International Scholarly Research Network
ISRN Mathematical Physics
Volume 2012, Article ID 869070, 19 pages
doi:10.5402/2012/869070

*Research Article*

# Generalizing the Multimodal Method for the Levitating Drop Dynamics

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Received 23 April 2012; Accepted 30 July 2012

Academic Editors: S. Ansoldi, K. Netocny, K.-E. Thylwe, and G. F. Torres del Castillo



The present paper extends the multimodal method, which is well known for liquid sloshing problems, to the free-surface problem modeling the levitating drop dynamics. The generalized Lukovsky-Miles modal equations are derived. Based on these equations an approximate modal theory is constructed to describe weakly-nonlinear axisymmetric drop motions. Whereas the drop performs almost-periodic oscillations with the frequency close to the lowest natural frequency, the theory takes a finite-dimensional form. Periodic solutions of the corresponding finite-dimensional modal system are compared with experimental and numerical results obtained by other authors. A good agreement is shown.

## 1. Introduction

Drops levitating in an ullage gas appear in chemical industry [1–3] and space technology [4–6]. The levitation is provided by weightless conditions or/and acoustic and/or electromagnetic fields created in the gas. When these external fields do not affect the static spherical drop shape, namely, there is no flattening caused by the fields as described, for example, in [7–9], one can assume that the drop dynamics is primary determined by the surface tension as if the drop levitates in zero-gravity conditions.

The relative drop dynamics with respect to the static spherical shape driven by the surface tension has been studied, experimentally [10–12] and theoretically [13]. Small-amplitude (linear) drop motions were *analytically* described by Lord Rayleigh [14, 15] in 1879. He found the corresponding natural (linear eigen) modes and frequencies in terms of the spherical harmonics. Theoretical studies of the nonlinear drop dynamics were mainly done *numerically* by employing different discretization schemes [16–18]. An alternative approach to numerical simulations could be theoretical methods developed, for example, in [19–21] where a Fourier approximation by the Rayleigh natural modes was combined with variational and asymptotic methods. This approach looks similar to nonlinear multimodal methods elaborated in the 70 s for liquid sloshing dynamics.

The multimodal methods took their canonical form in the pioneering papers by Lukovsky [22] and Miles [23] and, furthermore, were generalized by others. An extensive review on the multimodal methods can be found in the book by Lukovsky [24] and Faltinsen and Timokha [25]. The Lukovsky-Miles version of the multimodal methods makes it possible to derive a well-structured infinite-dimensional (modal) system of nonlinear ordinary differential (modal) equations with respect to generalized coordinates and velocities which, under certain circumstances, is fully equivalent to the original free-surface problem. Naturally, the generalized coordinates in sloshing correspond to the natural sloshing modes. Being "asymptotically-detuned" to a class of liquid sloshing phenomena the Lukovsky-Miles modal equations reduce to a rather compact, finite-dimensional form. The detuning suggests postulating a series of asymptotic relations between the generalized coordinates. Representative examples of such "asymptotic modal equations" can be found in the aforementioned books [24, 25] as well as in [26–30] and references therein.

The present paper generalizes the Lukovsky-Miles multimodal method for the free-surface problem describing the nonlinear dynamics of a levitating drop. This generalization includes derivation of general modal equations of Lukovsky-Miles' type as well as examples of asymptotic modal equations. The paper plan is as follows. In Section 2, we present both differential and variational formulations of the problem. Following Lukovsky and Miles as well as recalling [19], the variational formulation is based on the Bateman-Luke principle (see, also, [25], Ch. 7). In Section 3, we rederive the Rayleigh-type eigensolution to show that, from a mathematical point of view, the set of natural modes is not complete and extra four spherical harmonics should be included into the modal solution. In Section 4, we derive the general modal equations analogous to those in [24, 25, 31]. An approximate form of these equations describing the weakly-nonlinear axisymmetric drop dynamics is constructed in Section 5. These equations keep up to third-order polynomial terms as it has been in [27, 32] for sloshing problems. Based on these approximate equations, we derive in Section 6 a finite-dimensional system of "asymptotic" modal equations modeling the weakly-nonlinear almost-periodic drop motions with the frequency close to the lowest natural frequency. Periodic solutions of these equations are compared with experimental [12] and numerical [19, 33, 34] results. A good agreement is shown.

## 2. Statement of the Problem

We consider a levitating drop $Q(t)$ of an ideal incompressible liquid that performs oscillatory motions as illustrated in Figure 1. Due to the surface tension, the drop takes spherical shape in its hydrostatic state. We choose the radius $R_0$ of the sphere as the characteristic length and introduce the characteristic time $t_* = \sqrt{\rho R_0^3/T_s}$ ($T_s$ is the surface tension coefficient). The nondimensional drop dynamics is considered in the spherical coordinate system $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$, $(r \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi)$ so that the free surface $\Sigma(t)$ is described by the equation

$$r = \zeta(\theta,\varphi,t) = 1 + \xi(\theta,\varphi,t). \tag{2.1}$$

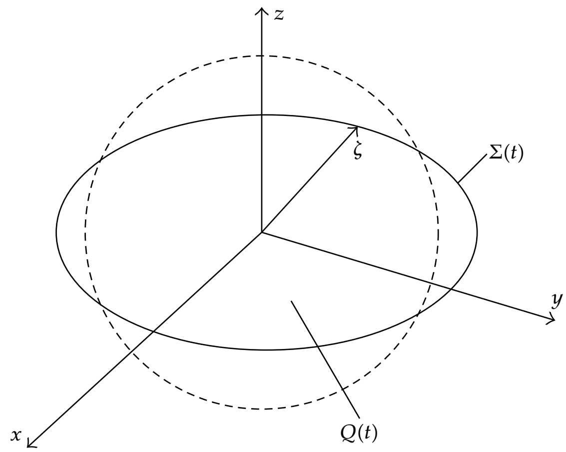


**Figure 1:** Geometric notations.

According to (2.1), perturbations of $\Sigma(t)$ relative to the static spherical shape are subject to the volume conservation condition

$$V_l = \int_{Q(t)} dQ = \frac{4}{3}\pi \Longrightarrow \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{3}\xi^3 + \xi^2 + \xi\right)\sin\theta d\theta d\varphi = 0 \tag{2.2}$$

playing the role of a *holonomic constraint*.

The free-surface problem describing the nonlinear drop dynamics couples the function $\zeta$ and the velocity potential $\Phi$ (see, e.g., [19]):

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0 \quad \text{in } Q(t), \tag{2.3a}$$

$$\zeta\Phi_r - \Phi_\theta\zeta_\theta - \frac{\Phi_\varphi\zeta_\varphi}{\sin\theta} = \zeta\zeta_t \quad \text{on } \Sigma(t), \tag{2.3b}$$

$$\begin{aligned}\left(\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2\right) + \Bigg[&\frac{2 + (\zeta_\theta/\zeta)^2 + (\zeta_\varphi/(\zeta\sin\theta))^2}{\sqrt{\zeta^2 + \zeta_\theta^2 + (\zeta_\varphi/\sin\theta)^2}} - \frac{1}{\zeta^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\zeta\zeta_\theta\sin\theta}{\sqrt{\zeta^2+\zeta_\theta^2+(\zeta_\varphi/\sin\theta)^2}}\right)\\ &-\frac{1}{\zeta^2\sin^2\theta}\frac{\partial}{\partial\varphi}\left(\frac{\zeta\zeta_\varphi}{\sqrt{\zeta^2 + \zeta_\theta^2 + (\zeta_\varphi/\sin\theta)^2}}\right)\Bigg] + \overline{p}_0(t) = 0 \quad \text{on } \Sigma(t),\end{aligned} \tag{2.3c}$$

subject to the volume conservation condition (2.2). Here, the Laplace equation (2.3a) and the Neumann boundary condition (2.3b) constitute together the *kinematic* subproblem and (2.3c) is the so-called *dynamic* boundary condition in which the square bracket term is the sum of the principal curvatures $[k_1 + k_2]$. The dynamic boundary condition expresses the pressure balance on the free surface assuming that the ullage pressure is a constant value and using the Bernoulli equation written for an incompressible inviscid liquid with irrotational flow. The time-dependent function $\overline{p}_0(t)$ implies the difference of the mean pressure between liquid and gas domains caused by the surface tension.

The free-surface problem (2.3a), (2.3b), and (2.3c) requires either the initial conditions

$$\zeta(\theta,\varphi,0) = \zeta_0(\theta,\varphi), \qquad \Phi(r,\theta,\varphi,0) = \Phi_0(r,\theta,\varphi) \tag{2.4}$$

defining initial drop shape and velocity field or the periodicity condition $\zeta(\theta,\varphi,t) = \zeta(\theta,\varphi,t+T)$, $\Phi(r,\theta,\varphi,t) = \Phi(r,\theta,\varphi,t+T)$, where $T = 2\pi/\sigma$ is a fixed period.

