Proceedings

of the Third International Conference

SYMMETRY

in Nonlinear Mathematical Physics

SYMMETRY

KYIV 1999

Institute of Mathematics
National Academy of Sciences of Ukraine
Kyiv, Ukraine

Proceedings of Institute of Mathematics of the National Academy of Sciences of Ukraine **Volume 30**

SERIES Mathematical Physics

Proceedings

of the Third International Conference

SYMMETRY

in Nonlinear Mathematical Physics

Kyiv, Ukraine *12–18 July 1999*

Part 1

Institute of Mathematics of NAS of Ukraine

Kyiv • 2000

УДК 517.95:517.958:512.81(06)

Симетрія у нелінійній математичній фізиці // Праці Інституту математики НАН України. — Т. 30. — Ч. 1. — Київ: Інститут математики НАН України, 2000 / Ред.: А.Г. Нікітін, В.М. Бойко. — 264 с.

Цей том “Праць Інституту математики НАН України” є збірником статей учасників Третьої міжнародної конференції “Симетрія у нелінійній математичній фізиці”. Збірник складається з двох частин, кожна з яких видана окремою книгою.

Дане видання є першою частиною і включає оригінальні праці, присвячені подальшому розвитку та застосуванню теоретико-групових методів у сучасній математичній фізиці та теорії диференціальних рівнянь. Поряд з аналізом симетрії та побудовою точних розв’язків складних багатовимірних нелінійних рівнянь, ці методи дозволяють будувати адекватні математичні моделі у фізиці, механіці, математичній біології та інших природничих науках.

Розраховано на наукових працівників, аспірантів, які цікавляться симетрійними методами аналізу і побудови точних розв’язків нелінійних рівнянь.

Symmetry in Nonlinear Mathematical Physics // Proceedings of Institute of Mathematics of NAS of Ukraine. — V. 30. — Part 1. — Kyiv: Institute of Mathematics of NAS of Ukraine, 2000 / Eds.: A.G. Nikitin, V.M.Boyko. — 264 p.

This volume of the Proceedings of Institute of Mathematics of NAS of Ukraine includes papers of participants of the Third International Conference “Symmetry in Nonlinear Mathematical Physics”. The collection consists of two parts which are published as separate issues.

This issue is the first part which is devoted to further development and applications of group-theoretical methods in modern mathematical physics and theory of differential equations. In addition to the analysis of symmetries and construction of exact solutions of complicated multidimensional nonlinear equations, these methods allow to formulate adequate mathematical models in physics, mechanics, biology, and other natural sciences.

The book may be useful for researchers and post graduate students who are interested in symmetry analysis and construction of exact solutions of nonlinear equations.

Затверджено до друку вченою радою Інституту математики НАН України

Редактори: *А.Г. Нікітін, В.М. Бойко*
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ISBN 966–02–1444–8

ISBN 966–02–1401–4 (Part 1)

Preface

The Third International Conference “Symmetry in Nonlinear Mathematical Physics” continued the series of the scientific meetings started in 1995 due to efforts of Professor Wilhelm Fushchych. And we believe that these series will be continued and consider such continuation as our duty with respect to the memory of our teacher. The conference was organized by the Institute of Mathematics of the National Academy of Sciences of Ukraine and M. Dragomanov National Pedagogical University. It was held in Kyiv, July 12–18, 1999.

The papers included into the Proceedings can be divided in two parts. The first part includes ones devoted to the topics which are traditional for our conferences, i.e., analysis of symmetries of nonlinear equations, symmetry reduction and construction of exact solutions of partial differential equations. In this part the classical Lie methods as well as the modern trends in symmetry analysis such as non-local, conditional, higher, super- and parasupersymmetries, are represented in more than 30 papers which are collected in the first volume.

The second volume includes papers devoted to the representation theory and applications of classical and deformed Lie algebras, super- and parasuperalgebras to some fundamental and applied problems of modern mathematical physics. It should be emphasized that such separation is rather conventional, since some papers can be related to both parts and some to neither of them. But we believe that all papers present a valuable contribution to the development of symmetry analysis of equations of mathematical physics.

Anatoly NIKITIN

December, 1999

Third International Conference

SYMMETRY IN NONLINEAR MATHEMATICAL PHYSICS

July 12–18, 1999, Kyiv, Ukraine

Organized by

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Third International Conference

SYMMETRY IN NONLINEAR MATHEMATICAL PHYSICS

July 12–18, 1999, Kyiv, Ukraine

TOPICS

* Classical Lie Analysis of Equations of Mathematical Physics
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* Symmetry in Nonlinear Quantum Mechanics, Quantum Fields, Gravity, Fluid Mechanics, Mathematical Biology, Mathematical Economics
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We invite everybody to participate in the next Conference planned for July, 2001.

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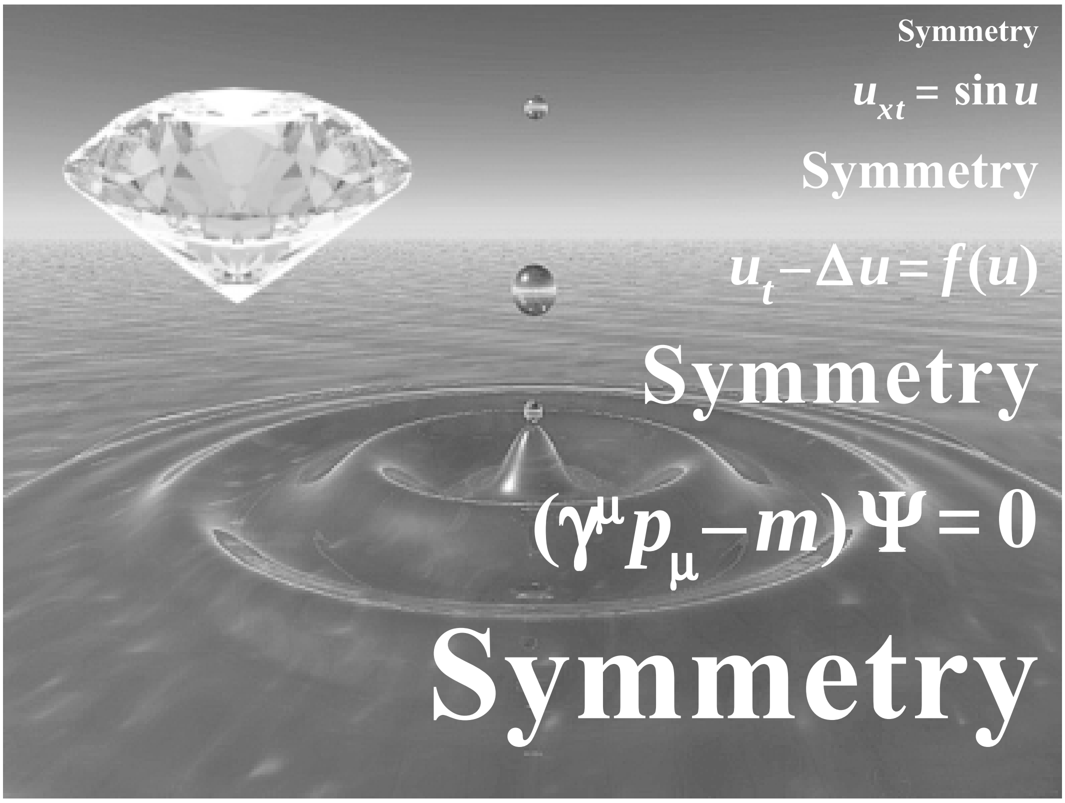

Symmetry in Nonlinear Mathematical Physics

Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 17–24.

Matrix Methods of Searching for Lax Pairs and a Paper by Estevez

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Lax pairs are useful in studying nonlinear partial differential equations, although finding them is often difficult. A standard approach for finding them was developed by Wahlquist and Estabrook [1]. It was designed to apply to for equations with two independent variables and generally produces incomplete Lie algebras (called "prolongation structures"), which can be written as relations among certain matrices and their commutators. Extending the method to three variable problems is more difficult. One still gets matrix equations, but now with a more complicated structure. Exploration of a Lax pair in a paper by Estevez [3] suggested a variation of the method. This paper will discuss how that can be used to obtain her Lax pair.

P.G. Estevez [3] published a recent paper dealing with a particular nonlinear partial differential equation (NLPDE):

$$0 = m_y^2(n_{yt} - m_{xxxy}) + m_{xy}\left(n_y^2 - m_{xy}^2\right) + 2m_y(m_{xy}m_{xxy} - n_y n_{xy}) - 4m_y^3 m_{xx} \tag{1}$$

with $m_t = n_x$. In this paper, subscripts mean derivatives.

This equation was a reformulation of a set of equations

$$\begin{aligned} 0 &= v_y - (uw)_x, \\ 0 &= \lambda u_t + u_{xx} - 2uv, \\ 0 &= \lambda w_t - w_{xx} + 2vw \end{aligned} \tag{2}$$

which were obtained earlier by other authors. These equations have the Painleve property, which was used by Estevez in an investigation of Eq. (1) by the singular manifold method in which she, among other things, found a Lax pair. Her treatment is fairly complicated. It is not obvious from the original equations that a Lax pair exists.

Study of that Lax pair led this author to try a matrix approach to try to find the same result. This is basically a version of the Wahlquist–Estabrook method [1] that this author spoke about at the first Kyiv Conference "Symmetry in Nonlinear Mathematical Physics" four years ago [4]. The matrix equations are quite complicated but can be simplified, with guidance from already known results. Here some earlier results are reviewed, with particular attention to the use of matrices.

Lax pairs have been known since 1968, when Lax discussed them in terms of operators in his paper of that year on the KdV equation (the 12 included here did not occur in Lax's version) [6]

$$u_t + 12uu_x + u_{xxx} = 0. \tag{3}$$

In the later treatment by Wahlquist and Estabrook [1, 2] (WE), the Lax pair may be expressed in terms of linear matrix equations for two auxiliary variables, with coefficients involving the variable u, whose integrability condition gives the KdV equation.

Wahlquist and Estabrook used differential forms in analyzing partial differential equations. We show here the definition of new variables z and p, introduced to reduce equations to first derivatives, with the KdV equation using the new variables:

$$z = u_x, \qquad p = z_x, \qquad u_t + p_x + 12uz = 0. \tag{4}$$

Then we write these three equations in terms of three differential forms in the five variables x, t, u, p, z (the set of these is called the ideal I of forms, $I = \{\alpha, \beta, \gamma\}$:

$$\begin{aligned} &\alpha = du\,dt - z\,dx\,dt, \\ &\beta = dz\,dt - p\,dx\,dt, \\ &\gamma = -du\,dx + dp\,dt + 12uz\,dx\,dt, \end{aligned} \tag{5}$$

where the hook product $\wedge$ between basis forms such as du and dt is understood. (If one now assumes that the field variables u, z, p are functions of x and t and requires these differential forms to vanish, one recovers the original equations.)

Next, WE assume the existence of a variable y and an auxiliary 1-form, called a prolongation form,

$$\omega = -dy + f(y, u, p, z)\,dx + g(y, u, p, z)\,dt. \tag{6}$$

The exterior derivative of this form is to lie in the "augmented" ideal of forms $I' = \{I, \omega\}$:

$$d\omega \subset \{I, \omega\} \tag{7}$$

so that when I and ω vanish, this amounts to an integrability condition.

Now take y to be a column vector and assume f and g to be linear in the components of y; then we can rewrite (6) as a matrix equation:

$$\omega = -dy + \alpha y, \tag{8}$$

where

$$\alpha = F\,dx + G\,dt \tag{9}$$

is a matrix 1-form and where F and G are matrices and are functions of u, z and p. The integrability condition (7) is then expressed by

$$\begin{aligned} d\omega &= d\alpha\,y - \alpha \wedge dy, \\ &= d\alpha\,y - \alpha \wedge (-\omega + \alpha y), \\ &= (d\alpha - \alpha \wedge \alpha)y \bmod \omega \end{aligned} \tag{10}$$

so that

$$d\alpha - \alpha \wedge \alpha \subset I \tag{11}$$

or

$$G_x - F_t - [F, G] = 0 \tag{12}$$

which is to be satisfied if the original field equations hold.

The use of differential forms gives some insight into how the problem might be formulated and presents an elegant structure. However, the problem can be formulated without forms. Write

$$y_x = F\,y, \qquad y_t = G\,y \tag{13}$$

then

$$\begin{aligned} y_{xt} &= F_t\, y + F y_t = F_t\, y + FGy \\ &= G_x\, y + G y_x = G_x\, y + GFy, \end{aligned} \tag{14}$$

giving Eq. (12) as before. For the KdV case this equation becomes simply:

$$\begin{aligned} &F_p = F_z = 0, \qquad G_p + F_u = 0, \\ &z\, G_u + p\, G_z + 12uz\, F_u = [F, G]. \end{aligned} \tag{15}$$

Solution of these equations leads eventually to the relations

$$\begin{aligned} &F = Au^2 + Bu + C, \\ &G = -p(2uA + B) + z^2 A + 6zD + K(u), \end{aligned} \tag{16}$$

where

$$[B, C] = 6D, \qquad [A, B] = [A, C] = 0 \tag{17}$$

and

$$K(u) = 2u^3([A, D] - 4A) + 3u^2([B, D] - 2B) + 6u[C, D] + E \tag{18}$$

and where there are six more equations among A, B, C, D and E, which are constant matrices. These equations involve commutators of commutators and will not be given here.

The set of equations for A, B, C, D and E constitutes an incomplete Lie algebra (called a "prolongation structure" by WE). It is of interest in its own right; however, we wish to find a representation in order to find the Lax pair. Closure of the algebra can be achieved by Ansatz, as WE show. A two-dimensional representation for these five matrices is, where λ is constant:

$$A = 0, \quad B = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}, \quad D = 1/3 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = -4\lambda C, \tag{19}$$

giving

$$F = \begin{bmatrix} 0 & \lambda - 2u \\ 1 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} -2z & 4(u+\lambda)(2u-\lambda) + 2p \\ -4(u+\lambda) & 2z \end{bmatrix}, \tag{20}$$

with a Lax pair written as differentials of the components of y:

$$\begin{aligned} &dy_1 = (\lambda - 2u) y_2\, dx + \{[4(u+\lambda)(2u-\lambda) + 2p] y_2 - 2zy_1\}\, dt, \\ &dy_2 = y_1\, dx + [2zy_2 - 4(u+\lambda) y_1]\, dt. \end{aligned} \tag{21}$$

It should be noted here that WE use the auxiliary variables in the Lax pair equations (which they call "pseudopotentials") to help derive Bäcklund transformations.

One can see that this method is most suited for differential equations in two independent variables, since the prolongation form is simply a 1-form. The three independent variable case is much harder. We can see why, from a differential form standpoint, by noting that the ideal of forms representing equations with n independent variables generally requires n-forms (although there are exceptions). As an example, we give the KP equation

$$3u_{tt} + 6(uu_x)_x + u_{xxx} + 3u_{xy} = 0, \tag{22}$$

and with new variables p, r, z and w defined by

$$p = u_x, \qquad r = p_x, \qquad z = w_x = -(3/4)u_t \tag{23}$$

this becomes

$$w_t = (3/2)up + (1/4)r_x + (3/4)u_y. \tag{24}$$

An ideal of 3-forms representing the KP equation is (where the $\wedge$ is suppressed):

$$\begin{aligned}
&(du\,dt - p\,dx\,dt)\,dy,\\
&(dp\,dt - r\,dx\,dt)\,dy,\\
&(dw\,dt - z\,dx\,dt)\,dy,\\
&(dp\,dx - (4/3)\,dz\,dt)\,dy,\\
&(dw\,dx + (3/2)up\,dx\,dt + (1/4)\,dr\,dt)\,dy + (3/4)\,du\,dx\,dt.
\end{aligned} \tag{25}$$

The difficulty in such cases arises in trying to construct a prolongation form. If we simply write it as a 1-form in three variables, then its exterior derivative is a 2-form, and its vanishing cannot be achieved with the 3-forms in the ideal.

