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# MODAL MODELLING OF THE NONLINEAR RESONANT FLUID SLOSHING IN A RECTANGULAR TANK I: A SINGLE-DOMINANT MODEL

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Steady-state nonlinear resonant fluid sloshing in moving tanks can be characterized by a finite set of natural modes (leading modes). Approximate solutions of the original free boundary value problem can be found from a system of nonlinear ordinary differential equations (modal system) coupling time-dependent amplitudes of these leading modes. The derivation of the modal systems combines projective and asymptotic methods. The work presents an extensive survey of the literature and examines bifurcations of periodic (steady-state) solutions of the single-dominant modal system based on Moiseyev asymptotic ordering. It describes two-dimensional resonant fluid sloshing in a rectangular tank due to its horizontal harmonic oscillations. The periodicity condition yields a two-point boundary value problem that allows both asymptotic and numerical treatments within the framework of the perturbed bifurcation theory. A secondary bifurcation that is found for response curves shows the flaws of the single-dominant modal modelling. Modifications of the single-dominant model should account for internal (secondary) resonance in the mechanical system leading to amplification of higher modes. Part II will consider modal systems which take into account the internal resonance.

*Keywords*: Fluid sloshing; modal systems; resonant waves; bifurcations; steady solutions.

AMS Subject Classification: 76B15, 34B60, 37M20

## 1. Introduction

Sloshing in moving tanks has many applications in the automotive, aerospace and shipbuilding industries. It is typically studied as part of the overall structural dynamics of carrying objects (their stability, safety and control). In the past few years a new technique in studying sloshing motions has been developed. It is known as *modal modelling* and has many specific advantages in comparison with the traditional numerical tools. Modal modelling reduces the sloshing free boundary problem to a system of nonlinear ordinary differential equations (modal system) which can easily be incorporated into dynamic equations of the whole object. The modal systems have a small dimension and provide CPU-efficient simulations of both transient and steady-state wave regimes. Numerous scientific publications in hydrodynamical, computational and physical journals have investigated special modal systems. However, a disadvantage of the modal modelling is that each fixed modal system is typically applicable only for a limited set of physical parameter, tank shapes and fluid fillings (depths). Due to a lack of mathematical work establishing the links and differences between distinct nonlinear modal systems and their solutions, a quantification of these sets of parameters is usually based on comparisons with experimental data. This paper makes the first steps to a more rigorous mathematical analysis of the modal systems as well as to a mathematical understanding of their applicability. The investigations focus on periodic (steady-state) solutions of asymptotic and pseudo-spectral multidimensional nonlinear modal systems derived by Faltinsen *et al.*,[25] La Rocca *et al.*[40,41] and Faltinsen & Timokha[18,20] for describing two-dimensional fluid sloshing due to horizontal harmonic excitations of a rectangular tank.

Bearing in mind interests of mathematically-oriented readers who are not familiar with this class of applied mathematical problems, we begin Part I with an extensive survey of the literature on modal modelling in fluid sloshing problems. The remaining sections concern the asymptotic modal system developed by Faltinsen *et al.*[25] based on the Moiseyev[55] single-dominant modal ordering. In Sec. 3, this system is subjected to periodicity conditions and reformulated. The corresponding periodic solutions are governed by the perturbed operator equation $T(B,\lambda,\tau)=0$, where $B(t)=\{\beta_i(t),\ i\geq 1\}$ is a $(2\pi)$-periodic function, $\lambda\in(-\infty,1)$ is the bifurcation parameter implying a relation between the actual and lowest natural frequencies, and $0\leq\tau\ll 1$ is the perturbation parameter (non-dimensional excitation amplitude). The unperturbed operator equation $T(B,\lambda,0)=0$ which determines free nonlinear standing waves, is examined in Sec. 4. Section 5 considers the perturbed bifurcations with $\tau>0$ (steady-state forced waves). The asymptotic analysis of primary perturbed bifurcations in Secs. 4 and 5 is consistent with well-known results on steady-state solutions by Moiseyev,[55] Ockendon & Ockendon[61] and Faltinsen,[16] as $\lambda,\tau\to 0$. However, in contrast to those results, the periodic solutions are characterized by secondary bifurcations which have not been previously detected. The modal system also has infinite periodic solutions for some $\lambda$ bounded away from zero. These are associated with the internal (secondary) resonance. Part II will concentrate on multidimensional modal systems capturing the amplifications of higher modes due to this resonance phenomenon.

## 2. Nonlinear Modal Systems

Two typical physical assumptions characterizing nonlinear modal modelling are that the fluid is incompressible with irrotational flows and that there are no overturning waves. The applicability of the inviscid fluid model has been validated for smooth tanks (without internal structures, baffles) and non-shallow fluid depths. In addition to these well-accepted conditions we also consider the fluid sloshing under earth-based conditions. This causes large Bond numbers which implies that the surface tensions together with related conditions on the moving contact curve between the surface and the wall can be neglected (see, Billingham[4]). Although the modal approach can be developed for tanks of arbitrary shape (see surveys by Lukovsky,[44] Limarchenko & Yasinsky,[43] Lukovsky & Timokha[47,48] and Gavrilyuk *et al.*[32]), we restrict ourselves to vertical cylindrical tanks. Special emphasis is also placed on tanks with a rectangular base.

### 2.1. *Original free boundary problem*

The free boundary problem, which describes an inviscid fluid sloshing in a rigid vertical cylindrical tank, takes the following "canonical form" (see its derivation by Moiseyev & Rumyantsev[56] and Narimanov *et al.*[60])

$$\begin{aligned}
&\Delta\Phi = 0 \quad \text{in } Q(t); \quad \int_{Q(t)} dQ = V = \text{const.}\\
&\frac{\partial\Phi}{\partial \mathbf{n}} = \mathbf{v}_0\cdot\mathbf{n} + \boldsymbol{\omega}\cdot(\mathbf{r}\times\mathbf{n}) \quad \text{on } S(t),\\
&\frac{\partial\Phi}{\partial \mathbf{n}} = \mathbf{v}_0\cdot\mathbf{n} + \boldsymbol{\omega}\cdot(\mathbf{r}\times\mathbf{n}) - \frac{f_t}{\sqrt{1+(\nabla f)^2}} \quad \text{on } \Sigma(t),\\
&\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - \nabla\Phi\cdot(\mathbf{v}_0 + \boldsymbol{\omega}\times\mathbf{r}) + U = 0 \quad \text{on } \Sigma(t),
\end{aligned} \tag{2.1}$$

where the two unknowns are the time-varying domain $Q(t)$ of the constant volume $V$ and the velocity potential $\Phi(x,y,z,t)$ which is defined inside the $Q(t)$. The domain $Q(t)$ is confined to the free boundary $\Sigma(t)$ determined by the equation $z = f(x,y,t)$ and the wetted internal tank surface $S(t)$, $U(x,y,z,t)$ is the gravity potential, $\mathbf{n}$ is the outer normal. The tank motions are described by the pair of known time-dependent vectors $\mathbf{v}_O(t) = \dot{\eta}(t)$ and $\omega(t) = \dot{\Psi}(t)$ representing instantaneous translatory and angular velocities of the mobile Cartesian coordinate system $Oxyz$ relative to an absolute coordinate system $O'x'y'z'$ (a dot over $\Psi$ and $\eta$ denotes the time-derivative). The $Oxyz$-coordinate system is rigidly fixed with the tank so that the hydrostatic free surface $\Sigma_0$ lies in the $Oxy$-plane. Since any absolute position vector $\mathbf{P}'(t) = (x', y', z')$ can be decomposed into the sum of $\mathbf{P}'_O(t) = \overrightarrow{\mathrm{O'O}}$ and the relative position vector $\mathbf{P} = (x,y,z)$, the gravity potential $U(x,y,z,t) = -\mathbf{g}\cdot\mathbf{P}'$, $\mathbf{P}' = \mathbf{P}'_O + \mathbf{P}$, where $g$ is the gravity acceleration vector. The nomenclature is illustrated in Fig. 1.

Problem (2.1) requires either initial or (for the periodic vector-functions $\mathbf{v}_0(\mathrm{t})$ and $\omega(t)$) periodicity conditions. Physically, solutions of the initial value problem determine transient waves which are caused by combined effects of both ($\mathbf{v}_O(\mathrm{t})$, $\omega(\mathrm{t})$) and the initial perturbations of $f$, but the periodicity conditions imply the so-called steady-state waves.

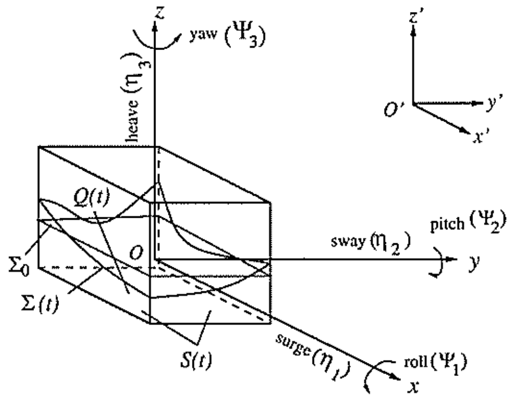


Fig. 1. Sketch of the fluid sloshing in a moving rectangularly-based tank with $\mathbf{v}_0 = \frac{d}{dt}(\eta_1, \eta_2, \eta_3)^{\mathrm{T}}; \omega = \frac{d}{dt}(\Psi_1, \Psi_2, \Psi_3)^{\mathrm{T}}$; $\Sigma(t) : z = f(x, y, t)$ and $\Sigma_0 : z = 0$.