Following Lukovsky and Miles [24, 25], we employ the Bateman-Luke variational formulation which states that the free-surface problem (2.3a), (2.3b), and (2.3c) follows from the necessary extrema condition of the action

$$A(\Phi,\zeta) = \int_{t_1}^{t_2} \mathrm{BL}(\Phi,\zeta)\,dt \tag{2.5}$$

within arbitrary instants $t_1$ and $t_2$ ($t_1 < t_2$) and independent variables $\zeta$ and $\Phi$ restricted to

$$\delta\Phi|_{t_1,t_2} = 0, \qquad \delta\zeta|_{t_1,t_2} = 0, \tag{2.6}$$

where the Lagrangian reads as

$$\mathrm{BL}(\Phi,\zeta) = -\int_{Q(t)}\left(\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2\right)dQ - |\Sigma(t)| - \overline{p}_0\left(\int_{Q(t)} dQ - V_l\right). \tag{2.7}$$

Here, $|\cdot|$ defines the area and $\overline{p}_0$ is the Lagrange multiplier (a time-dependent function) caused by the holonomic constraint (2.2).

The Bateman-Luke variational principle is based on the sum of the pressure-integral and potential energy associated with the surface tension. In addition, there is the Lagrange multiplier $\overline{p}_0$ which is the same as the mean pressure difference. Equivalence of the Bateman-Luke variational formulation and free-surface problems in fluid dynamics is, for instance, proven in [24, 25] and the book by Berdichevsky [35].

## 3. Linear Eigensolution and Natural Modes

Let us consider small-amplitude drop oscillations with respect to its static spherical shape by linearizing the volume conservation as well as kinematic (2.3b) and dynamic (2.3c) boundary conditions in terms of $\Phi$ and $\xi$. The linearized volume conservation condition (2.2) takes the form

$$\int_0^{2\pi}\int_0^{\pi} \xi \sin\theta\, d\theta\, d\varphi = 0, \tag{3.1}$$

but the linearized boundary conditions

$$\frac{\partial\Phi}{\partial r}=\frac{\partial\xi}{\partial t},\qquad \frac{\partial\Phi}{\partial t}+\left\{-2\xi-\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\xi}{\partial\theta}\right]+\frac{1}{\sin^2\theta}\frac{\partial^2\xi}{\partial\varphi^2}\right)\right\}=0\quad(r=1)\tag{3.2}$$

can be combined to exclude $\xi$ as follows:

$$\frac{\partial^2\Phi}{\partial t^2}-\left\{2\frac{\partial\Phi}{\partial r}+\frac{\partial}{\partial r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi}\right]\right\}=0\quad(r=1).\tag{3.3}$$

Postulating $\Phi(r,\theta,\varphi,t)=\phi(r,\theta,\varphi)\exp(i\sigma t)$ where $\sigma$ is the so-called *natural (linear eigen) frequency* leads to the spectral boundary problem

$$\begin{gathered}\nabla^2\phi=0\quad(r<1),\qquad \int_0^{2\pi}\int_0^{\pi}\left.\frac{\partial\phi}{\partial r}\right|_{r=1}\sin\theta d\theta d\varphi=0,\\ -\sigma^2\phi=\left\{2\frac{\partial\phi}{\partial r}+\frac{\partial}{\partial r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}\right]\right\}\quad(r=1)\end{gathered}\tag{3.4}$$

with respect to spectral parameter $\sigma^2$ and eigenfunction $\phi$.

The spectral boundary problem (3.4) can be solved by separating the spatial variables $\phi(r,\theta,\varphi)=\overline{Y}_{lm}(r,\theta,\varphi)=r^lY_{lm}(\theta,\varphi)$, $l\geq 0$ which leads to the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2Y_{lm}}{\partial\varphi}=-l(l+1)Y_{lm}.\tag{3.5}$$

The analytical *eigensolution* follows from (3.5) and consists of the eigenfrequencies

$$\sigma^2=\sigma_{lm}^2=l(l-1)(l+2),\quad l=0,1,\ldots,\ m=0,\ldots,l\tag{3.6}$$

and the eigenfunctions

$$\phi_{lm}=\overline{Y}_{lm}(r,\theta,\varphi)=N_{lm}r^lP_l^{(m)}(\cos\theta)\begin{cases}\cos m\varphi,\\ \sin m\varphi,\end{cases}\qquad N_{lm}=\begin{cases}\sqrt{\dfrac{(2l+1)(l-m)!}{4\pi(l+m)!}}, & m=0,\\ \sqrt{\dfrac{(2l+1)(l-m)!}{2\pi(l+m)!}}, & m\geq 1,\end{cases}\tag{3.7}$$

where $P_l^{(m)}$ are the associated Legendre polynomials.

Four eigenfunctions with $l=0$ and 1 imply the zero eigenfrequencies and, from the physical point of view, these eigenfunctions do not belong to the set of *natural modes* by Lord Rayleigh [14, 15]. The case $l=1$ with $m=0$ gives $\phi_{10}=z=r\cos\theta$ that describes a translatory drop motion (as a solid body) along $Oz$, but $l = 1$ and $m = 1$ yield $y = r\sin\theta\sin\varphi$ and $x = r\sin\theta\cos\varphi$ describing the same translatory motions but along $Oy$ and $Ox$, respectively. The case $l = 0$ corresponds to $\phi_{00} = 1/2\sqrt{\pi}$. Excluding these four eigenfunctions, that is, concentrating on the Rayleigh solution, makes the functional basis (3.7) incomplete from the mathematical point of view [36, 37].

## 4. Nonlinear Modal Equations

The Lukovsky-Miles multimodal method suggests the modal solution of the free-surface problem (2.3a), (2.3b), and (2.3c) as follows:

$$\zeta(\theta,\varphi,t) = 1 + \sum_I \beta_I(t) f_I(\theta,\varphi), \qquad \Phi(r,\theta,\varphi,t) = \sum_N F_N(t)\phi_N(r,\theta,\varphi), \tag{4.1}$$

where $\{f_I\}$ and $\{\phi_N\}$ are the complete sets of functions to define admissible shapes $Q(t)$ satisfying the volume conservation condition and approximating the velocity field, respectively. Dealing with the star-shaped domains $Q(t)$, the solid harmonics (3.7) provide the completeness [36, 37] so that we can write down

$$\begin{gathered}\phi_l = N_{l0} r^l P_l(\cos\theta), \quad l \geq 0,\\ \phi_{lm,c} = \phi_{lm}(r,\theta)\cos m\varphi = N_{lm} r^l P_l^{(m)}(\cos\theta)\cos m\varphi, \quad l \geq 1,\ m = 1,\ldots,l,\\ \phi_{lm,s} = \phi_{lm}(r,\theta)\sin m\varphi = N_{lm} r^l P_l^{(m)}(\cos\theta)\sin m\varphi, \quad l \geq 1,\ m = 1,\ldots,l,\end{gathered} \tag{4.2a}$$

$$\begin{gathered}f_l = N_{l0} P_l(\cos\theta), \quad l \geq 0,\\ f_{lm,c} = f_{lm}(\theta)\cos m\varphi = N_{lm} P_l^{(m)}(\cos\theta)\cos m\varphi, \quad l \geq 1,\ m = 1,\ldots,l,\\ f_{lm,s} = f_{lm}(\theta)\sin m\varphi = N_{lm} P_l^{(m)}(\cos\theta)\sin m\varphi, \quad l \geq 1,\ m = 1,\ldots,l.\end{gathered} \tag{4.2b}$$

This transforms the modal solution (4.1) to the form

$$\zeta(\theta,\varphi,t) = 1 + \sum_{l=0}^{\infty}\beta_l(t) f_l(\theta) + \sum_{l=1}^{\infty}\sum_{m=1}^{l}\left(\beta_{c,lm}(t)\cos m\varphi + \beta_{s,lm}(t)\sin m\varphi\right) f_{lm}(\theta), \tag{4.3a}$$

$$\Phi(r,\theta,\varphi,t) = \sum_{l=0}^{\infty} F_l(t)\phi_l(r,\theta) + \sum_{l=1}^{\infty}\sum_{m=1}^{l}\left(F_{c,lm}(t)\cos m\varphi + F_{s,lm}(t)\sin m\varphi\right)\phi_{lm}(r,\theta). \tag{4.3b}$$

Accounting for (4.3a) in (2.2) gives the holonomic constraint

$$2\sqrt{\pi}\beta_0 + \sum_{i=0}^{\infty}\beta_i^2 + \sum_{l=1}^{\infty}\sum_{m=1}^{l}\left(\beta_{c,lm}^2 + \beta_{s,lm}^2\right) + \widetilde{G}_3(\beta_i,\beta_{c,lm},\beta_{s,lm}) = 0, \tag{4.4}$$

where $\widetilde{G}_3$ implies the cubic, fourth, and so forth polynomial terms. Using the implicit function theorem resolves $\beta_0$ as follows:

$$\beta_0 = G(\beta_i,\ \beta_{c,lm},\ \beta_{s,lm},\ i \geq 1,\ l \geq 1) = -\frac{1}{2\sqrt{\pi}}\left[\sum_{i=1}\beta_i^2 + \sum_{l=1}^{\infty}\sum_{m=1}^{l}\left(\beta_{c,lm}^2 + \beta_{s,lm}^2\right)\right] - \frac{1}{2\sqrt{\pi}}G_3(\beta_i,\ \beta_{c,lm},\ \beta_{s,lm},\ i \geq 1,\ l \geq 1) \tag{4.5}$$