H. Morris' approach [5], motivated by the WE method and here called the MWE method, does not use forms. He proceeded by assuming the equations

$$\begin{aligned}
&\zeta_x = -F\zeta - A\zeta_y,\\
&\zeta_t = -G\zeta - B\zeta_y,
\end{aligned} \tag{26}$$

where A and B are constant matrices, with F and G being matrix functions of u, p, r, z and w, and by assuming integrability. After writing out the integrability condition and substituting from Eq. (26) where possible, he set the coefficients of ζ, ζ_y and ζ_{yy} to zero, yielding the following equations (to be taken modulo the field equations, in other words to be satisfied if the field equations are satisfied):

$$\begin{aligned}
&[A,B] = 0,\\
&[G,A] + [B,F] = 0,\\
&F_t - G_x + [G,F] + BF_y - AG_y = 0.
\end{aligned} \tag{27}$$

Note that the equations are now more complicated than just relations among commutators.

Morris' approach suggested to the author an approach to the three-variable problem using differential forms [4]. While this has some interest, it appears rather artificial. It is not needed here.

A solution of Morris' equations, given by himself and corrected in [4], has this set of matrices, where k is a constant:

$$\begin{aligned}
&A = (3/4)\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{bmatrix}, \qquad B = -(3/4)\begin{bmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{bmatrix},\\
&F = \begin{bmatrix} 0 & -1 & 0\\ 3u/4 & 0 & -1\\ w-k & 3u/4 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} u/4 & 0 & 1\\ -w+k+p/4 & -u/2 & 0\\ r/4 + 9u^2/16 & -w+k-p/4 & u/4 \end{bmatrix}.
\end{aligned} \tag{28}$$

Equation (26), with these matrices, now constitutes a Lax pair for the KP equation.

We now go back to Estevez' paper and equation. We write her Lax pair in matrix form, defining new variables:

$$C_t = M(C_{xx} + 2qC), \qquad C_{xy} = QC_y - pC, \tag{29}$$

with

$$q = m_x, \quad r = m_{xy}, \quad p = m_y, \quad z = n_y = \int m_{ty}\, dx, \quad s = r_x \tag{30}$$

and

$$Q = (r1 + zM)/(2p), \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{31}$$

where 1 in Q is the 2×2 unit matrix and C is a 2-component column matrix. We note that her original equation, in the new variables, is

$$0 = p^2(z_t - s_x) + r\left(z^2 - r^2\right) + 2p(rs - zz_x) - 4p^3 q_x, \tag{32}$$

where $z_x = p_t$.

Let us now attempt to use the MWE method to find this Lax pair. We assume equations exactly like Eq. (26),

$$\begin{aligned} \zeta_x &= -A\zeta_y - F\zeta, \\ \zeta_t &= -B\zeta_y - G\zeta, \end{aligned} \tag{33}$$

where F and G are matrix functions of m, n, p, q, r, s and z. The equations easily show that F is independent of n, q, r and s and is linear in z. Continuing the process eventually leads to a trivial solution. Interchanging independent variables in Eq. (33) does not lead to a solution either. Thus, the MWE method does not work, indicating that there is not a Lax pair of the form (33).

So we attempt to generalize the MWE method (denote this by GMWE). We try the following, noting that it uses a particular assumed structure, motivated by knowing the answer already!

Assume a pair of differential equations for the column vector C from above:

$$\begin{aligned} C_t &= FC_{xx} + GC_x + HC, \\ C_{xy} &= KC_x + LC_y + NC, \end{aligned} \tag{34}$$

where F, G, H, K, L and N are matrix functions of m, n, p, q, r, s and z, and assume integrability: $(C_t)_{xy} = (C_{xy})_t$, substituting for C_t and C_{xy} from these equations, wherever possible. One gets a complicated matrix equation with terms linear in C_{xxx}, C_{xx}, C_x, C_y and C. We equate the coefficients to zero. After some simplification, such as substitution of F_y from the first of these into other equations, we get

$$\begin{aligned} F_y &= [K, F], \\ G_y &= [K, G] + [LK + N, F] - FK_x - K_xF, \\ L_t &= [H + GL + FL^2, L] + F_x(L_x + L^2) + F(L_{xx} + 2L_xL) \\ &\quad + [F, L]L_x + G_xL + GL_x + H_x, \\ N_t &= (2FL_x + F_xL)N + [H, N] + [F, L](N_x + LN) + FN_{xx} \\ &\quad + F_xN_x + G_xN + GN_x + H_{xy} - LH_y - KH_x + [G, L]N, \\ K_t &= [G, N + LK] + [F, LN + L^2K] + [H, K] + K_xG + (N + LK - K_x)F_x \\ &\quad + (L_xK + 2LK_x - K_{xx} + N_x)F + F(N_x + L_xK) + H_y. \end{aligned} \tag{35}$$

Obviously further simplification is needed. We take some hints from the known answer; choose $K = G = 0$. Then $F_y = 0$, yielding

$$0 = F_m p + F_n z + F_p p_y + F_q r + F_r r_y + F_s s_y + F_z z_y. \tag{36}$$

The coefficients of p, z, etc., must vanish, giving all derivatives of F zero, so that F is constant. The remaining equations become

$$\begin{aligned} 0 &= [F, N], \\ 0 &= H_y + N_x F + F N_x + [F, LN], \\ L_t &= F(L_{xx} + 2L_x L + L L_x) - LFL_x + H_x + [FL^2 + H, L], \\ N_t &= F(N_{xx} + LN_x + 2L_x N) - L(FN_x + H_y) + H_{xy} + [F, L]LN + [H, N]. \end{aligned} \tag{37}$$

The second of these suggests that N and H be taken as functions of separate variables whose x- and y-derivatives are equal. p and q appear to be the obvious variables. We write $H = H(q)$ and $N = N(p)$, substitute, and cancel $q_y (= p_x)$. Then the second equation yields

$$H' = -N'F - FN', \qquad [F, LN] = 0. \tag{38}$$

Since F is constant, the fact that it commutes with N also means that it commutes with N'. We assume F to have an inverse. Then separation of variables in the first of Eq. (38) and dropping matrix integration constants gives

$$H = aq, \qquad N = cp, \tag{39}$$

where a and c are constant matrices and $c = -(1/2)aF^{-1}$. We note that F, a and c now all mutually commute. We assume that c has an inverse; then F commutes with L and with L_x as well.

Substitution of these expressions into the last of Eq. (37) gives, after simplification,

$$2FL_x cp = -(Fc + a)p_{xx} + Lam_{xy} + cz_x. \tag{40}$$

Multiplying by the inverses of F and c gives the equation

$$2pL_x = 1p_{xx} - 2Lp_x + F^{-1}z_x \tag{41}$$

which can be integrated and solved for L, giving

$$L = (1r + F^{-1}z + U)/(2p), \tag{42}$$

where U is a matrix integration constant satisfying $[F, U] = 0$.

The third of Eq. (37) now becomes, after using $[F, L] = 0$,

$$F(L_{xx} + 2L_x L) + H_x + [H, L] = L_t. \tag{43}$$

We substitute for L from Eq. (42) and find, after substituting for various derivatives, for z_t from Eq. (32), multiplying by $2p^3$, and canceling some terms,

$$\begin{aligned} & F\left[-3prs + 2r^3 + \left(2r^2 - ps\right)U\right] + 1\left(-2prz_x - psz + 2zr^2\right) \\ &\quad + \left[F\left(ps - r^2\right) + 1(pz_x - rz) - rFU\right]\left(1r + F^{-1}z + U\right) + 2p^3 q_x a + p^2 q[a, U] \\ &\quad = -1prz_x - pz_x U + F^{-1}\left(4p^3 q_x + pzz_x - 2prs + r^3 - rz^2\right). \end{aligned} \tag{44}$$

By comparing terms we see immediately that $F = F^{-1}$ and $a = 2F$, so that $c = -1$. The z_x term shows that $U = 0$; then the remaining terms cancel identically. If one now takes

$$F = M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{45}$$

one gets Estevez' Lax pair Eq. (29).

What could be done to try simplifying Eq. (35) in some other way? We can assume that all matrices commute. Then F is constant for the same reason as before. However, one gets this equation for G and K:

$$G_y = -FK_x - K_xF. \tag{46}$$

By the same argument used for H and N above, we may write

$$K = Ap, \qquad G = -2FAq, \tag{47}$$

where A is constant. We assume that F and A have inverses. The remaining equations are

$$\begin{aligned} K_t &= K_xG + 2KFL_x + 2LFK_x + F(2N_x - K_{xx}) + H_y, \\ L_t &= F(L_{xx} + 2LL_x) + LG_x + GL_x + H_x, \\ N_t &= F(N_{xx} + 2NL_x) + NG_x + GN_x + H_{xy} - KH_x - LH_y. \end{aligned} \tag{48}$$

Motivated by the first of these equations we take $H = H(q)$ since y-derivatives of other variables cannot be expressed in terms of the variables we are using. We expand N_x in terms of derivatives with respect to z, p, q and r and set coefficients of z_x and q_x to zero, giving expressions for N_z and N_q. Integration of those equations yields

$$N = (1/2)F^{-1}Az - pAL + W(p, r). \tag{49}$$

Substitution into the remaining part of the equation yields an equation linear in s. Setting the coefficients equal to zero gives finally

$$\begin{aligned} N &= (1/2)F^{-1}Az - pAL + rA - pC + E, \\ H &= FA^2q^2 + 2FCq + D, \end{aligned} \tag{50}$$

where C, D and E are constant.

Substitution of these results into the equation for N_t with elimination of L_t and z_t (from Eq. (32)) gives an equation which could be integrated on x to give L, were it not for a term $-2F^{-1}Apq_x$. This fact seems to show that A must be zero after all, giving a contradiction. Thus, at the least, A does not have an inverse and perhaps should be taken to be zero, leading to the previous case.

We can approach this from a slightly different point of view. Let us ask what NLPDE is consistent with the Lax pair (34) we have assumed. To simplify we will assume that all matrices commute, as before. We get a constant F as before. We assume some basic field m with $q = m_x$, $p = m_y$. We get $G = -2qFA + B$ and $K = qA + C$, where B and C are constant, similar to previous results. For reasons similar to the previous ones, we take $A = 0$. Thus G and K are constant. Furthermore, it seems appropriate and useful to take $K = 0$. Calculation for H, N and L proceeds much as before. We finally get an equation where, in order to make all terms proportional to the same matrix, we merely need to assume F^{-1} is proportional to F and c is proportional to 1, giving $F^2 = \lambda 1$ and $c = \mu 1$, where λ and μ are constants. μ can be absorbed by change of variables. So we have a slightly more general equation than Estevez:

$$\begin{aligned} 0 = &-m_y^2 m_{xxxy} + m_{xy}\left(2m_y m_{xxy} - m_{xy}^2\right) - 4m_y^3 m_{xx} \\ &+\lambda^{-1}\left(m_{xy}n_y^2 - 2n_y m_y m_{yt} + m_y^2 n_{yt}\right). \end{aligned} \tag{51}$$

We can also generalize by taking $H = H(q, m)$, $N = N(p)$ when solving for those two quantities. We get

$$N = cp, \qquad H = -(Fc + cF)q + Q(m). \tag{52}$$

We assume that F and c have inverses, and this enables explicit solution for several quantities. This all reduces eventually to the same equation as before.

One can ask what the most general equation is that is consistent with a generalized Lax pair of the type, say,

$$\begin{aligned} &C_t = FC_{xx} + GC_x + HC + AC_{yy} + BC_y, \\ &C_{xy} = KC_x + LC_y + NC. \end{aligned} \tag{53}$$

The equations resulting from this are very complicated and nothing has been done with them.

In summary, one can see that trial of a Lax pair of the generalized form (34) or something like it could perhaps work for some equations, as a generalization of (33). Chances are that any particular guess will not work for a new NLPDE that one might have; but this at least gives some suggestions for how one might look for a Lax pair using matrices. Defining new variables might motivate the linear structure that one might try. A general approach is not available.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 25–34.

Transformations of Ordinary Differential Equations: Local and Nonlocal Symmetries

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The brief review of new methods of factorization, autonomization and exact linearization of the ordinary differential equations is represented. These methods along with the method of the group analysis based on using both point and nonpoint, local and nonlocal transformations are effective tools for study of nonlinear autonomous and nonautonomous dynamical systems. Thus a scope of exactly solvable problems of the Nonlinear analysis is extended.

Introduction

This paper is devoted to the analytical aspect of the problem of integrability of ordinary differential equations (ODE). There are two approaches to the problem one of which is related to the changes of variables and another to implied algebraic concepts. However, an application of substitutions as a rule had the heuristic nature. Such powerful methods as factorization were hardly extendable to differential equations, even linear ones; besides, they were inefficient. Plenty of expectations was connected to an application of Lie group and Lie algebra theory to differential equations (the group analysis), and it was not in vain. Conceptual and uniformizing role of this theory is now universally recognized. Algebraic approach became especially fruitful in solving mechanical and physical fundamental equations since invariance principles are background for the construction of these equations. However, its capability does not allow "to close" the integrability problem.

Being concerned with the integrability problem for ODE, the author has concluded that the key to its comprehension is contained in the ideas of **factorization and transformation** and in realizing the necessity of their combined application since the summarized results exceed the effect of a single idea. The uniform theory of a factorization and transformations of ODE allows to investigate structurally nonlinear and non-stationary problems of technology and natural sciences, what is especially important in connection with a continuous delinearization of Science in general and Physics in particular.

For a first time the author presented the factorization method for differential operators in connection with a transformation theory in 1967 [1]. Further logical development of this method has led to the extension of the factorization to the nonlinear equations and the creation of effective algorithms for searching of transformations. The author [2] incorporated the fundamentals of theories of factorization and transformations of n-th order ODE to uniform theory which structurally permitted to solve the problems of equivalence of various classes, i.e. the problems of their reduction to given (including canonical) prescribed form (see also [3–5]).

In the present paper the special attention is paid to autonomizable and linearizable equation classes.

The paper is organized as follows. In Section 1 we present the new method of exact linearization of nonlinear ODEs. We consider in detail the linearization of autonomous equations with the help of nonlocal transformation of variables. In Section 2 the example of the exponential nonlocal symmetry is given. In Section 3 we consider the class of linearizable equations of the third order and present some examples.

1 A new method for exact linearization of ODE

Theorem 1.1 [4]. *The equation*

$$y^{(n)} - f\left(x, y, y', \dots, y^{(n-1)}\right) = 0 \tag{1.1}$$

is reducible to the linear autonomous form

$$M_n z \equiv z^{(n)}(t) + \sum_{k=1}^{n} \binom{n}{k} b_k z^{(n-k)}(t) = 0, \qquad b_k = \text{const.} \tag{1.2}$$

by means of the reversible transformation

$$y = v(x,y)z, \qquad dt = u_1(x,y)dx + u_2(x,y)dy, \tag{1.3}$$

where v, u_1, and u_2 are sufficiently smooth functions and $v(u_1 + u_2 y') \neq 0$ in a domain $\Gamma(x,y)$ iff (1.1) admits the noncommutative factorization

$$\prod_{k=n}^{1} \left[D - \frac{v_x + v_y y'}{v} - (k-1)\frac{D(u_1 + u_2 y')}{u_1 + u_2 y'} - r_k(u_1 + u_2 y')\right] y = 0, \tag{1.4}$$

or the commutative one

$$\begin{gathered}\prod_{k=1}^{n} \left[\frac{1}{u_1 + u_2 y'} D - \frac{v_x + v_y y'}{v(u_1 + u_2 y')} - r_k\right] y = 0, \\ D = d/dx, \qquad v_x = \partial v/\partial x, \qquad v_y = \partial v/\partial y,\end{gathered} \tag{1.5}$$

where $D = d/dx$, and r_k are the roots of the characteristic equation

$$M_n(r) \equiv \sum_{k=0}^{n} \binom{n}{k} b_k r^{n-k} = 0, \qquad b_0 = 1. \tag{1.6}$$

Necessity. Factorizing (1.2) we obtain

$$\prod_{k=n}^{1} (D_t - r_k) z = 0, \qquad D_t = d/dt. \tag{1.7}$$

We apply the transformation, inverse to (1.3), to (1.7):

$$z = v^{-1} y, \qquad dx = 1/(u_1 + u_2 y')dt :$$

$$\prod_{k=n}^{1} \left[\frac{1}{u_1 + u_2 y'} D - r_k\right] \frac{y}{v} = \prod_{k=n}^{2} \left[\frac{1}{u_1 + u_2 y'} D - r_k\right] \left[\frac{1}{u_1 + u_2 y'} D - r_k\right] \frac{y}{v}$$

$$= \prod_{k=n}^{2} \left[\frac{1}{u_1 + u_2 y'} D - r_k\right] \frac{1}{v} \left[\frac{1}{u_1 + u_2 y'} D - \frac{Dv}{v(u_1 + u_2 y')} - r_1\right] y = 0, \quad Dv = v_x + v_y y'.$$

Using the operator identity

$$\left(\frac{1}{u_1+u_2y'}D-r_k\right)\frac{1}{v}=\frac{1}{v}\left[\frac{1}{u_1+u_2y'}D-\frac{Dv}{v(u_1+u_2y')}-r_k\right],$$

we obtain the expression

$$\frac{1}{v}\prod_{k=1}^{n}\left[\frac{1}{u_1+u_2y'}D-\frac{v_x+v_yy'}{v(u_1+u_2y')}-r_k\right]y=0,$$

that corresponds to the factorization (1.4). The factorization (1.5) can be obtained from (1.4) as follows. If we apply an easily verifiable identity

$$\left[\frac{1}{u_1+u_2y'}D-\frac{v_x+v_yy'}{v(u_1+u_2y')}-r_s\right]\frac{1}{(u_1+u_2y')^{s-1}}$$

$$=\frac{1}{(u_1+u_2y')^s}\left[D-\frac{v_x+v_yy'}{v}-(s-1)\frac{D(u_1+u_2y')}{u_1+u_2y'}-r_s(u_1+u_2y')\right],$$

$s=\overline{1,n}$, we get a noncommutative factorization

$$\frac{1}{v(u_1+u_2y')^n}\prod_{k=n}^{1}\left[D-\frac{Dv}{v}-(k-1)\frac{D(u_1+u_2y')}{u_1+u_2y'}-r_k(u_1+u_2y')\right]y,$$

that corresponds to (1.5).

