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MULTIMODAL METHOD IN SLOSHING

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The multimodal method reduces the sloshing problem with free surface to a (modal) system of nonlinear ordinary differential equations. The method was originally proposed for nonimpulsive hydrodynamic loads. However, recently it has been successfully extended to the case of sloshing-induced slamming. In the 1950–1960's, this method was used in the computational fluid dynamics (CFD) but later was replaced by the algorithms developed in the 1990-2000's. At present, the method plays a dual role: *first*, as a unique analytic tool for the investigation of nonlinear sloshing regimes, their stability, and chaos as well as for simulations when traditional CFD fails (e.g., in the case of containers with perforated screen) and, *second*, as a source of construction of the modal systems, which are analogs of the Korteweg–de-Vries, Boussinesq, and other equations but for bounded volumes of liquid. We present a survey of the state-of-the-art of the problem, describe the existing modal systems, and formulate open problems.

1. Introduction

Coupled rigid tank-and-sloshing is a *hybrid mechanical system* in which the tank moves, as a rule, with six degrees of freedom governed by a system of ordinary differential equations (ODEs) but sloshing is described by a problem of free surface suggesting infinitely many degrees of freedom. The multimodal method introduces *hydrodynamic generalized coordinates* (HGCs) and derives a (modal) system (MS) of nonlinear ODEs with respect to these coordinates. The MSs facilitate the analytic studies of the resonant sloshing regimes, their stability, secondary resonances, chaos, etc., which look questionable with the use of traditional CFD.

The present survey is based on the plenary lecture delivered by the authors to experts in the field of differential equations at the Bogolyubov Readings (DIFF-2013, June, 2013, Sevastopol, Ukraine), where, parallel with historical aspects, the ideas and open problems of the multimodal method, the structure, dimensions and features of the obtained nonlinear MSs (NMSs), their solutions *vs.* the container shape, liquid filling (depth), and forcing were discussed. The paper is schematically split into three sections representing the past, the present, and the future of the multimodal method, respectively. The readers interested in Faraday waves are referred to [92, 93]. Newbies in the field of sloshing are recommended to have one of the textbooks [58, 91, 144] and use subject indexes therein in order to understand the terminology.

2. The Past

Most likely, the word *"multimodal"* comes from [50]. However, the multimodal method was first proposed 40 years earlier, in the 1950–1660's, when the researchers were first really challenged by the phenomenon of sloshing in aircraft, spacecraft and marine containers. An enthusiastic atmosphere of these years is described in the memoir article by Abramson [3] who headed the corresponding NASA program. The scientific heritage of the pioneering studies was systematized in [2, 34, 67, 89, 203, 209, 210] (USA) and [1, 66, 103, 110, 154, 155, 165, 166, 168, 190] (USSR). An emphasis was made, primarily, on *theoretical linear* (small-amplitude) sloshing, *experimental*

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nonlinear sloshing phenomena, and slosh-suppressing devices. The American collection of experiments was best reviewed in [2]. The experiments of the Soviet scientists were carried out mainly in Moscow, Kyiv [110, 144], Dnepropetrovsk [16], and Tomsk [17, 19] but remained, for the most part, unpublished.

The **linear multimodal method** (see the original [174, 181, 189], canonical [66, 145, 154, 214] and contemporary [58, 170] descriptions) was proposed in the 1950's to reduce the linear sloshing problem to an infinite set of linear oscillators (ODEs) called, all together, the *linear* MS (LMS) in which the inhomogeneous terms are functions of the six generalized coordinates (degrees of freedom) of motion of the rigid body but the unknowns are HGCs responsible for the amplifications (relative to the hydrostatic shape) of natural sloshing modes. The method interprets sloshing as a conservative mechanical system with infinitely many degrees of freedom. It is necessary to know *(a priori)* the *natural sloshing modes* φ_n and the frequencies σ_n, as well as the linear *Stokes–Joukowski potentials*,

$$\mathbf{\Omega}_0 = (\Omega_{01}, \Omega_{02}, \Omega_{03}).$$

The coupling of the LMS with the dynamic equations of the carrying rigid container is facilitated by the linearized *Lukovsky formulas,* which express the hydrodynamic forces and moments via the HGCs. The hydrodynamic coefficients in both LMS and Lukovsky formulas are integrals over $\mathbf{\Omega}_0$, φ_n, and their derivatives. This means that if we know $\mathbf{\Omega}_0$ and φ_n, then, by using the analytic and/or numerical methods of ODEs, it it possible to find the *semianalytic solution* of any linear sloshing problem (see Chap. 5 of [58] and [59, 101, 102, 226]) and, thereby, describe the *linear coupled "rigid tank–sloshing" dynamics*.

