Appendix to "A multimodal method for liquid sloshing in a two-dimensional circular tank"

By O. M. Faltinsen & A. N. Timokha

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Appendix A. The Trefftz approximate natural frequencies and modes

The Trefftz method reduces the original spectral boundary problem to computation of integrals (3.21) and solving the spectral matrix problem (3.20). The integrands in eqs. (3.21) may behave as $(\tau - \tau_{end})^{\alpha-1}$, $1 < \alpha < 2$ at the interval ends caused by the first-order derivatives of ϕ'_i. This requires special quadrature rules. We employ the sinc quadrature formula (Stenger 1983, p. 160, eq. (4.2.37) with conditions (4.2.36)), providing an exponential convergence.

Representation (3.22) uses the singular trial functions ϕ'_i which are required to capture the polar and log-type asymptotics at the corner points and, thereby, improve local convergence to the natural sloshing modes. The presence of these functions yields a formal mathematical conflict, because $\{W'_i(y', z')\}$ is already complete set of harmonic functions in domain Q'_0, without additional functions $\phi'_j(y', z')$ (see, theorems on the completeness by Vekua (1953, 1967) for the so-called star-shaped [relative to the origin O'] domains). This means that solely using the harmonic polynomials should theoretically provide a convergence to the natural sloshing modes in a mean square-root metrics on Σ'_0 and, therefore, the infinite functional set $\{\Phi'_i\}$ is overdetermined so that, formally, there exists a nonunique representation of φ'_i. However, bearing in mind that this mathematical conflict occurs only for $q_1 = \infty$, namely, with infinite number of the harmonic polynomials $W'_k(y', z')$, but q_1 and q_2 are, in practice, finite, and q_2 is not very large, the use of representation (3.22) does not lead to the mentioned nonuniqueness.

The coordinate functions $\{\Phi'_i\}$ are not orthogonal on Σ'_0. This means that the Gramian matrix B may be ill-posed and/or ill-conditioned. To avoid the corresponding numerical problems, the Gram-Schmidt orthogonalization of Φ'_i is implemented for each finite set of the used trial functions.

A.1. *Results on the natural sloshing frequencies (eigenvalues $\bar{\kappa}_i$)*

The Trefftz solution exactly satisfies the Laplace equation and the zero-Neumann boundary condition of eq. (2.5), thus, only the spectral boundary condition should be approximated. This approximation is expected in the mean square-root metrics

$$\epsilon_i = \sqrt{\frac{\int_{\Sigma_0} \left(\frac{\partial \varphi_i}{\partial z} - \bar{\kappa}_i \varphi_i\right)^2 \mathrm{d}S}{\int_{\Sigma_0} (\varphi_i)^2 \, \mathrm{d}S}}, \tag{A 1}$$

assuming $\epsilon_i \to 0$ with increasing q_1 and q_2 in representation (3.22). This means, that the method provides convergence to integral characteristics over the approximate eigenfunctions. Bearing in mind the Rayleigh quotient

$$\bar{\kappa}_i = \frac{\int_{Q_0} (\nabla \varphi_i)^2 \mathrm{d}Q}{\int_{\Sigma_0} \varphi_i^2 \mathrm{d}S} \tag{A 2}$$

$\bar{h} = 0.4$

	antisymmetric				symmetric			
q_1	$\bar{\kappa}_1$	$\bar{\kappa}_3$	$\bar{\kappa}_5$	$\bar{\kappa}_7$	$\bar{\kappa}_2$	$\bar{\kappa}_4$	$\bar{\kappa}_6$	$\bar{\kappa}_8$
5	1.09698	4.93704	9.00771	12.9922	2.89054	6.99059	11.0018	14.9717
6	1.09698	4.93704	9.00752	12.9842	2.89054	6.99058	11.0014	14.9604
7	1.09698	4.93704	9.00750	12.9836	2.89054	6.99058	11.0014	14.9596
8	1.09698	4.93704	9.00749	12.9835	2.89054	6.99058	11.0013	14.9595
9	1.09698	4.93704	9.00749	12.9835	2.89054	6.99058	11.0013	14.9595
	Results by McIver (1989)							
	1.09698	4.93704	9.00749	12.9835	2.89054	6.99058	11.0013	14.9595

