen}$ is the complement of $\mathcal{W}$ in $\mathcal{W}_{\rm gen}$, $\mathcal{W} = \mathcal{W}_{\rm gen} \setminus \mathcal{W}_{\rm lin}$. The reason why we separate nonlinear and linear equations of the form (4.16) is that they are not mixed by point transformations (see Remark 4.8 below) and have quite different Lie-symmetry properties. Although linear wave and elliptic equations with two independent variables were already extensively investigated within the framework of classical symmetry analysis, see, e.g., [37, 38, 186, 226, 227], we discuss specific transformational and symmetry properties of equations from $\mathcal{W}_{\rm lin}$ in Remark 4.19 below,

[4.1] Since we work within the local framework, auxiliary inequalities on arbitrary elements are interpreted as satisfied for all values of arguments of arbitrary elements on the relevant domain.

relating them to equations from $\mathcal{W}$. The sign of f is not too essential in the course of group classification of the class $\mathcal{W}$. In fact, we classify both the subclass of hyperbolic equations for which $f > 0$ and the subclass of elliptic equations with $f < 0$. Hyperbolic and elliptic equations are also not mixed by point transformations. Note that the consideration is local and all values are real throughout the paper although the transition to the complex case needs only minor modifications.

Following [7, 132], a so-called partial preliminary group classification problem [21, 72] for the class $\mathcal{W}$ has been considered in [272]. Specifically, the authors selected a six-dimensional subalgebra $\mathfrak{g}_6$ of the infinite-dimensional equivalence algebra $\mathfrak{g}^\sim$ of the class $\mathcal{W}$ and tried to only classify one-dimensional subalgebras of the subalgebra $\mathfrak{g}_6$ up to the equivalence generated by the corresponding six-dimensional subgroup G_6 of the infinite-dimensional equivalence (pseudo)group $G^\sim$ of the class (4.16). The G_6-equivalence is much weaker than the $G^\sim$-equivalence. This is why the classification in [272] led to an excessively large list of 24 G_6-equivalent simplest classification cases of one-dimensional Lie-symmetry extensions most of which are $G^\sim$-equivalent to each other and, up to $G^\sim$-equivalence, fit into the first four cases of Table 4.6 below. Moreover, a number of classification cases were missed even within the posed partial preliminary group classification problem.

We enhance and substantially generalize the results of [272]. The class $\mathcal{W}$ is neither normalized nor semi-normalized in any sense (the usual, the generalized or the extended ones). It cannot be partitioned into normalized or semi-normalized subclasses that are not related by point transformations. There is no mapping of it by families of point transformations to a class with better transformational properties. This is why Lie symmetries of equations from the class $\mathcal{W}$ cannot be exhaustively classified by the existing versions of the algebraic method of group classification, which are explicitly [21, 24, 72, 178, 222, 239, 248] or implicitly [17, 18, 96, 101, 109, 124, 125, 179, 181, 182, 192] based on certain normalization properties of classified classes. (Note that most of the above papers are devoted to group classifications of various classes of single (1+1)-dimensional evolution equations.) On the other hand, the class $\mathcal{W}$ is not convenient to be considered within the framework of the direct method of group classification [7, 38, 71, 131, 226, 227], including its advanced versions like the method of furcate splitting suggested in [209]. The last method is especially efficient for classes of differential equations with arbitrary elements depending on single arguments [122, 123, 138, 224, 245, 289, 297], although it has also been applied to classes whose arbitrary elements depend on two arguments [23, 209]. Various specific algebraic techniques were suggested for group classification of classes such that sets of certain objects related to Lie symmetries of equations from these classes can be endowed with Lie-algebra structures [23, 207, 244] but this is not applicable for the class $\mathcal{W}$.

This is why to efficiently solve the complete group classification problem for the class $\mathcal{W}$, we develop a new version of the algebraic method of group classification for non-normalized classes of (systems of) differential equations, which is based on classifying admissible transformations of the class under study up to their equivalence generated by the equivalence group of this class. In Chapter 1 we revisited the general framework of the classification of admissible transformations via modifying its basic notion of equivalent admissible transformations and introducing the notion of generating sets for equivalence groupoids. Several new techniques for classifying admissible transformations of non-normalized classes are also suggested. More specifically, we show that the method of furcate splitting and the algebraic method for computing the complete point or contact symmetry groups of single systems of differential equations [127–129] and the complete equivalence groups of classes of such systems [22] (including discrete symmetry and equivalence transformations) can be extended to admissible transformations. We also use the unexpected opportunity of describing admissible transformations via establishing of a functor between the equivalence groupoids of classes that are not related by families of point transformations. Revisiting the algebraic method of group classification, we introduce the notions of regular and singular Lie-symmetry extensions for a class of differential equations, $\mathcal{L}|_\mathcal{S}$. Regular Lie-symmetry extensions are associated with subalgebras of the equivalence algebra of the class $\mathcal{L}|_\mathcal{S}$. They are the extensions that can be constructed by the algebraic method in the course of the complete preliminary group classification [21,72] of $\mathcal{L}|_\mathcal{S}$. Singular Lie-symmetry extensions can involve only systems from $\mathcal{L}|_\mathcal{S}$ being sources of admissible transformations of $\mathcal{L}|_\mathcal{S}$ that are not generated by equivalence transformations of $\mathcal{L}|_\mathcal{S}$. As a result, the group classification problem for the class $\mathcal{W}$ that originated the above studies turns into a proof-of-concept example for the new methods designed for its solution.

4.2.1. Preliminary Study of Admissible Transformations To find the complete point equivalence group $G^\sim$ of the class (4.16) (including both continuous and discrete equivalence transformations) and the equivalence groupoid $\mathcal{G}^\sim$ of this class, it is necessary to apply the direct method of computing point transformations that relate systems of differential equations. We will start our consideration with a preliminary study of admissible transformations of the superclass $\mathcal{W}_{\rm gen}$, which constitute the groupoid $\mathcal{G}^\sim_{\rm gen}$ of $\mathcal{W}_{\rm gen}$. This will also give relevant information on the group $G^\sim$ and the groupoid $\mathcal{G}^\sim$.

Denote by $\mathcal{L}_\theta$ the equation in the class $\mathcal{W}_{\rm gen}$ that corresponds to a fixed value of the arbitrary-element tuple $\theta=(f,g)$. An admissible transformation of the class $\mathcal{W}_{\rm gen}$ is a triple $(\theta,\Phi,\tilde\theta)$, where $\theta=(f,g)$ and $\tilde\theta=(\tilde f,\tilde g)$ are respectively the source and target arbitrary-element tuples for $\mathcal{T}$, and

$$\Phi\colon\quad \tilde t=T(t,x,u),\quad \tilde x=X(t,x,u),\quad \tilde u=U(t,x,u) \tag{4.17}$$

with nonvanishing Jacobian $J:=\partial(T,X,U)/\partial(t,x,u)$ is a point transformation in the space with the coordinates (t, x, u) that maps the equation $\mathcal{L}_\theta$ to the equation $\mathcal{L}_{\tilde\theta}$. Therefore, we should directly seek for all point transformations mapping a fixed equation $\mathcal{L}_\theta$ of the form (4.16) to an equation $\mathcal{L}_{\tilde\theta}$ of the same form, $\tilde u_{\tilde t\tilde t} = \tilde f(\tilde x, \tilde u)\tilde u_{\tilde x\tilde x} + \tilde g(\tilde x, \tilde u)$.

To carry out the transformation Φ in practice, it is necessary to find its prolongation to derivatives of u up to order two. For this we act by the total derivative operators D_t and D_x, respectively, on the expression $\tilde u(\tilde t, \tilde x) = U(t, x, u)$, assuming $\tilde t = T(t, x, u)$ and $\tilde x = X(t, x, u)$. This gives

$$\begin{aligned}
&\tilde u_{\tilde t}\mathrm{D}_tT + \tilde u_{\tilde x}\mathrm{D}_tX - \mathrm{D}_tU = 0, \quad \tilde u_{\tilde t}\mathrm{D}_xT + \tilde u_{\tilde x}\mathrm{D}_xX - \mathrm{D}_xU = 0,\\
&\tilde u_{\tilde t\tilde t}(\mathrm{D}_tT)^2 + 2\tilde u_{\tilde t\tilde x}(\mathrm{D}_tX)(\mathrm{D}_tT) + \tilde u_{\tilde x\tilde x}(\mathrm{D}_tX)^2 + \tilde u_{\tilde t}\mathrm{D}_t^2T + \tilde u_{\tilde x}\mathrm{D}_t^2X = \mathrm{D}_t^2U,\\
&\tilde u_{\tilde t\tilde t}(\mathrm{D}_xT)^2 + 2\tilde u_{\tilde t\tilde x}(\mathrm{D}_xX)(\mathrm{D}_xT) + \tilde u_{\tilde x\tilde x}(\mathrm{D}_xX)^2 + \tilde u_{\tilde t}\mathrm{D}_x^2T + \tilde u_{\tilde x}\mathrm{D}_x^2X = \mathrm{D}_x^2U,
\end{aligned}$$

cf. [21]. Solving the last two equations for u_{tt} and u_{xx}, respectively, and substituting the derived expressions into (4.16), we obtain

$$\begin{aligned}
&\tilde u_{\tilde t\tilde t}(\mathrm{D}_tT)^2 + 2\tilde u_{\tilde t\tilde x}(\mathrm{D}_tT)(\mathrm{D}_tX) + \tilde u_{\tilde x\tilde x}(\mathrm{D}_tX)^2 + \tilde u_{\tilde t}V^tT + \tilde u_{\tilde x}V^tX\\
&\quad - V^tU = f\big[\tilde u_{\tilde t\tilde t}(\mathrm{D}_xT)^2 + 2\tilde u_{\tilde t\tilde x}(\mathrm{D}_xT)(\mathrm{D}_xX) + \tilde u_{\tilde x\tilde x}(\mathrm{D}_xX)^2\\
&\quad + \tilde u_{\tilde t}V^xT + \tilde u_{\tilde x}V^xX - V^xU\big] - g(\tilde u_{\tilde t}T_u + \tilde u_{\tilde x}X_u - U_u),
\end{aligned} \tag{4.18}$$

where we use the notation $V^t := \partial_{tt} + 2u_t\partial_{tu} + u_t^2\partial_{uu}$ and $V^x := \partial_{xx} + 2u_x\partial_{xu} + u_x^2\partial_{uu}$. The substitution $\tilde u_{\tilde t\tilde t} = \tilde f\tilde u_{\tilde x\tilde x} + \tilde g$ in view of $\mathcal{L}_{\tilde\theta}$ into (4.18) wherever $\tilde u_{\tilde t\tilde t}$ occurs leads to an identity with respect to $\tilde u_{\tilde t\tilde x}$ and $\tilde u_{\tilde x\tilde x}$. In particular, we can collect the coefficients of $\tilde u_{\tilde t\tilde x}$ in (4.18), which results in

$$(T_t + T_uu_t)(X_t + X_uu_t) = f(T_x + T_uu_x)(X_x + X_uu_x). \tag{4.19}$$

The equation (4.19) involves only original quantities (without tilde) and is a polynomial in u_t and u_x. Therefore, we can split it with respect to u_t and u_x by collecting the coefficients of different powers of these derivatives. As a result, we get

$$T_uX_u = 0, \tag{4.20}$$

$$T_u X_t + T_t X_u = 0, \tag{4.21}$$
$$T_u X_x + T_x X_u = 0, \tag{4.22}$$
$$T_t X_t = f T_x X_x. \tag{4.23}$$

We multiply the equation (4.21) by T_u (resp. X_u) and, in view of the equation (4.20), derive that $T_u X_t = 0$ (resp. $T_t X_u = 0$). We apply the same trick also to the equation (4.22) to have the equations $T_u X_x = 0$ and (resp. $X_u T_x = 0$). The system $T_u X_t = 0$, $T_u X_x = 0$, $T_u X_u = 0$ (resp. $X_u T_t = 0$, $X_u T_x = 0$, $X_u T_u = 0$) implies that $T_u = 0$ (resp. $X_u = 0$) since otherwise the Jacobian J of the point transformation (4.17) vanishes. The condition $T_u = X_u = 0$ means that any admissible point transformation of the class (4.16) is fiber-preserving. In view of this condition, the equations (4.20)–(4.22) are identically satisfied, and the splitting of (4.18) with respect to $\tilde u_{\tilde x\tilde x}$ gives the equations

$$\tilde f {T_t}^2 + {X_t}^2 = f(\tilde f {T_x}^2 + {X_x}^2), \tag{4.24}$$

$$\begin{aligned}&\tilde g T_t^2 + \tilde u_{\tilde t} T_{tt} + \tilde u_{\tilde x} X_{tt} - (U_{tt} + 2U_{tu}u_t + U_{uu}u_t^2)\\ &\qquad = f(\tilde g T_x^2 + \tilde u_{\tilde t} T_{xx} + \tilde u_{\tilde x} X_{xx} - (U_{xx} + 2U_{xu}u_x + U_{uu}u_x^2)) + gU_u.\end{aligned}$$

Splitting the last equation with respect to $\tilde u_{\tilde t}$ and $\tilde u_{\tilde x}$ gives the equations $U_{uu} = 0$ and

$$T_{tt} - 2\frac{U_{ut}}{U_u}T_t = f\left(T_{xx} - 2\frac{U_{ux}}{U_u}T_x\right), \tag{4.25}$$
$$X_{tt} - 2\frac{U_{ut}}{U_u}X_t = f\left(X_{xx} - 2\frac{U_{ux}}{U_u}X_x\right), \tag{4.26}$$
$$\tilde g T_t^2 - U_{tt} + 2\frac{U_{ut}}{U_u}U_t = f\left(\tilde g T_x^2 - U_{xx} + 2\frac{U_{ux}}{U_u}U_x\right) + gU_u. \tag{4.27}$$

The equation $U_{uu} = 0$ implies the representation $U = U^1(t,x)u + U^0(t,x)$. The additional condition to keep in mind is the nondegeneracy of Φ, which in view of the conditions $T_u = X_u = 0$ reduces to the inequality $U_u(T_tX_x - T_xX_t) \neq 0$, and hence $T_tX_x - T_xX_t \neq 0$ and $U^1 := U_u \neq 0$. Note that the equations $T_u = X_u = U_{uu} = 0$ for admissible point transformations within the class $\mathcal{W}_{\rm gen}$ can also be derived using item (c) of Theorem 4.4b in [160]. Rewriting the equation (4.24) as $\tilde f({T_t}^2 - f{T_x}^2) + {X_t}^2 - f{X_x}^2 = 0$ shows that both the expressions ${T_t}^2 - f{T_x}^2$ and ${X_t}^2 - f{X_x}^2$ are nonzero,

$$ {T_t}^2 - f{T_x}^2 \neq 0, \quad {X_t}^2 - f{X_x}^2 \neq 0. $$

Indeed, otherwise $f > 0$, $T_t = \varepsilon_1 f^{1/2} T_x \neq 0$, where $\varepsilon_1 = \pm 1$, and hence the equation (4.23) would imply that $X_t = \varepsilon_1 f^{1/2} X_x$ but this contradicts the nondegeneracy of Φ. Thus, $\tilde f_{\tilde u} = 0$ if $f_u = 0$. Conversely, supposing $\tilde f_{\tilde u} = 0$ and using the same argumentation for the inverse of $\mathcal{T}$, we derive that $f_u = 0$. Therefore, $\tilde f_{\tilde u} = 0$ if and only if $f_u = 0$.

In view of nonvanishing the expression ${T_t}^2 - f{T_x}^2$, the equation (4.27) similarly implies that $\tilde f_{\tilde u} = \tilde g_{\tilde u \tilde u} = 0$ if and only if $f_u = g_{uu} = 0$.

Remark 4.8. Preserving the constraint $f_u = g_{uu} = 0$ by admissible transformations of the class $\mathcal{W}_{\rm gen}$ means that the equivalence groupoid $\mathcal{G}^\sim_{\rm gen}$ of the class $\mathcal{W}_{\rm gen}$ is the disjoint union of the equivalence groupoids $\mathcal{G}^\sim$ and $\mathcal{G}^\sim_{\rm lin}$ of the subclasses $\mathcal{W}$ and $\mathcal{W}_{\rm lin}$, $\mathcal{G}^\sim_{\rm gen} = \mathcal{G}^\sim \sqcup \mathcal{G}^\sim_{\rm lin}$. In other words, equations from the class $\mathcal{W}$ are not related to equations from the class $\mathcal{W}_{\rm lin}$ by point transformations. This justifies the exclusion of the class $\mathcal{W}_{\rm lin}$ from the consideration, which has been mentioned in the introduction.

4.2.2. Equivalence Group and Equivalence Algebra At this point, we continue the consideration by computing the equivalence group of the class (4.16) as it is needed both for the exhaustive description of admissible transformations and for the analysis of the determining equations for components of Lie-symmetry vector fields. In the course of computing equivalence transformations, the arbitrary elements f and g of the class should be varied. We can therefore split the equations (4.23)–(4.27) with respect to these arbitrary elements. The equation (4.23) and the nondegeneracy constraint $T_tX_x - T_xX_t \neq 0$ imply that either $T_t = X_x = 0$ and $T_xX_t \neq 0$ or $T_x = X_t = 0$ and $T_tX_x \neq 0$.

For $T_t = X_x = 0$, the equation (4.24) is simplified to $X_t^2 = f\tilde{f}T_x^2$, or $T_x^2 f = X_t^2/\tilde{f}$. Differentiating the last equation with respect to t and splitting the arising equation with respect to $\tilde{f}$ and its derivatives implies $X_t = 0$, which contradicts the nondegeneracy condition.

Therefore, we necessarily have $X_t = T_x = 0$ and thus $T = T(t)$, $X = X(x)$, where $T_t X_x \neq 0$. Then the equation (4.24) reduces to $\tilde{f}T_t^2 = fX_x^2$ and the differentiation of this equation with respect to t yields

$$2T_tT_{tt}\tilde{f} + T_t^2U_t\tilde{f}_{\tilde{u}} = 0. \tag{4.28}$$

Since the equation (4.28) holds for all $\tilde{f}$, we can split it with respect to $\tilde{f}$ and $\tilde{f}_{\tilde{u}}$ and derive $T_{tt} = 0$ and $U_t = 0$. The equation (4.25) is identically satisfied in view of the above equations. The equation (4.26) reduces to the equation $(U_u^2/X_x)_x = 0$ and the equation (4.27) yields the transformation relation for g.

Integrating the derived equations in view of the nondegeneracy condition J $\neq 0$, we prove the following theorem.

Theorem 4.9. *The equivalence (pseudo)group $G^{\sim}$ of the class* (4.16) *consists of the transformations*

$$\begin{gathered}\tilde{t} = c_1t + c_0, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = c_2|\varphi_x|^{1/2}u + \psi(x), \quad \tilde{f} = \frac{\varphi_x^2}{c_1^2}f, \\ \tilde{g} = \frac{c_2}{c_1^2}|\varphi_x|^{1/2}g - \frac{1}{c_1^2}\left(c_2\frac{2\varphi_{xxx}\varphi_x - 3\varphi_{xx}^2}{4|\varphi_x|^{3/2}}u + \psi_{xx} - \frac{\varphi_{xx}}{\varphi_x}\psi_x\right)f,\end{gathered} \tag{4.29}$$

where c_0, c_1 and c_2 are arbitrary constants satisfying the condition $c_1c_2 \neq 0$, and φ and ψ run through the set of smooth functions of x with $\varphi_x \neq 0$.

Corollary 4.10. *The class* (4.16) *admits exactly three discrete equivalence transformations that are independent up to combining with each other and with continuous equivalence transformations of this class. These are involutions alternating signs of variables, $(t, x, u, f, g) \mapsto (-t, x, u, f, g)$, $(t, x, u, f, g) \mapsto (t, -x, u, f, g)$ and $(t, x, u, f, g) \mapsto (t, x, -u, f, -g)$.*

In contrast to [272], we have proved that there are no other independent discrete equivalence transformations for the class (4.16).

Theorem 4.9 implies that any transformation $\mathcal{T}$ from $G^{\sim}$ can be represented as the composition $\mathcal{T}=\mathcal{P}^t(c_0)\mathcal{D}^t(c_1)\mathcal{G}(\psi)\mathcal{D}(\varphi)\mathcal{D}^u(c_2)$ of elementary equivalence transformations, each of which belongs to a family of equivalence transformations parameterized by a single constant or functional parameter,

$$
\begin{aligned}
&\mathcal{P}^t(c_0)\colon && \tilde t=t+c_0,\ \tilde x=x,\ \tilde u=u,\ \tilde f=f,\ \tilde g=g,\\
&\mathcal{D}^t(c_1)\colon && \tilde t=c_1t,\ \tilde x=x,\ \tilde u=u,\ \tilde f=c_1^{-2}f,\ \tilde g=c_1^{-2}g,\\
&\mathcal{D}(\varphi)\colon && \tilde t=t,\ \tilde x=\varphi,\ \tilde u=|\varphi_x|^{1/2}u,\ \tilde f=\varphi_x^{2}f,\ \tilde g=|\varphi_x|^{1/2}g+\alpha^\varphi(x)uf,\\
&\mathcal{D}^u(c_2)\colon && \tilde t=t,\ \tilde x=x,\ \tilde u=c_2u,\ \tilde f=f,\ \tilde g=c_2g,\\
&\mathcal{Z}(\psi)\colon && \tilde t=t,\ \tilde x=x,\ \tilde u=u+\psi,\ \tilde f=f,\ \tilde g=g-\psi_{xx}f,
\end{aligned}
$$

where the parameters are described in Theorem 4.9, and

$$\alpha^\varphi(x):=\frac{2\varphi_{xxx}\varphi_x-3\varphi_{xx}^2}{4|\varphi_x|^{3/2}}.$$

These transformations are shifts and scalings in t, arbitrary transformations in x, scalings of u and shifts of u with arbitrary functions of x, respectively.

The equivalence algebra $\mathfrak{g}^{\sim}$ of the class (4.16) can be easily derived as the set of vector fields that generate local one-parametric subgroups of the equivalence group $G^{\sim}$. It is spanned by the vector fields

$$
\begin{aligned}
&\mathcal{P}^t=\partial_t,\quad \mathcal{D}^t=t\partial_t-2f\partial_f-2g\partial_g,\quad \mathcal{D}^u=u\partial_u+g\partial_g,\\
&\mathcal{D}(\zeta)=\zeta\partial_x+\tfrac12\zeta_xu\partial_u+2\zeta_xf\partial_f+\tfrac12(\zeta_xg-\zeta_{xxx}uf)\partial_g,\\
&\mathcal{Z}(\chi)=\chi\partial_u-\chi_{xx}f\partial_g,
\end{aligned}
\tag{4.30}
$$

where $\zeta=\zeta(x)$ and $\chi=\chi(x)$ run through the set of smooth functions of x. The nonvanishing commutation relations between these vector fields are exhausted by

$$
\begin{aligned}
&[\mathcal{P}^t,\mathcal{D}^t]=\mathcal{P}^t,\quad [\mathcal{Z}(\chi),\mathcal{D}^u]=\mathcal{Z}(\chi),\\
&[\mathcal{D}(\zeta^1),\mathcal{D}(\zeta^2)]=\mathcal{D}(\zeta^1\zeta^2_x-\zeta^1_x\zeta^2),\quad [\mathcal{D}(\zeta),\mathcal{Z}(\chi)]=\mathcal{Z}(\zeta\chi_x-\tfrac12\zeta_x\chi).
\end{aligned}
$$

4.2.3. Determining Equations for Lie Symmetries. Suppose that a vector field $Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$ belongs to the maximal Lie invariance algebra $\mathfrak{g}^{\max}$ of an equation $\mathcal{L}$: $L = 0$ from the class (4.16), i.e., it is the generator of a one-parameter Lie-symmetry group of $\mathcal{L}$.

$\tau_u = \xi_u = \eta_{uu} = 0$ or $\tau = \tau(t,x)$, $\xi = \xi(t,x)$ and $\eta = \eta^1(t,x)u + \eta^0(t,x)$. the system of determining equations

$$\xi_t = \tau_x f, \tag{4.31}$$

$$\tau_{tt} - \tau_{xx} f = 2\eta_{tu}, \tag{4.32}$$

$$\xi_{tt} - \xi_{xx} f = -2\eta_{xu} f, \tag{4.33}$$

$$\xi f_x + \eta f_u = 2(\xi_x - \tau_t) f, \tag{4.34}$$

$$\xi g_x + \eta g_u = (\eta_u - 2\tau_t) g - \eta_{xx} f + \eta_{tt}. \tag{4.35}$$

In order to derive the kernel algebra $\mathfrak{g}^\cap$ of the class (4.16), we further split the determining equations with respect to the arbitrary elements and their derivatives. This immediately gives that $\mathfrak{g}^\cap = \langle \partial_t \rangle$. Consequently, the Lie symmetries admitted by each equation from the class (4.16) are exhausted by transformations of the form $(t,x,u) \mapsto (t + c_0, x, u)$, where c_0 is an arbitrary constant.

4.2.4. Results of Classifications. For convenience, we collect, in a single table, the Lie-symmetry classification cases derived below and formulate the final result of group classification of the class (4.16) as a theorem.

Theorem 4.11. *All $G^\sim$-inequivalent (resp. point-inequivalent) cases of Lie-symmetry extensions of the kernel algebra $\mathfrak{g}^\cap = \langle \partial_t \rangle$ in the class* (4.16) *are exhausted by cases presented in Table 4.6.*

In each case of Table 4.6 we present only vector fields which extend the basis (∂_t) of $\mathfrak{g}^\cap$ into a basis of the corresponding Lie invariance algebra. The spans of $\mathfrak{g}^\cap$ and the vector fields given in cases of Table 4.6 that are parameterized by functions $\hat f$ or $\hat g$ are the maximal Lie invariance algebras

Table 4.6: $G^{\sim}$-inequivalent Lie-symmetry extensions of $\mathfrak{g}^{\cap} = \langle \partial_t \rangle$ for the class (4.16).

N	f	g	Basis of extension
1	$\hat f(\omega)\lvert u\rvert^p$	$\hat g(\omega)\lvert u\rvert^p u$	$-pt\partial_t + 2\delta\partial_x + 2u\partial_u$
2	$\hat f(u)e^x$	$\hat g(u)e^x$	$t\partial_t - 2\partial_x$
3	$\hat f(x)e^u$	$\hat g(x)e^u$	$t\partial_t - 2\partial_u$
4	$\hat f(u)$	$\hat g(u)$	∂_x
5a	ε	$\hat g(u)$	$\partial_x,\ x\partial_t + \varepsilon t\partial_x$
5b	1	$\hat g(u)e^{-2x}$	$\mathcal{R}(e^{x+t}),\ \mathcal{R}(e^{x-t})$
5c	-1	$\hat g(u)e^{-2x}$	$\mathcal{R}(e^x\cos t),\ \mathcal{R}(e^x\sin t)$
6a	ε	$\hat g(u)x^{-2}$	$t\partial_t + x\partial_x,\ (t^2+\varepsilon x^2)\partial_t + 2tx\partial_x$
6b	1	$\hat g(u)\cos^{-2}x$	$\mathcal{R}(\cos t\,\cos x),\ \mathcal{R}(\sin t\,\cos x)$
6c	1	$-\hat g(u)\cosh^{-2}x$	$\mathcal{R}(e^t\cosh x),\ \mathcal{R}(e^{-t}\cosh x)$
6d	1	$\hat g(u)\sinh^{-2}x$	$\mathcal{R}(e^t\sinh x),\ \mathcal{R}(e^{-t}\sinh x)$
6e	-1	$\hat g(u)\cos^{-2}x$	$\mathcal{R}(e^t\cos x),\ \mathcal{R}(e^{-t}\cos x)$
6f	-1	$\hat g(u)\sinh^{-2}x$	$\mathcal{R}(\cos t\,\sinh x),\ \mathcal{R}(\sin t\,\sinh x)$
7	-1	$\hat g(u)\cosh^{-2}x$	$\mathcal{R}(\cos t\,\cosh x),\ \mathcal{R}(\sin t\,\cosh x)$
8a	εu^{-4}	$\mu(x)u^{-3}$	$2t\partial_t + u\partial_u,\ t^2\partial_t + tu\partial_u$
8b	εu^{-4}	$\mu(x)u^{-3} + u$	$e^{2t}(\partial_t + u\partial_u),\ e^{-2t}(\partial_t - u\partial_u)$
8c	εu^{-4}	$\mu(x)u^{-3} - u$	$\cos(2t)\partial_t - \sin(2t)u\partial_u,\ \sin(2t)\partial_t + \cos(2t)u\partial_u$
9	$\varepsilon e^x\lvert u\rvert^p$	$\nu e^x\lvert u\rvert^p u$	$p\partial_x - u\partial_u,\ t\partial_t - 2\partial_x$
10	$\varepsilon x^2 e^u$	νe^u	$x\partial_x,\ t\partial_t - 2\partial_u$
11	$\hat f(u)$	0	$\partial_x,\ t\partial_t + x\partial_x$
12	εe^u	$\varepsilon' e^{qu}$	$\partial_x,\ qt\partial_t + (q-1)x\partial_x - 2\partial_u$
13	$\varepsilon\lvert u\rvert^p$	$\varepsilon'\lvert u\rvert^q$	$\partial_x,\ (1-q)t\partial_t + (1+p-q)x\partial_x + 2u\partial_u$
14a	εu^{-4}	$\varepsilon' u^{-3}$	$2t\partial_t + u\partial_u,\ t^2\partial_t + tu\partial_u,\ \partial_x$
14b	εu^{-4}	$\varepsilon' u^{-3} + u$	$e^{2t}(\partial_t + u\partial_u),\ e^{-2t}(\partial_t - u\partial_u),\ \partial_x$
14c	εu^{-4}	$\varepsilon' u^{-3} - u$	$\cos(2t)\partial_t - \sin(2t)u\partial_u,\ \sin(2t)\partial_t + \cos(2t)u\partial_u,\ \partial_x$
14d	εu^4	$\varepsilon' u$	$\partial_x,\ 2x\partial_x + u\partial_u,\ x^2\partial_x + xu\partial_u$

Table 2.1: Continuation.

15a	εu^{-4}	$\nu x^{-2}u^{-3}$	$2t\partial_t+u\partial_u,\ t^2\partial_t+tu\partial_u,\ 2x\partial_x-u\partial_u$
15b	εu^{-4}	$\nu x^{-2}u^{-3}+u$	$e^{2t}(\partial_t+u\partial_u),\ e^{-2t}(\partial_t-u\partial_u),\ 2x\partial_x-u\partial_u$
15c	εu^{-4}	$\nu x^{-2}u^{-3}-u$	$\cos(2t)\partial_t-\sin(2t)u\partial_u,\ \sin(2t)\partial_t+\cos(2t)u\partial_u,\ 2x\partial_x-u\partial_u$
16	$\varepsilon\|u\|^p$	0	$\partial_x,\ t\partial_t+x\partial_x,\ px\partial_x+2u\partial_u$
17	εe^u	0	$\partial_x,\ t\partial_t+x\partial_x,\ x\partial_x+2\partial_u$
18a	ε	$\varepsilon'\|u\|^q$	$\partial_x,\ t\partial_x+\varepsilon x\partial_t,\ (q-1)t\partial_t+(q-1)x\partial_x-2u\partial_u$
18b	1	$\varepsilon'\|u\|^q e^{-2x}$	$\mathcal{R}(e^{x+t}),\ \mathcal{R}(e^{x-t}),\ (q-1)\partial_x+2u\partial_u$
18c	-1	$\varepsilon'\|u\|^q e^{-2x}$	$\mathcal{R}(e^x\cos t),\ \mathcal{R}(e^x\sin t),\ (q-1)\partial_x+2u\partial_u$
19a	εu^{-4}	0	$2t\partial_t+u\partial_u,\ t^2\partial_t+tu\partial_u,\ \partial_x,\ 2x\partial_x-u\partial_u$
19b	εu^{-4}	u	$e^{2t}(\partial_t+u\partial_u),\ e^{-2t}(\partial_t-u\partial_u),\ \partial_x,\ 2x\partial_x-u\partial_u$
19c	εu^{-4}	$-u$	$\cos(2t)\partial_t-\sin(2t)u\partial_u,\ \sin(2t)\partial_t+\cos(2t)u\partial_u,\ \partial_x,\ 2x\partial_x-u\partial_u$
19d	εu^4	0	$\partial_x,\ t\partial_t+x\partial_x,\ 2x\partial_x+u\partial_u,\ x^2\partial_x+xu\partial_u$
20	ε	$\varepsilon' e^u$	$\tau\partial_t+\xi\partial_x-2\tau_t\partial_u$

Here $\varepsilon,\varepsilon'=\pm1 \bmod G^{\sim}$, $\delta\in\{0,1\} \bmod G^{\sim}$, p, q and ν are arbitrary constants with $p\neq0$ and $\nu\neq0$. Additionally, $p\neq\pm4$ in Case 16, and $q\neq0,1$ in Cases 18a–18c. In Case 1, $\omega:=x-\delta\ln|u|$. $\mathcal{R}(\Phi):=\Phi_x\partial_t+\Phi_t\partial_x$. The tuple (τ,ξ) of smooth functions depending on (t,x) runs through the solution set of the system $\tau_t=\xi_x$, $\xi_t=\varepsilon\tau_x$.

of the corresponding equations for the general values of these parameter functions $\hat f$ and $\hat g$, but for certain values of $\hat f$ and $\hat g$ additional extensions are possible, which are equivalent to other cases of Table 4.6. Thus, $\hat f\neq$ const in Case 4 since otherwise up to $G^{\sim}$-equivalence we obtain Case 5a. There are also constraints for constant parameters that are imposed by the condition of inequivalence of the corresponding extensions or the condition of their maximality.

Depending on the dimension of Lie-symmetry extension (one, two, three, four or infinity), we split the cases of Table 4.6 into groups separated by horizontal lines. Note that all Lie-symmetry extensions of maximal and submaximal dimensions (infinity and four) for equations from the class (4.16) are not associated with subalgebras of the equivalence algebra $\mathfrak{g}^\sim$, i.e., they are singular.

The usage of two-level numeration for classification cases listed in Table 4.6 is justified by the presence of additional equivalences among them. Namely, numbers with the same Arabic numerals and different Roman letters correspond to cases that are $G^\sim$-inequivalent but equivalent with respect to additional equivalence transformations. To construct all additional equivalence transformations among $G^\sim$-inequivalent classification cases and thus to solve the group classification problem for the class (4.16) up to $\mathcal{G}^\sim$-equivalence, we need to classify admissible transformations of this class up to $G^\sim$-equivalence. This classification is presented in the following theorem, which is proved in Section 4.2.5.

Theorem 4.12. *A generating (up to $G^\sim$-equivalence) set $\mathcal{B}$ of admissible transformations for the class $\mathcal{W}$, which is minimal and self-consistent with respect to $G^\sim$-equivalence, is the union of the following families of admissible transformations, where $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$:*

T1. $f_x = g_x = 0,\ \ f_u \neq 0$ *or* $f = 1,\ \ \tilde f = 1/f,\ \ \tilde g = -g/f,\ \ \Phi\colon\ \tilde t = x,\ \ \tilde x = t,\ \ \tilde u = u;$

T2. $f = \varepsilon u^{-4},\ \ g = \mu(x)u^{-3} + \sigma u,\ \ \sigma \in \{-1, 0, 1\},\ \ \tilde f = \varepsilon \tilde u^{-4},\ \ \tilde g = \mu(\tilde x)\tilde u^{-3},$ *μ runs through the set of smooth functions of x with* $\mu_x \neq 0,$

  a. $\Phi\colon\ \tilde t = t^{-1},\ \ \tilde x = x,\ \ \tilde u = t^{-1}u$ *if* $\sigma = 0;$

  b. $\Phi\colon\ \tilde t = \frac{1}{2}e^{2t},\ \ \tilde x = x,\ \ \tilde u = e^t u$ *if* $\sigma = 1;$

  c. $\Phi\colon\ \tilde t = \tan t,\ \ \tilde x = x,\ \ \tilde u = u\cos t$ *if* $\sigma = -1;$

T3. $f = 1,\ \ g = e^{-2x}g^2(u),\ \ \tilde f = 1,\ \ \tilde g = g^2(\tilde u),$ $\Phi\colon\ \tilde t = e^{-x}\sinh t,\ \ \tilde x = e^{-x}\cosh t,\ \ \tilde u = u;$

T4. $f = 1,\ \ g = g^1(x)g^2(u),\ \ \tilde f = 1,\ \ \tilde g = \tilde x^{-2}g^2(\tilde u),$

  a. $g^1(x) = x^{-2},\ \ \Phi\colon\ \tilde t = \dfrac{t}{x^2 - t^2},\ \ \tilde x = \dfrac{x}{x^2 - t^2},\ \ \tilde u = u;$

b. $g^1(x)=\cos^{-2}x$, Φ: $\tilde t=\dfrac{\cos t}{\sin t+\sin x}$, $\tilde x=\dfrac{\cos x}{\sin t+\sin x}$, $\tilde u=u$;

c. $g^1(x)=-\cosh^{-2}x$, Φ: $\tilde t=e^t\sinh x$, $\tilde x=e^t\cosh x$, $\tilde u=u$;

d. $g^1(x)=\sinh^{-2}x$, Φ: $\tilde t=e^t\cosh x$, $\tilde x=e^t\sinh x$, $\tilde u=u$;

T5. $f=-1$, $g=e^{-2x}g^2(u)$, $\tilde f=-1$, $\tilde g=g^2(\tilde u)$,
Φ: $\tilde t=e^{-x}\sin t$, $\tilde x=e^{-x}\cos t$, $\tilde u=u$;

T6. $f=-1$, $g=g^1(x)g^2(u)$, $\tilde f=-1$, $\tilde g=\tilde x^{-2}g^2(\tilde u)$,

a. $g^1(x)=x^{-2}$, Φ: $\tilde t=\dfrac{t}{x^2+t^2}$, $\tilde x=\dfrac{x}{x^2+t^2}$, $\tilde u=u$;

b. $g^1(x)=\cos^{-2}x$, Φ: $\tilde t=e^t\sin x$, $\tilde x=e^t\cos x$, $\tilde u=u$;

c. $g^1(x)=\sinh^{-2}x$, Φ: $\tilde t=\dfrac{\sin t}{\cos t+\cosh x}$, $\tilde x=\dfrac{\sinh x}{\cos t+\cosh x}$, $\tilde u=u$;

T7. $f=-1$, $g=g^2(u)\cosh^{-2}x$, $\tilde f=-1$, $\tilde g=g^2(\tilde u)\cosh^{-2}\tilde x$,

Φ: $\tilde t=\arctan\dfrac{\sin\gamma\,\sinh x+\cos\gamma\sin t}{\cos t}$,

$\tilde x=\operatorname{arctanh}\dfrac{\cos\gamma\,\sinh x-\sin\gamma\sin t}{\cosh x}$, $\tilde u=u$, $\gamma\in(0,2\pi)$;

T8. a. $f=\tilde f=1$, $g_x=0$, $\tilde g=g$,
Φ: $\tilde t=t\cosh\gamma+x\sinh\gamma$, $\tilde x=t\sinh\gamma+x\cosh\gamma$, $\tilde u=u$, $\gamma\in\mathbb{R}_{\neq 0}$;

b. $f=\tilde f=-1$, $g_x=0$, $\tilde g=g$, Φ: $\tilde t=t\cos\gamma-x\sin\gamma$,
$\tilde x=t\sin\gamma+x\cos\gamma$, $\tilde u=u$, $\gamma\in(0,2\pi)$;

T9. $f=\varepsilon$, $g=\varepsilon' e^u$, $\tilde f=\varepsilon$, $\tilde g=\varepsilon' e^{\tilde u}$, Φ: $\tilde t=T(t,x)$, $\tilde x=X(t,x)$, $\tilde u=u+\ln|T_t^2-\varepsilon T_x^2|$, *where* $\varepsilon,\varepsilon'=\pm1$, $\varepsilon'=1$ *if* $\varepsilon=1$, (T,X) *runs through a complete set of representatives of solution cosets of the system* $T_t=X_x$, $X_t=\varepsilon T_x$ *with* $(T_{tt},T_x)\neq(0,0)$ *with respect to the action of the group constituted by the transformations of the form* $\hat t=c_1t+c_0$, $\hat x=c_1x+c_2$, $\hat T=\tilde c_1T+\tilde c_0$, $\hat X=\tilde c_1X+\tilde c_2$, *where* c_0, c_1, c_2, $\tilde c_0$, $\tilde c_1$ *and* $\tilde c_2$ *are arbitrary constants with* $c_1\tilde c_1\neq 0$.

Throughout the rest of the paper, we use the notation TN for the admissible transformation given in item N of Theorem 4.12, where N is the corresponding (one- or two-level) number.

Remark 4.13. The transformational part Φ of admissible transformation T4b can be represented as

$$\text{T4b: } \tilde t = \cot\frac{x+t}{2} + \tan\frac{x-t}{2}, \quad \tilde x = \cot\frac{x+t}{2} - \tan\frac{x-t}{2}, \quad \tilde u = u.$$

The transformational parts Φ of admissible transformations T4c and T4d can be replaced by alternative ones, which are analogous to that of T4b,

$$\text{T4c: } \tilde t = \coth\frac{x+t}{2} - \tanh\frac{x-t}{2}, \quad \tilde x = \coth\frac{x+t}{2} + \tanh\frac{x-t}{2}, \quad \tilde u = u;$$

$$\text{T4d: } \tilde t = \tanh\frac{x+t}{2} - \tanh\frac{x-t}{2}, \quad \tilde x = \tanh\frac{x+t}{2} + \tanh\frac{x-t}{2}, \quad \tilde u = u.$$

There also exist similar alternatives for transformational parts of other admissible transformations in the class $\mathcal{W}$. The counterparts of modified admissible transformations T4b–T4d and of admissible transformation T3 for linear equations from the class $\mathcal{W}_{\rm lin}$ were presented in Notes 1 and 2 of [334].

As a result, we obtain the following independent additional equivalence transformations among classification cases given in Table 4.6. (Below we do not indicate the corresponding parameters if they are not changed.)

T1: (a) $4_{\hat f,\hat g} \to 4_{1/\hat f,-\hat g/\hat f}$, $5\mathrm{a}_{\hat g} \to 5\mathrm{a}_{-\hat g}$ if $\varepsilon=1$, $11_{\hat f} \to 11_{1/\hat f}$,
$12_{q,\varepsilon,\varepsilon'} \circ (u \to -u) \to 12_{1-q,\varepsilon,-\varepsilon\varepsilon'}$, $13_{p,q,\varepsilon,\varepsilon'} \to 13_{-p,q-p,\varepsilon,-\varepsilon\varepsilon'}$,
$14\mathrm{d}_{\varepsilon,\varepsilon'} \to 14\mathrm{a}_{\varepsilon,-\varepsilon\varepsilon'}$, $16_{p,\varepsilon} \to 16_{-p,\varepsilon}$, $18\mathrm{a}_{q,\varepsilon,\varepsilon'} \to 18\mathrm{a}_{q,\varepsilon,-\varepsilon\varepsilon'}$,
$19\mathrm{d} \to 19\mathrm{a}$, $20_{\varepsilon,\varepsilon'} \to 20_{\varepsilon,-\varepsilon\varepsilon'}$.