($G_3$ also denotes the cubic and other higher-order polynomial terms in $\beta_*$) and transforms (4.3a) to the form

$$\zeta(\theta,\varphi,t) = 1 - \frac{1}{4\pi}\left(\sum_{i=1}^{\infty}\beta_i^2 + \sum_{l=1}^{\infty}\sum_{m=1}^{l}\left(\beta_{c,lm}^2 + \beta_{s,lm}^2\right) + G_3\right) + \sum_{l=1}^{\infty}\beta_l(t)f_l(\theta) + \sum_{l=1}^{\infty}\sum_{m=1}^{l}(\beta_{c,lm}(t)\cos m\varphi + \beta_{s,lm}(t)\sin m\varphi)f_{lm}(\theta) \tag{4.6}$$

defining the free surface as a function of $\beta_i$, $\beta_{c,lm}$, $\beta_{s,lm}$, $i \geq 1$, $l \geq 1$. Representation (4.6) *automatically satisfies* the volume conservation condition and, as a consequence, the Lagrange multiplier in (2.7) should equal to zero, that is, $\overline{p}_0 = 0$.

The generalized velocity $F_0$ can also be excluded from consideration due to the identity

$$\frac{2}{3}\sqrt{\pi}\int_{t_1}^{t_2}\delta\dot{F}_0 dt = \frac{2}{3}\sqrt{\pi}[\delta F_0(t_2) - \delta F_0(t_1)] = 0, \tag{4.7}$$

provided by (2.6) or, more precisely, by

$$\delta\beta_i(t_1) = \delta\beta_i(t_2) = \delta\beta_{c,lm}(t_1) = \delta\beta_{c,lm}(t_2) = \delta\beta_{s,lm}(t_1) = \delta\beta_{s,lm}(t_2) = \delta F_i(t_1) = \delta F(t_2) = \delta F_{c,lm}(t_1) = \delta F_{c,lm}(t_2) = \delta F_{s,lm}(t_1) = \delta F_{s,lm}(t_2) = 0. \tag{4.8}$$

Substituting (4.3b) into (2.7) yields the Lagrangian as a function of generalized coordinates and velocities

$$\begin{aligned} \mathrm{BL} = &-\sum_{i=1}^{\infty} A_i\dot{F}_i - \sum_{l=1}^{\infty}\sum_{m=1}^{l} A_{c,lm}\dot{F}_{c,lm} - \sum_{l=1}^{\infty}\sum_{m=1}^{l} A_{c,lm}\dot{F}_{c,lm} - \frac{1}{2}\sum_{n,k=1}^{\infty} A_{n,k}F_nF_k \\ &- \frac{1}{2}\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2} A_{(c,l_1m_1),(c,l_2m_2)}F_{(c,l_1m_1)}F_{(c,l_2m_2)} \\ &- \frac{1}{2}\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2} A_{(s,l_1m_1),(s,l_2m_2)}F_{(s,l_1m_1)}F_{(s,l_2m_2)} \end{aligned}$$

$$-\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2} A_{(c,l_1m_1),(s,l_2m_2)}F_{(c,l_1m_1)}F_{(s,l_2m_2)} - \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{l} A_{n,(s,lm)}F_nF_{(s,lm)}$$
$$-\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{l} A_{n,(c,lm)}F_nF_{(c,lm)} - TS = 0, \tag{4.9}$$

where

$$A_n = \int_{Q(t)} \phi_n dQ = \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} \phi_n r^2 \sin\theta dr\, d\theta\, d\varphi, \tag{4.10a}$$

$$A_{c,lm} = \int_{Q(t)} \phi_{lm}\cos(m\varphi) dQ = \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} \phi_{lm}\cos(m\varphi) r^2 \sin\theta dr\, d\theta\, d\varphi, \tag{4.10b}$$

$$A_{s,lm} = \int_{Q(t)} \phi_{lm}\sin(m\varphi) dQ = \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} \phi_{lm}\sin(m\varphi) r^2 \sin\theta dr\, d\theta\, d\varphi, \tag{4.10c}$$

$$A_{n,k} = A_{k,n} = \int_{Q(t)} (\nabla\phi_n \cdot \nabla\phi_k) dQ = \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} (\nabla\phi_n \cdot \nabla\phi_k) r^2 \sin\theta dr\, d\theta\, d\varphi, \tag{4.11a}$$

$$\begin{aligned} A_{n,(c,lm)} = A_{(c,lm),n} &= \int_{Q(t)} (\nabla\phi_n \cdot \nabla[\phi_{lm}\cos m\varphi]) dQ \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} (\nabla\phi_n \cdot \nabla[\phi_{lm}\cos m\varphi]) r^2 \sin\theta dr\, d\theta\, d\varphi, \end{aligned} \tag{4.11b}$$

$$\begin{aligned} A_{n,(s,lm)} = A_{(s,lm),n} &= \int_{Q(t)} (\nabla\phi_n \cdot \nabla[\phi_{lm}\sin m\varphi]) dQ \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} (\nabla\phi_n \cdot \nabla[\phi_{lm}\sin m\varphi]) r^2 \sin\theta dr\, d\theta\, d\varphi, \end{aligned} \tag{4.11c}$$

$$\begin{aligned} A_{(c,l_1m_1),(s,l_2m_2)} = A_{(s,l_2m_2),(c,l_1m_1)} &= \int_{Q(t)} (\nabla[\phi_{l_1m_1}\cos m_1\varphi] \cdot \nabla[\phi_{l_2m_2}\sin m_2\varphi]) dQ \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} (\nabla[\phi_{l_1m_1}\cos m_1\varphi] \cdot \nabla[\phi_{l_2m_2}\sin m_2\varphi]) r^2 \sin\theta dr\, d\theta\, d\varphi, \end{aligned} \tag{4.11d}$$

$$\begin{aligned} A_{(c,l_1m_1),(c,l_2m_2)} = A_{(c,l_2m_2),(c,l_1m_1)} &= \int_{Q(t)} (\nabla[\phi_{l_1m_1}\cos m_1\varphi] \cdot \nabla[\phi_{l_2m_2}\cos m_2\varphi]) dQ \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} (\nabla[\phi_{l_1m_1}\cos m_1\varphi] \cdot \nabla[\phi_{l_2m_2}\cos m_2\varphi]) r^2 \sin\theta dr\, d\theta\, d\varphi, \end{aligned} \tag{4.11e}$$

$$A_{(s,l_1m_1),(s,l_2m_2)} = A_{(s,l_2m_2),(s,l_1m_1)} = \int_{Q(t)} \left(\nabla\left[\phi_{l_1m_1}\sin m_1\varphi\right]\cdot\nabla\left[\phi_{l_2m_2}\sin m_2\varphi\right]\right)dQ$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta}\left(\nabla\left[\phi_{l_1m_1}\sin m_1\varphi\right]\cdot\nabla\left[\phi_{l_2m_2}\sin m_2\varphi\right]\right)r^2\sin\theta dr\,d\theta\,d\varphi, \tag{4.11f}$$

$$TS = \int_{\Sigma(t)} dS = \int_0^{2\pi}\int_0^{\pi}\zeta\sqrt{\zeta^2+\zeta_\theta^2+\frac{\zeta_\varphi^2}{\sin^2\theta}}\sin\theta d\theta d\varphi. \tag{4.12}$$

Performing a variation of independent generalized velocities $F_i$, $F_{c,lm}$, $F_{s,lm}$, $i \ge 1$, $l \ge 1$ in the action (2.5) within the Lagrangian (4.9) leads to the equations

$$\frac{dA_n}{dt} = \sum_{k=1}^{\infty}A_{n,k}F_k + \sum_{k=1}^{\infty}\sum_{m=1}^{k}\left(A_{n,(c,km)}F_{c,km} + A_{n,(s,km)}F_{s,km}\right), \quad n \ge 1, \tag{4.13a}$$

$$\frac{dA_{c,lm}}{dt} = \sum_{k=1}^{\infty}A_{(c,lm),k}F_k + \sum_{k=1}^{\infty}\sum_{n=1}^{k}\left(A_{(c,lm),(c,kn)}F_{c,kn} + A_{(c,lm),(s,kn)}F_{s,kn}\right), \tag{4.13b}$$

$$\frac{dA_{s,lm}}{dt} = \sum_{k=1}^{\infty}A_{(s,lm),k}F_k + \sum_{k=1}^{\infty}\sum_{n=1}^{k}\left(A_{(s,lm),(c,kn)}F_{c,kn} + A_{(s,lm),(s,kn)}F_{s,kn}\right), \tag{4.13c}$$

($l \ge 1$, $m = 1,\dots,l$). Derivations leading to (4.13a), (4.13b), and (4.13c) are quite tedious but, under certain circumstances, these are similar to those in [24, 25] for sloshing problems.

