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ASYMPTOTIC NONLINEAR MULTIMODAL MODELING OF LIQUID SLOSHING IN AN UPRIGHT CIRCULAR CYLINDRICAL TANK. I. MODAL EQUATIONS

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Combining the Lukovsky–Miles variational method and the Narimanov–Moiseev asymptotics, we deduce a nonlinear modal system describing the resonant liquid sloshing in an upright circular cylindrical tank. The sloshing occurs due to a small-amplitude periodic or an almost-periodic excitation with forcing frequency close to the lowest natural sloshing frequency. In contrast to the existing nonlinear modal systems based on the Narimanov–Moiseev asymptotic intermodal relations, the derived modal equations (i) contain all necessary (infinitely many) generalized coordinates of the second and third orders and (ii) include exclusively nonzero hydrodynamic coefficients, for which (iii) fairly simple computational formulas are found. As a consequence, the modal equations can be used in *analytical studies* of nonlinear sloshing phenomena, which will be demonstrated in the forthcoming Part II.

1. Introduction

Taking account of liquid-sloshing loads is of importance for designing the engineering constructions carrying a liquid cargo. Problems related to the safety, reliability, stability, and control analysis of liquid-containing structures were extensively studied in the context of aircraft and spacecraft applications, for cargo tanks of automotive vehicles, for offshore platforms, and for the seismic analysis of elevated water tanks. The studies require the comprehensive quantitative and qualitative examination of the coupled fluid–structure dynamics and its modeling and simulation on the real-time scale. The liquid-sloshing response becomes most severe under resonance conditions when the carrying structure oscillates with a frequency close to the lowest natural sloshing frequency. These resonant free-surface motions are strongly nonlinear and must be described by solving an evolution free-boundary problem in which both instant shapes of the free surface $\Sigma(t)$ and the velocity field in the liquid domain $Q(t)$ are the unknowns.

Under certain circumstances, one can distinguish *three* different approaches to solving the *nonlinear liquid-sloshing* problem. The *first approach* is computational fluid dynamics (CFD). A broad variety of numerical methods exist, which can be divided into two subclasses comprising the potential-flow method, the Navier–Stokes method, and, sometimes, their hybrids, typically based on the domain decomposition method [8, 39]. The CFD methods are universal, accurate, and efficient, especially on the short-time scale when the focus is on transient waves. Their drawback is that they are, generally speaking, computational time consuming, especially for three-dimensional problems. Furthermore, their applicability can be rather limited when the task is to simulate and classify the so-called steady-state wave regimes occurring on the long-time scale and, therefore, requiring long-time simulations with different initial scenarios.

The *second approach* is purely analytical. It was developed for studying the steady-state (periodic) solution expected for prescribed small-amplitude harmonic tank excitations. The analytical approach employs asymptotic methods created by great mathematicians of the 19th century in the theory of nonlinear ocean waves [3]. An extension of these methods to nonlinear resonant sloshing in closed basins is often attributed to the pioneering paper

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by Moiseev [32]. More mathematical details on constructing the steady-state asymptotic solution by solving a series of recurrence boundary-value problems and deducing the so-called secularity condition (necessary solvability condition) that couples the forcing frequency and the dominant response amplitude can be found in [5, 36, 37, 11, 12, 8]. The asymptotic steady-state solution technique changes with the mean liquid depth. For finite liquid depths, the Taylor expansion of nonlinear free-surface conditions with respect to the mean (unperturbed) free surface leads to cubic algebraic secularity equations and yields the so-called third-order Moiseev asymptotics causing the dominant response amplitude to be of the order $O(\epsilon^{1/3})$ where ϵ is the nondimensional forcing amplitude. The asymptotic solution methods are generally inapplicable to transient waves and to modeling the fluid–structure interaction. Furthermore, the asymptotic steady-state solution is only valid within a matching forcing-frequency range and for a relatively small forcing amplitude. Forcing frequencies out of this range and an increase in the forcing amplitude can lead to the so-called internal (secondary, combinatory) resonance, thereby causing a failure of the Moiseev intermodal asymptotic relations (Moiseev asymptotics).

The *third approach* is associated with nonlinear multimodal methods whose application assumes an ideal liquid with irrotational flow and no overturning and breaking waves allowed. In this paper, we follow this approach to deduce an infinite-dimensional system of nonlinear asymptotic-type modal equations for sloshing in an upright circular cylindrical tank by combining the Lukovsky–Miles variational multimodal method [19, 20, 29] and the Narimanov–Moiseev intermodal asymptotic relations [34, 35, 32], which may follow from the second approach or may simply be postulated. The specific features of this combined variational–asymptotic version of multimodal methods and its difference from the others are outlined in [27, 11]. Readers interested in using other versions of multimodal methods for liquid sloshing in an upright cylindrical tank are referred to [34, 4, 35, 20, 10] (the Narimanov modal-type perturbation method), [33, 38, 15] (a completely nonlinear (nonasymptotic) multimodal (Perko-type) method), [16, 17, 18] (a combination of the Lagrange variational principle and the perturbation method), [20, 7, 8] (a combination of the Bateman–Luke variational principle and the perturbation method), and references therein.