T2: (b) 8b → 8a, 14b → 14a, 15b → 15a, 19b → 19a;
(c) 8c → 8a, 14c → 14a, 15c → 15a, 19c → 19a.

T3: 5b → $5\mathrm{a}_{\varepsilon=1}$, 18b → $18\mathrm{a}_{\varepsilon=1}$.

T4: (b) 6b $\to$ 6a$_{\varepsilon=1}$, (c) 6c $\to$ 6a$_{\varepsilon=1}$, (d) 6d $\to$ 6a$_{\varepsilon=1}$.

T5: 5c $\to$ 5a$_{\varepsilon=-1}$, 18c $\to$ 18a$_{\varepsilon=-1}$.

T6: (b) 6e $\to$ 6a$_{\varepsilon=-1}$, (c) 6f $\to$ 6a$_{\varepsilon=-1}$.

Remark 4.14. In Table 4.6, only Cases 1–4, 9–13, 14d, 16, 17 and 19d present regular Lie-symmetry extensions in the class $\mathcal{W}$. Therefore, the regular Cases 14d and 19d are $G^\sim$-inequivalent but $\mathcal{G}^\sim$-equivalent to the singular Cases 14a and 19a, respectively.

Consider the subclass $\mathcal{W}_\mathrm{c}$ of the class $\mathcal{W}$ singled out by the additional constraints $f_x = g_x = 0$ for the arbitrary-element tuple $\theta = (f, g)$, i.e., the class of equations of the general form

$$u_{tt} = f(u)u_{xx} + g(u), \tag{4.36}$$

where $(f_u, g_{uu}) \neq (0, 0)$. Cases 4, 5a, 11, 12, 13, 14a–14d, 16, 17, 18a, 19a–19d and 20 of Table 4.6 are related to the subclass $\mathcal{W}_\mathrm{c}$. The kernel Lie invariance algebra $\mathfrak{g}_\mathrm{c} = \langle \partial_t, \partial_x \rangle$ of equations from the subclass $\mathcal{W}_\mathrm{c}$ is given by Cases 4, which is the general case within this subclass. It is obvious from Theorem 4.11 and the above list of additional equivalence transformations that a complete list of $\mathcal{G}^\sim_\mathrm{c}$-inequivalent Lie-symmetry extensions within the subclass $\mathcal{W}_\mathrm{c}$, where $\mathcal{G}^\sim_\mathrm{c}$ is its equivalence groupoid, is exhausted by Cases 5a, 11, 12, 13, 14a, 16, 17, 18a, 19a and 20 of Table 4.6, where additionally $q > 1/2$ in Case 12, $p > 0$ in Cases 13 and 16, and $\varepsilon' = 1$ in Cases 18a and 20 with $\varepsilon = 1$. The group classification of the subclass $\mathcal{W}_\mathrm{c}$ up to equivalence generated by its equivalence group $G^\sim_\mathrm{c}$ is more delicate. The group $G^\sim_\mathrm{c}$ is generated by transformations of the form (4.29) with $\varphi_{xx} = \psi_x = 0$, which constitute the intersection $G^\sim_\mathrm{c} \cap G^\sim$, and one more (discrete) equivalence transformation $\tilde t = x$, $\tilde x = t$, $\tilde u = u$, $\tilde f = 1/f$, $\tilde g = -f/g$ of $\mathcal{W}_\mathrm{c}$, which generates the family T1 of admissible transformations within $\mathcal{W}$. This is why some $G^\sim$-inequivalent Lie-symmetry extensions can be $G^\sim_\mathrm{c}$-equivalent, which occurs for Cases 14a and 14d as well as for Cases 19a and 19d. The converse situation is not possible since the subgroupoid of $\mathcal{G}^{\sim}_{\mathrm{c}}$ generated by $G^{\sim}$ is contained in the subgroupoid generated by $G^{\sim}_{\mathrm{c}}$. This results in the following assertion.

Corollary 4.15. *A complete list of $G^{\sim}_{\mathrm{c}}$-inequivalent Lie-symmetry extensions within the subclass $\mathcal{W}_{\mathrm{c}}$ is exhausted by Cases 5a, 11, 12, 13, 14a–14c, 16, 17, 18a, 19a–19c and 20 of Table 4.6, where additionally $q > 1/2$ in Case 12, $p > 0$ in Cases 13 and 16, and $\varepsilon' = 1$ in Cases 18a and 20 with $\varepsilon = 1$.*

4.2.5. Equivalence Groupoid and Singular Lie Symmetry Extensions The equivalence groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{W}$ contains admissible transformations that are not generated by elements of $G^{\sim}$, i.e., this class is not normalized. Nevertheless, we can describe the groupoid $\mathcal{G}^{\sim}$, classifying admissible transformations within the class $\mathcal{W}$ up to the $G^{\sim}$-equivalence; see [248] for posing the general problem on classifying admissible transformations. More precisely, proving Theorem 4.12, we construct the generating (up to $G^{\sim}$-equivalence) subset $\mathcal{B}$ of $\mathcal{G}^{\sim}$ with the simultaneous classification of singular Lie-symmetry extensions within the class $\mathcal{W}$.

Since the class $\mathcal{W}$ is a subclass of $\mathcal{W}_{\mathrm{gen}}$, the transformational part Φ of any admissible transformation $\mathcal{T} = (\theta, \Phi, \tilde{\theta})$ in the class $\mathcal{W}$ takes the form

$$\tilde{t} = T(t,x), \quad \tilde{x} = X(t,x), \quad \tilde{u} = U = U^1(t,x)u + U^0(t,x)$$

with $T_tX_x - T_xX_t \neq 0$ and $U^1 \neq 0$, and additionally the system (4.23)–(4.27) is satisfied. Here $\theta = (f,g)$ and $\tilde{\theta} = (\tilde{f},\tilde{g})$ are respectively the source and target arbitrary-element tuples for $\mathcal{T}$, $\mathcal{L}_\theta, \mathcal{L}_{\tilde{\theta}} \in \mathcal{W}$.

By $\mathcal{W}_0$ and $\mathcal{W}_1$ we respectively denote the subclasses of $\mathcal{W}$ singled out by the constraints $f_u = 0$ and $f_u \neq 0$. The partition $\mathcal{W} = \mathcal{W}_0 \sqcup \mathcal{W}_1$ of the class $\mathcal{W}$ induces the partition of the equivalence groupoid $\mathcal{G}^{\sim}$ of this class since $\tilde{f}_{\tilde{u}} = 0$ if and only if $f_u = 0$, cf. the end of Section 4.2.1. In other words, equations in the subclass $\mathcal{W}_0$ are not related by point transformations to equations in the subclass $\mathcal{W}_1$. This claim can be nicely reformulated in terms of equivalence groupoids.

Proposition 4.16. *The equivalence groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{W}$ is the disjoint union of the equivalence groupoids $\mathcal{G}_0^{\sim}$ and $\mathcal{G}_1^{\sim}$ of the subclasses $\mathcal{W}_0$ and $\mathcal{W}_1$, $\mathcal{G}^{\sim} = \mathcal{G}_0^{\sim} \sqcup \mathcal{G}_1^{\sim}$.*

We describe the equivalence groupoids $\mathcal{G}_0^{\sim}$ and $\mathcal{G}_1^{\sim}$ separately.

Lemma 4.17. *The usual equivalence group of the subclass $\mathcal{W}_0$ coincides with $G^{\sim}$. A generating (up to $G^{\sim}$-equivalence) set $\mathcal{B}_0$ of admissible transformations for the class $\mathcal{W}_0$, which is minimal and self-consistent with respect to $G^{\sim}$-equivalence, is the union of the restriction of the family* T1 *to $\mathcal{W}_0$ and the families* T3–T9.

Lemma 4.18. *A complete list of $G^{\sim}$-inequivalent singular Lie-symmetry extensions for equations from the class $\mathcal{W}_0$, which are not related to appropriated subalgebras of $\mathfrak{g}^{\sim}$, is exhausted by Cases* 5a–7, 18a–18c *and* 20 *within the subclass of equations with arbitrary elements of the form $f = \varepsilon$ and $g = g^1(x)g^2(u)$.*

Proof. We will simultaneously prove Lemmas 4.17 and 4.18. Let $\mathcal{T} \in \mathcal{G}_0^{\sim}$, i.e., $f_u = 0$, $g_{uu} \neq 0$, $\tilde{f}_{\tilde{u}} = 0$ and $\tilde{g}_{\tilde{u}\tilde{u}} \neq 0$. We express ${X_t}^2$ from the equation (4.24), substitute this expression into the squared equation (4.23). After factorizing the resulting equation, we obtain the equation

$$({T_t}^2 - f{T_x}^2)(f{X_x}^2 - \tilde{f}{T_t}^2) = 0,$$

which implies in view of ${T_t}^2 - f{T_x}^2 \neq 0$ that $f{X_x}^2 = \tilde{f}{T_t}^2$. Then the equation (4.24) yields ${X_t}^2 = f\tilde{f}{T_x}^2$. This means that f and $\tilde{f}$ have the same sign and up to $G^{\sim}$-equivalence we can assume that $f = \tilde{f} = \varepsilon$, where $\varepsilon = \pm 1$, i.e., ${X_x}^2 = {T_t}^2$ and ${X_t}^2 = {T_x}^2$. More precisely, gauging of f and $\tilde{f}$ can be realized via transformations $\mathcal{D}(\varphi)$ and $\mathcal{D}(\tilde{\varphi})$ of x and $\tilde{x}$, respectively. Taking into account the equation (4.23) and alternating the sign of t (this transformation belongs to the kernel group of the class (4.16)), we can set

$X_x = T_t$, $X_t = \varepsilon T_x$. Since these equations give $T_{tt} = \varepsilon T_{xx}$ and $X_{tt} = \varepsilon X_{xx}$, the pair of the equations (4.25) and (4.26) reduce to the system of linear homogeneous algebraic equations $T_tU_{ut} - \varepsilon T_xU_{ux} = 0$, $X_tU_{ut} - \varepsilon X_xU_{ux} = 0$ with respect to U_{ut} and U_{ux}. The determinant of the associated matrix is nonzero, $\varepsilon(T_tX_x - T_xX_t) \neq 0$. Hence $U_{ut} = U_{ux} = 0$, i.e., U_u is a nonzero constant and using a transformation $\mathcal{D}^u(c_2)$ we can set $U_u = 1$. In view of the above conditions, the equation (4.27) takes the form

$$(X_x{}^2 - \varepsilon X_t{}^2)\tilde g = g + U^0_{tt} - \varepsilon U^0_{xx}. \tag{4.37}$$

Sequentially acting on the equation (4.37) by the operators $(X_x{}^2 - \varepsilon X_t{}^2)^{-1}\partial_t$ and $\partial_{\tilde t}$, we obtain two differential consequences of (4.37),

$$X_t\tilde g_{\tilde x} + U^0_t\tilde g_{\tilde u} + \frac{(X_x{}^2 - \varepsilon X_t{}^2)_t}{X_x{}^2 - \varepsilon X_t{}^2}\tilde g = \frac{(U^0_{tt} - \varepsilon U^0_{xx})_t}{X_x{}^2 - \varepsilon X_t{}^2}, \tag{4.38}$$

$$(X_t)_{\tilde t}\tilde g_{\tilde x} + (U^0_t)_{\tilde t}\tilde g_{\tilde u} + \left(\frac{(X_x{}^2 - \varepsilon X_t{}^2)_t}{X_x{}^2 - \varepsilon X_t{}^2}\right)_{\tilde t}\tilde g = \left(\frac{(U^0_{tt} - \varepsilon U^0_{xx})_t}{X_x{}^2 - \varepsilon X_t{}^2}\right)_{\tilde t}. \tag{4.39}$$

Studying the consistency of the equations (4.38) and (4.39) as first order quasilinear partial differential equations with respect to $\tilde g$, we consider different cases depending on whether the matrix of coefficients of the derivatives $\tilde g_{\tilde x}$ and $\tilde g_{\tilde u}$ in the system of these equations is degenerate or nondegenerate.[4.2]

1. Suppose first that this matrix is nondegenerate, $X_t(U^0_t)_{\tilde t} - U^0_t(X_t)_{\tilde t} \neq 0$. We solve the system (4.38)–(4.39) as a system of linear algebraic equations with respect to $\tilde g_{\tilde x}$ and $\tilde g_{\tilde u}$,

$$\tilde g_{\tilde x} = \alpha^1\tilde g + \alpha^0, \quad \tilde g_{\tilde u} = \beta^1\tilde g + \beta^0. \tag{4.40}$$

Here the coefficients α^0, α^1, β^0 and β^1 are functions of $(\tilde t, \tilde x)$ whose explicit expressions in terms of X and U^0 are not essential for the further consideration. Differentiating the equations (4.40) with respect to $\tilde t$, we derive

[4.2] This procedure and the previous partition of $\mathcal{W}$ into $\mathcal{W}_0$ and $\mathcal{W}_1$ fits well into the framework of the method of furcate splitting [209, 224, 289, 297]. This method of furcate splitting was used to describe the equivalence groupoid of the class of general Burgers–Korteweg–de Vries equations with space-dependent coefficients via classifying maximal conditional equivalence groups of this class [221, 222]. The further consideration is the first construction of a generating set of admissible transformations using this method.

the consequences $\alpha^1_{\tilde t}\tilde g+\alpha^0_{\tilde t}=0$, $\beta^1_{\tilde t}\tilde g+\beta^0_{\tilde t}=0$, which implies in view of $g_u\neq 0$ that $\alpha^1_{\tilde t}=\alpha^0_{\tilde t}=\beta^1_{\tilde t}=\beta^0_{\tilde t}=0$. The cross-differentiation of the equations (4.40) with respect to $\tilde x$ and $\tilde u$ leads to the compatibility conditions for these equations, which are $\beta^1_{\tilde x}=0$ and $\beta^0_{\tilde x}=\alpha^1\beta^0-\alpha^0\beta^1$. Therefore, β^1 is a constant. Since $\tilde g_{\tilde u\tilde u}\neq 0$, the second equation in (4.40) implies that $\beta^1\neq 0$. Using the equivalence transformation $\mathcal{D}^u(1/\beta^1)$, we can gauge β^1 to 1. Then the second equation in (4.40) integrates to $\tilde g=\tilde g^0(\tilde x)e^{\tilde u}+\tilde g^1(\tilde x)$ with $\tilde g^1=-\beta^0$. The equation (4.37) implies that the function g is of similar form in the initial variables, $g=g^0(x)e^u+g^1(x)$, and $({X_x}^2-\varepsilon {X_t}^2)e^{U^0}\tilde g^0=g^0$, $({X_x}^2-\varepsilon {X_t}^2)\tilde g^1=g^1+U^0_{tt}-\varepsilon U^0_{xx}$. Using equivalence transformations of the form $\mathcal{G}(\psi)$ in both the old and new variables, we set $g^0=\varepsilon'$ and $\tilde g^0=\tilde\varepsilon'$ with $\varepsilon',\tilde\varepsilon'=\pm1$. Then $({X_x}^2-\varepsilon {X_t}^2)e^{U^0}\tilde\varepsilon'=\varepsilon'$, i.e.,

$$U^0=-\ln|{X_x}^2-\varepsilon {X_t}^2|$$

and thus $U^0_{tt}-\varepsilon U^0_{xx}=0$ since $X_{tt}-\varepsilon X_{xx}=0$. The equation (4.37) takes the form

$$\frac{g^1(x)}{{X_x}^2-\varepsilon {X_t}^2}=\tilde g^1(X). \tag{4.41}$$

In other words, in this case it suffices to classify admissible transformations within the subclass $\mathcal{W}_{00}$ of equations of the form

$$u_{tt}=\varepsilon u_{xx}+\varepsilon' e^u+g^1(x) \tag{4.42}$$

up to the subgroup $G^\sim_{00}$ of $G^\sim$ singled out by the constraints $\varphi_x=\pm c_1$, $c_2=|c_1|^{-1/2}$ and $\psi=0$. This reduces to deriving possible $G^\sim_{00}$-inequivalent expressions for $X=X(t,x)$, $g^1=g^1(x)$ and $\tilde g^1=\tilde g^1(\tilde x)$ satisfying the joint system of the equation (4.41) and the equation $X_{tt}=\varepsilon X_{xx}$. We change the independent variables in this system, $y=x+\iota t$ and $z=x-\iota t$, where $\iota=1$ or $\iota=i$ if $\varepsilon=1$ or $\varepsilon=-1$, respectively, i is the imaginary unit, $i^2=-1$. Hence $\iota^2=\varepsilon$. In the variables y and z the equation $X_{tt}=\varepsilon X_{xx}$ takes the form $X_{yz}=0$ and its general solution is represented as $X=Y(y)+Z(z)$, where $Z(z)$ coincides with the conjugate value of $Y(y)$ if $\varepsilon=-1$. Then the equation (4.41) can be rewritten as

$$\frac{1}{4Y_yZ_z}\,g^1(x)=\tilde g^1(X),\quad \text{assuming}\quad x=\frac{y+z}{2},\quad X=Y+Z. \tag{4.43}$$

Excluding the parameter function $\tilde g^1$ via acting by the operator $Y_y\partial_z-Z_z\partial_y$ on the equation (4.43), we reduce this equation, after the expansion and algebraic transformations, to

$$2g^1(\partial_y+\partial_z)\big(Y_y^{-1}-Z_z^{-1}\big)=-g^1_x\big(Y_y^{-1}-Z_z^{-1}\big).$$

The last equation integrates to $Y_y^{-1}-Z_z^{-1}=\iota h^0h^1$, where h^0 is a (real-valued) smooth function of t, $h^1:=|g^1|^{-1/2}\neq0$ and thus h^1 is a (real-valued) smooth function of x. We act on the integration result by the operator $\partial_t^2-\varepsilon\partial_x^2=-4\varepsilon\partial_y\partial_z$ to get $h^0_{tt}h^1=\varepsilon h^0h^1_{xx}$.

If $h^0=0$, then $Y_y^{-1}=Z_z^{-1}=\text{const}\in\mathbb R$ and thus $Y_y=Z_z=\text{const}\in\mathbb R$, which implies $X_{xx}=0$. Therefore, $T_x=T_{tt}=0$ as well. This means that the admissible transformation $\mathcal T$ is induced by an element of $G^\sim$.

The case $h^1=g^1=0$ corresponds to the Liouville equation. The sign of ε' is alternated by the corresponding admissible transformation from the family T1 if $\varepsilon=1$ and cannot be alternated in view of the equation $({X_x}^2-\varepsilon{X_t}^2)e^{U^0}\tilde\varepsilon'=\varepsilon'$ if $\varepsilon=-1$. The equivalence group $G^\sim$ induces the subgroup H of the complete point symmetry group of the Liouville equation for each fixed value of $(\varepsilon,\varepsilon')$, which is constituted by the transformations of the form $\tilde t=c_1t+c_0$, $\tilde x=c_1x+c_2$, where c_0, c_1 and c_2 are arbitrary constants with $c_1\neq0$. For the minimality of the set of admissible transformations to be constructed, we should takes a single representative in each coset of $G^\sim$-equivalent elements of the corresponding vertex group. This gives the family T9 of admissible transformations.

Further we assume that $h^0h^1\neq0$ and thus $g^1\neq0$ as well. The separation of variables in the equation $h^0_{tt}h^1=\varepsilon h^0h^1_{xx}$ implies that $h^0_{tt}/h^0=\varepsilon h^1_{xx}/h^1$ is a constant, which can be assumed, modulo scalings from $G^{\sim}$ preserving the constraint $f = \varepsilon$, to take values from the set $\{-1,0,1\}$. Up to shifts of x and alternating the sign of x, we have

$$g^1 \in \mathcal{C} := \big\{\nu,\ \nu x^{-2},\ \nu\cos^{-2}x,\ -\nu\cosh^{-2}x,\ \nu\sinh^{-2}x,\ \varepsilon'' e^{-2x} \mid \\ \nu\in\mathbb{R}, \nu\neq 0, \varepsilon''=\pm1\big\}.$$

Using the same arguments for the inverse of the admissible transformation $\mathcal{T}$, we obtain that the function $\tilde g^1$ also belongs to the set $\mathcal{C}$ (up to replacing the argument x by $\tilde x$).

We first present a complete set of $G^{\sim}$-inequivalent (independent up to inversion and composing with each other) non-identity admissible transformations for g^1 running through the set $\mathcal{C}$ and then explain the derivation of this list. It is exhausted by the family T8$|_{\mathcal{W}_{00}}$ and the following families:

T3′. $f=1,\ \ g=e^u+\varepsilon'' e^{-2x},\ \ \tilde f=1,\ \ \tilde g=e^u+\varepsilon''$,
Φ: $\ \tilde t=e^{-x}\sinh t,\ \ \tilde x=e^{-x}\cosh t,\ \ \tilde u=u+2x$;

T4′. $f=1,\ \ g=\varepsilon' e^u+g^1(x),\ \ \tilde f=1,\ \ \tilde g=\tilde\varepsilon' e^u+\nu\tilde x^{-2},\ \ \nu\in\mathbb{R}_{\neq0}$,

a. $g^1(x)=\nu x^{-2},\ \ \tilde\varepsilon'=\varepsilon',\ \ \Phi$: $\ \tilde t=\dfrac{t}{x^2-t^2},\ \ \tilde x=\dfrac{x}{x^2-t^2}$,
$\tilde u=u+2\ln|x^2-t^2|$;

b. $g^1(x)=\nu\cos^{-2}x,\ \ \tilde\varepsilon'=\varepsilon'$,
Φ: $\ \tilde t=\dfrac{\cos t}{\sin t+\sin x},\ \ \tilde x=\dfrac{\cos x}{\sin t+\sin x},\ \ \tilde u=u+2\ln|\sin t+\sin x|$;

c. $g^1(x)=-\nu\cosh^{-2}x,\ \ \tilde\varepsilon'=-\varepsilon',\ \ \Phi$: $\ \tilde t=e^t\sinh x,\ \ \tilde x=e^t\cosh x$,
$\tilde u=u-2t$;

d. $g^1(x)=\nu\sinh^{-2}x,\ \ \tilde\varepsilon'=\varepsilon',\ \ \Phi$: $\ \tilde t=e^t\cosh x,\ \ \tilde x=e^t\sinh x$,
$\tilde u=u-2t$;

T5′. $f=-1,\ \ g=\varepsilon' e^u+\varepsilon'' e^{-2x},\ \ \tilde f=-1,\ \ \tilde g=\varepsilon' e^u+\varepsilon''$,
Φ: $\ \tilde t=e^{-x}\sin t,\ \ \tilde x=e^{-x}\cos t,\ \ \tilde u=u+2x$;

T6′. $f=-1,\ \ g=\varepsilon' e^u+g^1(x),\ \ \tilde f=-1,\ \ \tilde g=\varepsilon' e^u+\nu\tilde x^{-2},\ \ \nu\in\mathbb{R}_{\neq0}$,

a. $g^1(x)=\nu x^{-2}$, Φ: $\tilde t=\dfrac{t}{x^2+t^2}$, $\tilde x=\dfrac{x}{x^2+t^2}$, $\tilde u=u+2\ln|x^2+t^2|$;

b. $g^1(x)=\nu\cos^{-2}x$, Φ: $\tilde t=e^t\sin x$, $\tilde x=e^t\cos x$, $\tilde u=u-2t$;

c. $g^1(x)=\nu\sinh^{-2}x$,

Φ: $\tilde t=\dfrac{\sin t}{\cos t+\cosh x}$, $\tilde x=\dfrac{\sinh x}{\cos t+\cosh x}$,

$\tilde u=u+2\ln|\cos t+\cosh x|$;

T7′. $f=-1$, $g=\varepsilon' e^u+\nu\cosh^{-2}x$, $\tilde f=-1$, $\tilde g=\varepsilon' e^{\tilde u}+\nu\cosh^{-2}\tilde x$, $\nu\in\mathbb{R}_{\neq 0}$,

Φ: $\tilde t=\arctan\dfrac{\sin\gamma\,\sinh x+\cos\gamma\sin t}{\cos t}$,

$\tilde x=\operatorname{arctanh}\dfrac{\cos\gamma\,\sinh x-\sin\gamma\sin t}{\cosh x}$,

$\tilde u=u+\ln\left|\cosh^2 x-(\cos\gamma\,\sinh x-\sin\gamma\sin t)^2\right|$, $\gamma\in(0,2\pi)$.

The direct way of checking which elements of the set $\mathcal{C}$ are related via admissible transformation is to fix an element g^1 in $\mathcal{C}$, thus defining $h^1:=|g^1|^{-1/2}\neq 0$, to solve the equation $h^0_{tt}=\lambda h^0$ with $\lambda:=\varepsilon h^1_{xx}/h^1=\mathrm{const}$, to find Y and Z by separating variables y and z in the equation $Y_y^{-1}-Z_z^{-1}=\iota h^0h^1$ and further integration, and finally to determine $\tilde g^1$ from (4.41).

We follow an optimized strategy. In the above way, we find the mappings $\nu e^{-2x}\mapsto\nu$ by T3′ if $f=1$ and by T5′ if $f=-1$, $\nu\cos^{-2}x\mapsto\nu x^{-2}$ by T4′b if $f=1$ and by T6′b if $f=-1$, $-\nu\cosh^{-2}x\mapsto\nu x^{-2}$ by T4′c if $f=1$, $\nu\sinh^{-2}x\mapsto\nu x^{-2}$ by T4′d if $f=1$ and by T6′c if $f=-1$. The sign of ε' is alternated only in T4′c. For $f=-1$, the value $g^1=-\nu\cosh^{-2}x$ is mapped to the value $g^1=\nu x^{-2}$ by an admissible point transformation only over the complex field.

The maximal Lie invariance algebras of the equations of the form (4.42) with values of (f,g^1) that have not been reduced to other ones are

$$(f,g^1)=(1,\nu):\quad \mathfrak{g}_\theta=\langle\partial_t,\ \partial_x,\ x\partial_t+t\partial_x\rangle,$$

$$
\begin{aligned}
&(f,g^1)=(-1,\nu)\colon\quad \mathfrak{g}_\theta=\langle\partial_t,\,\partial_x,\,x\partial_t-t\partial_x\rangle,\\
&(f,g^1)=(\varepsilon,\nu x^{-2})\colon\\
&\qquad\qquad \mathfrak{g}_\theta=\langle\partial_t,\,t\partial_t+x\partial_x-2\partial_u,\,(t^2+\varepsilon x^2)\partial_t+2tx\partial_x-4t\partial_u\rangle,\\
&(f,g^1)=(-1,\nu\cosh^{-2}x)\colon\quad \mathfrak{g}_\theta=\langle\partial_t,\,\mathcal{R}'(\cos t\,\cosh x),\,\mathcal{R}'(\sin t\,\cosh x)\rangle,
\end{aligned}
$$

where $\mathcal{R}'(\Phi):=\Phi_x\partial_t+\Phi_t\partial_x-2\Phi_{tx}\partial_u$. These invariance algebras are given in Cases $5\mathrm{a}_{\varepsilon=1}$ and $5\mathrm{a}_{\varepsilon=-1}$ of Table 4.6 and are associated with Cases 6a and 7 of the same table, respectively. They are realizations of the Poincaré algebra $\mathrm{p}(1,1)$, the Euclidian algebra $\mathrm{e}(2)$, the real special linear algebra $\mathrm{sl}(2,\mathbb{R})$ and the orthogonal algebra $\mathrm{o}(3)$, which are not isomorphic to each other. At the same time, systems of differential equations are related by point transformations only if their maximal Lie invariance algebras are isomorphic.

Therefore, we need to classify admissible transformations within the four subclasses of equations of the form (4.42), where for each of these subclasses the tuple (f,g^1) is of a fixed form in $\{(1,\nu),\,(-1,\nu),\,(\varepsilon,\nu x^{-2})\,(-1,\nu\cosh^{-2})\}$, and ν runs through $\mathbb{R}_{\neq0}$. For this purpose, we apply for the first time an extension of the algebraic method to finding admissible transformations. This method was suggested by Hydon in [127–129] for computing discrete symmetries and extended to equivalence transformations in [22].

We in detail consider only the first subclass. Let $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ be two fixed equations of the form (4.42) with $f=\tilde f=1$, $\tilde g^1=\nu$, $g^1=\tilde\nu$ and some $\varepsilon',\tilde\varepsilon'=\pm1$. These equations have the same maximal Lie invariance algebra, $\mathfrak{g}_\theta=\mathfrak{g}_{\tilde\theta}=\langle\partial_t,\partial_x,t\partial_x+x\partial_t\rangle$, which is given in Case $5\mathrm{a}_{f=1}$ of Table 4.6 and is a realization of the Poincaré algebra $\mathrm{p}(1,1)$. Therefore, the pushforward of vector fields by Φ induces an automorphism of $\mathfrak{g}_\theta$ associated with an automorphism of $\mathrm{p}(1,1)$. Recall that the transformation Φ is completely defined by its t- and x-components. Inner automorphisms of $\mathrm{p}(1,1)$ correspond to continuous point transformations generated by vector fields from $\mathfrak{g}_\theta$. Such transformations are symmetries of $\mathcal{L}_\theta$, i.e., they do not change the parameters ν and ε'. Up to shifts of t and x, which are induced by elements of $G^\sim$, we obtain the family T8a$|_{\mathcal{W}_{00}}$ of admissible transformations. There are only two outer automorphisms of $\mathrm{p}(1,1)$ that are independent up to composing to each other and to inner automorphisms.[4.3] The corresponding transformations are the alternation of the sign of t, which is a discrete symmetry of $\mathcal{L}_\theta$ induced by $\mathcal{D}^t(-1)$, and the permutation of t and x, which belongs to the family T1. There is no point transformation that satisfies the restriction for Φ and induces the identity automorphism of $\mathfrak{g}_\theta$.

The other three subclasses are considered in a similar way. Each of the algebras $\mathrm{e}(2)$ and $\mathrm{sl}(2,\mathbb{R})$ possesses a single independent outer automorphism, which is here related, e.g., to alternating the sign of t. The algebra $\mathrm{o}(3)$ admits no outer automorphism but alternating the sign of t generates the identity automorphism of the corresponding algebra $\mathfrak{g}_\theta$. Factoring out shift and scaling symmetries of related equations, which are induced by elements of $G^\sim$, we construct the families T8b$|_{\mathcal{W}_{00}}$, T4′a and T7′, respectively. The last family consists of the non-identity transformations generated by the Lie-symmetry vector field $\mathcal{R}'(\cos t\,\cosh x)$ of the equation (4.42) with $(f,g^1)=(-1,\nu\cosh^{-2})$.

It is obvious that $G^\sim_{00}$-inequivalent singular cases of Lie-symmetry extensions within the subclass $\mathcal{W}_{00}$ are exhausted by those with $g^1\in\mathcal{C}$.

2. Now we suppose that the matrix of coefficients of the derivatives $\tilde g_{\tilde x}$ and $\tilde g_{\tilde u}$ in the system (4.38), (4.39) is degenerate,

$$X_t(U^0_t)_{\tilde t}-U^0_t(X_t)_{\tilde t}=0. \tag{4.44}$$

If $X_t=0$ then $T_x=X_{xx}=T_{tt}=0$ and the equation (4.37) implies in view of the condition $\tilde g_{\tilde u\tilde u}\neq0$ that $U^0_t=0$, i.e., the admissible transformation $\mathcal{T}$ is generated by an element of $G^\sim$. This is why in what follows we assume

[4.3] See [79, 243] for necessary facts on automorphisms of low-dimensional Lie algebras.

that $X_t \neq 0$. Representing the equation (4.44) in the form $(U^0_t/X_t)_{\tilde t} = 0$, we integrate it by $\tilde t$, which yields $U^0_t = V^0(X)X_t$ for some smooth function $V^0 = V^0(\tilde x)$. Then we integrate by t, obtaining $U^0 = V^1(X) + V^2(x)$, where V^1 is an antiderivative of V^0, $V^1_{\tilde x}(\tilde x) = V^0(\tilde x)$, and $V^2 = V^2(x)$ is a smooth function of x. Therefore, up to $G^{\sim}$-equivalence (namely, up to composing the transformation $\mathcal{T}$ with transformations from the subgroup $\{\mathcal{G}(\psi)\}$), we can set $U^0 = 0$ and thus $U = u$. We then rewrite the equation (4.38) as

$$\frac{\tilde g_{\tilde x}}{\tilde g} = -\frac{1}{X_t}\frac{({X_x}^2 - \varepsilon {X_t}^2)_t}{{X_x}^2 - \varepsilon {X_t}^2}.$$

The left- and right-hand sides of the last equations do not depend on $\tilde t$ and $\tilde u$, respectively, and hence they are equal to a function of only $\tilde x$. Solving the equation with respect to $\tilde g$ gives the representation of $\tilde g$ as the product of functions of different arguments, $\tilde g = \tilde g^1(\tilde x)\tilde g^2(\tilde u)$. Since $\tilde u = u$ and $g = ({X_x}^2 - \varepsilon {X_t}^2)\tilde g$, the function g admits the similar representation $g = g^1(x)g^2(u)$, where $g^2(u) = \tilde g^2(u)$. As a result, we again obtain the equation (4.41). Therefore, equations of the form

$$u_{tt} = \varepsilon u_{xx} + g^1(x)g^2(u) \quad \text{with} \quad g^2_u g^2_{uuu} \neq (g^2_{uu})^2 \tag{4.45}$$

with coinciding values of the parameter function g^2 are related by a point transformation if and only if the equations of the form (4.42) with the same values of the parameters ε and g^1 and some values of ε' are related by a point transformation. The inequality $g^2_u g^2_{uuu} \neq (g^2_{uu})^2$, which is equivalent to the linear independence of g^2_u, g^2 and 1, is imposed for excluding the intersection of the subclass (4.45) and the subclass of equations of the form $u_{tt} = \varepsilon u_{xx} + g^0(x)e^u + g^1(x)$ with $g^0 \neq 0$, which are reduced by equivalence transformations to equations from the subclass (4.42). To properly translate the classification of admissible transformations within the subclass (4.42) to those within the subclass (4.45), we take into account the condition $g^1 \neq 0$ for the subclass (4.45) and replace the u-components of all transformational parts by $\tilde u = u$. Since the coefficients ν and $\tilde\nu$ coincide, they can just be absorbed by g^2. In total, this gives the restrictions of the families T1, T3–T7 and T8 to the subclass (4.45). Here the families T3–T7 respectively correspond to the families T3′–T7′.

To complete the classification of admissible transformations in the subclass $\mathcal W_0$, we map each equation from the subclass (4.42) with $g^1\in\mathcal C$ by the equivalence transformation $\mathcal Z(\ln\hat g^1)$ to the equation $u_{tt}=\varepsilon u_{xx}+\hat g^1(x)g^2(u)$ with $g^2(u)=\varepsilon' e^u+\hat\nu$. Here $\hat g^1$ is of the same form as g^1 but with fixed $\nu=1$, $\hat\nu=2\varepsilon\varepsilon''$, $\varepsilon''=-1$ for $\hat g^1=\cosh^{-2}x$ and $\varepsilon''=1$ otherwise. As a result, the families T3′–T7′ are mapped into the families T3–T7. The completion of the latter families allows us to neglect the auxiliary inequality $g^2_ug^2_{uuu}\neq(g^2_{uu})^2$ of the subclass (4.45) for these families.

The obtained set $\mathcal B_0$ of admissible transformations of the subclass $\mathcal W_0$ is a generating set for $\mathcal G^\sim_0$ up to $G^\sim$-equivalence by construction. No element of $\mathcal G^{G^\sim}_0$ relates different values of θ from $\mathrm s(\mathcal B_0)\cup\mathrm t(\mathcal B_0)$. No element of $\mathcal B_0$ can be represented as the composition of a finite number of other elements of $\mathcal B_0$ or their inverses. Therefore, the generating set $\mathcal B_0$ is minimal and self-consistent with respect to $G^\sim$-equivalence for $\mathcal G^\sim_0$.

Up to $G^\sim$-equivalence, singular Lie-symmetry extensions in the subclass (4.45) are possible only for $g^1\in\mathcal C$, which gives Cases 5a–5c, 6a–6f and 7. Since Case 7 reduces to Case 6a over the complex field, for further extensions it suffices to check only equations with $g^1=1$ (Case 5a) and with $g^1=x^{-2}$ (Case 6a). For $g^1=1$, we obtain only Case 18a with two $\mathcal G^\sim$-equivalent Cases 18b and 18c. There are no further Lie-symmetry extensions for $g^1=x^{-2}$. □

Remark 4.19. A generating set of admissible point transformations within the class $\mathcal W_{\rm lin}$ of linear equations of the form (4.16) and the group classification of this class can be easily derived from the computation of their counterparts for the class $\mathcal W_0$. For this purpose, we need to consider the essential subgroupoid $\mathcal G^{\sim\rm ess}_{\rm lin}$ of the equivalence groupoid $\mathcal G^\sim_{\rm lin}$ of $\mathcal W_{\rm lin}$ and essential Lie invariance algebras of equations from $\mathcal{W}_{\rm lin}$, respectively factoring out transformations and Lie-symmetry vector fields related to the linear superposition of solutions, cf. [249, Section 2] and [178]. Elements of $\mathcal{W}_{\rm lin}$ take the form $u_{tt}=f(x)u_{xx}+g^1(x)u+g^0(x)$. Their kernel point symmetry group is generated by the transformations $\pi_*\mathcal{P}^t(c_0)$ and $\pi_*\mathcal{D}^u(c_2)$, and their kernel invariance algebra is $\mathfrak{g}^\cap_{\rm lin}=\langle\partial_t,u\partial_u\rangle$. The equivalence group of $\mathcal{W}_{\rm lin}$ coincides with $G^\sim$. Using equivalence transformations, we can gauge the parameter functions f and g^0 to $\varepsilon\in\{-1,1\}$ and 0, respectively. As a result, we map the class $\mathcal{W}_{\rm lin}$ onto its subclass $\mathcal{W}_{\rm lin'}$ of linear wave and elliptic equations with x-dependent potentials, which are of the form

$$u_{tt}=\varepsilon u_{xx}+g^1(x)u,$$

cf. [334]. Each equation $\mathcal{L}_\theta$ from $\mathcal{W}_{\rm lin'}$ admits the (pseudo)group $G^{\rm lin}_\theta$ of point symmetries associated with the linear superposition of solutions, $G^{\rm lin}_\theta=\{\Phi:\tilde t=t,\ \tilde x=x,\ \tilde u=u+h(t,x)\mid h\in\mathcal{L}_\theta\}$, where the notation "$h\in\mathcal{L}_\theta$" means that the function h runs through the solution set of $\mathcal{L}_\theta$. The corresponding Lie algebra is $\mathfrak{g}^{\rm lin}_\theta=\langle h(t,x)\partial_u\mid h\in\mathcal{L}_\theta\rangle$. Let $\mathcal{G}^{\sim\rm lin}_{\rm lin'}$ be the subgroupoid of the equivalence groupoid $\mathcal{G}^\sim_{\rm lin'}$ of the class $\mathcal{W}_{\rm lin'}$ that is constituted by the admissible transformations related the linear superposition of solutions, i.e., $\mathcal{G}^{\sim\rm lin}_{\rm lin'}$ is the union of $G^{\rm lin}_\theta$ as subgroups of vertex groups in $\mathcal{G}^\sim_{\rm lin'}$ for all θ with $\mathcal{L}_\theta\in\mathcal{W}_{\rm lin'}$. The essential equivalence groupoid $\mathcal{G}^{\sim\rm ess}_{\rm lin'}$ of $\mathcal{W}_{\rm lin'}$, which is the complement of $\mathcal{G}^{\sim\rm lin}_{\rm lin'}$ in $\mathcal{G}^\sim_{\rm lin'}$, is naturally isomorphic to the equivalence groupoid of the class of equations of the form (4.42) with a fixed value of ε'. Therefore, a generating set for $\mathcal{G}^{\sim\rm ess}_{\rm lin'}$ and, thus, for the essential equivalence groupoid $\mathcal{G}^{\sim\rm ess}_{\rm lin}$ of $\mathcal{W}_{\rm lin}$, which is defined similarly to $\mathcal{G}^{\sim\rm ess}_{\rm lin'}$, consists of the counterparts of the families T3–T9 for linear equations. That is, one should substitute $g^2=u$, $\tilde g^2=\tilde u$ into T3–T7 and $g=\varepsilon''u$, $\tilde g=\varepsilon''\tilde u$ into T8 and replace g, $\tilde g$ and the u-component of Φ in T9 by $g=\varepsilon''u$, $\tilde g=\varepsilon''\tilde u$ and $\tilde u=u$. A complete list of $G^\sim$-inequivalent essential Lie-symmetry extensions in the class $\mathcal{W}_{\rm lin}$ (i.e., extensions of $\mathfrak{g}^\cap_{\rm lin}\dotplus\mathfrak{g}^{\rm lin}_\theta$) are exhausted by Cases 5a–7 of Table 4.6 with $\hat g=\varepsilon''u$ and the counterpart of Cases 20, where $g=0$ and the extension is spanned by $\tau\partial_t+\xi\partial_x$ with the same condition on (τ,ξ). Additional equivalence transformations between the classification cases are exhausted by the counterparts of those for Cases 5a–6f.[4.4]

It now remains to study the equivalence groupoid $\mathcal{G}_1^\sim$ of the subclass $\mathcal{W}_1$, which is singled out from the class (4.16) by the constraint $f_u\neq 0$. By $\mathcal{W}_{11}$ and $\mathcal{W}_{12}$ we respectively denote the subclasses of $\mathcal{W}_1$ that is associated with the additional auxiliary condition $f_x=g_x=0 \bmod G^\sim$ and that consists of the equations $G^\sim$-equivalent to equations of the form

$$u_{tt}=\varepsilon u^{-4}u_{xx}+\mu(x)u^{-3}+\sigma u, \tag{4.46}$$

where μ runs through the set of smooth functions of x, ε and σ are constants, $\varepsilon\neq 0$ and hence $\varepsilon=\pm 1 \bmod G^\sim$ and $\sigma\in\{-1,0,1\} \bmod G^\sim$.

Lemma 4.20. *The usual equivalence group of the subclass $\mathcal{W}_1$ coincides with $G^\sim$. Any admissible transformation in $\mathcal{W}_1\setminus(\mathcal{W}_{11}\cup\mathcal{W}_{12})$ is generated by a transformation from $G^\sim$. A generating (up to $G^\sim$-equivalence) set $\mathcal{B}_1$ of admissible transformations for the class $\mathcal{W}_1$, which is minimal and self-consistent with respect to $G^\sim$-equivalence, is the union of the restriction of the family* T1 *to $\mathcal{W}_{11}$ and the family* T2*, which acts within $\mathcal{W}_{12}$.*

Proof. For $\mathcal{T}\in\mathcal{G}_1^\sim$, the equation (4.23) immediately implies that $T_tX_t=T_xX_x=0$ for admissible transformations within the subclass $\mathcal{W}_1$.