$\bar{h} = 0.6$

	antisymmetric				symmetric			
q_1	$\bar{\kappa}_1$	$\bar{\kappa}_3$	$\bar{\kappa}_5$	$\bar{\kappa}_7$	$\bar{\kappa}_2$	$\bar{\kappa}_4$	$\bar{\kappa}_6$	$\bar{\kappa}_8$
5	1.16268	4.69867	8.19917	11.6551	2.88924	6.46067	9.92725	13.3802
6	1.16268	4.69867	8.19888	11.6514	2.88924	6.46065	9.92646	13.3732
	...	...	...	...	...	...	...	...
9	1.16268	4.69867	8.19875	11.6491	2.88924	6.46064	9.92611	13.3692
10	1.16268	4.69867	8.19875	11.6490	2.88924	6.46064	9.92610	13.3691
11	1.16268	4.69867	8.19875	11.6490	2.88924	6.46064	9.92610	13.3691
	Results by McIver (1989)							
	1.16268	4.69867	8.19875	11.6490	2.88924	6.46064	9.92610	13.3691

$\bar{h} = 0.8$

	antisymmetric				symmetric			
q_1	$\bar{\kappa}_1$	$\bar{\kappa}_3$	$\bar{\kappa}_5$	$\bar{\kappa}_7$	$\bar{\kappa}_2$	$\bar{\kappa}_4$	$\bar{\kappa}_6$	$\bar{\kappa}_8$
5	1.24606	4.60679	7.85429	11.0893	2.93248	6.23620	9.46563	12.7018
6	1.24606	4.60672	7.85377	11.0749	2.93247	6.23614	9.46502	12.6823
	...	...	...	...	...	...	...	...
9	1.24606	4.60670	7.85373	11.0741	2.93246	6.23613	9.46500	12.6814
10	1.24606	4.60670	7.85373	11.0741	2.93246	6.23613	9.46500	12.6813
11	1.24606	4.60670	7.85373	11.0741	2.93246	6.23613	9.46500	12.6813
	Results by McIver (1989)							
	1.24606	4.60670	7.85373	11.0741	2.93246	6.23613	9.46499	12.6813

TABLE 1. Convergence of the Trefftz method based on harmonic polynomials W_j' (the ϕ_j'-type components are not used in representation (3.22) for smaller liquid depths. Increasing q_1 is stopped after the six significant figures of the eight eigenvalues are stabilized.

which computes the eigenvalues $\bar{\kappa}_i$ by using integrals over φ_i, an indication of the numerical convergence can also be a stabilization of the significant figures in $\bar{\kappa}_i$ with increasing q_1 and q_2 in representation (3.22). This stabilization is shown in Tables 1–4 for different liquid depths.

For the *lower liquid depths,* $0 < \bar{h} \lesssim 0.8$, the natural sloshing modes φ_i' in the transformed plane have continuous second-order derivatives in Q_0' as well as on boundaries Σ_0' and S_0' including the corner point between them. This means that the natural sloshing modes should be effectively approximated by the Trefftz solution (3.22) based on the harmonic polynomials. Table 1 confirms this preliminary expectation. We see, that the method leads to fast stabilization of the six significant figures of $\bar{\kappa}_i,\ i = 1, ...8$, with ten trial functions W_i'. These eight eigenvalues are in ideal agreement with benchmark numerical values by McIver (1989). The order of the mean square-root errors $\epsilon_i,\ i = 1, ..., 8$, vary for the table cases from 10^{-5} for $\bar{\kappa}_1$ to 10^{-2} for $\bar{\kappa}_8$. For these lower liquid depths,