The three Stokes–Joukowski velocity potentials, Ω_{0i}, $i = 1, 2, 3$, specify the inhomogeneous forcing terms associated with three angular degrees of freedom of the rigid tank. They are solutions of the Neumann boundary problems in the mean (hydrostatic) liquid domain Q_0 (the linear Stokes–Joukowski potential problems, LSJPPs) and were obtained for the first time by Nikolai Joukowski (1885) [99] in examining a rigid body moving in the space and containing a cavity completely filled with an ideal incompressible liquid. The exact analytic Ω_{0i} form a rare exception (see Chap. 5 in [58]).

The natural sloshing modes are the eigenfunctions of a spectral boundary problem (natural sloshing problem, NSP) in Q_0. The spectral parameter κ appears in the mean free-surface (Σ_0) boundary condition and

$$\sigma_n = \sqrt{\kappa_n g}$$

(g is the gravitational acceleration). The traces $\varphi_n|_{\Sigma_0}$ determine the standing wave patterns which were first described for an upright circular basin by Mikhail Ostrogradsky. His manuscript [182] was submitted to the Paris Academy of Sciences in 1826 and later revisited by Poisson and Rayleigh [35] for tanks of other shapes. The rigorous mathematical theory of the NSP was created in the 1960's (Chap. VI in [66] and [39, 170]). It states, in particular, that (i) the spectrum consists solely of positive eigenvalues κ_n with the only limiting point at infinity (unlike the problem of external water waves, which gives a continuous spectrum) and (ii) $\varphi_n|_{\Sigma_0}$ form, together with a nonzero constant, a Fourier basis on Σ_0. The fact (i) is important for understanding why the Korteweg–de Vries, Boussinesq, etc. equations (sea waves) and MSs (contained liquid) follow from the same free-surface problem but are of the different mathematical nature. The fact (ii) is a key stone for introducing the HGCs. In the 1960–1670's, S. Krein [106, 109] (see also [28, 29]) generalized these results to the case of *viscous incompressible liquid*. At the same time, Kopachevskii studied the case of *capillary liquid* (see Part II in [172] and [106, 107]).

In the 1950–1980's, the research focus was on the construction of *analytically approximate solutions of NSP and LSJPP*. Brilliant ideas were proposed (see a collection of these ideas in [145] and an amazing solution for a circular/spherical tank in [23]). Due to the condition of conservation of the volume, these solutions were normally obtained by the Trefftz method suggesting an analytic harmonic functional basis exemplified by the harmonic polynomials [145] whose completeness in the star-shaped domains was proved in [216, 217]. The Trefftz solutions become especially efficient and provide uniform convergence when the corner-point singularity [66, 104, 105, 223, 224] is taken into account [59, 65]. At present, the harmonic polynomials are extensively used in numerical methods, e.g., in the method of harmonic polynomial cell (HPC) [197, 198]. The improving computer facilities in the 1990–2000's made the problem of finding φ_n and $\mathbf{\Omega}_0$ *quite easy*. As an exception, we can probably mention nonsmooth tanks equipped with baffles, screens, etc., in which the behavior of $\nabla\varphi_n$ may become strongly singular [59–62, 65, 79].

The ***theoretical nonlinear sloshing*** was discovered by Penny and Price [184], Moiseev [167], Narimanov [175], and Perko [169, 185] in the 1950–1960's. By adopting the technique of perturbation theory [184], ***Moiseev*** [167] showed how to construct, in the analytic form, an *asymptotic steady-state* (*periodic*, frequency-domain) solution of the nonlinear sloshing problem in a rigid tank performing prescribed horizontal and/or angular small-amplitude harmonic motions with a forcing frequency σ close to the lowest natural sloshing frequency σ_1. He assumed that the liquid depth of an ideal incompressible liquid with irrotational flows is finite and proved that if the dimensionless forcing amplitude (scaled by the tank breadth) is a small parameter $\varepsilon \ll 1$, then the primary excited mode(s) is (are) of the order $O(\varepsilon^{1/3})$ and the matching resonant asymptotics (i.e., the so-called Moiseev detuning) is