Supposing $T_x\neq 0$, we obtain that $X_x=0$, $X_t\neq 0$ and hence $T_t=0$. Up to $G^\sim$-equivalence of admissible transformations we can assume that $T=x$ and $X=t$ as under the above restrictions the transformation $\mathcal{T}$ is represented as the composition of $\mathcal{D}(T)$, a transformation permuting t and x and $\mathcal{D}(X)$. For $T=x$ and $X=t$, the equations (4.24), (4.25)

[4.4]Therefore, a complete list of $\mathcal{G}_{\rm lin}^\sim$-inequivalent Lie-symmetry extensions within the class $\mathcal{W}_{\rm lin}$ are exhausted by the equations from the class $\mathcal{W}_{\rm lin'}$ with $g^1=0$ (the (1+1)-dimensional wave equation for $\varepsilon=1$ and the two-dimensional Laplace equation for $\varepsilon=-1$), $(g^1,\varepsilon)=(-1,1)$ (the (1+1)-dimensional Klein–Gordon equation), $(g^1,\varepsilon)=(\varepsilon',-1)$ (the two-dimensional Helmholz equation) and $g^1=\nu x^{-2}$ (the (1+1)-dimensional wave equation with the potential νx^{-2} for $\varepsilon=1$ and the two-dimensional Laplace equation with the potential νx^{-2} for $\varepsilon=-1$).

and (4.26) reduce to the simple equations $\tilde{f}f = 1$, and $U_{ux} = U_{ut} = 0$, i.e. $U_u = \text{const}$. By a scaling of u, which belongs to $G^{\sim}$, the constant U_u can be set equal to 1. Differentiating the equation $\tilde{f} = 1/f$ with respect to t and then, assuming $(\tilde{t}, \tilde{x}, \tilde{u})$ as basic variables, with respect to $\tilde{t}$, we derive the equation $U^0_{tx} = 0$. Therefore, $U^0 = \psi(x) + \tilde{\psi}(t)$ and the reduced transformation $\mathcal{T}$ with $T = x$, $X = t$ and $U_u = 1$ can be represented as the composition of the transformations $\mathcal{G}(\psi)$, $t \leftrightarrow x$ and $\mathcal{G}(\tilde{\psi})$, where $t \leftrightarrow x$ denotes the transformation which only permutes t and x: $\tilde{t} = x$, $\tilde{x} = t$ and $\tilde{u} = u$. This means that up to $G^{\sim}$-equivalence of admissible transformations the transformation $\mathcal{T}$ coincides with $t \leftrightarrow x$. The corresponding transformation components for the arbitrary elements f and g follow from the equations (4.24) and (4.27). They read $\tilde{f} = 1/f$ and $\tilde{g} = -g/f$. Since the left-hand (resp. right-hand) sides of these equalities do not depend on $\tilde{t} = x$ (resp. $\tilde{x} = t$), the arbitrary elements of equations from the class (4.16) that are connected by the transformation $t \leftrightarrow x$ satisfy the additional auxiliary constraints $f_x = g_x = 0$ and $\tilde{f}_{\tilde{x}} = \tilde{g}_{\tilde{x}} = 0$. In other words, admissible transformations of the case under consideration are generated by transformations from the equivalence group $G^{\sim}$ of the entire class (4.16) and the equivalence transformation $t \leftrightarrow x$ of the subclass $\mathcal{U}$ which is singled out from the class (4.16) by the additional auxiliary constraints $f_x = g_x = 0$ and $f_u \neq 0$. In particular, the equivalence group of the subclass $\mathcal{U}$ consists of the transformations of the form (4.29) with $\varphi_{xx} = \psi_x = 0$ and the compositions of these transformations with $t \leftrightarrow x$.

Now we consider the case $T_x = 0$ for which $T_t \neq 0$, $X_t = 0$ and $X_x \neq 0$. Then the equations (4.24)–(4.26) reduce to $\tilde{f}{T_t}^2 = f{X_x}^2$, $({U_u}^2/T_t)_t = 0$, $({U_u}^2/X_x)_x = 0$. From the first equation we can conclude that $\tilde{f}_{\tilde{u}} \neq 0$ if and only if $f_u \neq 0$. Solving the other two equations with respect to U_u, equating the expressions obtained and separating variables in this equality, we derive that $U^1 := U_u = \varkappa\sqrt{|T_tX_x|}$, where $\varkappa$ is a nonzero constant. Differentiating the equation $\tilde{f}{T_t}^2 = f{X_x}^2$ with respect to t results in the consequence

$$\frac{T_{tt}}{T_t}\Big((\tilde{u} - U^0)\tilde{f}_{\tilde{u}} + 4\tilde{f}\Big) + 2U^0_t\tilde{f}_{\tilde{u}} = 0. \tag{4.47}$$

If $T_{tt} = 0$, the equation (4.47) implies that $U^1/\sqrt{|X_x|} = \text{const}$ and $U^0_t = 0$ and, therefore, the transformation $\mathcal{T}$ belongs to the equivalence group $G^{\sim}$.

Further we assume that $T_{tt} \neq 0$. By fixing a value of t, we derive from the equation (4.47) that the arbitrary element $\tilde{f}$ is a solution of an ordinary differential equation of the general form $(\tilde{u} + \tilde{\beta}(\tilde{x}))\tilde{f}_{\tilde{u}} + 4\tilde{f} = 0$, where the variable $\tilde{x}$ plays the role of a parameter and $\tilde{\beta}$ is a smooth function of $\tilde{x}$. This implies that $\tilde{f} = \tilde{\alpha}(\tilde{x})(\tilde{u}+\tilde{\beta}(\tilde{x}))^{-4}$ for some smooth function $\tilde{\alpha} = \tilde{\alpha}(\tilde{x})$. Combining the equation $\tilde{f}{T_t}^2 = f{X_x}^2$ with the expression for $\tilde{f}$ yields

$$f = \frac{T_t^2}{X_x^2}\frac{\tilde{\alpha}(X)}{(U^1u + U^0 + \tilde{\beta}(X))^4} = \frac{\alpha(x)}{(u + \beta(x))^4},$$

where $\alpha(x) := (\varkappa X_x)^{-4}\tilde{\alpha}(X)$ and $\beta(x) := (\tilde{\beta}(X) + U^0)/U^1$. Furthermore, upon using transformations from the equivalence group $G^{\sim}$, we can set $\tilde{\beta} = \beta = 0$, which consequently implies that $U^0 = 0$. By means of equivalence transformations, we can also set $\alpha, \tilde{\alpha} \in \{-1, 1\}$ and as the multiplier relating α and $\tilde{\alpha}$ is strictly positive, we have that $\tilde{\alpha} = \alpha =: \varepsilon \in \{-1, 1\}$. Then X_x is a constant and we can set $X = x$ and $\varkappa = 1$ using a scaling and a translation of x and a scaling of u, which belong to $G^{\sim}$. Therefore, $U = \omega u$, where $\omega := \sqrt{|T_t|}$ and hence $\omega_t \neq 0$. After taking into account all the conditions derived, we reduce the equation (4.27) to the form $\omega^3\tilde{g} + \omega(\omega^{-1})_{tt}u = g$. Differentiating the last equation with respect to t and dividing the result by $\omega^2\omega_t$, we obtain $\tilde{u}\tilde{g}_{\tilde{u}} + 3\tilde{g} = 4\tilde{\sigma}\tilde{u}$, where $\tilde{\sigma} := -\big(\omega(\omega^{-1})_{tt}\big)_t/(4\omega^3\omega_t)$ is a constant. The general solution of the equation for $\tilde{g}$ is $\tilde{g} = \tilde{\mu}(x)\tilde{u}^{-3} + \tilde{\sigma}\tilde{u}$. The expression for g is similar: $g = \mu(x)u^{-3} + \sigma u$, where $\mu = \tilde{\mu}$, and $\sigma := \tilde{\sigma}\omega^4 + \omega(\omega^{-1})_{tt}$ is, like $\tilde{\sigma}$, a constant. We rewrite the relation defining σ as an ordinary differential equation for ω, $(\omega^{-1})_{tt} = \sigma\omega^{-1} - \tilde{\sigma}\omega^3$. Up to scalings from $G^{\sim}$ there are only three essentially different values of σ (resp. $\tilde\sigma$), $\sigma,\tilde\sigma\in\{-1,0,1\}$. Finally, from the class (4.16) we single out the subclass of equations of the general form (4.46).

For each pair of values of σ, the corresponding equations from the subclass (4.46) with the same value of the parameter function μ are related by a point transformation. This is why within this subclass it suffices to classify admissible transformations with $\tilde\sigma=0$. We solve the equation $(\omega^{-1})_{tt}=\sigma\omega^{-1}$ with respect to ω and then construct T using the relation $T_t=\omega^2 \bmod G^\sim$. We find

$$T=\begin{cases}(a_1t+a_0)/(a_3t+a_2) & \text{if}\quad \sigma=0,\\ (a_1e^{2t}+a_0)/(a_3e^{2t}+a_2) & \text{if}\quad \sigma=1,\\ b_1\tan(t+b_0)+b_2 & \text{if}\quad \sigma=-1,\end{cases}$$

where a_0, …, a_3, are constants with $a_1a_2-a_0a_3\neq0$ that are determined up to a common nonvanishing multiplier, and b_0, b_1 and b_2 are constants with $b_1\neq0$.

In the case $\sigma=0$ we obtain a subgroup of the complete point symmetry group of the corresponding equation. This group is obviously isomorphic to $\mathrm{PGL}(2,\mathbb R)$. The condition $T_{tt}\neq0$ is equivalent to $a_3\neq0$ and we can assume $a_3=1$ due to the indeterminacy up to a constant multiplier. Then $a_0-a_1a_2\neq0$ and we gauge a_2, a_0 and a_1 to 0, 1 and 0 using the s-action of $\mathcal P^t(a_2)$ and the t-action of $\mathcal P^t(-a_1)\circ\mathcal D^t(c_2^2)\circ\mathcal D^u(c_2)$ with $c_2:=(a_0-a_1a_2)^{-1}$. All the above transformations from the equivalence group $G^\sim$ induce point symmetries of the equation under consideration. Therefore, we can assume that $T=t^{-1} \bmod G^\sim$, obtaining the family T2a of admissible transformations.

In the same way we derive that $T=\frac12e^{2t} \bmod G^\sim$ and $T=\tan t \bmod G^\sim$ if $\sigma=1$ and $\sigma=-1$, which gives the family T2b and T2c of admissible transformations, respectively.

We set $\mu_x\neq0$ and $\tilde\mu_{\tilde x}\neq0$ for admissible transformations from the family T2 since similar admissible transformations with $\mu_x=0$ are $G^\sim$-

equivalent to admissible transformations from the restriction of the family T1 to $\mathcal{W}_{11}\cap\mathcal{W}_{12}$. The equations of the form (4.46) with $\mu_x\neq0$ are not related to those with $\mu_x=0$ by point transformations.

Following the argumentation for the generating set $\mathcal{B}_0$ of $\mathcal{G}_0^\sim$ from the end of the simultaneous proof of Lemmas 4.17 and 4.18, we can show that the singled out set $\mathcal{B}_1$ of admissible transformations of the subclass $\mathcal{W}_1$ is a minimal self-consistent generating set with respect to $G^\sim$-equivalence for $\mathcal{G}_1^\sim$. □

The equivalence groups of the subclasses $\mathcal{W}_0$ and $\mathcal{W}_1$ coincide with the equivalence group $G^\sim$ of the entire class $\mathcal{W}$, and $\mathcal{G}^\sim=\mathcal{G}_0^\sim\sqcup\mathcal{G}_1^\sim$. Therefore, after uniting the generating (up to $G^\sim$-equivalence) sets $\mathcal{B}_0$ and $\mathcal{B}_1$ of $\mathcal{G}_0^\sim$ and $\mathcal{G}_1^\sim$, which are minimal and self-consistent with respect to $G^\sim$-equivalence within the corresponding groupoids, we get the generating (up to $G^\sim$-equivalence) set $\mathcal{B}$ of $\mathcal{G}^\sim$, which is minimal and self-consistent with respect to $G^\sim$-equivalence within $\mathcal{G}^\sim$.

Lemma 4.21. *A complete list of $G^\sim$-inequivalent Lie-symmetry extensions for equations of the general form* (4.46) *is exhausted by the following cases:*

1a–1c. *general* μ: $\quad\mathfrak{g}_\theta=\mathfrak{g}^\cap_\sigma$,

2a–2c. $\mu=\pm1$: $\quad\mathfrak{g}_\theta=\mathfrak{g}^\cap_\sigma+\langle\partial_x\rangle$,

3a–3c. $\mu=\nu x^{-2},\ \nu\neq0$: $\quad\mathfrak{g}_\theta=\mathfrak{g}^\cap_\sigma+\langle2x\partial_x-u\partial_u\rangle$,

4a–4c. $\mu=0$: $\quad\mathfrak{g}_\theta=\mathfrak{g}^\cap_\sigma+\langle\partial_x,\,2x\partial_x-u\partial_u\rangle$

with

a. $\sigma=0$: $\quad\mathfrak{g}^\cap_0=\langle\partial_t,\,2t\partial_t+u\partial_u,\,t^2\partial_t+tu\partial_u\rangle$,

b. $\sigma=1$: $\quad\mathfrak{g}^\cap_1=\langle\partial_t,\,e^{2t}(\partial_t+u\partial_u),\,e^{-2t}(\partial_t-u\partial_u)\rangle$,

c. $\sigma=-1$: $\quad\mathfrak{g}^\cap_{-1}=\langle\partial_t,\,\cos(2t)\partial_t-\sin(2t)u\partial_u,\,\sin(2t)\partial_t+\cos(2t)u\partial_u\rangle$.

The cases $\sigma=1$ and $\sigma=-1$ reduce to the case $\sigma=0$ with the same value of the parameter function $\mu=\mu(x)$ by the additional equivalence transformations $\tilde t=\frac12e^{2t}$, $\tilde x=x$, $\tilde u=e^tu$ and $\tilde t=\tan t$, $\tilde x=x$, $\tilde u=u\cos t$, respectively.

Proof. It follows from Lemma 4.20 that it suffices to classify only equations of the form (4.46) with $\sigma = 0$. Spitting the system of determining equations (4.31)–(4.35) for a Lie-symmetry vector field Q of the equation $\mathcal{L}_\theta$ with $\theta = (f, g) = (\varepsilon u^{-4}, \mu(x)u^{-3})$ with respect to u, we derive that the components of Q are of the form $\tau = \tau(t)$, $\xi = 2c_1x + c_0$, $\eta = (\frac{1}{2}\tau_t - c_1)u$, where $\tau_{ttt} = 0$, and c_0 and c_1 are constants with $(2c_1x + c_0)\mu_x = -4c_1\mu$. The four cases for μ from the lemma's statement arise in the course of analysis of the last equation. □

Lemmas 4.20 and 4.21 jointly imply that there are no more singular Lie-symmetry extensions within the class $\mathcal{W}_1$.

Remark 4.22. The groupoid of the class of equations of the form (4.46) with $\varepsilon = \pm 1$ and $\sigma \in \{-1, 0, 1\}$ can be represented in the form (1.1), where the parameter σ plays the role of γ, Φ_0 is the identity transformation of (t, x, u), and Φ_1 and Φ_{-1} are transformational parts of T2a and T2b, respectively. Since this class is a subclass of $\mathcal{W}_{12}$ that is obtained by gauging the arbitrary elements $\mathcal{W}_{12}$ with equivalence transformations of $\mathcal{W}_{12}$, then the equivalence groupoid of $\mathcal{W}_{12}$ is of similar structure. The analogue of the last claim also holds for the intermediate class of equations of the form (4.46) with $\varepsilon \in \mathbb{R}_{\neq 0}$ and $\sigma \in \mathbb{R}$.

4.2.6. Classification of Appropriate Subalgebras. The equivalence group $G^\cap$ and the equivalence algebra $\mathfrak{g}^\sim$ admit related representations in the form of a semi-direct product and a semi-direct sum, $G^\sim = \hat{G}^\cap \rtimes G^\sim_{\rm ess}$ and $\mathfrak{g}^\sim = \hat{\mathfrak{g}}^\cap \niplus \mathfrak{g}^\sim_{\rm ess}$, respectively. Here $\hat{G}^\cap = \{\mathcal{P}^t(c_0) \mid c_0 \in \mathbb{R}\}$ is the normal subgroup of $G^\sim$ associated with the kernel group $G^\cap$ of the class (4.16), $G^\sim_{\rm ess}$ is the subgroup of $G^\sim$ that consists of the transformations of the form (4.29) with $c_0 = 0$ and thus effectively acts on the class (4.16), $\hat{\mathfrak{g}}^\cap = \langle \mathcal{P}^t \rangle$ is the ideal of $\mathfrak{g}^\sim$ corresponding to the kernel algebra $\mathfrak{g}^\cap$ and $\mathfrak{g}^\sim_{\rm ess} = \langle \mathcal{D}^u, \mathcal{D}^t, \mathcal{D}(\zeta), \mathcal{Z}(\chi) \rangle$ is a subalgebra of $\mathfrak{g}^\sim$, which is the "essential" part of $\mathfrak{g}^\sim$ from the point of view of Lie-symmetry extensions within the class (4.16). Denote by π the projection from the space with the coordinates (t,x,u,f,g) to the space with the coordinates (t,x,u), and by π_*Q the pushforward of a projectable vector field Q in the space with the coordinates (t,x,u,f,g) by π. A subalgebra $\mathfrak a$ of $\mathfrak g^\sim$ is called appropriate if its projection $\pi_*\mathfrak a$ is the maximal Lie invariance algebra $\mathfrak g_\theta$ of an equation $\mathcal L_\theta$ from the class (4.16). Any appropriate subalgebra $\mathfrak a$ of $\mathfrak g^\sim$ should contain $\hat{\mathfrak g}^\cap$ as an ideal. Hence it can also be represented in the form of the semi-direct sum $\mathfrak a=\hat{\mathfrak g}^\cap\mathbin{\supset\!\!\!\!+}\mathfrak s$, where $\mathfrak s$ is a subalgebra of $\mathfrak g^\sim_{\rm ess}$. We call a subalgebra $\mathfrak s$ of $\mathfrak g^\sim_{\rm ess}$ *appropriate* if $\mathfrak s=\mathfrak g^\sim_{\rm ess}\cap\mathfrak a$ for an appropriate subalgebra $\mathfrak a$ of $\mathfrak g^\sim$. Appropriate subalgebras $\mathfrak a_1$ and $\mathfrak a_2$ of $\mathfrak g^\sim$ are $G^\sim$-equivalent if and only if the corresponding subalgebras $\mathfrak s_1$ and $\mathfrak s_2$ of $\mathfrak g^\sim_{\rm ess}$ are $G^\sim_{\rm ess}$-equivalent. As a result, the classification of Lie-symmetry extensions induced by subalgebras of $\mathfrak g^\sim$ up to $G^\sim_{\rm ess}$-equivalence reduces to the classification of appropriate subalgebras of $\mathfrak g^\sim_{\rm ess}$ up to $G^\sim_{\rm ess}$-equivalence.

For the latter classification, we need to compute the adjoint action of the group $G^\sim_{\rm ess}$ on the algebra $\mathfrak g^\sim_{\rm ess}$. Since this algebra is infinite-dimensional, it is convenient to realize this computation via pushing forward the vector fields $\mathcal D^u$, $\mathcal D^t$, $\mathcal D(\zeta)$ and $\mathcal Z(\chi)$, which span $\mathfrak g^\sim_{\rm ess}$, by elementary equivalence transformations from $G^\sim_{\rm ess}$, i.e., by $\mathscr D^t(c_1)$, $\mathscr G(\psi)$, $\mathscr D(\varphi)$ and $\mathscr D^u(c_2)$, cf. Section 4.2.2. In other words, the usual transformation rule of vector fields under point transformations will be used [21,72]. This yields the following non-identity actions:

$$
\begin{aligned}
&\mathscr G_*(\psi)\mathcal D^u=\mathcal D^u-\mathcal Z(\psi),\quad \mathscr G_*(\psi)\mathcal D(\zeta)=\mathcal D(\zeta)+\mathcal Z(\zeta\psi_x-\tfrac12\zeta_x\psi),\\
&\mathscr D^u_*(c_2)\mathcal Z(\chi)=c_2\mathcal Z(\chi),\quad \mathscr D_*(\varphi)\mathcal Z(\chi)=\mathcal Z\big(|\hat\varphi_x|^{-1/2}\chi(\hat\varphi)\big),\\
&\mathscr D_*(\varphi)\mathcal D(\zeta)=\mathcal D\big(\zeta(\hat\varphi)/\hat\varphi_x\big),
\end{aligned}
$$

where $\hat\varphi=\hat\varphi(x)$ is the inverse of the function φ.

All vector fields from $\pi_*\mathfrak g^\sim_{\rm ess}$ identically satisfy the determining equations for Lie symmetries of equations from the class (4.16), except the equations (4.34) and (4.35). The latter two equations imply restrictions on appropriate subalgebras of $\mathfrak g^\sim_{\rm ess}$.

Lemma 4.23. $\mathfrak{s}\cap\langle\mathcal{D}^u,\mathcal{Z}(\chi)\rangle=\mathfrak{s}\cap\langle\mathcal{D}^t\rangle=\{0\}$ *for any appropriate subalgebra* $\mathfrak{s}$.

Proof. Suppose that an appropriate subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\sim}_{\rm ess}$ contains a vector field $Q=b\mathcal{D}^u+\mathcal{Z}(\chi)$, where the constant b or the function $\chi=\chi(x)$ does not vanish. Then π_*Q is a Lie-symmetry vector field for an equation $\mathcal{L}_\theta$ from the class (4.16). Substituting the components of the vector field π_*Q into the determining equations (4.34) and (4.35) implies the following conditions for the arbitrary-element tuple $\theta=(f,g)$:

$$(bu+\chi)f_u=0,\quad (bu+\chi)g_u=bg-\chi_{xx}f.$$

Then $f_u=0$ and $g_{uu}=0$ if $b\neq0$ or $\chi\neq0$. This contradicts the definition of the class (4.16).

Analogously, the condition $\pi_*\mathcal{D}^t\in\mathfrak{g}_\theta$ gives the equation $f=0$, which is also inconsistent with the definition of the class (4.16).

Therefore, any appropriate subalgebra contains no vector fields of the forms considered. □

Lemma 4.24. $\dim\big(\mathfrak{s}\cap\langle\mathcal{D}(\zeta),\mathcal{Z}(\chi)\rangle\big)\in\{0,1,3\}$ *for any appropriate subalgebra* $\mathfrak{s}$.

Proof. Suppose that $\mathfrak{s}$ is an appropriate subalgebra of $\mathfrak{g}^{\sim}_{\rm ess}$ and $\dim\big(\mathfrak{s}\cap\langle\mathcal{D}(\zeta),\mathcal{Z}(\chi)\rangle\big)\geqslant2$. This means that the subalgebra $\mathfrak{s}$ contains (at least) two vector fields $Q^i=\mathcal{D}(\zeta^i)+\mathcal{Z}(\chi^i)$, where the functions ζ^i, $i=1,2$, should be linearly independent in view of Lemma 4.23. In other words, the projections π_*Q^i of Q^i simultaneously are Lie-symmetry vector fields of an equation from the class (4.16). By W we denote the Wronskian of the functions ζ^1 and ζ^2, $W=\zeta^1\zeta^2_x-\zeta^2\zeta^1_x$. $W\neq0$ as the functions ζ^1 and ζ^2 are linearly independent.

Plugging the coefficients of π_*Q^i into the equation (4.34) gives two equations with respect to f only,

$$2\zeta^if_x+(\zeta^i_xu+2\chi^i)f_u=4\zeta^i_xf.\tag{4.48}$$

We multiply the equation (4.48) with $i=1$ by ζ^2 and subtract it from the equation (4.48) with $i=2$ multiplied by ζ^1. Dividing the resulting equation by W, we obtain the ordinary differential equation $(u+\beta)f_u=4f$, where $\beta=\beta(x):=2(\zeta^1\chi^2_x-\zeta^2\chi^1_x)/W$ and the variable x plays the role of a parameter. It is possible to set $\beta=0$ by means of an equivalence transformation, $\mathcal{G}(-\beta)$. Indeed, this transformation preserves the form of the vector fields Q^i, only changing the values of the functional parameters χ^i. In particular, it does not affect the linear independency of the functions ζ^i. The integration of the above equation for $\beta=0$ yields that $f=\alpha u^4$, where $\alpha=\alpha(x)$ is a nonvanishing function of x. In view of the derived form of f, splitting of equations (4.48) with respect to u leads to $\zeta^i\alpha_x=0$ and $\chi^i\alpha=0$, i.e., $\alpha_x=0$ and $\chi^i=0$. The constant α can be scaled to $\alpha=\pm1$ by an equivalence transformation.

In a similar manner, consider the equation (4.35), taking into account the restrictions set on parameter functions and the form of f. For each Q^i, the equation (4.35) gives an equation with respect to g,

$$2\zeta^i g_x+\zeta^i_x u g_u=\zeta^i_x g-\zeta^i_{xxx}\alpha u^5. \tag{4.49}$$

Again, we multiply the equation (4.49) with $i=1$ by ζ^2 and subtract it from the equation (4.49) with $i=2$ multiplied by ζ^1, divide the resulting equation by W and thereby obtain that $ug_u=g+\mu^0u^5$, where $\mu^0=\mu^0(x):=-\alpha(\zeta^1\zeta^2_{xxx}-\zeta^2\zeta^1_{xxx})/W$ and the variable x again plays the role of a parameter. Integrating the last equation for g directly gives $g=\mu^0u^5/4+\mu^1u$, where $\mu^1=\mu^1(x)$ is a smooth function of x. The parameter function μ^1 can be set equal to zero by the equivalence transformation $\mathcal{D}(\varphi)$, where the function $\varphi=\varphi(x)$ is a solution of the equation $\alpha(2\varphi_{xxx}\varphi_x-\varphi_{xx}^{\ 2})+\mu^1\varphi_x^{\ 2}=0$. Substituting the derived form of g into equations (4.49) and splitting with respect to u, we find that $\mu^1_x=0$, $\zeta^i_{xxx}=0$.

Summing up, we have proved that any equation of the class (4.16) admitting (at least) two linearly independent vector fields π_*Q^i in fact possesses exactly three linearly independent vector fields of this form and is $G^{\sim}$-equivalent to an equation of the form $u_{tt} = \pm u^4 u_{xx} + \mu^1 u$, where μ^1 is a constant which can be scaled to ± 1 if it is not zero. $\square$

The equation $u_{tt} = \pm u^4 u_{xx}$, for which $\mu^1 = 0$, admits an additional Lie-symmetry extension.

Corollary 4.25. *There are only two $G^{\sim}$-inequivalent cases of Lie-symmetry extensions in the class* (4.16) *where the corresponding Lie invariance algebras contain at least two linearly independent vector fields of the form $\pi_* Q^i$ with $Q^i = \mathcal{D}(\zeta^i) + \mathcal{Z}(\chi^i)$,*

14d. $u_{tt} = \varepsilon u^4 u_{xx} + \varepsilon' u\colon \quad \mathfrak{g}^{\max} = \mathfrak{g}^{\cap} + \pi_* \big\langle \mathcal{D}(1), \mathcal{D}(x), \mathcal{D}(x^2) \big\rangle$,

19d. $u_{tt} = \varepsilon u^4 u_{xx}\colon \quad \mathfrak{g}^{\max} = \mathfrak{g}^{\cap} + \pi_* \big\langle \mathcal{D}(1), \mathcal{D}(x), \mathcal{D}(x^2), \mathcal{D}^u - 2\mathcal{D}^t \big\rangle$

with $\varepsilon, \varepsilon' = \pm 1$.

Corollary 4.25 gives the classification of appropriate subalgebras of $\mathfrak{g}^{\sim}_{\rm ess}$ the dimensions of whose intersections with $\langle \mathcal{D}(\zeta), \mathcal{Z}(\chi) \rangle$ are not less than two. Hence we should continue with the computation of inequivalent appropriate subalgebras of $\mathfrak{g}^{\sim}_{\rm ess}$ that contain at most one linearly independent vector field of the form $\mathcal{D}(\zeta) + \mathcal{Z}(\chi)$, where $\zeta = \zeta(x)$ is a nonvanishing function. In view of Lemma 4.23 it is obvious that the dimension of such subalgebras cannot be greater than three. Here we select candidates for such subalgebras using only restrictions on appropriate subalgebras presented in Lemma 4.23. Since there exist specific restrictions for two- and three-dimensional appropriate subalgebras, we will make an additional selection of appropriate subalgebras from the set of candidates directly in the course of the construction of invariant equations.

The result of the classification is formulated in the subsequent lemmas.

Lemma 4.26. *A complete list of $G^{\sim}_{\rm ess}$-inequivalent appropriate one-dimensional subalgebras of $\mathfrak{g}^{\sim}_{\rm ess}$ is given by*

$$\langle 2\mathcal{D}^u - q\mathcal{D}^t + 2\mathcal{D}(\delta) \rangle, \quad \langle \mathcal{D}^t - \mathcal{D}(2) \rangle, \quad \langle \mathcal{D}^t - \mathcal{Z}(2) \rangle, \quad \langle \mathcal{D}(1) \rangle, \tag{4.50}$$

where $\delta \in \{0,1\}$ and q is an arbitrary constant.

Remark 4.27. In Lemma 4.26 and in the next two lemmas, we choose such values of parameters in basis elements of appropriate subalgebras among possible ones up to $G^{\sim}$-equivalence that the corresponding equations from the class $\mathcal{W}$ have a simple form.

Lemma 4.28. *Up to $G^{\sim}_{\rm ess}$-equivalence, any appropriate two-dimensional subalgebra of $\mathfrak{g}^{\sim}_{\rm ess}$ which contains at most one linearly independent vector field of the form $\mathcal{D}(\zeta)+\mathcal{Z}(\chi)$ belongs to the following list:*

$$\begin{gathered}\langle \mathcal{D}^u-\mathcal{D}(p),\,\mathcal{D}^t-\mathcal{D}(2)\rangle,\quad \langle \mathcal{D}^u-2\mathcal{D}(x),\,\mathcal{D}^t-\mathcal{Z}(2)\rangle,\\ \langle a_1\mathcal{D}^u+a_2\mathcal{D}^t+a_3\mathcal{D}(x)+\mathcal{Z}(\delta),\,\mathcal{D}(1)\rangle,\end{gathered}\tag{4.51}$$

where p, a_1, a_2, a_3 and δ are constants with $p\neq 0$, $(a_1,a_2)\neq(0,0)$, $(a_2,a_3)\neq(0,0)$ and $(a_1,a_3,\delta)\neq(0,0,0)$. Due to scaling of the first basis element and $G^{\sim}_{\rm ess}$-equivalence we can also assume that one of a's equals 1, $(2a_1+a_3)\delta=0$, and $\delta\in\{0,1\}$.

Lemma 4.29. *Up to $G^{\sim}_{\rm ess}$-equivalence, any appropriate three-dimensional subalgebra of $\mathfrak{g}^{\sim}_{\rm ess}$, which contains at most one linearly independent vector field of the form $\mathcal{D}(\zeta)+\mathcal{Z}(\chi)$, has one of the forms*

$$\begin{gathered}\langle \mathcal{D}^u+p_1\mathcal{D}(x),\,\mathcal{D}^t+p_2\mathcal{D}(x),\,\mathcal{D}(1)\rangle,\\ \langle \mathcal{D}^u-2\mathcal{D}(x)+\mathcal{Z}(d),\,\mathcal{D}^t-\mathcal{Z}(2),\,\mathcal{D}(1)\rangle,\end{gathered}\tag{4.52}$$

where p_1, p_2 and d are constants such that $p_1p_2\neq 0$.

4.2.7. Regular Lie Symmetry Extensions For each vector field $\mathcal{Q}$ from $\mathfrak{g}^{\sim}$, the substitution of the components of $\pi_*\mathcal{Q}$ into the system (4.34)–(4.35) results in the condition on the arbitrary-element tuple $\theta=(f,g)$ for the equation $\mathcal{L}_\theta$ to be invariant with respect to $\pi_*\mathcal{Q}$. This is why equations from the class (4.16) that are invariant with respect to the projection $\pi_*\mathfrak{s}$ of an appropriate subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\sim}$ can be described by the following way: For each basis element $\mathcal{Q}$ of $\mathfrak{s}$, we substitute the components of $\pi_*\mathcal{Q}$ into the equations (4.34) and (4.35). Collecting all the equations derived from the entire basis $\mathfrak{s}$ leads to a system of first-order (quasi)linear partial differential equations in the arbitrary elements f and g to be solved. Simultaneously we check whether the projection $\pi_*\mathfrak{s}$ is really the maximal Lie invariance algebra of the equation $\mathcal{L}_\theta$ for obtained values of the arbitrary-element tuple $\theta=(f,g)$.

Each of the algebras listed in Lemma 4.26 is really an appropriate one-dimensional subalgebra of $\mathfrak{g}^\sim_{\rm ess}$ and results in a simple uncoupled system of two first-order linear differential equations in f and g. The corresponding list of equations from the class (4.16), which possess one-dimensional Lie-symmetry extensions of $\mathfrak{g}^\cap$ related to $\mathfrak{g}^\sim$, reads

1. $2\mathcal{D}^u-q\mathcal{D}^t+2\mathcal{D}(\delta)\colon\quad u_{tt}=|u|^q(\hat f(\omega)u_{xx}+\hat g(\omega)u),$
2. $\mathcal{D}^t-\mathcal{D}(2)\colon\quad u_{tt}=e^x(\hat f(u)u_{xx}+\hat g(u)),$
3. $\mathcal{D}^t-\mathcal{Z}(2)\colon\quad u_{tt}=e^u(\hat f(x)u_{xx}+\hat g(x)),$
4. $\mathcal{D}(1)\colon\quad u_{tt}=\hat f(u)u_{xx}+\hat g(u),$

where $\omega:=x-\delta\ln|u|$, $\delta\in\{0,1\}$ and q is an arbitrary constant. Here and in what follows, in each case we present only vector fields that extend the basis $(\mathcal{P}^t)$ of the ideal $\hat{\mathfrak{g}}^\cap$ of $\mathfrak{g}^\sim$ into a basis of the corresponding subalgebra of $\mathfrak{g}^\sim$.

The computation related to two-dimensional extensions are more complicated. We first present its result and then give some explanations.

9. $\mathcal{D}^u-\mathcal{D}(p),\ \mathcal{D}^t-\mathcal{D}(2),\ p\neq0\colon\quad u_{tt}=\pm e^x|u|^p(u_{xx}+\nu u),$
10. $\mathcal{D}^u-2\mathcal{D}(x),\ \mathcal{D}^t-\mathcal{Z}(2)\colon\quad u_{tt}=\pm x^2e^uu_{xx}+\nu e^u,$
11. $-\mathcal{D}^u+2\mathcal{D}^t+2\mathcal{D}(x),\ \mathcal{D}(1)\colon\quad u_{tt}=\hat f(u)u_{xx},$
12. $(1-q)\mathcal{D}^u+2q\mathcal{D}^t-2(1-q)\mathcal{D}(x)-\mathcal{Z}(4),\ \mathcal{D}(1)\colon$
    $u_{tt}=\pm e^uu_{xx}+\varepsilon'e^{qu},$
13. $(3-p+q)D^u+2(1-q)\mathcal{D}^t+2(1+p-q)\mathcal{D}(x),\ \mathcal{D}(1)\colon$
    $u_{tt}=\pm|u|^pu_{xx}+\varepsilon'|u|^q.$

Constraints for constant and functional parameters that are imposed by the maximality condition for the corresponding extensions and their inequivalence are discussed after Theorem 4.11.

Cases 9 and 10 are associated with the first and second families of subalgebras listed in Lemma 4.28, respectively. In both the cases, ν is an arbitrary constant. Note that an arbitrary nonzero constant multiplier in the expression for the arbitrary element f, which arises in the course of integrating the equation for f, can always be set to ± 1, e.g., by scaling of t.

The third span from Lemma 4.28 in fact represents a multiparametric series of candidates for appropriate extensions, which is partitioned in the course of the construction of invariant equations into Cases 11–13. Not all values of series parameters give appropriate extensions. Additional constraints for parameters follow from the compatibility conditions of the associated system in the arbitrary elements,

$$
\begin{aligned}
&f_x = 0, \qquad ((a_1 + \tfrac{1}{2}a_3)u + \delta)f_u = pf,\\
&g_x = 0, \qquad ((a_1 + \tfrac{1}{2}a_3)u + \delta)g_u = qg,
\end{aligned}
$$

with the inequalities $f \neq 0$ and $(f_u, g_{uu}) \neq (0, 0)$ and the requirement that the dimension of extensions should not exceed two. Here we introduce the notation $p = 2(a_3 - a_2)$ and $q = a_1 + \frac{1}{2}a_3 - 2a_2$.

The above partition is carried out in the following way. If $a_3 = -2a_1$ and $\delta = 0$, the inequality $f \neq 0$ implies that $p = 0$, i.e., $a_2 = a_3$. Since a_1, a_2 and a_3 cannot simultaneously be zero, we obtain that $q \neq 0$ and hence $g = 0$. Multiplying the first basis element by $-a_1^{-1}$, we set $a_1 = -1$. This gives Case 11. For $a_3 = -2a_1$ and $\delta = 1$ we have $a_2 = -q/2$, $a_3 = (p-q)/2$ and $a_1 = -(p-q)/4$. The parameter p should be nonzero since otherwise we obtain the Liouville equation whose maximal Lie invariance algebra is infinite-dimensional. We additionally multiply the first basis element by -4 and scale p with $\mathcal{D}^u(c_2)$ for some c_2 to 1 and obtain Case 12. Case 13 corresponds to the condition $a_3 \neq -2a_1$. Scaling the first basis element allows us to set $a_1 + \frac{1}{2}a_3 = 4$. Then $a_2 = 2(1-q)$, $a_3 = 2(1+p-q)$ and $a_3 = (3-p+q)$. In both Cases 12 and 13 the parameter ε' is nonzero (otherwise the extension dimension is greater than two) and can be gauged to ±1 by simultaneous scaling of t and x.

Consider the candidates for three-dimensional appropriate extensions listed in Lemma 4.29. The compatibility of the associated systems in the arbitrary elements, supplemented with the inequality $f \neq 0$, implies $p_1 = 2(p_2-1)$ and $d=-4$ for the first and the second span of Lemma 4.29, respectively. The general solutions of these systems up to $G^{\sim}$-equivalence are $(f,g) = (\pm|u|^p, 0)$ and $(f,g) = (\pm e^u, 0)$. This gives the following cases of Lie-symmetry extensions:

16. $(p-4)\mathcal{D}^u - 2p\mathcal{D}(x)$, $(p-4)\mathcal{D}^t - 4\mathcal{D}(x)$, $\mathcal{D}(1)$, $p \neq 0, 4$:
    $u_{tt} = \pm|u|^p u_{xx}$,
17. $\mathcal{D}^u - 2\mathcal{D}(x) - \mathcal{Z}(4)$, $\mathcal{D}^t - \mathcal{Z}(2)$, $\mathcal{D}(1)$:
    $u_{tt} = \pm e^u u_{xx}$.

Here $p := 4(p_2-1)/p_2 \neq 4$ since for $p=4$ the corresponding equations admits the Lie-symmetry vector fields $\pi_*\mathcal{D}(x)$ and $\pi_*\mathcal{D}(x^2)$. Equations from the class (4.16) which are invariant with respect to two linearly independent vector fields of the form π_*Q^i, where $Q^i = \mathcal{D}(\zeta^i) + \mathcal{Z}(\chi^i)$, are classified in Corollary 4.25. Therefore, $G^{\sim}$-inequivalent regular Lie-symmetry extensions in the class (4.16) are exhausted by Cases 1–4, 9–13, 14d, 16, 17 and 19d.

4.2.8. Concluding Remarks. The complete group classification of the class $\mathcal{W}$ of (1+1)-dimensional nonlinear wave and elliptic equations of the form (4.16) is performed up to both $G^{\sim}$- and $\mathcal{G}^{\sim}$-equivalences using the new version of the algebraic method of group classification for non-normalized classes of differential equations. The results of the classification are collected in Theorem 4.11. The key ingredient of the classification procedure is the construction of a generating set for the equivalence groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{W}$ modulo $G^{\sim}$-equivalence. This generating set is given in Theorem 4.12. In view of the partition $\mathcal{G}^{\sim}_{\rm gen} = \mathcal{G}^{\sim} \sqcup \mathcal{G}^{\sim}_{\rm lin}$ of the equivalence groupoids $\mathcal{G}^{\sim}_{\rm gen}$ of the superclass $\mathcal{W}_{\rm gen}$ of all equations of the form (4.16) with $f \neq 0$, cf. Remark 4.8, we can merge the results on $\mathcal{W}$ with the analogous results from Remark 4.19 on the class $\mathcal{W}_{\rm lin}$ of linear equations of the form (4.16) to those for $\mathcal{W}_{\rm gen}$. Thus, we have also obtained the complete group classifications of the classes $\mathcal{W}_{\rm lin}$ and $\mathcal{W}_{\rm gen}$ and the classifications of admissible transformations of these classes.

Below we compare this paper's results with some similar results existing in the literature for related classes of differential equations. The problem of group classification for the class of semilinear wave equations of the general form

$$u_{tt} = u_{xx} + g(t, x, u, u_x) \tag{4.53}$$

was solved in [181, 182]. The class (4.53) was partitioned into four (normalized) subclasses, and each of these subclasses was classified separately. One of these subclasses, which we denote by $\mathcal{K}$, is singled out from the class (4.53) by the constraint $g_{u_x} = 0$. The group classification of the subclass $\mathcal{K}$ was carried out in Section 6 of [182] and the major part of classification results was collected in Table 1 therein, see also Section V and Table I in [181]. Cases $1_{\delta=1,p=2,\hat f=1}$, $2_{\hat f=1}$, $5\mathrm{a}_{\varepsilon=1}$, $6\mathrm{a}_{\varepsilon=1}$ and $18\mathrm{a}_{\varepsilon=1}$ of Table 4.6 in the present paper correspond to Cases 3, 2, 8, 5 and 9 of Table 1 in [182], whereas the Liouville equation is given as Case 20 in Table 4.6 of the present paper and as the equation (5.4) in [182], and this exhausts all possible analogous cases. The counterpart of Case $1_{\delta=1,p\neq2,\hat f=1}$ of our Table 4.6 is missed in [181, 182]. In fact, each of Cases 3 and 4 of Table 1 in [182] should contain one more constant parameter, which cannot be removed by equivalence transformations of the subclass $\mathcal{K}$. In [179,180], Lahno and Spichak classified the semilinear elliptic equations of the rather general form

$$u_{tt} + u_{xx} = F(t, x, u, u_t, u_x)$$

whose maximal Lie invariance algebras are finite-dimensional. Cases 6a, 7, 5a and 18a of Table 4.6 are the restrictions of the first cases of Theorems 3.1 and 3.2 from [179] and of the cases "$A_{3.8}$-Invariant Equations (1)" and "$A_{4.10}$-Invariant Equations (3)" from [180] to the class $\mathcal{W}$, respectively. There are no other related cases in [179, 180] and the present paper.