The sloshing of an ideal liquid with irrotational flow introduces a nonlinear free-boundary problem with two unknowns, which are the instant free-surface shapes and the velocity potential. According to the *concept* of multimodal methods for liquid sloshing in tanks with upright walls, the instant free-surface shapes should be defined by a Fourier-type solution with unknown time-dependent coefficients $q_i(t)$ (*generalized coordinates*) of $f_i(y,z) = \varphi_i(0,y,z)$, i.e.,

$$x = f(y,z,t) = \sum_i q_i(t) f_i(y,z), \tag{1}$$

where $\varphi_i(x,y,z)$ are the so-called natural sloshing modes. An analogous Fourier-type solution involving $\varphi_i(x,y,z)$ is used for the velocity potential. Even though there exist different versions of nonlinear multimodal methods, all of them are developed to transform the original problem into an infinite-dimensional system of nonlinear ordinary differential (modal) equations that couple the generalized coordinates q_i. However, since the derivation of nonlinear multimodal equations is a difficult mathematical task, each version proposes a proper analytical way pursuing modal equations of desirable structure.

Except for Perko-type methods, the derivation requires a postulation of asymptotic relations between the generalized coordinates q_i assuming a small set of dominant generalized coordinates. Neglecting the nonlinear terms in q_i whose order is higher than the forcing input signal associated with the highest-order term $(O(\epsilon))$, one obtains the so-called *asymptotic nonlinear modal equations*. The asymptotic modal equations help one to avoid physically-unrealistic higher harmonics, which give a negligible contribution to the liquid response but may cause the stiffness of the differential (modal) equations as is observed in the case of Perko-type simulations [15].

The Narimanov–Moiseev asymptotics [34, 35, 32] is the most often accepted system of asymptotic relations used for the derivation of asymptotic nonlinear modal systems. They follow from the Moiseev asymptotic solution (second approach) or can be postulated as in the classical works of Narimanov [34, 35]. The application of asymp-

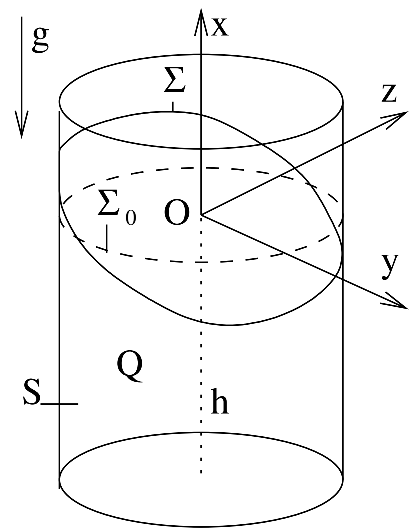

Fig. 1. Sketch of an upright circular cylindrical tank and adopted nomenclature. The geometrical and physical characteristics are scaled in our analysis by the dimensional tank radius R_0 so that, e.g., h in the figure is the ratio between the mean liquid depth and R_0.

totic relations reduces the problem to the calculation of the *nonzero* hydrodynamic coefficients of polynomial-type quantities in the asymptotic modal equations. Usually, the number of nonzero coefficients is rather limited. Taking account of *analytical studies* based on the asymptotic nonlinear modal equations, i.e., the consideration of nonlinear sloshing as an object of either *applied mathematics* or *theoretical mechanics*, strongly requires one to exclude the zeros from the modal equations as well as to provide simple and compact formulas for the nonzero hydrodynamic coefficients. Of course, the use of the asymptotic nonlinear multimodal equations as a computational tool [18, 7], i.e., the consideration of the multimodal method as a CFD approach, does not require the analytical extraction of the zeros.

For a rectangular cross-section, the Narimanov–Moiseev asymptotic relations lead, due to trigonometric relations between the natural sloshing modes f_i, to a nine-dimensional nonlinear asymptotic modal system. This system was explicitly deduced and analytically studied in [6, 8]. Other cylindrical tank shapes yield, generally speaking, infinite-dimensional asymptotic multimodal systems. The latter is also true for the case of a circular cross-section. The literature presents various analytically given finite-dimensional asymptotic modal systems [20–22], but these systems couple only a few of second-order and third-order generalized coordinates. To the best authors' knowledge, the present paper for the first time deduces the infinite-dimensional Narimanov–Moiseev asymptotic modal system *in an analytical form* for a circle-based tank, providing modal equations that (i) contain all necessary generalized coordinates of the second and third order that follow from the Narimanov–Moiseev asymptotic intermodal relations and (ii) include exclusively nonzero hydrodynamic coefficients, for which (iii) fairly simple computational formulas are found. In the forthcoming Part II, we will present analytical studies of nonlinear resonant sloshing based on the derived modal system and compare the analytical results with experiments.

2. Statement of the Problem

2.1. Free-Boundary Problem. We consider an upright circular cylindrical tank partially filled with an inviscid incompressible liquid with irrotational flow. Figure 1 introduces the basic notation. No overturning waves are assumed. The time-dependent liquid domain $Q(t)$ is bounded by the free surface $\Sigma(t)$ and wetted tank surface $S(t)$. The mean liquid depth is equal to h. In what follows, in all mathematical expressions, we assume that the liquid characteristics, including h and the gravity acceleration g, are scaled by the tank radius R_0 so that the theoretical radius of the tank is nondimensional and equal to 1.

The liquid motions are considered in the tank-fixed coordinate system $Oxyz$ whose origin is located at the center of the mean free surface Σ_0. The Ox-axis is superposed with the tank symmetry axis. For brevity, we concentrate on the case where the tank moves translatorily with velocity $\mathbf{v}_0(t)$ relative to an absolute Earth-fixed coordinate system $Ox'y'z'$. Small-magnitude angular forcing terms can also be taken into account by assuming that these terms are of the highest order in the Narimanov–Moiseev asymptotic ordering. The latter procedure was extensively discussed in [8].