$\bar{h} = 1.0$ (semicircle)

	antisymmetric				symmetric			
q_1	$\bar{\kappa}_1$	$\bar{\kappa}_3$	$\bar{\kappa}_5$	$\bar{\kappa}_7$	$\bar{\kappa}_2$	$\bar{\kappa}_4$	$\bar{\kappa}_6$	$\bar{\kappa}_8$
5	1.35580	4.65305	7.84283	11.1700	3.03344	6.24321	9.43474	12.8197
6	1.35576	4.65184	7.82712	11.0309	3.03326	6.24074	9.40891	12.6329
7	1.35574	4.65142	7.82261	10.9913	3.03319	6.23990	9.40126	12.5754
	...	...	...	...	...	...	...	...
19	1.35573	4.65106	7.81988	10.9718	3.03311	6.23920	9.39670	12.5457
20	1.35573	4.65106	7.81987	10.9718	3.03311	6.23920	9.39670	12.5457
21	1.35573	4.65106	7.81987	10.9718	3.03311	6.23920	9.39669	12.5457
22	1.35573	4.65106	7.81987	10.9718	3.03311	6.23920	9.39669	12.5457
	Results by McIver (1989)							
	1.35573	4.65105	7.81986	10.9718	3.03310	6.23920	9.39668	12.5457

TABLE 2. Convergence of the Trefftz method based on the harmonic polynomials in representation (3.22). Increasing q_1 is stopped after the six significant figures of the eight eigenvalues $\bar{\kappa}_i$ are stabilized. Larger dimensions q_1 do not affect the stabilized six significant figures. Calculation with $q_1 > 70$ may become unstable due to the ill-conditioned matrix A (within the framework of our double-precision FORTRAN-code using 16 digits).

the order of the maximum mean square-root error, $\epsilon = \max_{i=1,\dots,8} |\epsilon_i|$, becomes 10^{-6} with increasing q_1 to 30. For $0.6 \lesssim \bar{h} \lesssim 0.8$, the same maximum mean square-root error can be achieved with a lower dimension q_1 by using singular trial functions in representation (3.22).

For the *middle liquid depths,* $0.8 \lesssim \bar{h} \lesssim 1.25$, the Trefftz method gives satisfactory results on the eigenvalues by employing the harmonic polynomials. Table 2 illustrates typical convergence (for the half-filled circular tank, $\bar{h} = 1$). Comparing this table with results in Table 1 shows that the middle liquid depths require a larger number of the harmonic polynomials to stabilize the same six significant figures of $\bar{\kappa}_i$. The order of the mean square-root errors varies from 10^{-4} to 10^{-3} with $q_1 = 70$. When $q_1 > 70$, computations require special care of the matrix A in eq. (3.20) which becomes ill-conditioned within the framework of our double-precision FORTRAN-code with 16 digits accuracy.

Accounting for two-three singular trial functions in representation (3.22) can significantly improve convergence for the middle liquid depths. The numerical eigenvalues are then in ideal agreement with those by McIver (1989). We illustrate this fact in Table 3 for $\bar{h} = 1.2$ by comparing convergence of the Trefftz method based on the harmonic polynomials (strategy (a)) and using, in addition, several singular trial harmonic functions (strategy (b), $q_2 = 2$ in our calculations). Case (a) in Table 3 exhibits the convergence similar to that in Table 2 for $\bar{h} = 1$. The order of the mean square-root errors with $q_1 = 50$ varies from 10^{-3} for $\bar{\kappa}_1$ to 10^{-1} for $\bar{\kappa}_8$. Increasing q_1 does not change the stabilized significant figures and the mean square-root errors; the calculations become unstable with $q_1 > 60$. Strategy (b) in this table provides stabilization of the six significant figures with $q_1 = 10$ and $q_2 = 2$. The order of the mean square-root error with $q_1 = 50$ and $q_2 = 2$ is then 10^{-6} for $\bar{\kappa}_1$ and 10^{-4} for $\bar{\kappa}_8$.