The differentiation rule

$$\frac{dA_n}{dt} = \sum_{i=1}^{\infty}\frac{\partial A_n}{\partial\beta_i}\dot\beta_i + \sum_{l=1}^{\infty}\sum_{m=1}^{l}\left(\frac{\partial A_n}{\partial\beta_{c,lm}}\dot\beta_{c,lm} + \frac{\partial A_n}{\partial\beta_{s,lm}}\dot\beta_{s,lm}\right), \tag{4.14a}$$

$$\frac{dA_{c,lm}}{dt} = \sum_{i=1}^{\infty}\frac{\partial A_{c,lm}}{\partial\beta_i}\dot\beta_i + \sum_{j=1}^{\infty}\sum_{n=1}^{j}\left(\frac{\partial A_{c,lm}}{\partial\beta_{c,jn}}\dot\beta_{c,jn} + \frac{\partial A_{c,lm}}{\partial\beta_{s,jn}}\dot\beta_{s,jn}\right), \tag{4.14b}$$

$$\frac{dA_{s,lm}}{dt} = \sum_{i=1}^{\infty}\frac{\partial A_{s,lm}}{\partial\beta_i}\dot\beta_i + \sum_{j=1}^{\infty}\sum_{n=1}^{j}\left(\frac{\partial A_{s,jn}}{\partial\beta_{c,jn}}\dot\beta_{c,jn} + \frac{\partial A_{s,lm}}{\partial\beta_{s,jn}}\dot\beta_{s,jn}\right), \tag{4.14c}$$

shows that (4.13a), (4.13b), and (4.13c) is a system of nonlinear ordinary differential equations with respect to generalized coordinates where the mass-matrix depends on $\beta_*$. On the other hand, relations (4.13a), (4.13b), and (4.13c) can be considered as a system of algebraic equations with respect to generalized velocities $F_i$, $F_{c,lm}$, $F_{s,lm}$, $i \ge 1$, $l \ge 1$, where $A_{n,k}$ are nonlinear functions of generalized coordinates $\beta_i$, $\beta_{c,lm}$, $\beta_{s,lm}$, $i \ge 1$, $l \ge 1$ but the left-hand side $dA_n/dt$, $dA_{c,lm}/dt$, $dA_{s,lm}/dt$ implies expressions with respect to generalized coordinates $\beta_i$, $\beta_{c,lm}$, $\beta_{s,lm}$, $i \ge 1$, $l \ge 1$ and their first derivative. Equations (4.13a), (4.13b), and (4.13c) are interpreted as *kinematic equations* or a nonholonomic constraint.

The Euler-Lagrange equations follow from the extrema condition of the action with respect to generalized coordinates $\beta_i$, $\beta_{c,lm}$, $\beta_{s,lm}$, $i \geq 1$, $l \geq 1$. They are often called the *dynamic* modal equations and take the form

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\partial A_n}{\partial \beta_\mu} \dot{F}_n + \sum_{l=1}^{\infty} \sum_{m=1}^{l} \left( \frac{\partial A_{c,lm}}{\partial \beta_\mu} \dot{F}_{c,lm} + \frac{\partial A_{c,lm}}{\partial \beta_\mu} \dot{F}_{c,lm} \right) + \frac{1}{2} \sum_{n,k=1}^{\infty} \frac{\partial A_{n,k}}{\partial \beta_\mu} F_n F_k \\
&\quad + \sum_{n,l=1}^{\infty} \sum_{m=1}^{l} F_n \left( \frac{\partial A_{n,(c,lm)}}{\partial \beta_\mu} F_{c,lm} + \frac{\partial A_{n,(s,lm)}}{\partial \beta_\mu} F_{s,lm} \right) \\
&\quad + \sum_{l_1,l_2=1}^{\infty} \sum_{m_1,m_2=1}^{l_1,l_2} F_{c,l_1 m_1} F_{s,l_2 m_2} \frac{\partial A_{(c,l_1 m_1),(s,l_2 m_2)}}{\partial \beta_\mu} \\
&\quad + \frac{1}{2} \sum_{l_1,l_2=1}^{\infty} \sum_{m_1,m_2=1}^{l_1,l_2} F_{c,l_1 m_1} F_{c,l_2 m_2} \frac{\partial A_{(c,l_1 m_1),(c,l_2 m_2)}}{\partial \beta_\mu} \\
&\quad + \frac{1}{2} \sum_{l_1,l_2=1}^{\infty} \sum_{m_1,m_2=1}^{l_1,l_2} F_{s,l_1 m_1} F_{s,l_2 m_2} \frac{\partial A_{(s,l_1 m_1),(s,l_2 m_2)}}{\partial \beta_\mu} + \frac{\partial TS}{\partial \beta_\mu} = 0, \quad \mu \geq 1,
\end{aligned} \tag{4.15a}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\partial A_n}{\partial \beta_{c,\mu\nu}} \dot{F}_n + \sum_{l=1}^{\infty} \sum_{m=1}^{l} \left( \frac{\partial A_{c,lm}}{\partial \beta_{c,\mu\nu}} \dot{F}_{c,lm} + \frac{\partial A_{c,lm}}{\partial \beta_{c,\mu\nu}} \dot{F}_{c,lm} \right) + \frac{1}{2} \sum_{n,k=1}^{\infty} \frac{\partial A_{n,k}}{\partial \beta_{c,\mu\nu}} F_n F_k \\
&\quad + \sum_{n,l=1}^{\infty} \sum_{m=1}^{l} F_n \left( \frac{\partial A_{n,(c,lm)}}{\partial \beta_{c,\mu\nu}} F_{c,lm} + \frac{\partial A_{n,(s,lm)}}{\partial \beta_{c,\mu\nu}} F_{s,lm} \right) \\
&\quad + \sum_{l_1,l_2=1}^{\infty} \sum_{m_1,m_2=1}^{l_1,l_2} F_{c,l_1 m_1} F_{s,l_2 m_2} \frac{\partial A_{(c,l_1 m_1),(s,l_2 m_2)}}{\partial \beta_{c,\mu\nu}} \\
&\quad + \frac{1}{2} \sum_{l_1,l_2=1}^{\infty} \sum_{m_1,m_2=1}^{l_1,l_2} F_{c,l_1 m_1} F_{c,l_2 m_2} \frac{\partial A_{(c,l_1 m_1),(c,l_2 m_2)}}{\partial \beta_{c,\mu\nu}} \\
&\quad + \frac{1}{2} \sum_{l_1,l_2=1}^{\infty} \sum_{m_1,m_2=1}^{l_1,l_2} F_{s,l_1 m_1} F_{s,l_2 m_2} \frac{\partial A_{(s,l_1 m_1),(s,l_2 m_2)}}{\partial \beta_{c,\mu\nu}} + \frac{\partial TS}{\partial \beta_{c,\mu\nu}} = 0, \quad \mu \geq 1, \; n = 1, \ldots, \mu,
\end{aligned} \tag{4.15b}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\partial A_n}{\partial \beta_{s,\mu\nu}} \dot{F}_n + \sum_{l=1}^{\infty} \sum_{m=1}^{l} \left( \frac{\partial A_{c,lm}}{\partial \beta_{s,\mu\nu}} \dot{F}_{c,lm} + \frac{\partial A_{c,lm}}{\partial \beta_{s,\mu\nu}} \dot{F}_{c,lm} \right) + \frac{1}{2} \sum_{n,k=1}^{\infty} \frac{\partial A_{n,k}}{\partial \beta_{s,\mu\nu}} F_n F_k \\
&\quad + \sum_{n,l=1}^{\infty} \sum_{m=1}^{l} F_n \left( \frac{\partial A_{n,(c,lm)}}{\partial \beta_{s,\mu\nu}} F_{c,lm} + \frac{\partial A_{n,(s,lm)}}{\partial \beta_{s,\mu\nu}} F_{s,lm} \right) \\
&\quad + \sum_{l_1,l_2=1}^{\infty} \sum_{m_1,m_2=1}^{l_1,l_2} F_{c,l_1 m_1} F_{s,l_2 m_2} \frac{\partial A_{(c,l_1 m_1),(s,l_2 m_2)}}{\partial \beta_{s,\mu\nu}}
\end{aligned}$$

$$+\frac{1}{2}\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2}F_{c,l_1m_1}F_{c,l_2m_2}\frac{\partial A_{(c,l_1m_1),(c,l_2m_2)}}{\partial\beta_{s,\mu\nu}}$$
$$+\frac{1}{2}\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2}F_{s,l_1m_1}F_{s,l_2m_2}\frac{\partial A_{(s,l_1m_1),(s,l_2m_2)}}{\partial\beta_{s,\mu\nu}}+\frac{\partial TS}{\partial\beta_{s,\mu\nu}}=0,\quad \mu\geq 1,\ n=1,\ldots,\mu. \tag{4.15c}$$