$$|\sigma^2 - \sigma_1^2|/\sigma_1^2 = O(\varepsilon^{2/3}).$$

The Moiseev technique implicitly assumed that there are no so-called *secondary resonances* [22, 56–58, 225]. This was originally realized for a two-dimensional rectangular tank [44, 178]. Some other tank shapes were considered in [11, 89, 151, 202]. The construction of the Moiseev solution requires huge and cumbersome transformations. In the 1980's, looking for *almost periodic* sloshing, ***Miles*** [157, 158] generalized Moiseev's results by deducing the so-called *Miles equations* governing the slow-time variations of the predominant, $O(\varepsilon^{1/3})$, amplitudes of the primary excited HGC(s). He considered the case of horizontal harmonic excitation of an upright circular cylindrical tank and adopted the Moiseev asymptotic ordering and detuning. The separation of the fast and slow time scales was realized directly in the *Bateman–Luke variational* principle [9, 84, 126]. The Miles equations were later derived for an upright rectangular tank. The application of these equations is a quite popular approach in the applied mathematical studies of almost-periodic resonant sloshing, in detecting periodic orbits, and in clarifying the chaos [88, 157, 158]. Both horizontal and vertical (Faraday waves) harmonic excitations were in the focus of investigations in [70, 71, 85, 88, 158–163]. In [108, 109], the Miles equations were used to study the "rigid tank–contained liquid" system with limited power supply forcing.

By using the perturbation theory, ***Narimanov*** [175] deduced a *historically first* version of *weakly nonlinear modal systems* (WNMSs). Narimanov did not know Mosieev's results but he postulated certain asymptotic relationships between the HGCs and *hydrodynamic generalized velocities* (HGVs) as if these may follow from the Moiseev solution. The original presentation in [175] (the same in [204–207]) contained some algebraic errors that were later corrected by Lukovsky [136, 140, 144, 176]. Narimanov's technique leads to huge and cumbersome calculations dramatically increasing with the number of HGCs. For this reason, all existing Narimanov's modal systems are of low dimensions; they couple from two to five HGCs. These systems were obtained for upright tanks of circular, annular, and rectangular cross sections, conical and spherical tanks, as well as for an upright circular cylindrical tank with rigid annular baffle [80, 136, 140, 176]. At present, this *method is rarely employed* being replaced by the variational versions of the multimodal method [127–129] based, as a rule, on the Bateman–Luke variational principle [9, 58, 84, 126, 144] which gives, in a natural way, both the dynamic and kinematic relations of the sloshing problem [58, 140, 195, 221, 222].

Most probably, this variational multimodal method was first proposed in 1976 by ***Miles*** and ***Lukovsky*** [140, 156] who independently derived a *fully nonlinear modal system* (Miles–Lukovsky system; MLS) with respect to the HGCs and HGVs for sloshing in an upright tank performing a prescribed translatory motion. Later, Lukovsky derived the MLS for an arbitrary motion of rigid tank [50, 137], proposed the so-called *nonconformal mapping* *technique* to get the MLS for tanks with nonvertical walls [63, 75, 133, 136, 143], and derived the so-called *Lukovsky formulas* for the hydrodynamic forces and moments [140, 144] (Chap. 7 in [58] gives an alternative derivation). He also showed how to use the Bateman–Luke formalism in deducing the dynamic equations of the coupled "rigid tank–contained liquid" system [139, 140] and how to incorporate damping in the variational formulation [141, 144]. The application of asymptotic relationships between HGCs and HGVs reduces the MLS to a weakly nonlinear form, which yields systems now called *weakly nonlinear multimodal systems* (WNMSs, [58, 86, 130, 134, 144]). Both MLS and WNMs require that the *analytically approximate natural sloshing modes* φ_n and the *nonlinear Stokes–Joukowski potentials* $\mathbf{\Omega}$ must be analytically continuable over Σ_0. This and other restrictions are discussed in Chap. 7 of [58] and in [63, 86, 134].

The rare nonlinear sloshing simulations of the 1960–1980's were based on the Galerkin scheme, finite-difference (maker-and-cell etc.), and finite-element methods [6, 40, 41, 45, 171, 173, 180, 191, 200, 208]. By expanding the velocity potential in the Fourier series in natural sloshing modes, ***Perko*** [169, 185] proposed a *numerical multimodal method* aimed at simulating short-term transients. In the 1970–1980's, Perko's method was used, with certain modifications, by *Chakhlov* [18, 37, 38] and *Limarchenko* [114–117, 119, 121]. Since Limarchenko also used the Lagrange variational principle and the perturbation theory, he proposed, in fact, a *weakly nonlinear variational numerical multimodal method*, which is, generally speaking, inapplicable for analytic investigations but suitable for *ad hoc* simulations.