Strategy (c) in Table 3 assumes that we do not know how many singular trial functions are needed, and simply postulate $q_1 = q_2 = q$ in representation (3.22). Table 3 (c) illustrates convergence for this numerical strategy. The numerical results are similar to those with strategy (b); the order of the mean square-root error with $q = q_1 = q_2 = 6$ is 10^{-5} for $\bar{\kappa}_1$ and 10^{-3} for $\bar{\kappa}_8$.

Increasing liquid depths to the *larger values,* $1.25 \lesssim \bar{h} \lesssim 1.95$, makes the Trefftz method

$\bar{h} = 1.2$

(a) Results obtained with only harmonic polynomials W_j' in representation (3.22).

	antisymmetric				symmetric			
q_1	$\bar{\kappa}_1$	$\bar{\kappa}_3$	$\bar{\kappa}_5$	$\bar{\kappa}_7$	$\bar{\kappa}_2$	$\bar{\kappa}_4$	$\bar{\kappa}_6$	$\bar{\kappa}_8$
5	1.50805	4.86858	8.23415	12.0573	3.21944	6.50711	9.93949	13.9159
6	1.50782	4.85990	8.15389	11.6716	3.21811	6.48802	9.81497	13.4304
7	1.50770	4.85598	8.11849	11.4942	3.21745	6.47915	9.75670	13.1934
	...	...	...	...	...	...	...	...
49	1.50751	4.85091	8.07837	11.2932	3.21640	6.46748	9.68644	12.8990
50	1.50751	4.85091	8.07836	11.2932	3.21640	6.46748	9.68644	12.8990
51	1.50751	4.85091	8.07836	11.2932	3.21640	6.46748	9.68644	12.8990

(b) Results obtained with $q_2 = 2$

	antisymmetric				symmetric			
q_1	$\bar{\kappa}_1$	$\bar{\kappa}_3$	$\bar{\kappa}_5$	$\bar{\kappa}_7$	$\bar{\kappa}_2$	$\bar{\kappa}_4$	$\bar{\kappa}_6$	$\bar{\kappa}_8$
5	1.50751	4.85092	8.07872	11.3008	3.21640	6.46761	9.69028	12.9506
6	1.50751	4.85091	8.07842	11.2948	3.21640	6.46750	9.68723	12.9128
	...	...	...	...	...	...	...	...
9	1.50751	4.85091	8.07834	11.2932	3.21640	6.46747	9.68640	12.8990
10	1.50751	4.85091	8.07834	11.2932	3.21640	6.46747	9.68639	12.8988
	...	...	...	...	...	...	...	...
51	1.50751	4.85091	8.07834	11.2932	3.21640	6.46747	9.68639	12.8988

(c) Results with $q_1 = q_2 = q$ in representation (3.22)

	antisymmetric				symmetric			
q	$\bar{\kappa}_1$	$\bar{\kappa}_3$	$\bar{\kappa}_5$	$\bar{\kappa}_7$	$\bar{\kappa}_2$	$\bar{\kappa}_4$	$\bar{\kappa}_6$	$\bar{\kappa}_8$
3	1.50751	4.85093	8.07955	11.3169	3.21640	6.46805	9.70116	13.0138
4	1.50751	4.85091	8.07834	11.2933	3.21640	6.46748	9.68653	12.9019
5	1.50751	4.85091	8.07834	11.2932	3.21640	6.46747	9.68639	12.8988
6	1.50751	4.85091	8.07834	11.2932	3.21640	6.46747	9.68639	12.8988
Results by McIver (1989)								
	1.50751	4.85091	8.07834	11.2932	3.21640	6.46747	9.68639	12.8989

TABLE 3. Convergence of the Trefftz method to the eigenvalues $\bar{\kappa}_i,\ i = 1, \ldots, 8$, for $\bar{h} = 1.2$. The calculations are stopped after stabilizing the six significant figures. Case (a) presents calculations done with the harmonic polynomials in representation (3.22); these calculations become unstable for $q_1 > 60$. Case (b) demonstrates convergence when several singular trial functions ϕ_i' ($q_2 = 2$ in representation (3.22)) are added. These numerical results in case (b) are in ideal agreement with those by McIver (1989). Case (c) demonstrates calculations done with $q_1 = q_2 = q$ in representation (3.22).