Sufficiency. Let take place the factorization (1.5) takes place. We apply transformation (1.3), sequentially changing the dependent and independent variables: a) $y=vz$; b) $dt=(u_1+u_2y')dx$, $(D=(u_1+u_2y')D_t)$. Let $U=u_1+u_2y'$.

$$\prod_{k=n}^{1}\left[D-\frac{Dv}{v}-(k-1)\frac{DU}{U}-r_kU\right]vz=\prod_{k=n}^{2}\left[D-\frac{Dv}{v}-(k-1)\frac{DU}{U}-r_kU\right]v(D-r_1U)z.$$

By virtue of the identity

$$\left[D-\frac{Dv}{v}-(s-1)\frac{DU}{U}-r_sU\right]v=v\left[D-(s-1)\frac{DU}{U}-r_sU\right]v,\ \ s=\overline{1,n}$$

we obtain

$$\prod_{k=n}^{1}\left[D-\frac{Dv}{v}-(k-1)\frac{DU}{U}-r_kU\right]vz=v\prod_{k=n}^{1}\left[D-(k-1)\frac{DU}{U}-r_kU\right]z.$$

Further, changing the independent variable we get:

$$v\prod_{k=n}^{1}\left[D-(k-1)\frac{DU}{U}-r_kU\right]z=v\prod_{k=n}^{1}\left[UD_t-(k-1)\frac{DU}{U}-r_kU\right]z$$

$$=v\prod_{k=n}^{2}\left[UD_t-(k-1)\frac{DU}{U}-r_kU\right]U(D_t-r_1)z.$$

Applying the operator identity:

$$\left[D-(s-1)\frac{DU}{U}-r_sU\right]U^{s-1}=U^s(D_t-r_s),\qquad s=\overline{1,n}$$

we have as a result the factorization

$$\prod_{k=n}^{1}\left[D-\frac{Dv}{v}-(k-1)\frac{DU}{U}-r_kU\right]y=vU^n\prod_{k=n}^{1}(D_t-r_k)z=0,$$

that corresponds to the equation (1.2).

The transformation (1.3) encloses the following important ones: Kummer–Liouville transformation (KLT)

$$y(x)=v(x)z,\quad dt=u(x)dx,\quad vu\neq 0,\quad \forall\, x\in \mathbf{i_0}\subset \mathbf{i},\quad v,u\in \mathbf{C}^n(\mathbf{i_0}), \tag{1.8}$$

exact nonlocal linearization of nonlinear autonomous equations

$$y=v(y)z,\quad dt=u(y)dx,\quad u(y(x))v(y(x))\neq 0,\quad \forall\, x\in \mathbf{I}=\{x\mid a\leq x\leq b\}; \tag{1.9}$$

the general point linearization

$$t=f(x,y),\qquad z=\varphi(x,y),\qquad \det\left(\frac{t,z}{x,y}\right)=t_xz_y-t_yz_x\neq 0, \tag{1.10}$$

corresponding to (1.3) for $u_{1y}=u_{2x}$; the point linearization

$$t=f(x),\qquad z=\varphi(x,y), \tag{1.11}$$

preserving fibering; the linearization

$$y=v(x,y)z,\qquad dt=u(x,y)dx, \tag{1.12}$$

connected with arbitrary point Lie symmetry; and finally, the general nonlocal linearization (1.3).

Theorem 1.2. *After the sequential application the composition of the transformations (1.8) and (1.9), i.e. of the transformations*

$$y=v_1(x)v_2(y/v_1(x))z,\qquad dt=u_1(x)u_2(y/v_1(x))dx \tag{1.13}$$

the equation (1.1) is reducible to (1.2) iff the commutative factorization

$$\prod_{k=1}^{n}\left[\frac{1}{u_1u_2}D-\frac{v_1'v_2+v_1v_2^*Y'}{v_1v_2u_1u_2}-r_k\right]y=0,\qquad Y=\frac{y}{v_1},\quad (')=\frac{d}{dx},\quad (*)=\frac{d}{dY}; \tag{1.14}$$

or the noncommutative one

$$\prod_{k=n}^{1}\left[D-\frac{v_1'}{v_1}-(k-1)\frac{u_1'}{u_1}-\frac{v_2^*}{v_2}Y'-(k-1)\frac{u_2^*}{u_2}Y'-r_ku_1u_2\right]y=0, \tag{1.15}$$

takes place; and the diagram

$$\begin{array}{ccc}\mathbf{A} & \xrightarrow{f} & \mathbf{B}\\ \downarrow\varphi & & \downarrow g\\ \mathbf{C} & \xrightarrow{\psi} & \mathbf{D}\end{array}$$

is commutative, i.e. $g\circ f=\psi\circ\varphi$.

The formulas (1.14) and (1.15) easily follow from (1.4) and (1.5) by virtue of (1.13). a commutativity of the diagram or realization of the condition $f\circ g=\varphi\circ\psi$, is checked immediately.

The transformations f, g, φ and ψ have the following form

$$f:\ y=v_1(x)Y,\quad ds=u_1(x)dx;\qquad g:\ Y=v_2(Y)z,\quad dt=u_2(Y)ds;$$

$$\varphi:\ y=v_2(Y)P,\quad dq=u_2(Y)dx;\qquad \psi:\ P=V_1(q)z,\quad dt=U_1(q)dq.$$

Here **A** denotes the set of equations (1.1), (1.15), **B** denotes the set of nonlinear autonomous equations having the factorized form

$$\prod_{k=n}^{1}\left[D_s-\frac{v_2^*}{v_2}\frac{dY}{ds}-(k-1)\frac{u_2^*}{u_2}\frac{dY}{ds}-r_ku_2(Y)\right]Y=0,\qquad D_s=\frac{d}{ds},$$

C is a set of the linear nonautonomous reducible equations

$$\prod_{k=n}^{1}\left[D_q-\frac{1}{V_1}\frac{dV_1}{dq}-(k-1)\frac{1}{U_1}\frac{dU_1}{dq}-r_kU_1(q)\right]P=0,$$

$$V_1(q(x))=v_1(x),\qquad U_1(q(x))=u_1(x)$$

and **D** denotes the set of the linear equations (1.3), (1.7).

Remark 1.1. Theorems 1.1 and 1.2 were announced in [6]. a linearization through the transformation of unknown function was applied in [7], and through the transformation of independent variable was used in [8, 9]. The examples can be found in [10]. In cited works [7–10], as a rule, the considered equations had the second order. Linearization of equations of order $n>2$ is considered in [11]. Group analysis of ODE of the order $n>2$ is considered in [12].

It should be mentioned, that the fact of existence of the indicated factorizations for the differential equations allows to discover required transformations.

Theorem 1.3. *The equation*

$$y^{(n)}=F\left(y,y',\dots,y^{(n-1)}\right),\qquad n>2 \tag{1.16}$$

is reducible to the linear autonomous form

$$M_nz\equiv z^{(n)}(t)+\sum_{k=1}^{n}\binom{n}{k}b_kz^{(n-k)}+c=0,\qquad b_k,c=\mathrm{const}, \tag{1.2$'$}$$

by means of the transformation (1.9) iff (1.16) admits the noncommutative factorization

$$\prod_{k=n}^{1}\left[D-\left(\frac{1}{y}-\left(\log\int\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int fdy\right)dy\right)^*+(k-1)\frac{u^*}{u}\right)y'-r_ku\right]y$$
$$+\frac{c}{\beta}\varphi^{\frac{n^2+n-2}{2n}}\exp\left(-\int fdy\right)=0. \tag{1.17}$$

In expanded form it is written as

$$y^{(n)}+nf(y)y'y^{(n-1)}+\dots+nb_1\varphi(y)y^{(n-1)}+\cdots$$
$$+\sum_{m=1}^{n-1}\binom{n}{m}b_m\varphi^m\sum_{s_1+2s_2+\dots+(n-m)s_{n-m}=n-m}\psi^{12\dots n-m}_{s_1s_2\dots s_{n-m}}y^{(1)s_1}y^{(2)s_2}\cdots y^{(n-m)s_{n-m}} \tag{1.18}$$
$$+\varphi^{\frac{n^2+n-2}{2n}}\exp\left(-\int fdy\right)\left(b_n\int\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int fdy\right)dy+\frac{c}{\beta}\right)=0,$$

where the coefficients ψ are the differential expressions, depending from f and φ, $\psi^{12\dots n-m}_{00\dots 1}$=1.

In addition we have the linearized transformation

$$z = \beta \int \varphi^{\frac{n^2-n+2}{2n}} \exp\left(\int f(y)dy\right) dy, \qquad dt = \varphi(y)dx, \tag{1.19}$$

and also (for $c = 0$) the one-parameter set of solutions

$$\int \frac{\varphi^{\frac{n^2-3n+2}{2n}} \exp\left(\int f dy\right) dy}{\int \varphi^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) dy} = r_k x + C, \tag{1.20}$$

where r_k are distinct roots of the characteristic equation (1.6).

The equation (1.16) admits a factorization:

$$\prod_{k=n}^{1} \left[D - \left(\frac{v^*}{v} + (k-1)\frac{u^*}{u}\right) y' - r_k u\right] y + cu^n v = 0. \tag{1.21}$$

At first, writing down the product in (1.21), we obtain the expression

$$\left(1 - \frac{v^*}{v} y\right) y^{(n)} - \left[n\frac{v^{**}}{v} y + \left(1 - \frac{v^*}{v} y\right)\left(2n\frac{v^*}{v} + \frac{n^2-n+2}{2}\frac{u^*}{u}\right)\right] y' y^{(n-1)} + \cdots, \tag{1.22}$$

what is proved by induction for $n \geq 3$. Really, let the formula (1.22) hold for $n = m$. Then for $n = m+1$ we have:

$$\left[D - \left(\frac{v^*}{v} + m\frac{u^*}{u}\right) y'\right] \left\{\left(1 - \frac{v^*}{v} y\right) y^{(m)}\right.$$
$$\left. - \left[m\frac{v^{**}}{v} y + \left(1 - \frac{v^*}{v} y\right)\left(2m\frac{v^*}{v} + \frac{m^2-m+2}{2}\frac{u^*}{u}\right)\right] y' y^{(m-1)} + \cdots\right\}.$$

Collecting the terms at $y^{(m+1)}$ and $y'y^{(m)}$, we obtain the first terms of the new expression:

$$\left(1 - \frac{v^*}{v} y\right) y^{(m+1)} - \left[(m+1)\frac{v^{**}}{v} y\right.$$
$$\left. + \left(1 - \frac{v^*}{v} y\right)\left(2(m+1)\frac{v^*}{v} + \frac{m^2+m+2}{2}\frac{u^*}{u}\right)\right] y' y^{(m)} + \cdots.$$

This prove (1.22). Let us introduce the notation

$$n\frac{v^{**}}{v} y + \left(1 - \frac{v^*}{v} y\right)\left(2n\frac{v^*}{v} + \frac{n^2-n+2}{2}\frac{u^*}{u}\right) = -nf(y)\left(1 - \frac{v^*}{v} y\right).$$

We have the second order nonlinear nonautonomous equation for $v(y)$

$$v^{**} - \frac{2}{v} v^{*2} + \left(\frac{2}{y} - \frac{n^2-n+2}{2n}\frac{u^*}{u} - f\right) v^* + \left(\frac{n^2-n+2}{2n}\frac{u^*}{u} + f\right)\frac{1}{y} v = 0.$$

After the substitution $v = V^{-1}$ this equation is reduced to the linear nonautonomous equation

$$V^{**} + \left(\frac{2}{y} - \frac{n^2-n+2}{2n}\frac{u^*}{u} - f\right) V^* - \frac{1}{y}\frac{n^2-n+2}{2n}\frac{u^*}{u} V = 0,$$

admitting the factorization

$$\left(D_y + \frac{1}{y} - \frac{n^2-n+2}{2n}\frac{u^*}{u} - f\right)\left(D_y + \frac{1}{y}\right) V = 0, \qquad D_y = d/dy$$

and having the solution

$$V = \frac{1}{y}\beta \int u^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) dy.$$

Then we get

$$v(y) = y\left(\beta \int u^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) dy\right)^{-1}. \tag{1.23}$$

(In particular, for $u = \exp\left(-\dfrac{2n}{n^2-n+2}\displaystyle\int f dy\right)$ we get $v = y(\beta y + \gamma)^{-1}$, $\gamma = \text{const} \neq 0$.) Substituting (1.23) in (1.21), we obtain (1.17). Putting $u = \varphi(y)$, in accordance with (1.23) we get (1.19). Writing down explicitly the product in (1.17), we have (1.18). The equation (1.21) is consisted with the first order equation

$$\left(1 - \frac{v^*}{v}y\right) y' - r_k u y = 0 \tag{1.24}$$

for $c = 0$. Put $u = \varphi(y)$ and substitute (1.23) in (1.24), we obtain (1.20).

Remark 1.2. Rather wide class of n-th order nonlinear autonomous equations can be tested by the method of the exact linearization. The tests can be specializations of the theorem 1.3 for concrete values of n.

2 The example of nonlocal symmetry

Nonlocal symmetries are considered in [13–17] and other works.

Example 2.1 [17]. The equation

$$y'' = y^{-1}y'^2 + pg(x)y^p y' + g'(x)y^{p+1}, \tag{2.1}$$

where p is a nonzero constant and $g(x)$ a nonzero arbitrary function, does not possess a Lie point symmetries except special cases. However, it has the first integral $I = y'/y - g(x)y^p$. The equation (2.1) admits the factorization $D\left(y'/y - g(x)y^p\right) = 0$ and has the exponential nonlocal symmetry

$$G = y \exp\left(\int g(x)y^p dx\right) \frac{\partial}{\partial y}.$$

The author is not going to develop this theme in detail in this paper because he hopes to develop it in other papers.

