MULTIMODAL METHOD IN SLOSHING.
PART 1. IDEAS AND BACKGROUND

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ABSTRACT. Commonly, two lectures imply a "journey" over necessary background of nonlinear modal modelling in the fluid sloshing problems. Part 1 introduces general ideas, describes results on the linear sloshing and presents the Lagrangian formalism. Its main focus is on the pseudospectral approach, which makes it possible to reduce the free boundary problem to an infinite-dimensional system of ordinary differential equations. A test problem for validating the system consists of lateral/pitch resonant excitations of the lowest natural frequency, when, as it is shown in experiments, hydrodynamic loads on the tank reach their maxima. By considering small excitation amplitude, we can find asymptotic steady-state (periodic) solutions of the original problem, which coincide with corresponding asymptotic solutions of the infinite-dimensional system. These solution are attributed by an intermodal ordering. Assuming this ordering to be uniformly valid for transient waves truncates the system to a finite-dimensional form, to the **multidimensional asymptotic modal systems**. The forthcoming Part 2 will give numerous examples of those systems for both 2D and 3D cases.

1. INTRODUCTION

A fluid occupying partly either earth-fixed basins or moving tanks of rockets, nuclear reactors, tower- and bridge constructions, ships and liquefied natural gas carriers performs wave motions, the **sloshing**[0], that are caused by time-dependent and instantaneous perturbations of its hydrostatic equilibrium. Since the fluid sloshing disturbed by guidance and control systems commands, manoeuvres and structural vibrations of mobile vehicles generates significant hydrodynamic force- and moment loads on the tank, it becomes a danger for structural integrity and can produce a dramatical feedback sensed and responded to by the tank motions forming a closed loop that leads to an instability and even damage.

The structural instability hazard of mobile vehicles can physically be justified by the closeness of a control structural frequency to a fundamental sloshing frequency that yields, over and above the gust inputs, coupled resonant oscillations involving large fluid mass. Since the lowest fundamental sloshing mode is characterised by the weakest damping, design of slosh-suppressing devices should be concerned about shifting the lowest sloshing frequency away from the dominating structural frequency. By ensuring the splitting of these two effective frequency domains (sloshing and structural), installation of such a slosh-suppressing device guarantees the smallness of surface waves and, therefore, justifies applicability of the *linear sloshing theory*. In view of minimising the crucial loads, preventing structural failure and governing the fluid position within the tank, extensive experimental and theoretical studies have been undertaken from several decades ago and, as a result, numerous devices have been designed for suppressing the fluid mobility. Systematisations of various slosh-suppressing devices of passive nature have at different time and for different applications been done by Abramson (1969); Ibrahim et al. (2001).

Environmental concerns require a new design for tanks of ships, roads and storage systems. This suggests double walls, bottom and roof and may as a result increase the total structural weight. A way to avoid the weight penalty consists of removing the slosh-suppressing structures. This makes invalid the linear theories and, as a consequence, the coupled "structure-fluid" oscillations become strongly resonant that lead to a large-amplitude fluid sloshing so that the large fluid mass motions become a predominating factor. Analysis of the resonant sloshing should be based on a *fully nonlinear formulation.* This implies requirements in robust and accurate computer programs to predict the sloshing, in particular, and mathematical models for modelling coupled motions, in general.

1.1. **What is the current the state-of-the-art?** The majority of researches focuses on the Computational Fluid Dynamics (CFD) methods (see, Cariou & Casella, 1999; Ibrahim et al., 2001). Unfortunately, this is a traditional fallacy that successful implementation of the CFD methods in simulating streamline, propulsion etc. problems of aircraft and spacecraft applications means immediate success in solving engineering tasks associated with fluid sloshing. Progress of the CFD-methods in modelling the sloshing is still unsatisfactory and this is due to especial properties of this hydrodynamic system, which is *almost conservative, oscillatory (hyperbolic), stiff* and, generally speaking, very *sensitive to initial perturbations* and (in direct simulations) *to numerical damping.* In addition, two main engineering tasks consist usually of *coupling* with structural motions and of predicting dynamic response on *the long-time scale*: Even if the tank is exposed to regular (usually harmonic) excitations, the limiting fluid motions on the long-time scale (about 10-15min of the real time) can be not only periodic (almost periodic), steady-state, but also "chaotic" due to hydrodynamic instability. This yields a paradigm in a clear *distinguishing of hydrodynamic and numerical instability.*

1.2. **Historical view.** The maximum progress has been achieved in spacecraft problems of 60-70th (Abramson, 1966). And this progress is not connected with the CFD, but rather *analytically-oriented, modal* methods. The modal methods are, of course, less universal, but these are CPU-efficient to solve the engineering tasks. The coupling based on the modal methods appears to be very natural, because both solids (carrying object) and fluid dynamics can be modelled by a low-dimensional systems of ordinary differential equations, which appears as the *dynamic equations* of the whole object. Fluid and structure "speak" then on the same "language". Generally, this "language" is *the analytical mechanics.*

FOR NOTES:

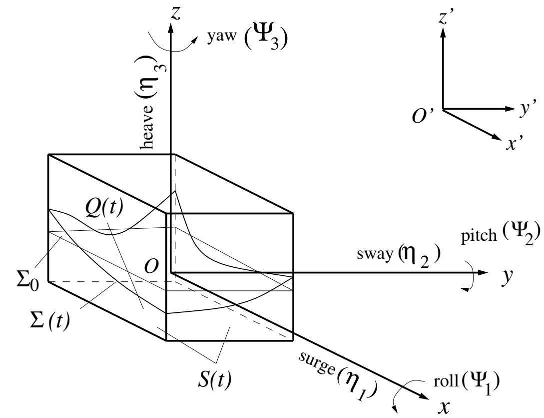

FIGURE 1. Sketch of s smooth tank moving in space. The vectors $\boldsymbol{P}'_O, \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\boldsymbol{v}_O = (v_{O1}, v_{O2}, v_{O3})$ are considered in the moving coordinate system $Oxyz$ framed with the rigid tank. $Q(t)$ is a piece-smooth, Lipschitzian domain.