More important than the solutions of the above specific classification problems are the development and modification of general concepts and techniques as well as their combinations that have been carried out in the course of solving these problems in the present paper.

Partitions of classes of differential equations into subclasses that induce partitions of the corresponding equivalence groupoids had already regularly been applied in the course of the study of equivalence groupoids [21, 239, 248]. We have made the two partitions of classes, $\mathcal{W}_{\rm gen}=\mathcal{W}\sqcup\mathcal{W}_{\rm lin}$ and $\mathcal{W}=\mathcal{W}_0\sqcup\mathcal{W}_1$, obtaining the partitions of the groupoids

$$\mathcal{G}^\sim_{\rm gen}=\mathcal{G}^\sim\sqcup\mathcal{G}^\sim_{\rm lin}\quad\text{and}\quad\mathcal{G}^\sim=\mathcal{G}^\sim_0\sqcup\mathcal{G}^\sim_1,$$

see Remark 4.8 and Proposition 4.16. All the above classes and subclasses have the same equivalence group $G^\sim$. Nevertheless, in contrast to the examples existing in the literature, the subclasses in these two partitions do not have better normalization properties than their superclasses. This is why no kind of normalization can be used for justifying the partitions, which are rather derived via the direct analysis of the determining equations for admissible transformations. Although the structure of the partition components is simpler than the entire groupoid for both the groupoid partitions, this becomes clear only after a comprehensive study of admissible transformations.

We have separately constructed generating sets $\mathcal{B}_0$ and $\mathcal{B}_1$ of the equivalence groupoids $\mathcal{G}^\sim_0$ and $\mathcal{G}^\sim_1$, which are constituted by the families T1$|_{\mathcal{W}_0}$ and T3–T9 and by the families T1$|_{\mathcal{W}_1}$ and T2 given in Theorem 4.12, respectively. Due to constructing $\mathcal{B}_0$ and $\mathcal{B}_1$ modulo $G^\sim$-equivalence, we can and should factor out elements of the action groupoid $\mathcal{G}^{G^\sim}$ from admissible transformations before including to these sets. This is realized via successively gauging arbitrary elements of singled out subclasses of equations that are sources or, equivalently, targets of elements from $\mathcal{G}^\sim \setminus \mathcal{G}^{G^\sim}$. In other words, mapping these subclasses onto smaller ones with simpler equivalence groupoids by families of equivalence transformations, we have factored out subgroups of $G^\sim$ and have simplified the consideration for the corresponding classification cases.

To classify admissible transformations of the class $\mathcal{W}_0$ in the optimal way, we split the construction of the generating set $\mathcal{B}_0$ in the simultaneous proof of Lemmas 4.17 and 4.18 into two cases depending on the number of independent constraints for arbitrary elements that arise in the course of the classification. In this way, we have extended for the first time the method of furcate splitting to the construction of generating sets of admissible transformations.

Moreover, we have found a bijective functor between two categories, which are the equivalence groupoids $\mathcal{G}^\sim_{00\varepsilon'}$ and $\mathcal{G}^\sim_{01g_2}$ of the subclasses $\mathcal{W}_{00\varepsilon'}$ and $\mathcal{W}_{01g_2}$ of equations of the forms (4.42) and (4.45) with $g^1 \neq 0$ and a fixed $\varepsilon' \in \{-1, 1\}$ and with a fixed g^2 satisfying $g^2_u g^2_{uuu} \neq (g^2_{uu})^2$, respectively. The isomorphism from $\mathcal{G}^\sim_{00\varepsilon'}$ to $\mathcal{G}^\sim_{01g_2}$ is given by

$$
\begin{aligned}
&\varepsilon \mapsto \check{\varepsilon} = \varepsilon, \quad g^1 \mapsto \check{g}^1 = g^1, \\
&\Phi\colon\ \tilde{t} = T,\ \tilde{x} = X,\ \tilde{u} = u - \ln |{X_x}^2 - \varepsilon {X_t}^2| \quad \mapsto \\
&\check{\Phi}\colon\ \tilde{t} = T,\ \tilde{x} = X,\ \tilde{u} = u.
\end{aligned}
$$

Fixing ε' and g^2 is natural since values of these parameters cannot be changed by admissible transformations in the entire classes $\mathcal{W}_{00}$ and $\mathcal{W}_{01}$ up to gauge equivalence transformations of moving a nonzero constant multiplier between g^1 and g^2 within $\mathcal{W}_{01}$, which can be neglected. That is, the partitions of the classes $\mathcal{W}_{00}$ and $\mathcal{W}_{01}$ into the subclasses associated with fixed values of ε' and of g^2, $\mathcal{W}_{00} = \sqcup_{\varepsilon'}\mathcal{W}_{00\varepsilon'}$ and $\mathcal{W}_{01} = \sqcup_{g^2}\mathcal{W}_{01g^2}$, induce the partition of the corresponding equivalence groupoids,

$$\mathcal{G}^{\sim}_{00} = \sqcup_{\varepsilon'} \mathcal{G}^{\sim}_{00\varepsilon'} \quad \text{and} \quad \mathcal{G}^{\sim}_{01} = \sqcup_{g^2} \mathcal{G}^{\sim}_{01g^2}.$$

No equations from $\mathcal{W}_{00\varepsilon'}$ are related to equations from $\mathcal{W}_{01g_2}$ by point transformations. In other words, the functor from $\mathcal{G}^{\sim}_{00\varepsilon'}$ to $\mathcal{G}^{\sim}_{01g_2}$ is not underlaid by a family of point transformations generating a mapping from $\mathcal{W}_{00\varepsilon'}$ onto $\mathcal{W}_{01g_2}$ or conversely. Nevertheless, it allows us to easily obtain the equivalence groupoid $\mathcal{G}^{\sim}_{01g_2}$ from the equivalence groupoid $\mathcal{G}^{\sim}_{00\varepsilon'}$.

A necessary preliminary step for finding the above functor is the proper selection of classes to be related via a functor. For the first (degenerate) case in the simultaneous proof of Lemmas 4.17 and 4.18, under the gauge $f=\varepsilon$ we derive the specific form $g=g^0(x)e^u+g^1(x)$ for values of g of the source and target equations of admissible transformations that are not generated by elements of $G^{\sim}$. There are two possibilities for a further gauging of parameters in the above form of g, either to $g^1=0$ or to $g^0=\varepsilon'$. The first possibility seems preferable since after gauging we obtain equations of the same general form as those in the class (4.45). In this way, the study can be reduced to describing the equivalence groupoid of the single class of equations of the form (4.45), where the auxiliary inequality $g^2_u g^2_{uuu} \neq (g^2_{uu})^2$ is neglected. At the same time, the structure of the subgroupoid of the above groupoid that is the equivalence groupoid of the subclass singled out by the constraint $g^2_u g^2_{uuu} = (g^2_{uu})^2$ is different from and more complicated than the structure of its complement, and thus this subgroupoid needs a separate consideration. As a result, the preferable gauge is in fact $g^0=\varepsilon'$. Although we then have to study two classes of equations of different forms, via excluding the evidently marked out value $g^1=0$, which corresponds to the Liouville equations giving rise to the family T9 of admissible transformations, and via fixing ε' and g^2 we have partitioned the corresponding equivalence groupoids into naturally isomorphic subgroupoids. Therefore, it suffices to describe only one of them.

It is convenient to construct a generating set for the equivalence groupoid $\mathcal{G}^{\sim}_{00\varepsilon'}$ up to the equivalence group of the subclass $\mathcal{W}_{00\varepsilon'}$ since then we can apply various algebraic techniques, including an original extension of Hydon's algebraic method to admissible transformations.[4.5] These techniques are based on knowing the maximal Lie invariance algebras of equations from the subclass $\mathcal{W}_{00\varepsilon'}$ whose efficient classification involves a preliminary knowledge on admissible transformations within the subclass $\mathcal{W}_{00\varepsilon'}$. This is why we have merged the proofs of Lemmas 4.17 and 4.18. Mapping the families T3′–T7′ into the families T3–T7 and uniting the restrictions of the families T3–T8 to $\mathcal{W}_{01}$ and to the class of equations of the same form with $g_u^2 g_{uuu}^2 = (g_{uu}^2)^2$ provide us with the presentation of the final results in Theorem 4.12 in a concise form.

An unexpected by-product of the proper additional gauging of the arbitrary elements for equations from the class $\mathcal{W}_0$ with $f = \varepsilon$ and $g = g^0(x)e^u + g^1(x)$ by transformations from the group $G^{\sim}$ is that this gauging is in accordance with the maximal natural gauging of the arbitrary elements within the class $\mathcal{W}_{\rm lin}$ by transformations from the same group, which leads to the subclass $\mathcal{W}_{\rm lin'}$ of $\mathcal{W}_{\rm lin}$. There exists a canonical isomorphism between the essential equivalence groupoid $\mathcal{G}^{\sim\rm ess}_{\rm lin'}$ of $\mathcal{W}_{\rm lin'}$ and the equivalence groupoid of the class of equations of the form (4.42) with a fixed value of ε', and it is the above concordance that makes this existence evident. As a result, the complete group classifications of the class $\mathcal{W}_{\rm lin}$ up to $G^{\sim}$- and $\mathcal{G}^{\sim}$-equivalences and the classification of admissible transformations within this class are carried out in the single Remark 4.19. This is one more demonstration of the efficiency of the functor method in classification problems of group analysis of differential equations. Note that analogously to the previous isomorphism between $\mathcal{G}^{\sim}_{00\varepsilon'}$ and $\mathcal{G}^{\sim}_{01g_2}$, this groupoid isomorphism is not induced by families of admissible point transformations within

[4.5] This consideration shows that the algebraic method can further be developed to the construction of the complete equivalence groupoids for classes of differential equations via applying the algebraic method to the corresponding equivalence algebroids, which are infinitesimal counterparts of the equivalence groupoids.

the superclass $\mathcal{W}_{\rm gen}$.

Necessary preliminaries for the classification of singular Lie-symmetry extensions within the subclass $\mathcal{W}_1$ have been given by the classification of admissible transformations within this subclass. As a result, the former classification can easily be completed by either the direct or the algebraic method. The classification of regular Lie-symmetry extensions has been carried out within the framework of the algebraic method and has reduced to the preliminary group classification of the subclass $\mathcal{W}$. We have used our optimized version of this method, which involves the classification of candidates for appropriate subalgebras of $\mathfrak{g}^{\sim}$ by taking into account the principal restrictions on the dimensions and structure of such subalgebras and the completion of selecting appropriate subalgebras in the course of constructing the corresponding equations possessing Lie-symmetry extensions.

4.3. Lie Symmetries of (2+1)-Dimensional Nonlinear Dirac Equations

In 2004/2005 the first truly two-dimensional solid state material, graphene, was created in the laboratory [210, 211, 331]. Later in 2010 the Nobel Prize in Physics was awarded to A. Geim and K. Novoselov "for groundbreaking experiments regarding the two-dimensional material graphen". It was noted in [210] that in graphene "electron transport is essentially governed by Dirac's (relativistic) equation." The authors of [331] has presented a new class of nonlinear phenomena in Bose–Einstein condensates in a honeycomb optical lattice, that can be described by a nonlinear Dirac equation (NLDE) in 2+1 dimensions. The form of the nonlinearity appeared as a natural physical result of binary interactions between bosons. It was shown that NLDE for Bose–Einstein condensates breaks the Lie symmetry governed by Poincaré algebra. After these works there were a number of papers with studies of NLDEs in two spatial dimensions. For example, some exact stationary state solutions of a nonlinear Dirac equation in 2+1 dimensions was constructed in [113] (see also [115]). The important results on symmetries of NLDEs can be found in [89, 95].

Inspired by the importance of (2+1)-dimensional NLDEs we investigate Lie symmetries of the following model equations

$$(i\sigma_2\partial_t + \sigma_1\partial_x - \sigma_3\partial_y)\Psi = \Phi, \quad \text{where} \quad \Psi = \begin{pmatrix} u \\ v \end{pmatrix}, \ \Phi = \begin{pmatrix} F \\ G \end{pmatrix}. \tag{4.54}$$

Here $u = u(t, x, y)$ and $v = v(t, x, y)$ are dependent variables, $F = F(u, v)$ and $G = G(u, v)$ are arbitrary smooth functions which are not linear in u and v simultaneously, σ_1, σ_2, and σ_3 are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Equation (4.54) can be rewritten as the coupled system of first-order partial differential equations (PDE system)

$$\begin{aligned} v_t + v_x - u_y &= F(u, v), \\ u_t - u_x - v_y &= -G(u, v). \end{aligned} \tag{4.55}$$

It is well known that the Dirac equation describes a complex wave function. However, for some physical problems it is possible to restrict ourselves to real solutions like it is done in theories of massive neutrino. In this case a special (Majorana) representation of the Dirac matrices with purely imaginary entries should be used. In the present paper we also restrict ourselves to the Dirac equation for real wave functions.

4.3.1. Lie symmetries. We study Lie symmetries of PDE systems from class (4.55) using the classical approach [217, 227]. We fix a system $\mathcal{L}$ from class (4.55) and search for vector fields of the form

$$X = \xi^t(t, x, y, u, v)\partial_t + \xi^x(t, x, y, u, v)\partial_x + \xi^y(t, x, y, u, v)\partial_y$$

$$+\,\eta^u(t,x,y,u,v)\partial_u + \eta^v(t,x,y,u,v)\partial_v$$

that generate one-parameter point symmetry groups of $\mathcal{L}$. These vector fields form the maximal Lie invariance algebra $A^{\max} = A^{\max}(\mathcal{L})$ of the PDE system $\mathcal{L}$. Any such vector field X satisfies the infinitesimal invariance criterion, i.e., we require that

$$\begin{aligned} &X^{(1)}\big(v_t + v_x - u_y - F(u,v)\big)\big|_{\mathcal{L}} = 0,\\ &X^{(1)}\big(u_t - u_x - v_y + G(u,v)\big)\big|_{\mathcal{L}} = 0. \end{aligned} \tag{4.56}$$

After the elimination of u_t and v_t by means of (4.55), equations (4.56) become identities in nine variables, t, x, y, u, v, u_x, u_y, v_x and v_y. In fact, equations (4.56) are polynomials in the variables u_x, u_y, v_x and v_y. The coefficients of different powers of these variables are zero, which gives twenty four determining equations for the coefficients ξ^t, ξ^x, ξ^y, η^u and η^v. The computations were verified using GeM software package for computation of symmetries and conservation laws of differential equations [57].

At first we solve those determining equations, which do not involve F and G, and find the form of the coefficients of the operator X:

$$\begin{aligned} \xi^t &= \tfrac{1}{2}\alpha_0(t^2+x^2+y^2) + (\delta + \alpha_1 x + \alpha_2 y)t + \beta_1 y + \beta_2 x + \rho_0,\\ \xi^x &= \tfrac{1}{2}\alpha_1(t^2+x^2-y^2) + (\delta + \alpha_0 t + \alpha_2 y)x - \beta_0 y + \beta_2 t + \rho_1,\\ \xi^y &= \tfrac{1}{2}\alpha_2(t^2-x^2+y^2) + (\delta + \alpha_0 t + \alpha_1 x)y + \beta_0 x + \beta_1 t + \rho_2,\\ \eta^u &= \lambda u + \varphi - \tfrac{1}{2}(\beta_2 + \alpha_0 x + \alpha_1 t)u\\ &\quad + \tfrac{1}{2}(\alpha_2(x-t) - (\alpha_0+\alpha_1)y - (\beta_0+\beta_1))v,\\ \eta^v &= \lambda v + \psi + \tfrac{1}{2}(\beta_2 + \alpha_0 x + \alpha_1 t)v\\ &\quad - \tfrac{1}{2}(\alpha_2(x+t) + (\alpha_0-\alpha_1)y + (\beta_0-\beta_1))u, \end{aligned}$$

where α_i, β_i, ρ_i, $i = 0,1,2$, and δ are arbitrary constants whereas λ, φ, and ψ are arbitrary smooth functions of the independent variables t, x, and y.

Therefore, the general form of the infinitesimal operator is

$$X = \chi + \alpha_i K^i + \beta_i J^i + \delta D + \rho_i P^i, \tag{4.57}$$

where α_i, β_i, ρ_i, $i=0,1,2$, and δ are arbitrary constants and

$$\begin{aligned}
&\chi=(\lambda u+\varphi)\partial_u+(\lambda v+\psi)\partial_v, \quad P^0=\partial_t, \quad P^1=\partial_x, \quad P^2=\partial_y,\\
&D=t\partial_t+x\partial_x+y\partial_y, \quad J^0=x\partial_y-y\partial_x-\tfrac{1}{2}v\partial_u+\tfrac{1}{2}u\partial_v,\\
&J^1=t\partial_y+y\partial_t-\tfrac{1}{2}v\partial_u-\tfrac{1}{2}u\partial_v, \quad J^2=t\partial_x+x\partial_t-\tfrac{1}{2}u\partial_u+\tfrac{1}{2}v\partial_v,\\
&K^0=\tfrac{1}{2}(t^2+x^2+y^2)\partial_t+tx\partial_x+ty\partial_y-\tfrac{1}{2}(xu+yv)\partial_u+\tfrac{1}{2}(xv-yu)\partial_v,\\
&K^1=tx\partial_t+\tfrac{1}{2}(t^2+x^2-y^2)\partial_x+xy\partial_y-\tfrac{1}{2}(tu+yv)\partial_u+\tfrac{1}{2}(yu+tv)\partial_v,\\
&K^2=ty\partial_t+xy\partial_x+\tfrac{1}{2}(t^2-x^2+y^2)\partial_y+\tfrac{1}{2}(x-t)v\partial_u-\tfrac{1}{2}(x+t)u\partial_v.
\end{aligned}$$

The usual summation convention, i.e., summation over repeated indices, is used in (4.57). The nonzero commutation relations are

$$\begin{aligned}
&[P^i,K^i]=D,\ [P^0,J^1]=P^2,\ [P^0,J^2]=P^1,\ [P^1,J^0]=P^2,\\
&[P^1,J^2]=P^0,\ [P^2,J^0]=-P^1,\ [P^2,J^1]=P^0,\ [J^0,J^1]=J^2,\\
&[J^0,J^2]=-J^0,\ [J^1,J^2]=-J^1,\ [P^0,K^1]=J^2,\ [P^0,K^2]=J^1,\\
&[P^1,K^0]=J^2,\ [P^1,K^2]=-J^0,\ [P^2,K^0]=J^1,\ [P^2,K^1]=J^0,
\end{aligned}$$

where $i=0,1,2$.

Then the remaining determining equations, which involve the functions F, G and their first-order partial derivatives with respect to u and v, are

$$\begin{aligned}
&\left[\left(\lambda-\tfrac{1}{2}(\beta_2+\alpha_0x+\alpha_1t)\right)u-\tfrac{1}{2}\left(\alpha_2(t-x)+(\alpha_0+\alpha_1)y+\beta_0+\beta_1\right)v+\varphi\right]F_u\\
&+\left[\tfrac{1}{2}\left(-\alpha_2(x+t)+(\alpha_1-\alpha_0)y+\beta_0-\beta_1\right)u+\left(\lambda+\tfrac{1}{2}(\beta_2+\alpha_0x+\alpha_1t)\right)v+\psi\right]F_v\\
&+\tfrac{1}{2}\left[\alpha_2(t-x)+(\alpha_1+\alpha_0)y+\beta_0+\beta_1\right]G\\
&+\left[(\alpha_0+\tfrac{1}{2}\alpha_1)t+(\alpha_1+\tfrac{1}{2}\alpha_0)x+\alpha_2y+\tfrac{1}{2}\beta_2+\delta-\lambda\right]F\\
&+(\lambda_y+\alpha_2)u-(\lambda_t+\lambda_x+\alpha_1+\alpha_0)v-\psi_t-\psi_x+\varphi_y=0,\\
&\left[\left(\lambda-\tfrac{1}{2}(\beta_2+\alpha_0x+\alpha_1t)\right)u-\tfrac{1}{2}\left(\alpha_2(t-x)+(\alpha_0+\alpha_1)y+\beta_0+\beta_1\right)v+\varphi\right]G_u\\
&+\left[\tfrac{1}{2}\left(-\alpha_2(x+t)+(\alpha_1-\alpha_0)y+\beta_0-\beta_1\right)u+\left(\lambda+\tfrac{1}{2}(\beta_2+\alpha_0x+\alpha_1t)\right)v+\psi\right]G_v\\
&+\tfrac{1}{2}\left[\alpha_2(x+t)-(\alpha_1-\alpha_0)y-\beta_0+\beta_1\right]F
\end{aligned}$$

$$+\left[(\alpha_0-\tfrac{1}{2}\alpha_1)t+(\alpha_1-\tfrac{1}{2}\alpha_0)x+\alpha_2 y-\tfrac{1}{2}\beta_2+\delta-\lambda\right]G$$
$$+(\lambda_t-\lambda_x-\alpha_1+\alpha_0)u-(\lambda_y+\alpha_2)v+\varphi_t-\varphi_x-\psi_y=0.$$

These equations are called the classifying equations or the classifying system. The main difficulty of group classification problem is that they should be solved for remaining uncertainties in the coefficients of X and the arbitrary elements of the class simultaneously.

4.3.2. The kernel algebra. In order to find Lie symmetries which are admitted by any PDE system (4.55) we should split classifying equations with respect to the functions F, G and their derivatives. This results in the conditions $\alpha_i=\beta_j=0$, $i,j=0,1,2$, and $\delta=\lambda=\varphi=\psi=0$, ρ^k, $k=0,1,2$, are arbitrary constants. The following statement is true.

Lemma 4.30. *The kernel of the maximal Lie invariance algebras of systems of equations from class* (4.55) *coincides with the three-dimensional algebra* $A^{\rm ker}=\langle P^0,\ P^1,\ P^2\rangle$.

One-dimensional extension of $A^{\rm max}$. As classifying equations are quite complicated we at first consider extension of the kernel algebra $A^{\rm ker}$ on one symmetry generator. For this purpose we take the general form of the admitted Lie symmetry operator (4.57) and require that X and operators from $A^{\rm ker}$ form a Lie algebra, i.e., we require $[A^{\rm ker},X]\in\langle A^{\rm ker},X\rangle$. This condition implies the equalities

$$[P^0,X]=\alpha_0 D+\alpha_1 J^2+\alpha_2 J^1+\chi_t=aX,$$
$$[P^1,X]=\alpha_0 J^2+\alpha_1 D-\alpha_2 J^0+\chi_x=bX,$$
$$[P^2,X]=\alpha_0 J^1+\alpha_1 J^0+\alpha_2 D+\chi_y=cX,$$

which result in the two possibilities for X

1. $(a,b,c)=(0,0,0)\quad\Rightarrow\quad X=\delta D+\beta_i J^i+\chi,$
2. $(a,b,c)\neq(0,0,0)\quad\Rightarrow\quad X=e^{at+bx+cy}\chi.$

Here a, b and c are arbitrary real constants, $\chi=(\lambda u+\varphi)\partial_u+(\lambda v+\psi)\partial_v$ with λ, φ, and ψ being arbitrary constants.

Case I. If $X=\delta D+\beta_i J^i+(\lambda u+\varphi)\partial_u+(\lambda v+\psi)\partial_v$, where λ, φ, and ψ are arbitrary constants, then the classifying equations take the form

$$\begin{aligned}&[(2\lambda-\beta_2)u-(\beta_0+\beta_1)v+2\varphi]\,F_u+[(\beta_0-\beta_1)u\\&\quad+(2\lambda+\beta_2)v+2\psi]\,F_v+(\beta_1+\beta_0)\,G+(2\delta-2\lambda+\beta_2)\,F=0,\\&[(2\lambda-\beta_2)u-(\beta_0+\beta_1)v+2\varphi]\,G_u+[(\beta_0-\beta_1)u\\&\quad+(2\lambda+\beta_2)v+2\psi]\,G_v+(2\delta-2\lambda-\beta_2)\,G+(\beta_1-\beta_0)\,F=0.\end{aligned}\tag{4.58}$$

This PDE system admits the point equivalence transformations

$$\begin{aligned}&u=a_1\tilde u+b_1\tilde v+c_1, &&v=a_2\tilde u+b_2\tilde v+c_2,\\&F=a_1\tilde F+b_1\tilde G, &&G=a_2\tilde F+b_2\tilde G.\end{aligned}$$

Here a_i, b_i and c_i, $i=1,2$, are arbitrary constants with $\Delta=a_1b_2-b_1a_2\neq 0$.

The constant parameters appearing in system (4.58) are changed under the action of these transformations as follows: $\tilde\lambda d=\tilde d\lambda$,

$$\begin{aligned}&\tilde\beta_0=\tfrac{1}{2\Delta}\tfrac{\tilde d}{d}\left((\beta_0-\beta_1)(a_1^2+b_1^2)+(\beta_0+\beta_1)(a_2^2+b_2^2)+2\beta_2(a_1a_2+b_1b_2)\right),\\&\tilde\beta_1=-\tfrac{1}{2\Delta}\tfrac{\tilde d}{d}\left((\beta_0-\beta_1)(a_1^2-b_1^2)+(\beta_0+\beta_1)(a_2^2-b_2^2)+2\beta_2(a_1a_2-b_1b_2)\right),\\&\tilde\beta_2=\tfrac{1}{\Delta}\tfrac{\tilde d}{d}\left((\beta_0-\beta_1)a_1b_1+(\beta_0+\beta_1)a_2b_2+\beta_2(a_1b_2+b_1a_2)\right),\\&\tilde\varphi=-\tfrac{1}{2\Delta}\tfrac{\tilde d}{d}\left[(\beta_0-\beta_1)b_1c_1+(\beta_0+\beta_1)b_2c_2+\beta_2(b_1c_2+c_1b_2)\right.\\&\quad\left.+2\lambda(b_1c_2-c_1b_2)+2b_1\psi-2b_2\varphi\right],\\&\tilde\psi=\tfrac{1}{2\Delta}\tfrac{\tilde d}{d}\left[(\beta_0-\beta_1)a_1c_1+(\beta_0+\beta_1)a_2c_2+\beta_2(a_1c_2+c_1a_2)\right.\\&\quad\left.+2\lambda(a_1c_2-c_1a_2)+2a_1\psi-2a_2\varphi\right],\end{aligned}$$

where $\Delta=a_1b_2-b_1a_2\neq 0$. Considering the possibilities of simplifying the coefficients we obtain that the nonzero triple $(\tilde\beta_0,\tilde\beta_1,\tilde\beta_2)$ has only three inequivalent values depending on the sign of $D=b_2^2+b_1^2-b_0^2$:

$$(0,0,1)\ \text{ if }\ D>0,\qquad (1,1,0)\ \text{ if }\ D=0,\qquad (1,0,0)\ \text{ if }\ D<0.$$

Therefore, it is sufficient to consider four inequivalent forms of the constant tuple $(\tilde\beta_0,\tilde\beta_1,\tilde\beta_2,\tilde\delta,\tilde\lambda,\tilde\varphi,\tilde\psi)$, namely, 1. $(0,0,1,\delta',\lambda',0,0)$,

2. $(1,0,0,\delta',\lambda',0,0)$, 3. $(1,1,0,\delta',\lambda',0,0)$, 4. $(1,1,0,\delta',0,\varphi',\psi')$ and the fifth tuple, namely, 5. $(0,0,0,\delta',\lambda',0,0)$ arises when $\beta_0=\beta_1=\beta_2=0$.

Consider the first case, where $\beta_0=\beta_1=\varphi=\psi=0$, $\beta_2=1$, δ and λ are arbitrary constants. Then system (4.58) becomes

$$\begin{aligned}&(2\lambda-1)uF_u+(2\lambda+1)vF_v+(2\delta-2\lambda+1)\,F=0,\\&(2\lambda-1)uG_u+(2\lambda+1)vG_v+(2\delta-2\lambda-1)\,G=0.\end{aligned}\tag{4.59}$$

If $\lambda\neq 1/2$, then the general solution of (4.59) is $F=u^{1+\frac{2\delta}{1-2\lambda}}\,\mathcal{F}\big(vu^{\frac{1+2\lambda}{1-2\lambda}}\big)$, $G=u^{1+\frac{2\delta-2}{1-2\lambda}}\,\mathcal{G}\big(vu^{\frac{1+2\lambda}{1-2\lambda}}\big)$ (Case 1 of Table 4.7). Here and below the functions $\mathcal{F}$ and $\mathcal{G}$ are arbitrary smooth functions of their variables.

If $\lambda=\frac{1}{2}$, then $F=v^{-\delta}\,\mathcal{F}(u)$, $G=v^{1-\delta}\,\mathcal{G}(u)$ (Case 2 of Table 4.7).

In the second case system (4.58) takes such a form which has no solutions over the real field $\mathbb{R}$ and can be solved over the field $\mathbb{C}$ only.

In the third and the fourth cases system (4.58) becomes

$$\begin{aligned}&(\lambda u-v+\varphi)\,F_u+(\lambda v+\psi)\,F_v+(\delta-\lambda)\,F+G=0,\\&(\lambda u-v+\varphi)\,G_u+(\lambda v+\psi)\,G_v+(\delta-\lambda)\,G=0,\end{aligned}\tag{4.60}$$

where either $\varphi=\psi=0$ or $\lambda=0$. The results of its integration are presented by Cases 3–5 of Table 4.7.

In the fifth case system (4.58) is of the form

$$uF_u+vF_v+(\delta/\lambda-1)F=0,\quad uG_u+vG_v+(\delta/\lambda-1)G=0,$$

whose general solution is $F=u^{1-\delta/\lambda}\,\mathcal{F}(v/u)$, $G=u^{1-\delta/\lambda}\,\mathcal{F}(v/u)$ (Case 6 of Table 4.7).

Case II. If $X=e^{at+bx+cy}\,((\lambda u+\varphi)\partial_u+(\lambda v+\psi)\partial_v)$, where a, b, c, λ, φ, and ψ are arbitrary constants with $(a,b,c)\neq(0,0,0)$, then system of the classifying equations becomes the uncoupled system

$$\begin{aligned}&(\lambda u+\varphi)F_u+(\lambda v+\psi)F_v-\lambda F+c(\lambda u+\varphi)-(a+b)(\lambda v+\psi)=0,\\&(\lambda u+\varphi)G_u+(\lambda v+\psi)G_v-\lambda G+(a-b)(\lambda u+\varphi)-c(\lambda v+\psi)=0\end{aligned}\tag{4.61}$$

If $\lambda\neq 0$, then the equivalence transformation $\tilde u=\lambda u+\varphi$, $\tilde v=\lambda v+\psi$,

Table 4.7: Lie symmetries of (2+1)-dimensional Dirac equations (4.55).

	Nonlinearities	Basis operators of $A^{\max}$
1 $\lambda\neq\frac{1}{2}$	$F=u^{1+\frac{2\delta}{1-2\lambda}}\,\mathcal{F}\big(vu^{\frac{1+2\lambda}{1-2\lambda}}\big)$, $G=u^{1+\frac{2\delta-2}{1-2\lambda}}\,\mathcal{G}\big(vu^{\frac{1+2\lambda}{1-2\lambda}}\big)$	P^0, P^1, P^2, $\delta D+J^2+\lambda(u\partial_u+v\partial_v)$
2	$F=v^{-\delta}\,\mathcal{F}(u)$, $G=v^{1-\delta}\,\mathcal{G}(u)$	P^0, P^1, P^2, $\delta D+J^2+\frac{1}{2}(u\partial_u+v\partial_v)$
3 $\lambda\neq 0$	$F=-\frac{1}{\lambda}\,\mathrm{W}(z)G$ $+\,e^{\frac{\lambda-\delta}{\lambda}\,\mathrm{W}(z)}\mathcal{F}(\ln v+\lambda u/v)$, $G=e^{\frac{\lambda-\delta}{\lambda}\,\mathrm{W}(z)}\mathcal{G}(\ln v+\lambda u/v)$	P^0, P^1, P^2, $\delta D+J^0+J^1+\lambda(\partial_u+v\partial_v)$
4 $\psi\neq 0$	$F=\dfrac{\varphi-v}{\psi}G$ $+e^{\frac{\delta}{\psi}(\varphi-v)}\,\mathcal{F}(\varphi v-\psi u-\frac{1}{2}v^2)$, $G=e^{\frac{\delta}{\psi}(\varphi-v)}\,\mathcal{G}(\varphi v-\psi u-\frac{1}{2}v^2)$	P^0, P^1, P^2, $\delta D+J^0+J^1+\varphi\partial_u+\psi\partial_v$
5	$F=\dfrac{u}{v-\varphi}G+e^{\frac{\delta u}{v-\varphi}}\,\mathcal{F}\,(v)$, $G=e^{\frac{\delta u}{v-\varphi}}\,\mathcal{G}\,(v)$	P^0, P^1, P^2, $\delta D+J^0+J^1+\varphi\partial_u$
6 $\lambda\neq 0$	$F=u^{1-\delta/\lambda}\,\mathcal{F}(v/u)$, $G=u^{1-\delta/\lambda}\,\mathcal{G}(v/u)$	P^0, P^1, P^2, $\delta D+\lambda(u\partial_u+v\partial_v)$
7	$F=((b+a)v-cu)\ln u+u\,\mathcal{F}(v/u)$, $G=((b-a)u+cv)\ln u+u\,\mathcal{G}(v/u)$	P^0, P^1, P^2, $e^{at+bx+cy}(u\partial_u+v\partial_v)$
8 $\varphi\neq 0$	$F=((b+a)/\varphi-c)\,u+\mathcal{F}(u-\varphi v)$, $G=(b-a+c/\varphi)\,u+\mathcal{G}(u-\varphi v)$	P^0, P^1, P^2, $\sigma(\omega)e^{at+bx+cy}(\varphi\partial_u+\partial_v)$
9	$F=(a+b)v+\mathcal{F}(u)$, $G=cv+\mathcal{G}(u)$	P^0, P^1, P^2, $\sigma(x-t)e^{at+bx+cy}\partial_v$
10	$F=-cu+\mathcal{F}(v)$, $G=(b-a)u+\mathcal{G}(v)$	P^0, P^1, P^2, $\sigma(x+t)e^{at+bx+cy}\partial_u$

Here δ, λ, φ, ψ, a, b and c are constants with $a^2+b^2+c^2\neq 0$, $\omega=\varphi^2(t+x)+2\varphi y+t-x$; σ is arbitrary smooth nonvanishing function of the indicated variable. $\mathrm{W}(z)=\mathrm{LambertW}(z)$ [64], where $z=-\lambda\frac{u}{v}e^{-\lambda\frac{u}{v}}$.

$\tilde F=\lambda F$, $\tilde G=\lambda G$ maps system (4.61) to the one with $\tilde\lambda=1$, $\tilde\varphi=\tilde\psi=0$, whose general solution is

$$F=((b+a)v-cu)\ln u+u\,\mathcal{F}(v/u)\,,$$
$$G=((b-a)u+cv)\ln u+u\,\mathcal{G}(v/u)$$

(see Case 7 of Table 4.7).

If $\lambda = 0$, then (φ, ψ) has the following inequivalent values $(\varphi, 1)$ with $\varphi \neq 0$ and $(1, 0)$. The corresponding general solutions are

$$\begin{aligned} F &= ((b+a)/\varphi - c)\, u + \mathcal{F}(u - \varphi v)\,, \\ G &= (b - a + c/\varphi)\, u + \mathcal{G}(u - \varphi v)\,, \end{aligned} \qquad \varphi \neq 0,\ \psi = 1,$$

$$\begin{aligned} F &= (a{+}b)v + \mathcal{F}(u), \\ G &= cv + \mathcal{G}(u), \end{aligned} \qquad \varphi = 0,\ \psi = 1,$$

$$\begin{aligned} F &= -cu + \mathcal{F}(v), \\ G &= (b{-}a)u + \mathcal{G}(v), \end{aligned} \qquad \varphi = 1,\ \psi = 0,$$

(Cases 8–10 of Table 4.7).

Let us note that the four-dimensional Lie symmetry algebras

$$\langle P^0,\, P^1,\, P^2,\, e^{at+bx+cy}(\varphi\partial_u + \psi\partial_v)\rangle, \quad (\varphi, \psi) \in \{(\varphi, 1), (1, 0)\}$$

are not maximal for nonlinearities presented in Cases 8–10. The corresponding systems (4.55) admit infinite-dimensional Lie symmetry algebras with basis operators adduced in Table 4.7.

Therefore we have found all nonlinearities F and G, for which (2+1)-dimensional real Dirac equations admit extension on one-symmetry generator. It should be noted that in most cases the maximal Lie invariance algebra becomes four-dimensional (Cases 1–7), but sometimes the extension operator appears to involve an arbitrary function and in this case the maximal Lie invariance algebra becomes infinite-dimensional (Cases 8–10).

Note 4.31. Table 4.7 consists some cases that are equivalent with respect to point transformations. Thus, Cases 9 and 10 are mapped to each other by the transformation

$$t \mapsto -t, \quad x \mapsto x, \quad y \mapsto -y, \quad u \mapsto v, \quad v \mapsto u, \quad F \mapsto G, \quad G \mapsto F,$$

that belongs to the equivalence group of class (4.55). The same transformation maps Case 2 to Case 1 with $\lambda = -1/2$.

Note 4.32. The Lie invariance algebras adduced in Table 4.7 are maximal Lie invariance algebras for arbitrary values of constants and functions appearing in corresponding nonlinearities F and G. For certain values of the parameters these algebras will not be maximal. For example, let $\varphi = 0$, $\mathcal{F}_v = \mathcal{G}_v = 0$ in Case 5 of Table 4.7, then the system (4.55) takes the form

$$
\begin{aligned}
&v_t + v_x - u_y = \left(\frac{u}{v}\kappa_1 + \kappa_2\right) e^{\frac{\delta u}{v}},\\
&u_t - u_x - v_y = -\kappa_1 e^{\frac{\delta u}{v}},
\end{aligned}
$$

where κ_1 and κ_2 are constants, $\kappa_1^2 + \kappa_2^2 \neq 0$. It admits additional Lie symmetry generated by the operator $D + u\partial_u + v\partial_v$. The maximal Lie invariance algebra is five-dimensional in this case.

4.3.3. Reduction Procedure. In this section we present a couple of examples of finding exact solutions of NLDEs from class (4.55) via Lie reduction method. Consider Case 6 of Table 4.7 with $\delta = \lambda \neq 0$, namely, PDE systems (4.55) of the form

$$
\begin{aligned}
&v_t + v_x - u_y = \mathcal{F}(v/u),\\
&u_t - u_x - v_y = -\mathcal{G}(v/u),
\end{aligned}
\tag{4.62}
$$

admitting the Lie invariance algebra $\mathfrak{g} = \langle P^0,\ P^1,\ P^2,\ D + u\partial_u + v\partial_v\rangle$.

Using one-dimensional subalgebras of $\mathfrak{g}$ we can reduce family of PDE systems (4.62) to PDE system in (1+1) dimensions. Consider, for example, the one-dimensional subalgebra $\langle D + u\partial_u + v\partial_v)\rangle$ of $\mathfrak{g}$. This subalgebra belongs to the optimal system of subalgebras of $\mathfrak{g}$ (a procedure of finding the optimal system is well described in [217] and classification of subalgebras of real three- and four-dimensional algebras can be found in [230]). The substitutions $u = t\,U(z,w)$, $v = t\,V(z,w)$, where $z = x/t$, $w = y/t$, reduce (4.62) to the (1+1)-dimensionsional PDE system

$$
\begin{aligned}
&(1 - z)V_z - wV_w - U_w + V = \mathcal{F}(V/U),\\
&(1 + z)U_z + wU_w + V_w - U = \mathcal{G}(V/U).
\end{aligned}
\tag{4.63}
$$

We have found simple particular solutions of this system for arbitrary $\mathcal{F} = \pm\mathcal{G}$. It is

$$U = \mp V = (w \mp 1)C_1 \mp \mathcal{F}(\mp 1), \quad \mathcal{F} = \pm\mathcal{G},$$

where C_1 is an arbitrary constant. Thus system (4.62) has the solutions

$$\begin{aligned} &u = -v = (y-t)C_1 - t\mathcal{F}(-1) \quad \text{if} \quad \mathcal{F} = \mathcal{G}, \\ &u = v = (y+t)C_1 + t\mathcal{F}(1) \quad \text{if} \quad \mathcal{F} = -\mathcal{G}, \end{aligned}$$

that are valid for any function $\mathcal{F}$ that is well-defined when its argument is equal to -1 (resp. to 1 in the second case).

To reduce system (4.62) to an ODE system a two-dimensional subalgebra should be used. Consider the two-dimensional subalgebra

$$\langle P^1 + \alpha_2 P^2 + \alpha_0 P^0,\ D + u\partial_u + v\partial_v\rangle, \quad \alpha_2, \alpha_0 \in \mathbb{R}.$$

The substitutions reducing system (4.62) to ODE system are

$$\begin{aligned} u &= (t-\alpha_0 x)U(z), \\ v &= (t-\alpha_0 x)V(z), \end{aligned} \qquad \text{where} \quad z = \frac{y-\alpha_2 x}{t-\alpha_0 x}.$$

The corresponding ODE system is

$$\begin{aligned} &((\alpha_0 - 1)z - \alpha_2)\, V_z - U_z + (1-\alpha_0)V = \mathcal{F}(V/U), \\ &((\alpha_0 + 1)z - \alpha_2)\, U_z + V_z - (1+\alpha_0)U = \mathcal{G}(V/U). \end{aligned} \tag{4.64}$$

If $\mathcal{F} = \mathcal{G}$ then the system has particular solution

$$U = -V = (\alpha_2 - 1 + (1-\alpha_0)z)C_1 - \mathcal{F}(-1)/(1-\alpha_0), \quad \alpha_0 \neq 1,$$

where C_1 is an arbitrary constant. Therefore the solution of system (4.62) with $\mathcal{F} = \mathcal{G}$ is

$$u = -v = \left[(t-\alpha_0 x)(\alpha_2 - 1) + (y - \alpha_2 x)(1-\alpha_0)\right] C_1 - \frac{(t-\alpha_0 x)\mathcal{F}(-1)}{(1-\alpha_0)}.$$

It is valid for any $\mathcal{F}$ that is well-defined when its argument is equal to -1.

Using the reduction method exact solutions can be constructed to other NLDEs (4.55) with nonlinearities presented in Table 4.7.

The preliminary group classification of nonlinear Dirac equations in two spatial dimensions for real wave functions is carried out in [303]. Namely, all forms of nonlinearities F and G such that the corresponding NLDE (4.55) admits one-dimensional extension of its Lie invariance algebra are described. The found symmetries are useful for construction of exact solutions for wide subclasses of such equations with nonlinearities presented in Table 4.7.