The mathematical validation of the initial value and periodic free boundary value problems is still an open question (even in the two-dimensional formulation). Being familiar with both former Soviet and Western literature, the present authors were not able to find rigorous mathematical results for the periodic problem. There is only a limited set of mathematical papers that report local existence theorems for the initial-boundary value problems. Almost all of these results are documented by Shinbrot,[73] Reeder & Shinbrot,[68,69] Ovsyannikov *et al.*,[64] Lukovsky,[46] Pawell & Günther[65] and Lukovsky & Timokha.[47]

Since there is not a proven theory on how to formulate initial and periodic conditions for the free boundary value problem (2.1), the majority of papers utilise conditions from the linear sloshing theory (Feschenko *et al.*[28]). In this case, the Cauchy conditions at $t = t_0$ are

$$f(x, y, t_0) = \tilde{f}_0(x, y), \quad \frac{\partial f}{\partial t}(x, y, t_0) = \tilde{f}_1(x, y), \tag{2.2}$$

and the periodicity conditions are

$$f(x, y, t + T) = f(x, y, t), \quad \frac{\partial f}{\partial t}(x, y, t + T) = \frac{\partial f}{\partial t}(x, y, t), \tag{2.3}$$

where, in the most general case, $T$ should be determined together with the $T$-periodic solutions of (2.1).

### 2.2. *Numerical methods and nonlinear modal modelling*

The free boundary value problem (2.1) appears in many "real world applications" including satellite, missile and tanker ship dynamics, safety of petroleum storage tanks on coastal terminals, microgravity technology etc. Reviews of experimental and theoretical studies dealing with fluid sloshing have been given by Abramson,[1] Abramson *et al.*,[2] Narimanov *et al.*,[60] Mikishev,[49] Mikishev & Rabinovich,[50] Ibrahim *et al.*,[39] Lukovsky & Timokha[47] and Faltinsen & Timokha.[25,18,19]

Due to the mathematical complexity of this problem, current studies of (2.1) have mainly concentrated on numerical methods (Computational Fluid Dynamics, CFD) and approximate analytical theories that are derivable from (2.1). Differences and advantages of various CFD methods have been discussed in comparative surveys by Solaas,[74] Moan & Berge,[54] Cariou and Casella,[11] Gerrits,[33] Ibrahim *et al.*,[39] Celebi & Akyildiz,[12] Sames *et al.*,[70] Aliabadi *et al.*[3] and Frandsen.[29] Investigations by Solaas,[74] Faltinsen & Rognebakke[17] and a recent conference presentation by Landrini *et al.*[42] (Smoothed particle hydromechanics methods in two-dimensional sloshing problems) have also focused on comparisons between analytically-oriented and pure numerical approaches. Under certain circumstances, almost all of the analytically-oriented approaches can be treated in terms of the modal modelling based on a generalised Fourier representation of $f(x, y, t)$ and $\Phi(x, y, z, t)$, where the time-dependent Fourier coefficients are interpreted as generalised coordinates (*modal functions*). This generic Fourier-like insight into analytically-oriented methods has been developed for cylindrical tanks by Miles,[52,53] Lukovsky,[45,46] Lukovsky & Timokha,[47] La Rocca *et al.*[41] and Faltinsen *et al.*[25]

Let us consider an open vertical cylindrical tank $Q_T = \Sigma_0 \times [-h, +\infty]$, where $\Sigma_0$ is the unperturbed planar free surface perpendicular to the $Oz$-axis and

$$f(x, y, t) = \sum_{i=1}^{\infty} \beta_i(t) f_i(x, y). \tag{2.4}$$

Here, $\{f_i(x, y)\}$ is a complete orthogonal system of functions (in $L_2(\Sigma_0)$) satisfying the volume conservation condition $\int_{\Sigma_0} f_i(x, y)dxdy = 0$. Along with (2.4), the modal methods introduce a Kirchhoff-type solution of the velocity potential, i.e.

$$\Phi(x, y, z, t) = \mathrm{v}_0 \cdot \mathrm{r} + \boldsymbol{\omega} \cdot \boldsymbol{\Omega} + \phi(x, y, z, t), \tag{2.5}$$

where the harmonic vector-function $\boldsymbol{\Omega} = \boldsymbol{\Omega}(x, y, z, t)$ is the Stokes–Zhukovsky potential determined from the Neumann boundary value problem

$$\begin{gathered} \Delta\boldsymbol{\Omega} = 0 \quad \text{in } Q(t); \quad \left.\frac{\partial \Omega_1}{\partial \mathrm{n}}\right|_{S(t)+\Sigma(t)} = yn_3 - zn_2, \\ \left.\frac{\partial \Omega_2}{\partial \mathrm{n}}\right|_{S(t)+\Sigma(t)} = zn_1 - xn_3; \quad \frac{\partial \Omega_3}{\partial \mathrm{n}}|_{S(t)+\Sigma(t)} = xn_2 - yn_1 \end{gathered} \tag{2.6}$$

and $\phi(x, y, z, t)$ is expanded into the Fourier series

$$\phi(x, y, z, t) = \sum_{n=1}^{\infty} R_n(t)\phi_n(x, y, z). \tag{2.7}$$

Due to the representation (2.4), the domain $Q(t)$ depends exclusively on the modal functions $\{\beta_i(t),\ i \geq 1\}$. Solutions of (2.6) as well as the term $\boldsymbol{\omega} \cdot \boldsymbol{\Omega}$ in (2.5) are therefore parametric functions of $\{\beta_i(t)\}$. This implies that inserting (2.4)–(2.7) into (2.1) (or into its variational formulation) and implementing a projective procedure (various suitable methods are reported by Perko,[66] Miles,[51] Lukovsky,[45,46] Lukovsky & Timokha,[47] Chern *et al.*,[13] Faltinsen *et al.*,[25] Ferrant & Touze,[27] Shankar & Kidambi[71] and La Rocca *et al.*[41]) lead to an infinite-dimensional system of nonlinear ordinary differential equations coupling the unknowns $\beta_i(t)$ and $R_i(t)$, $i \geq 1$. This system can be considered as a generic *nonlinear modal system* which is mathematically equivalent to (2.1).

**Remark 2.1.** Application of modal solution (2.7) is only possible if the set $\{\phi_i(x, y, z),\ i \geq 1\}$ is a harmonic basis in $H^1(Q(t))$ which satisfies a zero-Neumann boundary condition on the bottom and on the vertical walls for arbitrary $t$. The practical choice of $\{\phi_n\}$ and $\{f_i\}$ consists in using the solutions of the spectral problem for linear natural sloshing (natural modes)

$$\Delta\phi_n = 0 \text{ in } Q_0; \quad \frac{\partial \phi_n}{\partial \nu} = 0 \text{ on } S_0; \quad \frac{\partial \phi_n}{\partial \nu} = \kappa_n \phi_n \text{ on } \Sigma_0; \quad \int_{\Sigma_0} \phi_n dS = 0 \quad (2.8)$$

($Q_0$ is the hydrostatic, unperturbed fluid volume confined to $\Sigma_0$ and the wetted tank surface $S_0$) and setting

$$f_n(x, y) = \phi_n(x, y, 0), \quad n \geq 1. \quad (2.9)$$

Spectral theorems by Eastham[15] and Feschenko *et al.*[28] justify the completeness of $\{f_i\}$ from (2.9) in $L_2(\Sigma_0)$ satisfying $\int_{\Sigma_0} f_i dS = 0$. These theorems guarantee the completeness of $\{\phi_n\}$ in the hydrostatic fluid domain $Q_0$ (for harmonic functions in $H^1(Q_0)$ satisfying zero-Neumann boundary condition on $S_0$). However, as discussed by Faltinsen & Timokha[19] and Timokha,[77] even if eigenmodes from (2.8) are analytically expandable over $\Sigma_0$, these are not complete for arbitrary admissible instantaneous $Q(t)$. Timokha[77] gives a numerical treatment of this incompleteness.

**Remark 2.2.** For cylindrical tanks, the truncated series (2.4) based on (2.8) and (2.9) always determines the instantaneous free surfaces $\Sigma(t)$ which are smooth and perpendicular to the vertical walls. This fact can easily be established by separating spatial variables $z$ and $(x, y)$ in (2.8). Although modal methods are uniformly valid for sloshing problems, the 'non-perpendicular contact' between $\Sigma(t)$ and the walls indicates weak convergence of the Fourier series (2.4). Taking into account the experimental observations of $\Sigma(t)$-shapes for shallow fluid depths (see, photos by Faltinsen & Timokha[20]) and for angular tank motions, limitations of the small-dimensional nonlinear modal system utilising solutions of (2.8) are possible.

### 2.3. *Pseudo-spectral and asymptotic modal systems*

The practical accuracy of the truncated Fourier series (2.4) based on (2.7) (*modal approximations*) can be explained by an amazing physical fact: The nonlinear fluid sloshing with a finite depth is almost always characterised by wave patterns assembled from a limited set of natural modes (*leading modes*, $\{\beta_L\}$), while the remaining (higher) modes supply only a minor contribution to the fluid flow. The set of leading modes increases for decreasing fluid depths and large amplitude external forcing (see the survey and the detailed physical analysis by Ockendon *et al.*,[63] Faltinsen &

Timokha[20,18] and Hill[38]). In many cases, either asymptotic (Moiseyev,[55] Faltinsen & Timokha[18] etc.) or experimental (Mikishev,[49] Bogomaz & Sirota,[5] La Rocca *et al.*[41] etc.) analysis makes it possible to detect (estimate) the set of the leading modes and to consider naively truncated infinite-dimensional modal systems in $\{\beta_i(t)\}$ and $\{R_i(t)\}$. Such a technique has been proposed by Moore & Perko[57] and Perko.[66] It is often called the *pseudo-spectral* or *Perko-like* approach. Some modifications of the pseudo-spectral approach have been documented by La Rocca *et al.*[41], Shankar & Kidambi,[71] Ferrant & Le Touze[27] and Chern *et al.*[13] However, pseudo-spectral nonlinear modal systems are not often used in applied mathematical studies. Numerous analytical and computational difficulties caused by their large dimensions and stiffness are one reason. Another reason is the incompleteness of the linear natural basis $\{\phi_n\}$ in strongly perturbed instantaneous volumes (see Remark 2.1), which makes those methods "pseudo-asymptotic", i.e. these systems can only be applied if $Q(t)$ is asymptotically close to its hydrostatic shape $Q_0$.