The absolute velocity potential $\Phi(x, y, z, t)$ and free surface $\Sigma(t)$ are the two unknowns that should be found from the following nonlinear free-boundary problem:

$$\nabla^2 \Phi = 0, \quad \mathbf{r} \in Q(t), \tag{2}$$

$$\frac{\partial \Phi}{\partial \boldsymbol{\nu}} = \mathbf{v}_0 \cdot \boldsymbol{\nu} + \frac{f_t}{\sqrt{1 + |\nabla f|^2}}, \quad \mathbf{r} \in \Sigma, \tag{3}$$

$$\frac{\partial \Phi}{\partial \boldsymbol{\nu}} = \mathbf{v}_0 \cdot \boldsymbol{\nu}, \quad \mathbf{r} \in S(t), \tag{4}$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} |\nabla \Phi|^2 - \nabla \Phi \cdot \mathbf{v}_0 + U = 0, \quad \mathbf{r} \in \Sigma(t). \tag{5}$$

Here, $\boldsymbol{\nu}$ is the outer normal vector, $U = (\mathbf{g} \cdot \mathbf{r})$ is the gravity potential with $\mathbf{r} = (x, y, z)$, $\mathbf{g} = (-g, 0, 0)$ is the gravity acceleration vector, and $x = f(y, z, t)$ is the free-surface equation.

For the free-boundary problem (2), the typical initial conditions (at $t = 0$) define the initial liquid shape and velocity field and take the form

$$f(y, z, 0) = \xi_0(y, z), \quad \Phi(x, y, z, 0) = \Phi_0(x, y, z), \quad \mathbf{r} \in Q(0). \tag{6}$$

2.2. Variational Formulation. In 1976, Miles [29] and Lukovsky [19] independently proposed to use the Bateman–Luke variational principle for the derivation of nonlinear modal systems. The history of the Bateman–Luke principle dates back to 1908, when Hargreaves [13] noted that the pressure integral can play the role of the Lagrangian in variational formulations of diverse hydrodynamic problems. The canonical formulation of this principle for an incompressible ideal liquid was given by Bateman [1]. Later, this formulation was generalized by Luke [28] for ocean waves and by Lukovsky [20] for liquid sloshing in a tank performing arbitrary spatial motions. The Bateman–Luke principle for a compressible fluid can be found in [2, 24–26].

According to Lukovsky [20], the Bateman–Luke principle for (2)–(5) can be formulated as follows:

The free-boundary problem (2)–(5) is associated with the necessary extrema of the action

$$W = \int_{t_1}^{t_2} L \, dt, \tag{7}$$

where the Lagrangian L is defined by the pressure integral

$$L = \int\limits_{Q(t)} (p - p_o)\, dQ = -\rho \int\limits_{Q(t)} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} |\nabla \Phi|^2 - \nabla \Phi \cdot \mathbf{v}_0 + U \right] dQ \tag{8}$$

and the trial functions satisfy the conditions

$$\delta\Phi(x, y, z, t_1) = \delta\Phi(x, y, z, t_2) = \delta f(y, z, t_1) = \delta f(y, z, t_2) = 0. \tag{9}$$

3. Nonlinear Multimodal Modeling

The nonlinear multimodal modeling is based on the Fourier-type solution (1) in which $q_i(t)$ are treated as *generalized coordinates* of the considered hydromechanical system. Here, $f_i(x, y)$ is a complete orthogonal system of functions satisfying the volume conservation condition

$$\int\limits_{\Sigma_0} f_i(x, y) dx dy = 0.$$

In addition, one should introduce the Fourier-type representation of the velocity potential

$$\Phi(x, y, z, t) = \mathbf{v}_0 \cdot \mathbf{r} + \sum_n Q_n(t)\, \varphi_n(x, y, z), \tag{10}$$

where the complete set of harmonic functions $\varphi_n(x, y, z)$ satisfies both the Laplace equation in the whole tank domain and the zero Neumann boundary conditions on the wetted body surface.

Normally, φ_n and $f_n(x, y) = \varphi_n(x, y, 0)$ are the eigenfunctions (natural sloshing modes) of the spectral boundary-value problem

$$\nabla^2 \varphi_n = 0, \quad \vec{r} \in Q_0, \qquad \frac{\partial \varphi_n}{\partial \nu} = \kappa_n \varphi_n, \quad \vec{r} \in \Sigma_0, \qquad \frac{\partial \varphi_n}{\partial \nu} = 0, \quad \vec{r} \in S_0, \tag{11}$$

where Q_0 is the mean liquid domain and S_0 is the mean wetted tank surface. The natural sloshing frequencies are defined by the eigenvalues κ_n via $\sigma_n = \sqrt{g\kappa_n}$.

The aim of multimodal modeling is to deduce a system of ordinary differential equations (modal equations) with respect to the generalized coordinates $q_i(t)$. There are different analytical schemes (multimodal methods) for doing this; they are briefly outlined in Introduction. According to [19, 20, 29], the derivation can use the Bateman–Luke principle instead of the free-boundary problem (2).