2. Sloshing of heavy masses in moving containers (in context of marine applications)

After sketching definitions in Figure 1, let us consider a mobile tank partly filled by an incompressible fluid. The fluid volume bounded by the free surface $\Sigma(t)$ and the wetted tank surface $S(t)$ is denoted $Q(t)$. Let $O'x'y'z'$ be an absolute coordinate system and $Oxyz$ be a moving coordinate system fixed with respect to the tank. The origin of $Oxyz$ is in the unperturbed free surface and moves with the velocity $\boldsymbol{v}_0$ relatively to $O'x'y'z'$. The tank has an angular velocity $\boldsymbol{\omega}$ relatively to $O'x'y'z'$. The gravity field has the potential

$$U(x, y, z, t) = -\boldsymbol{g} \cdot \boldsymbol{r}'; \quad \boldsymbol{r}' = \boldsymbol{r}'_0 + \boldsymbol{r}, \tag{1}$$

where $\boldsymbol{r}'$ is the radius–vector of a point of the body–fluid system with respect to O', $\boldsymbol{r}'_0$ is the radius–vector of the point O with respect to O', $\boldsymbol{r}$ is the radius–vector with respect to O and $\boldsymbol{g}$ is the gravity acceleration vector.

Furthermore, $\boldsymbol{v}_O$ and $\boldsymbol{\omega}$ are known time-dependent functions, but we should bear in mind that these appear in dynamic equations of the tank as the unknowns, namely, these vectors are isomorphically equivalent to a set of generalised coordinates p_i and impulses q_i, $i = 1, \ldots, N$, where N is the number of the degrees of the freedom.

Keywords: **sloshing**[0] → sloshing as a **dynamic system**[1] →

Behaviour of the fluid flows is attributed by:

(1) Natural sloshing frequencies are much lower than eigenfrequencies of elastic vibrations of the tank walls/bottom, even if these vibrations are non-negligible. Conclusion: **assumption on rigid tank**.
(2) Unless the container includes internal structures (e.g. baffles) and the fluid fill is not small (shallowing), the Galileo numbers Ga=$\rho^2 Lg/\mu$ (an analogy

of the Reynolds numbers for sloshing problems) are large. Conclusion: **the fluid flows are inviscid**.

(3) Fluid domains are smooth (piece-smooth) and Lipschitzian. In the majority of realistic initial scenarios, the fluid flows are irrotational (due to corresponding theorems). Conclusion: **the fluid sloshing in smooth tanks is a conservative (almost conservative) dynamic system.**

(4) Large characteristic dimensions (sizes) of the tank and, therefore, large Bond numbers Bo=$\rho g L^2/\sigma$. Conclusion: **the surface tension can be neglected.**

NB: The fluid sloshing can be described in the framework of the surface wave theory (the same postulations).

Engineering TASKS associated with sloshing:

(1) Prediction of **hydrodynamic forces and moments on the long-time scale** due to almost periodic low-frequency loading on the tank. Pressure field is ONLY of particular interest, only for transients, when the loads reach maximum values.

(2) **Coupling.** Dynamic equations of the structure should be effectively coupled with dynamic equations of the fluid motions.

(3) **Other** tasks associated with accurate simulations of sloshing together with slamming etc.

Conclusions: Both the dynamic equations of the carrying structure and governing equations for the fluid sloshing **CAN** be treated as a **dynamic (Hamiltonian) system**[1]. This implies the set of generalised coordinates $p_i(t)$ and impulses $q_i(t),\ i = 1, \ldots, N$ (N is the number of the degrees of the freedom) for the structural dynamics as well as the possibility to introduce analogous generalised coordinates and impulses $\beta_i(t),\ R_i(t),\ i = 1, \ldots$ (generally, infinite number of the degrees of the freedom) for sloshing. Then, using the Lagrangian principle derives the Lagrange equation for the whole object. Furthermore, we focus on the problem how to get the **Lagrange equation for the sloshing problem**.

FOR NOTES:

3. Free boundary problem, Lagrangian formulation

Keywords: → **dynamic system**[1] → surface waves →

3.1. **Free boundary value problem.** When $\boldsymbol{v}_O(t)$ and $\boldsymbol{\omega}(t)$ [or $\boldsymbol{\psi}(t)$] are given functions (prescribed motions of the tank), the sloshing is described by the following free boundary problem

$$\text{(2a)} \qquad \Delta\Phi = 0 \quad \text{in } Q(t); \quad \frac{\partial \Phi}{\partial \nu} = \boldsymbol{v}_O \cdot \boldsymbol{\nu} + \boldsymbol{\omega} \cdot [\boldsymbol{P} \times \boldsymbol{\nu}] \quad \text{on } S(t),$$

$$\text{(2b)} \qquad \frac{\partial \Phi}{\partial \nu} = \boldsymbol{v}_O \cdot \boldsymbol{\nu} + \boldsymbol{\omega} \cdot [\boldsymbol{P} \times \boldsymbol{\nu}] - \boxed{\frac{\xi_t}{|\nabla \xi|}} \quad \text{on } \Sigma(t); \quad \int_{Q(t)} dQ = \text{const},$$

$$\text{(2c)} \qquad \boxed{\frac{\partial \Phi}{\partial t}} + \frac{1}{2}(\nabla \Phi)^2 - \nabla \Phi \cdot (\boldsymbol{v}_O + \boldsymbol{\omega} \times \boldsymbol{P}) + U = 0 \quad \text{on } \Sigma(t),$$

where the Neumann boundary value problem (2a)-(2b) ($\boldsymbol{\nu}$ is outward normal to $Q(t)$) expresses the so-called kinematic conditions.

The problem defines the absolute velocity potential $\Phi(x, y, z, t)$ in time-varying volume $Q(t)$ confined to the wetted body surface $S(t)$ and the free surface $\Sigma(t)$ versus $\Sigma(t) : \xi(x, y, z, t) = 0$. The boundary condition (2c) implies the pressure balance on the free surface (the later states that the pressure on the free surface is equal to a constant p_0), it follows from the Lagrange–Cauchy integral. The hydrodynamic pressure p in $Q(t)$ can be obtained by

$$\text{(3)} \qquad \frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla \Phi)^2 - \nabla \Phi \cdot (\boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}) + U + \frac{p - p_0}{\rho} = 0 \quad \text{in} \quad Q(t).$$

Here $\partial \Phi / \partial t$ is calculated in the moving coordinate system, i.e. for a point rigidly connected with the system $Oxyz$.