In contrast to the pseudo-spectral methods, an alternative means of achieving a finite-dimensional modal system consists of combining projective and asymptotic schemes. This idea was probably first proposed by Narimanov[59] and further developed by Dodge *et al.*,[14] Narimanov *et al.*,[60] Lukovsky,[46] Lukovsky & Timokha,[47] Faltinsen *et al.*[25] and La Rocca *et al.*[40] It suggests small deviations of the free surface $\Sigma(t)$ and therefore the smallness (relative to the characteristic tank size) of the generalised coordinates $\beta_i(t)$. Nonlinear terms of the asymptotic modal systems are calculated as integrals over $Q_0$ and $\Sigma_0$, where the natural modes $\{\phi_i\}$ constitute the complete harmonic basis.

The asymptotic schemes suggest "*dominating*" $\{\beta_D\}$ and "*driven*" $\{\beta_R\}$ leading modes, where $\{\beta_L\} = \{\beta_D\} \cup \{\beta_R\}$. The dominating modal functions (modes) are associated with the lowest possible asymptotic contributions, i.e. $\tau \ll |\beta_R| \ll |\beta_D| \ll 1$, where $\tau$ is typically defined as a non-dimensional excitation amplitude. Postulating a finite set of dominating modes one always obtains a finite-dimensional nonlinear modal system coupling all the leading modes. The infinite set of higher modal functions is then described in the framework of the linear sloshing theory.

### 2.4. *Averaged and multi-modal schemes*

Since the dominating modes are the principle contributors to global fluid motions, the problem of resonant fluid sloshing due to harmonic excitation allows a multi-timing scheme that reduces the asymptotic modal systems in $\{\beta_L\}$ to the Hamiltonian systems coupling slow-time evolutions of the dominating modes $\{\beta_D\}$. Between 1965 and 1995, this strategy was elaborated in the works of Miles,[52,53] Shemer,[72] Bridges,[6–8] Feng & Sengtna,[26] Nagata[58] and, recently, by Hill.[38] These works also present mathematical results, which provide the asymptotic classification of resonant steady solutions and their stability analysis as $\tau \to 0$. The results agree well with experiments published in the literature.

However, numerical simulations based on the Hamiltonian systems for small but not infinitesimal $\tau$ do not provide, in many cases, a satisfactory quantitative agreement with the experimental data. Although the driven modes $\{\beta_R\}$ are formally of higher order than $\{\beta_D\}$, their contribution is always of practical concern, sometimes up to 50% in computing the steady-state wave amplitude response (see practical examples by Faltinsen,[16] Gavrilyuk *et al.*[31] and Faltinsen *et al.*[25,21] for $\tau \approx 0.001-0.025$). By ignoring the driven modes, prominent nonlinear surface wave phenomena such as mobility of the nodal line are left unexplained. During the last decade, this motivated Gavrilyuk *et al.*,[31] La Rocca,[40,41] Faltinsen *et al.*[25,18,20,21] and many other researchers to turn back to either pseudo-spectral or multidimensional asymptotic modal modelling which use the full set of leading modes. Operating with finite-dimensional governing equations coupling $\{\beta_L\}$, the researchers used them as efficient tools for time-simulations of transient regimes, coupled "tank-fluid" motions and even randomly forced waves. Multidimensional modal systems for resonant sloshing were validated with realistic values of $\tau$. The results showed good agreement with experimental observations of the wave patterns, measurements of the wave elevations, hydrodynamic forces and moments acting on the tank etc.

Multidimensional modal systems have become increasingly popular in the literature of physics and computational sciences. Each year at least two to three newly derived models are published. As far as we know, however, an independent theoretical analysis of steady-state regimes and their stability based on the multidimensional modal systems has only been done for isolated cases. Many researchers simply refer to the theoretical prediction provided by the averaged Hamiltonian systems governing dominating modes (Lukovsky,[46] Faltinsen *et al.*[25,21] and Hill[38]). Since these predictions are only valid in the asymptotic limit $\tau \to 0$, this path may lead to wrong conclusions for small but realistic parameters $\tau$. Examples are given by Faltinsen & Timokha,[18,20] Faltinsen *et al.*[22]

## 3. A Modal System for Two-Dimensional Sloshing

As mentioned, this paper focuses on the multimodal modelling of two-dimensional sloshing in a rectangularly-based tank with finite depth. This type of sloshing is caused by a longitudinal excitation in the $Oxz$-plane. Thus, we have to set

$$\eta_1 = H\cos(\sigma t), \quad \eta_2 = \eta_3 \equiv 0; \quad \Psi_i \equiv 0, \;\; i = 1, 2, 3, \tag{3.1}$$

in the three-dimensional problem (2.1), where $H \neq 0$ is the dimensional forcing amplitude and $\sigma$ is the circular forcing frequency. Pitch harmonic forcing ($\Psi_2 \neq 0$) and the Faraday waves due to heave (vertical) excitation ($\eta_3 \sim \cos\sigma t$) have been extensively studied by Faltinsen *et al.*,[25,18] Perlin & Schultz[67] and Hill.[38]

### 3.1. *Relevance of two-dimensional sloshing*

Mathematically, to fix a unique two-dimensional solution $f = f(x,t), \Phi = \Phi(x,z,t)$ of (2.1), (3.1) the two-dimensional initial conditions (2.2), i.e. $f_0 = \tilde{f}_0(x)$,

$f_1 = \tilde{f}_1(x)$, are required. Physically, the relevance of the two-dimensional solutions in a rectangularly-based tank implies their uniform stability relative to small three-dimensional initial perturbations. The stability (instability) of two-dimensional solutions of (2.1), (3.1) has only been minimally investigated in the scientific literature. In numerical and physical publications, the majority of the authors refer to model tests. There exist experimental observations which confirm that, if the tank length (the size along the $Ox$-axis) is much longer than the breadth (along the $Oy$-axis), the two-dimensional waves are stable for almost all forcing frequencies $\sigma$. Recent theoretical studies by Faltinsen *et al.*[21,22,24] have shown that two-dimensional fluid sloshing becomes unstable for square base tanks. This instability leads to amazing wave patterns including the so-called "swirling" (rotary) waves. The appearance of three-dimensional waves for $H = 0$ (free nonlinear waves) and related bifurcations have also been shown by Bryant & Stiassnie[9,10] and Bridges.[7] Finally, some examples of the hydrodynamic instability of the two-dimensional solutions have been reported in the experimental book by Bogomaz & Sirota[5] and, theoretically, by Tsai *et al.*[79] (wave-maker problem) for the length/breadth ratios 2 and 3. An important result shown by Faltinsen *et al.*[22] and Bridges[8] is that the hydrodynamic instability of the two-dimensional solutions and the passage to three-dimensional wave regimes are strongly nonlinear problems. Moreover, damping plays only a minor role: it can reduce, but not remove the intervals of $\sigma$ in which the two-dimensional solutions are not stable. This implies that the Squire[75] theorem cannot be generalised to (2.1), (3.1).

### 3.2. *Preliminaries and the linear modal theory*

Figure 2 shows a rectangular tank with the length $l$ filled by a sloshing fluid with the mean fluid depth $h$. The dimensional problem (2.1) is normalised by $l$ ($x := x/l$ and $z := z/l$) and the characteristic time $1/\sigma$ ($t := \sigma t$), where $\sigma$ is the forcing circular frequency from (3.1). Denoting for simplicity $h := h/l$ (non-dimensional mean fluid depth) we define the normalised time-varying domain

$$Q(t) = \{(x, z) : -h < z < f(x, t); -1/2 < x < 1/2\},$$

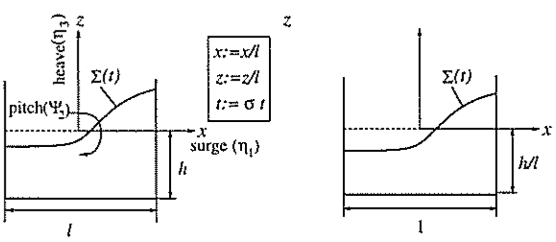


Fig. 2. Two-dimensional fluid sloshing in a rectangular tank (dimensional and non-dimensional statements) occurring for $\Psi_1 = \Psi_3 = \eta_2 \equiv 0$; $\Sigma(t) : z = f(x, t)$.

which has an unperturbed static shape $Q_0 = (-1/2, 1/2) \times (-h, 0)$. The non-dimensional forcing (3.1) is also transformed into the form

$$\eta_1 := \eta_1/l = \tau \cos t, \quad \eta_2 = \eta_3 = 0; \quad \Psi_i = 0, \; i = 1, 2, 3, \tag{3.3}$$

where $\tau = H/l \ll 1$ is the non-dimensional forcing amplitude which *becomes a small parameter* and can be treated as an imperfection.