3.1. Lukovsky–Miles Variational Method. Lukovsky [20] showed how to use the Bateman–Luke principle for the derivation of nonlinear modal equations that couple $q_i(t)$ and $Q_n(t)$. The result for translatory tank excitations is the following infinite-dimensional system of nonlinear ordinary differential equations:

$$\sum_i \frac{\partial A_n}{\partial q_i} \dot{q}_i - \sum_k A_{nk} Q_k = 0, \quad n = 1, 2, \ldots, \tag{12}$$

$$\sum_n \frac{\partial A_n}{\partial q_i}\dot{Q}_n + \frac{1}{2}\sum_{n,k}\frac{\partial A_{nk}}{\partial q_i}Q_n Q_k + \sum_{j=1}^{3}(\dot{v}_{Oj} - g_j)\frac{\partial l_j}{\partial q_i} = 0, \quad i = 1, 2, \ldots, \tag{13}$$

where

$$\frac{\partial l_1}{\partial q_i} = \int_{\Sigma_0} f_i^2 \, dS \, q_i, \quad \frac{\partial l_2}{\partial q_i} = \int_{\Sigma_0} y f_i \, dS, \quad \frac{\partial l_3}{\partial q_i} = \int_{\Sigma_0} z f_i \, dS, \tag{14}$$

$\mathbf{g} = (g_1, g_2, g_3) = (-g, 0, 0)$, and

$$A_n = \int_{Q(t)} \varphi_n \, dQ, \quad A_{nk} = \int_{Q(t)} \nabla\varphi_n \cdot \nabla\varphi_k \, dQ. \tag{15}$$

The nonlinear modal equations (12) and (13) are a complete analog of the original free-boundary problem. Direct simulations using the modal equations (12), (13) imply the so-called Perko numerical method (see Introduction). Lukovsky and Timokha [27] pointed out that these simulations can be stiff for resonant sloshing, and, therefore, a certain numerical time integration becomes numerically unstable. This physically unrealistic stiffness is caused by the amplification of higher harmonics, which, in reality, are highly damped due to different dissipative mechanisms. An alternative is to introduce an asymptotic relationship between generalized coordinates and thereby exclude ("filter") the unrealistically high harmonics.

3.2. Narimanov–Moiseev Asymptotic Intermodal Relations for a Circular-Base Tank. For a circular-base tank, the modal solution (1) can be rewritten in the cylindrical coordinate system as follows:

$$x = f(\xi, \eta, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(r_{m,n}(t)\sin(m\eta) + p_{m,n}(t)\cos(m\eta)) f_{mn}(\xi), \tag{16}$$

where

$$f_{mn} = \frac{J_m(k_{m,n}\xi)}{J_m(k_{m,n})},$$

$J_m(\cdot)$ is the Bessel function, and $J'_m(k_{m,n}) = 0$. The zeros of the last equation define the eigenvalues $\kappa_{m,n}$ and natural sloshing frequencies $\sigma_{m,n}$ by the formulas

$$\kappa_{m,n} = k_{m,n}\tanh(k_{m,n}h) \qquad \text{and} \qquad \sigma_{m,n}^2 = g\kappa_{m,n}. \tag{17}$$

The generalized coordinates q_i, as well as $r_{m,n}$ and $p_{m,n}$, are nondimensional (scaled by the tank radius), and one can introduce asymptotic relations between them. When the forcing frequency σ is close to the lowest natural frequency $\sigma_{1,1}$ associated with the two generalized coordinates $p_{1,1}(t)$ and $r_{1,1}(t)$, the Narimanov–Moiseev asymptotics [32, 20, 30, 31, 27] requires the asymptotic relation

$$p_{1,1} \sim r_{1,1} = O(\epsilon^{1/3}), \tag{18}$$

where $\epsilon \ll 1$ implies the nondimensional forcing magnitude.

Postulating (18) and using trigonometric algebra with respect to the angular coordinate η, one can establish the second- and third-order generalized coordinates:

$$p_{0,n} \sim p_{2,n} \sim r_{2,n} = O(\epsilon^{2/3}), \qquad p_{3,n} \sim r_{3,n} = O(\epsilon), \quad n = 1, 2, \ldots,$$
$$p_{1,m} \sim r_{1,m} = O(\epsilon), \quad m = 2, 3, \ldots . \tag{19}$$

The remaining generalized coordinates are of order $o(\epsilon)$ and can be neglected in our nonlinear asymptotic multimodal analysis.

4. Nonlinear Asymptotic Multimodal Equations

The most general analytical scheme for combining the Lukovsky–Miles variational method and the Narimanov–Moiseev asymptotics is described in [27]. With regard for (18)–(19), the scheme suggests the following steps:

1. Using the Taylor expansion, one should find polynomial expressions (in terms of nondimensional generalized coordinates q_i) for $\partial A_n/\partial q_k$ and A_{nk} keeping terms up to the order $O(\epsilon^{2/3})$ and $\partial A_{nk}/\partial q_i$ keeping terms up to the order $O(\epsilon^{1/3})$.

2. One should find the asymptotic solution $Q_i = F(q_k, \dot{q}_k)$ from the modal equations (12) by substituting the previously found asymptotic expressions for $\partial A_n/\partial q_k$ and A_{nk}. This solution should neglect the terms of order $o(\epsilon)$.

3. One should substitute the expressions $Q_i = F(q_k, \dot{q}_k)$ from the previous step into modal equations (13) and keep terms up to the order $O(\epsilon)$. This will give the desirable asymptotic modal equations.

The scheme was *completely* realized only for upright cylindrical tanks of rectangular shape. By generalizing [20], the paper [23] showed that this scheme can also be applied to a circular cylindrical tank. It is implemented in the present paper to obtain the required *analytically* given asymptotic nonlinear modal equations.