The evolutional free boundary problem (2) should be completed by either initial or periodicity conditions. The initial (Cauchy) conditions require

$$\text{(4)} \qquad \xi(x, y, z, t_0) = \xi_0(x, y, z); \quad \left.\frac{\partial \Phi}{\partial \nu}\right|_{\Sigma(t_0)} = \Phi_0(x, y, z)$$

to be known at $t = t_0$. The periodicity conditions are in many applied problems associated with periodicity of wave pattern and velocity field, i.e.

$$\text{(5)} \qquad \xi(x, y, z, t + T) = \xi(x, y, z, t); \quad \nabla \Phi(x, y, z, t + T) = \nabla \Phi(x, y, z, t).$$

This requires that $Q(t + T) = Q(t)$. Further, the last equality in Eq. (5) is justified by the first one (for ξ) establishing the equivalence of instantaneous fluid shapes at t and $t + T$.

NB: The framed terms of (2) make it possible to consider the sloshing as a dynamic system, a hyperbolic dynamic system.
NB: The standard physical treatment of ξ and Φ consists in generalisation of coordinates and impulses, but NOT as it is required in the Hamiltonian mechanics.

Keywords: → surface waves → Hamiltonian mechanics →

3.2. **Lagrangian formulation.** The dynamic system (2) admits the classical Lagrangian formulation (Whitham, 1967), but, in contrast to the solid mechanics, this formulation derives only the *dynamic condition* (2c), while (2a)-(2b) should be satisfied a priory. It reads as

Lagrange variational principle: *The boundary value problem given by (2) can be described by examining the necessary conditions for the extrema of the functional*

$$W = \int_{t_1}^{t_2} L dt, \tag{6}$$

where the Lagrangian L is as follows

$$L = \rho \int_{Q(t)} \left(\frac{1}{2} (\nabla \Phi)^2 - U \right) dQ \tag{7}$$

and Φ satisfies (2a)-(2b); the test functions ξ are as follows

$$\delta\xi(x, y, z, t_1) = 0, \quad \delta\xi(x, y, z, t_2) = 0. \tag{8}$$

This formulation is inconvenient, because it does not provide a projective scheme for (2a)-(2b). That is why, most analytical methods in sloshing operate with the Bateman–Luke variational principle based on the pressure-integral formulation. The idea of the pressure integral as the Lagrangian in hydrodynamic problems was first proposed by Hargneaves (1908). The canonical formulation of this principle is given by Bateman and Luke (for gravity surface waves in ocean). We use the formulation from the book by Lukovsky & Timokha (1995) and paper by Faltinsen et al. (2000).

Pressure–integral Lagrangian variational principle. *The boundary value problem given by (2) can be described by examining the necessary conditions for the extrema of the functional (6), where the Lagrangian L is the pressure integral*

$$L = \int_{Q(t)} (p - p_0) dQ = -\rho \int_{Q(t)} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 - \nabla \Phi \cdot (\boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}) + U \right] dQ; \tag{9}$$

and the test functions satisfy

$$\delta\Phi(x, y, z, t_1) = 0, \ \ \delta\Phi(x, y, z, t_2) = 0; \ \ \delta\xi(x, y, z, t_1) = 0, \ \ \delta\xi(x, y, z, t_2) = 0. \tag{10}$$

Now, the problem consists of introducing $\beta_i(t)$ and $R_i(t)$.

FOR NOTES:

4. Modal equations and modal methods

Keywords: → **generalised coordinates and impulses**[2] → **modes**[3] →

4.1. Generalised coordinates and impulses. The Kirchhoff solution. One obvious way to introduce the generalised coordinates β_i consists of separating variables by defining

$$\xi = z - f(x,y,t); \quad f = \sum_{i=1}^{\infty} \beta_i(t) f_i(x,y), \tag{11}$$

where $\{f_i\}$ is a complete system of functions, let's say, a Fourier basis, where each f_i determines a form of motions, a **mode**[3]. Note, that (11) is *only possible for cylindrical tanks*, but more complicated geometry needs special modifications (Lukovsky and Timokha, 2002).

Analogous representation for the velocity potential utilises the Kirchhoff solution $K(x,y,z,t) = K(\beta_i,t)$, which has first been found by Zhukovsky for cavities which are completely occupied by fluid (there are no free surface)

$$\Phi(x,y,z,t) = \underbrace{\boldsymbol{v}_0 \cdot \boldsymbol{r} + \boldsymbol{\omega} \cdot \boldsymbol{\Omega}}_{K(\beta_i,t)} + \sum_{k=1}^{\infty} R_k(t)\varphi_k(x,y,z), \tag{12}$$

where $\{\varphi_k\}$ is also a complete system of harmonic functions and $\{R_k\}$ implies the generalised impulses.

Remark: The the vector-function $\boldsymbol{\Omega} = (\Omega_1,\Omega_2,\Omega_3)$ (Stokes-Zhukovsky potentials) is the solution of the following Neumann boundary value problem

$$\Delta \boldsymbol{\Omega} = 0 \quad \text{in} \quad Q(t); \quad \frac{\partial \Omega_1}{\partial \nu}|_{S(t)+\Sigma(t)} = y\nu_3 - z\nu_2,$$

$$\frac{\partial \Omega_2}{\partial \nu}|_{S(t)+\Sigma(t)} = z\nu_1 - x\nu_3, \quad \frac{\partial \Omega_3}{\partial \nu}|_{S(t)+\Sigma(t)} = x\nu_2 - y\nu_1, \tag{13}$$

where ν_1,ν_2,ν_3 are the projections of the outer normal $\boldsymbol{\nu}$ onto the $Oxyz$ axes.

NB: Forces and moments are also functions of generalised coordinates β_i!!! Example for forces is

$$\boldsymbol{F} = m\boldsymbol{g} - m\left[\dot{\boldsymbol{v}}_0 + \boldsymbol{\omega}\times\boldsymbol{v}_0 + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_C) + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}_C + 2\boldsymbol{\omega}\times\dot{\boldsymbol{r}}_C + \ddot{\boldsymbol{r}}_C\right]. \tag{14}$$

Here m is the fluid mass, $\boldsymbol{r}_C$ is the radius–vector of mass centre in mobile coordinate system (directly computed by β_i) and $m\boldsymbol{g}$ is the fluid weight. The terms in square brackets mean: $\dot{\boldsymbol{v}}_0$ is the acceleration of the origin O, $\boldsymbol{\omega}\times\boldsymbol{v}_0$ is the tangential acceleration, $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_C)$ is the centripetal acceleration, $2\boldsymbol{\omega}\times\dot{\boldsymbol{r}}_C$ is Coriolis acceleration, $\ddot{\boldsymbol{r}}_C$ is the relative acceleration. Obviously, $\boldsymbol{r}_C$ is an explicit function of β_i.