3 Linearization of the autonomous equations of the third order

Proposition 3.1. *a third order autonomous equation in the form*

$$y''' + f_5(y)y'y'' + f_4(y)y'' + f_3(y)y'^3 + f_2(y)y'^2 + f_1(y)y' + f_0(y) = 0 \tag{3.1}$$

is linearizable by the transformation (1.9)

$$\dddot{z} + 3b_1\ddot{z} + 3b_2\dot{z} + b_3 z + c = 0, \qquad b_1, b_2, b_3, c = \text{const}, \tag{3.2}$$

iff it can be represented in the form

$$
\begin{aligned}
& y''' + 3f(y)y'y'' + \left(\frac{1}{3}\frac{\varphi^{**}}{\varphi} - \frac{5}{9}\frac{\varphi^{*2}}{\varphi^2} - \frac{1}{3}f\frac{\varphi^*}{\varphi} + f^2 + f^*\right)y'^3 \\
& \quad +3b_1\varphi\left[y'' + \left(f + \frac{1}{3}\frac{\varphi^*}{\varphi}\right)y'^2\right] + 3b_2\varphi^2 y' \\
& \quad +\varphi^{5/3}\left(b_3\exp\left(-\int f dy\right)\int \varphi^{4/3}\exp\left(\int f dy\right)dy + \frac{c}{\beta}\right) = 0,
\end{aligned}
\tag{3.3}
$$

which is reduced to (3.2) by the substitution

$$
z = \beta\int\varphi^{4/3}\exp\left(\int f dy\right)dy, \qquad dt = \varphi(y)dx \tag{3.4}
$$

and we have one-parameter families of solutions as $c = 0$

$$
\int \frac{\varphi^{1/3}\exp\left(\int f dy\right)dy}{\int\varphi^{4/3}\exp\left(\int f dy\right)dy} = r_k x + C, \tag{3.5}
$$

where r_k *are the distinct roots of the characteristic equation*

$$
r^3 + 3b_1 r^2 + 3b_2 r + b_3 = 0. \tag{3.6}
$$

Remark 3.1. Equations of the type

$$
y''' + \varphi(y)y'y'' + \psi(y)y'' + \sum_{k=0}^{3} f_k(y)y'^k = 0 \tag{3.7}
$$

can be tested by the method of the exact linearization.

Example 3.1. It is known that the *sin-Gordon* equation

$$
u_{xt} = \sin u \tag{3.8}
$$

has a generalized nonlocal symmetry of the third order $\left(u_{xxx} + \frac{1}{2}u_x^3\right)\partial_u$, which is connected (see, for example, [18], p. 117–119) with ODE

$$
y''' + 1/2y'^3 = 0. \tag{3.9}
$$

About this equation it is said: "Unfortunately, determination of general solutions for higher order ODE is very complicated problem". But the equation (3.9) can be integrated. It is a special case of (3.3), admits the factorization

$$
2iy\left(y''' + \frac{1}{2}y'^3\right) \equiv \left[D - \left(i + \frac{1}{y}\right)y'\right]\left[D + \left(\frac{1}{2}i - \frac{1}{y}\right)y'\right]\left[D + \left(2i - \frac{1}{y}\right)y'\right]y = 0
$$

and is linearized to $\dddot{z} = 0$ by the substitution $z = \exp(2iy)$, $dt = \exp\left(\frac{3}{2}iy\right)dx$. The general solution of (3.9) in the parametric form is $x = \int\left(c_1 + c_2 t + c_3 t^2\right)^{-3/4}dt$, $y = -1/2i\ln(c_1 + c_2 t + c_3 t^2)$.

Consider third order equation

$$
y''' = F(x, y, y', y''), \tag{3.10}
$$

which by the transformation of the form

$$
y = v_1(x)v_2(y/v_1(x))z, \qquad dt = u_1(x)u_2(y/v_1(x))dx \tag{3.11}
$$

can be reduced to the linear autonomous form (3.2).

About an integration of the generalized Emden–Fowler equation (EFE)

For example, let us consider one of possible generalizations of the EFE, i.e.

$$y''' + bx^s y^n = 0, \qquad n \neq 0, \quad n \neq 1. \tag{3.12}$$

We use a test of the autonomization. Equation (3.12) by means of the transformation $y = x^{\frac{s+3}{1-n}} z$, $dt = x^{-1}dx$ is reduced to an autonomous form

$$\dddot{z} + (3k-3)\ddot{z} + [k(k-1) + k(k-2) + (k-1)(k-2)]\dot{z} + k(k-1)(k-2)z + bz^n = 0$$

and has the exact solutions

$$y = \rho x^k, \qquad k(k-1)(k-2)\rho + b\rho^n = 0, \qquad k = \frac{s+3}{1-n}.$$

So, the equation (3.12) is reduced to the autonomous form

$$Y'''(\tau) + bY^n = 0 \tag{3.13}$$

by the transformations $y = x^2 Y$, $d\tau = x^{-2}dx$. For thus obtained equation we apply the test of the linearization, i.e. we use the proposition 3.1. Equation (3.13) can be related to the class of (3.3) iff it can be represented in the form

$$Y''' + \varphi^{5/3}\left(b_3 \int \varphi^{4/3} dY + \frac{c}{\beta}\right) = 0, \tag{3.14}$$

where $b_1 = b_2 = 0$ in (3.3) and φ satisfies the equation

$$\frac{1}{3}\frac{\varphi^{**}}{\varphi} - \frac{5}{9}\frac{\varphi^{*2}}{\varphi^2} = 0. \tag{3.15}$$

The solution of equation (3.15) is a function $\varphi = Y^{-3/2}$. Then equation (3.16) is in the form

$$Y''' - b_3 Y^{-7/2} + \frac{c}{\beta} Y^{-5/2} = 0, \qquad \beta = -1. \tag{3.16}$$

Two cases are possible: $b_3 = 0$, $c \neq 0$ and $b_3 \neq 0$, $c = 0$. Let us consider the first one:

$$Y''' - cY^{-5/2} = 0, \qquad (b_3 = 0). \tag{3.17}$$

At $n = -5/2$ we have $-4 = s - 5$, then $s = 1$. The input equation is:

$$y''' + bxy^{-5/2} = 0, \qquad (c = -b). \tag{3.18}$$

Let $c = 0$. Then equation (3.16) takes the form

$$Y''' + bY^{-7/2} = 0, \qquad (b_3 = -b). \tag{3.19}$$

For $n = -7/2$ we obtain $s = -4 - 2(-7/2) = 3$. Then the equation (3.12) gets the form:

$$y''' + bx^3 y^{-7/2} = 0. \tag{3.20}$$

Let us apply to equations (3.18) and (3.20) the following substitutions

$$y = x^2 Y, \qquad d\tau = x^{-2}dx, \qquad Y = Y^2 z, \qquad dt = Y^{-3/2}d\tau$$

or the resulting substitutions in the transformed form

$$y = x^2 Y^2 z, \quad dt = x^{-2}Y^{-3/2}dx, \quad \left(Y = yx^{-2}\right), \quad \text{i.e.} \quad y = x^{-2}y^2 z, \quad dt = xy^{-3/2}dx,$$

we obtain respectively $\dddot{z} - b = 0$, $\dddot{z} - bz = 0$. The factorizations of equations (3.18) and (3.20) have respectively the forms:

$$-\left(D + \frac{y'}{y}\right)\left(D + \frac{1}{x} - \frac{1}{2}\frac{y'}{y}\right)\left(D + \frac{2}{x} - \frac{2y'}{y}\right)y + bxy^{-5/2} = 0,$$

$$\left(D + \frac{y'}{y} - r_3xy^{-3/2}\right)\left(D + \frac{1}{x} - \frac{1}{2}\frac{y'}{y} - r_2xy^{-3/2}\right)\left(D + \frac{2}{x} - \frac{2y'}{y} - r_1xy^{-3/2}\right)y = 0,$$

where $r_k, k = \overline{1,3}$, satisfies to a characteristic equation $r^3 - b = 0$.

Now let us consider the linear equation $y''' + bx^s y = 0$, $s \neq 0$, i.e. equation (3.12) for $n = 1$. Then we get two values for s: $s = -3$, $s = -6$. At $s = -3$ we have Euler's equation $y''' + bx^{-3}y = 0$, and we have the Halphen's equation $y''' + bx^{-6}y = 0$ for $s = -6$.

Let us note that the asymptotic solutions of the equation (3.14) were considered in ([19], p. 261–265).

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 35–46.

Computer Package for Investigation of the Complete Integrability

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The problems concerned with the complete integrability of the partial differential systems with two independent variables are considered. The algorithms and the Maple V procedures for the investigation of complete integrability and some examples are presented.

1 General information

This paper describes a new computer package for the investigation of partial differential systems with two independent variables. We called this package JET because of the jet space language is used. The package includes twenty eight basic procedures in Maple V language and fifteen auxiliary procedures. Our main aim was to create the full collection of the instruments for the investigation of completely integrable systems. But the package can be used for other purposes as well.

For independent variables we used the fixed (global) names **t** and **x**. And besides, **t** is the temporal variable and **x** is the spatial one. For the dependent variables one may use any names that must be fixed in the list with the global name **vard**. For example, if we deal with the jet space $J^\infty(R, R^2)$ with the local coordinates $(x,\, u_i,\, v_i)$ then we must assign **vard:=[u, v]:**. Then the coordinates in all programs will be denoted as $x, u0, v0, u1, v1, \ldots$ and so on.

Here is the list of the basic procedures:

```
dif, INT, depend, DF, DN, ED, EU, ord, pot, defeq, SU, part, chn,
cho, L_E,  recursion, Frechet, Noether, INoether, implectic,
symplectic, com, Jac, evsub, struct, Cmetric, Killing, triada
```

We comment this list in the subsequent sections. And here we mention only that all procedures work in **interactive mode**. The automatic mode is impossible in view of two following reasons. First, the computation of the higher conserved densities or Lie–Bäcklund higher symmetries of nonlinear **systems** leads us to very cumbersome partial differential systems whose solutions are unknown to science. Second, these systems often contain dozens of thousands terms. Solving such systems is an art but not a mechanical process.

2 Differentiation and integration

Two first names in the previous list, **dif** and **INT**, are names of procedures for differentiation and integration. There are built-in procedures **diff** and **int** for differentiation and integration in Maple. Nevertheless we wrote our own procedures in order to make all expressions more compact. Everybody who computes Lie–Bäcklund symmetries or conserved densities knows that you are forced to deal with a lot of arbitrary functions. Equations arising in such problems often are very long. The procedure **depend** enables to omit all arguments of all functions. The following example shows the difference between the built-in and our procedures:

```
> vard:=[u, v]: depend(f(u0,v0,u1,v1)):
> a:=dif(f,v0)*dif(f,u0$3,v0),
> b:=diff(f(u0,v0,u1,v1),v0)*diff(f(u0,v0,u1,v1),u0$3,v0);
```

$$a := \frac{\partial f}{\partial v0}\,\frac{\partial^4 f}{\partial u0^3 \partial v0}, \qquad b := \frac{\partial f(u0,v0,u1,v1)}{\partial v0}\,\frac{\partial^4 f(u0,v0,u1,v1)}{\partial u0^3 \partial v0}$$

The first expression is 3–4 times shorter than the second one. It is very important if you deal with a long expression. The procedures `dif` and `INT` possess the same facilities as the built-in `diff` and `int`. Moreover, `INT` possesses many powerful facilities for operating with arbitrary functions. In continuation of the previous input dialog we give the next examples

```
> INT(a,u0);
```

$$\frac{\partial f}{\partial v0}\,\frac{\partial^3 f}{\partial u0^2 \partial v0} - \frac{1}{2}\left(\frac{\partial^2 f}{\partial u0 \partial v0}\right)^2$$

```
> INT(u0^2*dif(f,u0$3), u0);
```

$$u0^2\,\frac{\partial^2 f}{\partial u0^2} - 2\,u0\,\frac{\partial f}{\partial u0} + 2\,f$$

and so on. The built-in procedure `int` returns such integrals without having them evaluated.

The next procedure **DF** calculates the total derivative with respect to x on a jet space and **DN(f,n)** calculates the n-th total derivative of f:

```
> vard:=[u]: depend(f(x,u0,u1), g(u0) ):
> DF(f), DF(g), DN(g,2);
```

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u0}\,u1 + \frac{\partial f}{\partial u1}\,u2, \qquad \frac{\partial g}{\partial u0}\,u1, \qquad \frac{\partial g}{\partial u0}\,u2 + \frac{\partial^2 g}{\partial u0^2}\,u1^2$$

The procedure **ED** computes the evolution derivative

$$ED(F) \to D_t(F) = \frac{\partial F}{\partial t} + \sum_{i,\alpha} \frac{\partial F}{\partial u_i^\alpha} D^i K^\alpha,$$

where D is the total derivative with respect to x and K^α are the right hand sides of an evolution system

$$u_t^\alpha = K^\alpha(u). \tag{1}$$

One has to input the vector field K beforehand as the list **sys** (`sys` is the global name). For example, if you deal with the KdV equation $u_t = u_{xxx} + 6\,u\,u_x$ you must enter the following commands:

```
> vard:=[u]: sys:=[u3+6*u0*u1]:
```

3 Symmetries and conservation laws

The determining equation for Lie–Bäcklund symmetries of the system $u_t = K$ takes the following form (see [1, 2] or [3] for instance):

$$(D_t - K')F = 0, \tag{2}$$

where the prime denotes the Fréchet derivative

$$(K')^{\alpha}_{\beta} = \frac{\partial K^{\alpha}}{\partial u^{\beta}_{i}} D^{i}. \tag{3}$$

Here and below the summation rule over the repeated indices is implied. The procedure **Frechet** calculates the Fréchet derivative in different forms for scalar and vector cases. Let us consider the examples.

```
> vard:=[u]: depend(f(u0,u1) ):
> Frechet(u3+f);
```

$$array\left(0..3, \left[\ (0) = \frac{\partial f}{\partial u0} \quad (1) = \frac{\partial f}{\partial u1} \quad (2) = 0 \quad (3) = 1\ \right]\right)$$

Here we obtained a 1-dimensional array with scalar elements $(\partial F/\partial u_i)$. But in the vector case the elements of this array are square matrices:

```
> vard:=[u,v]: depend(f(u0,v0,u1,v1),g(u0,v0,u1,v1) ):
> Frechet([u2+f, -v2+g]);
```

$$array\left(0..2, \left[\ (0) = \begin{bmatrix} \frac{\partial f}{\partial u0} & \frac{\partial f}{\partial v0} \\ \frac{\partial g}{\partial u0} & \frac{\partial g}{\partial v0} \end{bmatrix} \quad (1) = \begin{bmatrix} \frac{\partial f}{\partial u1} & \frac{\partial f}{\partial v1} \\ \frac{\partial g}{\partial u1} & \frac{\partial g}{\partial v1} \end{bmatrix} \quad (2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\ \right]\right)$$

To obtain the left hand side of equation (2) you do not need to use the procedure `Frechet`. More simple way is provided by the procedure **defeq**. For example, in order to compute the third order Lie–Bäcklund symmetries for the KdV equation you must enter the following commands:

```
> vard:=[u]: depend(F(t,x,u0,u1,u2,u3) ):
> sys:=[u3+6*u0*u1]: flag:=0: a:=defeq([F],1);
```

$$a := ED(F) - 6\,u1\,F - 6\,u0\,D(F) - D^{(3)}(F)$$

Here **flag** is the control variables, $D(F)$ is `DF(F)` and $D^{(3)}(F)$ is `DN(F,3)`. If `flag=0` then the expressions `ED(F)`, `DF(F)` and `DN(F,n)` are not expanded. But if one assigns `flag:=1` or nothing (`flag:='flag':`) then all expressions will be expanded. Let us continue our example:

```
> flag:=1: a:=a: nops(");
```

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This means that the expression a consists of 62 terms and there is no need to look through it. In order to know which variables this expression contains, we use the procedure **ord**:

```
> ord(a);
```

[5]

This means that the order of a is equal to 5, that is, expression a contains u5. For the systems ord(a) returns a list [m, n, ...]. If vard=[u,v] and ord(a)=[2,3] for example, then the expression a contains u2 and v3 and does not contain u3, u4, ..., or v4, v5, ...