The derivative by $\beta_*$ is done assuming that (4.6) accounts for the volume conservation condition so that, for instance,

$$\begin{aligned}\frac{\partial TS}{\partial\beta_\mu}&=\int_0^{2\pi}\int_0^{\pi}(k_1+k_2)\zeta^2\left[f_\mu-\frac{1}{2\pi}\beta_\mu-\frac{1}{4\pi}\frac{\partial G_3}{\partial\beta_\mu}\right]\sin\theta d\theta\, d\varphi\\&=\int_{\Sigma(t)}\frac{\zeta(k_1+k_2)}{\sqrt{\zeta^2+\zeta_\theta^2+(\zeta_\varphi/\sin\theta)^2}}\left[f_\mu-\frac{1}{2\pi}\beta_\mu-\frac{1}{4\pi}\frac{\partial G_3}{\partial\beta_\mu}\right]dS,\end{aligned} \tag{4.16a}$$

$$\begin{aligned}\frac{\partial TS}{\partial\beta_{c,\mu\nu}}&=\int_0^{2\pi}\int_0^{\pi}(k_1+k_2)\zeta^2\left[f_{\mu\nu}\cos(\nu\varphi)-\frac{1}{2\pi}\beta_{c,\mu\nu}-\frac{1}{4\pi}\frac{\partial G_3}{\partial\beta_{c,\mu\nu}}\right]\sin\theta d\theta\, d\varphi\\&=\int_{\Sigma(t)}\frac{\zeta(k_1+k_2)}{\sqrt{\zeta^2+\zeta_\theta^2+(\zeta_\varphi/\sin\theta)^2}}\left[f_{\mu\nu}\cos(\nu\varphi)-\frac{1}{2\pi}\beta_{c,\mu\nu}-\frac{1}{4\pi}\frac{\partial G_3}{\partial\beta_{c,\mu\nu}}\right]dS,\end{aligned} \tag{4.16b}$$

$$\begin{aligned}\frac{\partial TS}{\partial\beta_{s,\mu\nu}}&=\int_0^{2\pi}\int_0^{\pi}(k_1+k_2)\zeta^2\left[f_{\mu\nu}\sin(\nu\varphi)-\frac{1}{2\pi}\beta_{s,\mu\nu}-\frac{1}{4\pi}\frac{\partial G_3}{\partial\beta_{s,\mu\nu}}\right]\sin\theta d\theta\, d\varphi\\&=\int_{\Sigma(t)}\frac{\zeta(k_1+k_2)}{\sqrt{\zeta^2+\zeta_\theta^2+(\zeta_\varphi/\sin\theta)^2}}\left[f_{\mu\nu}\sin(\nu\varphi)-\frac{1}{2\pi}\beta_{s,\mu\nu}-\frac{1}{4\pi}\frac{\partial G_3}{\partial\beta_{s,\mu\nu}}\right]dS.\end{aligned} \tag{4.16c}$$

In summary, the Lukovsky-Miles modal equations (4.13a), (4.13b), (4.13c), (4.15a), (4.15b), and (4.15c) constitute an infinite-dimensional system of nonlinear ordinary differential equations with respect to generalized coordinates and velocities. A direct Runge-Kutta simulation with this system (Perko-type method [25, 26]) is possible adopting appropriate initial conditions following from (2.4). However, when the goal consists of analytical studies and/or description of almost-periodic motions, it would be better to reduce the system to a simpler approximate form by postulating asymptotic relationships between generalized coordinates and velocities and neglecting the higher-order terms. The reduced (asymptotic) modal system may in particular cases possess a finite-dimensional form.

## 5. Weakly-Nonlinear Modal Equations for Axisymmetric Drop Motions

For the axisymmetric drop dynamics, the velocity potential takes the form

$$\Phi(r,\theta,\varphi,t)=\sum_{l=1}^{\infty}F_l(t)\phi_l(r,\theta) \tag{5.1}$$

and the free-surface equation is as follows:

$$\zeta(\theta,\varphi,t) = 1 - \frac{1}{4\pi}\left(\sum_{i=1}^{\infty}\beta_i^2 + \frac{1}{3}\sum_{i,j,k=1}^{\infty}\Lambda_{ijk}^{(3)}\beta_i\beta_j\beta_k + G_5\right) + \sum_{l=1}^{\infty}\beta_l(t)f_l(\theta), \tag{5.2}$$

where

$$\Lambda_{ijm}^{(3)} = 2\pi\int_0^{\pi} f_i f_j f_m \sin\theta d\theta = \frac{1}{2}\sqrt{\frac{(2i+1)(2j+1)}{\pi(2m+1)}}\left(C_{i0,j0}^{m0}\right)^2 \tag{5.3}$$

and $C_{i0,j0}^{m0}$ are the Clebsch-Gordan coefficients [38].

Henceforth, adopting ideas from [27, 32], we will construct a weakly-nonlinear, third-order modal equations by postulating the relationships

$$\beta_l \sim F_l = O\left(\epsilon^{1/3}\right), \quad \epsilon \ll 1, \tag{5.4}$$

and neglecting the $o(\epsilon)$-terms in the Lukovsky-Miles modal equations (4.13a), (4.13b), (4.13c), (4.15a), (4.15b), and (4.15c).

Accounting for (4.14a), kinematic modal equations (4.13a) read as

$$\sum_{i=1}^{\infty}\frac{\partial A_n}{\partial\beta_i}\dot{\beta}_i = \sum_{k=1}^{\infty}A_{nk}F_k, \quad n \geq 1, \tag{5.5}$$

where neglecting the $o(\epsilon)$-terms implies that $\partial A_n/\partial\beta_i$ and $A_{nk}$ keep only the second-order polynomial quantities, that is,

$$\begin{aligned}\frac{\partial A_n}{\partial\beta_i} &= \delta_{ni} + (2+n)\sum_{j=1}^{\infty}\Lambda_{nij}^{(3)}\beta_j + \frac{(n+1)(n+2)}{2}\sum_{j,k=1}^{\infty}\Lambda_{nijk}^{(4)}\beta_j\beta_k \\ &\quad - \frac{2+n}{4\pi}\left[\delta_{ni}\sum_{j=1}^{\infty}\beta_j^2 + 2\beta_i\beta_n\right] = \delta_{ni} + \sum_{j=1}^{\infty}\chi_{n,i,j}^{(1)}\beta_j + \sum_{j,k=1}^{\infty}\chi_{n,i,jk}^{(2)}\beta_j\beta_k,\end{aligned} \tag{5.6}$$

$$\begin{aligned}A_{nk} &= n\delta_{nk} + \sum_{j=1}^{\infty}\left[nk\Lambda_{knj}^{(3)} + \Lambda_{nk,j}^{(-3)}\right]\beta_j + \frac{n+k}{2}\sum_{i,j=1}^{\infty}\left[nk\Lambda_{knij}^{(4)} + \Lambda_{nk,ij}^{(-4)}\right]\beta_i\beta_j \\ &\quad - \frac{n(n+k+1)}{4\pi}\delta_{nk}\sum_{j=1}^{\infty}\beta_j^2 = n\delta_{nk} + \sum_{j=1}^{\infty}\prod_{nk,j}^{(1)}\beta_j + \sum_{i,j=1}^{\infty}\prod_{nk,ij}^{(2)}\beta_i\beta_j.\end{aligned} \tag{5.7}$$

Here, $\Lambda^{(3)}_{ijk}$ is defined by (5.3) and $\Lambda^{(4)}_{ijkm}$ is also expressed via the Clebsch-Gordan coefficients

$$\begin{aligned}\Lambda^{(4)}_{ijkm} &= 2\pi \int_0^{\pi} f_i f_j f_k f_m \sin\theta d\theta \\ &= \frac{\sqrt{(2i+1)(2j+1)(2k+1)(2m+1)}}{4\pi} \sum_{n=\max(|i-j|,|k-m|)}^{\min(i+j,k+m)} \frac{1}{2n+1}\left(C^{n0}_{i0,j0}\cdot C^{n0}_{k0,m0}\right)^2.\end{aligned} \tag{5.8}$$

Furthermore,

$$\begin{gathered}\Lambda^{(-2)}_{nk} = 2\pi \int_0^{\pi} \frac{\partial f_n}{\partial\theta}\frac{\partial f_k}{\partial\theta}\sin\theta d\theta = n(n+1)\delta_{nk}, \\ \Lambda^{(-3)}_{in,k} = 2\pi \int_0^{\pi} \frac{\partial f_i}{\partial\theta}\frac{\partial f_n}{\partial\theta} f_k \sin\theta d\theta = -\frac{1}{2}\sqrt{\frac{i(i+1)(2i+1)n(n+1)(2n+1)}{\pi(2k+1)}}C^{k0}_{i0,n0}C^{k0}_{i(-1),n1}, \\ \Lambda^{(-4)}_{in,kj} = 2\pi \int_0^{\pi} \frac{\partial f_i}{\partial\theta}\frac{\partial f_n}{\partial\theta} f_k f_j \sin\theta d\theta = \frac{1}{4\pi}\sqrt{i(i+1)(2i+1)n(n+1)(2n+1)} \\ \times\sqrt{k(k+1)(2k+1)j(j+1)(2j+1)} \sum_{m=\max(|i-n|,|k-j|)}^{\min(i+n,k+j)} \frac{1}{2m+1}C^{m0}_{i0,n0}C^{m0}_{i(-1),n1}C^{m0}_{k0,j0}C^{m0}_{k(-1),j1}.\end{gathered} \tag{5.9}$$