Thus, introducing generalised coordinates and impulses needs the basic functions $\{f_i\}$ and $\{\varphi_k\}$, which are called the **modes**[3]. The typical expressions for these modes will be given. Now, let us consider how to obtain the Lagrange equation coupling the generalised coordinates.

Keywords: → **modes**[3] → **Lagrange equation**[4] → **modal method**[5] →

4.2. **Modal method.** By substituting (12) into (9), the Bateman-Luke functional L leads (see, Appendix A) to the following infinite-dimensional Lagrange equations coupling modal functions $R_n(t)$ and $\beta_i(t)$

$$\sum_i \boxed{\dot{\beta}_i} \frac{\partial A_n}{\partial \beta_i} - \sum_k R_k A_{nk} = 0, \quad n = 1, 2, \ldots; \tag{15}$$

$$\sum_n \boxed{\dot{R}_n} \frac{\partial A_n}{\partial \beta_i} + \frac{1}{2} \sum_n \sum_k \frac{\partial A_{nk}}{\partial \beta_i} R_n R_k + \dot{\omega}_1 \frac{\partial l_{1\omega}}{\partial \beta_i} + \dot{\omega}_2 \frac{\partial l_{2\omega}}{\partial \beta_i} + \dot{\omega}_3 \frac{\partial l_{3\omega}}{\partial \beta_i} + \omega_1 \frac{\partial l_{1\omega t}}{\partial \beta_i} + \omega_2 \frac{\partial l_{2\omega t}}{\partial \beta_i} +$$

$$+ \omega_3 \frac{\partial l_{3\omega t}}{\partial \beta_i} - \frac{d}{dt} \left(\omega_1 \frac{\partial l_{1\omega t}}{\partial \dot{\beta}_i} + \omega_2 \frac{\partial l_{2\omega t}}{\partial \dot{\beta}_i} + \omega_3 \frac{\partial l_{3\omega t}}{\partial \dot{\beta}_i} \right) + (\dot{v}_{01} - g_1 + \omega_2 v_{03} - \omega_3 v_{02}) \frac{\partial l_1}{\partial \beta_i} +$$

$$+ (\dot{v}_{02} - g_2 + \omega_3 v_{01} - \omega_1 v_{03}) \frac{\partial l_2}{\partial \beta_i} + (\dot{v}_{03} - g_3 + \omega_1 v_{02} - \omega_2 v_{01}) \frac{\partial l_3}{\partial \beta_i} - \frac{1}{2} \omega_1^2 \frac{\partial J_{11}^1}{\partial \beta_i} - \frac{1}{2} \omega_2^2 \frac{\partial J_{22}^1}{\partial \beta_i} -$$

$$- \frac{1}{2} \omega_3^2 \frac{\partial J_{33}^1}{\partial \beta_i} - \omega_1 \omega_2 \frac{\partial J_{12}^1}{\partial \beta_i} - \omega_1 \omega_3 \frac{\partial J_{13}^1}{\partial \beta_i} - \omega_2 \omega_3 \frac{\partial J_{23}^1}{\partial \beta_i} = 0, \tag{16}$$

or, more generally,

$$[M(\boldsymbol{\beta})] \, \dot{\boldsymbol{\beta}} = [F_1(\boldsymbol{\beta})] \boldsymbol{R} \tag{17}$$

(kinematic equation),

$$[M(\boldsymbol{\beta})]^T \, \dot{\boldsymbol{R}} = F_2(\boldsymbol{\beta}, \boldsymbol{R}, \boldsymbol{v}_O, \boldsymbol{\omega}, \dot{\boldsymbol{v}}_O, \dot{\boldsymbol{\omega}}) \tag{18}$$

(dynamic equation), where $\boldsymbol{\beta} = \{\beta_i\}, \boldsymbol{R} = \{R_i\}$.

If $\{f_i\}$ and $\{\varphi_i\}$ and some their derivatives are explicitly given, the "mass matrix" $[M(\boldsymbol{\beta})]$ and the function $F_i, \ i = 1, 2$ are easily derivable analytically from the formulae of Appendix A. In order to exclude $\boldsymbol{R}$ and get equations in $\boldsymbol{\beta}$, (17) should be resolved in $\boldsymbol{R}$ and the corresponding expression substituted in (18). This leads to an infinite set of equations of the second order.

Concluding remarks:

- The constructed infinite–dimensional system of equations (15) (16) is fully equivalent to the original free boundary problem. It is applicable to any type of rigid body motions, but one is necessary that $f(x, y, t)$ is single-valued. This means that plunging breakers cannot be described. There are no other restrictions on the type of surface waves that can be studied.
- **modal (Perko-like) method**[5] consists of the naive truncating of (15) (16). How efficient is this approach? Everything depends on the modes and actual type of wave motions. If the real wave patterns are accurately approximated by a finite set of the modes, the method works well.
- The system should be applicable for waves of small magnitude, i.e. it should be *valid for linear sloshing problems.*

FOR NOTES:

5. Natural modes

The main question of the approach above is what should be the complete sets $\{f_i(x,y)\}$ and $\{\varphi_i(x,y,z)\}$. Theoretically, the approach admits any arbitrary basis, but, physically, there is no warranty that these basic systems will approximate accurately the fluid sloshing. Such an example for $\{\varphi_i\}$ is a system harmonic polynomials which provide the completeness, but is not applicable for linear sloshing problem.