More detailed information about the expression a can be obtained with the help of the built-in procedure **indets**. Let us mention that the obtained expression `a` is a polynomial with respect to the highest order variables u_i, and therefore the built-in procedure **degree** is useful as well. To extract the terms with u5 one can use the procedure **chn** (CHoose Name), but the better way is to use the following command:

```
> b:=factor(chn(a,u5));
```

$$b := -3\,u5\,\left(\frac{\partial^2 F}{\partial x \partial u3} + \frac{\partial^2 F}{\partial u3^2}\,u4 + \frac{\partial^2 F}{\partial u3 \partial u0}\,u1 + \frac{\partial^2 F}{\partial u3 \partial u1}\,u2 + \frac{\partial^2 F}{\partial u3 \partial u2}\,u3\right)$$

It is easy to see that $b = -3\,u5\,D(\partial F/\partial u3)$. Hence the equation $b = 0$ implies $\partial F/\partial u3 = f_1(t)$ or $F = f_1(t)\,u3 + f_2(t, x, u0, u1, u2)$. To continue the computation you must enter the following commands:

```
> depend(f1(t), f2(t,x,u0,u1,u2) ): F:=f1*u3+f2:
> a:=expand(eval(subs(Diff=dif,a))):
```

The last command is necessary for the recomputation of all derivatives because the procedure `dif` returns the result in the inert form, for example `dif(f,u0)` $\rightarrow$ `Diff(f,u0)`.

The next problem that we consider is the computation of conserved currents. The vector function (ρ, θ) on the jet space is called the conserved current if it solves the equation

$$D_t\,\rho = D\,\theta, \tag{4}$$

where D_t is the evolution derivative along the trajectories of the system $u_t = K(u)$. The function ρ is said to be the conserved density and θ is said to be the density current. The current $(D\,f, D_t\,f)$ is conserved for any system and it is called the trivial conserved current. A trivial current may be added to any conserved current and result will be the conserved current again.

Equation (4) can be investigated with the help of the Euler operator E

$$E_\alpha = (-D)^n \frac{\partial}{\partial u_n^\alpha} \tag{5}$$

that possesses an important property: $E\,f = 0$ if and only if $f = D(F)$ [4]. Applying the operator E to equation (4), we obtain the following equation for the conserved densities

$$E\,D_t\,\rho = 0. \tag{6}$$

The package JET contains the procedure **EU** that performs the computation according to formula (5). To obtain the left hand side of equation (6) you must call

```
> EU(ED(rho),k);
```

where $k = 1, 2, \ldots, m$, and m is the number of the dependent variable (or number of entry of the list `vard`). These equations can be solved by the same method as it was demonstrated above for Lie–Bäcklund symmetries.

Another way of the computation of the conserved currents is given by the procedure **pot** (potential) that calculates a function f if the function $\phi = D\,f$ is known: `pot`(ϕ)=f. Hence if $(\rho,\ \theta)$ is a conserved current then θ =pot(ED(ρ)). Let us consider the zero order conserved densities for the KdV equation:

```
> pot(ED(u0)),  pot(ED(u0^2));
```

$$u2 + 3\,u0^2, \qquad 2\,u0\,u2 - u1^2 + 4\,u1^3$$

Now let us take the expression $\rho = u0^3$ that is not a conserved density, of course:

```
> th:=pot(ED(u0^3)), rm;
```

$$Break,\ ord(rm) = [1]$$

$$th := 3\,u0^2\,u2 - 3\,u0\,u1^2 + \frac{9}{2}u0^4, \qquad 3\,u1^3$$

This result means that `ED`($u0^3$)=`DF(th) + rm`, where `rm`=$3\,u1^3$. `rm` is the global name for a remainder when the `pot` is called.

When the zero order conserved density ρ exists, one can perform the following contact transformation $(t,\,x,\,u(t,\,x)) \to (t,\,y,\,U(t,\,y))$:

$$d\,y = \rho\,d\,x + \theta\,d\,t, \qquad U(t,\,y) = u(t,\,x). \tag{7}$$

This transformation is analogous to the transformation between Lagrange and Euler variables in the fluid dynamics. Therefore the procedure executing transformation (7) was called `L_E`. Let us transform the KdV equation, for example:

```
> L_E(u0,[U]);
```

$$[U\_t = 3\,U0^2\,U1\,U2 + U0^3\,U3 + 3\,U0^2\,U1]$$

Here the second argument of `L_E` must be a list of new dependent variables. And besides the procedure `L_E` may be called with three arguments: `L_E`(ρ,θ,`VARD`), where (ρ,θ) is a conserved current and `VARD` is the list of new dependent variables. In this case the procedure works slightly faster because θ is entered but is not evaluated.

4 Canonical conserved densities

In the paper [5] the necessary conditions of the complete integrability for evolution systems were introduced. Later these conditions were explained and generalized in [6] for a wide class of systems with two independent variables. Let the system

$$F(u) = 0 \tag{8}$$

be transformable to the Cauchy–Kowalewski normal form with the help of transformation of independent variables. Let us denote $\Phi(D_t,\,D_x) = F'^{+}$, where D_t and D_x are the total differentiation operators and $+$ is the symbol of the formal conjugation. Then let us consider the following system

$$\Phi(D_t + \theta,\, D_x + \rho)\psi = 0, \qquad (c,\psi) = 1, \tag{9}$$

where (c,ψ) is the Euclidean scalar product and c is an arbitrary constant vector. The main result is as follows.

If system (8) is integrable by the inverse spectral transform method then system (9) possesses a formal solution of the following form:

$$\rho=\sum_{i=-n}^{\infty}\rho_i\,k^i,\qquad \theta=\sum_{i=-n}^{\infty}\theta_i\,k^i,\qquad \psi=\sum_{i=0}^{\infty}\psi_i\,k^i, \tag{10}$$

where k is a parameter, $n>0$, $\rho_{-n}\neq 0$ or $\theta_{-n}\neq 0$ and $(\rho_i,\,\theta_i)$, $i=-n,-n+1,\dots$ are local or weakly nonlocal conserved currents of system (8).

System (9) and expansions (10) imply a recursion relation for ρ_i, θ_i and ψ_i. Therefore, the continuity equations $D_t\rho_i=D_x\theta_i$ give the constraints for system (8).

Let us consider the example

$$u_t=u_3+f(u,u_1). \tag{11}$$

A simple calculation gives $F'^{+}=D_t-D^3+f_0-D\,f_1$, where $f_0=\partial f/\partial u_0,\ f_1=\partial f/\partial u_1$. Hence equation (9) takes the form $[\theta-(D+\rho)^3+f_0-(D+\rho)\,f_1]\,1=0$, or

$$\theta-\rho^3+f_0-f_1\,\rho-D\left(\frac{3}{2}\,\rho^2-D\rho-f_1\right)=0.$$

Setting

$$\rho=k^{-1}+\sum_{i=0}^{\infty}\rho_i\,k^i,\qquad \theta=k^{-3}+\sum_{i=0}^{\infty}\theta_i\,k^i,$$

we obtain the required recursion formula

$$\begin{aligned}3\,\rho_{i+2}&=\theta_i-3\sum_{j=0}^{i+1}\rho_j\,\rho_{i-j+1}-\sum_{j,k=0}^{i}\rho_j\,\rho_k\,\rho_{i-j-k}-D^2(\rho_i)\\&\quad-\frac{3}{2}\,D\left(2\,\rho_{i+1}+\sum_{j=0}^{i}\rho_j\,\rho_{i-j}\right)+(f_0-D(f_1))\delta_{i0}-f_1\,\delta_{i,-1}-f_1\,\rho_i,\end{aligned} \tag{12}$$

where $i=-2,-1,\dots$ It is obvious that

$$\rho_0=0,\qquad \theta_0=0,\qquad \rho_1=-\frac{1}{3}\,f_1,\quad \dots$$

The conserved densities of system (8) produced by means of formula (9) are called canonical conserved densities. The canonical conserved densities of the KdV equation defined in (12) can be easily obtained, using the following program:

```
> r:=proc(n)
> local i;
> i:=n-2; if n <= 0 then RETURN(0) fi;
> if n = 1 then RETURN(-1/3*dif(f,u1)) fi;
> th.i/3-SU(r,r,0,i+1)-1/3*SU(r,r,r,0,i) - 'DF'(r(i+1))
> -1/2*'DF'(SU(r,r,0,i))-'DN'(r(i),2)/3+(dif(f,u0)-DF(dif(f,u1)))
>  *DLT(i,0)/3-dif(f,u1)*DLT(i,-1)/3-dif(f,u1)*r(i)/3
> end;
```

Here **DLT** is an auxiliary procedure for the Kronecker δ-symbol and **SU** is the procedure for the multiple sums. For example, the call `SU(A,B,C,n,m)` returns the sum of the monomials `A(i)*B(j)*C(k)` where $i, j, k \geq n$ and besides $i + j + k = m$. Number of arguments of `SU` may be arbitrary, and arguments of `SU` may be under `DF` or `DN` operators. So expressions of the type `SU(A,DF(B),DN(C,p),n,m)` are admissible. Moreover we assume that the expressions θ_0, $\theta_1, \ldots$ must be saved under the names `th0,th1,...`

For systems of two and more equations canonical densities may consist of dozens of hundreds terms. The evolution derivatives of such long expressions consist of dozens thousands terms. Processing a large expression requires very long time. And, moreover, if the number of addends in an expression is more than 40000 then Maple finishes the computation and informs: *Object is too large*. To solve this problem we apply the procedure **part**. The command `z:=part(F,n)` returns the list `z` with n entries so that each entry contains a part of the expression `F` and `z[1]+z[2]+...+z[n] =F`. Then one can perform the required operations with each element `z[i]` separately and obtain the final result. Another method is based on using the procedure **cho** (CHoose Order). The call `cho(F, 3)` for example, collects and returns those terms from the expression `F` whose orders ≥ 3. As the terms with the greatest order are interesting almost always then the procedure `cho` is very useful.

5 Zero curvature representations

Let us consider the following linear overdetermined system

$$\Psi_x = U\,\Psi, \qquad \Psi_t = V\,\Psi, \tag{13}$$

where Ψ is a column, U and V are the square matrices depending on the jet space coordinates t, x, u_n^α and a parameter λ. System (13) is compatible if and only if the following equation holds

$$U_t - V_x + [\,U,\, V\,] = 0. \tag{14}$$

If equation (13) is satisfied on the solutions manifold of an evolution partial differential system (1) but not identically then it is said that system (1) possesses the zero curvature representation.

Systems (13) and (14) are covariant under the gauge transformation:

$$\Psi \to \tilde{\Psi} = S\Psi, \qquad U \to \tilde{U} = S\,U\,S^{-1} + S_x S^{-1}, \qquad V \to \tilde{V} = S\,V\,S^{-1} + S_t S^{-1}.$$

This transformation may be used for simplification of the matrices U and V.

To investigate equation (14) in JET-package you must enter the following commands

```
> depend(U(...), V(...) ): matrices:={U,V}:
> z:=ED(U) - DF(V) + com(U, V);
```

and solve the equation $z = 0$. Here **matrices** is the global name of the set of symbolic matrices names, **com** is the name of a procedure for commutator. Arguments of `com` may be both symbolic matrices (names) and arrays. Procedure `com` knows all properties of commutators. For example,

```
> com(U, 2*U+3*V}, com(V,U), dif(com(U,V),u0);
```

$$3\,[\,U,\, V\,],\ -[\,U,\, V\,],\ \left[\frac{\partial U}{\partial u0},\, V\right] + \left[U,\, \frac{\partial V}{\partial u0}\right]$$

Ordering is performed automatically in the alphabetical order. Integration of `com(A,B)` is possible only if `A` and `B` are constants, but it suffices the analysis of equations (14). The procedure **Jac** transforms the nested commutators according to the Jacobi identity:

```
> matrices:={U,V,A,B,C,E}:
> z1:=com(A, com(B, E)) + com(C, com(A, B)),
```

$$z1 := [A, [B, E]] + [C, [A, B]]$$

```
> Jac(z1,A,B,C), Jac(z1,A,B,B);
```

$$[A, [B, E]] - [A[B, C]] + [B, [A, C]], \qquad [C, [A, B]] + [B, [A, E]] - [E, [A, B]]$$

`Jac` searches for the first nested commutator containing the 2nd, 3rd and 4th arguments of `Jac`, transforms it and returns the result. That is why different results are obtained. Here is one more example

```
> z2:=com(A, com(B, C)) + com(E,com(C, com(A, B)));
```

$$z2 := [A, [B, C]] + [E, [C, [A, B]]]$$

```
> Jac(z2,A,B,C,2); Jac(z1,A,B,C,yes,2);
```

$$[A, [B, C]] + [C, [E, [A, B]]] + [[A, B], [C, E]]$$
$$[A, [B, C]] - [E, [A, [B, C]]] + [E, [B, [A, C]]]$$

The 5th argument 2 in the first case makes `Jac` to begin the search from the second addend. The 5th argument `yes` make `Jac` perform the transformation within the external commutator. The call `Jac(z1,A,B,C,yes)` makes `Jac` to perform the transformation within the external commutator in the first addend and the error message will be returned. The call `Jac(z1,A,B,C,2,yes)` is the mistake as well, both parameters 2 and `yes` will be ignored and the first term will be transformed in this case.

When equation (14) is solved, the next problem is to construct the Lie algebra. Let us consider the KdV equation as an example. After some simple calculations one can obtain the matrices $U = A_1\, u_0 + A_2$ and

$$V = A1\, u_2 - [A1,\, A2]\, u_1 + 3\, A1\, u_0^2 - 1/2\, u_0^2\, [A1,\, [A1,\, A2]] - [A2,\, [A1,\, A2]]\, u_0,$$

where A_i are constant matrices. Moreover the following equations are obtained

$$\begin{aligned} &[A1,\, [A2,\, A3]] + 2\, A3 = 0, \qquad [A1,\, A2] = A3, \\ &[A1,\, [A1,\, A3]] = 0, \qquad [A2,\, [A2,\, A3]] = 0. \end{aligned} \tag{15}$$

There are different ways to solve this system. For example, one can choose one of the matrices in the Jordan normal form and try to solve the equations directly. But this way is difficult for large algebras and the better way is investigation of equations (15) in the spirit of the ideas by H.D. Wahlquist and F.B. Estabrook [7]. There is very useful procedure **struct** for obtaining the closed algebra in this approach . The procedure `struct` constructs the adjoint representation of a Lie algebra and returns the equations for unknown structural constants if the input algebra is not closed. In our example we can use `struct` at once, but it is necessary to enter the basis of the algebra beforehand. Let us assume that $A1$, $A2$ and $A3$ form the basis and enter the commands:

```
> s:={com(A1,A2)=A3}: bas:=[A1,A2,A3]:
> struct(bas,s,x);
```

Structural constants are given by array C[i][kj]=C^k_{ij}
*Table of commutators [e_i,e_n]=C^k_{in}*e_k is given by set EQ*
Substitutions bas[i]=C[i] are set S, and constraints are:
z:=[[-x1 x6+x3 x4, x3 x1+x2 x6, x1+x5]]

Here the first parameter `bas` is the list of basis elements, the second parameter `s` must be a set of commutation relations and the third parameter of `struct` must be a name x so that *x1, x2,* $\ldots$, are free variables. These variables are used in the table of commutators:

$$EQ = \{[A1,\, A2] = A3, \qquad [A1,\, A3] = x1\, A1 + x2\, A2 + x3\, A3,$$
$$[A2,\, A3] = x4\, A1 + x5\, A2 + x6\, A3\}.$$

The names `C, EQ` and `S` are global. The obtained list z contains the left hand sides of the equations

$$-x1\, x6 + x3\, x4 = 0, \qquad x3\, x1 + x2\, x6 = 0, \qquad x1 + x5 = 0.$$

Solving these equations we obtain the closed table of commutators EQ. Setting for example $x1 = -x5 = 2$, $x2 = k$, $x3 = x4 = x6 = 0$, where k is a parameter we obtain the standard algebra $sl(2)$ for the KdV:

```
> EQ;
```

$$\{[A1,\, A2] = A3, \;\; [A1,\, A3] = 2\, A1 + k\, A2, \;\; [A2,\, A3] = -2\, A2\} \tag{16}$$

This algebra solves all equations (15). Constructing then a representation of the obtained Lie algebra we can find an explicit form of the matrices U and V.

For solving the problems considered in this section the following procedures are useful: **evsub, Cmetric** and **Killing**. The command `evsub(A)` is used for evaluation and simplification of the elements of 2-dimensional array `A`. The call `evsub(s,A)` where `s` is the set of substitutions is used for performing the substitutions into 2-dimensional array `A`. The call `Cmetric()` returns the Cartan metric tensor g_{ij} of a Lie algebra. And the call `Cmetric(y)` returns the quadratic form $g_{ij}\, y^i\, y^j$. The command `Killing(A,B)` returns the value of the Killing form $\langle A, B\rangle$=trace(ad A ad B) for the pair of elements A, B of a Lie algebra.