The ***CFD simulations*** of nonlinear sloshing arose *en masse* only in the 1990's. The success was initially associated with the FLOW-3D, which is a famous Navier–Stokes commercial solver of the 1990-2000's (numerous examples were given by Solaas [201] who discussed its advantages and drawbacks). The Volume of Fluid (VoF), Smoothed Partitions Hydromechanics (SPH), and their modifications provided, using parallel computations, rather accurate, efficient, and robust simulations. The readers are referred to [25], which reviews the numerical sloshing of the 1980–1990's.

3. The Present

In the 2000's, the *theoretical nonlinear sloshing* split into almost independent *numerical* and *analytic* directions. The recent advances in the ***numerical sloshing*** are reported in [192] (see also the introductions in [24, 42, 82, 211, 219, 227, 228]). The best CFD solvers are based on viscous and fully nonlinear statement and enable one, by conducting simulations with different initial scenarios (conditions), to model some special free-surface phenomena: fragmentation, wave breaking, overturning, roof and wall impacts, and flip-through. These phenomena cannot be accurately described within the framework of physical (mathematical) models adopted by the ***analytic sloshing*** that are typically of the weakly nonlinear nature, assume ideal potential flows, and smooth instant free-surface patterns. In the XIX century, the same splitting occurred in the theory of sea waves which is now investigated almost independently, with CFD and approximate analytic models. The indicated models are exemplified by the Korteweg–de Vries and Boussinesq, etc. partial differential equations [183]. The approximate analytic models are typically not used in practical computations but rather focus on quantifying and classifying the surface waves, their stability, chaos, and other characteristics depending on the initial scenarios and input parameters in the cases, where the CFD is not very efficient.

Since the *Perko-type method* pursues direct numerical simulations but employs simplified mathematical models of analytic sloshing, it lost the context both of the CFD (in the accurate computing of hydrodynamic loads and in the description of the above-mentioned free-surface phenomena) and of the *analytic sloshing* (in the analysis of the input/initial parameters). This explains why this method is now rarely used being restricted to a few fully nonlinear simulations by MLS for a rectangular tank [111, 196] and weakly nonlinear simulations in [47, 55, 118–120, 122–124]. Another drawback of the Perko-type simulations is that they are unrealistically *stiff* so that artificial damping terms should be incorporated in order to damp the rising parasitic higher harmonics [47, 55] and, hence, to facilitate computations on relatively long time scales. Conducting the simulations may be justified for the screen-equipped tanks when the screen-induced damping is very high and there are no reasons to introduce artificial damping terms [122–124].

The *reincarnation of the multimodal method* as a powerful tool for the investigation of *analytic sloshing* is, to a certain extent, explained by the growing practical interest in smooth (without internal structures) tanks for Liquefied Natural Gas (LNG) . The revolutionary paper [50] rederived the MLS and initiated a series of publications on WNMSs, mostly for *upright tanks with rectangular and circular (annular) cross sections*, when exact analytic φ_n and $\mathbf{\Omega}_0$ exist and the depths of liquid are finite. The WNMSs were applied for the description of steady-state and transient resonant waves; they were validated by model tests.

The ***two-dimensional*** weakly nonlinear resonant sloshing in ***smooth rectangular tanks*** with ***finite depths of liquid*** was studied by using diverse WNMSs [50, 56, 68, 86, 87, 94, 95, 98]. The forcing was small, $O(\varepsilon)$, and the forcing frequency σ was close to the lowest natural sloshing frequency σ_1. The system in [50] was based on the Narimanov–Moiseev asymptotics. The transient and steady-state predictions were validated both for the prescribed harmonic excitations [50] and for coupling with external surface waves (floating tank) [112, 113, 193]. The steady-state resonant sloshing was characterized by the Duffing-like response curves with theoretically soft spring for the *depth-to-breadth ratio* $> 0.3368\ldots$ and a hard spring for $< 0.3368\ldots$, so that the *theoretical dimensionless critical depth* $= 0.3368\ldots$ [58, 220]. A pure mathematical analysis of the WNMS was reported in [68, 86, 87].