Kinematic equations (5.5) can be considered as linear algebraic equations with respect to $F_k$ whose asymptotic solution (neglecting $o(\epsilon)$) should admit the form

$$F_l = \frac{\dot\beta_l}{l} + \sum_{i,j=1}^{\infty} V^{(2)}_{l,i,j}\dot\beta_i\beta_j + \sum_{i,j,k=1}^{\infty} V^{(3)}_{l,i,j,k}\dot\beta_i\beta_j\beta_k, \quad l \ge 1. \tag{5.10}$$

Substituting (5.10) into (5.5) and gathering all the similar polynomial terms give

$$V^{(2)}_{n,i,j} = \frac{\chi^{(1)}_{n,i,j} - \Pi^{(1)}_{ni,j}/i}{n}, \qquad V^{(3)}_{n,i,j,k} = \frac{\chi^{(2)}_{n,i,j,k} - \Pi^{(2)}_{ni,jk}/i - \sum_{l=1}^{\infty} V^{(2)}_{l,i,j}\Pi^{(1)}_{nl,k}}{n}. \tag{5.11}$$

For the axisymmetric drop dynamics, the dynamic modal equations (4.15a), (4.15b), and (4.15c) take the form

$$\sum_{n=1}^{\infty} \frac{\partial A_n}{\partial\beta_\mu}\dot F_n + \frac{1}{2}\sum_{n,k=1}^{\infty}\frac{\partial A_{n,k}}{\partial\beta_\mu}F_nF_k + \frac{\partial TS}{\partial\beta_\mu} = 0, \quad \mu \ge 1. \tag{5.12}$$

Pursuing the announced weakly-nonlinear theory, we should employ (5.6), (5.7) and expand (4.16a) up to the third-order polynomial terms, that is,

$$
\begin{aligned}
\frac{\partial TS}{\partial \beta_\mu} &= 2\pi \int_0^\pi \zeta^2 \sin\theta \left( \frac{2 + (\zeta_\theta/\zeta)^2}{\sqrt{\zeta^2 \zeta_\theta^2}} - \frac{1}{\zeta^2 \sin\theta} \frac{\partial}{\partial\theta} \left( \frac{\zeta\zeta_\theta \sin\theta}{\sqrt{\zeta^2\zeta_\theta^2}} \right) \right) \\
&\quad \times \left[ f_\mu - \frac{1}{2\pi}\beta_\mu - \frac{1}{4\pi} \sum_{i,j=1}^{\infty} \Lambda_{ij\mu} \beta_i \beta_j \right] d\theta \\
&= 2\pi \int_0^\pi (2 + 2\xi) \left[ f_\mu - \frac{1}{2\pi}\beta_\mu - \frac{1}{4\pi} \sum_{i,j=1}^{\infty} \Lambda_{ij\mu} \beta_i \beta_j \right] \sin\theta d\theta + 2\pi \int_0^\pi \left[ \xi_\theta - \frac{1}{2}\xi_\theta^3 \right] \frac{\partial f_\mu}{\partial \theta} \sin\theta d\theta \\
&= (\mu + 2)(\mu - 1)\beta_\mu + \sum_{i,j=1}^{\infty} T_{ij}^{(2\mu)} \beta_i \beta_j + \sum_{i,j,k=1}^{\infty} T_{i,j,k}^{(3\mu)} \beta_i \beta_j \beta_k,
\end{aligned}
\tag{5.13}
$$

where

$$
T_{ij}^{(2\mu)} = -2\Lambda_{ij\mu}^{(3)}, \qquad T_{i,j,k}^{(3\mu)} = -\frac{1}{2}\Lambda_{ijk\mu}^{(-4\theta)} + \frac{1}{\pi}\delta_{i\mu}\delta_{jk}, \qquad \Lambda_{ijk\mu}^{(-4\theta)} = 2\pi \int_0^\pi \frac{\partial f_i}{\partial\theta} \frac{\partial f_j}{\partial\theta} \frac{\partial f_k}{\partial\theta} \frac{\partial f_\mu}{\partial\theta} \sin\theta d\theta.
\tag{5.14}
$$

Substituting (5.6), (5.7), (5.10), and (5.13) into (5.12) and neglecting the $o(\epsilon)$-terms yield the following weakly-nonlinear modal equations:

$$
\begin{aligned}
&\sum_{i=1}^{\infty} \left[ \delta_{\mu i} + \sum_{j=1}^{\infty} d_{i,j}^{1,\mu} \beta_j + \sum_{j,k=1}^{\infty} d_{i,j,k}^{2,\mu} \beta_j \beta_k \right] \ddot{\beta}_i + \sum_{n,k=1}^{\infty} \left[ t_{n,k}^{0,\mu} + \sum_{m=1}^{\infty} t_{n,k}^{1,\mu} \beta_m \right] \dot{\beta}_n \dot{\beta}_k + \sigma_\mu^2 \beta_\mu \\
&\quad + \sum_{i,j=1}^{\infty} \left[ \mu T_{ij}^{2\mu} \right] \beta_i \beta_j + \sum_{i,j,k=1}^{\infty} \left[ \mu T_{i,j,k}^{3\mu} \right] \beta_i \beta_j \beta_k = 0, \quad \mu \geq 1,
\end{aligned}
\tag{5.15}
$$

where $\sigma_\mu = \sigma_{\mu 0}$ are the nondimensional frequencies (3.6) and

$$
\begin{aligned}
d_{i,j}^{1,\mu} &= \mu \left[ \frac{\chi_{i,\mu,j}^{(1)}}{i} + V_{\mu,i,j}^{(2)} \right], \qquad d_{i,j,k}^{2,\mu} = \mu \left[ \frac{\chi_{i,\mu,j,k}^{(2)}}{i} + \sum_{\alpha=1}^{\infty} \chi_{\alpha,\mu,j}^{(1)} V_{\alpha,i,k}^{(2)} + V_{\mu,i,j,k}^{(3)} \right], \\
t_{n,k}^{0,\mu} &= \mu \left[ V_{\mu,n,k}^{(2)} + \frac{\Pi_{nk,\mu}^{(1)}}{nk} \right], \\
t_{n,k,m}^{1,\mu} &= \mu \left[ \overline{V}_{\mu,n,k,m}^{(3)} + \frac{\Pi_{nk,\mu m}^{(2)}}{nk} + \sum_{\alpha=1}^{\infty} \left( \chi_{\alpha,\mu,m}^{(1)} V_{\alpha,n,k}^{(2)} + \frac{\Pi_{\alpha k,\mu}^{(1)} V_{\alpha,n,m}^{(2)}}{k} + \frac{\Pi_{\alpha n,\mu}^{(1)} V_{\alpha,k,m}^{(2)}}{n} \right) \right].
\end{aligned}
\tag{5.16}
$$

We see that the weakly-nonlinear modal equations (5.15) constitute an infinite-dimensional system of ordinary differential equations (with respect to generalized coordinates $\beta_n$) that are not resolved relative to the highest derivative. Again, one can use (5.15) for direct numerical simulations (see, e.g., [32]). However, as in [27] and other analytical papers on sloshing, the derived weakly-nonlinear modal equations can reduce to a finite-dimensional form by employing a Duffing-type asymptotics. Using this asymptotics in sloshing problems implicitly suggests that we look for almost-periodic solutions with the frequency close to the lowest natural frequency of the mechanical system. For the studied case within no external excitations, these periodic solutions mean the nonlinear eigenoscillations of a freely-levitating drop.