Keywords: → **modes**[3] → **natural** frequencies and modes[6] →
→ **pseudospectral method**[7]

How to find appropriate modes? Once again, the test, "detuning" problem is the linear sloshing, because modal method must be efficient in the linear case. Let us assume $\boldsymbol{v} = \boldsymbol{\omega} \equiv 0$ and $\Phi := \tilde{\Phi} + o(||\tilde{\Phi}||)$. After linearisation in terms of $||\tilde{\Phi}||$ (relative to hydrostatic shape), we arrive at

$$\Delta\tilde{\Phi} = 0 \text{ in } Q_0; \quad \frac{\partial\tilde{\Phi}}{\partial\nu} = 0 \text{ on } S_0 \text{ and } \Gamma; \quad \int_{\Sigma_0} \frac{\partial\tilde{\Phi}}{\partial z} dS = 0, \tag{19}$$

$$\frac{\partial\tilde{\Phi}}{\partial z} = \frac{\partial\tilde{f}}{\partial t}; \qquad \frac{\partial\tilde{\Phi}}{\partial t} + g\tilde{f} = 0 \text{ on } \Sigma_0, \tag{20}$$

where Q_0 is the static fluid domain, S_0 is the statically wetted tank surface, Σ_0 coincides with the mean fluid surface, ν is the outward normal to Q_0, the function $\tilde{f}(x,y,t)$ defines small-amplitude elevations of the free surface $(z = \tilde{f}(x,y,t))$ and $\tilde{\Phi}(x,y,z,t)$ denotes the linear velocity potential.

Solutions of the linear problem (19)-(20) are associated with a special class of spectral problems with spectral parameter in boundary conditions. Reduction to these spectral problems suggests the substitution

$$\tilde{\Phi}(x,y,z,t) = \varphi(x,y,z) \, \exp(\mathrm{i}\omega t), \; \mathrm{i}^2 = -1, \tag{21}$$

which defines natural frequencies ω and modes (complex amplitudes) $\varphi(x,y,z)$. By rewriting the boundary conditions (20) to the form

$$\frac{\partial^2\tilde{\Phi}}{\partial t^2} + g\frac{\partial\tilde{\Phi}}{\partial z} = 0; \quad \tilde{f} = -\frac{1}{g}\frac{\partial\tilde{\Phi}}{\partial t} \text{ on } \Sigma_0 \tag{22}$$

and introducing $\kappa = \omega^2/g$, the evolutional problem (19), (22) is transformed to the spectral problem

$$\begin{aligned} &\Delta\varphi = 0 \quad \text{in } Q_0; \quad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } S_0 \text{ and } \Gamma, \\ &\frac{\partial\varphi}{\partial z} = \kappa\varphi \quad \text{on } \Sigma_0; \quad \int_{\Sigma_0} \varphi dS = 0. \end{aligned} \tag{23}$$

Main point: As established by Eastham (1962), the spectral problem (23) has the real positive pointer spectrum $\{\kappa_i\}$, $\kappa_i \to +\infty$ and $\{f_i = \varphi_i(x,y,0)\}$ constitute an orthogonal, **natural, Fourier basis**[6] for any functions, which satisfy the integral condition of (23). These spectral theorems deduce that eigenfunctions of (23) form, via formula (21), fundamental solutions of the evolutional problem (19)-(20).

Particular cases, when natural modes can be found analytically:

1: The rectangular base basin. In the scaled form (the width is equal to 1, and the aspect ratio is r), we have

$$(24) \quad \varphi_{i,j}(x,y,z) = f_i^{(1)}(x) f_j^{(2)}(y) \frac{\cosh(\lambda_{i,j}(z+h))}{\cosh(\lambda_{i,j}h)},$$
$$f_i^{(1)}(x) = \cos(\pi i(x+1/2)); \quad f_j^{(2)}(y) = \cos(\pi r j(y+1/2r)),$$
$$\lambda_{i,j} = \pi\sqrt{i^2+r^2j^2}, \quad \sigma_{i,j}^2 = g\lambda_{i,j}\tanh(\lambda_{i,j}h), \quad i,j \geq 0, \quad i+j \neq 0$$

and $\sigma_{i,j}$ are the natural frequencies, h is the depth to width ratio (scaled depth). Projections of $\varphi_{i,j}$ on the mean free surface $z=0$ introduce the shapes of standing waves $f_{i,j}(x,y) = f_i^{(1)}(x) f_j^{(2)}(y)$.

2: The rectangular basin (2D flows). Particular case of (24) with $j = 0$. For brevity, we write down
(25)

$$\lambda_i = \pi i \tanh(i\pi h); \quad f_i(x) = \cos(\pi i(x+1/2)); \quad \varphi_i(x,z) = f_i(x)\frac{\cosh(\pi i(z+h))}{\cosh(\pi i h)}.$$

and now

$$(26) \quad f(x,t) = \sum_{i=1}^{\infty} \beta_i(t) f_i(x),$$
$$\Phi(x,0,z,t) = v_{0x}x + v_{0z}z + \omega_2(t)\Omega(x,z,t) + \sum_{n=1}^{\infty} R_n(t)\varphi_n(x,z),$$

3: The circular base basin. In the scaled form (the radius is equal to 1), we obtain in the circular coordinate system

$$(27) \quad \varphi)_{m,i}(z,\theta,r)_{m,i} = J_m(\lambda_{m,i}r)\frac{\cosh(\lambda_{m,i}(z+h))}{\cosh(\lambda_{m,i}h)},$$
$$\sigma_{i,j}^2 = g\lambda_{m,i}\tanh(\lambda_{m,i}h), \quad m \geq 0, \quad i \geq 1,$$

where $J_m(\cdot)$ is the Bessel function of first kind and $J_m'(\lambda_{m,i}) = 0$, namely, $\lambda_{m,i}$ is the ith root.

For brevity, we will concentrate on the case **2**!!! However, the next lecture will describe the modal systems for 3D case.

FOR NOTES:

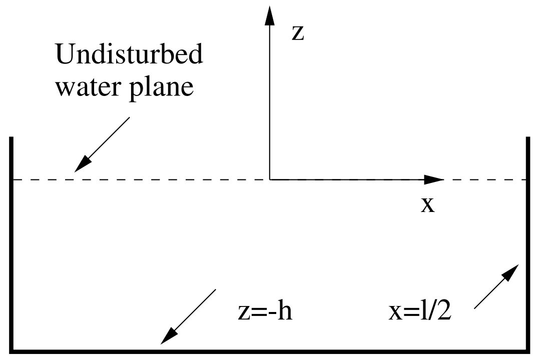

FIGURE 2. Coordinate system.

6. An asymptotic modal system for lateral excitations in a rectangular tank (2D flows)

Either simplification of the pseudospectral system is related to special properties on β_i, R_i. Our *idea* is to use existing asymptotic solutions, to treat them as a modal solutions and, finally, to assume that the pseudospectral system models numerical solutions, which are close to the analytical one.