The dimensionless representation of (2.1), (3.3) suggests normalised modal representations (2.4) and (2.5) ($\beta_i$ and $R_i$ admit an asymptotic treatment in the usual sense) as follows:

$$f = \sum_{i=1}^{\infty} \beta_i(t) f_i(x); \quad \Phi = -\tau x \sin t + \sum_{i=1}^{\infty} R_i(t)\phi_i(x, z), \tag{3.4}$$

where the natural modes

$$f_i(x) = \cos(\pi i(x + 1/2)); \quad \phi_i(x, z) = f_i(x)\frac{\cosh(\pi i(z + h))}{\cosh(\pi i h)} \tag{3.5}$$

are the solutions of the following two-dimensional formulation of problem (2.8):

$$\begin{aligned} &\Delta\phi_i = 0 \quad \text{in } Q_0; \quad \left.\frac{\partial \phi_i}{\partial x}\right|_{x=\pm\frac{1}{2}} = 0; \quad \left.\frac{\partial \phi_i}{\partial z}\right|_{z=-h} = 0; \\ &\frac{\partial \phi_i}{\partial z} = \kappa_i \phi_i \;\; (z = 0); \quad \int_{\Sigma_0} \phi_i dS = 0. \end{aligned} \tag{3.6}$$

Moreover, the dimensional natural circular frequencies $\sigma_i$ (see, e.g., Faltinsen *et al.*[16,25])

$$\sigma_i^2 = \frac{g}{l}\kappa_i; \quad \kappa_i = \pi i \tanh(\pi i h), \tag{3.7}$$

($g \approx 9.81\,\mathrm{m/s^2}$ is the gravity acceleration) can be written in a normalised form as

$$\bar{\sigma}_i = \frac{\sigma_i}{\sigma}.$$

For the resonant condition $\sigma \to \sigma_i$, this implies $\bar{\sigma}_i \to 1, \; i \geq 1$.

### 3.3. *Governing equations*

#### 3.3.1. *Linearised problem and resonant forcing with $\tau \neq 0$*

Using the assumption $\beta_i \sim \tau, \; i \geq 1$, and keeping the terms up to the order $O(\tau)$ in Eq. (2.1), the following linear normalised modal system results

$$\ddot{\beta}_i + \bar{\sigma}_i^2 \beta_i + P_i \tau \cos t = 0; \quad R_i(t) = \frac{\dot{\beta}_i(t)}{\kappa_i} \;\; i \geq 1, \tag{3.9}$$

where

$$P_i = \frac{2\tanh(\pi i h)}{\pi i}[(-1)^i - 1] \tag{3.10}$$

and $\kappa_i, \; i \geq 1$ are defined in (3.7).

The system (3.9) can be subjected to the initial conditions

$$\beta_i(0) = \alpha_i^0; \quad \dot{\beta}_i(0) = \alpha_i^1, \qquad i \geq 1. \tag{3.11}$$

Here the constants $|\alpha_i^0| \sim |\alpha_i^1| \sim \tau$ appear as the coefficients of the Fourier series

$$\sum_{i=1}^{\infty} \alpha_i^0 f_i(x) = \tilde{f}_0(x); \quad \sum_{i=1}^{\infty} \alpha_i^1 f_i(x) = \tilde{f}_1(x),$$

where $\tilde{f}_0$ and $\tilde{f}_1$ are defined by (2.2).

For two-dimensional fluid sloshing, the infinite-dimensional Cauchy problem (3.9), (3.11) is equivalent to the linearised evolution problem (2.1), (2.2), (3.1). Solutions of the evolution problem describe two-dimensional waves with a small amplitude in rectangularly-based tanks occurring due to horizontal excitation and initial disturbances.

An alternative approach consists in the use of the following periodicity conditions (which is equivalent to (2.3) with $T = 2\pi/\sigma$):

$$\beta_i(2\pi) = \beta_i(0); \quad \dot{\beta}_i(2\pi) = \dot{\beta}_i(0), \qquad i \geq 1. \tag{3.12}$$

Linear modal systems similar to (3.9) (in a truncated form) are widely used in the structural analysis of numerous applied mechanical systems modelling tanks filled with a fluid (see, for instance, results for tanks of various shape by Abramson,[1] Feschenko *et al.*,[28] Ibrahim *et al.*,[39] Mikishev & Rabinovich[50] and Moiseyev & Rumyantsev[56]). Its applicability is strongly restricted to the non-resonant case. The resonant solutions of (3.9) ($\bar{\sigma}_i \to 1$ for odd numbers $i$) have a $2\pi$-periodic component that may tend to infinity, in what follows, that the basic assumption $\beta_i \sim \tau,\ i \geq 1$, is invalid. However, the even modes cannot be resonantly forced in the framework of the linear theory. Furthermore, we will show that an appropriate nonlinear mechanism of their activation is associated with the so-called secondary (internal) resonance.

Resonant excitation of the lowest frequency $\bar{\sigma}_1 \to 1$ (primary resonance) is most dangerous for structural stability, because here the resulting fluid response has the largest amplitude. This is the object of numerous mathematical and physical studies. The limit $\bar{\sigma}_1 \to 1$ is typically interpreted in terms of the parameter

$$\lambda = 1 - \bar{\sigma}_1^2 \to 0, \tag{3.13}$$

which measures how close $\sigma$ is to $\sigma_1$ (Moiseyev[55] and Ockendon & Ockendon[61]). Our studies do not restrict $\lambda$ to be small. We treat $\lambda$ as a *bifurcation parameter* (since $\sigma > 0$, $-\infty < \lambda < 1$).

### 3.3.2. *Single-dominant modal system*

The dominating character of periodic solutions illustrated by the linear modal theory motivates us to focus on studying the steady-state (periodic) waves and their stability. Moiseyev[55] was probably the first to find an asymptotic periodic solution of (2.1), (3.1) for a two-dimensional rectangular tank with infinite depth, i.e. in the asymptotic limit $h \to \infty$ and $\lambda, \tau \to 0$. Ockendon & Ockendon[61] and Faltinsen[16] derived similar periodic solutions for $h = O(1)$ and $\lambda, \tau \to 0$. In these papers the periodic solutions are obtained under the Moiseyev asymptotic detuning which links the infinitesimal numbers $\lambda$ and $\tau$ as follows:

$$\tau^{2/3} \sim |\lambda|. \tag{3.14}$$

Such an asymptotic relationship is a typical attribute of various asymptotic theories. Examples are given by Miles,[52,53] Ockendon *et al.*,[61,63] Feng & Senthna,[26] Hill[38] and Shemer,[72] where this detuning asymptotics appears as a necessary condition.

A simple analysis by Faltinsen[16] shows that the asymptotic detuning (3.14) leads to the following modal ordering

$$R_1 \sim \beta_1 = O(\tau^{1/3}); \quad R_2 \sim \beta_2 = O(\tau^{2/3}); \quad R_i \sim \beta_i \leq O(\tau), \ \ i \geq 3, \tag{3.15}$$

where the notations of representation (3.4) are used. Faltinsen *et al.*[25] have shown that the single-dominant asymptotic modal system, which is based on (3.15), can be derived without the Moiseyev detuning (3.14) between $\tau$ and $\lambda$. As a consequence, the resulting single-dominant modal system adopts arbitrary initial perturbations including those for higher (not only dominating) modes and may be used for studying complex transient waves. This model makes it possible to consider $\lambda \in (-\infty, 1)$ and small $\tau$. The modal system by Faltinsen *et al.*[25] takes the dimensionless form

$$\begin{aligned} &\ddot{\beta}_1 + (1 - \delta_1(\lambda))\beta_1 + d_1(\ddot{\beta}_1\beta_2 + \dot{\beta}_1\dot{\beta}_2) \\ &\quad + d_2(\ddot{\beta}_1\beta_1^2 + \dot{\beta}_1^2\beta_1) + d_3\ddot{\beta}_2\beta_1 + P_1\tau\cos t = 0; && (3.16) \\ &\ddot{\beta}_2 + (4 - \delta_2(\lambda))\beta_2 + d_4\ddot{\beta}_1\beta_1 + d_5\dot{\beta}_1^2 = 0; && (3.17) \\ &\ddot{\beta}_3 + (9 - \delta_3(\lambda))\beta_3 + q_1\ddot{\beta}_1\beta_2 + q_2\ddot{\beta}_1\beta_1^2 + q_3\ddot{\beta}_2\beta_1 \\ &\quad + q_4\dot{\beta}_1^2\beta_1 + q_5\dot{\beta}_1\dot{\beta}_2 + P_3\tau\cos t = 0. && (3.18) \end{aligned}$$

The equations for $\beta_i(t)$, $i \geq 4$, are linear and coincide with (3.9). The coefficients $d_i, q_i, i \geq 1$, are known functions of $h$ (in Appendix A explicit formulas for their computation are given) and it holds

$$\delta_i = \delta_i(\lambda) = i^2 - \mu_i^2(1 - \lambda), \ \ i \geq 1; \ \ (\delta_1 \equiv \lambda), \tag{3.19}$$

where

$$\mu_i = \frac{\bar{\sigma}_i}{\bar{\sigma}_1} \tag{3.20}$$

is the dispersion relationship. Physically, the formulas (3.16)–(3.18) can only be used if the assumptions (3.15) are satisfied. These conditions may fail when $h \to 0$ and $\tau \not\ll 1$. Moreover, its failure is expected at $h \approx h_R = 0.337\ldots$, where experimental response curves (Tadjbakhsh & Keller,[76] Fultz[30] and Waterhouse[81]) demonstrate a transition from "hard spring" to "soft spring" behaviour.

Note, the nonlinear differential equation (3.18) is linear in terms of the modal function $\beta_3$ and the other differential equations (3.16), (3.17) do not contain $\beta_3$. Once $\beta_1$ and $\beta_2$ are determined from (3.16), (3.17), $\beta_3$ can be generated from (3.18) by simple calculations. This means that a nonlinear bifurcation analysis of periodic (steady-state) solutions can focus on the differential equations (3.16)–(3.17) and the periodic boundary conditions (3.12).