The Narimanov–Moiseev WNMS [50] becomes physically inapplicable as *the forcing amplitude increases* and for the *critical and small (shallow) depths of liquid* due to the ***secondary resonance***,

$$n\sigma \approx \sigma_n,$$

observed for some n, which leads to a nonlinearity-driven amplification of the $n\sigma$ harmonics and the energy transfer from primary (σ_1) to secondary (σ_n) excited HGCs. The analysis of the secondary resonance for a finite depth of liquid requires the so-called *adaptive modal systems* (AMSs) [56, 87] suggesting a few extra predominant (secondarily excited) HGCs $\sim O(\varepsilon^{1/3})$ for which the contribution of predominant higher harmonics is theoretically predicted ($n\sigma \approx \sigma_n$). The concept of adaptive multimodal method was extensively validated by the experiments [55–57]. In the presence of the secondary resonance, the response curves are characterized by double peaks in the zone of primary resonance; the peaks grow as the forcing amplitude increases. The AMSs and their structure are extensively discussed in Chap. 8 [58]. Based on the results of AMS modeling, in [87], it was shown that the *critical depth* is a function of the *forcing amplitude* ($0.3368\ldots$ is obtained in the limit as $\varepsilon \to 0$) and explains the experimental value equal to 0.28 presented in [69]. The AMSs [94, 95, 98] use a *sophisticated asymptotic ordering* based on the experimental observations of the surface wave patterns but not on the concept of secondary resonance.

The commensurate (almost commensurate) spectrum of ***shallow (small depth)*** liquid sloshing leads to a hydrodynamic jump [218] (multiple secondary resonances [57]). In deducing the WNMS [57] for small depths of liquid in a rectangular tank, it is necessary to have a Boussinesq fourth-order asymptotic ordering (combining the Moiseev and the Korteweg–de-Vries asymptotics [177–179]), where all HGCs and the dimensionless depth have the same order $O(\varepsilon^{1/4})$. By truncating this infinite-dimensional system and incorporating the ***linear damping*** terms (due to the laminar boundary layer and bulk viscosity; Chap. 6 in [58] and [27, 100, 152, 164, 215]), it is possible to get good agreement with the experiments [32, 33, 57] for both steady-state and transient sloshing. Thus, as in Chester's experiments [32, 33, 57], the theoretical response curves [5, 57] have a fingers-like shape with numerous peaks in the primary resonance zone ($\sigma \approx \sigma_1$). The increase in the excitation amplitude and/or decrease in the depth of liquid make the small-depth WNMS [57] physically inapplicable due to the breaking and overturning of waves, bores, and the free-surface fragmentation responsible for enormously large damping. A detailed experimental classification of the shallow water sloshing (four different types of waves were described), in general, and the indicated phenomena, in particular, is presented in Chap. 8 of [58]. The phenomenon of damping caused by the aforementioned free-surface phenomena is similar to that observed for the ***roof impact*** whose effect was included in the WNMSs of [50, 56] in [49] by using the Wagner theory. In the shallow-water case, the problem of taking damping into account is a challenge.

The effect of damping caused by the separation of flow through a ***perforated screen*** was included in the WNMSs [46–48, 122] for sloshing in a rectangular two-dimensional tank of finite depth. For smaller solidity ratios of the screen,

$$0 < Sn < 0.5,$$

and a relatively small forcing, the use of the pressure-drop condition [15] adds an integral term to the existing modal systems [46, 122]; the modified WNMSs reveal satisfactory agreement with the experiments. Higher solidity ratios,

$$0.5 < Sn < 1,$$

also modify the natural sloshing modes and frequencies [60] and, thereby, both linear [48] and nonlinear (increasing excitation amplitudes) [47] WNMSs changed their analytic structure. The secondary resonance becomes, in this case, evident and, hence, the secondary resonant peaks in the primary resonant zone differ from those observed for the smooth rectangular tank. Numerous WNMSs were derived, studied, and validated for various tanks with perforated screens (see [83, 122–125]).