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Appendix A

Lie and Nonclassical Symmetries of Variable Coefficient Reaction–Diffusion Equations

In this appendix we collect the results on classification of Lie and nonclassical (conditional and potential) reduction operators for classes of (1+1)-dimensional reaction–diffusion equations with coefficients dependent on spatial variable.

Section A.1 is devoted to the group classification of diffusion equations with a nonlinear source $u_t = u_{xx} + h(x)B(u)$, $hB_{uu} \neq 0$. The achieve complete classification we use a conditional equivalence group found in the course of the study of admissible point transformation within the class.

In section A.2 Lie and nonclassical reduction operators of variable coefficient semilinear reaction-diffusion equations with exponential source $f(x)u_t = (g(x)u_x)_x + h(x)e^{mu}$ are investigated using the algorithm involving a mapping between classes of differential equations, which is generated by a family of point transformations. Special attention is paid to check whether reduction operators are inequivalent to Lie symmetry operators. The derived reduction operators are applied to construction of closed-form solutions. The similar study on nonclassical reduction operators of variable coefficient semilinear reaction-diffusion equations with power source $f(x)u_t = (g(x)u_x)_x + h(x)u^m$ is performed in Section A.3.

In Section A.4 we derive potential symmetries of variable coefficient nonlinear diffusion equations of the form $f(x)u_t = (g(x)u^nu_x)_x$, $fgn \neq 0$.

The results collected in this appendix are published in the following papers [11*,23*,25*,26*,30*].

A.1. Group Classification of a Class of Quasilinear Reaction–Diffusion Equations

In this section we solve the group classification problem for the class of variable coefficient semilinear reaction–diffusion equations of the form

$$u_t = u_{xx} + h(x)B(u), \tag{A.1}$$

where $h = h(x)$ and $B = B(u)$ are arbitrary smooth functions of their variables, $hB_{uu} \neq 0$. Linear equations singled out from class (A.1) by the condition $B_{uu} = 0$ are excluded from consideration since group classification of all second-order linear PDEs in two dimensions was performed by Lie (see [186]). Equations (A.1) are used to model various phenomena such as microwave heating, problems in population genetics, etc. (see, e.g., [44] and references therein). Lie symmetries of certain subclasses of (A.1) are known. The group classification of constant coefficient equations from class (A.1) was carried out by Dorodnitsyn [70] (the results are adduced in handbook [131]). Class (A.1) includes the generalized Huxley equations $u_t = u_{xx} + h(x)u^2(1-u)$, whose Lie symmetries were studied in [43, 136]. There exists also a certain intersection with the results on group classification of the classes

$$u_t = u_{xx} + H(x)u^m + F(x)u, \quad m \neq 0, 1,\ H \neq 0, \quad \text{and} \tag{A.2}$$

$$u_t = u_{xx} + H(x)u^2 + G(x), \quad H \neq 0, \tag{A.3}$$

which were obtained in [300]. Firstly we present the results on the study of admissible transformations in class (A.1).

Theorem A.1. *The usual equivalence group $G^{\sim}$ of class* (A.1) *consists of the transformations*

$$\tilde t = \delta_1^2 t + \delta_2, \quad \tilde x = \delta_1 x + \delta_3, \quad \tilde u = \delta_4 u + \delta_5, \quad \tilde h = \frac{\delta_4}{\delta_1^2\delta_0} h, \quad \tilde B = \delta_0 B,$$

where δ_j, $j = 0, \ldots, 5$, are arbitrary constants with $\delta_0\delta_1\delta_4 \neq 0$.

Table A.1: The group classification of the class $u_t = u_{xx} + h(x)B(u)$, $hB_{uu} \neq 0$.

no.	$B(u)$	$h(x)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_t
1	$\forall$	δx^{-2}	$\partial_t,\quad 2t\partial_t + x\partial_x$
2	$\forall$	δ	$\partial_t,\quad \partial_x$
3	u^m	δx^s	$\partial_t,\quad 2(m-1)t\partial_t + (m-1)x\partial_x - (s+2)u\partial_u$
4	u^m	δe^x	$\partial_t,\quad (1-m)\partial_x + u\partial_u$
5	u^m	δ	$\partial_t,\quad \partial_x,\quad 2(m-1)t\partial_t + (m-1)x\partial_x - 2u\partial_u$
6	e^u	δx^s	$\partial_t,\quad 2t\partial_t + x\partial_x - (s+2)\partial_u$
7	e^u	$\delta e^{\pm x^2}$	$\partial_t,\quad \partial_x \mp 2x\partial_u$
8	e^u	δ	$\partial_t,\quad \partial_x,\quad 2t\partial_t + x\partial_x - 2\partial_u$
9	$u\ln u$	δ	$\partial_t,\quad \partial_x,\quad e^{\delta t}u\partial_u,\quad e^{\delta t}(\partial_x - \frac{\delta}{2}xu\partial_u)$

Here δ, m and s are arbitrary constants, $m \neq 0, 1$, $s \neq 0$, $\delta = \pm 1 \bmod G^{\sim}$.

It appears that there exist point transformations between equations from (A.1) which do not belong to $G^{\sim}$ and form a conditional equivalence group. Moreover, this group is not usual but a generalized extended one.

Theorem A.2. *The generalized extended equivalence group $\hat{G}^{\sim}_{\exp}$ of the subclass*

$$u_t = u_{xx} + h(x)(e^{nu} + r) \tag{A.4}$$

of class (A.1) is formed by the transformations

$$\tilde{t} = {\delta_1}^2 t + \delta_2, \quad \tilde{x} = \delta_1 x + \delta_3, \quad \tilde{u} = \delta_4 u + \varphi(x),$$
$$\tilde{h} = \frac{\delta_4}{{\delta_1}^2} e^{-\frac{n}{\delta_4}\varphi} h, \quad \tilde{n} = \frac{n}{\delta_4}, \quad \tilde{r} = e^{\frac{n}{\delta_4}\varphi}\left(r - \frac{\varphi_{xx}}{\delta_4 h}\right),$$

where r and δ_j, $j = 1, \ldots, 4$, are arbitrary constants with $\delta_1\delta_4 \neq 0$. The transformation component for r can be interpreted as the constraint for φ,

$$\varphi_{xx} = \delta_4 h(r - \tilde{r}e^{-\frac{n}{\delta_4}\varphi}).$$

Theorem A.2 implies that class (A.4) reduces to the class

$$\tilde u_t = \tilde u_{xx} + \tilde h(x)e^{n\tilde u} \tag{A.5}$$

by the transformation $\tilde t = t$, $\tilde x = x$, $\tilde u = u + \varphi(x)$, where $\tilde h(\tilde x) = e^{-\varphi(x)}h(x)$ and $\varphi_{xx} = rh(x)$. Class (A.4) is normalized. Therefore, the equivalence group of class (A.4) with $r = 0$ can be found setting $\tilde r = r = 0$ in transformations from the group $\hat G^{\sim}_{\rm exp}$.

Corollary A.3. *The usual equivalence group $G^{\sim}_{\rm exp}$ of the class*

$$u_t = u_{xx} + h(x)e^{nu}$$

consists of the transformations

$$\tilde t = \delta_1^2 t + \delta_2, \quad \tilde x = \delta_1 x + \delta_3, \quad \tilde u = \delta_4 u + \delta_5 x + \delta_6,$$
$$\tilde h = \frac{\delta_4}{\delta_1^2}e^{-\frac{n}{\delta_4}(\delta_5 x + \delta_6)}h, \quad \tilde n = \frac{n}{\delta_4},$$

where δ_j, $j = 1, \dots, 6$, are arbitrary constants with $\delta_1\delta_4 \neq 0$.

In the course of the study of Lie symmetries we use the derived equivalence transformations for the simplification of calculations and for presenting the final results in a concise form. We study Lie symmetries of equations from class (A.1) using the classical approach [227] in a combination with the method of furcate splitting [245]. The results on group classification of class (A.1) are summarized in Table A.1. It is important to note that group classification of subclass (A.4) is carried out up to the $\hat G^{\sim}_{\rm exp}$-equivalence, whereas all other cases are classified up to the usual $G^{\sim}$-equivalence. The complete proofs can be found in [308].

The derived results on group classification can be applied for searching closed form solutions via the classical reduction method. The knowledge of Lie symmetries is also necessary for finding nonclassical symmetries (called also Q-conditional symmetries or reduction operators) of equations (A.1).

A.2. Lie and Nonclassical Reduction Operators of a Class of Semilinear Diffusion Equations with an Exponential Source

In this section we use the method of mappings between classes to find non-classical reduction operators of the variable coefficient reaction-diffusion equations with exponential nonlinearity

$$f(x)u_t = (g(x)u_x)_x + h(x)e^{mu}. \tag{A.6}$$

Here f, g and h are arbitrary smooth functions of the variable x, $fgh \neq 0$ and m is an arbitrary nonvanishing constant.

Lie Symmetries and Equivalence Transformations. Class (A.6) has complicated transformational properties. An indicator of this is that it possesses the nontrivial generalized extended equivalence group, which does not coincide with its usual equivalence group, cf. Theorem A.4 below. To produce the group classification of class (A.6) it is necessary to gauge arbitrary elements of this class with equivalence transformations and a subsequent mapping of it onto a simpler class [288, 300]. It appears that the preimage set of each equation from the imaged class is a biparametric family of equations from the initial class (A.6). Moreover preimages of the same equation belong to the same orbit of the equivalence group of the initial class. It allows one to look only for the simplest representative of the preimage to obtain its symmetries, solution etc., and then to reproduce these results for a two-parametric family of equations from the initial class using equivalence transformations.

Theorem A.4. *The generalized extended equivalence group $\hat{G}^{\sim}_{\rm exp}$ of class (A.6) consists of the transformations*

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \delta_3 u + \psi(x),$$

$$\tilde{f} = \frac{\delta_0\delta_1}{\varphi_x} f, \quad \tilde{g} = \delta_0\varphi_x g, \quad \tilde{h} = \frac{\delta_0\delta_3}{\varphi_x} \exp\left(-\frac{m}{\delta_3}\psi\right) h, \quad \tilde{m} = \frac{m}{\delta_3},$$

where φ is an arbitrary nonconstant smooth function of x, $\psi = \delta_4 \int \frac{dx}{g(x)} + \delta_5$ and δ_j, $j = 0, 1, \dots, 5$, are arbitrary constants such that $\delta_0\delta_1\delta_3 \neq 0$.

Corollary A.5. *The usual equivalence group of class (A.6) is the subgroup of $\hat{G}^{\sim}_{\exp}$ singled out by the condition $\delta_4 = 0$.*

The transformations from $\hat{G}^{\sim}_{\exp}$ associated with varying the parameter δ_0 in fact do not change equations from class (A.6) and hence form the gauge equivalence group of this class. The values of arbitrary elements connected by a such transformation correspond to different representations of the same equation.

We firstly map class (A.6) onto its subclass

$$f(x)u_t = (f(x)u_x)_x + h(x)e^u \tag{A.7}$$

(we omit tildes over the variables) using the family of equivalence transformations parameterized by the arbitrary elements f, g and m,

$$\tilde{t} = \mathrm{sign}(f(x)g(x))\, t, \quad \tilde{x} = \int \left|\frac{f(x)}{g(x)}\right|^{\frac{1}{2}} dx, \quad \tilde{u} = m\, u. \tag{A.8}$$

The new arbitrary elements are expressed via the old ones in the following way:

$$\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) = \mathrm{sign}(g(x))|f(x)g(x)|^{\frac{1}{2}}, \quad \tilde{h}(\tilde{x}) = m\left|\frac{g(x)}{f(x)}\right|^{\frac{1}{2}} h(x), \quad \tilde{m} = 1.$$

The next step is to change the dependent variable in class (A.7):

$$v(t,x) = u(t,x) + G(x), \quad \text{where} \quad G(x) = \ln|f(x)^{-1}h(x)|. \tag{A.9}$$

Finally we obtain the class

$$v_t = v_{xx} + F(x)v_x + \varepsilon e^{\,v} + H(x), \tag{A.10}$$

where $\varepsilon = \mathrm{sign}(f(x)h(x))$ and the new arbitrary elements F and H are expressed via the arbitrary elements of class (A.10) according to the formulas

$$F = f_x f^{-1} \quad \text{and} \quad H = -G_{xx} - G_x F. \tag{A.11}$$

Table A.2: The group classification of the class A.10.

no.	$F(x)$	$H(x)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_t
1	$\alpha x^{-1}+\mu x$	$\beta x^{-2}+2\mu$	$\partial_t,\quad e^{-2\mu t}(\partial_t-\mu x\partial_x+2\mu\partial_v)$
2	αx^{-1}	βx^{-2}	$\partial_t,\quad 2t\partial_t+x\partial_x-2\partial_v$
3	μx	γ	$\partial_t,\quad e^{-\mu t}\partial_x$
4	λ	γ	$\partial_t,\quad \partial_x$
5	μx	2μ	$\partial_t,\quad e^{-\mu t}\partial_x,\quad e^{-2\mu t}(\partial_t-\mu x\partial_x+2\mu\partial_v)$
6	λ	0	$\partial_t,\quad \partial_x,\quad 2t\partial_t+(x-\lambda t)\partial_x-2\partial_v$

Here $\lambda\in\{0,1\}\bmod G^{\sim}_{\exp}$, $\mu=\pm1\bmod G^{\sim}_{\exp}$; α,β and γ are arbitrary constants, $\alpha^2+\beta^2\neq0$. We also have $\gamma\neq2\mu$ and $\gamma\neq0$ in Cases 3 and 4, respectively.

All results on Lie symmetries and solutions of class (A.10) can be extended to class (A.7) by the inversion of transformation (A.9).

The arbitrary elements f and h of class (A.10) are expressed via the functions F and H in the following way:

$$
\begin{aligned}
&f=c_0\exp\left(\textstyle\int F dx\right), \qquad h=\varepsilon c_0\exp\left(\textstyle\int F dx+G\right),\\
&\text{where}\quad G=\int e^{-\int F dx}\left(c_1-\int He^{\int F dx}dx\right)dx+c_2.
\end{aligned}
\tag{A.12}
$$

Here c_0, c_1 and c_2 are arbitrary constants, $c_0\neq0$. The constant c_0 is inessential and can be set to the unity by an obvious gauge equivalence transformation. The equations from class (A.7), that have the same image in class (A.10) with respect to transformation (A.9), i.e. the arbitrary elements of which are given by (A.12) and differ only by values of constants c_1 and c_2, are $\hat G^{\sim}_{\exp}$-equivalent. The equivalence transformation

$$
\tilde t=t,\quad \tilde x=x,\quad \tilde u=u+c_1\int e^{-\int F dx}dx+c_2 \tag{A.13}
$$

maps an equation (A.10) having f and h of the form (A.12) with $c_1^2+c_2^2\neq0$ to the one with $c_1=c_2=0$. Hence up to $\hat G^{\sim}_{\exp}$-equivalence we can consider, without loss of generality, only equations from class (A.7) that have the arbitrary elements determined by (A.12) with $c_1 = c_2 = 0$.

Theorem A.6. *The generalized extended equivalence group $G^{\sim}_{\rm exp}$ of class (A.10) coincides with its usual equivalence group and is formed by the transformations*

$$\tilde t = \delta_1^2 t + \delta_2, \quad \tilde x = \delta_1 x + \delta_3, \quad \tilde v = v - \ln \delta_1^2, \quad \tilde F = \delta_1^{-1} F, \quad \tilde H = \delta_1^{-2} H,$$

where δ_j, $j = 1, 2, 3$, are arbitrary constants, $\delta_1 \neq 0$.

The *kernel* of the maximal Lie invariance algebras of equations from class (A.10) is the one-dimensional algebra $\langle \partial_t \rangle$. It means that any equation from class (A.10) is invariant with respect to translations by t and there are no more common Lie symmetries.

Theorem A.7. *$G^{\sim}_{\rm exp}$-inequivalent cases of extension of the maximal Lie invariance algebras in class (A.10) are exhausted by those presented in Table A.2.*

The corresponding results on group classification of class (A.6) up to $\hat G^{\sim}_{\rm exp}$-equivalence were derived in [288] and collected in Table A.3. The first number of each case indicates the associated case of Table A.2.

Additional equivalence transformations between $G^{\sim}_{\rm exp}$-inequivalent cases of Lie symmetry extension are also constructed. The pairs of point-equivalent cases from Table A.2 and the corresponding transformations are exhausted by the following:

$$\begin{aligned} &1 \mapsto \tilde 2, \quad 5 \mapsto \tilde 6|_{\tilde\lambda=0}\colon\ \tilde t = \frac{1}{2\mu} e^{2\mu t}, \quad \tilde x = e^{\mu t} x, \quad \tilde v = v - 2\mu t, \\ &4 \mapsto \tilde 4|_{\tilde\lambda=0}, \quad 6 \mapsto \tilde 6|_{\tilde\lambda=0}\colon\ \tilde t = t, \quad \tilde x = x + \lambda t, \quad \tilde v = v. \end{aligned} \tag{A.14}$$

The inequivalence of other different cases of Table A.2 can be proved using differences in properties of the corresponding maximal Lie invariance algebras, which should coincide for similar equations. Thus the dimensions of the maximal Lie invariance algebras are one, three and two in the general

Table A.3: The group classification of class (A.7)

no.	$f(x)$	$h(x)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_t
1	$x^\alpha e^{\frac{\mu}{2}x^2}$	$\delta x^\alpha e^{\frac{\mu}{2}x^2+\omega^1}$	$\partial_t,\quad e^{-2\mu t}\left[\partial_t - \mu x\partial_x + \mu\left(2 + x\omega^1_x\right)\partial_u\right]$
2.1	x^α	$\delta x^{\alpha+\frac{\beta}{1-\alpha}}$	$\partial_t,\quad 2t\partial_t + x\partial_x - \left(2+\frac{\beta}{1-\alpha}\right)\partial_u$
2.2	x	$\delta x^{1-\frac{\beta}{2}\ln x}$	$\partial_t,\quad 2t\partial_t + x\partial_x - (2-\beta\ln x)\partial_u$
3	$e^{\frac{\mu}{2}x^2}$	$\delta e^{\frac{\mu}{2}x^2+\omega^3}$	$\partial_t,\quad e^{-\mu t}\partial_x - e^{-\mu t}\omega^3_x\partial_u$
4.1	e^x	$\delta e^{\rho x}$	$\partial_t,\quad \partial_x + (1-\rho)\partial_u$
4.2	1	$\delta e^{-\frac{\gamma}{2}x^2}$	$\partial_t,\quad \partial_x + \gamma x\partial_u$
5	$e^{\frac{\mu}{2}x^2}$	$\delta e^{\frac{\mu}{2}x^2+\omega^5}$	$\partial_t,\quad e^{-\mu t}\partial_x - e^{-\mu t}\omega^5_x\partial_u,$ $e^{-2\mu t}\left[\partial_t - \mu x\partial_x + \mu\left(2 + x\omega^5_x\right)\partial_u\right]$
6.1	e^x	δe^x	$\partial_t,\quad \partial_x,\quad 2t\partial_t + (x-t)\partial_x - 2\partial_u$
6.2	1	δ	$\partial_t,\quad \partial_x,\quad 2t\partial_t + x\partial_x - 2\partial_u$

Here $\delta=\pm1$, $\mu=\pm1 \bmod \hat{G}^{\sim}_1$; α,β,γ,ρ are arbitrary constants, $\rho\neq1$, $\alpha^2+\beta^2\neq0$. In case 2.1 $\alpha\neq1$. In case 3 $\gamma\neq2\mu$. In case 4.2 $\gamma\neq0$; $\omega^1=-\int x^{-\alpha}e^{-\frac{\mu}{2}x^2}\int(\beta x^{-2}+2\mu)x^\alpha e^{\frac{\mu}{2}x^2}dx\,dx$, $\omega^3=-\gamma\int e^{-\frac{\mu}{2}x^2}\int e^{\frac{\mu}{2}x^2}dx\,dx$, $\quad \omega^5=\omega^3|_{\gamma=2\mu}$, $\quad \omega^i_x=\frac{d\omega^i}{dx}$, i=1,3,5.

case (Case 0), Cases 5 and 6 and the other cases, respectively. In contrast to Cases 1–3, the algebra of Case 4 is commutative. The derivative of the algebra of Case 3 has the zero projection onto the space of t and this is not the case for Cases 1 and 2. Possession of the zero (resp. nonzero) projection onto the space of t is an invariant characteristic of Lie algebras of vector fields in the space of the variables t, x and v with respect to point transformations connecting a pair of evolution equations since for any such transformation the expression of the transformed t is well known to depend only on t [160, 192].

A more difficult problem is to prove that there are no more additional equivalences within a parameterized case of Table A.2. (In fact all the cases are parameterized.) A description of the set of admissible transformations of the class (A.10) is given by the following statements.

Proposition A.8. *Any admissible point transformation in the class* (A.10) *has the form*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta\sqrt{T_t}\,x + X(t), \quad \tilde{v} = v - \ln T_t,$$

where $\delta = \pm 1$ *and* T *and* X *are arbitrary smooth functions of* t *such that* $T_t > 0$. *The corresponding values of the arbitrary elements are related via the formulas*

$$\tilde{F} = \frac{\delta}{\sqrt{T_t}}F - \frac{\delta}{2}\frac{T_{tt}}{\sqrt{T_t}^3}x - \frac{X_t}{T_t}, \quad \tilde{H} = \frac{1}{T_t}H - \frac{T_{tt}}{T_t{}^2}.$$

Corollary A.9. *Only equations from the class* (A.10) *the arbitrary elements of which have the form*

$$F = \mu x + \lambda + \frac{\alpha}{x+\kappa}, \quad H = \gamma + \frac{\beta}{(x+\kappa)^2}, \tag{A.15}$$

where α, β, γ, κ *and* μ *are constants, possess admissible transformations that are not generated by transformations from the equivalence group* $G^{\sim}_{\rm exp}$. *The subclass of the class* (A.10), *singled out by condition* (A.15), *is closed under any admissible transformation within class* (A.10). *The (constant) parameters of the representation* (A.15) *are transformed by an admissible transformation in the following way:*

$$\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta, \qquad \tilde{\kappa} = \delta\sqrt{T_t}\kappa - X \quad \text{if} \quad (\alpha,\beta) \neq (0,0),$$
$$\tilde{\gamma} = \frac{\gamma}{T_t} - \frac{T_{tt}}{T_t^2}, \quad \tilde{\mu} = \frac{\mu}{T_t} - \frac{1}{2}\frac{T_{tt}}{T_t^2}, \quad \tilde{\lambda} = -\tilde{\mu}X - \frac{X_t}{T_t} + \frac{\delta\lambda}{\sqrt{T_t}}.$$

In particular $T_{tt} = 0$ *if* $\gamma \neq 2\mu$.

Finally we can formulate the assertion on group classification with respect to the set of admissible transformations.

Theorem A.10. *Up to point equivalence cases of extension of the maximal Lie invariance algebras in class* (A.10) *are exhausted by Cases 0, 2, 3,* $4_{\lambda=0}$ *and* $6|_{\lambda=0}$ *of Table A.2.*

A.2.1. Nonclassical Reduction Operators and Solutions. Reduction operators of equations from class (A.7) are easily found from reduction operators of corresponding equations from (A.10) using the formula

$$\tilde{Q} = \tau\partial_t + \xi\partial_x + (\eta - \xi G_x)\,\partial_u. \tag{A.16}$$

Here τ, ξ and η, respectively, are the coefficients of ∂_t, ∂_x and ∂_v in a reduction operator of an equation from class (A.10). The function G is defined in (A.12).

In [299, 300] we discussed two ways to use mappings between classes of equations in the investigation of reduction operators and their usage to find solutions. The preferable way is based on the implementation of reductions in the imaged class and preimaging of the obtained solutions instead of preimaging the corresponding reduction operators.

Following the above algorithm we look for $G^{\sim}_{\rm exp}$-inequivalent reduction operators with nonvanishing coefficient of ∂_t for the equations from the imaged class (A.10). Up to the usual equivalence of reduction operators we need to consider only the operators of the form

$$Q = \partial_t + \xi(t,x,v)\partial_x + \eta(t,x,v)\partial_v.$$

Applying conditional invariance criterion to equation (A.10) we obtain a third-degree polynomial of v_x with coefficients depending on t, x and v which has to identically equal zero. Separation respect to different powers of v_x results in the following determining equations for the coefficients ξ and η:

$$\begin{gathered}\xi_{vv} = 0, \quad \eta_{vv} = 2(\xi_{xv} - \xi\xi_v - F\xi_v), \quad \xi_t - \xi_{xx} + 2\xi_x\xi \\ + 3\xi_v\left(H + \varepsilon e^v\right) + 2\eta_{vx} - 2\xi_v\eta + F\xi_x + \xi F_x = 0, \\ \eta_t - \eta_{xx} + 2\xi_x\eta = \xi H_x + F\eta_x + (2\xi_x - \eta_v)H + \varepsilon e^v\left(\eta + 2\xi_x - \eta_v\right).\end{gathered} \tag{A.17}$$

Integration of the first two equations of (A.17) gives us the expressions for ξ and η with an explicit dependence on v:

$$\xi = av + b, \quad \eta = -\frac{1}{3}a^2v^3 + (a_x - ab - aF)v^2 + cv + d, \tag{A.18}$$

where $a = a(t,x)$, $b = b(t,x)$, $c = c(t,x)$ and $d = d(t,x)$ are smooth functions of t and x.

Substituting the expressions (A.18) for ξ and η into the third and forth equations of (A.17) and collecting the coefficients of different powers of v in the resulting equations, we derive the conditions $a = c = 0$, $d = -2b_x$ and two classifying equations, which contain both the coefficient $b = b(t,x)$ and the arbitrary elements $F = F(x)$ and $H = H(x)$. Summarising the above consideration we have the following assertion.

Proposition A.11. *Any regular reduction operator of an equation from the imaged class* (A.10) *is equivalent to an operator of the form*

$$Q = \partial_t + b\partial_x - 2b_x\partial_v, \tag{A.19}$$

where the coefficient $b = b(t,x)$ satisfies the overdetermined system of partial differential equations

$$\begin{aligned} &b_t - b_{xx} + 2bb_x + Fb_x + bF_x = 0,\\ &bH_x + 2b_xH - 4bb_{xx} - 2(Fb)_{xx} - 2Fb_{xx} = 0 \end{aligned} \tag{A.20}$$

with the corresponding values of the arbitrary elements $F = F(x)$ and $H = H(x)$.

We were not able to completely study all the cases of integration of system (A.20) depending upon values of F and H. This is why we try to solve this system under different additional constraints imposed either on b or on (F,H).

The most interesting results are obtained for the constraint $b_t = 0$. Then F and H are expressed, after a partial integration of (A.20), via the function $b = b(x)$ that leads to the following statement.

Theorem A.12. *For an arbitrary smooth function $b = b(x)$ the equation from class* (A.10) *with the arbitrary elements*

$$F = \frac{1}{b}\left(b_x + k_1 - b^2\right), \quad H = \frac{2}{b^2}\left(k_2 + b_x(k_1 - b^2) + bb_{xx}\right), \tag{A.21}$$

where k_1 and k_2 are constants, admits the reduction operator (A.19) *with the same* b.

An Ansatz constructed by the reduction operator (A.19) with $b_t = 0$ has the form

$$v = z(\omega) - 2\ln|b|, \quad \text{where} \quad \omega = t - \int \frac{dx}{b}.$$

The substitution of the Ansatz into equation (A.10) leads to the reduced ODE

$$z_{\omega\omega} - k_1 z_\omega + \varepsilon e^{\,z} + 2k_2 = 0. \tag{A.22}$$

For $k_1 = 0$ the general solution of (A.22) is written in the implicit form

$$\textstyle\int (c_1 - 4k_2 z - 2\varepsilon e^{\,z})^{-\frac{1}{2}}\, dz = \pm(\omega + c_2). \tag{A.23}$$

Up to similarity of solutions of equation (A.10) the constant c_2 is inessential and can be set to equal zero by a translation of ω, which is always induced by a translation of t.

Setting additionally $k_2 = 0$ in (A.23), we are able to integrate (A.23) in closed form and to write explicitly the general solution of (A.22). If $\varepsilon = 1$, then $c_1 > 0$ and (A.23) gives the following expression for e^z:

$$e^z = \frac{2s_1^2}{\cosh^2(s_1\omega + s_2)}.$$

Here and below $s_1 = \sqrt{|c_1|}/2$ and $s_2 = c_2 s_1$. If $\varepsilon = -1$, the integration leads to

$$e^z = \begin{cases} \dfrac{2s_1^2}{\sinh^2(s_1\omega + s_2)}, & c_1 > 0, \\[2ex] \dfrac{2s_1^2}{\cos^2(s_1\omega + s_2)}, & c_1 < 0, \\[2ex] \dfrac{2}{(\omega + c_2)^2}, & c_1 = 0. \end{cases}$$

As a result, for the equation from class (A.10) of the form

$$v_t = v_{xx} + \frac{1}{b}\left(b_x - b^2\right) v_x + \varepsilon e^v + \frac{2}{b}\left(b_{xx} - bb_x\right) \tag{A.24}$$

with $\varepsilon = -1$, we construct three families of closed-form solutions

$$\begin{aligned}
v &= -2\ln\left|\frac{\sqrt{2}}{2s_1} b \sinh\left(s_1 t - s_1 \int \frac{dx}{b} + s_2\right)\right|, \\
v &= -2\ln\left|\frac{\sqrt{2}}{2s_1} b \cos\left(s_1 t - s_1 \int \frac{dx}{b} + s_2\right)\right|, \\
v &= -2\ln\left|\frac{\sqrt{2}}{2} b \left(t - \int \frac{dx}{b} + c_2\right)\right|,
\end{aligned} \tag{A.25}$$

where s_1, s_2 and c_2 are arbitrary constants, $s_1 \neq 0$. Also we obtain a family of solutions

$$v = -2\ln\left|\frac{\sqrt{2}}{2s_1} b \cosh\left(s_1 t - s_1 \int \frac{dx}{b} + s_2\right)\right| \tag{A.26}$$

of the equation (A.24) with $\varepsilon = 1$.

We continue the consideration by studying whether the equations from class (A.10) possessing nontrivial Lie symmetry properties, i.e. having the maximal Lie invariance algebras of dimension two or three, have nontrivial (i.e. inequivalent to Lie ones) regular reduction operators. It has been already remarked that constant coefficient equations from class (A.10) do not admit such reduction operators [13,61]. Hence it is needless to consider Cases 4 and 6 of Table A.2 as well as Case 5 connected with Case 6 by point transformation (A.14). As Case 1 reduces to Case 2 with the same transformation (A.14), we have to study only two cases, namely Cases 2 and 3. We substitute the pairs of values of the parameter-functions F and H corresponding to Cases 2 and 3 into system (A.20) in order to find relevant values for b. We ascertain that $b_t = 0$ is a necessary condition for existing non-Lie regular reduction operators for equations with the above values of (F, H). This is why we can use equations (A.21) instead of (A.20) for further studying.

The investigation of Case 3 of Table A.2 leads to the conclusion that there are no non-Lie regular reduction operators for this case.

The functions F and H presented in Case 2 of Table A.2 satisfy (A.21) if and only if $\beta = 2(1-\alpha)$, i.e., they have the form $F = \alpha x^{-1}$, $H = 2(1-\alpha)x^{-2}$, and $k_1 = k_2 = 0$. The corresponding value of b is $b = -(1+\alpha)x^{-1}$. Hence $\alpha \neq -1$ since otherwise $b = 0$. Substituting the derived form of the function b into the formulas (A.25) and (A.26), we find that the equation

$$v_t = v_{xx} + \frac{\alpha}{x} v_x + \varepsilon e^v + \frac{2(1-\alpha)}{x^2} \tag{A.27}$$

has the families of solutions

$$v = -2\ln\left|\frac{\sqrt{2}(1+\alpha)}{2s_1 x}\cosh\left(s_1 t + \frac{s_1 x^2}{2(1+\alpha)} + s_2\right)\right|$$

if $\varepsilon = 1$ and

$$v = -2\ln\left|\frac{\sqrt{2}(1+\alpha)}{2s_1 x}\sinh\left(s_1 t + \frac{s_1 x^2}{2(1+\alpha)} + s_2\right)\right|,$$

$$v = -2\ln\left|\frac{\sqrt{2}(1+\alpha)}{2s_1 x}\cos\left(s_1 t + \frac{s_1 x^2}{2(1+\alpha)} + s_2\right)\right|,$$

$$v = -2\ln\left|\frac{\sqrt{2}(1+\alpha)}{2\,x}\left(t + \frac{x^2}{2(1+\alpha)} + c_2\right)\right|$$

if $\varepsilon = -1$. Recall that s_1, s_2 and c_2 are arbitrary constants with $s_1 \neq 0$.

As a representative of the preimage of equation (A.46) with respect to the transformation (A.9) we can choose the equation

$$x^\alpha u_t = (x^\alpha u_x)_x + \varepsilon x^{\alpha+2} e^u. \tag{A.28}$$

Solutions of this equation can be easily constructed from the above solutions of equation (A.46) using the transformation $u = v - 2\ln|x|$. If $\alpha = 1$, the chosen equation (A.28) can be replaced, e.g., by $xu_t = (xu_x)_x + \varepsilon x e^u$ which is just another representation of equation (A.46).

Non-Lie solutions of the equation

$$v_t = v_{xx} + \left(\frac{\alpha}{x} + \mu x\right) v_x + \varepsilon e^v + \frac{2(1-\alpha)}{x^2} + 2\mu,$$

where $\alpha \neq -1$ (Case 1 of Table A.2), can be easily obtained from exact solutions of the equation (A.28) using the transformation (A.14). The corresponding reduction operator has the form (A.19) with $b = -(1+\alpha)x^{-1} - \mu x$.

We also prove the following assertions.

Proposition A.13. *Equations from class* (A.10) *with* $F = \mathrm{const}$ *or* $H = \mathrm{const}$ *may admit only nontrivial regular reduction operators that are equivalent to operators of the form* (A.19), *where the function* b *does not depend upon the variable* t.

Proposition A.14. *Any reduction operator of an equation from class* (A.10), *having the form* (A.19) *with* $b_{xx} = 0$, *is equivalent to a Lie symmetry operator of this equation.*

A.3. Nonclassical Reduction Operators of a Class of Semilinear Diffusion Equations With Power Source

In [300] simultaneous usage of equivalence transformations and mappings between classes allowed us to carry out group classification of the class of variable coefficient semilinear reaction–diffusion equations with power nonlinearity

$$f(x)u_t = (g(x)u_x)_x + h(x)u^m, \tag{A.29}$$

where $f = f(x)$, $g = g(x)$ and $h = h(x)$ are arbitrary smooth functions of the variable x, $f(x)g(x)h(x) \neq 0$, m is an arbitrary constant ($m \neq 0, 1$).

At first the gauge $f = g$ was performed using equivalence transformations of the class, so class was reduced to its subclass A.29

$$f(x)u_t = (f(x)u_x)_x + h(x)u^m, \quad fh \neq 0, \quad m \neq 0, 1. \tag{A.30}$$

The next step was to make the change of the dependent variable

$$v(t,x) = \sqrt{|f(x)|}u(t,x) \tag{A.31}$$

in class (A.30). As a result, we obtain the class of related equations

$$v_t = v_{xx} + H(x)v^m + F(x)v, \tag{A.32}$$

where the new arbitrary elements F and H are connected with the old ones via the formulas

$$F(x) = -\frac{(\sqrt{|f(x)|})_{xx}}{\sqrt{|f(x)|}}, \quad H(x) = \frac{h(x)\operatorname{sign} f(x)}{(\sqrt{|f(x)|})^{m+1}}. \tag{A.33}$$

We study nonclassical reduction operators for equations (A.32) and then use them to construct exact solutions of equations (A.30).

Reduction Operators for General Values of m. We look for $G^{\sim}_{FH}$-inequivalent reduction operators of the imaged class (A.32). Here reduction operators have the general form $Q = \tau\partial_t + \xi\partial_x + \eta\partial_v$, where τ, ξ and η are functions of t, x and v, and $(\tau, \xi) \neq (0,0)$. Since (A.32) is an evolution equation, there are two principally different cases of finding Q: $\tau \neq 0$ and $\tau = 0$ [92, 173, 333]. The singular case $\tau = 0$ was exhaustively investigated for general evolution equation in [173, 333].

If $\tau \neq 0$, we can assume $\tau = 1$ up to the usual equivalence of reduction operators. Then the determining equations are of the form

$$\begin{gathered}
\xi_{vv} = 0, \quad \eta_{vv} = 2(\xi_{xv} - \xi\xi_v), \\
\eta_t - \eta_{xx} + 2\xi_x\eta = \\
\xi\left(H_x v^m + F_x v\right) + (2\xi_x - \eta_v)\left(Hv^m + Fv\right) + \eta\left(F + Hv^{m-1}m\right), \\
3\xi_v\left(Hv^m + Fv\right) + 2\xi_x\xi + \xi_t + 2\eta_{vx} - \xi_{xx} - 2\xi_v\eta = 0.
\end{gathered} \tag{A.34}$$

Integration of first two equations of system (A.34) gives us the following expressions for ξ and η

$$\begin{aligned} &\xi = av + b, \\ &\eta = -\tfrac{1}{3}a^2v^3 + (a_x - ab)v^2 + cv + d, \end{aligned} \tag{A.35}$$

where $a = a(t,x)$, $b = b(t,x)$, $c = c(t,x)$ and $d = d(t,x)$.

Substituting ξ and η from (A.35) into the third and forth equations of (A.34), we obtain the classifying equations which include both the residuary uncertainties in coefficients of the operator and the arbitrary elements of the class under consideration.

Since the functions a, b, c, d, F and H do not depend on the variable v, the classifying equations should be split with respect to different powers of v. Two principally different cases $a = 0$ and $a \neq 0$ should be considered separately.

If $a = 0$ then for any $m \neq 0, 1, 2$ the splitting results in the system of five equations

$$\begin{aligned} &mHd = 0, \quad d_t - d_{xx} + 2b_x d - Fd = 0, \\ &b_t - b_{xx} + 2bb_x + 2c_x = 0, \\ &bH_x + \left(c(m-1) + 2b_x\right) H = 0, \\ &bF_x + 2b_x F + c_{xx} - c_t - 2b_x c = 0. \end{aligned} \tag{A.36}$$

Since $mH \neq 0$ then $d = 0$ and the second equation of (A.36) becomes identity.

Finding the general solution of the other three equations from (A.36) appears to be a very difficult problem. But it is easy to construct certain particular solutions setting, e.g., $b_t = 0$. This supposition implies that $c_t = 0$. Then the integration of (A.36) gives the expressions of c, F and H via the function $b(x) \neq 0$

$$c = -\frac{1}{2}b^2 + \frac{1}{2}b_x + k_1,$$

$$F = -\frac{1}{4}b^2 + k_1 + k_2 b^{-2} + b_x + \frac{1}{4}\left(\frac{b_x}{b}\right)^2 - \frac{1}{2}\frac{b_{xx}}{b}, \tag{A.37}$$

$$H = k_3 b^{-\frac{m+3}{2}} \exp\left[(m-1)\int\left(\frac{b}{2} - \frac{k_1}{b}\right)\mathrm{d}x\right], \tag{A.38}$$

where k_1, k_2 and k_3 are arbitrary constants, $k_3 \neq 0$.

Theorem A.15. *The equations from class (A.32) with the arbitrary elements given by formulas (A.37) and (A.38) admit reduction operators of the form*

$$Q = \partial_t + b\partial_x + \left(-\frac{1}{2}b^2 + \frac{1}{2}b_x + k_1\right)v\partial_v, \tag{A.39}$$

where $b = b(x)$ is an arbitrary smooth function and k_1 is an arbitrary constant.

Note A.16. *Theorem A.15 is true for any $m \in \mathbb{R}$, including $m \in \{0, 1, 2\}$.*

We present the illustrative example, by considering specific of the function b.

Example A.17. Consider $b = x^{-1}$. In view of theorem A.15 the equations from class (A.32) with the arbitrary elements

$$F = k_1 + k_2 x^2 - 2x^{-2}, \quad H = k_3 x^{m+1} e^{\frac{1}{2}(1-m)k_1 x^2} \tag{A.40}$$

admit the reduction operator

$$Q = \partial_t + x^{-1}\partial_x + \left(k_1 - x^{-2}\right)v\partial_v.$$

The ansatz constructed with this operator is $v = x^{-1}e^{k_1 t}z(\omega)$, where $\omega = x^2 - 2t$, and the reduced equation reads

$$4z_{\omega\omega} + k_3 e^{\frac{1}{2}(1-m)k_1\omega} z^m + k_2 z = 0.$$

If $k_1 = k_2 = 0$, the reduced equation has the particular solution

$$z = \begin{cases} \left(\pm\frac{m-1}{2}\sqrt{-\frac{k_3}{2(m+1)}}\,\omega\right)^{\frac{2}{1-m}}, & m \neq -1, \\ \exp\left\{-\left[\operatorname{erf}^{-1}\left(\pm\frac{\sqrt{2}}{2}\sqrt{\frac{k_3}{\pi}}\,\omega\right)\right]^2\right\}, & m = -1. \end{cases}$$

Substituting the obtained z to the ansatz, we construct exact solutions of equations from class (A.32) with the arbitrary elements (A.40) for the values $k_1 = k_2 = 0$.

The preimaged equation $x^4u_t = (x^4u_x)_x + k_3\,x^{3(m+1)}u^m$ has the exact solution

$$u = \begin{cases} x^{-3}\left(\pm\frac{m-1}{2}\sqrt{-\frac{k_3}{2(m+1)}}\,(x^2-2t)\right)^{\frac{2}{1-m}}, & m \neq -1, \\ x^{-3}\exp\left\{-\left[\operatorname{erf}^{-1}\left(\pm\frac{\sqrt{2}}{2}\sqrt{\frac{k_3}{\pi}}\,(x^2-2t)\right)\right]^2\right\}, & m = -1. \end{cases}$$

More examples can be found in [299]. We have shown the applicability of theorem A.15 for construction of non-Lie exact solutions of equations from classes (A.32) and (A.30). Moreover, using these solutions one can find exact solutions for other equations from (A.32) and (A.30) with the help of equivalence transformations from the corresponding equivalence groups.

In the case $m = 3$ we are able to construct more exact solutions of equations from class (A.32) whose coefficients are given by (A.37)–(A.38) with $k_1 = 0$, namely, for the equations

$$\begin{aligned} v_t &= v_{xx} + k_3 b^{-3} e^{\int b\,\mathrm{d}x} v^3 \\ &\quad + \left(\frac{k_2}{b^2} - \frac{1}{4}b^2 + b_x + \frac{1}{4}\left(\frac{b_x}{b}\right)^2 - \frac{1}{2}\frac{b_{xx}}{b}\right)v, \end{aligned} \tag{A.41}$$

where $b = b(x)$, $k_3 \neq 0$.