### 3.3.3. *Operator form of the periodic (steady-state) boundary value problem*

When setting $B = (\beta_1, \beta_2)^{\mathrm{T}}$, the system (3.16)–(3.17) can be rewritten as

$$M(B)\ddot{B} = G^M(t, B, \dot{B}; \lambda, \tau), \tag{3.21}$$

where

$$M = \begin{pmatrix} 1 + d_1\beta_2 + d_2\beta_1^2 & d_3\beta_1 \\ d_4\beta_1 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2} \tag{3.22}$$

is an invertible matrix and $G^M = (G_1^M, G_2^M) \in \mathbb{R}^2$, with

$$\begin{aligned} G_1^M &= -(1-\lambda)\beta_1 - d_1\dot{\beta}_1\dot{\beta}_2 - d_2\dot{\beta}_1^2\beta_1 - P_1\tau\cos t, \\ G_2^M &= -(4 - \delta_2(\lambda))\beta_2 - d_5\dot{\beta}_1^2. \end{aligned}$$

Inverting $M$ leads to the normal form

$$T(B; \lambda, \tau) = \ddot{B} - G(t, B, \dot{B}; \lambda, \tau) = 0, \tag{3.23}$$

where $G = M^{-1}G^M$ and $G : D_G := [0, 2\pi] \times D_B \times D_{\dot{B}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$, $0 \in D_G, D_{\dot{G}}$; $G \in C^p(D_f)$, $p \geq 4$. Employing the periodic boundary condition (3.12) for $\beta_i, i = 1, 2, 3$, gives

$$l_0(B) = B(0+) - B(2\pi-) = 0, \quad l_1(\dot{B}) = \dot{B}(0+) - \dot{B}(2\pi-). \tag{3.24}$$

The pair (3.23), (3.24) defines a parametrised nonlinear two-point boundary value problem for a second-order differential equation. It permits the following operator formulation

$$T(B; \lambda; \tau) = 0, \qquad B \in X, \quad \lambda, \tau \in \mathbb{R}, \tag{3.25}$$

where the operator $T : Z := X \times \mathbb{R} \times \mathbb{R} \to Y$ and the Banach spaces $X, Y$ are defined as follows:

$$\begin{aligned} X &:= BC^2([0, 2\pi], \mathbb{R}^2) := \{B \in C^2([0, 2\pi], \mathbb{R}^2) : l_0(B) = 0, l_1(\dot{B}) = 0\}, \\ Y &:= C([0, 2\pi], \mathbb{R}^2) \end{aligned} \tag{3.26}$$

with the usual norms

$$\begin{aligned} \|B\|_X &= \|B : BC^2\| = \|B : C^2\| = \sup_{t\in[0,2\pi]}(|B(t)| + |\dot{B}(t)| + |\ddot{B}(t)|), \\ \|B\|_Y &= \|B : C\| = \sup_{t\in[0,2\pi]} |B(t)|. \end{aligned} \tag{3.27}$$

**Remark 3.1.** Explicit representations of $X$ and $Y$ may be given by means of the Fourier series

$$X = \left\{ B = \left( \sum_{m=0}^{\infty} c_m^{(1)} \cos(m(t+\theta_m^{(1)})), \sum_{m=0}^{\infty} c_m^{(2)} \cos(m(t+\theta_m^{(2)})) \right), \theta_m^{(1)}, \theta_m^{(2)} \in \mathbb{R}, \right.$$
$$\left. \sum_{m=0}^{\infty} m^{2-\epsilon}(|c_m^{(1)}| + |c_m^{(2)}|),\ \forall\, \epsilon > 0 \right\},$$
$$Y = \left\{ B = \left( \sum_{m=0}^{\infty} c_m^{(1)} \cos(m(t+\theta_m^{(1)})), \sum_{m=0}^{\infty} c_m^{(2)} \cos(m(t+\theta_m^{(2)})) \right), \theta_m^{(1)}, \theta_m^{(2)} \in \mathbb{R}, \right.$$
$$\left. \sum_{m=0}^{\infty} m^{-\epsilon}(|c_m^{(1)}| + |c_m^{(2)}|),\ \forall\, \epsilon > 0 \right\}. \tag{3.28}$$

The solutions of the operator equation (3.25) depend on the two numbers $\lambda$ and $\tau$, where $\tau$, $0 \le \tau \ll 1$, belongs to an interval $I_\tau$ (interpreted as imperfection) and the parameter $\lambda$, $-\infty < \lambda < 1$, is a control (bifurcation) parameter. Our goal is to obtain a global bifurcation picture. Following the general scheme by Hermann,[34] Wallisch & Hermann[80] and Hermann & Ullrich,[35] we want to determine a part of the solution set

$$\mathcal{M} = \{(B, \lambda, \tau) : -\infty < \lambda < 1; \tau \in I_\tau; T(B, \lambda, \tau) = 0\}. \tag{3.29}$$

## 4. Unperturbed Bifurcations, the Case $\tau = 0$

Physically, $\tau = 0$ in Eq. (3.25) implies free nonlinear standing waves relative to the hydrostatic equilibrium $z = 0$. The equilibrium corresponds to the trivial solution $C_{\text{triv}} = \{(0, \lambda, 0) : -\infty < \lambda < 1\}$ of (3.25). Although each nontrivial solution $B(t) = (\beta_1(t), \beta_2(t))^{\mathrm{T}} \in \mathbb{R}^2$ of (3.25) determines a unique orbit in $\mathbb{R}^2$, the operator equation is not uniquely solvable because of the phase shift invariance: if $B(t)$ is a solution then $B(t) := B(t+\theta)$, for all $\theta$, is also a solution of (3.25). Such an invariance is a typical property of periodic solutions of nonlinear ordinary differential equations modelling conservative mechanical systems.

Let us rewrite the unperturbed operator equation (3.25) in the following form

$$T(B; \lambda, 0) = T_0(B; \lambda) = T_B^0[\lambda]B + \tilde{T}_0(B; \lambda) = 0, \quad T_0 : X \times \mathbb{R} \to Y, \tag{4.1}$$

where $T_B^0[\lambda]B$ is the linear part of $T_0$ represented by the Fréchet derivative $T_B^0[\lambda] = \partial T_0(0,0)/\partial B$. The corresponding differential expression is

$$\ddot{\varphi}^i + (i^2 - \delta_i(\lambda))\varphi^i, \quad i = 1, 2, \tag{4.2}$$

where $\varphi = (\varphi^1, \varphi^2)^{\mathrm{T}} \in X$. It can be easily shown that $T_B^0[\lambda]$ is a self-adjoint Fredholm operator (on a suitable set of functions from $L_2(0, 2\pi)$). The linear operator equation

$$T_B^0[\lambda]\varphi = 0, \quad \varphi \in X, \tag{4.3}$$

has nontrivial solutions for $\delta_i(\lambda) = i^2 - k^2,\ k \in \mathbb{N},\ i = 1, 2$. From this relation we get the critical values

$$\lambda = \lambda_0(i,k) = 1 - k^2/\mu_i^2, \qquad k \in \mathbb{N}, \quad i = 1, 2. \tag{4.4}$$

The values $\lambda_0(i,k)$ define the primary bifurcation points on the trivial solution curve $C_{\text{triv}}$. Since $\mu_1^2 = 1$ and $2 < \mu_2^2 < 4$ (see the definition (3.20)), there do not exist two integers $k_1$ and $k_2$ such that $\lambda_0(1,k_1) = \lambda_0(2,k_2)$. This implies that the kernels $\mathcal{N}(T_B^0[\lambda_0(i,k)])$ have dimension 2 and are spanned by either $\varphi_1[1,k] = (\sin(kt), 0), \varphi_2[1,k] = (\cos(kt), 0)$ or $\varphi_1[2,k] = (0, \cos(kt)), \varphi_2[2,k] = (0, \sin(kt))$. Moreover, the kernel $\mathcal{N}(T_B^0[\lambda_0(i,k)]^*)$ of the adjoint operator $T_B^0[\lambda_0(i,k)]^*$ has dimension 2 as well and is spanned by $\psi_m[i,k] = \varphi_m[i,k],\ m, i = 1, 2;\ k \in \mathbb{N}$. As noted above, dimension 2 is caused by the phase-shift invariance, i.e. the kernels have the following representation

$$\mathcal{N}(T_B^0[\lambda_0(i,k)]) = \begin{cases} (\cos(k(t+\theta)), 0), & \text{for } i = 1, \\ (0, \cos(k(t+\theta))), & \text{for } i = 2, \end{cases} \tag{4.5}$$

where $\theta \in \mathbb{R}$. Since a variation of $\theta$ does not affect the geometric orbits $B(t) \in \mathbb{R}^2$, we study, for simplicity, only solutions for a fixed $\theta$.

### 4.1. *Bifurcations at* $\lambda_0(1,k),\ k \in \mathbb{N}$

Performing either the well-known Lyapunov–Schmidt reduction in a neighbourhood of $\lambda_0(1,k),\ k \in \mathbb{N}$, or the multi-bifurcation analysis by Wallisch & Hermann[80] (Chap. 2), we get the following local curves which branch at the bifurcation points $\lambda_0(1,k)$

$$\begin{aligned} \beta_1(t) &= s\cos(k(t+\theta)) - s^3 \frac{n_1 k^2}{9k^2 - 1 + \lambda_0(1,k)} \cos(3k(t+\theta)) + O(s^5); \\ \beta_2(t) &= s^2[p_0 + h_0 \cos(2k(t+\theta))] + O(s^4); \\ \lambda &= \lambda_0(1,k) + s^2\lambda_2(1,k) + O(s^4) \quad \text{for } |s| \ll 1, \end{aligned} \tag{4.6}$$

where

$$p_0 = \frac{d_4 - d_5}{2\mu_2^2}, \quad h_0 = \frac{d_4 + d_5}{2(\mu_2^2 - 4)}, \quad n_1 = \frac{1}{2}d_2 + h_0\left(\frac{3}{2}d_1 + 2d_3\right). \tag{4.7}$$

Here, the parameter $\lambda_2(1,k) = O(1)$ should be calculated from the following equation

$$\lambda_2(1,k) = k^2 m_1[h], \tag{4.8}$$

where

$$m_1[h] = -\frac{1}{2}d_2 - d_1\left(p_0 - \frac{1}{2}h_0\right) - 2h_0 d_3 \tag{4.9}$$

depends only on the mean fluid depth $h$. Equation (4.8) is the necessary resolvability condition.