Since most dangerous ones are resonant excitations, usual way is to find asymptotic steady-state (periodic) solutions, but computations should describe transient waves close to these periodic solutions.

Keywords: $\rightarrow$ surface waves due to resonant excitation $\rightarrow$ steady-state wave motions $\rightarrow$
asymptotic modal ordering[8] $\rightarrow$

6.1. Asymptotic steady-state solution.

We consider a mobile rectangular rigid tank filled partly by an inviscid incompressible fluid. The **scaled mean water depth** is $h := h/l$ (l is the tank breadth). The origin of the coordinate system is in the mean free surface at the centre plane of the tank. The equation $z = f(x,t)$ determines the perturbed free surface $\Sigma(t)$. The fluid domain is

$$Q(t) = \{(x,z) : -h < z < f(x,t); -1/2 < x < 1/2\}. \tag{28}$$

The complete (to within a constant) orthogonal system of functions $\{f_i(x)\}$ should satisfy the volume conservation condition

$$\int_{-1/2}^{1/2} f_i(x)dx = 0. \tag{29}$$

The theory of steady–state solutions of nonlinear sloshing problem in a rectangular tank was created by Faltinsen (1974) based on Moiseev's method. The constructed asymptotic modal theory makes it possible to re-derive the main relations of this theory. For lateral excitation we pose $\boldsymbol{v}_0$ as $(-\epsilon\sigma\sin(\sigma t), 0, 0)$, set $\omega = \psi \equiv 0$, where ϵ is the dimensionless forcing amplitude. In this paper, the periodic solutions are obtained under the Moiseyev asymptotic detuning. A simple analysis shows that the asymptotic solutions can be treated in terms of the modal representation leading to

(30) $$R_1 \sim \beta_1 = O(\epsilon^{1/3});\quad R_2 \sim \beta_2 = O(\epsilon^{2/3});\quad R_i \sim \beta_i \leq O(\epsilon),\ i \geq 3.$$

The relationship (30) is a suitable **asymptotic modal ordering**[8], which is fulfilled at least for a class of fluid motions. No warranty that this relationship hold true for arbitrary sloshing, but we *can assume this.*

THUS, we realise the formula

modal ordering[8] + **pseudospectral method**[7] = **multimodal method**[10]

6.2. **Asymptotic modal systems.** Now, for general motions of the tank, pursuing $O(\epsilon)$ in the pseudospectral system gets the following system of ordinary differential equations describing modal oscillations of a fluid in rectangular tank performing arbitrary small magnitude motions (keeping terms up to third order in nonlinear equations)

(31) $$(\ddot\beta_1 + \sigma_1^2\beta_1) + d_1(\ddot\beta_1\beta_2 + \dot\beta_1\dot\beta_2) + d_2(\ddot\beta_1\beta_1^2 + \dot\beta_1^2\beta_1) + d_3\ddot\beta_2\beta_1 + {} + P_1(\dot v_{0x} - S_1\dot\omega - g\psi) + Q_1\dot v_{0z}\beta_1 = 0,$$

(32) $$(\ddot\beta_2 + \sigma_2^2\beta_2) + d_4\ddot\beta_1\beta_1 + d_5\dot\beta_1^2 + Q_2\dot v_{0z}\beta_2 = 0,$$

(33) $$(\ddot\beta_3 + \sigma_3^2\beta_3) + d_6\ddot\beta_1\beta_2 + d_7\ddot\beta_1\beta_1^2 + d_8\ddot\beta_2\beta_1 + d_9\dot\beta_1\dot\beta_2 + d_{10}\dot\beta_1^2\beta_1 + {} + P_3(\dot v_{0x} - S_3\dot\omega - g\psi) + Q_3\dot v_{0z}\beta_3 = 0.$$

The linear equations describing higher modes are

(34) $$\ddot\beta_i + \sigma_i^2\beta_i + P_i(\dot v_{0x} - S_i\dot\omega - g\psi) + Q_i\dot v_{0z}\beta_i = 0,\quad i \geq 4.$$

Here v_{0x} and v_{0z} are projections of translational velocity onto axes of Oxz, $\omega(t)$ is the value of angular velocity of coordinate system $Oxyz$ with respect to $O'x'y'z'$.

The generalised coordinates are often called **modal functions**.

The introduced coefficients are calculated by formulae

(35) $$\sigma_i^2 = 2giE_i;\quad P_{2i-1} = -\frac{8E_{2i-1}}{\pi^2(2i-1)},\quad P_{2i} = 0;\quad Q_i = 2iE_i,$$
$$S_i = \frac{2}{\pi i}\tanh\left(i\pi h/2\right),\quad i \geq 1,$$

where σ_i is natural frequency of mode i. Further,

$$d_1 = 2\frac{E_0}{E_1} + E_1,\quad d_2 = 2E_0\left(-1 + \frac{4E_0}{E_1E_2}\right),\quad d_3 = -2\frac{E_0}{E_2} + E_1,$$
$$d_4 = -4\frac{E_0}{E_1} + 2E_2,\quad d_5 = E_2 - 2\frac{E_0E_2}{E_1^2} - \frac{4E_0}{E_1},\quad d_6 = 3E_3 - \frac{6E_0}{E_1},$$
$$d_7 = 9E_0 - 12\frac{E_0E_4}{E_1} - 6E_3E_4 + 24\frac{E_0^2}{E_1E_2} + 3\frac{E_0E_3}{E_1},\quad d_8 = -6\frac{E_0}{E_2} + 3E_3,$$
$$d_9 = -6\frac{E_0}{E_1} - 6\frac{E_0}{E_2} - 6\frac{E_0E_3}{E_1E_2} + 3\frac{E_3E_1}{E_2},$$
$$d_{10} = 18E_0 - 2E_4\frac{12E_0 + 6E_1E_3}{E_1} + \frac{72E_0^2}{E_1E_2} + 12E_0\left(\frac{E_3}{E_1} - \frac{E_1}{E_2}\right),$$

with

$$E_0 = \frac{\pi^2}{8}; \quad E_i = \frac{\pi}{2}\tanh(\pi i h).$$

The first two nonlinear equations couple β_1 with β_2 and do not depend on β_3. The third mode component is excited by rigid body motions and the first and the second modes of sloshing. The second mode response becomes infinite if the excitation has frequency content at the natural frequency for the second mode. Similarly can be said about the third and higher modes. The first mode will be finite if it is excited at the natural frequency for the first mode. This is caused by nonlinear effects and will become more evident in the next section on steady–state response.