According to theorem A.15, equation (A.41) admits the reduction operator (A.39) (with $k_1 = 0$). An ansatz constructed with this operator has the form

$$v = z(\omega)\sqrt{|b|}\,e^{-\frac{1}{2}\int b\,\mathrm{d}x}, \quad \text{where} \quad \omega = t - \int\frac{\mathrm{d}x}{b},$$

and reduces (A.41) to the second-order ODE

$$z_{\omega\omega} = -k_3 z^3 - k_2 z.$$

It is interesting that the reduced ODE does not depend on the function $b(x)$. Multiplying this equation by z_ω and integrating once, we obtain the equation

$$z_\omega^2 = -\frac{k_3}{2}z^4 - k_2 z^2 + C_1.$$

Its general solution is expressed via Jacobian elliptic functions depending on values of the constants k_2, k_3 and C_1.

For example, if $k_2 = 1+\mu^2$, $k_3 = -2\mu^2$ and $C_1 = 1$ $(0<\mu<1)$ we find two exact solutions of equation (A.41)

$$v = \mathrm{sn}\left(t - \textstyle\int \frac{1}{b}\mathrm{d}x, \mu\right)\sqrt{|b|}\, e^{-\frac{1}{2}\int b\,\mathrm{d}x}, \quad v = \mathrm{cd}\left(t - \textstyle\int \frac{1}{b}\mathrm{d}x, \mu\right)\sqrt{|b|}\, e^{-\frac{1}{2}\int b\,\mathrm{d}x},$$

where $\mathrm{sn}(\omega,\mu)$, $\mathrm{cd}(\omega,\mu)$ are Jacobian elliptic functions [316].

The second case to be considered is $a \neq 0$. Then after substitution of ξ and η from (A.35) to system (A.34) its last equation takes the form

$$\begin{aligned} &\frac{2}{3}a^3v^3 + 2a(ab-2a_x)v^2 + (a_t + 3a_{xx} + 3aF - 2(ab)_x - 2ac)v \\ &\qquad + b_t + 2b_xb - b_{xx} - 2ad + 2c_x + 3aHv^m = 0. \end{aligned} \tag{A.42}$$

It is easy to see that $a \neq 0$ if and only if $m = 3$.

Specific Reduction Operators for the Cubic Nonlinearity. Splitting equation (A.42) in the case $m=3$ and $a \neq 0$ with respect to u, we obtain that the functions a, b c and d do not depend on the variable t and are expressed via the functions F and H in the following way

$$\begin{aligned} &a = \frac{3}{2}\sqrt{2}\,\varepsilon\sqrt{-H}, \quad b = \frac{H_x}{H}, \\ &c = \frac{1}{8}\left(12F - 2\left(\frac{H_x}{H}\right)_x - \left(\frac{H_x}{H}\right)^2\right), \\ &d = \frac{\sqrt{2}\,\varepsilon}{2\sqrt{-H}}\left(F_x + \frac{1}{2}\frac{H_x}{H}\left(\frac{H_x}{H}\right)_x - \frac{1}{2}\left(\frac{H_x}{H}\right)_{xx}\right), \end{aligned} \tag{A.43}$$

where $\varepsilon = \pm 1$. If $H<0$ the corresponding reduction operators have real coefficients.

Then splitting of the third equation of system (A.34) for $m=3$ results in the system of two ordinary differential equations

$$
\begin{aligned}
&H^3H_{xxxx}-13\,H_x^4+2\,F_xH^3H_x+22\,HH_x^2H_{xx}-4\,FH^2H_x^2\\
&-4\,H^2H_{xx}^2-6\,H^2H_xH_{xxx}+4\,FH^3H_{xx}-6\,F_{xx}H^4=0,\\
&16F_{xxx}H^5+16\,H^2H_xH_{xx}^2+3\,H^2H_x^2H_{xxx}-4\,F_xH^4H_{xx}\\
&-6\,H^3H_{xx}H_{xxx}-18\,HH_x^3H_{xx}-8\,FF_xH^5+2\,F_xH^3H_x^2\\
&-20\,FH^2H_x^3-12\,FH^4H_{xxx}+5\,H_x^5+32\,FH^3H_xH_{xx}=0.
\end{aligned}
\tag{A.44}
$$

The following statement is true.

Theorem A.18. *The equations from class (A.32) with $m=3$ and the arbitrary elements satisfying system (A.44) admit reduction operators of the form*

$$
\begin{aligned}
Q=\partial_t&+\left(\frac{3}{2}\sqrt{2}\,\varepsilon\sqrt{-H}\,v+\frac{H_x}{H}\right)\partial_x\\
&+\left[\frac{3}{2}Hv^3+\frac{3}{4}\sqrt{2}\,\varepsilon\frac{H_x}{\sqrt{-H}}\,v^2+\frac{1}{8}\left(12F-2\left(\frac{H_x}{H}\right)_x-\left(\frac{H_x}{H}\right)^2\right)v\right.\\
&\quad\left.+\frac{\sqrt{2}\,\varepsilon}{2\sqrt{-H}}\left(F_x+\frac{1}{2}\frac{H_x}{H}\left(\frac{H_x}{H}\right)_x-\frac{1}{2}\left(\frac{H_x}{H}\right)_{xx}\right)\right]\partial_v,
\end{aligned}
\tag{A.45}
$$

where $\varepsilon=\pm1$.

System (A.44) consists of two nonlinear fourth- and third-order ODEs. Unfortunately we were not able to find its general solution. Nevertheless, we tested the six pairs of functions F and H appearing in table 1 in order to check whether they satisfy system (A.44). In the case of positive answer the corresponding reduction operator is easily constructed via formula (A.45). It appears that system (A.44) is satisfied by F and H from cases 1, 2 and 6 and by those from cases 3 and 4 for special values of the constants k and a_2, namely, $(k,a_2)\in\left\{\left(-3,\frac{9}{4}\right),\left(-\frac{3}{2},\frac{3}{16}\right)\right\}$.

Example A.19. Class (A.32) contains equations with cubic nonlinearity, which are not reduced to constant-coefficient ones by point transformations and admit reduction operators of the form (A.45). One of them is the equation with the coefficients F and H presented by case 3 of table **??** with $k=-3$, $a_2=\frac{9}{4}$ and $\delta=-1$, namely,

$$v_t = v_{xx} - x^{-3}v^3 + \frac{9}{4}x^{-2}v. \tag{A.46}$$

According to theorem A.18 this equation admits two similar reduction operators ($\varepsilon=\pm1$)

$$Q_\pm = \partial_t + \frac{3}{2}\sqrt{2}\left(\varepsilon x^{-\frac{3}{2}}v - \sqrt{2}\,x^{-1}\right)\partial_x - \\ \frac{3}{4}\sqrt{2}\left(\sqrt{2}\,x^{-3}v^3 - 3\varepsilon x^{-\frac{5}{2}}v^2 - \sqrt{2}\,x^{-2}v + 4\varepsilon x^{-\frac{3}{2}}\right)\partial_v.$$

They lead to the solutions differing only in their signs. Since equation (A.46) is invariant with respect to the transformation $v\mapsto -v$, we consider in detail only the case $\varepsilon=1$. For all expressions to be correctly defined, we have to restrict ourself with values $x>0$. (Another way is to replace x by $|x|$.)

For convenient reduction we apply the hodograph transformation

$$\tilde t = v, \quad \tilde x = x, \quad \tilde v = t$$

which maps equation (A.46) and the reduction operator Q_+ to the equation

$$\tilde v_{\tilde t}^{\,2}\,\tilde v_{\tilde x\tilde x} + \tilde v_{\tilde x}^{\,2}\,\tilde v_{\tilde t\tilde t} - 2\,\tilde v_{\tilde t}\,\tilde v_{\tilde x}\,\tilde v_{\tilde t\tilde x} + \tilde v_{\tilde t}^{\,2} + \frac{\tilde t^3}{\tilde x^3}\,\tilde v_{\tilde t}^{\,3} - \frac{9}{4}\frac{\tilde t}{\tilde x^2}\,\tilde v_{\tilde t}^{\,3} = 0 \tag{A.47}$$

and its reduction operator

$$\tilde Q_+ = -\frac{3}{4}\sqrt{2}\left(\sqrt{2}\,\tilde x^{-3}\tilde t^3 - 3\tilde x^{-\frac{5}{2}}\tilde t^2 - \sqrt{2}\,\tilde x^{-2}\tilde t + 4\tilde x^{-\frac{3}{2}}\right)\partial_{\tilde t} + \\ \frac{3}{2}\sqrt{2}\left(\tilde x^{-\frac{3}{2}}\tilde t - \sqrt{2}\,\tilde x^{-1}\right)\partial_{\tilde x} + \partial_{\tilde v},$$

respectively. An ansatz constructed with the operator $\tilde Q_+$ has the form

$$\tilde v = \frac{1}{24}\tilde x^2\frac{\tilde t+\sqrt{2\tilde x}}{\tilde t-\sqrt{2\tilde x}} - \frac{1}{12}\tilde x^2 + z(\omega), \quad \text{where} \quad \omega = \tilde x^2\frac{\tilde t-\sqrt{2\tilde x}}{\tilde t+\sqrt{2\tilde x}},$$

and reduces (A.47) to the simple linear ODE $\omega z_{\omega\omega}+2z_\omega=0$ whose general solution $z=\tilde c_1+\tilde c_2\omega^{-1}$ substituted to the ansatz gives the exact solution

$$\tilde v=\frac{\tilde x^4+24\tilde c_2}{24\tilde x^2}\,\frac{\tilde t+\sqrt{2\tilde x}}{\tilde t-\sqrt{2\tilde x}}-\frac{1}{12}\tilde x^2+\tilde c_1$$

of equation (A.47). Applying the inverse hodograph transformation and canceling the constant $\tilde c_1$ by translations with respect to t, we construct the non-Lie solution

$$v=\sqrt{2x}\,\frac{3x^4+24tx^2+c_2}{x^4+24tx^2-c_2} \tag{A.48}$$

of equation (A.46). The solution (A.48) with $c_2=0$ is a Lie solution invariant with respect to the dilatation operator $D=4t\partial_t+2x\partial_x+v\partial_v$ from the maximal Lie invariance algebra of equation (A.46). However, it is much harder to find this solution by the reduction with respect to the operator D. The corresponding ansatz $v=\sqrt{x}z(\omega)$, where $\omega=t^{-1}x^2$, has a simple form but the reduced ODE $4\omega^2z_{\omega\omega}+\omega(\omega+4)z_\omega+2z-z^3=0$ is nonlinear and complicated.

This example justifies the observation made by W. Fushchych [85] that "ansatzes generated by conditional symmetry operators often reduce an initial nonlinear equation to a linear one. As a rule, a Lie reduction does not change the nonlinear structure of an equation." We can also formulate the more general similar observation that a complicated non-Lie ansatz may lead to a simple reduced equation while a simple Lie ansatz may give a complicated reduced equation which is difficult to be integrated.

One of the preimages of equation (A.46) with respect to transformation (A.31) is the equation

$$x\sin^2(\sqrt2\ln x)u_t=\Big(x\sin^2(\sqrt2\ln x)u_x\Big)_x-x^{-1}\sin^4(\sqrt2\ln x)\,u^3,$$

having the non-Lie exact solution

$$u=\sqrt{\frac{2}{x}}\,|\operatorname{cosec}(\sqrt2\ln x)|\frac{3x^4+24tx^2+c_2}{x^4+24tx^2-c_2}.$$

Example A.20. Consider the equation from the imaged class (A.32)

$$v_t = v_{xx} - x^{-\frac{3}{2}}v^3 + \frac{3}{16}\frac{v}{x^2} \tag{A.49}$$

for the values $x > 0$. It admits the reduction operator of form (A.45)

$$Q_+ = \partial_t + \frac{3}{2}\left(\sqrt{2}\,x^{-\frac{3}{4}}v - x^{-1}\right)\partial_x - \frac{3}{8}\left(4x^{-\frac{3}{2}}v^3 - 3\sqrt{2}x^{-\frac{7}{4}}v^2 + x^{-2}v\right)\partial_v.$$

Usage of the same technique as in the previous example gives the non-Lie exact solution of (A.49)

$$v = \frac{1}{2}5\sqrt{2}\,x^{\frac{1}{4}}\frac{3\,t + x^2}{\sqrt{x}(15\,t + x^2) + c_2}. \tag{A.50}$$

Applying the transformation $v = \sqrt{x}(b_1x^{\frac{1}{4}} + b_2x^{-\frac{1}{4}})\,u$ to solution (A.50), we obtain a non-Lie solution of the equation

$$x(b_1x^{\frac{1}{4}} + b_2x^{-\frac{1}{4}})^2u_t = \Big(x(b_1x^{\frac{1}{4}} + b_2x^{-\frac{1}{4}})^2u_x\Big)_x - \sqrt{x}(b_1x^{\frac{1}{4}} + b_2x^{-\frac{1}{4}})^4u^3$$

from class (A.30), where b_1 and b_2 are arbitrary constants, $b_1^2 + b_2^2 \neq 0$.

More examples and nonclassical reduction operators of those equations from class (A.30) that are reducible to constant coefficient equations from class (A.32) can be found in [299].

A.4. Potential Symmetries of a Class of Inhomogeneous Diffusion Equations

In this section we consider variable coefficient nonlinear diffusion equations of the form

$$f(x)u_t = (g(x)u^nu_x)_x, \quad fgn \neq 0. \tag{A.51}$$

Using the transformation $\tilde t = t$, $\tilde x = \int\frac{dx}{g(x)}$, $\tilde u = u$, we can reduce equation (A.51) to $\tilde f(\tilde x)\tilde u_{\tilde t} = (\tilde u^n\tilde u_{\tilde x})_{\tilde x}$, where $\tilde f(\tilde x) = g(x)f(x)$ and $\tilde g(\tilde x) = 1$.

That is why, without loss of generality, we restrict ourselves to the investigation of equations having the form

$$f(x)u_t = (u^n u_x)_x, \quad fn \neq 0. \tag{A.52}$$

The equivalence group $G^{\sim}$ of class (A.52) has a simple structure and consists of the transformations

$$\tilde{t} = \delta_1 t + \delta_4, \quad \tilde{x} = \delta_2 x + \delta_5, \quad \tilde{u} = \delta_3 u,$$
$$\tilde{f} = \delta_1 \delta_2^{-2} \delta_3^n f, \quad \tilde{n} = n,$$

where δ_i, $i = 1, \ldots, 5$, are arbitrary constants, $\delta_1\delta_2\delta_3 \neq 0$. At the same time, class (A.52) possesses a generalized equivalence group which is wider than $G^{\sim}$.

Theorem A.21. *The generalized equivalence group $\hat{G}^{\sim}$ of class (A.52) under the condition $n \neq -1$ consists of the transformations*

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \frac{\delta_3 x + \delta_4}{\delta_5 x + \delta_6}, \quad \tilde{u} = \delta_7 |\delta_5 x + \delta_6|^{-\frac{1}{n+1}} u,$$
$$\tilde{f} = \delta_1 {\delta_7}^n |\delta_5 x + \delta_6|^{\frac{3n+4}{n+1}} f, \quad \tilde{n} = n,$$

where δ_j, $j = 1, \ldots, 7$, are real constants, $\delta_1\delta_7 \neq 0$ and $\delta_3\delta_6 - \delta_4\delta_5 = \pm 1$.

In the case $n = -1$ transformations from the group $\hat{G}^{\sim}$ take the form

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x + \delta_4, \quad \tilde{u} = \delta_5 e^{\delta_6 x} u, \quad \tilde{f} = \delta_1 \delta_3^{-2} \delta_5^{-1} e^{-\delta_6 x} f,$$

where δ_j, $j = 1, \ldots, 6$, are arbitrary constants, $\delta_1\delta_3\delta_5 \neq 0$.

Since the parameter n is an invariant of all admissible (point) transformations in class (A.52), this class can be presented as the union of disjoint subclasses where each from the subclasses corresponds to a fixed value of n. This representation allows us to give the interpretation of the generalized equivalence group $\hat{G}^{\sim}$ as a family of the usual conditional equivalence groups of the subclasses parameterized with n, and the value $n = 0$ and $n = -1$ being singular.

The conservation laws for the class (A.52) are found in [137, 140]. The space of local conservation laws of any equation (A.52) with $n \neq 0$ is two-dimensional and spanned by conservation laws with the conserved vectors $(fu, -u^n u_x)$ and $(\,xfu,\ -xu^n u_x + \int u^n du\,)$.

Up to $G^{\sim}$-equivalence, these conservation laws give rise to the following inequivalent potential systems for equations (A.52):

1. $\quad v^1_x = fu, \quad v^1_t = u^n u_x;$

2. $\quad v^2_x = xfu, \quad v^2_t = xu^n u_x - \int u^n du;$

3. $\quad v^1_x = fu, \quad v^1_t = u^n u_x, \quad v^2_x = xfu, \quad v^2_t = xu^n u_x - \int u^n du.$

Systems **1** and **2** are associated with the conservation laws having the characteristics 1 and x, respectively. The united system **3** corresponds to the whole space of conservation laws. The generalized equivalence group $\hat{G}^{\sim}$ prolonged to potentials establishes additional equivalence between potential systems. Thus, in the case $n \neq -1$ the transformation

$$\begin{aligned} &\tilde{t} = t, \quad \tilde{x} = x^{-1}, \quad \tilde{u} = |x|^{-\frac{1}{n+1}} u, \\ &\tilde{v}^1 = -(\operatorname{sign} x) v^2, \quad \tilde{v}^2 = -(\operatorname{sign} x) v^1 \end{aligned} \tag{A.53}$$

maps systems **1** and **2** to systems **2** and **1** in the tilde variables with $\tilde{f} = |\tilde{x}|^{-\frac{3n+4}{n+1}} f(\tilde{x}^{-1})$, respectively. Systems **1** and **2** are $\hat{G}^{\sim}$-inequivalent for an arbitrary pair of values of f iff $n = -1$.

Potential symmetries of equation (A.52), associated with system **1**, were first obtained in [274, 275], see also [7, 38, 39] for the constant coefficient case $f = 1$. There exist two inequivalent equations of form (A.52) admitting such nonlocal symmetries. Below we adduce the values of arbitrary elements together with bases of the corresponding maximal Lie invariance algebras.

1.1. $f = 1$, $n = -2$:

$$\langle \partial_t,\ \partial_{v^1},\ 2t\partial_t + u\partial_u + v^1\partial_{v^1},\ x\partial_x - u\partial_u,\ -v^1 x\partial_x + (xu + v^1)u\partial_u + 2t\partial_{v^1},$$
$$4t^2\partial_t - ((v^1)^2 + 2t)x\partial_x + ((v^1)^2 + 6t + 2xuv^1)u\partial_u + 4tv^1\partial_{v^1},\ \varphi\partial_x - \varphi_{v^1} u^2\partial_u \rangle;$$

1.2. $f = x^{-4/3}$, $n = -2$:

$$\langle \partial_t,\ \partial_{v^1},\ 2t\partial_t + u\partial_u + v^1\partial_{v^1},\ 3x\partial_x - u\partial_u - 2v^1\partial_{v^1},$$
$$3xv^1\partial_x - (v^1 + 3x^{-1/3}u)u\partial_u - (v^1)^2\partial_{v^1}\rangle.$$

Here and below $\varphi = \varphi(t, v^1)$ is an arbitrary solution of the linear heat equation $\varphi_t = \varphi_{v^1v^1}$.

Potential symmetries of equation (A.52) associated with system **2** were first investigated in [276] (see also [141]). Up to the equivalence group $G^\sim$, there exist exactly two cases of equations in class (A.52) admitting such potential symmetries:

2.1. $f = x^{-2}$, $n = -2$:

$$\langle \partial_t,\ \partial_{v^2},\ x\partial_x,\ 2t\partial_t + u\partial_u + v^2\partial_{v^2},\ v^2x\partial_x - u^2\partial_u + 2t\partial_{v^2},$$
$$4t^2\partial_t + ((v^2)^2 + 2t)x\partial_x + 2(2t - uv^2)u\partial_u + 4tv^2\partial_{v^2},\ x^2\psi\partial_x - xu(\psi + \psi_{v^2}u)\partial_u\rangle.$$

2.2. $f = x^{-2}(c_1 + c_2x^{-1})^{-4/3}, \quad c_2 \neq 0, \quad n = -2$:

$$\langle \partial_t,\ \partial_{v^2},\ 2t\partial_t + u\partial_u + v^2\partial_{v^2},\ 3(c_1x + c_2)x\partial_x - (2c_2 + 3c_1x)u\partial_u + 2c_2v^2\partial_{v^2},$$
$$3v^2(c_1x + c_2)x\partial_x - (3x^{4/3}(c_1x + c_2)^{-1/3}u + (2c_2 + 3c_1x)v^2)u\partial_u + c_2(v^2)^2\partial_{v^2}\rangle;$$

Here and below $\psi = \psi(t, v^2)$ is an arbitrary solution of the linear heat equation $\psi_t = \psi_{v^2v^2}$. By transformation (A.53), cases 2.1 and 2.2 are reduced to cases 1.1 and 1.2, respectively. For the precise reduction 2.2 $\to$ 1.2 the transformation $\hat{t} = \tilde{t}$, $\hat{x} = c_1 + c_2\tilde{x}$, $\hat{u} = c_2^{-1}\tilde{u}$ from $G^\sim$ has to be additionally carried out.

The united system **3** is equivalent to the second-level potential system

$$v^1_x = fu, \quad w_x = v^1, \quad w_t = \textstyle\int u^n du \tag{A.54}$$

constructed from system **1** using its conserved vector $(v, -\int u^n du)$, and $w = xv^1 - v^2$. Nontrivial $G^\sim$-inequivalent cases of potential symmetries associated with system (A.54) are new and exhausted by the following ones:

3.1. $f = 1$, $n = -2$:

$$\langle \partial_t,\ \partial_w,\ \partial_{v^1} + x\partial_w,\ 2t\partial_t + u\partial_u + v^1\partial_{v^1} + w\partial_w,\ x\partial_x - u\partial_u + w\partial_w,$$

$$(w-2v^1x)\partial_x+(2xu+v^1)u\partial_u+2t\partial_{v^1}+(2t-(v^1)^2)x\partial_w,$$

$$4t^2\partial_t+(2v^1w-3x(v^1)^2-6tx)\partial_x+(6xuv^1-2uw+(v^1)^2+10t)u\partial_u+4tv^1\partial_{v^1}$$

$$+((v^1)^2w-2tw-2x(v^1)^3)\partial_w,\ \varphi_{v^1}\partial_x-\varphi_tu^2\partial_u+(v^1\varphi_{v^1}-\varphi)\partial_w\rangle;$$

3.2. $f=1$, $n=-2/3$:

$$\langle\partial_t,\ \partial_x,\ \partial_w,\ \partial_{v^1}+x\partial_w,\ 2t\partial_t+3u\partial_u+3v^1\partial_{v^1}+3w\partial_w,$$

$$x\partial_x-3u\partial_u-2v^1\partial_{v^1}-w\partial_w,\ w\partial_x-3uv^1\partial_u-(v^1)^2\partial_{v^1}\rangle;$$

3.3. $f=x^{-2}$, $n=-2$:

$$\langle\partial_t,\ \partial_w,\ \partial_{v^1}+x\partial_w,\ x\partial_x-v^1\partial_{v^1},\ 2t\partial_t+u\partial_u+v^1\partial_{v^1}+w\partial_w,$$

$$x(2xv^1-w)\partial_x-u(xv^1+2u)\partial_u+v^1(w-xv^1)\partial_{v^1}+(x^2(v^1)^2-2t)\partial_w,$$

$$4t^2\partial_t+x(6t+3x^2(v^1)^2-4xv^1w+w^2)\partial_x$$
$$+2u(2t-3xuv^1+2uw-x^2(v^1)^2+xv^1w)\partial_u$$

$$+(2xv^1w-2t-x^2(v^1)^2-w^2)v^1\partial_{v^1}+2(2tw+x^3(v^1)^3-x^2(v^1)^2w)\partial_w,$$

$$x^2\psi_{v^2}\partial_x+(\psi_{v^2}+u\psi_t)xu\partial_u-\psi\partial_{v^1}+x(xv^1\psi_{v^2}-\psi)\partial_w\rangle;$$

3.4. $f=x^{-6}$, $n=-2/3$:

$$\langle\partial_t,\ \partial_w,\ \partial_{v^1}+x\partial_w,\ 2t\partial_t+3u\partial_u+3v^1\partial_{v^1}+3w\partial_w,\ x\partial_x+6u\partial_u+v^1\partial_{v^1}+2w\partial_w,$$

$$x^2\partial_x+3xu\partial_u+(w-xv^1)\partial_{v^1}+xw\partial_w,$$
$$xw\partial_x-3(xv^1-2w)u\partial_u-(xv^1-w)v^1\partial_{v^1}+w^2\partial_w\rangle.$$

Here $v^2=xv^1-w$. By transformation (A.53) which is rewritten for v^1 and w as $\tilde v^1=w\operatorname{sign}x-|x|v^1$ and $\tilde w=|x|^{-1}w$, cases 3.3 and 3.4 are reduced to cases 3.1 and 3.2, respectively.

Appendix B

Group Analysis of $\mathbf{K(m, n)}$, Benjamin–Bona–Mahony–Burgers and KdV-Like Equations

In this appendix we present the results on group analysis of $\mathbf{K(m, n)}$, Benjamin–Bona–Mahony–Burgers and generalized Korteweg–de Fries equations of third and fifth orders. Usage of transformations from the equivalence groupoids of the classes plays the crucial role in complete solution of group classification problems for all the classes under consideration.

In Section B.1 we perform the group classification of the variable coefficient Gardner equations $u_t + k(t)uu_x + f(t)u^2u_x + g(t)u_{xxx} = 0$, $fg \neq 0$. We show that even the use of usual equivalence group allows one to get the complete result. At the same time utilizing wider generalized extended equivalence group provides more simplification and therefore is preferable.

The exhaustive group classification of a class of variable coefficient generalized KdV equations $u_t+u^nu_x+h(t)u+g(t)u_{xxx} = 0$, $ng \neq 0$, is presented in Section B.2. The found Lie symmetries are applied in order to reduce the initial and boundary value problem for the generalized KdV equations to an initial value problem for nonlinear third-order ODEs.

Sections B.3 and B.4 are devoted to the exhaustive group classification of generalized fKdV equations with time dependent coefficients of the general form $u_t + u^nu_x + \alpha(t)u + \beta(t)u_{xxxxx} = 0$, $n\beta \neq 0$. The cases $n = 1$ and $n \neq 1$ differ by their transformational properties and therefore treated separately in sections B.3 and B.4, respectively.

In Section B.5 we investigate Lie symmetry properties of variable coefficient $K(m,n)$ equations $u_t + g(t)(u^m)_x + f(t)(u^n)_{xxx} = 0$, $fn \neq 0$. Group classification is presented up to widest possible equivalence groups, the usual equivalence group of the whole class for the general case and the conditional equivalence groups for special values of the exponents m and n.

Using the method of mapping between classes we present the complete group classification of BBMB equations $u_t + f(t)u_x + g(t)uu_x + k(t)u_{xx} + h(t)u_{xxt} = 0$, $ghk \neq 0$, in Section B.6. As a by-product of this approach we also get the group classification of a related class of BBMB equations with a forcing term $u_t + uu_x + K(t)u_{xx} + H(t)u_{xxt} = F(t)$, $HK \neq 0$.

For many equations from the considered classes exact solutions are constructed. The results adduced in this appendix are published in papers [5*,8*,12*,14*,15*,16*].

B.1. Enhanced Group Classification of Gardner Equations with Time Dependent Coefficients

In this section we perform the group classification of the variable coefficient Gardner equations

$$u_t + k(t)uu_x + f(t)u^2u_x + g(t)u_{xxx} = 0, \qquad fg \neq 0, \tag{B.1}$$

Here k, f, and g are smooth functions of the variable t. The particular results on Lie symmetries of such equations were derived in [200]. We achieve the exhaustive classification using the groups of equivalence transformations of class (B.1). We show that even the use of usual equivalence group allows one to get the complete result. At the same time utilizing wider generalized extended equivalence group provides more simplification and therefore is preferable. This is illustrated in the process of finding Lie symmetries of equations (B.1).

Firstly we investigate admissible transformations in class (B.1). The following statements are true.

Theorem B.1. *The generalized extended equivalence group* $\hat{G}^{\sim}$ *of class* (B.1) *is formed by the transformations*

$$\tilde{t} = \alpha(t), \quad \tilde{x} = \delta_1 x + \tfrac{\delta_1\delta_3}{\delta_2^2}\int(\delta_2 k(t) - \delta_3 f(t))\mathrm{d}t + \delta_4, \quad \tilde{u} = \delta_2 u + \delta_3,$$

$$\tilde{k}(\tilde{t}) = \frac{\delta_1}{\delta_2\alpha_t}\left(k(t) - 2\frac{\delta_3}{\delta_2}f(t)\right), \quad \tilde{f}(\tilde{t}) = \frac{\delta_1}{\delta_2^2\alpha_t}f(t), \quad \tilde{g}(\tilde{t}) = \frac{\delta_1^3}{\alpha_t}g(t),$$

where δ_i, $i = 1,\dots,4$, *are arbitrary constants with* $\delta_1\delta_2 \neq 0$, α *is an arbitrary smooth function with* $\alpha_t \neq 0$.

The usual equivalence group $G^{\sim}$ *of class* (B.1) *consists of the above transformations with* $\delta_3 = 0$.

Theorem B.2. *The entire set of admissible transformations (equivalence groupoid) of class* (B.1) *is generated by the transformations from the group* $\hat{G}^{\sim}$. *Class* (B.1) *is normalized in the generalized extended sense.*

Thus, there are no other point transformations between equations from class (B.1) than those transformations from the group $\hat{G}^{\sim}$. To deduce which variable coefficient equations of the form (B.1) is reducible to their constant coefficient counterparts we assume $\tilde{k}$ and $\tilde{f}$ are constant in the transformation components for arbitrary elements in $\hat{G}^{\sim}$, this results in the following statement.

Proposition B.3. *A variable coefficient equation from class* (B.1) *is reducible to constant coefficient equation from the same class if and only if the coefficients* f, g *and* k *satisfy the conditions*

$$(f/k)_t = (g/k)_t = 0.$$

As there is one arbitrary function, $\alpha(t)$, in the transformations from the group $\hat{G}^{\sim}$, we can set one of the arbitrary elements of class (B.1) to a nonzero constant value. We choose the gauging $g = 1$ and perform it using the transformation

$$\tilde{t} = \int g(t)\mathrm{d}t, \quad \tilde{x} = x, \quad \tilde{u} = u. \tag{B.2}$$

Then, any equation from the class (B.1) is mapped to one from its subclass singled out by the condition $g = 1$. Old forms of the arbitrary elements are connected with new ones via the formulae $\tilde{k} = k/g$ and $\tilde{f} = f/g$.

The most general form of transformation that maps an equation from class (B.1) to another equation from the same class with $g = 1$ is

$$\begin{aligned} &\tilde{t} = \delta_1^3 \int g(t)\mathrm{d}t + \delta_0, \quad \tilde{u} = \delta_2 u + \delta_3, \\ &\tilde{x} = \delta_1 x + \frac{\delta_1\delta_3}{\delta_2^2} \int (\delta_2 k(t) - \delta_3 f(t))\mathrm{d}t + \delta_4, \end{aligned} \tag{B.3}$$

where δ_i, $i = 0, \ldots, 4$, are constants with $\delta_1\delta_2 \neq 0$.

Without loss of generality, we can restrict ourselves to the study of the class

$$u_t + k(t)uu_x + f(t)u^2u_x + u_{xxx} = 0. \tag{B.4}$$

As class (B.1) is normalized in the generalized extended sense, in order to derive the equivalence group for its subclass with $g = 1$ it is enough to set $\tilde{g} = g = 1$ in the transformations from the group $\hat{G}^{\sim}$ presented in Theorem 1. This leads to the equation for α: $\alpha_t = \delta_1^3$, resulting in $\alpha = \delta_1^3 t + \delta_0$, where δ_0 is an arbitrary constant. The following statement is true.

Theorem B.4. *The generalized extended equivalence group $\hat{G}_1^{\sim}$ of class (B.4) comprises the transformations*

$$\begin{aligned} &\tilde{t} = \delta_1^3 t + \delta_0, \quad \tilde{x} = \delta_1 x + \frac{\delta_1\delta_3}{\delta_2^2} \int (\delta_2 k(t) - \delta_3 f(t))\mathrm{d}t + \delta_4, \\ &\tilde{u} = \delta_2 u + \delta_3, \quad \tilde{k}(\tilde{t}) = \frac{\delta_2 k(t) - 2\delta_3 f(t)}{\delta_1^2\delta_2^2}, \quad \tilde{f}(\tilde{t}) = \frac{f(t)}{\delta_1^2\delta_2^2}, \end{aligned}$$

where δ_i, $i = 0, \ldots, 4$, are arbitrary constants with $\delta_1\delta_2 \neq 0$.

The usual equivalence group $G_1^{\sim}$ of class (B.4) consists of the above transformations with $\delta_3 = 0$.

Table B.1: The group classification of class (B.1) up to $G^{\sim}$-equivalence.

no.	$f(t)$	$k(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_x
I.1	$\lambda_1 t^\rho$	$\lambda_2 t^{\frac{3\rho-2}{6}}+\delta t^\rho$	∂_x, $3t\partial_t+\left(x-\varkappa\frac{3\rho+2}{2}\left(\frac{6\lambda_2}{3\rho+4}t^{\frac{3\rho+4}{6}}+\frac{\delta}{\rho+1}t^{\rho+1}\right)\right)\partial_x$ $-\frac{3\rho+2}{2}(u+\varkappa)\,\partial_u$
I.2	$\lambda_1 t^{-\frac{2}{3}}$	$t^{-\frac{2}{3}}\ln\lvert t\rvert$	∂_x, $2\lambda_1 t\partial_t+\left(\frac{2}{3}\lambda_1 x-3\left(\ln\lvert t\rvert-3\right)t^{\frac{1}{3}}\right)\partial_x-\partial_u$
I.3	$\lambda_1 t^{-1}$	$\lambda_2 t^{-\frac{5}{6}}+\delta t^{-1}$	∂_x, $3t\partial_t+\left(x+\varkappa\left(3\lambda_2 t^{\frac{1}{6}}+\frac{1}{2}\delta\ln\lvert t\rvert\right)\right)\partial_x+\frac{1}{2}(u+\varkappa)\,\partial_u$
I.4	$\lambda_1 t^{-\frac{4}{3}}$	$\lambda_2 t^{-1}+\delta t^{-\frac{4}{3}}$	∂_x, $3t\partial_t+\left(x+\varkappa\left(\lambda_2\ln\lvert t\rvert-3\delta t^{-\frac{1}{3}}\right)\right)\partial_x+(u+\varkappa)\,\partial_u$
II	$\lambda_1 e^t$	$\lambda_2 e^{\frac{t}{2}}+\delta e^t$	∂_x, $2\partial_t-\varkappa\left(2\lambda_2 e^{\frac{t}{2}}+\delta e^t\right)\partial_x-(u+\varkappa)\,\partial_u$
III	λ_1	t	∂_x, $2\lambda_1\partial_t-\frac{1}{2}t^2\partial_x-\partial_u$
IV	ε	δ	∂_x, ∂_t, $3t\partial_t+(x-\varkappa\,\delta t)\partial_x-(u+\varkappa)\,\partial_u$

Here $g=1 \bmod G^{\sim}$; λ_i, $i=1,2$, and ρ are arbitrary constants with $\lambda_1\neq 0$, $\rho\neq-\frac{4}{3},-1$; $\delta\in\{0,1\} \bmod G^{\sim}$, $\varepsilon=\pm1 \bmod G^{\sim}$, and $\varkappa=\frac{1}{2}\delta/\lambda_1$. In Case I.1 $(\rho,\delta)\neq(0,0)$.

We note that class (B.4) is normalized in the generalized extended sense. From Proposition B.3 we deduce that there are no variable coefficient equations (B.4) that are reducible to constant coefficient equations from the same class by point transformations.

In the next section we demonstrate usage of the found equivalence transformations in the process of group classification. Simplifications by usual and generalized equivalence groups will be compared.

Classification of Lie Symmetries. Using the classical Lie-Ovsiannikov technique, we have proved the following statement.

Theorem B.5. *The kernel of the maximal Lie invariance algebras of equations from class* (B.1) *coincides with the one-dimensional algebra* $\langle\partial_x\rangle$*. All possible* $G^{\sim}$*-inequivalent (resp.* $\hat{G}^{\sim}$*-inequivalent) cases of extension of the maximal Lie invariance algebras are exhausted by Cases I–IV of Table B.1 (resp. Table B.2).*

Table B.2: The group classification of class (B.1) up to $\hat{G}^{\sim}$-equivalence.

no.	$f(t)$	$k(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_x
I.1	$\lambda_1 t^\rho$	$\delta t^{\frac{3\rho-2}{6}}$	$\partial_x,\ 3t\partial_t + x\partial_x - \frac{3\rho+2}{2}u\partial_u$
I.2	$\lambda_1 t^{-\frac{2}{3}}$	$t^{-\frac{2}{3}}\ln\lvert t\rvert$	$\partial_x,\ 2\lambda_1 t\partial_t + \left(\frac{2}{3}\lambda_1 x - 3\left(\ln\lvert t\rvert - 3\right)t^{\frac{1}{3}}\right)\partial_x - \partial_u$
II	$\lambda_1 e^t$	$\delta e^{\frac{t}{2}}$	$\partial_x,\ 2\partial_t - u\partial_u$
III	λ_1	t	$\partial_x,\ 2\lambda_1\partial_t - \frac{1}{2}t^2\partial_x - \partial_u$
IV	ε	0	$\partial_x,\ \partial_t,\ 3t\partial_t + x\partial_x - u\partial_u$

Here $g = 1 \bmod \hat{G}^{\sim}$; λ_1 and ρ are arbitrary constants with $\lambda_1 \neq 0$, $\delta \in \{0,1\} \bmod \hat{G}^{\sim}$, and $\varepsilon = \pm 1 \bmod G^{\sim}$. In Case I.1 $(\rho,\delta) \neq (0,0)$.

In order to get the most general forms of arbitrary elements of class (B.1) (not simplified by equivalence transformations) we should apply transformation (B.3) to the equations (B.4) with k and f presented in Table B.1 or even simpler transformation (B.3) with $\delta_3 = 0$ to the equations (B.4) with k and f presented in Table B.2. The complete results are presented in [297], were we also used the method of mappings between classes to perform the group classification of the related class of equations $u_t + F(t)u^2u_x + u_{xxx} = L(t)$, $F \neq 0$.

We have found that, besides the usual equivalence group $G^{\sim}$, class (B.1) admits the wider generalized extended equivalence group $\hat{G}^{\sim}$. Although the exhaustive group classification of class (B.1) can be achieved even using the usual equivalence group, we have shown that the generalized extended equivalence group provides more simplification and allows one to write down the classification list in a simple and concise form (compare Table B.2 with Table B.1).

We note that the case of the three-dimensional maximal Lie symmetry algebra was not indicated in [200].

B.2. Application of Lie Symmetries to Boundary Value Problems for Variable Coefficient Generalized KdV Equations

In applications, one is usually interested in a solution of a given PDE satisfying some initial condition or/and a boundary condition. In a recent paper [307], Lie symmetries were successfully applied to solve an initial and boundary value problem (IBVP) for a generalized Burgers equation arising in nonlinear acoustics. Namely, the IBVP for a generalized Burgers equation was reduced to an initial value problem (IVP) for a related nonlinear second-order ODE. As a result, a closed-form solution of the IBVP for the generalized Burgers equation was found. Motivated by that work, we intend to apply Lie symmetries to construct solutions for IBVP for the variable coefficient generalized Korteweg–de Vries (KdV) equation,

$$u_t + u^n u_x + h(t)u + g(t)u_{xxx} = 0, \quad ng \neq 0, \tag{B.5}$$

arising in several applications (see [266] and references therein). For this purpose we carry out an exhaustive group classification of equations from this class. In other words, we at first find a Lie invariance algebra admitted by any equation in the class, the so-called kernel algebra, and then classify all possible cases of extension of Lie invariance algebras of such equations with respect to the equivalence group of the class [227]. Some cases of Lie symmetry extension for (B.5) were found in [266], namely, the cases $h = \text{const}$ and $h = 1/(at+b)$ with a and b being constants. Here we present a complete group classification taking advantage of the use of equivalence transformations (this opportunity was neglected in [266]). We point out that complete group classifications of class (B.5) for $n = 1$ and $n = 2$ were obtained in [251, 289] (see also [290]). The results were presented there in two ways: with respect to corresponding equivalence groups and using no equivalence. We would like to mention that in [109] group classifications for more general classes that include class (B.5) were carried out. However it seems to be inconvenient to derive group classifications for class (B.5) using those results obtained up to a very wide equivalence group.

Note that the more general class of the form

$$u_t + f(t)u^n u_x + h(t)u + g(t)u_{xxx} = 0, \quad nfg \neq 0, \tag{B.6}$$

reduces to class (B.5) via the change of the variable t. That is why without loss of generality it is sufficient to study class (B.5), since all results on exact solutions, symmetries, conservation laws, etc. for class (B.6) can be derived form those obtained for (B.5).

Admissible Transformations. We study admissible transformations in class (B.5) using the direct method [160]. We omit the details of calculations and present the result only.

Theorem B.6. *The equivalence group $G^{\sim}$ of class* (B.5) *consists of the transformations*

$$\begin{gathered}\tilde t = \varepsilon_1 \textstyle\int \theta^{-n} dt + \varepsilon_2, \quad \tilde x = \varepsilon_1 x + \varepsilon_0, \quad \tilde u = \theta u, \\ \tilde g = \varepsilon_1^2\, \theta^n\, g, \quad \tilde h = \dfrac{\theta^n}{\varepsilon_1}\left(h - \dfrac{\theta_t}{\theta}\right), \quad \tilde n = n,\end{gathered} \tag{B.7}$$

where ε_j, $j = 0, 1, 2$, are arbitrary constants with $\varepsilon_1 \neq 0$; $\theta = \theta(t)$ is an arbitrary nonvanishing smooth function.

Now we can use equivalence transformations (B.7) to gauge one of the arbitrary elements g or h to a simple constant value. It was shown in [251] that the parameter-function h in (B.5) can be set equal to zero by the point transformation

$$\tilde t = \textstyle\int e^{-n\int h(t)\,dt}\, dt, \quad \tilde x = x, \quad \tilde u = e^{\int h(t)\,dt} u, \tag{B.8}$$

and the transformed value of the arbitrary element g is $\tilde g(\tilde t) = e^{n\int h(t)\,dt} g(t)$. This transformation can be easily found from Theorem B.6 setting $\tilde h = 0$ in (B.7) and solving the obtained equation for θ. The fact that the arbitrary element h can always be set to zero means that fixing the arbitrary element h cannot lead to cases of equations (B.5) with special symmetry properties. So, without loss of generality we can restrict our investigation to the class

$$u_t + u^n u_x + g(t)u_{xxx} = 0, \quad ng \neq 0. \tag{B.9}$$

There are no other point transformations except (B.7) that link equations from class (B.5) with $n \neq 0, 1$, therefore, this class is normalized. It means that an equivalence group for (B.9) can be derived from Theorem B.6 by simply setting $\tilde{h} = h = 0$. Then we get that $\theta = \varepsilon_3$ is an arbitrary nonzero constant and the following statement is true.