The implementation of the analytical scheme with $3N$ $(N \to \infty)$ generalized coordinates of the second order and $4N$ generalized coordinates of the third order leads to the nonlinear asymptotic modal equations including the following two differential equations for the lowest-order generalized coordinates $p_{1,1}$ and $r_{1,1}$:

$$\begin{aligned}
\mu_{1,1}\left[\ddot{p}_{1,1} + \sigma_{1,1}^2 p_{1,1}\right] &+ p_{1,1}\sum_{n=1}^{N} d_{0,n}^{(2)}\ddot{p}_{0,n} + \sum_{n=1}^{N} d_{0,n}^{(3)}\left(\ddot{p}_{1,1}p_{0,n} + \dot{p}_{1,1}\dot{p}_{0,n}\right)\\
&+ d_1\left(p_{1,1}^2\ddot{p}_{1,1} + p_{1,1}\dot{p}_{1,1}^2 + r_{1,1}p_{1,1}\ddot{r}_{1,1} + p_{1,1}\dot{r}_{1,1}^2\right)\\
&+ d_2\left(r_{1,1}^2\ddot{p}_{1,1} + 2r_{1,1}\dot{r}_{1,1}\dot{p}_{1,1} - r_{1,1}p_{1,1}\ddot{r}_{1,1} - 2p_{1,1}\dot{r}_{1,1}^2\right)\\
&+ \sum_{n=1}^{N} d_{2,n}^{(3)}\left(\ddot{p}_{1,1}p_{2,n} + \ddot{r}_{1,1}r_{2,n} + \dot{p}_{1,1}\dot{p}_{2,n} + \dot{r}_{1,1}\dot{r}_{2,n}\right)\\
&+ \sum_{n=1}^{N} d_{2,n}^{(2)}\left(p_{1,1}\ddot{p}_{2,n} + r_{1,1}\ddot{r}_{2,n}\right) = -\frac{\mu_{1,1}\kappa_{1,1}}{k_{1,1}^2 - 1}\dot{v}_{01},
\end{aligned} \tag{20a}$$

$$\mu_{1,1}\left[\ddot r_{1,1}+\sigma_{1,1}^2 r_{1,1}\right]+r_{1,1}\sum_{n=1}^{N} d_{0,n}^{(2)}\ddot p_{0,n}+\sum_{n=1}^{N} d_{0,n}^{(3)}\left(\ddot r_{1,1}p_{0,n}+\dot r_{1,1}\dot p_{0,n}\right)$$
$$+d_1\left(r_{1,1}^2\ddot r_{1,1}+r_{1,1}\dot r_{1,1}^2+r_{1,1}p_{1,1}\ddot p_{1,1}+r_{1,1}\dot p_{1,1}^2\right)$$
$$+d_2\left(p_{1,1}^2\ddot r_{1,1}+2p_{1,1}\dot r_{1,1}\dot p_{1,1}r_{1,1}p_{1,1}\ddot p_{1,1}-2r_{1,1}\dot p_{1,1}^2\right)$$
$$+\sum_{n=1}^{N} d_{2,n}^{(3)}\left(\ddot p_{1,1}r_{2,n}-\ddot r_{1,1}p_{2,n}+\dot p_{1,1}\dot r_{2,n}-\dot r_{1,1}\dot p_{2,n}\right)$$
$$+\sum_{n=1}^{N} d_{2,n}^{(2)}\left(p_{1,1}\ddot r_{2,n}-r_{1,1}\ddot p_{2,n}\right)=-\frac{\mu_{1,1}\kappa_{1,1}}{k_{1,1}^2-1}\dot v_{02}. \tag{20b}$$

These equations contain both the lowest- and second-order generalized coordinates, but the third-order generalized coordinates are absent here. The notation for $k_{m,n}$ (the roots of the equation $J'_m(k_{m,n})=0$), $\kappa_{m,n}$ [see Eq. (17)], $\sigma_{m,n}$ (natural sloshing frequency), and the translatory velocity components $\dot v_{01}(t)$ and $\dot v_{02}(t)$ has been explained before. The nondimensional hydrodynamic coefficients of the nonlinear terms are defined at the end of this section.

The differential equations for finding the second-order generalized coordinates $p_{0,n}$, $p_{2,n}$, and $r_{2,n}$ take the form

$$2\mu_{0,n}\left[\ddot p_{0,n}+\sigma_{0,n}^2 p_{0,n}\right]+d_{0,n}^{(1)}\left(\dot p_{1,1}^2+\dot r_{1,1}^2\right)+d_{0,n}^{(2)}\left(\ddot p_{1,1}p_{1,1}+\ddot r_{1,1}r_{1,1}\right)=0, \tag{21a}$$

$$\mu_{2,n}\left[\ddot p_{2,n}+\sigma_{2,n}^2 p_{2,n}\right]+d_{2,n}^{(1)}\left(\dot p_{1,1}^2-\dot r_{1,1}^2\right)+d_{2,n}^{(2)}\left(\ddot p_{1,1}p_{1,1}-\ddot r_{1,1}r_{1,1}\right)=0, \tag{21b}$$

$$\mu_{2,n}\left[\ddot r_{2,n}+\sigma_{2,n}^2 r_{2,n}\right]+2d_{2,n}^{(1)}\dot r_{1,1}\dot p_{1,1}+d_{2,n}^{(2)}\left(\ddot p_{1,1}r_{1,1}+\ddot r_{1,1}p_{1,1}\right)=0. \tag{21c}$$