The generalization of the two-dimensional results [50] to the case of ***three-dimensional rectangular*** tanks was performed in [52]. A focus was made on the *nearly square cross sections* leading to the degenerating Stokes natural sloshing modes including the two modes for the lowest natural frequency σ_1. The corresponding Narimanov–Moiseev WNMS in [52, 54] has nine degrees of freedom with two predominant, $O(\varepsilon^{1/3})$, HGCs. The system provided an accurate *classification* of steady-state (planar, diagonal, nearly diagonal, and swirling) regimes for both longitudinal and diagonal harmonic excitations of the tank [51, 52]. The asymptotically periodic solution of the WNMS gives an amazing 3D bifurcation diagram, especially when the cross-sectional aspect ratio differs from one [20, 21, 54]; a good qualitative agreement with experiments was revealed, including the estimates of the region of chaos. On the other hand, the theoretical transient and steady-state wave response was not quantitatively supported by experiments due to the effect of secondary resonance, which becomes especially evident for swirling, even if the excitation amplitude is sufficiently small. The analysis of the secondary resonance effect leads to the multidimensional AMSs [53, 55], which results in good predictions of swirling and its stability ranges. The numerical stability also shows that the two stable periodic predominant HGCs may coexist with the unstable higher-order HGCs demonstrating an irregular character. This is an additional source of discrepancy. Another source is the fact that damping is satisfactorily predicted for the lowest HGCs (Chap. 6 of [58] and [52, 55, 100, 152, 164]) but not for the higher HGCs. The latter type of damping is strongly affected by the wave breaking and overturning observed in [51, 52, 55]. The adaptive WNMs can also be based on the sophisticated asymptotic ordering deduced from the experiments. As an example, we can mention the paper [97], where the authors studied the resonant sloshing subjected to obliquely horizontal harmonic excitation.

In the 1980's, Lukovsky [131, 139, 140] derived the five-dimensional WNMS for sloshing in ***upright circular cylindrical tanks***, constructed its asymptotic periodic (planar and swirling) solutions, studied their stability by the first Lyapunov method, established the presence of chaos within a certain frequency range, and validated these classification results by experiments [4]. In [96], this system was rederived, the steady-state regimes were classified once again, and the Runge–Kutta simulations were performed (as in [139, 144]). The same WNMS and the constructed periodic solutions were also revisited in [77] and in Chap. 9 of [58]. The paper [77] examined the patterns of nodal curves and theoretically justified that these are not moving straight lines. In Chap. 9 of [58], it is shown that the theoretical classification of the periodic regimes in [140, 144] is, in general, supported by the model tests from [194]. The fifth-order asymptotic ordering [$O(\varepsilon^{1/5})$ for the two predominant HGCs] gives a negligible contribution to this classification [148].

The Narimanov–Moiseev asymptotics theoretically requires an infinite set of the second and third order HGCs included in the WNMSs for ***axisymmetric tanks*** [63, 134]. For the upright circular cylindrical tank, this "complete" WNMS was deduced and analyzed in [130]. The infinite set of higher-order HGCs does not affect the Lukovsky classification results on the periodic steady-state solutions, except for the isolated liquid depths and forcing frequencies for which the presence of secondary resonance is expected (see the depths listed in [22, 80]). Unfortunately, the modal systems from [140] and [130] are not able to quantify the steady-state wave amplitudes measured in [194]. This discrepancy can be explained by the above-mentioned secondary resonance (requiring the AMSs, which have not yet been deduced for this tank shape) and by the effect of surface tension which is neglected in the multimodal analysis.

Lukovsky also derived and analyzed the corresponding five-dimensional WNMS [140, 186, 187] for ***upright annular cylindrical tanks***. It was rederived and modified by adding some extra third-order HGCs in [213], where new model tests were also performed. In [213], by comparing the experimental and theoretical response curves, it was shown that a satisfactory agreement exists for the planar wave regime but, even despite the fact that speculative damping ratios were included to fit the experimental data, a discrepancy was still evident in the case of swirling. An attempt to deduce a Narimanov–Moiseev WNMS for the noncentral pile position was realized in [212]. The problem of sloshing in upright ***compartment*** tanks of circular and annular cross sections was studied in [149, 150] by using the same research scheme as in [140, 144].

For tanks with ***nonvertical walls***, there are no exact analytic natural sloshing modes and the normal presentation of the free surface fails. The latter problem can be resolved by utilizing the *nonconformal mapping technique* proposed by Lukovsky [135, 136, 138, 176] in 1975. The technique was combined with the Narimanov scheme [136, 137] used in the Perko-type simulations [119–121] and incorporated in the Miles–Lukovsky variational method [63, 140, 144]. However, the analytically given multidimensional WNMSs were obtained only for *conical* and *spherical* tanks. The main difficulty in the subsequent extension to the other tank shapes is connected with the absence of the required analytically approximate natural sloshing modes, which exactly satisfy the Laplace equation and the conditions of slip on the nonvertical walls and admit *analytic continuations over the mean free surface* (see the limitations of the multimodal method discussed in [134, 144] and Chap. 9 of [58]).