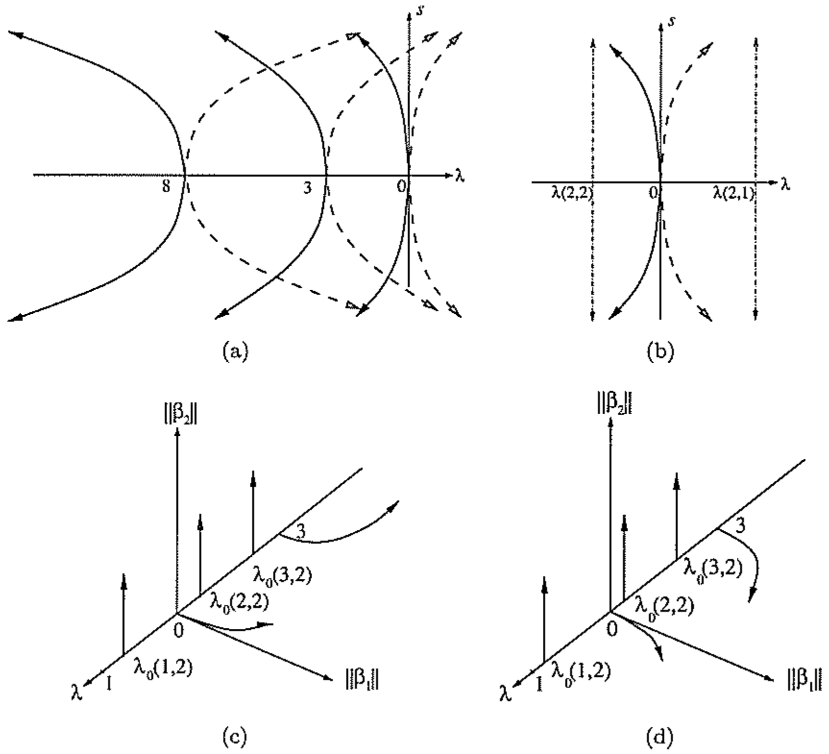


Fig. 3. (a) The local branches at $\lambda_0(1,k) = 1 - k^2, k \in \mathbb{N}$, which have been computed by formulae (4.6), in the $(\lambda, s)$-plane. Solid lines correspond to $h > h_R$, dashed lines represent the case $h < h_R$, where $h_R = 0.3368\ldots$. (b) the local branches in the neighbourhood of the origin $(0,0)$, imposed always by three bifurcation points $\lambda_0(1,1) = 0$ (nonlinear standing waves dominated by the mode $f_1(x)$ with the frequency close to the lowest natural tone) and $\lambda_0(2,1), \lambda_0(2,2)$ (standing waves associated with the second mode $f_2(x)$); (c) and (d) the three-dimensional picture of the local branches in the $(\lambda, \|\beta_1\|, \|\beta_2\|)$-space for $h > h_R$ and $h < h_R$, respectively.

Since the asymptotic solution (4.6) has the "asymptotic" norm $\|B\| = O(s)$, the locally bifurcating branches can be interpreted in the $(\lambda, s)$-plane. Two types of so-called backbones (solid and dashed lines) are shown in Fig. 3(a). These types depend on the sign of $m_1$, namely, negative $m_1$ determine the "soft-spring" behaviour (solid lines), but positive $m_1$ imply the "hard-spring" behaviour (dashed lines). The sign of $m_1$ depends on $h$. Calculations show that $m_1[h] < 0,\ h > h_R$, $m_1[h] > 0,\ h < h_R$, where $h_R = 0.3368\ldots$ is the already mentioned critical depth.

### 4.2. *Bifurcations at* $\lambda_0(2,k),\ k \in \mathbb{N}$

Since the differential equation (3.17) is linear in $\beta_2(t)$, the formal procedure based on the Lyapunov–Schmidt reduction leads to the following local solutions

$$\beta_1(t) \equiv 0; \quad \beta_2(t) = s\cos(k(t+\theta)); \quad \lambda \equiv \lambda_0(2,k) \quad \text{for any } |s| \ll 1. \qquad (4.10)$$

In the $(\lambda, s)$-plane, the corresponding local curves bifurcating at $\lambda_0(2,k)$ consist of a family of vertical lines $\lambda = \lambda_0(2,k)$.

### 4.3. *Local bifurcating curves*

The relationship between the two families of primary bifurcation points $\lambda_0(1,k)$ and $\lambda_0(2,k)$ changes with $h$ and, therefore, their order along the $\lambda$-axis is not very predictable, in general. However, it can be shown that the three lowest values from the resulting set $\{|\lambda_0(i,k)|, i = 1,2;\ k \in \mathbb{N}\}$ are $\lambda(1,1) = 0$, $\lambda(2,2) \in (-1,0)$ and $\lambda(2,1) \in (1/2, 3/4)$, and they are linked as $\lambda_0(2,2) < \lambda < \lambda_0(2,1)$. Moreover, $\lambda(2,2) \to 0$ as $h \to 0$, but it is bounded away from zero for finite depths $h = O(1)$. The local branching behavior related to these three bifurcation points is shown in Fig. 3(b).

We found it useful to give a three-dimensional presentation of the local curves by operating with the two normed values $\|\beta_1\|$ and $\|\beta_2\|$ independently. The corresponding local branching behaviour in the $(\lambda, \|\beta_1\|, \|\beta_2\|)$-space is shown in Figs. 3(c), (d), where (c) and (d) correspond to $h > h_R$ and $h < h_R$, respectively. One important conclusion, which is based on the three-dimensional bifurcation diagrams, is that the closeness of $\lambda$ to either $\lambda_0(1,k)$ or $\lambda_0(2,k)$ leads, in the lowest asymptotic order, to the bifurcations in the $(\lambda, \|\beta_1\|)$- or $(\lambda, \|\beta_2\|)$-plane, respectively. Another important point is that, while the branches at $\lambda_0(2,k)$ are of "linear nature" with vertical strain lines in the $(\lambda, \|\beta_1\|, \|\beta_2\|)$-space, the curves bifurcating at $\lambda_0(1,k)$ become more three-dimensional with increasing $s$. This constitutes a very interesting mathematical problem, and, as shown by Timokha & Hermann,[78] poses a new set of numerical problems. However, having based our studies on physical treatments of realistic free-standing waves, we do not investigate this problem. Even a very small amount of dissipation, always present in a realistic sloshing condition, prevents the large-amplitude free standing waves associated with increasing $s$.

## 5. Perturbed Bifurcations

When $\tau > 0$, the perturbed operator equation (3.25) describes the forced steady-state waves. In this case, the smallness of $\tau$ makes it possible to perform the Lyapunov–Schmidt reduction and to analyse perturbations of the local branches at $\lambda_0(i,k)$, $i = 1,2$; $k \in \mathbb{N}$. The study below shows that the $\tau$-perturbations lead to a quite different behaviour of the local branches for distinct indexes $i$ and $k$.

### 5.1. *Local analysis at* $\lambda_0(i,k)$, $i = 1,2$, $k \in \mathbb{N}$

Consider the local solutions of the unperturbed problem (4.1) which are defined parametrically for $S = (s, \bar{s})^{\mathrm{T}} \in \mathbb{R}^2, \|S\| \ll 1$. By introducing small perturbations $\tau \ll \|S\|$ into the given problem, we can study the influence of imperfections on the corresponding solutions. As shown before, we can perform a Lyapunov–Schmidt reduction to analyse the bifurcation properties and to find exact analytical expressions for the local solutions.

5.1.1. *Perturbed bifurcations of* (3.25) *at* $\lambda_0(1,k)$

By assuming $|\lambda-\lambda_0(1,k)| \ll 1$, $k \in \mathbb{N}$, taking into account $\dim \mathcal{N}(T_B^0[\lambda_0(1,k)]) = 2$ and considering the unperturbed solutions from Sec. 4.1, the Lyapunov–Schmidt reduction deduces that the $\tau$-perturbations can either *preserve* or *destroy* the bifurcations. The result depends on $k$. For $k \neq 1$, the $\tau$-perturbations preserve the bifurcating solutions (4.6). In that case, $S = (s, 0)$ and any small $\tau \ll S$ does not effect the dominating asymptotic terms in (4.6) which take now the following form

$$\begin{aligned} \beta_1(t) &= s\cos(k(t+\theta)) - \frac{P_1\tau}{k^2-1}\cos t + O(s^3); \\ \beta_2(t) &= s^2[p_0 + h_0\cos(2k(t+\theta))] + O(s\tau); \\ \lambda &= \lambda_0(1,k) + s^2\lambda_2(1,k) + O(s\tau), \end{aligned} \tag{5.1}$$

where $p_0$, $h_0$ and $\lambda_2(1,k)$ are governed by (4.7) and (4.8).