6 Recursion operators

The recursion operator Λ of evolution system (1) satisfies the following equation

$$[D_t - K',\, \Lambda] = 0 \tag{17}$$

by definition (see [1–3]). There are two procedures in JET for the computation of the recursion operator.

If you know the zero curvature representation for your system, try call the procedure **triada** that uses the algorithm published in [8, 9]. For example, the KdV equation possesses algebra (16) and we have

```
> triada(U,s);
```

$$\left[\frac{\partial^3 g}{\partial x^3} + 4\, u0\, \frac{\partial g}{\partial x} - k\, \frac{\partial g}{\partial x} + 2\, u1\, g\right]$$

Now you should transform this equation (or system in the vector case) to the following form $L\,g = k\,g$. Then $\Lambda = L^+$ [8, 9]. In our example this gives the well-known Lenard operator

$$\Lambda = D^2 + 4\,u_0 + 2\,u_1\,D^{-1}.$$

When the matrices U and V are embedded in the Lie algebra of a small dimension then this approach is acceptable. Otherwise the equation $L(u,k)g = 0$ is too large object. In this case you can try calculate the recursion operator or Noether operators directly. If we set

$$\Lambda = \sum_{i=0}^{n} F_{n-i}(u)D^i + \Sigma(u)D^{-1}\Gamma(u), \tag{18}$$

then equation (17) implies that the columns of Σ are symmetries and the rows of Γ are gradients of conserved densities. That is, Σ satisfies equation (2) and Γ^T satisfies the adjoint equation

$$(D_t + K'^{+})\Gamma^T = 0.$$

The coefficients F_i, Σ and Γ satisfy a cumbersome system that can be obtained with the help of procedure **recursion**. It can be called with one or two input parameters:

```
> sys:=[u3+u0*u1]:
> recursion(0); recursion(1); recursion(1,2);
```

$$(F0\&*K3) - (K3\&*F0)$$

$$(F1\&*K3) - (K3\&*F1) + (F0\&*K2) - (K2\&*F0) + n\,(F0\&*D(K3))$$
$$-3\,(K3\&*D(F0)) - (rsys(3+n)\&*(\Sigma\&*\Gamma)) + (\Sigma\&*(\Gamma\&*rsys(3+n)))$$

$$(F1\&*K3) - (K3\&*F1) + (F0\&*K2) - (K2\&*F0) + n\,(F0\&*D(K3))$$
$$-3\,(K3\&*D(F0))$$

Here $\&*$ is a symbol of the matrix multiplication, $F0$, $F1$ etc., Σ and Γ are exactly the coefficients of operator (18), $K0$, $K1$ etc. are the coefficients of the operator

$$K' = \sum_{i=0}^{N} K_i\,D^i,$$

`rsys(i)`$=K_i$ if $0 \le i \le N$ and otherwise `rsys(i)`$=0$. The number N is determined by the list `sys` ($N = 3$ in our example) and n is a nonnegative parameter. The call `recursion(1,2)` means that we assume $n \ge 2$. Then the result is shorter. The number of equations returned by procedure `recursion` is $N + n + 1$. To solve the equations for $F0$, $F1, \ldots$ you must substitute there these matrices with undetermined coefficients and the matrices $K0$, $K1, \ldots$ that one can obtain with help of the procedure `Frechet`. Matrices Σ and Γ must be calculated beforehand. If you solve the first equation and find $F0$, try enter $n = 0$ or $n = 1$ and solve next equations. If such solution does not exist then you can call `recursion(i,2)`, $i = 0, 1, \ldots$ and solve these equations with arbitrary n (but $n \ge 2$). Then you can enter $n = 2$ and so on.

7 Noether operators

Let us consider a pair of operators Θ and J satisfying the following equations

$$(D_t - K')\Theta = \Theta(D_t + K'^{+}), \tag{19}$$

$$(D_t + K'^{+})J = J(D_t - K'). \tag{20}$$

The operator Θ is called a Noether operator and J is called the inverse Noether operator [10, 11]. Of course if Θ satisfies equation (19) then Θ^{-1} satisfies equation (20). But one cannot find Θ^{-1} or J^{-1} explicitly as a rule. If an evolution system admits two Noether operators Θ_1 and Θ_2 and Θ_2 is invertible then $\Theta_1\,\Theta_2^{-1}$ is the recursion operator. If two inverse Noether operators J_1 and J_2 exist and J_2 is invertible then $J_2^{-1}J_1$ is the recursion operator. Sometimes system (1) admits Noether operator Θ and inverse Noether operator J ($\neq \Theta^{-1}$) then ΘJ is the recursion operator [11].

The most general form of the Noether and inverse Noether operators known today is

$$\Theta = \sum_{i=0}^{n} \theta_{n-i} D^i + A\,D^{-1}B, \tag{21}$$

$$J = \sum_{i=0}^{n} J_{n-i} D^i + G\,D^{-1}H. \tag{22}$$

Here the columns of A and rows of B are Lie–Bäcklund symmetries of system (1); the columns of G and rows of H are gradients of the conserved densities of system (1). It happens that $A = 0$ or $G = 0$ for some systems.

The procedure **Noether** returns the equations for the matrices θ_i, A and B of operator (21). The procedure **INoether** returns the equations for the matrices J_i, G and H of operator (22). Both procedures have the same syntax as the procedure `recursion`: `Noether(m)` or `Noether(m,k)`. Here m is a number of the returned equation, the second parameter k is used if you know that the order of Θ or J is greater than or equal to k.

The Noether operator of an integrable evolution system is an implectic operator and the inverse Noether operator is a symplectic operator as a rule.

The operator Θ is called implectic if it is antisymmetric ($\Theta^+ = -\Theta$) and the bracket $\{f, g, h; \Theta\} = \langle f,\, \Theta'[\Theta\, g]h\rangle$ satisfies the Jacobi identity

$$\{f, g, h; \Theta\} + \{g, h, f; \Theta\} + \{h, f, g; \Theta\} = 0. \tag{23}$$

The operator J is called symplectic if it is antisymmetric ($J^+ = -J$) and the bracket $[f, g, h; J] = \langle f,\, J'[g]h\rangle$ satisfies the Jacobi identity

$$[f, g, h; J] + [g, h, f; J] + [h, f, g; J] = 0. \tag{24}$$

The procedure **implectic** checks the identities $\Theta^+ = -\Theta$ and (23). The syntax is `implectic(L,n)`. Here L=$[\theta_0, \theta_1, \ldots, \theta_n, A, B]$ is the list of the coefficients of operator (21), the second parameter n is the order of Θ.

The procedure **symplectic** checks the identities $J^+ = -J$ and (24). The syntax is `symplectic(L,n)`. Here L=$[J_0, J_1, \ldots, J_n, G, H]$ is the list of the coefficients of operator (22), the second parameter n is the order of J.

Both procedures `implectic` and `symplectic` return the text information: "Antisymmetry – OK" or "Antisymmetry is not valid, reminder is saved as rm" Then these procedures simplify the left hand sides of identities (23) and (24) as much as possible and return them as the results.

Conclusion

We are going to prepare the help file for our package and place it in Internet.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 47–59.

Symmetries of Systems of Nonlinear Reaction-Diffusion Equations

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We present the complete analysis of classical Lie symmetries of systems of two nonlinear diffusion equations with $1+m$ independent variables $t, x_1, \ldots, x_m$, whose nonlinearities do not depend on t and x.

1 Introduction

Coupled systems of nonlinear diffusion equations have many important applications in mathematical physics, chemistry and biology. These systems are very complex in nature and admit fundamental particular solutions (for example, traveling waves and spiral waves) which have a clear group-theoretical interpretation and which can be obtained using the classical Lie approach. The existence of such solutions predetermines an important role for the group theoretical approach in the analysis of systems of reaction diffusion equations. However, to the best of our knowledge, a comprehensive group analysis has not been undertaken previously although analyses of some special cases do exist.

In the present paper we investigate Lie symmetries of equations in the general form

$$\frac{\partial u}{\partial t} - \Lambda \sum_i \frac{\partial^2 u}{\partial x_i^2} - f(u) = 0, \tag{1.1}$$

where the dependent variable $u = \mathrm{column}(u_1, u_2, \ldots, u_n)$ is a n-component vector-function, dependent on $m+1$ variables $t, x_1, x_2, \ldots, x_m$. Also, $f = \mathrm{column}(f_1, f_2, \ldots, f_n)$ is an arbitrary vector-function of u and Λ is a $n \times n$ constant matrix which is non-singular.

Classical Lie symmetries of equation (1.1) with, $n = m = 1$, were investigated by Ovsiannikov [1] whose results were completed by Dorodnitsyn [2]. The related conditional (nonclassical) symmetries were described by Fushchych and Serov [3] and Clarkson and Mansfield [4]. Symmetries of equation (1.1) with $m > 1$ and (or) $n > 0$ were partly investigated in papers [5–7]. We notice that it was equation (1.1) for $m = n = 1$, $f \equiv 0$, was the subject of a group analysis by Sophus Lie [8].

An investigation of the symmetries of the general equation (1.1) can be undertaken within the framework of the classical Lie algorithm (see, for example, [9, 10]) which reduces the problem of determining symmetry to the solution the systems of linear over-determined equations for the coefficients of the symmetry operators. We will show that when applied to systems (1.1), this algorithm admits a rather simple formulation which may also be applied to an extended class of partial differential equations.

2 Determining equations for symmetries of the system (1.1)

We require form-invariance of the system of reaction diffusion equations (1.1) with respect to the one-parameter group of transformations:

$$t \to t'(t, x, \varepsilon), \qquad x \to x'(t, x, \varepsilon), \qquad u \to u'(t', x', \varepsilon), \tag{2.1}$$

where ε is a group parameter. In other words, we require that $u'(t', x', \varepsilon)$ satisfies the same equation, as $u(t, x)$:

$$L'u' = f(u'), \qquad L' = \frac{\partial}{\partial t'} - \frac{\partial^2}{\partial x'^2}. \tag{2.2}$$

From the infinitesimal transformations:

$$\begin{aligned} &t \to t' = t + \Delta t = t + \varepsilon\eta, \qquad x_a \to x'_a = x_a + \Delta x_a = x_a + \varepsilon\xi^a, \\ &u_a \to u'_a = u_a + \Delta u_a = u_a + \varepsilon\pi_a \end{aligned} \tag{2.3}$$

we obtain the following representation for the operator L'.

$$L' = \left[1 + \varepsilon\left(\eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a}\right)\right] L \left[1 - \varepsilon\left(\eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a}\right)\right] + O\left(\varepsilon^2\right). \tag{2.4}$$

Using the classical Lie algorithm, it is possible to find the determining equations for the functions η, ξ_a and π_a, which specify the generator X of the symmetry group:

$$X = \eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a} - \pi^a\frac{\partial}{\partial u_a}, \tag{2.5}$$

where a summation from 1 to m is assumed over repeated indices. This system will not be reproduced here but we note that three of the equations are:

$$\frac{\partial\eta}{\partial u_a} = 0, \qquad \frac{\partial\xi^a}{\partial u_b} = 0, \qquad \frac{\partial^2\pi^a}{\partial u_c\partial u_b} = 0. \tag{2.6}$$

So from (2.6), η and ξ^a are functions of t and x_a and π^a is linear in u_a. Thus:

$$\pi^a = -\pi^{ab}u_b - \omega^a, \tag{2.7}$$

where π^{ab} and ω^a are functions of t and $x = (x_1, x_2, \ldots x_m)$.

From (2.6) it is possible to deduce all the remaining determining equations. Indeed, substituting (2.3), (2.7) into (2.4), using (1.1) and neglecting the terms of order ε^2, we find that

$$[Q, L]\,u + L\omega = \pi f + \frac{\partial f}{\partial u_a}\left(-\pi^{ab}u_b - \omega^a\right), \qquad Q = \eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a} + \pi, \tag{2.8}$$

and π is a matrix whose elements π^{ab} are defined by the relation (2.7).

To guarantee that equation (2.8) is compatible with (1.1) and does not impose new nontrivial conditions for u in addition to (1.1), it is necessary to suppose that the commutator $[Q, L]$ admits the representation:

$$[Q, L] = \Lambda L + \varphi(t, x), \tag{2.9}$$

where Λ and φ are $n \times n$ matrices dependent on (t, x_a).

Substituting (2.9) into (2.8) the following determining equations for f are obtained:

$$\left(\Lambda^{kb} - \pi^{kb}\right) f^b + \varphi^{kb}u^b + L\omega^k = -\left(\omega^a + \pi^{ab}u_b\right)\frac{\partial f^k}{\partial u^a}. \tag{2.10}$$

Thus, to find all nonlinearities f^k generating Lie symmetries for equation (1.1) it is necessary to solve the operator equation (2.9) for L, Q given in (2.2), (2.8) and determine the corresponding matrices Λ, π, φ and functions η and ξ. In the second step the nonlinearities f^a may be found by solving the system of first order equations (2.10) with their known coefficients.

Equation (2.9) is a straight forward generalization of the invariance condition for the *linear* system of diffusion equations (1.1) with $f(u) = 0$, so that:

$$[Q, L] = \Lambda L$$

which may readily be solved. By means of this "linearization" the problem of investigating symmetries of systems of nonlinear diffusion equations is reduced to the, rather simple, application of elements of matrix calculus in order to classify non-equivalent solutions of the determining equations.

We note also that calculations of the *conditional* (nonclassical) symmetries for the system (1.1) may be reduced to the solution of the determining equations (2.10) where now Λ, π, φ, η and ξ are defined as solutions of the following relationship:

$$[Q, L] = \Lambda L + \varphi(t, x) + \mu(t, x)Q, \tag{2.11}$$

where $\mu(t, x)$ is an unknown function of the independent variables.

3 General form of symmetry operators

We now determine the general solutions for matrices Λ, φ, π and also the functions ξ, η, π which satisfy (2.10), (2.9). Evaluating the commutator in (2.9) and equating the coefficients for linearly independent differential operators, we obtain the determining equations:

$$2A\xi^a_b = -\delta_{ab}(\Lambda A + [A, \pi]), \tag{3.1}$$

$$\dot{\eta}_a = 0, \qquad \dot{\eta} = \Lambda, \tag{3.2}$$

$$\dot{\xi}^a - 2A\pi_a - A\xi^a_{nn} = 0, \tag{3.3}$$

$$\varphi = A\pi_{nn} - \dot{\pi}. \tag{3.4}$$

Here the dots denotes derivatives with respect to t and subscripts denote derivatives with respect to the spatial variables, so for example, $\eta_a = \frac{\partial \eta}{\partial x_a}$.

From (3.2) Λ is proportional to the unit matrix, $\Lambda = \lambda I$. Moreover, it follows from (3.1) that $[A, \pi] \equiv 0$. Indeed, choosing in (3.1) $a = b$ we obtain

$$\pi - A^{-1}\pi A = (2\xi^a_a - \lambda)I. \tag{3.5}$$

The trace of the left hand side of (3.5) is equal to zero, and so $2\xi^a_a - \lambda \equiv 0$ and $A\pi - \pi A = 0$.

Equations (3.1)–(3.4) contain matrices which commute, and so they may easily be integrated using, for example, the method of characteristics. The general solution of (3.1)–(3.4) is:

$$\begin{gathered}
\xi^a = C^{[ab]}x_b + \dot{d}x^a + g^a, \qquad \eta = -2d, \\
\pi = \frac{1}{2}A^{-1}\left(\frac{\ddot{d}}{2}x^2 + \dot{g}^a x^a\right) + C, \qquad \Lambda = -2\dot{d}I, \\
\varphi = \frac{m}{2}\ddot{d} - \dot{C} - \frac{1}{2}A^{-1}\left(\frac{\dddot{d}}{2}x^2 + \ddot{g}^a x^a\right),
\end{gathered} \tag{3.6}$$

where d, g^a are arbitrary functions of t and C is a t-dependent matrix commuting with A.