unstable without adding a sufficient number of singular trial functions. Strategy (b) from Table 3 with $2 \leqslant q_2 \leqslant 3$ is generally applicable, but only for $1.25 \lesssim \bar{h} \lesssim 1.6$. When increasing the liquid depth, one must add more and more singular trial functions for each new larger depth. The required number of these trial functions is *a priori* unknown and, therefore, numerical experiments with q_2 are needed. A way to avoid these experiments consists of using strategy (c) from Table 3 implying $q = q_1 = q_2$. The typical numerical result on convergence to the eight lower eigenvalues is demonstrated in Table 4. The obtained numerical eigenvalues are consistent with those by McIver (1989).

$\bar{h} = 1.4$

	antisymmetric				symmetric			
q	$\bar{\kappa}_1$	$\bar{\kappa}_3$	$\bar{\kappa}_5$	$\bar{\kappa}_7$	$\bar{\kappa}_2$	$\bar{\kappa}_4$	$\bar{\kappa}_6$	$\bar{\kappa}_8$
3	1.73463	5.27693	8.72739	12.2081	3.53753	7.00202	10.4660	13.9593
4	1.73463	5.27678	8.72223	12.1599	3.53751	7.00002	10.4408	13.8989
5	1.73463	5.27678	8.72206	12.1571	3.53751	6.99993	10.4389	13.8724
6	1.73463	5.27678	8.72206	12.1571	3.53751	6.99993	10.4388	13.8722
7	1.73463	5.27678	8.72206	12.1571	3.53751	6.99993	10.4388	13.8722
	Results by McIver (1989)							
	1.73463	5.27678	8.72206	12.1571	3.53751	6.99993	10.4388	13.8722

$\bar{h} = 1.6$

	antisymmetric				symmetric			
q	$\bar{\kappa}_1$	$\bar{\kappa}_3$	$\bar{\kappa}_5$	$\bar{\kappa}_7$	$\bar{\kappa}_2$	$\bar{\kappa}_4$	$\bar{\kappa}_6$	$\bar{\kappa}_8$
3	2.12372	6.14006	10.0942	14.0680	4.14334	8.10782	12.0710	16.3311
4	2.12372	6.13940	10.0831	14.0415	4.14329	8.10410	12.0568	16.0645
5	2.12372	6.13932	10.0808	14.0153	4.14328	8.10320	12.0428	15.9866
6	2.12372	6.13932	10.0807	14.0138	4.14328	8.10315	12.0419	15.9751
7	2.12372	6.13932	10.0807	14.0138	4.14328	8.10314	12.0419	15.9749
8	2.12372	6.13932	10.0807	14.0138	4.14328	8.10314	12.0419	15.9749
	Results by McIver (1989)							
	2.12372	6.13932	10.0807	14.0138	4.14328	8.10314	12.0419	15.9749

$\bar{h} = 1.8$

	antisymmetric				symmetric			
q	$\bar{\kappa}_1$	$\bar{\kappa}_3$	$\bar{\kappa}_5$	$\bar{\kappa}_7$	$\bar{\kappa}_2$	$\bar{\kappa}_4$	$\bar{\kappa}_6$	$\bar{\kappa}_8$
3	3.02142	8.31596	13.6102	19.5977	5.62729	10.9276	16.6444	24.6065
4	3.02140	8.31530	13.5797	18.8751	5.62712	10.9135	16.1997	21.6793
5	3.02140	8.31416	13.5656	18.8530	5.62699	10.9083	16.1850	21.5254
6	3.02140	8.31391	13.5603	18.8088	5.62695	10.9064	16.1628	21.4449
7	3.02140	8.31388	13.5596	18.8003	5.62695	10.9062	16.1589	21.4070
8	3.02140	8.31388	13.5596	18.7997	5.62694	10.9061	16.1586	21.4033
9	3.02140	8.31388	13.5596	18.7997	5.62694	10.9061	16.1586	21.4033
	Results by McIver (1989)							
	3.02140	8.31388	13.5596	18.7997	5.62694	10.9061	16.1586	21.4033