Here, $n=1,\dots,N$, i.e., there are $3N$ ordinary differential equations for these generalized coordinates. Note that Eqs. (21) contain $p_{1,1}$ and $r_{1,1}$ defined by (21), and, therefore, one can say that the first- and second-order generalized coordinates are nonlinearly coupled by our modal equations. However, the third-order generalized coordinates $p_{3,n}$ and $r_{3,n}$ are absent from (21). The equations for these generalized coordinates take the form

$$\mu_{3,n}\left[\ddot r_{3,n}+\sigma_{3,n}^2 r_{3,n}\right]+d_3\left(r_{1,1}\dot p_{1,1}^2+2p_{1,1}\dot p_{1,1}\dot r_{1,1}-r_{1,1}\dot r_{1,1}^2\right)$$
$$+d_4\left(p_{1,1}^2\ddot r_{1,1}+2r_{1,1}p_{1,1}\ddot p_{1,1}-r_{1,1}^2\ddot r_{1,1}\right)+\sum_{n=1}^{N} d_{3,n}^{(1)}\left(\dot p_{1,1}\dot r_{2,n}+\dot r_{1,1}\dot p_{2,n}\right)$$
$$+\sum_{n=1}^{N} d_{3,n}^{(2)}\left(p_{1,1}\ddot r_{2,n}+r_{1,1}\ddot p_{2,n}\right)+\sum_{n=1}^{N} d_{3,n}^{(3)}\left(\ddot p_{1,1}r_{2,n}+\ddot r_{1,1}p_{2,n}\right)=0, \tag{22a}$$

$$\mu_{3,n}\left[\ddot{p}_{3,n}+\sigma_{3,n}^2 p_{3,n}\right]+d_3\left(p_{1,1}\dot{p}_{1,1}^2-2r_{1,1}\dot{p}_{1,1}\dot{r}_{1,1}-p_{1,1}\dot{r}_{1,1}^2\right)$$

$$+d_4\left(p_{1,1}^2\ddot{p}_{1,1}-2p_{1,1}r_{1,1}\ddot{r}_{1,1}-r_{1,1}^2\ddot{p}_{1,1}\right)+\sum_{n=1}^{N}d_{3,n}^{(1)}\left(\dot{p}_{1,1}\dot{p}_{2,n}-\dot{r}_{1,1}\dot{r}_{2,n}\right)$$

$$+\sum_{n=1}^{N}d_{3,n}^{(2)}\left(p_{1,1}\ddot{p}_{2,n}-r_{1,1}\ddot{r}_{2,n}\right)+\sum_{n=1}^{N}d_{3,n}^{(3)}\left(\ddot{p}_{1,1}p_{2,n}-\ddot{r}_{1,1}r_{2,n}\right)=0, \tag{22b}$$

$$\mu_{1,n}\left[\ddot{r}_{1,n}+\sigma_{1,n}^2 r_{1,n}\right]+d_5\left(\ddot{r}_{1,1}r_{1,1}^2+r_{1,1}p_{1,1}\ddot{p}_{1,1}\right)$$

$$+d_6\left(r_{1,1}\dot{r}_{1,1}^2+r_{1,1}\dot{p}_{1,1}^2\right)+d_7\left(\ddot{r}_{1,1}p_{1,1}^2-r_{1,1}p_{1,1}\ddot{p}_{1,1}\right)$$

$$+d_8\left(\dot{r}_{1,1}\dot{p}_{1,1}p_{1,1}-r_{1,1}\dot{p}_{1,1}^2\right)+\sum_{n=1}^{N}d_{4,n}^{(1)}\left(\dot{p}_{1,1}\dot{r}_{2,n}-\dot{r}_{1,1}\dot{p}_{2,n}\right)$$

$$+\sum_{n=1}^{N}d_{4,n}^{(2)}\left(p_{1,1}\ddot{r}_{2,n}-r_{1,1}\ddot{p}_{2,n}\right)+\sum_{n=1}^{N}d_{4,n}^{(3)}\left(\ddot{p}_{1,1}r_{2,n}-\ddot{r}_{1,1}p_{2,n}\right)$$

$$+\dot{r}_{1,1}\sum_{n=1}^{N}d_{5,n}^{(1)}\dot{p}_{0,n}+r_{1,1}\sum_{n=1}^{N}d_{5,n}^{(2)}\ddot{p}_{0,n}+\ddot{r}_{1,1}\sum_{n=1}^{N}d_{5,n}^{(3)}p_{0,n}=-\frac{\mu_{1,n}\kappa_{1,n}}{k_{1,n}^2-1}\dot{v}_{02}, \tag{22c}$$

$$\mu_{1,n}\left[\ddot{p}_{1,n}+\sigma_{1,n}^2 p_{1,n}\right]+d_5\left(\ddot{p}_{1,1}p_{1,1}^2+r_{1,1}p_{1,1}\ddot{r}_{1,1}\right)$$

$$+d_6\left(p_{1,1}\dot{p}_{1,1}^2+p_{1,1}\dot{r}_{1,1}^2\right)+d_7\left(\ddot{p}_{1,1}r_{1,1}^2-r_{1,1}p_{1,1}\ddot{r}_{1,1}\right)$$

$$+d_8\left(\dot{r}_{1,1}\dot{p}_{1,1}p_{1,1}-p_{1,1}\dot{r}_{1,1}^2\right)+\sum_{n=1}^{N}d_{4,n}^{(1)}\left(\dot{p}_{1,1}\dot{p}_{2,n}+\dot{r}_{1,1}\dot{r}_{2,n}\right)$$

$$+\sum_{n=1}^{N}d_{4,n}^{(2)}\left(r_{1,1}\ddot{r}_{2,n}+p_{1,1}\ddot{p}_{2,n}\right)+\sum_{n=1}^{N}d_{4,n}^{(3)}\left(\ddot{p}_{1,1}p_{2,n}+\ddot{r}_{1,1}r_{2,n}\right)$$

$$+\dot{p}_{1,1}\sum_{n=1}^{N}d_{5,n}^{(1)}\dot{p}_{0,n}+p_{1,1}\sum_{n=1}^{N}d_{5,n}^{(2)}\ddot{p}_{0,n}+\ddot{p}_{1,1}\sum_{n=1}^{N}d_{5,n}^{(3)}p_{0,n}=-\frac{\mu_{1,n}\kappa_{1,n}}{k_{1,n}^2-1}\dot{v}_{01}, \tag{22d}$$

where $n=1,\ldots,N$. Equations (22) are linear in $p_{3,n}$ and $r_{3,n}$ and contain nonlinear quantities in terms of the first- and second-order generalized coordinates.