## 6. Nonlinear Axisymmetric Eigenoscillation

We consider almost-periodic oscillations of an axisymmetric drop with the frequency $\sigma$ close to the lowest linear eigenfrequency (natural frequency) $\sigma_{20}$ subject to the Duffing-type third-order asymptotics implying the dominant character of the primary generalized coordinate $\beta_2 = O(\epsilon^{1/3})$. Analyzing the nonzero coefficients in (5.15) shows that this asymptotics yields

$$\beta_2 = O\left(\epsilon^{1/3}\right), \qquad \beta_4 = O\left(\epsilon^{2/3}\right), \qquad \beta_6 = O(\epsilon), \qquad \beta_l = o(\epsilon), \quad l \neq 2,4,6 \tag{6.1}$$

so that neglecting the $o(\epsilon)$-terms in (5.15) leads to the finite-dimensional system of nonlinear modal equations

$$\begin{aligned} &\ddot{\beta}_2 + \sigma_2^2\beta_2 + d_1\ddot{\beta}_2\beta_4 + d_2\ddot{\beta}_4\beta_2 + d_3\dot{\beta}_2\dot{\beta}_4 + d_4\ddot{\beta}_2\beta_2^2 + d_5\dot{\beta}_2^2\beta_2 + t_1\beta_2^2 + t_2\beta_2\beta_4 + t_3\beta_2^3 \\ &\quad + c_1\ddot{\beta}_2\beta_2 + c_2\dot{\beta}_2^2 = 0, \end{aligned} \tag{6.2a}$$

$$\ddot{\beta}_4 + \sigma_4^2\beta_4 + d_6\ddot{\beta}_2\beta_2 + d_7\dot{\beta}_2^2 + t_4\beta_2^2 + t_5\beta_2\beta_4 + c_3\ddot{\beta}_4\beta_2 + c_4\dot{\beta}_4\dot{\beta}_2 = 0, \tag{6.2b}$$

$$\ddot{\beta}_6 + \sigma_6^2\beta_6 + d_8\ddot{\beta}_2\beta_4 + d_9\ddot{\beta}_4\beta_2 + d_{10}\dot{\beta}_4\dot{\beta}_2 + d_{11}\ddot{\beta}_2\beta_2^2 + d_{12}\dot{\beta}_2^2\beta_2 + t_6\beta_2\beta_4 + t_7\beta_2^3 = 0, \tag{6.2c}$$

where $\sigma_2 = \sigma_{20}, \sigma_4 = \sigma_{40}, \sigma_6 = \sigma_{60}$ and

$$\begin{gathered} d_1 = \frac{24}{4\sqrt{\pi}}, \qquad d_2 = \frac{15}{14\sqrt{\pi}}, \qquad d_3 = \frac{75}{14\sqrt{\pi}}, \qquad d_4 = -\frac{67}{98\pi}, \qquad d_5 = -\frac{585}{196\pi}, \\ d_6 = \frac{15}{7\sqrt{\pi}}, \qquad d_7 = -\frac{9}{7\sqrt{\pi}}, \qquad d_8 = \frac{105\sqrt{65}}{286\sqrt{\pi}}, \qquad d_9 = \frac{30\sqrt{65}}{143\sqrt{\pi}}, \qquad d_{10} = -\frac{75\sqrt{65}}{143\sqrt{\pi}}, \\ d_{11} = \frac{135\sqrt{65}}{1001\pi}, \qquad d_{12} = \frac{135\sqrt{65}}{2002\pi}; \qquad t_1 = -\frac{4\sqrt{5}}{7\sqrt{\pi}}, \qquad t_2 = -\frac{24}{7\sqrt{\pi}}, \qquad t_3 = -\frac{76}{7\pi}, \\ t_4 = -\frac{24}{7\sqrt{\pi}}, \qquad t_5 = -\frac{160\sqrt{5}}{77\sqrt{\pi}}, \qquad t_6 = -\frac{180\sqrt{65}}{143\sqrt{\pi}}, \qquad t_7 = \frac{540\sqrt{65}}{143\pi}; \qquad c_1 = \frac{9\sqrt{5}}{14\sqrt{\pi}}, \\ c_2 = \frac{4\sqrt{5}}{7\sqrt{\pi}}, \qquad c_3 = \frac{75\sqrt{5}}{154\sqrt{\pi}}, \qquad c_4 = \frac{185\sqrt{5}}{154\sqrt{\pi}}. \end{gathered} \tag{6.3}$$

Other modal equations in (5.15) do not include nonlinear terms.

To construct a periodic asymptotic solution of (6.2a), (6.2b), and (6.2c) we assume, as usually [26], the asymptotic closeness condition between $\sigma$ and $\sigma_2$:

$$\frac{\sigma - \sigma_2}{\sigma_2} = O\left(\epsilon^{2/3}\right) \tag{6.4}$$

(the nonlinear eigenfrequency $\sigma$ is unknown). The wanted periodic solution takes then the form

$$\begin{gathered} \beta_2 = A\cos(\sigma t) + A^2(E_1 + E_2\cos(2\sigma t)) + O\left(A^3\right), \\ \beta_4 = A^2(E_3 + E_4\cos(2\sigma t)) + O\left(A^3\right), \qquad \beta_6 = O\left(A^3\right), \qquad A = O\left(\epsilon^{1/3}\right), \end{gathered} \tag{6.5}$$

where $A$ is the unknown dominant amplitude and substituting (6.5) into (6.2a) and (6.2b) and accounting for (6.4) give

$$E_1 = \frac{-t_1 + (c_1 - c_2)\sigma^2}{2\sigma_2^2} = \frac{-t_1 + (c_1 - c_2)\sigma_2^2}{2\sigma_2^2} + O\left(A^2\right), \tag{6.6a}$$

$$E_2 = \frac{-t_1 + (c_1 + c_2)\sigma^2}{2(\sigma_2^2 - 4\sigma^2)} = \frac{t_1 - (c_1 + c_2)\sigma_2^2}{6\sigma_2^2} + O\left(A^2\right), \tag{6.6b}$$

$$E_3 = \frac{-t_4 + (d_6 - d_7)\sigma^2}{2\sigma_4^2} = \frac{-t_4 + (d_6 - d_7)\sigma_2^2}{2\sigma_4^2} + O\left(A^2\right), \tag{6.6c}$$

$$E_2 = \frac{-t_4 + (d_6 + d_7)\sigma^2}{2(\sigma_4^2 - 4\sigma_2^2)} = \frac{-t_4 + (d_6 + d_7)\sigma_2^2}{2(\sigma_4^2 - 4\sigma_2^2)} + O\left(A^2\right). \tag{6.6d}$$

Gathering the $A^3$-order terms at the first harmonics in (6.2a) and using (6.4) lead to the secular equation to find the dependence between the normalized nonlinear eigenfrequency $(\sigma - \sigma_2)/\sigma_2 = O(\epsilon^{2/3})$ and the nondimensional amplitude parameter $A^2 = O(\epsilon^{2/3})$

$$\frac{\sigma - \sigma_2}{\sigma_2} + \frac{6347}{7840\pi}A^2 = 0. \tag{6.7}$$

Secular equation (6.7) determines the so-called "soft-type" spring behavior suggesting that amplitude $A$ increases with decreasing frequency $\sigma$.

Figure 2 compares our asymptotic result (6.7) with experimental data from [12] (see, also [19]) where dependence between $(\sigma - \sigma_2)/\sigma_2$ and the maximum ratio $(H/W)$ between the instant drop height and width were reported. The experimental data for $R_0 = 0.49$ cm are denoted by $\bullet$, but $\circ$ marks measurements made for $R_0 = 0.62$ cm. In the lowest-order approximation, the Legendre polynomials properties deduce that

$$\frac{H}{W} = \frac{1 + \sqrt{5/(4\pi)}A}{1 - (1/2)\sqrt{5/(4\pi)}A}, \tag{6.8}$$

where $A$ is defined by (6.7).

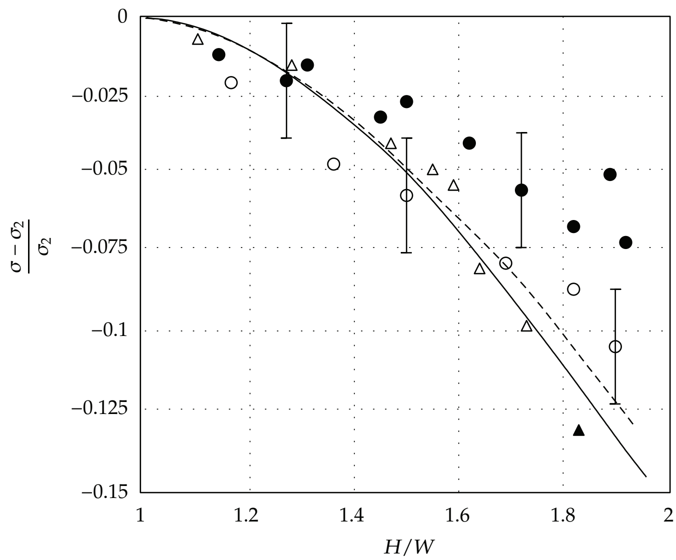


**Figure 2:** Theoretical ((6.8), solid line)) and experimental dependence between $(\sigma - \sigma_2)/\sigma_2$ ($\sigma_2$ is the first linear eigenfrequency, $\sigma$ is the corresponding nonlinear eigenfrequency) and the maximum ratio between the drop heights and width ($H/W$). Experimental values [12] are for $R_0$ =0.49 cm (•) and $R_0$ =0.62 cm (∘). Theoretical results do not depend on $R_0$ and $T_s$ within the framework of an ideal incompressible liquid model. The figure also shows numerical results from [19] (dashed line), [33] (△) and [34] (▲).

Figure 2 shows that (6.8) provides a good agreement with the numerical values from [19] (dashed line), [33] (△), and [34] (▲) in which the studies were also based on the incompressible ideal liquid model (the results do not depend on $R_0$). Our theoretical values are in good agreement with experimental measurements for $R_0 = 0.63$ cm. For lower drop radius, an qualitative agreement with experimental measurements is established. The discrepancy can be related to viscous effects discussed, for example, in [19].