TABLE 4. Typical convergence of the Trefftz method for $1.25 < \bar{h} < 1.95$ based on representation (3.22) with $q = q_1 = q_2$. The calculations are stopped after stabilizing the six significant figures of $\bar{\kappa}_i$. The method needs special care of the ill-conditioned matrix A for $q > 12$ (within the framework of our double-precision FORTRAN-code using 16 digits).

A.2. *Uniform approximation of the natural sloshing modes*

We consider convergence to the natural sloshing modes in the uniform metrics

$$\bar{\chi}_i = \max_{y\in[-\bar{y}_0,\bar{y}_0]} |\chi_i(y)|, \quad \chi_i(y) = \frac{\frac{\partial\varphi_i}{\partial z} - \bar{\kappa}_i\varphi_i}{\varphi_j(-\bar{y}_0, \bar{z}_0)}. \tag{A 3}$$

For the *lower liquid depths*, $0 < \bar{h} \lesssim 0.8$, calculations based on the harmonic polynomials provide a fast practical convergence of the Trefftz solution (3.22) to the natural sloshing modes, φ_i, in both mean square-root, ϵ_i, and uniform, χ_i, metrics. The approximate modes are then characterized by a *clearly dominant contribution* of the W_i-function to the φ_i-mode, i.e. $|c_i| \gg |c_j|,\ j \neq i$.

Situation changes for the *middle liquid depths,* $0.8 \lesssim \bar{h} \lesssim 1.25$, when the Trefftz solu-