By considering the x-dependence of functions (3.6) it is convenient to represent a still unknown function ω, occuring in (2.10), as:

$$\omega_a = \omega_2^a x^2 + \omega_1^{ab} x_b + \omega_0^a + \mu^a, \tag{3.7}$$

where ω_2^a, ω_1^{ab}, ω_0^a are functions of t, and μ^a is a function of t and x, $x^2 = x_1^2 + x_2^2 + \cdots + x_m^2$. Without loss of generality we suppose that all terms in the right hand side of (3.7) are linearly independent. Then comparing with (2.10), (3.6) the functions μ^k have to satisfy:

$$L\mu^k = \lambda^{kb}\mu^b + \xi_0^k + \xi_1^{kb} x_b + \xi_2^k x^2, \tag{3.8}$$

where λ^{kb} are constants and ξ_0^k, ξ_1^{kb}, ξ_2^k are functions of t.

The final step is to substitute (3.5), (3.7) into (2.10) and equate coefficients for all different powers of x_a. As a result we obtain the system of equations:

$$\ddot{d}\left(A^{-1}\right)^{kb} f^b + \dddot{d}(A^{-1})^{kb} u^b - \ddot{d}\left(A^{-1}\right)^{ab} u^b \frac{\partial f^k}{\partial u^a} = 4\left(\dot{\omega}_2^k + \xi_2^k - \omega_2^b \frac{\partial f^k}{\partial u_b}\right) = 0, \tag{3.9}$$

$$\dot{g}^a \left(A^{-1}\right)^{kb} f^b + \ddot{g}^a \left(A^{-1}\right)^{kb} u^b - \dot{g}^a \left(A^{-1}\right)^{kb} u^b \frac{\partial f^k}{\partial u^a} = 2\left(\dot{\omega}_1^{ka} + \xi_1^k a - \omega_1^{ba} \frac{\partial f^k}{\partial u_a}\right) = 0, \tag{3.10}$$

$$\left(2\dot{d}\delta^{kb} + C^{kb}\right) f^b + \left(\dot{C}^{kb} - \frac{m}{2}\ddot{d}\delta^{kb}\right) u^b - \left(\omega_0^a + C^{ab} u^b\right)\frac{\partial f^k}{\partial x_a} = \omega_0^k - 2mA^{kb}\omega_2^b - \omega_0^b \frac{\partial f^k}{\partial u_b}, \tag{3.11}$$

$$\frac{\partial f^k}{\partial u_b}\mu^b = \lambda^{kb}\mu^b. \tag{3.12}$$

Thus, the general form of symmetry group generators for equation (1.1) is given by relations (2.5), (3.6), (3.7), where d, g^a, C^{ab}, ω_0^k, ω_1^{kb}, ω_2^a, μ^a are functions of t to be specified using equations (3.9)–(3.12).

4 Nonlinearities and symmetries

We will not give the detailed calculations but present the general solution of relations (3.9)–(3.12) in the form of the following Tables 1–3.

In Table 1 the Greek letters denote arbitrary coefficients while D_μ, G_a^i and $\bar{G}_a^i$, X_A, Y_a, $\hat{F}$, $\hat{B}$ are various types of dilatation, Galilei and special transformation generators as follows:

$$\begin{gathered}
D_0 = 2t\frac{\partial}{\partial t} + x_a\frac{\partial}{\partial x_a}, \qquad D_1 = D_0 - \frac{2}{k}\hat{F}, \qquad D_2 = D_0 - \frac{2s}{\sqrt{k^2+s^2}}\left(u\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right), \\
D_3 = D_0 - \frac{2}{k}\left(\frac{\partial}{\partial u_1} - 2nu_1\frac{\partial}{\partial u_1}\right), \qquad D_4 = D_0 - \frac{2}{k}\omega_a\frac{\partial}{\partial u_a}, \\
G_a = t\frac{\partial}{\partial x_a} - \frac{1}{2}x_a\left(A_1^{-1}\right)^{nb} u_b\frac{\partial}{\partial u_n}, \qquad \hat{G}_a = \exp(nt)\left(\frac{\partial}{\partial x_a} - \frac{1}{2}nx_a\left(A_1^{-1}\right)^{nb} u_b\frac{\partial}{\partial u_n}\right), \\
X_0 = \alpha\frac{\partial}{\partial t} + \beta_a\frac{\partial}{\partial x_a} + \nu^{[a,b]}x_a\frac{\partial}{\partial x_b}, \qquad \nu^{[a,b]} = -\nu^{[b,a]}, \qquad Y_1 = nt\hat{F} - \hat{B}, \\
Y_2 = \exp(st)\left(u_1\frac{\partial}{\partial u_1} + n\frac{\partial}{\partial u_2}\right), \qquad Y_2 = u_1\frac{\partial}{\partial u_2} - n\frac{\partial}{\partial u_1}, \qquad Y_3 = u_1\frac{\partial}{\partial u_2} - 2n\frac{\partial}{\partial u_2}, \\
Y_4 = \exp(kt)\left(u_1\frac{\partial}{\partial u_2} + \frac{nx^2}{2m}\frac{\partial}{\partial u_2}\right),
\end{gathered} \tag{4.1}$$

where k, n, s are parameters used in the definitions of the nonlinear terms,

$$\hat{F} = F^{ab} u_b \frac{\partial}{\partial u_a}, \qquad \hat{B} = B^{ab} u_b \frac{\partial}{\partial u_a}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \tag{4.2}$$

and B is one of the matrices:

$$\begin{aligned} &I. \quad B = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}; \qquad && II. \quad B = \begin{pmatrix} d & -1 \\ 1 & d \end{pmatrix}; \\ &IIIa. \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \qquad && IIIb. \quad B = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}. \end{aligned} \tag{4.3}$$

In Tables 2 and 3 we form a two-dimensional Lie algebra based upon the matrices, F and B, classified the categories:

$$\begin{aligned} &I. \quad F = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \qquad && B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \\ &IIa. \quad F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad && B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \\ &IIb. \quad F = \begin{pmatrix} d & -1 \\ 1 & d \end{pmatrix}, \qquad && B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \\ &IIIa. \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad && B = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}; \\ &IIIb. \quad F = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}, \qquad && B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{4.4}$$

Here, $\kappa = \text{const}$, $\Delta = k_0 n_1 - n_0 k_1$, $\delta = \frac{1}{4}(k_0 - n_1)^2 + k_1 n_0$ and:

$$\left(\frac{\partial}{\partial t} - \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} \right) \psi_n = n \psi_n, \qquad n = 0, b. \tag{4.5}$$

The generators D_μ, $\hat{A}_\alpha$, G_a, $\hat{G}_a$, X_ν, Y_s when not specified in (4.1) are given by:

$$\hat{A}_0 = t^2 \frac{\partial}{\partial t} + t x_a \frac{\partial}{\partial x_a} - \frac{1}{4} x^2 \left(A^{-1}\right)^{ab} u_b \frac{\partial}{\partial u_a} - \frac{m}{2} + \left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right),$$

$$\hat{A}_1 = \hat{A}_0 + n t^2 \hat{F} - \frac{m}{2} \hat{B}, \qquad D_5 = D_0 - \frac{m}{2} \left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right),$$

$$D_6 = D_0 - 2tn\hat{F} - \frac{2}{k} \hat{B}, \qquad D_7 = D_0 + \frac{2}{r} \left(\frac{kst}{r} - 1 \right) u_1 \frac{\partial}{\partial u_1} - \frac{2st}{r} \frac{\partial}{\partial u_2},$$

$$D_8 = D_0 - 2st(u_2 - 1) \frac{\partial}{\partial u_1} - 2 \frac{\partial}{\partial u_2}, \qquad D_9 = D_0 - \left(\frac{p}{m} x^2 + 2qt \right) \frac{\partial}{\partial u_1} - 2 \frac{\partial}{\partial u_2},$$

$$D_{10} = D_0 - \frac{1}{n} \left(u_1 \frac{\partial}{\partial u_1} + 2 u_2 \frac{\partial}{\partial u_2} \right) + \frac{t}{2sn} \frac{\partial}{\partial u_1} - \frac{t}{s} u_1 \frac{\partial}{\partial u_2},$$

$$D_{11} = D_0 - \frac{1}{k} \left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} - s(2k+1) x^2 \frac{\partial}{\partial u_1} \right),$$

$$X_1 = \mu \exp(\lambda^+ t)\hat{\mathcal{F}}_1 + \nu \exp(\lambda^- t)\hat{\mathcal{F}}_2, \qquad \mathcal{F}_1 = \left[\frac{1}{2}(k_o - n_1) + \sqrt{\delta}\right] F + n_0 B, \qquad n = \lambda^+,$$

$$\mathcal{F}_2 = k_1 F - \left[\frac{1}{2}(k_o - n_1) + \sqrt{\delta}\right] B, \qquad n = \lambda^-, \qquad \lambda^\pm = \frac{1}{2}(k_0 + n_1) \pm \delta,$$

$$X_2 = \mu \exp(nt)\hat{\mathcal{F}}_3 + \nu\hat{\mathcal{F}}_4, \qquad n = k_0 + n_1, \qquad \mathcal{F}_3 = k_0 F + n_0 B, \qquad \mathcal{F}_4 = k_0 B - k_1 F;$$

$$X_3 = \mu \exp(nt)\hat{\mathcal{F}}_5 + \nu\hat{\mathcal{F}}_6, \qquad n = k_0 + n_1, \qquad \mathcal{F}_5 = k_1 F + n_0 B, \qquad \mathcal{F}_6 = n_1 F - n_0 B,$$

$$X_4 = \exp(nt)\hat{\mathcal{F}}_7, \qquad n = \frac{1}{2}(k_0 + n_1), \qquad \mathcal{F}_7 = \mu F + \nu B,$$

$$X_5 = \exp(nt)\left[\mu(k_1 t\hat{F} + \hat{B}) + \nu\hat{F}\right],$$

$$X_6 = \exp(nt)\left[\mu\left(\hat{F} + n_0 t\hat{B}\right) + \nu\hat{B}\right], \qquad n = \frac{1}{2}(k_0 + n_1),$$

$$X_7 = \exp(nt)\mu\left[\left(\sqrt{-n_0 k_1}\,t + 1\right)\hat{F} + n_0 t\hat{B}\right] + \nu\left[\left(k_1\hat{F} - \sqrt{-n_0 k_1}\,\hat{B}\right)t + \hat{B}\right],$$

$$n = \frac{1}{2}(k_0 + n_1),$$

$$X_8 = \nu\left[k_1 t\hat{F} + (1 - k_0 t)\hat{B} + \mu\left(k_1\hat{F} - k_0\hat{B}\right)\right],$$

$$X_9 = \exp(nt)\left[\frac{1}{2}(n_1 - k_0)\cos(\omega t) + \omega\sin(\omega t)\right]\hat{F} - n_0\cos(\omega t)\hat{B},$$

$$X_{10} = \exp(nt)\left[\omega\sin(\omega t) + \frac{1}{2}(k_0 - n_1)\cos\omega t\right]\hat{B} - k_1\cos(\omega t)\hat{F}, \qquad n = \frac{1}{2}(k_0 - n_1),$$

$$Y_5 = \exp(nt)\left(u_1\frac{\partial}{\partial u_2} - \frac{q}{2p}\left(\frac{sx^2}{2m}\frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1}\right)\right), \qquad Y_6 = \exp(nt)\left(u_1\frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1}\right),$$

$$Y_7 = u_1\frac{\partial}{\partial u_1} + nt\frac{\partial}{\partial u_2}, \qquad Y_8 = \exp(kt)\left(u_1\frac{\partial}{\partial u_1} + nt\frac{\partial}{\partial u_2}\right),$$

$$Y_9 = \exp(kt)\left(u_1\frac{\partial}{\partial u_1} + \frac{n}{k-b}\frac{\partial}{\partial u_2}\right), \qquad Y_{10} = ku_1\frac{\partial}{\partial u_1} - r\frac{\partial}{\partial u_2},$$

$$Y_{11} = \exp(nt)\left(\sin(pt)u_1\frac{\partial}{\partial u_1} + \cos(pt)\frac{\partial}{\partial u_2}\right),$$

$$Y_{12} = \exp(nt)\left(\cos(pt)u_1\frac{\partial}{\partial u_1} - \sin(pt)\frac{\partial}{\partial u_2}\right),$$

$$Y_{13} = \exp(nt)\left(ptu_1\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right), \qquad Y_{14} = \exp(nt)u_1\frac{\partial}{\partial u_1}.$$

Table 1. Nonlinearities with arbitrary functions

No	Nonlinear terms	Type of matrix B (4.3)	Arguments of φ_1, φ_2	Conditions for parametrs	Symmetries $A^{-1} \neq \kappa B$	Symmetries $A^{-1} = \kappa B$
1.	$f^1 = u_1^{k+1}\varphi_1$ $f^2 = u_1^{k+d}\varphi_2$	I	$\frac{u_2}{u_1^d}$	$k \neq 0$	$X_0 + \nu D_1$	$X_0 + \nu D_1$
2.	$f^1 = \exp(k\theta)[\varphi_1 u_2 + \varphi_2 u_1]$ $f^2 = \exp(k\theta)[\varphi_1 u_1 - \varphi_2 u_2]$	II	$R\, e^{-d\theta}$	$k \neq 0$	$X_0 + \nu D_1$	$X_0 + \nu D_1$
3.	$f^1 = \varphi_1 u_1^{k+1}$ $f^2 = (\varphi_1 \ln u_1 + \varphi_2)\, u^{k+1}$	$IIIb$, $d = 1$	$u_1 e^{-\frac{u_2}{u_1}}$	$k \neq 0$	$X_0 + \nu D_1$	$X_0 + \nu D_1$
4.	$f^1 = \exp\left(k\frac{u_2}{u_1}\right)\varphi_1 u_1$ $f^2 = \exp\left(k\frac{u_2}{u_1}\right)(\varphi_1 u_2 + \varphi_2)$	$IIIa$	u_1	$k \neq 0$	$X_0 + \nu D_1$	
5.	$f^1 = u_1(n \ln u_1 + \varphi_1)$ $f^2 = u_2(n \ln u_2 + \varphi_2)$	I	$\frac{u_2}{u_1^d}$	$n \neq 0$	$X_0 + \mu e^{nt}\hat{B}$	$X_0 + \nu_a \tilde{G}_a + \mu e^{nt}\hat{B}$
				$n = 0$	$X_0 + \mu\hat{B}$	$X_0 + \mu\hat{B} + \nu_a G_a$
6.	$f^1 = \varphi_1 u_2 + \varphi_2 u_1$ $\quad + \frac{n}{2}\left(\frac{1}{d}\ln R + \theta\right)(du_1 - u_2)$ $f^2 = \varphi_1 u_1 - \varphi_2 u_2$ $\quad + \frac{n}{2}\left(\frac{1}{d}\ln R + \theta\right)(du_2 + u_1)$ $R^2 = u_1^2 + u_2^2$, $\theta = \arctan\left(\frac{u_2}{u_1}\right)$	II, $d \neq 0$	$R\, e^{-d\theta}$	$n \neq 0$	$X_0 + \mu e^{nt}\hat{B}$	$X_0 + \nu_a \hat{G}_a + \mu e^{nt}\hat{B}$
				$n = 0$	$X_0 + \mu\hat{B}$	$X_0 + \mu\hat{B} + \lambda_a G_a$
7.	$f^1 = (\varphi_1 - n\theta)u_2 + \varphi_2 u_1$ $f^2 = \varphi_1 u_1 - \varphi_2 u_2$	II, $d = 0$	R	$n \neq 0$	$X_0 + \mu e^{nt}\hat{B}$	$X_0 + \nu_a \hat{G}_a + \mu e^{nt}\hat{B}$
				$n = 0$	$X_0 + \mu\hat{B}$	$X_0 + \mu\hat{B} + \lambda_a G_a$
8.	$f^1 = \varphi_1 u_1 + n u_2$ $f^2 = (\varphi_1 u_2 + u_1\varphi_2) + n u_2\left(1 + \frac{u_2}{u_1}\right)$	$IIIb$, $d = 1$	$\frac{u_2}{u_1} - \ln u_1$	$n \neq 0$	$X_0 + \mu e^{nt}\hat{B}$	$X_0 + \nu_a \hat{G}_a + \mu e^{nt}\hat{B}$
				$n = 0$	$X_0 + \mu\hat{B}$	$X_0 + \mu\hat{B} + \lambda_a G_a$