Abilities of this and others modal systems in modelling the sloshing will be presented in Part 2.

FOR NOTES:

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Appendix A. Derivation of the Lagrange equation

By substituting (12) into (9) the Bateman-Luke functional L takes the following form

$$L = -\rho \int_{Q(t)} \Big[\dot{\boldsymbol{v}}_0 \cdot \boldsymbol{r} + \frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\boldsymbol{\Omega}) + \frac{1}{2}\nabla(\boldsymbol{\omega}\cdot\boldsymbol{\Omega})\cdot\nabla(\boldsymbol{\omega}\cdot\boldsymbol{\Omega}) - \boldsymbol{\omega}\cdot(\boldsymbol{r}\times\nabla(\boldsymbol{\omega}\cdot\boldsymbol{\Omega})) -$$

$$- \frac{1}{2}v_0^2 - \boldsymbol{\omega}\cdot(\boldsymbol{r}\times\boldsymbol{v}_0) - \boldsymbol{\omega}\cdot(\boldsymbol{r}\times\nabla\varphi) + \nabla(\boldsymbol{\omega}\cdot\boldsymbol{\Omega})\cdot\nabla\varphi \Big] dQ + L_r, \tag{36}$$

where

$$L_r = -\rho \int_{Q(t)} \left[\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + U \right] dQ. \tag{37}$$

The two last integrand terms in square brackets of (36) cancel each other from Green's formula, i.e.

$$\int_{Q(t)} \nabla(\boldsymbol{\omega}\cdot\boldsymbol{\Omega})\cdot\nabla\varphi - (\boldsymbol{\omega}\times\boldsymbol{r})\cdot\nabla\varphi dQ = \int_{S(t)+\Sigma(t)} \left(\frac{\partial(\boldsymbol{\omega}\cdot\boldsymbol{\Omega})}{\partial\nu} - (\boldsymbol{\omega}\times\boldsymbol{r})\cdot\boldsymbol{\nu} \right)\varphi dS = 0.$$

We introduce also the quadratic symmetric inertia tensor $\boldsymbol{J}^1$ with components J^1_{ij} defined by equality

$$-\rho\int_{Q(t)} \left(\frac{1}{2}\nabla(\boldsymbol{\omega}\cdot\boldsymbol{\Omega})\cdot\nabla(\boldsymbol{\omega}\cdot\boldsymbol{\Omega}) - \boldsymbol{\omega}\cdot(\boldsymbol{r}\times\nabla(\boldsymbol{\omega}\cdot\boldsymbol{\Omega})) \right) dQ =$$

$$= -\frac{1}{2}\omega_1^2 J^1_{11} - \frac{1}{2}\omega_2^2 J^1_{22} - \frac{1}{2}\omega_3^2 J^1_{33} - \omega_1\omega_2 J^1_{12} - \omega_1\omega_3 J^1_{13} - \omega_2\omega_3 J^1_{23}.$$

These components J^1_{ij} can be calculated by Green's formula, i.e.

$$J^1_{11} = \rho\int_{Q(t)} \left(y\frac{\partial\Omega_1}{\partial z} - z\frac{\partial\Omega_1}{\partial y} \right) dQ = \rho\int_{S(t)+\Sigma(t)} \Omega_1\frac{\partial\Omega_1}{\partial\nu} dS,$$

$$J^1_{22} = \rho\int_{Q(t)} \left(z\frac{\partial\Omega_2}{\partial x} - x\frac{\partial\Omega_2}{\partial z} \right) dQ = \rho\int_{S(t)+\Sigma(t)} \Omega_2\frac{\partial\Omega_2}{\partial\nu} dS,$$

$$J^1_{33} = \rho\int_{Q(t)} \left(x\frac{\partial\Omega_3}{\partial y} - y\frac{\partial\Omega_3}{\partial x} \right) dQ = \rho\int_{S(t)+\Sigma(t)} \Omega_3\frac{\partial\Omega_3}{\partial\nu} dS,$$

$$J^1_{12} = J^1_{21} = \rho\int_{Q(t)} \left(z\frac{\partial\Omega_1}{\partial x} - x\frac{\partial\Omega_1}{\partial z} \right) dQ = \rho\int_{Q(t)} \left(y\frac{\partial\Omega_2}{\partial z} - z\frac{\partial\Omega_2}{\partial y} \right) dQ =$$

$$= \rho\int_{S(t)+\Sigma(t)} \Omega_1\frac{\partial\Omega_2}{\partial\nu} dS = \rho\int_{S(t)+\Sigma(t)} \Omega_2\frac{\partial\Omega_1}{\partial\nu} dS,$$

$$J^1_{13} = J^1_{31} = \rho\int_{Q(t)} \left(x\frac{\partial\Omega_1}{\partial y} - y\frac{\partial\Omega_1}{\partial x} \right) dQ = \rho\int_{Q(t)} \left(y\frac{\partial\Omega_3}{\partial z} - z\frac{\partial\Omega_3}{\partial y} \right) dQ =$$

$$= \rho\int_{S(t)+\Sigma(t)} \Omega_1\frac{\partial\Omega_3}{\partial\nu} dS = \rho\int_{S(t)+\Sigma(t)} \Omega_3\frac{\partial\Omega_1}{\partial\nu} dS,$$

$$J^1_{23} = J^1_{32} = \rho\int_{Q(t)} \left(x\frac{\partial\Omega_2}{\partial y} - y\frac{\partial\Omega_2}{\partial x} \right) dQ = \rho\int_{Q(t)} \left(z\frac{\partial\Omega_3}{\partial x} - x\frac{\partial\Omega_3}{\partial z} \right) dQ =$$

$$= \rho\int_{S(t)+\Sigma(t)} \Omega_2\frac{\partial\Omega_3}{\partial\nu} dS = \rho\int_{S(t)+\Sigma(t)} \Omega_3\frac{\partial\Omega_2}{\partial\nu} dS. \tag{38}$$