Corollary B.7. *The equivalence group $G_0^\sim$ of class* (B.9) *is formed by the transformations*

$$\tilde{t} = \varepsilon_1 \varepsilon_3^{-n} t + \varepsilon_0, \quad \tilde{x} = \varepsilon_1 x + \varepsilon_2, \quad \tilde{u} = \varepsilon_3 u,$$
$$\tilde{g} = \varepsilon_1^2 \varepsilon_3^n g, \quad \tilde{n} = n,$$

where ε_j, $j = 0, \ldots, 3$, are arbitrary constants with $\varepsilon_1 \varepsilon_3 \neq 0$.

B.2.1. Lie Symmetries. The group classification of class (B.5) up to $G^\sim$-equivalence reduces to the group classification of class (B.9) up to $G_0^\sim$-equivalence. We carry out the group classification of class (B.9) using the classical algorithm [217].

Theorem B.8. *The kernel of the maximal Lie invariance algebras $A^{\max}$ of equations from class* (B.9) (*resp.* (B.5)) *coincides with the one-dimensional algebra $\langle \partial_x \rangle$. All possible $G_0^\sim$-inequivalent* (*resp. $G^\sim$-inequivalent*) *cases of extension of $A^{\max}$ are exhausted by Cases 1–3 of Table B.3.*

In Table B.4, we also adduce the results of the group classification of (B.5) without gauging of h and g by equivalence transformations. The extended classification list can be derived from that presented in Table B.3 using equivalence transformations (the detailed procedure is described in [289, 290]). It is easy to see that Table B.4 includes all cases presented in [266] as special cases.

Table B.3: The group classification of class (B.5) up to $G^{\sim}$-equivalence.

no.	$g(t)$	Basis of $A^{\max}$
0	$\forall$	∂_x
1	εt^ρ	$\partial_x,\ 3nt\partial_t + (\rho+1)nx\partial_x + (\rho-2)u\partial_u$
2	εe^t	$\partial_x,\ 3n\partial_t + nx\partial_x + u\partial_u$
3	ε	$\partial_x,\ \partial_t,\ 3nt\partial_t + nx\partial_x - 2u\partial_u$

Here $h = 0 \bmod G^{\sim}$, $\varepsilon = \pm 1 \bmod G^{\sim}$, ρ is an arbitrary nonzero constant.

Table B.4: The group classification of class (B.5) using no equivalence.

no.	$g(t)$	Basis of $A^{\max}$
0	$\forall$	∂_x
1	$\lambda\big(\int e^{-n\int h(t)\,dt}dt + \kappa\big)^{\rho} e^{-n\int h(t)dt}$	$\partial_x,\ H\partial_t + n(\rho+1)x\partial_x + (\rho - 2 - h(t)H)u\partial_u$
2	$\lambda\, e^{\int\left(me^{-n\int h(t)\,dt} - nh(t)\right)dt}$	$\partial_x,\ 3ne^{n\int h(t)dt}\partial_t + mnx\partial_x + \big(m - 3nh(t)e^{n\int h(t)dt}\big)u\partial_u$
3	$\lambda\, e^{-n\int h(t)dt}$	$\partial_x,\ e^{n\int h(t)dt}\left(\partial_t - h(t)u\partial_u\right),$ $H\partial_t + nx\partial_x - (2 + h(t)H)u\partial_u$

Here λ, κ, ρ, and m are arbitrary constants with $\lambda\rho m \neq 0$. The function $h(t)$ is arbitrary in all cases and $H = 3ne^{n\int hdt}\big(\int e^{-n\int hdt}dt + \kappa\big)$. In Case 3, $\kappa = 0$ in the formula for H.

Similarity Solutions of the Generalized KdV Equations. Lie symmetries provide us with an algorithmic technique for finding exact solutions using the reduction method [217, 227]. It was found by Lie that if one Lie symmetry generator of an ODE is known, than the order of this ODE can be reduced by one, and if we know a Lie symmetry generator for a n-dimensional PDE, then it can be reduced to a $n-1$-dimensional PDE. This is true for Lie symmetry generators corresponding to one-parameter Lie symmetry groups that act regularly and transversally on a manifold defined by this PDE [217]. So, in our case of $(1+1)$-dimensional PDEs it is enough to perform reductions with respect to one-dimensional subalgebras

Table B.5: Optimal systems of subalgebras of $A^{\max}$ presented in Table B.3.

no.	Optimal system
$1_{\rho\neq-1}$	$\mathfrak{g}_1 = \langle\partial_x\rangle,\quad \mathfrak{g}_{1.1} = \langle 3nt\partial_t + (\rho+1)nx\partial_x + (\rho-2)u\partial_u\rangle$
$1_{\rho=-1}$	$\mathfrak{g}_1 = \langle\partial_x\rangle,\quad \mathfrak{g}^a_{1.2} = \langle nt\partial_t + a\partial_x - u\partial_u\rangle$
2	$\mathfrak{g}_1 = \langle\partial_x\rangle,\quad \mathfrak{g}_2 = \langle 3n\partial_t + nx\partial_x + u\partial_u\rangle$
3	$\mathfrak{g}_1 = \langle\partial_x\rangle,\quad \mathfrak{g}^\sigma_{3.1} = \langle\partial_t + \sigma\partial_x\rangle,\quad \mathfrak{g}_{3.2} = \langle 3nt\partial_t + nx\partial_x - 2u\rangle$

In all cases $a\in\mathbb{R}$, $n\neq 0$, $\sigma\in\{-1,0,1\}$.

Table B.6: Similarity reductions of the class $u_t + u^n u_x + g(t)u_{xxx} = 0$ with $ng\neq 0$ that correspond to the subalgebras presented in Table B.5.

no.	$g(t)$	$\mathfrak{g}$	ω	Ansatz, u	Reduced ODE
1	$\varepsilon t^\rho,\ \rho\neq-1$	$\mathfrak{g}_{1.1}$	$xt^{-\frac{\rho+1}{3}}$	$t^{\frac{\rho-2}{3n}}\varphi(\omega)$	$\varepsilon\varphi''' + \left(\varphi^n - \frac{\rho+1}{3}\omega\right)\varphi' + \frac{\rho-2}{3n}\varphi = 0$
2	εt^{-1}	$\mathfrak{g}^a_{1.2}$	$x - \frac{a}{n}\ln t$	$t^{-\frac{1}{n}}\varphi(\omega)$	$\varepsilon\varphi''' + \left(\varphi^n - \frac{a}{n}\right)\varphi' - \frac{1}{n}f = 0$
3	εe^t	$\mathfrak{g}_2$	$xe^{-\frac{1}{3}t}$	$e^{\frac{1}{3n}t}\varphi(\omega)$	$\varepsilon\varphi''' + \left(\varphi^n - \frac{1}{3}\omega\right)\varphi' + \frac{1}{3n}\varphi = 0$
4	ε	$\mathfrak{g}^\sigma_{3.1}$	$x-\sigma t$	$\varphi(\omega)$	$\varepsilon\varphi''' + (\varphi^n - \sigma)\varphi' = 0$
5	ε	$\mathfrak{g}_{3.2}$	$xt^{-\frac{1}{3}}$	$t^{-\frac{2}{3n}}\varphi(\omega)$	$\varepsilon\varphi''' + \left(\varphi^n - \frac{1}{3}\omega\right)\varphi' - \frac{2}{3n}\varphi = 0$

In all cases $a\in\mathbb{R}$, $n\neq 0$, $\sigma\in\{-1,0,1\}$, $\varepsilon=\pm1 \bmod G^\sim$.

of the found maximal Lie invariance algebras to get reductions to ODEs.

Reductions should be performed using subalgebras from the optimal system [217]. Optimal systems of one-dimensional subalgebras of Lie invariance algebras from Table B.3 are presented in Table B.5.

We do not consider reductions associated with the subalgebra $\mathfrak{g}_1 = \langle\partial_x\rangle$ because they lead to constant solutions only. Ansatzes and reduced equations that are obtained for equations from class (B.9) by means of one-dimensional subalgebras from Table B.5 are collected in Table B.6.

It is possible to get exact traveling wave solutions of the equation $u_t +$

$u^n u_x + \varepsilon u_{xxx} = 0$ solving the reduced ODE from Case 4 of Table B.6,

$$\varepsilon\varphi''' + (\varphi^n - \sigma)\varphi' = 0. \tag{B.10}$$

If $\sigma \neq 0$, two partial solutions of this equation are, for example, $\varphi = \left(-\frac{\sigma(n+1)(n+2)}{2\sinh^2\left(\frac{n}{2}\sqrt{\frac{\sigma}{\varepsilon}}\omega+C\right)}\right)^{\frac{1}{n}}$, and $\varphi = \left(\frac{\sigma(n+1)(n+2)}{2\cosh^2\left(\frac{n}{2}\sqrt{\frac{\sigma}{\varepsilon}}\omega+C\right)}\right)^{\frac{1}{n}}$, where C is an arbitrary constant. They lead to the following traveling wave solutions

$$u = \left(-\frac{\sigma(n+1)(n+2)}{2\sinh^2 z}\right)^{\frac{1}{n}}, \quad u = \left(\frac{\sigma(n+1)(n+2)}{2\cosh^2 z}\right)^{\frac{1}{n}},$$

where $z = \frac{n}{2}\sqrt{\frac{\sigma}{\varepsilon}}(x-at)+C$, of the generalized KdV equation with constant coefficients, $u_t + u^n u_x + \varepsilon u_{xxx} = 0$. Note that some exact solutions were constructed in the literature for variable coefficient generalized KdV equations of the form $u_t+u^n u_x+\alpha(t)u+\varepsilon e^{-n\int\alpha(t)\,\mathrm{d}t}u_{xxx} = 0$, (see, e.g., [27,322]), but it is due to the fact that the latter equations are reduced to constant coefficient ones.

Boundary Value Problem for Generalized KdV Equations. There are several approaches for exploiting Lie symmetries to reduce boundary value problems (BVPs) for PDEs to those for ODEs. The classical technique is to require that both equation and boundary conditions are left invariant under the one-parameter Lie group of infinitesimal transformations. Of course, the infinitesimal approach is usually applied (see, e.g., [32, Section 4.4]). We apply this technique for an IBVP for variable coefficient generalized KdV equations with and without linear damping terms.

IBVP for Generalized KdV Without Linear Damping. We apply at first this procedure to a problem with the governing equation being the KdV of the form (B.9), i.e., we consider the following initial and boundary value problem

$$u_t + u^n u_x + g(t)u_{xxx} = 0, \quad t > 0, \quad x > 0, \tag{B.11}$$

$$\begin{aligned} &u(x,0)=0, && x>0,\\ &u(0,t)=q(t), && t>0,\\ &u_x(0,t)=0, && t>0,\\ &u_{xx}(0,t)=0, && t>0, \end{aligned} \tag{B.12}$$

where $q(t)$ is a nonvanishing smooth function of its variable.

The procedure starts by assuming a general symmetry of the form

$$Q=\sum_{i=1}^{n}\alpha_i Q_i, \tag{B.13}$$

where n is the number of basis operators of the maximal Lie invariance algebra of the given PDE and α_i, $i=1,\dots,n$, are constants to be determined.

We consider Case 1 of Table B.3. The general symmetry (B.13) takes the form

$$Q=\alpha_1\partial_x+\alpha_2\left[3nt\partial_t+(\rho+1)nx\partial_x+(\rho-2)u\partial_u\right].$$

Application of Q to the first boundary condition $x=0,\ u(t,0)=q(t)$ gives

$$\alpha_1=0 \quad \text{and} \quad q(t)=\gamma t^{\frac{\rho-2}{3n}}, \quad \gamma=\text{const}\,.$$

Using the second extension of Q,

$$\begin{aligned} Q^{(2)}=&\,3nt\partial_t+(\rho+1)nx\partial_x+(\rho-2)u\partial_u\\ &+(\rho-n\rho-n-2)u_x\partial_{u_x}+(\rho-2n\rho-2n-2)u_{xx}\partial_{u_{xx}}, \end{aligned}$$

where the unused terms have been ignored, it can be shown that it leaves the initial condition and the remaining three boundary conditions invariant. Finally, symmetry Q produces the ansatz

$$u=t^{\frac{\rho-2}{3n}}\varphi(\omega), \qquad \omega=xt^{-\frac{\rho+1}{3}}, \tag{B.14}$$

which reduces the problem (B.11) into the initial value problem

$$\begin{aligned} &\varepsilon\varphi'''+\varphi^n\varphi'-\tfrac{\rho+1}{3}\omega\varphi'+\tfrac{\rho-2}{3n}\varphi=0,\\ &\varphi(0)=\gamma, \qquad \varphi'(0)=0, \qquad \varphi''(0)=0. \end{aligned} \tag{B.15}$$

In Case 2 of Table B.3, the corresponding symmetry does not leave the boundary conditions invariant and for Case 3 we obtain the above results with $\rho = 0$.

IBVP for Generalized KdV With Linear Damping. We consider the IBVP for the generalized KdV equation with variable-coefficient linear damping

$$u_t + u^n u_x + \frac{j}{t} u + g(t) u_{xxx} = 0, \quad t > 0, \quad x > 0, \tag{B.16}$$

with initial and boundary conditions (B.12). In the previous section we have shown that the symmetry operator which is admitted by both an equation from class (B.5) and initial and boundary conditions (B.12) with q being a power function is the so-called dilatation operator, i.e., the operator corresponding to the one-parameter Lie group of scalings of the variables t, x and u. Equation (B.17) admits a Lie symmetry generator which keeps the boundary conditions invariant if and only if g is a power function or constant (Cases 1 and 3 of Table B.4). Substituting $h = j/t$ into the formulas for g and corresponding symmetry generators presented in Case 1 of Table B.4 (without loss of generality we set $\kappa = 0$) we find that the equation

$$u_t + u^n u_x + \frac{j}{t} u + \lambda t^{\rho(1-nj)-nj} u_{xxx} = 0, \tag{B.17}$$

admits the Lie symmetry generators ∂_x and

$$Q = \frac{3n}{1-nj} t\partial_t + n(\rho+1)x\partial_x + \left(\rho - 2 - \frac{3nj}{1-nj}\right) u\partial_u.$$

Boundary conditions (B.12) are left invariant with respect to the symmetry transformation generated by the operator Q if and only if $q = \gamma t^{\frac{(\rho-2)(1-nj)-3nj}{3n}}$, where $\gamma = \text{const}$. Therefore we can apply Lie symmetries to solve the following BVP for the generalized KdV equation with linear damping

$$u_t + u^n u_x + \frac{j}{t} u + \lambda t^{\rho(1-nj)-nj} u_{xxx} = 0, \quad t > 0, \quad x > 0, \tag{B.18}$$

$$\begin{aligned}
&u(x,0)=0, && x>0,\\
&u(0,t)=\gamma t^{\frac{\rho(1-nj)-nj-2}{3n}}, && t>0,\\
&u_x(0,t)=0, && t>0,\\
&u_{xx}(0,t)=0, && t>0.
\end{aligned} \tag{B.19}$$

The symmetry Q produces the transformation

$$u=t^{\frac{\rho(1-nj)-nj-2}{3n}}\varphi(\omega),\quad \omega=xt^{\frac{(\rho+1)(nj-1)}{3}},$$

which reduces the problem (B.18)–(B.19) to

$$\begin{aligned}
&\lambda\varphi'''+\varphi^n\varphi'+\tfrac{(\rho+1)(nj-1)}{3}\omega\varphi'+\tfrac{(\rho-2)(1-nj)}{3n}\varphi=0,\\
&\varphi(0)=\gamma,\quad \varphi'(0)=0,\quad \varphi''(0)=0.
\end{aligned} \tag{B.20}$$

For $j=1/2$ and $j=1$ (B.18) becomes generalized cylindrical and spherical KdV equations, respectively.

A group classification for variable coefficient generalized KdV equations (B.5) is carried out exhaustively. The results are presented in two ways: up to $G^{\sim}$-equivalence (Table B.3) and without simplification by equivalence transformations (Table B.4). Similarity solutions are classified (Table B.6). The derived Lie symmetries of a generalized KdV equation are employed to IBVP (B.11)–(B.12) transforming it into an IVP for an ODE (B.15) in [298]. The resulting nonlinear problem is solved numerically with the aid of a second-order finite-difference scheme with fixed-point iterations. The scheme was validated by applying it to similar problems in the literature which were solved using other methods and the results were found to be in excellent agreement. The effect of the governing parameters on the solutions of (B.11)–(B.12) was examined and solutions of the original PDE were constructed using the aforementioned transformations.

B.3. Group Analysis of Variable Coefficient Fifth-Order KdV Equations

The KdV equation arises, in particular, in the modeling of one-dimensional plane waves in cold quasi-neutral collision-free plasma propagating along the x-direction under the presence of a uniform magnetic field [149]. It appeared that, when the propagation angle of the wave relative to the external magnetic field becomes critical, the third-order (dispersion) term in the model equation should be replaced by the fifth-order one [150]. Namely, magneto-acoustic waves propagating along this critical direction are modeled by the simplest fifth-order KdV (fKdV) equation (called also quintic KdV equation),

$$u_t + uu_x + \mu u_{xxxxx} = 0, \quad \mu = \text{const}. \tag{B.21}$$

In [204] the equation (B.21) with $\mu = -1$ was shown to describe solitary waves in the nonlinear transmission line of a LC ladder type.

Later equation (B.21) and its generalizations were studied in a number of papers. Thus, an exact solitary wave solution of equation (B.21) in terms of Jacobi elliptic function cn was found in [152, 320]. Pulsating multiplet solutions of this equation were examined in [130]. Local conservation laws with the densities u, u^2 and $u^3 + \frac{3}{2}(u_{xx})^2$ were indicated therein. Note that the fKdV equation is not integrable by the inverse scattering transform method in contrast to the classical KdV equation [196]. Lie symmetries and the corresponding reductions of (B.21) to ordinary differential equations (ODEs) were found in [189].

An attempt of Lie symmetry classification of the generalized fKdV equations with time dependent coefficients, $u_t + u^n u_x + \alpha(t)u + \beta(t)u_{xxxxx} = 0$, was made in [311]. However, the results presented therein are incorrect in general. In the present paper we perform the correct and complete group classification of the class

$$u_t + uu_x + \alpha(t)u + \beta(t)u_{xxxxx} = 0, \quad \beta \neq 0, \tag{B.22}$$

where α and β are smooth functions of the variable t. To be able to reduce the number of variable coefficients and to proceed with Lie symmetry analysis in an optimal way, we at first find the admissible transformations [248] in class (B.22). Then classifications of Lie symmetries and similarity reductions are presented.

Admissible Transformations. The equivalence groupoid of class (B.22) is described in the following statement.

Theorem B.9. *The generalized extended equivalence group $\hat{G}^{\sim}$ of class (B.22) is formed by the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = (x+\delta_1)X^1 + \delta_2, \quad \tilde{u} = \frac{1}{T_t}\left(X^1 u + X^1_t(x+\delta_1)\right),$$

$$\tilde{\alpha}(\tilde{t}) = \frac{1}{T_t}\left(\alpha(t) - 2\frac{X^1_t}{X^1} + \frac{T_{tt}}{T_t}\right), \quad \tilde{\beta}(\tilde{t}) = \frac{(X^1)^5}{T_t}\beta(t),$$

where $X^1 = (\delta_3 \int e^{-\int \alpha(t)\mathrm{d}t}\mathrm{d}t + \delta_4)^{-1}$, δ_j, $j=1,\dots,4$, are arbitrary constants with $(\delta_3,\delta_4)\neq(0,0)$ and $T=T(t)$ is a smooth function with $T_t\neq 0$.

The entire set of admissible transformations of class (B.22) is generated by the transformations from the group $\hat{G}^{\sim}$.

Using this theorem we can formulate a criterion of reducibility of variable coefficient fKdV equations to constant coefficient ones.

Theorem B.10. *A variable coefficient equation from class (B.22) is reducible to the constant coefficient fKdV equation (B.21) if and only if its coefficients α and β are related by the formula*

$$\beta = e^{-\int \alpha(t)\mathrm{d}t}\left(c_1 \int e^{-\int \alpha(t)\mathrm{d}t}\mathrm{d}t + c_2\right)^3, \tag{B.23}$$

where c_1 and c_2 are arbitrary constants with $(c_1,c_2)\neq(0,0)$.

Using the equivalence transformation

$$\hat{t} = \int e^{-\int \alpha(t)\,\mathrm{d}t}\mathrm{d}t, \quad \hat{x} = x, \quad \hat{u} = e^{\int \alpha(t)\,\mathrm{d}t}u \tag{B.24}$$

from the group $\hat{G}^{\sim}$ we can set the arbitrary element α to the zero value. Indeed, this transformation maps class (B.22) to its subclass with $\hat{\alpha}=0$.

The arbitrary element $\hat{\beta}$ of a mapped equation is expressed in terms of α and β as $\hat{\beta} = e^{\int \alpha(t)\,dt}\beta$. Without loss of generality we can restrict ourselves to the investigation of the class

$$u_t + uu_x + \beta(t)u_{xxxxx} = 0. \tag{B.25}$$

We derive equivalence transformations in class (B.25) setting $\tilde{\alpha} = \alpha = 0$ in transformations presented in Theorem B.9.

Corollary B.11. *The usual equivalence group $G^{\sim}_{\alpha=0}$ of class (B.25) consists of the transformations*

$$\tilde{t} = \frac{at+b}{ct+d}, \quad \tilde{x} = \frac{e_2x+e_1t+e_0}{ct+d}, \quad \tilde{u} = \frac{e_2(ct+d)u - e_2cx - e_0c + e_1d}{\Delta},$$
$$\tilde{\beta} = \frac{{e_2}^5}{(ct+d)^3}\frac{\beta}{\Delta},$$

where a, b, c, d, e_0, e_1 and e_2 are arbitrary constants with $\Delta = ad - bc \neq 0$ and $e_2 \neq 0$, the tuple $(a, b, c, d, e_0, e_1, e_2)$ is defined up to a nonzero multiplier and hence without loss of generality we can assume that $\Delta = \pm 1$.

The entire set of admissible transformations of class (B.25) is generated by the transformations from the group $G^{\sim}_{\alpha=0}$.

The transformation components for t, x and u coincide with those obtained for the class of Burgers equations $u_t + uu_x + \beta(t)u_{xx} = 0$ [232] and the class of KdV equations $u_t + uu_x + \beta(t)u_{xxx} = 0$ [251].

Corollary B.12. *A variable coefficient equation from class (B.25) is reducible to the constant coefficient fKdV equation (B.21) if and only if $\beta = (c_1t+c_2)^3$, where c_1 and c_2 are arbitrary constants with $(c_1, c_2) \neq (0, 0)$.*

Lie Symmetries. We solve the group classification problem using the classical Lie–Ovsiannikov approach. The following assertion is true.

Theorem B.13. *The kernel of the maximal Lie invariance algebras of equations from class (B.25) is the two-dimensional Abelian algebra $A^{\mathrm{ker}} = \langle \partial_x,\ t\partial_x + \partial_u \rangle$. All possible $G^{\sim}_{\alpha=0}$-inequivalent cases of extension of the maximal Lie invariance algebras are exhausted by Cases 1–4 of Table B.7.*

Table B.7: The group classification of the class (B.25) up to $G^{\sim}_{\alpha=0}$-equivalence.

no.	$\beta(t)$	Basis of $A^{\max}$
0	$\forall$	$\partial_x,\quad t\partial_x+\partial_u$
1	t^ρ	$\partial_x,\quad t\partial_x+\partial_u,\quad 5t\partial_t+(\rho+1)x\partial_x+(\rho-4)u\partial_u$
2	e^t	$\partial_x,\quad t\partial_x+\partial_u,\quad 5\partial_t+x\partial_x+u\partial_u$
3	$(t^2+1)^{\frac{3}{2}}e^{5\nu\arctan t}$	$\partial_x,\quad t\partial_x+\partial_u,\quad (t^2+1)\partial_t+(t+\nu)x\partial_x+((\nu-t)u+x)\partial_u$
4	1	$\partial_x,\quad t\partial_x+\partial_u,\quad \partial_t,\quad 5t\partial_t+x\partial_x-4u\partial_u$

Here ρ and ν are real constants, $\rho\neq 0$. Up to $G^{\sim}_{\alpha=0}$-equivalence we can assume that $\rho\leqslant 3/2$, $\nu\geqslant 0$.

Remark B.14. A group classification list for class (B.22) up to $\hat{G}^{\sim}$-equivalence coincides with the list presented in Table B.7.

Remark B.15. An equation of the form (B.22) admits a four-dimensional Lie symmetry algebra if and only if it is point-equivalent to the constant coefficient fKdV equation (B.21).

In Table B.8 we present also the complete list of Lie symmetry extensions for class (B.22), where arbitrary elements are not simplified by point transformations. This is achieved using the equivalence-based approach [289].

Cases presented in Table B.8 give all equations (B.22) for which the classical method of Lie reduction can be effectively used.

Lie Symmetry Reductions. Lie symmetries provide one with the powerful tool for finding solutions of nonlinear PDEs reducing them to PDEs with fewer number of independent variables or even to ODEs. If a (1+1)-dimensional PDE admits a Lie symmetry operator, $Q=\tau\partial_t+\xi\partial_x+\eta\partial_u$, then the ansatz reducing this PDE to an ODE is found as a solution of the invariant surface condition $Q[u]:=\tau u_t+\xi u_x-\eta=0$ [217,227]. In practice, one has to solve the corresponding characteristic system $\frac{\mathrm{d}t}{\tau}=\frac{\mathrm{d}x}{\xi}=\frac{\mathrm{d}u}{\eta}$. To get inequivalent reductions one should use subalgebras from an optimal

Table B.8: The group classification of class (B.22) using no equivalence.

no.	$\beta(t)$	Basis of $A^{\max}$
0	$\forall$	$\partial_x,\quad T\partial_x+T_t\partial_u$
1	$\lambda T_t(aT+b)^{\rho}(cT+d)^{3-\rho}$	$\partial_x,\quad T\partial_x+T_t\partial_u,\quad 5T_t^{-1}(aT+b)(cT+d)\partial_t+\big[5acT$ $+ad(\rho+1)+bc(4-\rho)\big]x\partial_x+\Big(5acxT_t-\big[5acT$ $+5\alpha T_t^{-1}(aT+b)(cT+d)+bc(\rho+1)+ad(4-\rho)\big]u\Big)\partial_u$
2	$\lambda T_t(cT+d)^3\exp\left(\frac{aT+b}{cT+d}\right)$	$\partial_x,\quad T\partial_x+T_t\partial_u,\quad 5T_t^{-1}(cT+d)^2\partial_t+(5c(cT+d)+\Delta)x\partial_x$ $+\left[5c^2xT_t+\left(\Delta-5(cT+d)(c+\alpha(cT+d)T_t^{-1})\right)u\right]\partial_u$
3	$\lambda T_t e^{5\nu\arctan\left(\frac{aT+b}{cT+d}\right)}$ $\times\left((aT+b)^2+(cT+d)^2\right)^{\frac{3}{2}}$	$\partial_x,\quad T\partial_x+T_t\partial_u,\quad T_t^{-1}\left((aT+b)^2+(cT+d)^2\right)\partial_t$ $+\big[a(aT+b)+c(cT+d)+\nu\Delta\big]x\partial_x$ $+\big[(a^2+c^2)xT_t-\big(a(aT+b)+c(cT+d)-\nu\Delta$ $+\alpha T_t^{-1}\left((aT+b)^2+(cT+d)^2\right)\big)u\big]\partial_u$
4a	λT_t	$\partial_x,\quad T\partial_x+T_t\partial_u,\quad T_t^{-1}(\partial_t-\alpha u\partial_u),$ $5TT_t^{-1}\partial_t+x\partial_x-(4+5TT_t^{-1}\alpha)u\partial_u$
4b	$T_t(cT+d)^3$	$\partial_x,\quad T\partial_x+T_t\partial_u,$ $5T_t^{-1}(cT+d)\partial_t+4cx\partial_x-(c+5T_t^{-1}(cT+d)\alpha)u\partial_u,$ $T_t^{-1}(cT+d)^2\partial_t+c(cT+d)x\partial_x$ $+[c^2xT_t-(cT+d)(c+T_t^{-1}(cT+d)\alpha)u]\partial_u$

Here a, b, c, d, λ, ν, and ρ are arbitrary constants with $\lambda\neq 0$ and $\rho\neq 0,3$, $\Delta=ad-bc\neq 0$; The function $\alpha(t)$ is arbitrary in all cases, $T=\int e^{-\int\alpha(t)\mathrm{d}t}\mathrm{d}t$.

system (see Section 3.3 in [217]).

We have constructed optimal systems of one-dimensional subalgebras for all the maximal Lie invariance algebras presented in Table B.7. The results are summarized in Table B.9.

The reductions with respect to the subalgebra $\mathfrak{g}$ lead to constant solutions only. The reduction with respect to the subalgebra $\mathfrak{g}_{4.3}$ is not presented since it coincides with that performed using $\mathfrak{g}_{1.1}$ for $\rho=0$. Other reductions are listed in Table B.10.

Table B.9: Optimal systems of one-dimensional subalgebras of $A^{\max}$ from Table B.7.

Case	Optimal system
0	$\mathfrak{g} = \langle \partial_x \rangle, \quad \mathfrak{g}^a = \langle (t+a)\partial_x + \partial_u \rangle$
$1_{\rho\neq -1,4}$	$\mathfrak{g} = \langle \partial_x \rangle, \quad \mathfrak{g}^\sigma = \langle (t+\sigma)\partial_x + \partial_u \rangle, \quad \mathfrak{g}_{1.1} = \langle 5t\partial_t + (\rho+1)x\partial_x + (\rho-4)u\partial_u \rangle$
$1_{\rho=-1}$	$\mathfrak{g} = \langle \partial_x \rangle, \quad \mathfrak{g}^\sigma = \langle (t+\sigma)\partial_x + \partial_u \rangle, \quad \mathfrak{g}^a_{1.2} = \langle t\partial_t + a\partial_x - u\partial_u \rangle$
2	$\mathfrak{g} = \langle \partial_x \rangle, \quad \mathfrak{g}^0 = \langle t\partial_x + \partial_u \rangle, \quad \mathfrak{g}_2 = \langle 5\partial_t + x\partial_x + u\partial_u \rangle$
3	$\mathfrak{g} = \langle \partial_x \rangle, \quad \mathfrak{g}_3 = \langle (t^2+1)\partial_t + (t+\nu)x\partial_x + (x+(\nu-t)u)\partial_u \rangle$
4	$\mathfrak{g} = \langle \partial_x \rangle, \quad \mathfrak{g}_{4.1} = \langle \partial_t \rangle, \quad \mathfrak{g}^\sigma_{4.2} = \langle \sigma\partial_t + t\partial_x + \partial_u \rangle, \quad \mathfrak{g}_{4.3} = \langle 5t\partial_t + x\partial_x - 4u\partial_u \rangle$

Here a is a real constant, $\sigma \in \{-1, 0, 1\}$. Up to $G^\sim_{\alpha=0}$-equivalence we can assume that $\rho \leqslant 3/2$, $\nu \geqslant 0$.

Table B.10: Similarity reductions of the equations (B.25).

Case	$\mathfrak{g}$	ω	Ansatz, $u =$	Reduced ODE
0	$\mathfrak{g}^a$	t	$\varphi(\omega) + \dfrac{x}{t+a}$	$(\omega + a)\varphi' + \varphi = 0$
$1_{\rho\neq -1,4}$	$\mathfrak{g}_{1.1}$	$xt^{-\frac{\rho+1}{5}}$	$t^{\frac{\rho-4}{5}}\varphi(\omega)$	$\varphi'''' + \left(\varphi - \frac{\rho+1}{5}\omega\right)\varphi' + \frac{\rho-4}{5}\varphi = 0$
$1_{\rho=-1}$	$\mathfrak{g}^a_{1.2}$	$x - a\ln t$	$t^{-1}\varphi(\omega)$	$\varphi'''' + (\varphi - a)\,\varphi' - \varphi = 0$
2	$\mathfrak{g}_2$	$xe^{-\frac{1}{5}t}$	$e^{\frac{1}{5}t}\varphi(\omega)$	$\varphi'''' + \left(\varphi - \frac{1}{5}\omega\right)\varphi' + \frac{1}{5}\varphi = 0$
3	$\mathfrak{g}_3$	$\dfrac{xe^{-\nu\arctan t}}{\sqrt{t^2+1}}$	$\dfrac{e^{\nu\arctan t}}{\sqrt{t^2+1}}\varphi(\omega) + \dfrac{xt}{t^2+1}$	$\varphi'''' + (\varphi - \nu\omega)\varphi' + \nu\varphi + \omega = 0$
4.1	$\mathfrak{g}_{4.1}$	x	$\varphi(\omega)$	$\varphi'''' + \varphi\varphi' = 0$
4.2	$\mathfrak{g}^\sigma_{4.2}$	$x \pm \dfrac{t^2}{2}$	$\varphi(\omega) \mp t$	$\varphi'''' + \varphi\varphi' \mp 1 = 0$

Here a is an arbitrary constant.

Solving the first-order reduced equation from Table B.10 and subsequently applying to it transformation (B.28) we get a "degenerate" solution of equation (B.22),

$$u = \frac{x+b}{\int e^{-\int \alpha(t)\mathrm{d}t}\mathrm{d}t + a}\, e^{-\int \alpha(t)\mathrm{d}t},$$

that is valid for any smooth function α. Here a and b are arbitrary constants.

Using equivalence transformations it is possible to construct an exact solution for the equations (B.22) that are reducible to their constant coefficient counterparts, i.e., whose coefficients are related by (B.23). We take the known solution in terms of the Jacobi elliptic function cn from [320] for equation (B.21) and get the exact solution

$$u = \frac{\frac{105}{16}a\ \mathrm{cn}^4\Big(W(t,x); \frac{\sqrt{2}}{2}\Big) + c_1(x+d)}{e^{\int \alpha(t)\mathrm{d}t} Z},$$

for the variable coefficient fKdV equation,

$$u_t + uu_x + \alpha(t)u - e^{-\int \alpha(t)\mathrm{d}t} Z^3 u_{xxxxx} = 0,$$

where $W(t,x) = \frac{\sqrt{2}}{4}a^{\frac{1}{4}}\left(\frac{x+d}{Z} - \frac{21}{8}a\int \frac{e^{-\int \alpha(t)\mathrm{d}t}}{Z^2}\,\mathrm{d}t\right) + b$, $Z = c_1\int e^{-\int \alpha(t)\mathrm{d}t}\mathrm{d}t + c_2$, a is a positive constant, c_1, c_2, b and d are arbitrary constants with $(c_1, c_2) \neq (0,0)$.

One-dimensional subalgebras of the Lie symmetry algebras admitted by equations from class (B.22) are classified in Table B.9 and all inequivalent reductions with respect to such subalgebras are summarized in Table B.10. Performed reductions can be used for the construction of exact and/or numerical solutions. Examples of such constructions were given in [175] for the generalized Kawahara equations. Two simple solutions are constructed in this section.

B.4. Group Analysis of a Class of Generalized Fifth-Order Korteweg–de Vries Equations

In this section the class of generalized variable-coefficient fifth-order Korteweg–de Vries (fKdV) equations

$$u_t + u^n u_x + \alpha(t)u + \beta(t)u_{xxxxx} = 0 \tag{B.26}$$

is investigated from the Lie symmetry point of view. Here α and β are smooth nonvanishing functions of the variable t and n is a positive integer,

$n \geqslant 2$. This work is a natural continuation of the study undertaken by ourselves in [177], where the group classification of the equations (B.26) with $n = 1$ was carried out exhaustively. Lie symmetry analysis of the class (B.26) was initiated in [311]. We show that the results presented therein are incorrect. The case $n = 2$ was considered also in [312] but the complete group classification was not achieved.

Various generalizations of the Korteweg–de Vries equation appear in many physical models, including ones describing gravity waves, plasma waves and waves in lattices [144]. The equation (B.26) with $n = 1$, $\alpha = 0$ and $\beta =$ const models, for example, one-dimensional hydromagnetic waves in a cold quasi-neutral collision-free plasma propagating along the x-direction under the presence of a uniform magnetic field under some conditions, namely, when the propagation angle of the wave relative to the external magnetic field becomes special, critical angle [150]. More references on studies concerned with these equations can be found in [177]. We also discuss the incorrectnesses of the results obtained in [311, 312].

Admissible Transformations. We search for admissible transformations in class (B.26) using the direct method [160]. The following statement is true.

Theorem B.16. *The generalized equivalence group $G^{\sim}$ of the class (B.26) consists of the transformations*

$$\tilde t = T(t), \quad \tilde x = \delta_1 x + \delta_2, \quad \tilde u = \left(\frac{\delta_1}{T_t}\right)^{\frac{1}{n}} u,$$

$$\tilde\alpha(\tilde t) = \frac{\alpha}{T_t} + \frac{T_{tt}}{nT_t^2}, \quad \tilde\beta(\tilde t) = \frac{{\delta_1}^5}{T_t}\beta(t), \quad \tilde n = n,$$

where δ_j, $j = 1, 2$, are arbitrary constants, T is an arbitrary smooth function with $\delta_1 T_t > 0$.

The entire set of admissible transformations of the class (B.26) is generated by the transformations from the group $G^{\sim}$.

If we assume that the constant n varies in the class (B.26), then the equivalence group $G^{\sim}$ is generalized since n is involved explicitly in the transformation of the variable u. Since n is invariant under the action of transformations from the equivalence group, the class (B.26) can be considered as the union of its disjoint subclasses with fixed n. For each such subclass the equivalence group $G^{\sim}$ is usual one.

Using Theorem B.16 we derive a criterion of reducibility of variable-coefficient equations (B.26) to constant coefficient equations from the same class.

Theorem B.17. *A variable coefficient equation from the class (B.26) is reducible to the constant coefficient equation from the same class if and only if its coefficients α and β satisfy the equality*

$$n\,(\alpha/\beta)_t = (1/\beta)_{tt}\,. \tag{B.27}$$

Equivalence transformations from the group $G^{\sim}$ allow us to gauge one of the arbitrary element α or β to a simple constant value, for example, α can be set to zero or β to unity. The gauge $\alpha = 0$ leads to more essential simplification of the study than the gauge $\beta = 1$, therefore, the first one is preferable. Any equation from the class (B.26) can be mapped to an equation from the same class with $\tilde{\alpha} = 0$ by the equivalence transformation

$$\tilde{t} = \textstyle\int e^{-n\int \alpha(t)\,\mathrm{d}t}\mathrm{d}t, \quad \tilde{x} = x, \quad \tilde{u} = e^{\int \alpha(t)\,\mathrm{d}t}u. \tag{B.28}$$

Then the single variable coefficient in the transformed equation will be expressed via α and β as $\tilde{\beta} = e^{n\int \alpha(t)\,dt}\beta$. (Here and in what follows an integral with respect to t should be interpreted as a fixed antiderivative.) Therefore, we can restrict ourselves to the study of the class

$$u_t + u^n u_x + \beta(t) u_{xxxxx} = 0. \tag{B.29}$$

To derive the equivalence group for (B.29) we set $\tilde{\alpha} = \alpha = 0$ in the corresponding transformation presented in Theorem B.16 and deduce that the function T is linear with respect to t. The following assertion is true.

Table B.11: The group classification of class (B.29) up to $G_0^{\sim}$-equivalence.

no.	$\beta(t)$	Basis of $A^{\max}$
1	$\forall$	∂_x
2	εt^ρ	$\partial_x,\ 5nt\partial_t+(\rho+1)nx\partial_x+(\rho-4)u\partial_u$
3	εe^t	$\partial_x,\ 5n\partial_t+nx\partial_x+u\partial_u$
4	ε	$\partial_x,\ \partial_t,\ 5nt\partial_t+nx\partial_x-4u\partial_u$

Here $n\neq 0,1$, ρ is an arbitrary nonzero constant; $\varepsilon=\pm1 \bmod G_0^{\sim}$.

Corollary B.18. *The generalized equivalence group $G_0^{\sim}$ of the class (B.29) comprises the transformations*

$$\begin{aligned}&\tilde t=\delta_3 t+\delta_4,\ \ \tilde x=\delta_1 x+\delta_2,\ \ \tilde u=\left(\tfrac{\delta_1}{\delta_3}\right)^{\frac1n}u,\\&\tilde\beta(\tilde t)=\frac{{\delta_1}^5}{\delta_3}\beta(t),\ \ \tilde n=n,\end{aligned}\tag{B.30}$$

where δ_j, $j=1,2,3,4$, are arbitrary constants with $\delta_1\delta_3>0$.

The entire set of admissible transformations of the class (B.29) is generated by the transformations from the group $G_0^{\sim}$.

Lie symmetries. In the previous section we have shown that the group classification problem for the class (B.26) reduces to the similar problem for its subclass (B.29). We have proved the following statement.

Theorem B.19. *The kernel of the maximal Lie invariance algebras of nonlinear equations from the class (B.29) with $n\neq1$ coincides with the one-dimensional algebra $\langle\partial_x\rangle$. All possible $G_0^{\sim}$-inequivalent cases of extension of the maximal Lie invariance algebras are exhausted by those presented in Cases 2–4 of Table B.11.*

Proposition B.20. *A group classification list for the class (B.26) up to $G^{\sim}$-equivalence coincides with the list presented in Table B.11.*

Proposition B.21. *An equation of the form (B.26) admits a three-dimensional Lie symmetry algebra if and only if it is point-equivalent to*

Table B.12: The group classification of the class (B.26) with $n \neq 0, 1$ using no equivalence.

no.	$\beta(t)$	Basis of $A^{\max}$
1	$\forall$	∂_x
2	$\lambda T_t (T+\kappa)^\rho$	$\partial_x,\ 5n(T+\kappa)T_t^{-1}\partial_t + n(\rho+1)x\partial_x + (\rho - 4 - 5n\alpha(t)(T+\kappa)T_t^{-1})u\partial_u$
3	$\lambda T_t e^{mT}$	$\partial_x,\ 5nT_t^{-1}\partial_t + mnx\partial_x + \big(m - 5n\alpha(t)T_t^{-1}\big)u\partial_u$
4	λT_t	$\partial_x,\ T_t^{-1}(\partial_t - \alpha(t)u\partial_u),\ 5nTT_t^{-1}\partial_t + nx\partial_x - (4 + 5n\alpha(t)TT_t^{-1})u\partial_u$

Here λ, κ, ρ, and m are arbitrary constants with $\lambda\rho m \neq 0$, $T = T(t) = \int e^{-n\int \alpha(t)\,\mathrm{d}t}\mathrm{d}t$, and the function $\alpha(t)$ is arbitrary in all cases.

the constant-coefficient fKdV equation $u_t + u^n u_x + \varepsilon u_{xxxxx} = 0$ from the same class.