According to [10, 11, 36], the flat mean free surface in a ***circular conical*** tank was replaced by a spherical cap and, thereby, an approximate analytic solution of NSP was constructed in terms of the spherical functions. The corresponding multidimensional WNMS was derived in [146, 147]. The indicated result was improved (step-by-step) in [73, 74, 76, 81, 132, 140, 142] (see also the references therein). The analytically approximate natural sloshing modes without the indicated replacement were constructed in [73, 81, 132]. Based on these modes, the Narimanov–Moiseev-type five-dimensional WNMSs for *nontruncated and truncated circular V-conical tanks* were deduced and studied in [75, 81]. The systems contain additional (as compared with the upright circular tank) nonlinear terms expressing the so-called *geometric nonlinearity* but the steady-state analysis leads to the same qualitative results extracting the planar and swirling wave regimes, as well as the chaos in a certain frequency range. The paper [78] is focused on the motions of nodal lines generalizing the results from [77]. An emphasis was made on the expectations of secondary resonance. The theoretical analysis of the secondary resonance [75, 81] and the comparison of the theoretical results with the experiments [26, 74, 75, 153] showed that the higher harmonics caused by the secondary resonance are indeed important. The ANMs are required for almost all semiapex angles. This is a challenge.

The required analytically approximate natural sloshing modes for ***circular*** and ***spherical*** tanks were constructed in [7, 8, 61, 62]. Based on the corresponding solutions from [8, 61], a complete infinite-dimensional Narimanov–Moiseev type WNMS (a generalization of [130]) was explicitly derived in [63, 64] and its steady-state solutions were classified. The classification results were supported by the experiments [203] for the depth-to-diameter ratios ≤ 0.5. However, the presence of multiple secondary resonances and an experimentally observed fragmentation of the free surface (accompanied by the overturning waves) make this WNMS inapplicable for higher tank fillings and as the forcing amplitude increases. The Narimanov–Moiseev-type infinite-dimensional WNMS for circular tanks was constructed but not analyzed in [30]. The circular tank shape leads to the appearance of multiple secondary resonances for almost all tank fillings [16, 59]. This makes the results presented in [30] weakly applicable for strongly nonlinear sloshing.

4. The Future

We now discuss *extensive* and *intensive* challenges of the nonlinear multimodal method. Thus, ***extensive*** challenges are mainly associated with the generalization and extension of the multimodal method to new practically important sloshing problems (see, e.g., Chap. 1 in [58] and [31, 144]). At the same time, ***intensive*** challenges are connected with the improvement of the method and WNMSs (from the physical and mathematical points of view), as well as with the dedicated rigorous mathematical studies (uniquely presented by [68, 86, 87] dealing with the modal systems from [50, 56]). Obvious generalizations are connected with the investigation of tanks of complex shapes. It is necessary to construct the analytically approximate natural sloshing modes of special kind and develop the tensor algebra related to the curvilinear coordinates [144] used for these tank shapes. Finally, since the procedure of derivation of the WNMSs for complex tank shapes is especially cumbersome, a challenge is to code a computer algebra that would enable one to realize a computer-based derivation, as this was done for upright circular cylindrical tanks in [72].

As two important physical challenges, we can mention the necessity of adequate taking into account the phenomena of damping and surface tension. In Chap. 6 of [58], the authors reviewed the analytic methods that can be used for the incorporation of the damping-related terms in the WNMSs. However, these ideas were only realized for the case of linear damping caused by the laminar boundary layer and bulk viscosity; recently, this was also done for perforated screens [47, 48, 122]. Damping caused by baffles and piles should be the next goal. The big challenge is to take into account damping caused by the fragmentation of the free surface, overturning, and breaking waves. For nonlinear sloshing, the surface tension requires the analysis of the effect of dynamic contact angle [13, 14]. It is not clear yet, how to include this effect in the multimodal method.

Finally, we believe that the multimodal method may help to describe an attractive swirling-induced V-constant rotation of liquid [43], which was observed in the famous experiments [12, 89, 90, 188, 194] but has not been explained yet.

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