## 7. Concluding Remarks

The present paper gives a generalization of Lukovsky-Miles' multimodal method and derives the corresponding general nonlinear modal equations describing the nonlinear dynamics of a levitating drop. The modal equations are a full analogy of the original free-surface problem and look similar to those known for the nonlinear liquid sloshing problem [24, 25, 31].

The derived nonlinear modal equations are used to construct weakly-nonlinear modal equations for axisymmetric drop motions. The weakly-nonlinear equations possess a finite-dimensional form for the case of almost-periodic drop motions with the nonlinear eigenfrequency close to the lowest linear eigenfrequency. The latter case was studied by other authors, experimentally and numerically. To compare our analytical asymptotic results with earlier experimental [11, 12] and numerical [19, 33, 34] results, we constructed periodic solutions of the finite-dimensional modal system. All the results on periodic solutions from different sources are in a good agreement.

## References


[1] R. Eberhardt and B. Neidhart, "Acoustic levitation device for sample pretreatment in microanalysis and trace analysis," *Fresenius' Journal of Analytical Chemistry*, vol. 365, no. 6, pp. 475–479, 1999.

[2] E. H. Brandt, "Suspended by sound," *Nature*, vol. 413, no. 6855, pp. 474–475, 2001.

[3] F. Priego-Capote and L. de Castro, "Ultrasound-assisted levitation: lab-on-a-drop," *Trends in Analytical Chemistry*, vol. 25, no. 9, pp. 856–867, 2006.

[4] T. G. Wang, A. V. Anilkumar, and C. P. Lee, "Oscillations of liquid drops: results from USML-1 experiments in Space," *Journal of Fluid Mechanics*, vol. 308, pp. 1–14, 1996.

[5] C. P. Lee, A. V. Anilkumar, A. B. Hmelo, and T. G. Wang, "Equilibrium of liquid drops under the effects of rotation and acoustic flattening: results from USML-2 experiments in Space," *Journal of Fluid Mechanics*, vol. 354, pp. 43–67, 1998.

[6] Y. Liu, D. M. Zhu, D. M. Strayer, and U. E. Israelsson, "Magnetic levitation of large water droplets and mice," *Advances in Space Research*, vol. 45, no. 1, pp. 208–213, 2010.

[7] T. Shi and R. E. Apfel, "Oscillations of a deformed liquid drop in an acoustic field," *Physics of Fluids*, vol. 7, no. 7, pp. 1545–1552, 1995.

[8] W. J. Xie and B. Wei, "Dynamics of acoustically levitated disk samples," *Physical Review E*, vol. 70, Article ID 046611, pp. 1–11, 2004.

[9] C. L. Shen, W. J. Xie, and B. Wei, "Parametrically excited sectorial oscillation of liquid drops floating in ultrasound," *Physical Review E*, vol. 81, no. 4, Article ID 046305, pp. 1–10, 2010.

[10] E. Becker, W. J. Hiller, and T. A. Kowalewski, "Experimental and theoretical investigation of large-amplitude oscillations of liquid droplets," *Journal of Fluid Mechanics*, vol. 231, pp. 189–210, 1991.

[11] E. Trinh and T. G. Wang, "Large-amplitude free and driven drop-shape oscillations: experimental observations," *Journal of Fluid Mechanics*, vol. 122, pp. 315–338, 1982.

[12] E. Trinh and T. G. Wang, "Large amplitude drop shape oscillations," in *Proceedings of the 2nd International Colloquium on Drops and Bubbles*, pp. 82–87, JPL Publication, Pasadena, Calif, USA, 1982.

[13] E. Becker, W. J. Hiller, and T. A. Kowalewski, "Nonlinear dynamics of viscous droplets," *Journal of Fluid Mechanics*, vol. 258, pp. 191–216, 1994.

[14] Lord Rayleigh, "On the capillary phenomena of jets," *Proceedings of the Royal Society, London*, vol. 29, pp. 71–97, 1879.

[15] L. D. Landau and E. M. Lifschitz, *Hydrodynamics*, Verlag Harri Deutsch, Berlin, Germany, 1991.

[16] O. A. Basaran, "Nonlinear oscillations of viscous liquid drops," *Journal of Fluid Mechanics*, vol. 241, pp. 169–198, 1992.

[17] T. W. Patzek, R. E. Benner Jr., O. A. Basaran, and L. E. Scriven, "Nonlinear oscillations of inviscid free drops," *Journal of Computational Physics*, vol. 97, no. 2, pp. 489–515, 1991.

[18] X. Chen, T. Shi, Y. Tian, J. Jankovsky, R. G. Holt, and R. E. Apfel, "Numerical simulation of super-oscillations of a Triton-bearing drop in microgravity," *Journal of Fluid Mechanics*, vol. 367, pp. 205–220, 1998.

[19] J. A. Tsamopoulos and R. A. Brown, "Nonlinear oscillations of inviscid drops and bubbles," *Journal of Fluid Mechanics*, vol. 127, pp. 519–537, 1983.

[20] J. A. Tsamopoulos and R. A. Brown, "Resonant oscillations of inviscid charged drops," *Journal of Fluid Mechanics*, vol. 147, pp. 373–395, 1984.

[21] R. Natarajan and R. A. Brown, "Third-order effects and the nonlinear stability of drop oscillations," *Journal of Fluid Mechanics*, vol. 183, pp. 95–121, 1987.

[22] I. A. Lukovsky, "Variational method in the nonlinear problems of the dynamics of a limited liquid volume with free surface," in *Oscillations of Elastic Constructions with Liquid*, pp. 260–264, Volna, Moscow, Russia, 1976.

[23] J. W. Miles, "Nonlinear surface waves in closed basins," *Journal of Fluid Mechanics*, vol. 75, no. 3, pp. 419–448, 1976.

[24] I. A. Lukovsky, *Introduction to the Nonlinear Dynamics of Solid Bodies with Cavities Filled by a Liquid*, Naukova dumka, Kiev, Ukraine, 1990.

[25] O. M. Faltinsen and A. N. Timokha, *Sloshing*, Cambridge University Press, 2009.

[26] M. Hermann and A. Timokha, "Modal modelling of the nonlinear resonant fluid sloshing in a rectangular tank. I. A single-dominant model," *Mathematical Models and Methods in Applied Sciences*, vol. 15, no. 9, pp. 1431–1458, 2005.

[27] O. M. Faltinsen, O. F. Rognebakke, and A. N. Timokha, “Resonant three-dimensional nonlinear sloshing in a square-base basin,” *Journal of Fluid Mechanics*, vol. 487, pp. 1–42, 2003.

[28] I. Gavrilyuk, I. Lukovsky, Yu. Trotsenko, and A. Timokha, “Sloshing in a vertical circular cylindrical tank with an annular baffle. II. Nonlinear resonant waves,” *Journal of Engineering Mathematics*, vol. 57, no. 1, pp. 57–78, 2007.

[29] I. Lukovsky, D. Ovchynnykov, and A. Timokha, “Asymptotic nonlinear multimodal modeling of liquid sloshing in an upright circular cylindrical tank. I. Modal equations,” *Nonlinear Oscillations*, vol. 14, no. 4, pp. 512–525, 2012.

[30] H. Takahara and K. Kimura, “Frequency response of sloshing in an annular cylindrical tank subjected to pitching excitation,” *Journal of Sound and Vibration*, vol. 331, no. 13, pp. 3199–3212, 2012.

[31] O. M. Faltinsen, O. F. Rognebakke, I. A. Lukovsky, and A. N. Timokha, “Multidimensional modal analysis of nonlinear sloshing in a rectangular tank with finite water depth,” *Journal of Fluid Mechanics*, vol. 407, pp. 201–234, 2000.

[32] J. S. Love and M. J. Tait, “Non-linear multimodal model for tuned liquid dampers of arbitrary tank geometry,” *International Journal of Non-Linear Mechanics*, vol. 46, no. 8, pp. 1065–1075, 2011.

[33] G. B. Foote, “A numerical method for studying liquid drop behavior: simple oscillation,” *Journal of Computational Physics*, vol. 11, no. 4, pp. 507–530, 1973.

[34] C. T. Alonso, “The dynamics of colliding and oscillating drops,” in *Proceedings of the International Colloquium on Drops and Bubbles*, D. J. Collins, M. S. Plesset, and M. M. Saffren, Eds., p. 14, Jet Propulsion Laboratory, 1974.

[35] V. Berdichevsky, *Variational Principles of Continuum Mechanics*, Springer, 2010.

[36] I. N. Vekua, “On completeness of a system of harmonic polynomials in space,” *Doklady Akademii Nauk SSSR (NS)*, vol. 90, pp. 495–498, 1953.

[37] I. N. Vekua, *New Methods for Solving Elliptic Equations*, John Wiley & Sons, 1967.

[38] E. Zaarur, Y. Peleg, and R. Pnini, *Quantum Mechanics*, Schaums Easy Oulines Crash Course, Mc Graw Hill.