For convenience of further applications we present in Table B.12 the complete list of Lie symmetry extensions for the initial class (B.26), where arbitrary elements are not simplified by equivalence transformations (the detailed procedure of deriving such a list from a simplified one is described in [289]).

The obtained group classification results give all equations (B.26) for which the classical method of Lie reduction can be applied.

Symmetry Reductions and Construction of Exact Solutions. To find optimal systems of one-dimensional subalgebras for Lie algebras $A^{\max}$ presented in Table B.11, we firstly consider their structure, using notations of [230]. In Cases 2 and 3 the maximal Lie-invariance algebras are two-dimensional. In Case 2 with $\rho = -1$ it is Abelian ($2A_1$). The algebras adduced in Case 2 with $\rho \neq -1$ and Case 3 are non-Abelian (A_2). The three-dimensional algebra with basis operators presented in Case 4 is of the type $A^a_{3.5}$, where $a = 1/5$.

Therefore, optimal systems of one-dimensional subalgebras of the maximal Lie invariance algebras $A^{\max}$ presented in Table B.11 are the following:

$2_{\rho\neq-1}$: $\mathfrak{g}_0 = \langle\partial_x\rangle$, $\mathfrak{g}_{2.1} = \langle 5nt\partial_t + (\rho+1)nx\partial_x + (\rho-4)u\partial_u\rangle$;

Table B.13: Similarity reductions of the equations $u_t + u^n u_x + \beta(t) u_{xxxxx} = 0$.

no.	$\beta(t)$	$\mathfrak{g}$	ω	Ansatz	Reduced ODE
1	$\varepsilon t^\rho,\ \rho \neq -1$	$\mathfrak{g}_{2.1}$	$xt^{-\frac{\rho+1}{5}}$	$u = t^{\frac{\rho-4}{5n}}\varphi(\omega)$	$\varepsilon\varphi''''' + \left(\varphi^n - \frac{\rho+1}{5}\omega\right)\varphi' + \frac{\rho-4}{5n}\varphi = 0$
2	εt^{-1}	$\mathfrak{g}^a_{2.2}$	$x - \frac{a}{n}\ln t$	$u = t^{-\frac{1}{n}}\varphi(\omega)$	$\varepsilon\varphi''''' + \left(\varphi^n - \frac{a}{n}\right)\varphi' - \frac{1}{n}\varphi = 0$
3	εe^t	$\mathfrak{g}_3$	$xe^{-\frac{1}{5}t}$	$u = e^{\frac{1}{5n}t}\varphi(\omega)$	$\varepsilon\varphi''''' + \left(\varphi^n - \frac{1}{5}\omega\right)\varphi' + \frac{1}{5n}\varphi = 0$
4	ε	$\mathfrak{g}^\sigma_{4.1}$	$x - \sigma t$	$u = \varphi(\omega)$	$\varepsilon\varphi''''' + (\varphi^n - \sigma)\varphi' = 0$
5	ε	$\mathfrak{g}_{4.2}$	$xt^{-\frac{1}{5}}$	$u = t^{-\frac{4}{5n}}\varphi(\omega)$	$\varepsilon\varphi''''' + \left(\varphi^n - \frac{\omega}{5}\right)\varphi' - \frac{4}{5n}\varphi = 0$

Here a is an arbitrary constant, $\sigma \in \{-1, 0, 1\}$, $\varepsilon = \pm 1 \bmod G_0^\sim$, $n \neq 0, 1$.

$2_{\rho=-1}$: $\mathfrak{g}_0 = \langle \partial_x \rangle$, $\mathfrak{g}^a_{2.2} = \langle nt\partial_t + a\partial_x - u\partial_u \rangle$, where a is an arbitrary constant;

3: $\mathfrak{g}_0 = \langle \partial_x \rangle$, $\mathfrak{g}_3 = \langle 5n\partial_t + nx\partial_x + u\partial_u \rangle$;

4: $\mathfrak{g}_0 = \langle \partial_x \rangle$, $\mathfrak{g}^\sigma_{4.1} = \langle \partial_t + \sigma\partial_x \rangle$, $\mathfrak{g}_{4.2} = \langle 5nt\partial_t + nx\partial_x - 4u\partial_u \rangle$; $\sigma \in \{-1, 0, 1\}$.

We do not perform the reductions with respect to the subalgebra $\mathfrak{g}_0$ since they lead to constant solutions only. The reductions with respect to other one-dimensional subalgebras from the found optimal lists are presented in Table B.13.

It is possible to consider also reductions of the generalized fKdV equations to algebraic equations using two-dimensional subalgebras of their Lie invariance algebras. There is only one such subalgebra that leads to a nonconstant solution, it is the subalgebra

$$\langle \partial_t,\ 5nt\partial_t + nx\partial_x - 4u\partial_u \rangle$$

of the algebra $A^{\max}$ presented in Case 4 of Table B.11. The corresponding ansatz $u = Cx^{-\frac{4}{n}}$ reduces the equation

$$u_t + u^n u_x + \varepsilon u_{xxxxx} = 0 \tag{B.31}$$

to an algebraic equation on the constant C. We solve it and get the stationary solution

$$u = (-8\varepsilon(n+1)(n+2)(n+4)(3n+4))^{\frac{1}{n}}(nx)^{-\frac{4}{n}}.$$

of the equation (B.31). Using this solution and equivalence transformation (B.28) we construct simple nonstationary exact solution,

$$u = (-8\varepsilon(n+1)(n+2)(n+4)(3n+4))^{\frac{1}{n}}(nx)^{-\frac{4}{n}}e^{-\int\alpha(t)\,\mathrm{d}t},$$

for the fKdV equation with time-dependent coefficients

$$u_t + u^n u_x + \alpha(t)u + \varepsilon e^{-n\int\alpha(t)\,dt}u_{xxxxx} = 0, \qquad \text{(B.32)}$$

where α is an arbitrary nonvanishing smooth function.

If $n = 2$ the travelling wave solution

$$u = \pm 2\sqrt{-10\varepsilon}\left(3\tanh(x+24\varepsilon t)^2 - 2\right)$$

of the equation (B.31) is known [228]. Using (B.28) we get the exact solution of the equation (B.32) with $n = 2$,

$$u = \pm 2\sqrt{-10\varepsilon}\left(3\tanh\left(x + 24\varepsilon\textstyle\int e^{-2\int\alpha(t)\,\mathrm{d}t}\mathrm{d}t\right)^2 - 2\right)e^{-\int\alpha(t)\,\mathrm{d}t}.$$

It is worthy to note that the obtained reductions to ODEs can be used for construction of numerical solutions of the generalized fKdV equations, see [175, 298] for details.

Concluding remarks. Lie symmetry analysis of the class (B.26) was initiated in [311], and the case $n = 2$ was also treated separately in [312]. However, the results presented therein are either incorrect [311] or incomplete [312]. Here we discuss main lucks of the results obtained in those two papers.

In [312] only some cases of Lie symmetry extensions for equations of the form (B.26) with $n = 2$ were found, namely, the cases with $\alpha = \mathrm{const}$ and $\alpha = 1/t$. If one performs the group classification up to the corresponding equivalence transformations it is enough to consider the case $\alpha = 0$. If one wants to get the classification, where all equations admitting Lie symmetry extensions are presented, not only their inequivalent representatives, then all such equations will have the coefficient α being arbitrary, so the cases $\alpha = \text{const}$ and $\alpha = 1/t$ can be considered as particular examples only. Moreover, even studying these particular cases the authors of [312] missed one case of Lie symmetry extension for each value of α considered by them. For example, for the case $\alpha = 0$ this is $\beta = \varepsilon(t+\delta)^\rho$, where ε, δ and ρ are arbitrary constants with $\varepsilon\rho \neq 0$. Nevertheless, at least dimensions and basis operators of the found Lie symmetry algebras for those particular cases derived in [312] are correct in contrast to the results presented in [311].

In [311] the authors state that they find three cases of Lie symmetry extensions for equations (B.26) and in each derived case the corresponding Lie symmetry algebra is four-dimensional. This is a false assertion. In this paper and in [177] we show that equation (B.26) admits four-dimensional Lie symmetry algebra if and only if $n = 1$ and, moreover, the equation is point-equivalent to the simplest constant-coefficient fKdV equation $u_t + uu_x + \mu u_{xxxxx} = 0$, where $\mu = \text{const}$. So, the results of [311] are principally incorrect.

B.5. Group Classification of Variable Coefficient $K(m, n)$ Equations

In order to understand the role of nonlinear dispersion in the formation of patterns in liquid drops, Rosenau and Hyman [259] introduced a generalization of the KdV equation of the form

$$u_t + \varepsilon(u^m)_x + (u^n)_{xxx} = 0,$$

where $\varepsilon = \pm 1$. Such equations, that are known as $K(m, n)$ equations, have the property for certain values of m and n their solitary wave solutions are of compact support. In other words, they vanish identically outside a finite core region. Further study followed in the references [255–258].

Here we consider a class of variable coefficient $K(m,n)$ equations of the form

$$u_t + \varepsilon(u^m)_x + f(t)\,(u^n)_{xxx} = 0, \tag{B.33}$$

where f is an arbitrary nonvanishing function of the variable t, n and m are arbitrary constants with $n \neq 0$, and $\varepsilon = \pm 1$. Note that the more general class (appeared, e.g., in [324]) of the form

$$u_t + g(t)(u^m)_x + f(t)(u^n)_{xxx} = 0, \quad fn \neq 0, \tag{B.34}$$

reduces to class (B.33) via the transformation $\tilde{t} = \varepsilon \int g(t)\mathrm{d}t$, $\tilde{x} = x$, $\tilde{u} = u$. This transformation maps the class (B.34) into its subclass (B.33), where $\tilde{f} = \varepsilon f/g$. This is why without loss of generality it is sufficient to study class (B.33).

In this section we carry out the Lie group classification for the class (B.33). All point transformations that link equations from the class are described. Firstly we find equivalence group of the entire class and then derive three its subclasses that have nontrivial conditional equivalence groups. The obtained Lie symmetries are employed also to a specific boundary value problem.

Admissible transformations. The results on admissible transformations for equations from the class (B.33) are given in the following theorems, which give the description of the equivalence groupoid. We exclude linear equations, i.e., equations with $(n,m) \in \{(1,0),(1,1)\}$, from the consideration. The proofs of these theorems are omitted.

Theorem B.22. *The usual equivalence group $G^{\sim}$ of the class (B.33) is formed by the transformations*

$$\tilde{t} = \pm\delta_1\delta_3^{1-m}t + \delta_0, \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u,$$
$$\tilde{f} = \pm\delta_1^2\delta_3^{m-n}f, \quad \tilde{\varepsilon} = \pm\varepsilon, \quad \tilde{n} = n, \quad \tilde{m} = m,$$

where δ_j, $j = 0,1,2,3$, are arbitrary constants with $\delta_1\delta_3 \neq 0$.

It appears that if $(n,m) \in \{(n,0),(n,1),(1,2)\}$, then there exist nontrivial conditional equivalence groups of the class (B.33) that are wider than $G^{\sim}$, namely the following assertions are true.

Theorem B.23. *The class* (B.33) *with* $m=0$,

$$u_t + f(t)\,(u^n)_{xxx} = 0, \tag{B.35}$$

admits usual equivalence group $G^{\sim}_{(n,0)}$ *consisting of the transformations*

$$\tilde t = T(t), \quad \tilde x = \delta_1 x + \delta_2, \quad \tilde u = \delta_3 u, \quad \tilde f = \frac{\delta_1^3\delta_3^{1-n}}{T_t}\,f, \quad \tilde n = n,$$

where δ_j, $j=1,2,3$, *are arbitrary constants with* $\delta_1\delta_3 \neq 0$, $T(t)$ *is an arbitrary smooth function with* $T_t \neq 0$.

Theorem B.24. *The generalized equivalence group* $G^{\sim}_{(n,1)}$ *of the class* (B.33) *with* $m=1$,

$$u_t + \varepsilon u_x + f(t)\,(u^n)_{xxx} = 0, \tag{B.36}$$

comprises the transformations

$$\tilde t = T(t), \quad \tilde x = \delta_1(x-\varepsilon t) \pm \varepsilon T(t) + \delta_2, \quad \tilde u = \delta_3 u,$$
$$\tilde f = \frac{\delta_1^3\delta_3^{1-n}}{T_t}\,f, \quad \tilde\varepsilon = \pm\varepsilon, \quad \tilde n = n,$$

where δ_j, $j=1,2,3$, *are arbitrary constants with* $\delta_1\delta_3 \neq 0$, $T(t)$ *is an arbitrary smooth function with* $T_t \neq 0$.

Theorem B.25. *The generalized equivalence group* $G^{\sim}_{(1,2)}$ *of the class,*

$$u_t + \varepsilon(u^2)_x + f(t)u_{xxx} = 0, \tag{B.37}$$

consists of the transformations

$$\tilde u = \pm\frac{2\varepsilon\kappa(\gamma t+\delta)u - \kappa\gamma x + \mu_1\delta - \mu_0\gamma}{2\varepsilon(\alpha\delta-\beta\gamma)},$$
$$\tilde t = \frac{\alpha t+\beta}{\gamma t+\delta}, \quad \tilde x = \frac{\kappa x + \mu_1 t + \mu_0}{\gamma t+\delta}, \quad \tilde\varepsilon = \pm\varepsilon, \quad \tilde f = \frac{\kappa^3}{\alpha\delta-\beta\gamma}\frac{f}{\gamma t+\delta},$$

where $\alpha,\beta,\gamma,\delta,\mu_1,\mu_0$, *and* κ *are constants defined up to a nonzero multiplier,* $\kappa(\alpha\delta-\beta\gamma)\neq 0$.

Table B.14: Classification of the equations (B.33) with $n \neq 1$.

no.	n	m	$f(t)$	Basis of $A^{\max}$
1	$\forall$	$\forall$	$\forall$	∂_x
2	$\forall$	$\frac{n+2}{3}$	$\forall$	$\partial_x,\ x\partial_x + \frac{3}{n-1}u\partial_u$
3	$\forall$	0	1	$\partial_t,\ \partial_x,\ x\partial_x + \frac{3}{n-1}u\partial_u,\ 3t\partial_t + x\partial_x$
4	$-\frac{1}{2}$	0	1	$\partial_t,\ \partial_x,\ x\partial_x - 2u\partial_u,\ 3t\partial_t + x\partial_x,\ x^2\partial_x - 4xu\partial_u$
5	$\forall$	$\forall$	1	$\partial_t,\ \partial_x,\ (3m-n-2)t\partial_t + (m-n)x\partial_x - 2u\partial_u$
6_a	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$\partial_t,\ \partial_x,\ 3t\partial_t + 2u\partial_u,$ $\sin x\,\partial_x - 2\cos x\,u\partial_u,\ \cos x\,\partial_x + 2\sin x\,u\partial_u$
6_b	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$\partial_t,\ \partial_x,\ 3t\partial_t + 2u\partial_u,\ e^x\partial_x - 2e^x u\partial_u,\ e^{-x}\partial_x + 2e^{-x}u\partial_u$
7	$\forall$	$\forall$	t^k	$\partial_x,\ (3m-n-2)t\partial_t + (km-k+m-n)x\partial_x + (k-2)u\partial_u$
8	$\forall$	$\frac{n+2}{3}$	t^2	$\partial_x,\ x\partial_x + \frac{3}{n-1}u\partial_u,\ t\partial_t + x\partial_x$
9	$\forall$	$\forall$	e^t	$\partial_x,\ (3m-n-2)\partial_t + (m-1)x\partial_x + u\partial_u$

Here k is an arbitrary nonzero constant. In Cases 6_a and 6_b $\varepsilon = 1$ and $\varepsilon = -1$, respectively.

The equations from the class (B.36) can be reduced to ones (with tilded variables) from the class (B.35) by the additional equivalence transformation $\tilde{t} = t$, $\tilde{x} = x - \varepsilon t$, $\tilde{u} = u$. Therefore, the case $m = 1$ being equivalent to the case $m = 0$ will be excluded from the classification list.

Lie symmetries. We perform the group classification of class (B.33) within the framework of the classical approach [38, 217, 227]. There are two essentially distinguished cases $n \neq 1$ and $n = 1$.

Lie symmetries according to the forms of $f(t)$ of equations (B.33) with $n \neq 1$ are tabulated in Table B.14. All cases presented in Table B.14 except Cases 3 and 4 are classified up to $G^{\sim}$-equivalence. For Cases 3 and 4, where $m = 0$, we used the equivalence group $G^{\sim}_{(n,0)}$ that is wider than $G^{\sim}$. Thus, the equation (B.35) with $n \neq -1/2$ admits the four-dimensional Lie symmetry algebra with the basis operators

$$\frac{1}{f(t)}\partial_t, \quad \partial_x, \quad x\partial_x + \frac{3}{n-1}u\partial_u, \quad 3\frac{\int f(t)\mathrm{d}t}{f(t)}\partial_t + x\partial_x$$

Table B.15: Classification of the class (B.33) with $n = 1$.

no.	$f(t)$	Basis of $A^{\max}$
		$m \neq 2$
1	$\forall$	∂_x
2	1	$\partial_t,\ \partial_x,\ 3(m-1)t\partial_t + (m-1)x\partial_x - 2u\partial_u$
3	t^k	$\partial_x,\ 3(m-1)t\partial_t + (m-1)(k+1)x\partial_x + (k-2)u\partial_u$
4	e^t	$\partial_x,\ 3(m-1)\partial_t + (m-1)x\partial_x + u\partial_u$
		$m = 2$
5	$\forall$	$\partial_x,\ 2\varepsilon t\partial_x + \partial_u$
6	1	$\partial_t,\ 2\varepsilon t\partial_x + \partial_u,\ \partial_x,\ 3t\partial_t + x\partial_x - 2u\partial_u$
7	t^k	$\partial_x,\ 2\varepsilon t\partial_x + \partial_u,\ 3t\partial_t + (k+1)x\partial_x + (k-2)u\partial_u$
8	e^t	$\partial_x,\ 2\varepsilon t\partial_x + \partial_u,\ 3\partial_t + x\partial_x + u\partial_u$
9	$e^{k\arctan t}\sqrt{t^2+1}$	$\partial_x,\ 2\varepsilon t\partial_x + \partial_u,\ 6\varepsilon(t^2+1)\partial_t + 2\varepsilon(3t+k)x\partial_x + (2\varepsilon(k-3t)u + 3x)\partial_u$

Here k is an arbitrary constant satisfying the following constraints: $k \neq 0$ in Case 3, $k \neq 0, 1$ and $k \geqslant 1/2 \bmod G^{\sim}_{(1,2)}$ in Case 7, $k \geqslant 0 \bmod G^{\sim}_{(1,2)}$ in Case 9.

irrespectively of the form of the function f. Here and throughout the paper an integral with respect to t should be interpreted as a fixed antiderivative. If $n = -1/2$, then the Lie symmetry algebra of the equation (B.35) is five-dimensional spanned by the above operators and the additional operator $x^2\partial_x - 4xu\partial_u$. Using the equivalence transformation $\tilde t = \int f(t)\mathrm{d}t$, $\tilde x = x$, $\tilde u = u$ from the group $G^{\sim}_{(n,0)}$ we reduce these cases to ones with $f = 1$ (cf., Cases 3 and 4 of Table B.14).

Lie symmetries according to the forms of $f(t)$ of equations (B.33) with $n = 1$ are tabulated in Table B.15.

The group classification of the class (B.33) with $n = 1$ and $m \neq 2$ is performed up to $G^{\sim}$-equivalence. For the classification of Lie symmetries of the equations (B.33) with $n = 1$ and $m = 2$ we used the wider conditional equivalence group $G^{\sim}_{(1,2)}$. Since transformations from the group $G^{\sim}_{(1,2)}$ are quite complicated, we adduce also the additional cases of Lie symmetry extensions of equations (B.33) with $n=1$ and $m=2$ that are inequivalent with respect to the group $G^{\sim}$ to Cases 6–9 of Table B.15.

1. $f=(t+\beta)^k t^{1-k}$, $k\neq 0,1$, $\beta\neq 0$: $\langle\partial_x,\ 2\varepsilon t\partial_x+\partial_u,\ Q_3\rangle$, where $Q_3=6\varepsilon t(t+\beta)\partial_t+2\varepsilon\,(3t+\beta(2-k))\,x\partial_x+[3x-2\varepsilon(3t+\beta(k+1))u]\partial_u$;
2. $f=te^{\frac{1}{t}}$: $\langle\partial_x,\ 2\varepsilon t\partial_x+\partial_u,\ 6\varepsilon t^2\partial_t+2\varepsilon(3t-1)x\partial_x+(3x-2\varepsilon(3t+2)u)\partial_u\rangle$;
3. $f=t$: $\langle\partial_x,\ 2\varepsilon t\partial_x+\partial_u,\ 3t\partial_t+2x\partial_x-u\partial_u,$
$2\varepsilon t^2\partial_t+2\varepsilon tx\partial_x+(x-2\varepsilon tu)\partial_u\rangle$.

From the first sight it looks like the counterpart to Case 9 of Table B.14 is missed. At the same time it appears that the function $f=\lambda\exp\Big(k\arctan\frac{\alpha t+\beta}{\gamma t+\delta}\Big)$ locally coincides with the function $\check f=\check\lambda\exp(k\arctan(\check\alpha t+\check\beta))$, see [234] for details.

As an example for a reduction into an ordinary differential equation, we consider Case 7 of Table B.15 which corresponds to the variable coefficient KdV equation

$$u_t+\varepsilon(u^2)_x+t^k u_{xxx}=0 \tag{B.38}$$

that admits the three-dimensional Lie symmetry algebra

$$Q_1=\partial_x,\quad Q_2=2\varepsilon t\partial_x+\partial_u,\quad Q_3=3t\partial_t+(k+1)x\partial_x+(k-2)u\partial_u.$$

Depending on the value of k an optimal system of one-dimensional subalgebras of this Lie symmetry algebra consists of the subalgebras

$$\begin{array}{llll}\langle Q_1\rangle, & \langle Q_2+\sigma Q_1\rangle, & \langle Q_3\rangle & \text{if}\quad k\neq -1,2\\ \langle Q_1\rangle, & \langle Q_2+\sigma Q_1\rangle, & \langle Q_3+aQ_1\rangle & \text{if}\quad k=-1\\ \langle Q_1\rangle, & \langle Q_2+\sigma Q_1\rangle, & \langle Q_3+aQ_2\rangle & \text{if}\quad k=2.\end{array}$$

Here $\sigma\in\{-1,0,1\}$, $a\in\mathbb{R}$.

Reductions associated with the subalgebra $\langle Q_1\rangle$ are not considered since they lead to constant solutions only. The ansatz constructed with the subalgebra $\langle Q_2+\sigma Q_1\rangle$ has the form $u=\dfrac{x}{2\varepsilon t+\sigma}+\phi(\omega)$ with the similarity variable $\omega = t$. This ansatz reduces equation (7) to the ODE $(2\varepsilon\omega+\sigma)\phi_\omega + 2\varepsilon\phi = 0$ whose general solution is $\phi = \dfrac{c_1}{2\varepsilon\omega+\sigma}$, where c_1 is an arbitrary constant. The corresponding solution of (7) takes the form $u = \dfrac{x+c_1}{2\varepsilon t+\sigma}$. It is fair to note that this solution satisfies equations of the form (B.37) for arbitrary f. Other reductions depend on the value of the exponent k. We adduce the ansatzes together with the corresponding reduced equations.

$$k \neq -1, 2. \quad \langle Q_3\rangle\colon\ u = t^{\frac{k-2}{3}}\phi(\omega), \quad \omega = xt^{-\frac{k+1}{3}},$$
$$3\phi_{\omega\omega\omega} + 6\varepsilon\phi\phi_\omega - (k+1)\phi_\omega\omega + (k-2)\phi = 0;$$
$$k = -1. \quad \langle Q_3 + aQ_1\rangle\colon\ u = \frac{1}{t}\phi(\omega), \quad \omega = x - \frac{a}{3}\ln t,$$
$$3\phi_{\omega\omega\omega} + 6\varepsilon\phi\phi_\omega - a\phi_\omega - 3\phi = 0;$$
$$k = 2. \quad \langle Q_3 + aQ_2\rangle\colon\ u = \frac{a}{3}\ln t + \phi(\omega), \quad \omega = \frac{x}{t} - \frac{2a\varepsilon}{3}\ln t,$$
$$3\phi_{\omega\omega\omega} + 6\varepsilon\phi\phi_\omega - 3\phi_\omega\omega - 2a\varepsilon\phi_\omega + a = 0.$$

We note that the latter two cases are equivalent. Indeed, the equation $u_t + \varepsilon(u^2)_x + t^2u_{xxx} = 0$ is mapped to the equation $\tilde u_{\tilde t} + \varepsilon(\tilde u)^2{}_{\tilde x} + \tilde t^{-1}\tilde u_{\tilde x\tilde x\tilde x} = 0$ by the following transformation from the group $G^\sim_{(1,2)}$

$$\tilde t = \frac{1}{t}, \quad \tilde x = -\frac{x}{t}, \quad \tilde u = \frac{2\varepsilon tu - x}{2\varepsilon}.$$

As a result Lie symmetries of $K(m,n)$ equations with time-dependent coefficients are classified. Group classification is presented up to widest possible equivalence groups, the usual equivalence group of the whole class for the general case and the conditional equivalence groups for special values of the exponents m and n.

B.6. Enhanced Group Analysis of a Class of Benjamin–Bona–Mahony–Burgers Equations

The regularized long wave equation $u_t + u_x + uu_x - u_{xxt} = 0$ was proposed by Peregrine [231] and later by Benjamin et al. [20] to describe small-amplitude long waves on the surface of water in a channel. In order to take into account the mechanisms leading to the degradation of the wave the model including dissipative term u_{xx} was considered in [9], namely

$$u_t + u_x + uu_x - \nu u_{xx} - u_{xxt} = 0, \quad \nu \in \mathbb{R}^+.$$

Here $u = u(x,t)$ is a real-valued function of the two real variables x and t, which, in applications, are typically proportional to distance in the direction of propagation and to elapsed time, respectively. The dependent variable may represent a displacement of the underlying medium or a velocity [9]. The regularized long wave equation with a Burgers-type dissipative term appended is more frequently called the Benjamin–Bona–Mahony–Burgers (BBMB) equation [205].

The most general form of the BBMB equation with time-dependent coefficients is

$$u_t + f(t)u_x + g(t)uu_x + k(t)u_{xx} + h(t)u_{xxt} = 0, \quad ghk \neq 0, \tag{B.39}$$

where f, g, h, and k are smooth functions of the variable t.

In this paper we aim to investigate this class with the Lie symmetry point of view, namely, to present the complete group classification of this class of equations. A similar study was initiated in [170] but the complete and correct group classification was not achieved therein. Lie symmetries and conservation laws of equations (B.39) without dissipative term (i.e. with $k = 0$) were thoroughly investigated in [305].

We carry out the group classification of class (B.39) using the method of mapping between classes suggested in [300] and then successfully applied for several classes of variable coefficient PDEs, see, e.g., recent works [297, 305]. When the complete group classification is achieved Lie reductions of BBMB equations to ODEs are performed as well as some exact solutions are constructed.

Equivalence Groupoid and Group Classification. Consider firstly the transformational properties of class (B.39). We look for the admissible point transformations using the direct method [160]. The proofs of the statements below are similar to those presented in [305] for equations (B.39) with $k = 0$. Thus, we skip the details of calculations for the sake of brevity and present the final results only.

All the admissible transformations in class (B.39) are generated by equivalence transformations and therefore this class is normalized in the usual sense. The following statement is true.

Theorem B.26. *The usual equivalence group $G^{\sim}$ of class (B.39) consists of the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u + \delta_4, \quad \tilde{k}(\tilde{t}) = \frac{{\delta_1}^2}{T_t} k(t),$$
$$\tilde{f}(\tilde{t}) = \frac{\delta_1}{T_t \delta_3}(\delta_3 f(t) - \delta_4 g(t)), \quad \tilde{g}(\tilde{t}) = \frac{\delta_1}{T_t \delta_3} g(t), \quad \tilde{h}(\tilde{t}) = {\delta_1}^2 h(t),$$

where δ_j, $j = 1, 2, 3, 4$, are arbitrary constants with $\delta_1\delta_3 \neq 0$ and $T = T(t)$ is an arbitrary smooth function with $T_t \neq 0$. Class (B.39) is normalized in the usual sense.

Using this theorem, we can formulate the criterion of reducibility of variable coefficient BBMB equations from class (B.39) to their constant coefficient counterparts:

Proposition B.27. *A variable-coefficient equation from class (B.39) is reduced to a constant-coefficient equation from the same class by a point transformation if and only if the corresponding coefficients f, g, h, and k satisfy the conditions*

$$(f/g)_t = h_t = k_t = 0,$$

i.e., k and h are constants and the function f is proportional to g.

We note that the maximal Lie invariance algebra $A^{\max}$ of the constant coefficient BBMB equation was found in [46]. It is two-dimensional Abelian algebra $\langle \partial_t, \partial_x \rangle$ spanned by the operators of time and space translations.

Table B.16: The group classification of class (B.41) up to $G_1^\sim$-equivalence.

no.	$H(t)$	$K(t)$	$F(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	$\forall$	∂_x
1	εt^ρ	$\lambda t^{\rho-1}$	$\delta t^{\frac{\rho-4}{2}}$	$\partial_x,\ 2t\partial_t+\rho x\partial_x+(\rho-2)u\partial_u$
2	εe^t	λe^t	$\delta e^{\frac{1}{2}t}$	$\partial_x,\ 2\partial_t+x\partial_x+u\partial_u$
3	ε	λ	δ	$\partial_x,\ \partial_t$

Here ε, δ, λ, and ρ are arbitrary constants with $\varepsilon\lambda\neq 0$, $\varepsilon=\pm1 \bmod G_1^\sim$. In Case 3 $\delta=0,1 \bmod G_1^\sim$ and additionally $\lambda=-1 \bmod G_1^\sim$ if $\delta=0$.

The presence of four arbitrary elements in class (B.39) leads to difficulties in solving the group classification problem. Therefore, we firstly simplify the problem by reducing the number of arbitrary elements in the class. This can be done either via gauging of arbitrary elements by equivalence transformations or using the method of mapping between classes. We choose the second option.

The family of point transformations

$$\tilde t=\textstyle\int g(t)\mathrm{d}t,\quad \tilde x=x,\quad \tilde u=u+\dfrac{f(t)}{g(t)},\tag{B.40}$$

parameterized by two arbitrary elements of class (B.39), maps class (B.39) to the related class of variable coefficient BBMB equations with a forcing term

$$u_t+uu_x+K(t)u_{xx}+H(t)u_{xxt}=F(t),\quad HK\neq 0.\tag{B.41}$$

(Tildes in (B.41) are omitted.) The arbitrary elements of the initial class (B.39) and the imaged class (B.41) are related via the formulas

$$K(\tilde t)=\frac{k(t)}{g(t)},\quad H(\tilde t)=h(t),\quad F(\tilde t)=\frac{1}{g(t)}\left(\frac{f(t)}{g(t)}\right)_t.\tag{B.42}$$

Following the method of mapping between classes, we firstly classify Lie symmetries of the imaged class (B.41) and then use the family of point transformations (B.40) and the relations (B.42) to extend the result to the initial class (B.39).

In order to efficiently solve the group classification problem for class (B.41), we look for admissible transformations in this class using the direct method. Similarly to class (B.39) such transformations are exhausted by transformations from the usual equivalence group admitted by this class.

Theorem B.28. *The usual equivalence group $G_1^{\sim}$ of class* (B.41) *comprises the transformations*

$$\tilde t = \frac{\delta_1}{\delta_3}t + \delta_0, \quad \tilde x = \delta_1 x + \delta_2, \quad \tilde u = \delta_3 u,$$
$$\tilde K(\tilde t) = \delta_1\delta_3 K(t), \quad \tilde H(\tilde t) = {\delta_1}^2 H(t), \quad \tilde F(\tilde t) = \frac{{\delta_3}^2}{\delta_1}F(t),$$

where δ_j, $j = 0, 1, 2, 3$, are arbitrary constants with $\delta_1\delta_3 \neq 0$. Class (B.41) *is normalized in the usual sense.*

The results on group classification of the class (B.41) are summarized in Table B.16.

Note B.29. *Any constant coefficient BBMB equation $u_t + fu_x + guu_x + ku_{xx} + hu_{txx} = 0$ can be reduced to the equation $u_t + uu_x - u_{xx} + \varepsilon u_{txx} = 0$, where $\varepsilon = \mathrm{sign}(h)$, (Case 4 of Table B.17 with $\delta = 0$ and $\lambda = -1$) by the transformation $\tilde t = -\dfrac{k}{|h|}t$, $\tilde x = \dfrac{x}{\sqrt{|h|}}$, $\tilde u = -\dfrac{\sqrt{|h|}}{k}(gu + f)$ from the group $G^{\sim}$.*

The classification list for the initial class (B.39) can be obtained using the transformation (B.40), the relations (B.42) and the group classification results derived for the imaged class (B.41) (Table B.16). The transformation (B.40) can be considered as a composition of the equivalence transformation $\tau^{\sim}$: $\tilde t = \int g(\bar t)\mathrm{d}\bar t$, $\tilde x = \bar x$, $\tilde u = \bar u$ from the group $G^{\sim}$ and the transformation τ: $\bar t = t$, $\bar x = x$, $\bar u = u + \frac{f(t)}{g(t)}$ that does not belong to the group $G^{\sim}$. The transformation $\tau^{\sim}$ maps any equation from (1) to the

Table B.17: The group classification of class (B.39) up to $G^{\sim}$-equivalence.

no.	$h(t)$	$k(t)$	$f(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	$\forall$	∂_x
1	εt^ρ	$\lambda t^{\rho-1}$	$\delta t^{\frac{\rho-2}{2}}$	$\partial_x,\ 2t\partial_t+\rho x\partial_x+(\rho-2)u\partial_u$
2	εt^2	λt	$\delta\ln t$	$\partial_x,\ t\partial_t+x\partial_x-\delta\partial_u$
3	εe^t	λe^t	$\delta e^{\frac{1}{2}t}$	$\partial_x,\ 2\partial_t+x\partial_x+u\partial_u$
4	ε	λ	δt	$\partial_x,\ \partial_t-\delta\partial_u$

Here $g(t)=1 \bmod G^{\sim}$; ε, δ, λ, and ρ are arbitrary constants with $\varepsilon\lambda\neq 0$, $\varepsilon=\pm1 \bmod G^{\sim}$. In Case 4 $\delta=0,1 \bmod G^{\sim}$ and additionally $\lambda=-1 \bmod G^{\sim}$ if $\delta=0$.

equation from the same class with $g=1$. In order to obtain the group classification for class (B.39) up to the $G^{\sim}$-equivalence it is enough to consider the transformation (B.40) with $g=1$. Then the formulas (B.42) which connect arbitrary elements in classes (B.39) and (B.41) take the simple form $H=h$, $K=k$, $F=f_t$. We integrate the latter ODE for the values of F appearing in Table B.16. We, respectively, get the following forms of $f=f(t)$:

1. $f=\bar\delta t^{\frac{\rho-2}{2}}+C$ $(\bar\delta=2\delta/(\rho-2))$, if $\rho\neq 2$ and $f=\delta\ln t+C$, otherwise;
2. $f=\bar\delta e^{\frac{1}{2}t}+C$ $(\bar\delta=2\delta)$; 3. $f=\delta t+C$.

The integration constant $C=0 \bmod G^{\sim}$. The last step is to perform the change of variable $\tilde u=u+f(t)$ in basis operators of the maximal Lie invariance algebras presented in Table B.16. The results are summarized in Table B.17.

We also derive the complete list of Lie symmetry extensions for the entire class (B.39), where arbitrary elements are not simplified by point transformations (Table B.18).

Reductions and Exact Solutions. The classes of BBMB equations (B.39) and (B.41) are similar with respect to the transformations (B.40). If one has exact solutions for equations (B.41) the similar solutions for equations (B.39) can easily be recovered using (B.40).

Table B.18: The group classification of class (B.39) using no equivalence.

no.	$h(t)$	$k(t)$	$f(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	$\forall$	∂_x
1	$\mu_1(\varepsilon T+\kappa)^\rho$	$\mu_2 g(\varepsilon T+\kappa)^{\rho-1}$	$\mu_3 g(\varepsilon T+\kappa)^{\frac{\rho-2}{2}}+\mu_4 g$	$\partial_x,\ \dfrac{2}{g}(\varepsilon T+\kappa)\partial_t+\varepsilon\rho x\partial_x$ $+\varepsilon(\rho-2)(u+\mu_4)\partial_u$
2	$\mu_1(\varepsilon T+\kappa)^2$	$\mu_2 g(\varepsilon T+\kappa)$	$\mu_3 g\ln(\varepsilon T+\kappa)+\mu_4 g$	$\partial_x,\ \dfrac{1}{g}(\varepsilon T+\kappa)\partial_t+\varepsilon x\partial_x-\varepsilon\mu_3\partial_u$
3	$\mu_1\exp(\sigma T)$	$\mu_2 g\exp(\sigma T)$	$\mu_3 g\exp(\frac{1}{2}\sigma T)+\mu_4 g$	$\partial_x,\ \dfrac{2}{g}\partial_t+\sigma x\partial_x+\sigma(u+\mu_4)\partial_u$
4	μ_1	$\mu_2 g$	$\mu_3 gT+\mu_4 g$	$\partial_x,\ \dfrac{1}{g}\partial_t-\mu_3\partial_u$

Here g is an arbitrary nonvanishing smooth function, $T=\int g(t)\,\mathrm{d}t$; $\varepsilon=\pm1$; μ_i, $i=1,\dots,4$, σ, κ and ρ are arbitrary constants with $\sigma\mu_1\mu_2\neq 0$.

That is why it is convenient to perform classification of Lie reductions for class (B.41), where the number of inequivalent cases of Lie symmetry extension is smaller.

To perform the classification of Lie reductions we need the optimal systems of one-dimensional subalgebras of the maximal Lie invariance algebras of the BBMB equations (B.41). Such algebras are at most two-dimensional. The optimal system of a two-dimensional Lie algebra $\langle X_1, X_2\rangle$ is $\{\langle X_1\rangle, \langle X_2\rangle\}$ if the algebra is non-Abelian and $\{\langle X_1\rangle, \langle X_2+\alpha X_1\rangle\}$, where $\alpha\in\mathbb{R}$, if it is Abelian. We have non-Abelian algebras in Case 1 with $\rho\neq 0$ and Case 2 of Table B.16 and Abelian algebras in Case 1 with $\rho=0$ and Case 3 of this table. We note that reductions with respect to subalgebra $\langle X_1=\partial_x\rangle$ lead to trivial constant solutions. Therefore, we perform only the reductions with the second subalgebra. For each case we list the BBMB equation, the one-dimensional subalgebra, the Ansatz constructed with this subalgebra and the corresponding reduced ODE.

Case $1_{\rho\neq0}$.

$$u_t+uu_x+\lambda t^{\rho-1}u_{xx}+\varepsilon t^\rho u_{xxt}=\delta t^{\frac{\rho-4}{2}}, \tag{B.43}$$

$$\langle 2t\partial_t + \rho x\partial_x + (\rho - 2)u\partial_u \rangle, \quad u = t^{\frac{\rho}{2}-1}\varphi(\omega), \text{ where } \omega = xt^{-\frac{\rho}{2}},$$
$$\varepsilon\rho\,\omega\varphi''' + (\varepsilon(\rho+2) - 2\lambda)\varphi'' - 2\varphi\varphi' + \rho\,\omega\varphi' + (2-\rho)\varphi + 2\delta = 0.$$

This ODE has particular exact solution $\varphi = \omega + \frac{2\delta}{\rho}$, which gives the "degenerate" solution $u = \dfrac{x}{t} + \dfrac{2\delta}{\rho}t^{\frac{\rho}{2}-1}$ of equation (B.43) with $\rho \neq 0$.

Case $1_{\rho=0}$.

$$u_t + uu_x + \lambda t^{-1}u_{xx} + \varepsilon u_{xxt} = \delta t^{-2}, \tag{B.44}$$
$$\langle t\partial_t + \alpha\partial_x - u\partial_u \rangle, \quad u = \frac{1}{t}\varphi(\omega), \text{ where } \omega = x - \alpha\ln t,$$
$$\varepsilon\alpha\varphi''' + (\varepsilon - \lambda)\varphi'' - \varphi\varphi' + \alpha\varphi' + \varphi + \delta = 0.$$

For $\alpha = -\delta$ this ODE has particular solution $\varphi = \omega + c$, where c is an arbitrary constant. This leads to the "degenerate" solution $u = \dfrac{1}{t}(x + \delta\ln t + c)$ of the equation (B.44).

Case 2.

$$u_t + uu_x + \lambda e^t u_{xx} + \varepsilon e^t u_{xxt} = \delta e^{\frac{1}{2}t}, \tag{B.45}$$
$$\langle 2\partial_t + x\partial_x + u\partial_u \rangle, \quad u = e^{\frac{t}{2}}\varphi(\omega), \text{ where } \omega = xe^{-\frac{t}{2}},$$
$$\varepsilon\omega\varphi''' + (\varepsilon - 2\lambda)\varphi'' - 2\varphi\varphi' + \omega\varphi' - \varphi + 2\delta = 0.$$

Case 3.

$$u_t + uu_x + \lambda u_{xx} + \varepsilon u_{xxt} = \delta, \tag{B.46}$$
$$\langle \partial_t + \alpha\partial_x \rangle, \quad u = \varphi(\omega), \text{ where } \omega = x - \alpha t,$$
$$\varepsilon\alpha\varphi''' - \lambda\varphi'' - \varphi\varphi' + \alpha\varphi' + \delta = 0.$$

Up to the equivalence we can consider $\lambda = -1$. We found exact solutions for the case $\delta = 0$, these are $\varphi = -2\tanh\omega$, $\alpha = 0$; and

$$\varphi = \pm\frac{1 - 12\varepsilon}{10\varepsilon} - \frac{12}{5}\tanh\omega \pm \frac{6}{5}\tanh^2\omega, \quad \alpha = \pm\frac{1}{10\varepsilon}.$$

Thus, equation $u_t + uu_x - u_{xx} + \varepsilon u_{xxt} = 0$ admits the exact solutions $u = -2\tanh x$ and

$$u = \pm\frac{1 - 12\varepsilon}{10\varepsilon} - \frac{12}{5}\tanh\left(x \pm \frac{t}{10\varepsilon}\right) \pm \frac{6}{5}\tanh^2\left(x \pm \frac{t}{10\varepsilon}\right).$$

Using the method of mapping between classes we have presented the complete group classification of BBMB equations (B.39). As a by-product of this approach we also got the group classification of a related class of BBMB equations with a forcing term (B.41). For the convenience of applications we adduced the results in two ways: the classification list where only inequivalent equations are presented (Table B.17) and the list with their most general forms (Table B.18).