The most *important result* of the present paper is that the *nonzero* hydrodynamic coefficients in (20)–(22) can be effectively calculated by the following fairly simple formulas:

$$d_{0,n}^{(1)} = d_{0,n}^{(2)} - \frac{d_{0,n}^{(3)}}{2}, \quad d_{0,n}^{(2)} = \frac{\pi}{2}\left[2 - \frac{k_{0,n}^2}{\kappa_{0,n}\kappa_{1,1}}\right] j_{(0,n)(1,1)^2},$$

$$d_{0,n}^{(3)} = \pi\left[j_{(0,n)(1,1)^2} - \frac{1}{\kappa_{1,1}^2}\left(j_{(0,n)}^{(1,1)^2} + i_{(0,n)(1,1)^2}\right)\right], \quad d_{2,n}^{(1)} = d_{2,n}^{(2)} - \frac{d_{2,n}^{(3)}}{2},$$

$$d_{2,n}^{(2)} = \frac{\pi}{2}\left[j_{(2,n)(1,1)^2} - \frac{1}{\kappa_{2,n}\kappa_{1,1}}\left(j_{(1,1)}^{(2,n)(1,1)} + 2i_{(2,n)(1,1)^2}\right)\right],$$

$$d_{2,n}^{(3)} = \frac{\pi}{2}\left[j_{(2,n)(1,1)^2} - \frac{1}{\kappa_{1,1}^2}\left(j_{(2,n)}^{(1,1)^2} - i_{(2,n)(1,1)^2}\right)\right],$$

$$d_1 = \frac{\pi}{2\kappa_{1,1}}\left[\frac{k_{0,1}^4\left(j_{(0,1)(1,1)^2}\right)^2}{4\kappa_{01}\kappa_{1,1}}\frac{1}{j_{(0,1)^2}} + i_{(1,1)^4} - j_{(1,1)^2}^{(1,1)^2}\right] + d_2,$$

$$d_2 = \frac{\pi}{4\kappa_{1,1}}\left[\frac{\left(j_{(1,1)}^{(1,1)(2,1)} + 2i_{(2,1)(1,1)^2}\right)^2}{\kappa_{1,1}\kappa_{2,1}}\frac{1}{j_{(2,1)^2}} - 3i_{(1,1)^4} - j_{(1,1)^2}^{(1,1)^2}\right],$$

$$d_3 = \frac{\pi}{4\kappa_{1,1}}\frac{\left(j_{(1,1)}^{(1,1)(2,1)} + 2i_{(2,1)(1,1)^2}\right)}{\kappa_{1,1}\kappa_{2,1}}\frac{\left(2i_{(1,1)(2,1)(3,1)} - j_{(3,1)}^{(1,1)(2,1)}\right)}{j_{(2,1)^2}} + \frac{\pi}{4\kappa_{1,1}}\left[j_{(1,1)(3,1)}^{(1,1)^2} - i_{(1,1)^3(3,1)}\right] + 2d_4,$$

$$d_4 = \frac{\pi}{4\kappa_{1,1}}\frac{\left(j_{(1,1)}^{(1,1)(2,1)} + 2i_{(2,1)(1,1)^2}\right)}{\kappa_{2,1}\kappa_{3,1}}\frac{\left(6i_{(1,1)(2,1)(3,1)} + j_{(1,1)}^{(2,1)(3,1)}\right)}{j_{(2,1)^2}} - \frac{\pi}{4\kappa_{1,1}}\left[\frac{\left(\kappa_{1,1} + \kappa_{3,1}\right)}{2\kappa_{3,1}}\left(3i_{(1,1)^3(3,1)} + j_{(1,1)^2}^{(1,1)(3,1)}\right)\right],$$

$$d_{3,n}^{(1)} = d_{3,n}^{(2)} + d_{3,n}^{(3)} - \frac{\pi}{2}\left[j_{(1,1)(2,n)(3,1)} - \frac{j_{(3,1)}^{(1,1)(2,n)} - 2i_{(1,1)(2,n)(3,1)}}{\kappa_{1,1}\kappa_{2,n}}\right],$$

$$d_{3,n}^{(2)} = \frac{\pi}{2}\left[j_{(1,1)(2,n)(3,1)} - \frac{j_{(1,1)}^{(2,n)(3,1)} + 6i_{(1,1)(2,n)(3,1)}}{\kappa_{2,n}\kappa_{3,1}} \right],$$

$$d_{3,n}^{(3)} = \frac{\pi}{2}\left[j_{(1,1)(2,n)(3,1)} - \frac{j_{(2,n)}^{(1,1)(3,1)} + 3i_{(1,1)(2,n)(3,1)}}{\kappa_{1,1}\kappa_{3,1}} \right],$$

$$d_{4,n}^{(1)} = d_{4,n}^{(2)} - d_{3,n}^{(3)} - \frac{\pi}{2}\left[j_{(1,1)(2,n)(1,2)} - \frac{j_{(1,2)}^{(1,1)(2,n)} + 2i_{(1,1)(2,n)(1,2)}}{\kappa_{1,1}\kappa_{2,n}} \right],$$