On the other hand, for $k = 1$, the $\tau$-perturbations destroy the local bifurcating solution (4.6) and lead to a new necessary resolvability condition, which couples $S$ and $\tau$ in the following way

$$\begin{cases} s(-\lambda_2(1,1)(s^2+\bar{\alpha}\bar{s}^2) + m_1(s^2+\bar{s}^2)) + P_1\tau = 0, \\ \bar{s}(-\lambda_2(1,1)(s^2+\bar{\alpha}\bar{s}^2) + m_1(s^2+\bar{s}^2)) = 0, \end{cases} \tag{5.2}$$

where $m_1$ is defined by (4.9) and

$$\lambda = \lambda_2(1,1)(s^2+\bar{\alpha}\bar{s}^2) + O(\|S^3\|); \quad \lambda_2(1,1) = O(1). \tag{5.3}$$

Further, a simple analysis shows that the system (5.2) requires $\bar{s} = 0$. Therefore it can be transformed into the form

$$\bar{s} = 0; \quad \lambda_2(1,1) = m_1 + \frac{P_1\tau}{s^3}. \tag{5.4}$$

The last relationship should be compared with formula (4.8) which holds for the unperturbed problem. At first, we note that the necessary condition (5.4) makes unperturbed bifurcations associated with $\varphi_2[1,1] = (\sin t, 0)^{\mathrm{T}}$ impossible. Further, since $\lambda_2(1,1) = O(1)$, the system (5.2) needs the Moiseyev-like detuning $s^3 \sim \tau$, which gives here an additional asymptotic resolvability condition. The corresponding local solution reads

$$\begin{aligned} \beta_1(t) &= s\cos(t) + O(s^2); \quad \beta_2(t) = s^2(p_0 + h_0\cos 2t) + O(s^3), \\ \lambda &= \lambda_2(1,1)s^2 + O(s^3), \end{aligned} \tag{5.5}$$

where $h_0, p_0$ are given in (4.7). This solution is mathematically equivalent to the classical results by Moiseyev,[55] Ockendon & Ockendon[61] and Faltinsen[16] obtained by the asymptotic expansion directly applied to the original free boundary problem (2.1).

5.1.2. *Perturbed bifurcations of* (3.25) *at* $\lambda_0(2,k)$

This type of $\tau$-perturbations can also lead to either a preserving ($k \neq 2,\ k \in \mathbb{N}$) or a destroying ($k = 2$) of the unperturbed bifurcations associated with the local solutions (4.10). If $k \neq 2$, the Lyapunov–Schmidt reduction does not yield a new resolvability condition and, if $\tau \ll \|S\|$, the perturbed local solution

$$\begin{aligned} \beta_1(t) &= s_k \tau \cos t + O(\tau^2); \quad \beta_2(t) = s\cos(k(t+\theta)) + O(s^2), \\ \lambda &= \lambda_0(2,k); \quad s_k = -P_1/(k^2/\mu_2^2 - 1) = O(1) \end{aligned} \tag{5.6}$$

is equivalent to (4.10).

In the case $k = 2$ the situation changes. Now, the $\tau$-perturbations destroy the bifurcations appearing in the unperturbed problem and generate the following local solution

$$\begin{aligned} \beta_1(t) &= s_2 \tau \cos t + O(\tau^2); \quad \beta_2(t) = s_2^2 \frac{d_4 + d_5}{2\mu_2^2} \frac{\tau}{s} \cos(2t) + O\Big( \Big| \frac{\tau}{s} \Big|^2 \Big), \\ \lambda &= \lambda_0(2,2) + s, \end{aligned} \tag{5.7}$$

expressed in terms of $\tau$ and the local parametrisation $s$. The resolvability condition simply reads

$$\left| \frac{\tau}{s} \right| \ll 1. \tag{5.8}$$

Note that both perturbed solutions (5.6) and (5.7) can be characterised as linear and do not follow from Moiseyev's technique, because their occurrence is quantified away from $\lambda = 0$.

### 5.2. *The bifurcation destroying perturbations*

The local solutions (5.1), (5.5), (5.6) and (5.7) at $\lambda_0(i,k)$ represent the general structure of the solutions of the perturbed problem in the $(\lambda, \|B\|)$-plane. Physically, these solutions imply different levels of nonlinear resonant phenomena. The bifurcation preserving perturbations, associated with (5.1) and (5.6), do not introduce a new resolvability condition and, since the $s$-component in these solutions is independent of $\tau$, the $s$-terms can be related to the free nonlinear sloshing. In practise, these components disappear rapidly due to any small, even infinitesimal, dissipation. Without the $s$-terms, the solutions (5.1) and (5.6) are small enough to be computed by linear modal approximation (3.9); they do not represent forced resonant waves.

In contrast to the bifurcation preserving perturbations, the relations (5.5) and (5.7) couple $\tau$ and $s$. These solutions imply resonant waves that are the primary focus of our study. As mentioned above, the local solution (5.5) coincides with the Moiseyev resonant solution. The local solution (5.7) can be related to the so-called secondary (internal) resonance phenomena in the fluid sloshing problems studied by Ockendon *et al.*[63] and Faltinsen & Timokha.[18]

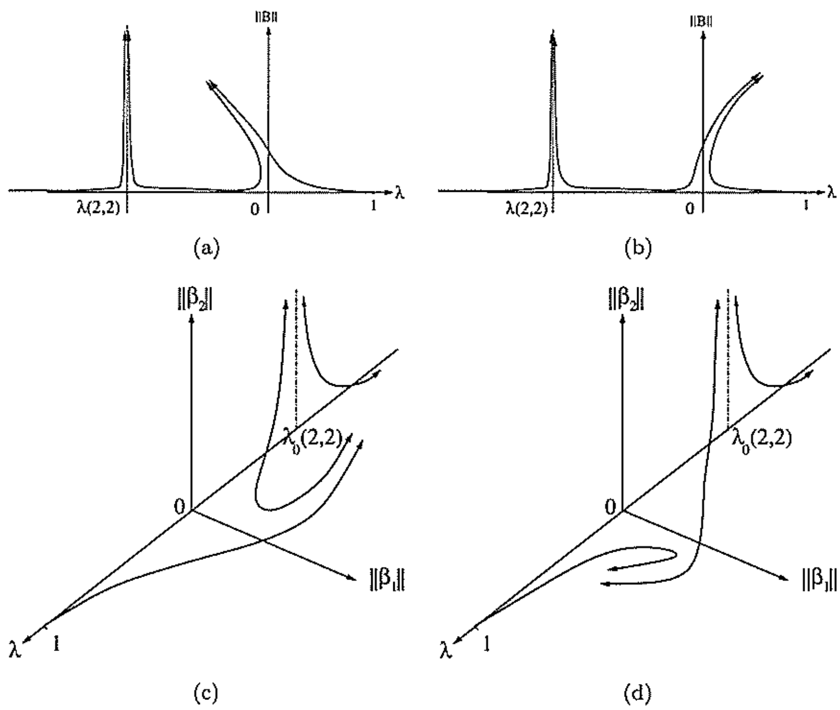


Fig. 4. Perturbed bifurcations in the $(\lambda, \|B\|)$-plane for $\|B\| \ll 1$: (a) $h > h_R$ and (b) $h < h_R$. The graphs (c) and (d) give a three-dimensional representation of (a) and (b) in the $(\lambda, \|\beta_1\|, \|\beta_2\|)$-coordinate system, respectively.

5.2.1. *Local analysis*

When ignoring the $s$-terms in (5.5) and (5.6) and continuously varying $\lambda \in (-\infty, 1)$, the four local solutions (5.1), (5.5), (5.6) and (5.7) can be transformed into each other. This indicates a general branching structure in the case of the bifurcation destroying perturbations and for relatively small norms $\|B\|$. It is shown in the $(\lambda, \|B\|)$-plane in Figs. 4(a), (b) for $h > h_R$ and $h < h_R$, respectively. Here the curves at the origin are associated with the local solutions (5.5) as $\lambda \to \lambda_0(1,1) = 0$. The linear-like resonant behaviour of $\|B\|$ as $\lambda \to \lambda_0(2,2)$ is described by the solutions (5.7).

The next, more accurate graphical interpretation of the perturbed solutions requires a three-dimensional view in the $(\lambda, \|\beta_1\|, \|\beta_2\|)$-system as shown schematically in Figs. 4(c), (d). This three-dimensional representation makes clear, whether the postulations (3.15) used in derivations of the governing modal system (3.16)–(3.18) are fulfilled, or not. Since $\|\beta_2\|$ dominates over $\|\beta_1\|$ as $\lambda \to \lambda_0(2,2)$, the ordering (3.15), which can be rewritten as

$$\|\beta_2\| \sim \|\beta_1\|^2, \tag{5.9}$$

becomes invalid even in a small neighbourhood of $\lambda_0(2,2)$.

The relationship (5.9) is true in the neighbourhood of the primary bifurcation point $\lambda_0(1,1) = 0$. Of interest then is the non-local behaviour of the solutions of the perturbed problem in the vicinity of this bifurcation point. To accomplish this task numerical path-following methods are required.

#### 5.2.2. *Non-local analysis*

Using the local solution (5.1) at $\lambda_0(1,1) = 0$, we implemented the RWPM-package developed by Hermann & Ulrich[35] and Hermann & Kaiser.[36] This package can be used to study parametrised two-point boundary value problems. It is based on two numerical shooting techniques (multiple shooting and stabilised march, see e.g. Hermann[37]) and enables the computation of isolated solutions of two-point boundary value problems as well as path-following and the detection and determination of turning and bifurcation points. The application of this package made it possible to test a wide range of fluid depths and the excitation amplitude $\tau$, to gain a global insight into the admissible steady-state solutions of the modal system (3.16)–(3.18) with increasing norms. The restriction $h \geq 0.27$ is due to the hypothesis given by Faltinsen *et al.*[25] The numerical experiments are illustrated in the Figs. 5(a)–(d).