The Lagrangian L (36) can be rewritten as

$$L = -[\dot{v}_{01}l_1 + \dot{v}_{02}l_2 + \dot{v}_{03}l_3 + \dot{\omega}_1 l_{1\omega} + \dot{\omega}_2 l_{2\omega} + \dot{\omega}_3 l_{3\omega} + \omega_1 l_{1\omega t} + \omega_2 l_{2\omega t} + \omega_3 l_{3\omega t} -$$

$$-\frac{1}{2}(\omega_1^2 J^1_{11} + \omega_2^2 J^1_{22} + \omega_3^2 J^1_{33}) - \omega_1\omega_2 J^1_{12} - \omega_1\omega_3 J^1_{13} - \omega_2\omega_3 J^1_{23} - \frac{1}{2}m_1(v_{01}^2 + v_{02}^2 + v_{03}^2) +$$

$$+(\omega_2 v_{03} - \omega_3 v_{02})l_1 + (\omega_3 v_{01} - \omega_1 v_{03})l_2 + (\omega_1 v_{02} - \omega_2 v_{01})l_3] + L_r, \tag{39}$$

where

$$m_1 = \rho \int_{Q(t)} dQ,\ l_{k\omega} = \rho \int_{Q(t)} \Omega_k dQ,\ l_{k\omega t} = \rho \int_{Q(t)} \frac{\partial \Omega_k}{\partial t} dQ,$$

$$l_1 = \rho \int_{Q(t)} x dQ,\ l_2 = \rho \int_{Q(t)} y dQ,\ l_3 = \rho \int_{Q(t)} z dQ. \tag{40}$$

The vectors $\boldsymbol{l} = \{l_k\}, \boldsymbol{l}_\omega = \{l_{k\omega}\}, \boldsymbol{l}_{\omega t} = \{l_{k\omega t}\}$ depend only on $\beta_i(t)$ and $\dot\beta_i(t)$.

It follows that

$$L_r = -\rho \int_{Q(t)} \left[\sum_{n=1} \dot R_n \varphi_n + \frac{1}{2} \sum_n \sum_k R_n R_k (\nabla\varphi_n, \nabla\varphi_k) + U \right] dQ =$$

$$= - \left[\sum_n A_n \dot R_n + \frac{1}{2} \sum_n \sum_k A_{nk} R_n R_k - g_1 l_1 - g_2 l_2 - g_3 l_3 - m_1 \boldsymbol{g} \cdot \boldsymbol{r}_0' \right], \tag{41}$$

where

$$A_n = \rho \int_{Q(t)} \varphi_n dQ, \quad A_{nk} = A_{kn} = \rho \int_{Q(t)} (\nabla\varphi_n, \nabla\varphi_k) dQ = \rho \int_{\Sigma(t)+S(t)} \varphi_n \frac{\partial \varphi_k}{\partial \nu} dS \tag{42}$$

are functions of $\beta_i(t)$.

The Lagrangian L is originally a function of two independent variables $f(x,y,z,t)$ and $\Phi(x,y,z,t)$. The independent variables become the time–varying functions $\beta_i(t), i \geq 1$ and $R_n(t), n \geq 1$ after substituting into the Lagrangian. The variations of the functional (6) by $\beta_i(t)$ and $R_n(t)$ for given $\boldsymbol{v}_0(t)$ and $\boldsymbol{\omega}(t)$ are

$$\delta W = \int_{t_1}^{t_2} \left[\sum_n A_n \delta \dot R_n + \sum_n \sum_k A_{nk} R_k \delta R_n + \sum_i \left(\sum_n \dot R_n \frac{\partial A_n}{\partial \beta_i} + \right.\right.$$

$$+\omega_1 \frac{\partial l_{1\omega t}}{\partial \beta_i} + \omega_2 \frac{\partial l_{2\omega t}}{\partial \beta_i} + \omega_3 \frac{\partial l_{3\omega t}}{\partial \beta_i} + \frac{1}{2} \sum_n \sum_k R_n R_k \frac{\partial A_{nk}}{\partial \beta_i} + \dot\omega_1 \frac{\partial l_{1\omega}}{\partial \beta_i} + \dot\omega_2 \frac{\partial l_{2\omega}}{\partial \beta_i} + \dot\omega_3 \frac{\partial l_{3\omega}}{\partial \beta_i} +$$

$$+(\dot v_{01} - g_1 + \omega_2 v_{03} - \omega_3 v_{02}) \frac{\partial l_1}{\partial \beta_i} + (\dot v_{02} - g_2 + \omega_3 v_{01} - \omega_1 v_{03}) \frac{\partial l_2}{\partial \beta_i} +$$

$$+(\dot v_{03} - g_3 + \omega_1 v_{02} - \omega_2 v_{01}) \frac{\partial l_3}{\partial \beta_i} - \frac{1}{2}\omega_1^2 \frac{\partial J_{11}^1}{\partial \beta_i} - \frac{1}{2}\omega_2^2 \frac{\partial J_{22}^1}{\partial \beta_i} - \frac{1}{2}\omega_3^2 \frac{\partial J_{33}^1}{\partial \beta_i} - \omega_1\omega_2 \frac{\partial J_{12}^1}{\partial \beta_i} -$$

$$\left.\left. - \omega_1\omega_3 \frac{\partial J_{13}^1}{\partial \beta_i} - \omega_2\omega_3 \frac{\partial J_{23}^1}{\partial \beta_i} \right) \delta\beta_i + \left(\omega_1 \frac{\partial l_{1\omega t}}{\partial \dot\beta_i} + \omega_2 \frac{\partial l_{2\omega t}}{\partial \dot\beta_i} + \omega_3 \frac{\partial l_{3\omega t}}{\partial \dot\beta_i} \right) \delta\dot\beta_i \right] dt = 0. \tag{43}$$

Finally, we obtain (15) (16), where the values $\partial l_k / \partial \beta_i$ are given by

$$\frac{\partial l_3}{\partial \beta_i} = \rho \int_{\Sigma_0} f_i^2 dS \beta_i = \lambda_{i3}\beta_i; \quad \frac{\partial l_2}{\partial \beta_i} = \rho \int_{\Sigma_0} y f_i dS = \lambda_{i2}; \quad \frac{\partial l_1}{\partial \beta_i} = \rho \int_{\Sigma_0} x f_i dS = \lambda_{i1}. \tag{44}$$