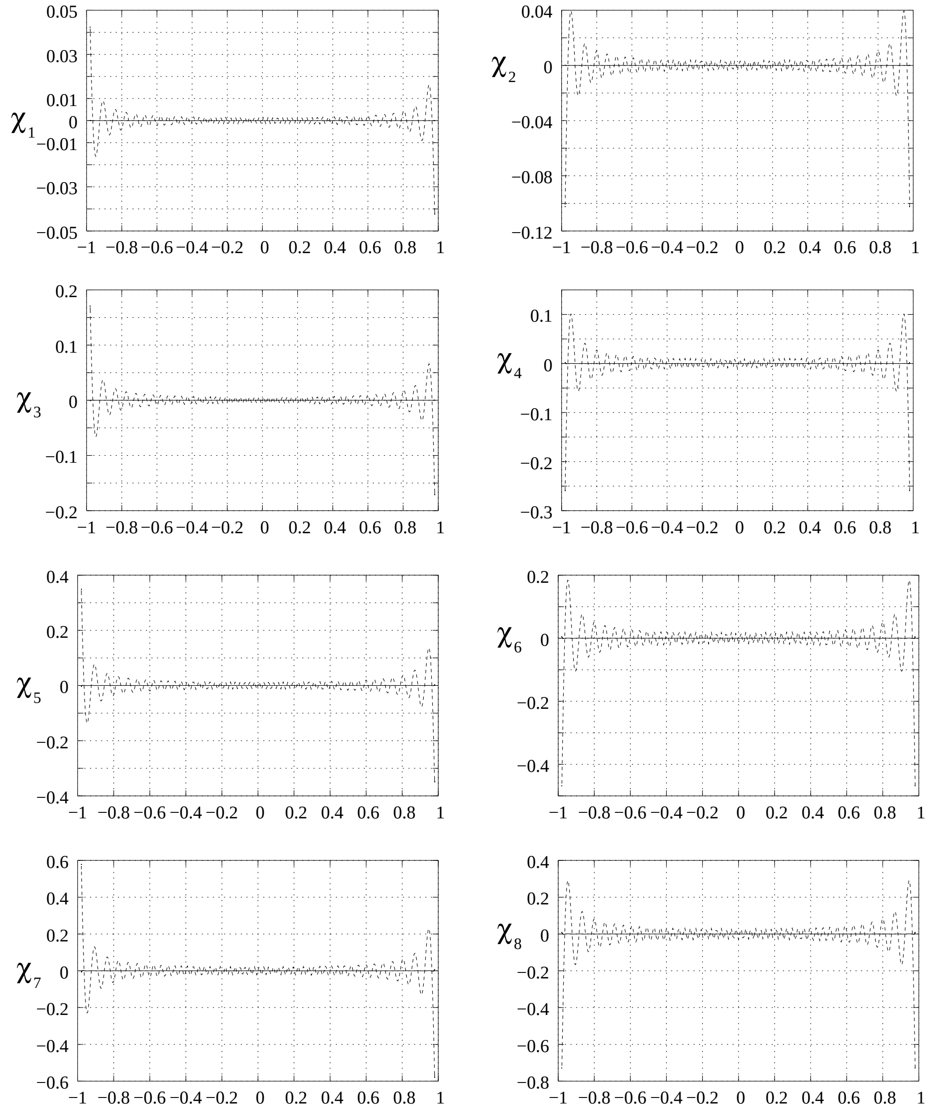

FIGURE 1. The relative error-functions, $\chi_i(y),\ i = 1, \ldots, 8$, by eq. (A 3) representing an error in satisfying the spectral boundary condition in each point of Σ_0. The cases (a) and (b) in Table 2; $\bar{h} = 1.2$. The dotted line presents the error-functions for the Trefftz solution based on the harmonic polynomials ($q_1 = 50$). The solid line shows the error-functions for the case $q_1 = 50$ and $q_2 = 2$ in representation (3.22). The uniform error of the order 10^{-3} is detected for the eight mode ($\bar{\chi}_8$), it can be hardly seen in the figure. The uniform error for the lowest mode, ($\bar{\chi}_1$), is of the order 10^{-5}.

tion based on the harmonic polynomials demonstrates convergence in the mean square-root metrics ϵ_i (see, previous section), but not in the χ_i-metrics. Convergence in the uniform metrics is provided by employing a number of singular trial functions in representation (3.22). This fact is demonstrated in figure 1 showing the error-functions $\chi_i(y)$ for the studied eight approximate eigenfunctions with $\bar{h} = 1.2$. The dotted line presents $\chi_i(y)$ for the approximate natural modes based on the harmonic polynomials, but the solid line corresponds to $\chi_i(y)$ for the Trefftz solution employing the singular trial functions ϕ'_j ($q_2 = 2$ in representation (3.22)). The uniform convergence can also be achieved for the *larger liquid depths,* $1.25 \lesssim \bar{h} \leqslant 1.95$, by using strategy (c) in Table 4 with $q = q_1 = q_2$ in representation (3.22). However, the approximations are less precise. For instance, the case $\bar{h} = 1.8$ in Table 4 establishes the order of the uniform error from 10^{-4} for $\bar{\chi}_1$ to 10^{-2} for $\bar{\chi}_8$.

A.3. *Final comments*

The Trefftz method is an efficient tool for getting approximate natural sloshing modes being adopted in the multimodal method (see, constraints I, II and III in Introduction). A requirement is specific sets of trial functions constructed in the present paper. Numerical experiments show very good consistency with benchmark numerical results by McIver (1989) for the natural sloshing frequencies. We demonstrate a uniform convergence of the Trefftz to the natural sloshing modes.

We also conduced numerical tests for the *passage* $1.95 < \bar{h} \to 2$. Using $q = q_1 = q_2$ in representation (3.22), the corresponding computations demonstrated an accurate approximation of two-three lowest eigenvalues for $1.95 < \bar{h} \leqslant 1.99$. However, these computations are generally unstable for larger $\bar{h}$. A reason is that the natural sloshing modes of the ice-fishing problem should exponentially decay from the mean free surface to the bottom, but the used trial functions do not capture this behavior. Additional trial functions accounting for this decay should be constructed.

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