$$d_{4,n}^{(2)} = \frac{\pi}{2}\left[j_{(1,1)(2,n)(1,2)} - \frac{j_{(1,1)}^{(2,n)(1,1)} + 2i_{(1,1)(2,n)(1,2)}}{\kappa_{2,n}\kappa_{1,2}} \right],$$

$$d_{4,n}^{(3)} = \frac{\pi}{2}\left[j_{(1,1)(2,n)(1,2)} - \frac{j_{(2,n)}^{(1,1)(1,2)} - i_{(1,1)(2,n)(1,2)}}{\kappa_{1,1}\kappa_{1,2}} \right],$$

$$d_{5,n}^{(1)} = d_{5,n}^{(2)} + d_{5,n}^{(3)} - \pi\left[j_{(0,n)(1,1)(1,2)} - \frac{j_{(1,2)}^{(0,n)(1,1)}}{\kappa_{1,1}\kappa_{0,n}} \right],$$

$$d_{5,n}^{(2)} = \pi\left[j_{(0,n)(1,1)(1,2)} - \frac{j_{(1,1)}^{(0,n)(1,2)}}{\kappa_{0,n}\kappa_{1,2}} \right],$$

$$d_{5,n}^{(3)} = \pi\left[j_{(0,n)(1,1)(1,2)} - \frac{j_{(0,n)}^{(1,1)(1,2)} + i_{(0,n)(1,1)(1,2)}}{\kappa_{1,1}\kappa_{1,2}} \right]$$

where, by definition,

$$j_{(a,b)}^{(c,d)} = \int \xi \left(\prod f_{a,b}(k_{a,b}\xi)\right)\left(\prod \frac{d}{d\xi} f_{c,d}(k_{c,d}\xi)\right) d\xi,$$

$$i_{(a,b)}^{(c,d)} = \int \frac{1}{\xi} \left(\prod f_{a,b}(k_{a,b}\xi)\right)\left(\prod \frac{d}{d\xi} f_{c,d}(k_{c,d}\xi)\right) d\xi$$

and there are special indexing rules for i and j exemplified by the formula

$$j_{(0,2)(2,2)(1,1)}^{(1,2)(0,1)(1,2)} = j_{(0,2)(1,1)(2,2)}^{(0,1)(1,2)^2}$$

$$= \int_0^1 \xi \left(f_{0,2}(k_{0,2}\xi) f_{1,1}(k_{1,1}\xi) f_{2,2}(k_{2,2}\xi) \right) \left(\frac{d}{d\xi} f_{0,1}(k_{0,1}\xi) \left(\frac{d}{d\xi} f_{1,2}(k_{1,2}\xi) \right)^2 \right) d\xi.$$

Equations (22c) and (22d) contain the coefficients d_5, d_6, d_7, and d_8, which are computed by the formulas

$$d_5 = -\frac{0.51201}{h_{1,1}} - \frac{0.16879}{h_{1,2}} + \frac{0.50224}{h_{1,2}h_{1,1}h_{0,1}} + \frac{0.17969}{h_{1,2}h_{2,1}h_{1,1}},$$

$$d_6 = -\frac{1.34899}{h_{1,1}} - \frac{0.3376}{h_{1,2}} + \frac{1.00448}{h_{1,2}h_{1,1}h_{0,1}} + \frac{0.37908}{h_{1,1}^2 h_{0,1}} + \frac{0.35938}{h_{1,2}h_{2,1}h_{1,1}} + \frac{0.23782}{h_{1,1}^2 h_{2,1}},$$

$$d_7 = -\frac{0.11748}{h_{1,1}} - \frac{0.00307}{h_{1,2}} + \frac{0.17969}{h_{1,2}h_{2,1}h_{1,1}},$$

$$d_8 = -\frac{0.68799}{h_{1,1}} - \frac{0.17186}{h_{1,2}} + \frac{0.37908}{h_{1,2}^2 h_{0,1}} + \frac{0.35938}{h_{1,2}h_{2,1}h_{1,1}} + \frac{0.50224}{h_{1,2}h_{1,1}h_{0,1}},$$

where $h_{m,n} = \tanh(k_{m,n}h)$ depends on the nondimensional depth.

It may be important for applications that the modal equations (20)–(22) can be rewritten in the following matrix form:

$$Q(\vec{q})\ddot{\vec{q}} + C\vec{q} + \vec{\Psi}(\vec{q};\dot{\vec{q}}) = V, \tag{23}$$

where

$$\vec{q} = (q_{1,1}; q_{1,2}; \ldots; q_{1,n}; q_{2,1}; q_{2,2}; \ldots; q_{2,n}; \ldots; q_{7,1}; q_{7,2}; \ldots; q_{7,n})^T.$$

5. Conclusions

Taking into account analytical studies of nonlinear resonant sloshing in an upright circular-base tank, the present paper analytically deduces a system of nonlinear ordinary differential equations (modal system) that facilitates the approximate modeling of sloshing phenomena. The derivation uses the Narimanov–Moiseev intermodal asymptotic relations, which cause, for this tank shape, an infinite number of generalized coordinates coupled by the system. In contrast to the existing *analytically given* modal equations, the derived system (i) contains all necessary generalized coordinates and (ii) includes exclusively nonzero hydrodynamic coefficients, for which (iii) fairly simple computational formulas are found. The use of the modal equations in *analytical studies* of nonlinear resonant sloshing will be demonstrated in the forthcoming Part II.

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