The detailed numerical analysis establishes that the periodic solutions of (3.16), (3.17) may be qualitatively different from the asymptotic prediction, even if $\tau$ is very small. If $h > h_R = 0.3368\ldots$, the numerically determined periodic solutions characterise not only the primary bifurcation (the turning point $T$) following from the Moiseyev-like asymptotic solution (5.5), but also the secondary bifurcations arising as two "twin"-like secondary turning points $S_1$ and $S_2$. These appear when $\|\beta_2\| \sim \|\beta_1\|$. The presence of $S_1$ and $S_2$ makes our numerical results similar to the fifth-order theory by Waterhouse[81] capturing the case of the critical depth $h \approx h_R$. This similarity is implicitly confirmed by the fact that the length between $T$ and $S_i$, $i = 1, 2$, diminishes as $h \to h_R+$. The secondary bifurcation points $S_i$, $i = 1, 2$, do not disappear for large $h$, but $T$ becomes closer to $S_1$ with increasing $\tau$. The last effect establishes the mathematical limits of the applicability of the single-dominant approximate modal system (3.16)–(3.18) as well as the restrictions on $\tau$ in Waterhouse's theory requiring also the single-dominant ordering with $\tau^{1/5} \sim \|\beta_1\|$.

## 6. Concluding Remarks and Open Questions

First we may conclude that periodic solutions of modal systems, in general, and the single-dominant modal system by Faltinsen *et al.*,[25] in particular, can be analysed by using the imperfect bifurcation theory. This is possible, because, in contrast to traditional single-dominant averaging asymptotic theories employing the Moiseyev ordering, the modal modelling does not need asymptotic links between the so-called Moiseyev detuning parameter $\lambda$ and the dimensionless excitation amplitude $\tau$. Therefore, the parameter $\lambda$ can be interpreted as the bifurcation parameter and the other non-dimensional parameter $\tau$ as an imperfection. Periodic solutions of the unperturbed modal system ($\tau = 0$, free-standing waves) as well as of the perturbed

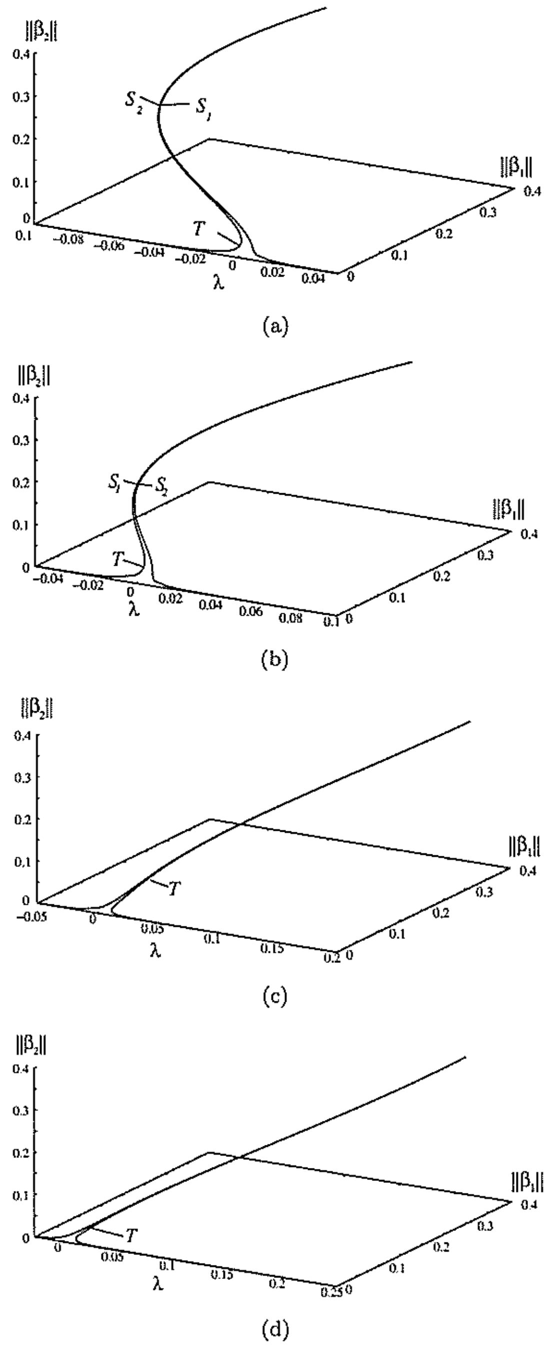


Fig. 5. Numerical results on the bifurcation destroying perturbations at $\lambda_0(1,1) = 0$ sketched in the $(\lambda, \|\beta_1\|, \|\beta_2\|)$-coordinate system. The calculations have been done for $\tau = 0.0001$ with the four values of $h = 1.0, 0.5, 0.3368$ and $0.3$ depicted in (a), (b), (c) and (d), respectively. $T$ denotes the turning point captured by the local solution. Two "twin"-secondary bifurcation points $S_1$ and $S_2$ appear for $h > h_R = 0.3368\ldots$ when $\|\beta_1\|$ becomes numerically of the same order as $\|\beta_2\|$.

system ($\tau \neq 0$, forced waves) are then considered as solutions of a suitable nonlinear operator equation. We assume that this can be generalised to other modal systems, published, for example, by Lukovsky,[46] Faltinsen & Timokha[18,20] and La Rocca *et al.*[40,41]

The next important conclusion is that some particular results of this paper are consistent with the well-known asymptotic solutions obtained by various authors by means of direct asymptotic expansions of the original free boundary problem. However, there is a difference to those classical results which show an infinite set of bifurcation points of the unperturbed problem for $\lambda$ away from 0 and, as a consequence, an infinite number of bifurcation points of the perturbed problem. The paper gives an asymptotic and numerical treatment of this difference. It is of interest to study these differences for modal systems of larger dimensions. This will be the principle aim of the forthcoming Part II.

Although, in contrast to traditional asymptotic results based on the Moiseyev asymptotic detuning, there is an infinite number of bifurcation points of the unperturbed problem, the local analysis of their perturbations uncovers two and only two points where small perturbations destroy the bifurcation. Perturbations around the other points preserve the bifurcation. These two points occur at $\lambda = 0$ (the primary resonance) and $\lambda = \lambda_0(2,2)$ (the secondary resonance). The physical treatment of the secondary resonance in nonlinear sloshing problems is given by Ockendon *et al.*,[63] Ockendon & Ockendon,[62] Faltinsen & Timokha[18] and Faltinsen *et al.*[22]

For small $\tau > 0$ and $h \geq 0.27$, we have used the RWPM-package in the non-local analysis of periodic solutions. This made it possible to quantify mathematically the applicability of the single-dominant model. A new discovery is that, even if $\tau$ is very small and $h > h_R = 0.3368\ldots$, the solution curves of the perturbed problem in the neighbourhood of $\lambda = 0$ indicate two "twin"-like secondary bifurcations. This switches "soft-spring" curves to the "hard-spring" character with increasing the periodic solution norms. The "hard-spring" behaviour for $h < h_R$ stays the same as in the traditional asymptotic analysis. Calculations show that the norm of the periodic solutions is still small in the neighbourhood of the secondary bifurcation points, but the basic Moiseyev ordering is violated and two modes are of the equal order, i.e., $\|\beta_1\| \sim \|\beta_2\|$. This may indicate failure of the single-dominant modal systems. Part II will extend this analysis to alternative, multi-dominant modal theories presented, for instance, by La Rocca[41] and Faltinsen & Timokha,[18] in which some higher modes are of the same order as the primary $\beta_1$.

Another problem to be studied in Part II is the case $h = h_R = 0.337\ldots$, where the local response curves of periodic solutions demonstrate a transition from "hard spring" to "soft spring" behaviour as the fluid depth passes through this value (Tadjbakhsh & Keller,[76] Fultz[30] and Waterhouse[81]). Waterhouse[81] proposed a fifth-order Moiseyev-like solution considering another asymptotic detuning $\tau \sim |\lambda|^{5/4} \sim |h - h_R|^{3/2} \to 0$, which orders the five lowest modal functions as $\beta_i = O(\tau^{i/5})$, $i = 1, \ldots, 5$. The response curves by Waterhouse demonstrate qualitatively the same secondary bifurcation phenomena. This requires re-examining of this case by using the perturbed bifurcation theory.

## Appendix A. Coefficients of the Single-Dominant Modal System

$$
\begin{aligned}
d_1 &= 2\frac{E_0}{E_1} + E_1, \quad d_2 = 2E_0\left(-1 + \frac{4E_0}{E_1E_2}\right), \quad d_3 = -2\frac{E_0}{E_2} + E_1,\\
d_4 &= -4\frac{E_0}{E_1} + 2E_2, \quad d_5 = E_2 - 2\frac{E_0E_2}{E_1^2} - \frac{4E_0}{E_1};\\
q_1 &= 3E_3 - \frac{6E_0}{E_1}, \quad q_2 = 9E_0 - 12\frac{E_0E_4}{E_1} - 6E_3E_4 + 24\frac{E_0^2}{E_1E_2} + 3\frac{E_0E_3}{E_1},\\
q_3 &= -6\frac{E_0}{E_2} + 3E_3, \quad q_4 = -6\frac{E_0}{E_1} - 6\frac{E_0}{E_2} - 6\frac{E_0E_3}{E_1E_2} + 3\frac{E_3E_1}{E_2},\\
q_5 &= 18E_0 - 2E_4\frac{12E_0 + 6E_1E_3}{E_1} + \frac{72E_0^2}{E_1E_2} + 12E_0\left(\frac{E_3}{E_1} - \frac{E_1}{E_2}\right),
\end{aligned}
$$

where

$$
E_0 = \frac{\pi^2}{8}, \quad E_i = \frac{\pi}{2}\tanh(\pi i h), \quad i \geq 1.
$$

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