Table 1 (continued). Nonlinearities with arbitrary functions

No	Nonlinear terms	Type of matrix B (4.3)	Arguments of φ_1, φ_2	Conditions for parametrs	Symmetries $A^{-1} \neq \kappa B$	Symmetries $A^{-1} = \kappa B$
9.	$f^1 = \varphi u_1$ $f^2 = \varphi u_2 - n u_1$	$IIIa$, $d = 0$	u_1		$X_0 + \mu \hat{B} + \nu Y_1$	
10.	$f^1 = \varphi u_1$ $f^2 = \varphi_1 u_2 + \varphi_2 u_1 + n u_2$	$IIIa$	u_1		$X_0 + \mu e^{nt} \hat{B}$	
11.	$f^1 = u_1^{\left(k + \frac{k^2}{\sqrt{k^2+s^2}}\right)}$ $\times \exp\left[\left(s + \frac{sk}{\sqrt{k^2+s^2}} u_2\right)\right] \varphi_1$ $f^2 = u_1^k \exp(s u_2) \varphi_2$	I, $d = 0$	$s \ln u_1 + k u_2$		$X_0 + \nu D_2$	
12.	$f^1 = \varphi_1 \exp\left(\frac{n u_2}{n^2+1}\right) u_1^{\frac{1}{n^2+1}}$ $+ \frac{s}{n^2+1} u_1 \left(n u_2 + \ln u_1\right)$ $f^2 = \varphi_2 + \frac{ns}{n^2+1}(n u_2 + \ln u_1)$	I, $d = 0$	$u_2 - n \ln u_1$		$X_0 + \nu Y_2$	
13.	$f^1 = \exp(k u_1) \varphi_1$ $f^2 = \exp(k u_1)(\varphi_2 - 2n \varphi_1 \ln u_1)$	$IIIa$	$n u_1^2 + u_2$	$k \neq 0$	$X_0 + \lambda D_3$	
14.	$f^1 = \varphi_1 + s u_1$ $f^2 = \varphi_2 - 2n \varphi_1 u_1 + 2 s u_2$	$IIIa$	$n u_1^2 + u_2$		$X_0 + \lambda Y_3$	
15.	$f^1 = n$ $f^2 = k u_2 + \varphi_2$	$IIIa$	u_1		$X_0 + \lambda Y_4 + \psi_k \frac{\partial}{\partial u_2}$	
16.	$f^\alpha = \exp\left[\frac{k}{\omega}(\omega_1 u_1 + \omega_2 u_2)\right] \varphi^\alpha$ $\alpha = 1, 2,\ \omega^2 = \omega_1^2 + \omega_2^2$	*any*	$\omega_1 u_2 - \omega_2 u_1$	$k \neq 0$	$X_0 + \nu D_4$	$X_0 + \nu D_4$
17.	$f^1 = \varphi_1$ $f^2 = \varphi_2$	*any*	(u_1, u_2)		X_0	X_0

Table 2. Non-linearities with arbitrary parameters which generate symmetry with respect to dilatation

No	Nonlinear terms	Conditions for parameters	Symmetries, $A^{-1} \neq \kappa F$	Symmetries, $A^{-1} = \kappa F$	Matrix class and symmetry generator parameters
1	$f^1 = \left(g u_1^q u_2^r - s\right) u_1$ $f^2 = \left(p u_1^q u_2^r - \frac{rs}{q}\right) u_2$	$s = 0,\ q \neq 0$ $q + r = \frac{4}{m}$	$X_0 + \mu\hat{F} + \nu D_5$	$X_0 + \mu\hat{F} + \nu D_5 + \sigma_a G_a + \lambda A_0$	$I,\ d = -\frac{q}{r}$
		$s = 0,\ q \neq 0,\ r \neq 0$ $0 \neq q + r \neq \frac{4}{m}$	$X_0 + \nu\hat{F} + \mu D_5$	$X_0 + \nu\hat{F} + \mu D_5 + \sigma_a G_a$	$I,\ k = r + q$ $d = -\frac{q}{r}$
		$s \neq 0,\ q \neq 0$	$X_0 + \nu\hat{F} + \mu D_6$	$X_0 + \nu\hat{F} + \mu D_6 + \sigma_a G_a$	$I, k = r$ $d = -\frac{q}{s}$
2	$f^1 = e^{q\theta} R^r \left(g u_1 - p u_2\right)$ $+s u_2 - l u_1$ $f^2 = e^{q\theta} R^r \left(g u_2 + p u_1\right)$ $-s u_1 - l u_2$ $R^2 = u_1^2 + u_2^2$ $\theta = \tan^{-1}\left(\frac{u_2}{u_1}\right)$	$s = l = 0$ $r = \frac{4}{m}$	$X_0 + \nu\hat{F} + \mu D_5$	$X_0 + \nu\hat{F} + \mu D_5 + \sigma_a G_a + \lambda A_0$	$IIb,\ k = r$ $d = -\frac{q}{r}$
		$r \neq \frac{4}{m},\ r \neq 0$ $s = l = 0$	$X_0 + \nu\hat{F} + \mu D_5$	$X_0 + \nu\hat{F} + \mu D_5 + \sigma_a G_a$	$IIb,\ k = \frac{4}{m}$ $d = -\frac{q}{r}$
		$l = sq\left(1 + \frac{1}{r}\right)$ $s \neq 0,\ r \neq 0$	$X_0 + \nu\hat{F} + \mu D_6$	$X_0 + \nu\hat{F} + \mu D_6 + \sigma_a G_a$	$IIb,\ k = r$ $n = sq$ $d = q\left(1 + \frac{1}{r}\right)$
		$s = 0,\ l \neq 0$ $q \neq 0,\ r = 0$	$X_0 + \nu\hat{F} + \mu D_6$	$X_0 + \nu\hat{F} + \mu D_6 + \sigma_a G_a$	$IIa,\ k = q$ $n = lq$
3	$f^1 = \left(p u_1^r e^{q\frac{u_2}{u_1}} - s\right) u^1$ $f^2 = e^{q\frac{u_2}{u_1}} \left(p u_2 + g u_1\right) u_1^r$ $-s\left(u_2 - \frac{r}{q} u_1\right)$	$r = -q = \frac{4}{m}$ $l = s = 0$	$X_0 + \nu\hat{F} + \mu D_5$	$X_0 + \nu\hat{F} + \mu D_5 + \alpha_a G_a + \lambda A_1$	$IIIb,\ d = 1$ $k = \frac{4}{m}$
		$-q = r \neq \frac{4}{m}$ $l = s = 0,\ r \neq 0$	$X_0 + \nu\hat{F} + \mu D_5$	$X_0 + \nu\hat{F} + \mu D_5 + \alpha_a G_a$	$IIIb,\ d = 1$ $k = r$
		$q \neq 0,\ s \neq 0$	$X_0 + \nu\hat{F} + \mu D_6$	$X_0 + \nu\hat{F} + \mu D_6 + \sigma_a G_a$	$IIIb,\ k = q$ $n = sq,\ d = -\frac{n}{q}$

Table 2 (continued). Non-linearities with arbitrary parameters which generate symmetry with respect to dilatation

No	Nonlinear terms	Conditions for parameters	Symmetries, $A^{-1} \neq \kappa F$	Symmetries, $A^{-1} = \kappa F$	Matrix class and symmetry generator parameters
4	$f^1 = pu_1^{k+1}$ $f^2 = u_1^k\left(pu_2 + qu_1^d\right) - \frac{n}{d+k-1}u_1$	$d+k \neq 1$ $k \neq 0,\ q \neq 0$	$X_0 + \nu\hat{F} + \mu D_6$		$IIIa$
		$k \neq 0,\ n = 0$ $q = 0$	$X_0 + \nu\hat{F} + \mu D_5 + \lambda\hat{B}$		$IIIa,\ d = 0$
		$k \neq 0,\ n = 0$ $q = 0$	$X_0 + \nu\hat{F} + \mu D_5 + \lambda Y_1$		$IIIa,\ d = 0$
5	$f^1 = pu_1^{k+1}$ $f^2 = pu_1^k u_2 + qu_1 + nu_1 \ln u_1$	$k \neq 0,\ n \neq 0$	$X_0 + \nu\hat{F} + \mu D_6$		$IIIa,\ d = 1-k$
6	$f^1 = qu_1^{r+1}e^{ku_2} + \frac{ks}{r^2}u_1$ $f^2 = pu_1^r e^{ku_2} + \frac{r}{s}$	$r \neq 0, -1\ p \neq 0$	$X_0 + \nu D_7 + \mu Y_7$		I
7	$f^1 = e^{u_2} + su_2 + q$ $f^2 = n$	$q = 0$	$X_0 + \nu D_7 + \mu Y_7 + \psi_n\frac{\partial}{\partial u_1}$		$I,\ k = -r = 1$
		$s = 0, q \neq 0$ $n \neq 0$	$X_0 + \nu D_1$ $+\mu\left(Y_7 - qt\frac{\partial}{\partial u_1}\right) + \psi_0\frac{\partial}{\partial u_1}$		$I, d = 0, k = 1$
		$q = n = 0$ $s \neq 0$	$X_0 + \nu D_8 + \mu\hat{F} + \psi_0\frac{\partial}{\partial u_1}$		$IIIa$
8	$f^1 = k_1 e^{u_2} - pA^{21}$ $f^2 = k_2 e^{u_2} - pA^{11} + q$		$X_0 + \lambda D_9 + \psi_0\frac{\partial}{\partial u_1}$	$X_0 + \nu D_9 + \psi_0\frac{\partial}{\partial u_1}$	any
9	$f^1 = p\left(u_2 + nu_1^2\right)^{s+\frac{1}{2}} + \frac{1}{2n(2s+1)}$ $f^2 = q\left(u_2 + nu_1^2\right)^{s+1} - \frac{1}{2s+1}u_1$ $-2npu_1\left(u_2 + nu_1^2\right)^{s+\frac{1}{2}}$	$s \neq 0,\ s \neq \frac{1}{2}$ $p \neq 0,\ n \neq 0$	$X_0 + \nu D_{10}$		I
10	$f^1 = pu_1^{2k+1} - 2msA^{11}$ $f^2 = qu_1^{2k+1} - 2msA^{21}$	$k \neq 0$	$X_0 + \nu D_{11} + \psi_0\frac{\partial}{\partial u_2}$		any

Table 3. Further non-linearities with arbitrary parameters

No	Nonlinear terms	Matrix Class (4.4)	Conditions for parameters	Symmetries, $A^{-1} \neq \kappa F$	Symmetries, $A^{-1} = \kappa F$
1	$f^1 = (k_0 \ln u_1 + k_1 \ln u_2 + q)\, u_1$ $f^2 = (n_0 \ln u_1 + n_1 \ln u_2 + p)\, u_2$	$I,\ d = 0$	$\delta > 0,\ \Delta \neq 0$ $\mathcal{F} = \mathcal{F}_1, \mathcal{F}_2$	$X_0 + \nu X_1$	$X_0 + \lambda X_1 + \sigma_a \hat{G}_a$
			$\delta > 0,\ \Delta = 0$ $n_1 = 0,\ k_1 \neq 0,\ \mathcal{F} = \mathcal{F}_3$	$X_0 + \lambda X_2$	$X_0 + \lambda X_2 + \sigma_a \hat{G}_a$
2	$f^1 = (k_0 u_1 - n_0 u_2) \ln R$ $+\theta\,(k_1 u_1 - n_1 u_2) + p u_1 - q u_2$ $f^2 = (k_0 u_2 + n_0 u_1) \ln R$ $+\theta\,(n_1 u_1 + k_1 u_2) + q u_1 + p u_2$	IIa	$\delta > 0,\ \Delta = 0$ $n_1 = 0,\ k_1 \neq 0,\ \mathcal{F} = \mathcal{F}_4$	$X_0 + \lambda X_2$	$X_0 + \lambda X_2 + \sigma_a G_a$
			$\delta > 0,\ \Delta = 0$ $n_1 \neq 0,\ \mathcal{F} = \mathcal{F}_5$	$X_0 + \lambda X_3$	$X_0 + \lambda X_3 + \sigma_a \hat{G}_a$
			$\delta > 0,\ \Delta = 0$ $n_1 \neq 0,\ \mathcal{F} = \mathcal{F}_6$	$X_0 + \lambda X_3$	$X_0 + \lambda X_3 + \sigma_a G_a$
3	$f^1 = (k_0 \ln u_1 + q)\, u_1 + k_1 u_2$ $f^2 = (n_0 u_1 + k_0 u_2) \ln u_1$ $+k_1 \frac{u_2^2}{u_1} + p u_1 + (n_1 + q)\, u_2$	$IIIa$	$\delta = 0,\ \Delta \neq 0$ $k_1 = n_0 = 0,\ \mathcal{F} = \mathcal{F}_7$	$X_0 + \lambda X_4$	$X_0 + \lambda X_4 + \sigma_a \hat{G}_a$
			$\delta = 0,\ \Delta \neq 0$ $n_0 = 0,\ k_1 \neq 0,\ \mathcal{F} = F$	$X_0 + \lambda X_5$	$X_0 + \lambda X_5 + \sigma_a \hat{G}_a$
			$\delta = 0,\ \Delta \neq 0$ $k_1 = 0,\ n_0 \neq 0,\ \mathcal{F} = B$	$X_0 + \lambda X_6$	$X_0 + \lambda X_6 + \sigma_a \hat{G}_a$
			$\delta = 0,\ \Delta \neq 0$ $n_0 k_1 < 0$	$X_0 + \lambda X_7$	
			$\delta = 0 = \Delta$ $k_1 = 0,\ n_0 = 0$ $\mathcal{F} = \alpha F + \mu B$	$X_0 + \nu \hat{F} + \mu \hat{B}$	$X_0 + \nu \hat{F} + \mu \hat{B} + \sigma_a G_a$
			$\delta = 0 = \Delta$ $k_1 = 0,\ n_0 \neq 0,\ \mathcal{F} = B$	$X_0 + \nu \hat{B}$ $+\mu \left(\hat{F} + n_0 t \hat{B}\right)$	$X_0 + \nu \hat{B}$ $+\mu \left(\hat{F} + n_0 t \hat{B}\right) + \sigma_a G_a$
			$\delta = 0, \Delta = 0$ $k_1 \neq 0,\ n_0 = 0,\ \mathcal{F} = F$	$X_0 + \nu \hat{F}$ $+\mu \left(\hat{B} + k_1 t \hat{F}\right)$	$X_0 + \nu \hat{F}$ $+\mu \left(\hat{B} + k_1 t \hat{F}\right) + \sigma_a G_a$
			$\delta = 0, \Delta = 0,\ n_0 k_1 \neq 0$ $\mathcal{F} = k_1 F - k_0 B$	$X_0 + \nu X_8$	$X_0 + \nu X_8 + \sigma_a G_a$
			$\delta = -\omega^2 < 0$	$X_0 + \nu X_9 + \mu X_{10}$	

Table 3 (continued). Further non-linearities with arbitrary parameters

No	Nonlinear terms	Matrix Class (4.4)	Conditions for parameters	Symmetries, $A^{-1} \neq \kappa F$	Symmetries, $A^{-1} = \kappa F$
4	$f^1 = qu_1$ $f^2 = nu_2 + pu_1^d su_1 + k$	$IIIa$	$d \neq 1, 2, k = 0$ $s = 0,\ n \neq q,\ \mathcal{F} = B$	$X_0 + \nu \hat{B} + \psi_n \frac{\partial}{\partial u_2}$	$X_0 + \nu \hat{B} + \psi_n \frac{\partial}{\partial u_2} + \sigma_a G_a$
			$d \neq 0, 2, n = 0$ $s = 0, \mathcal{F} = B$	$X_0 + \psi_0 \frac{\partial}{\partial u_2} + \nu \left(\hat{B} - dkt \frac{\partial}{\partial u_2} \right)$	$X_0 + \psi_0 \frac{\partial}{\partial u_2}$ $+\nu \left(\hat{B} - dkt \frac{\partial}{\partial u_2} \right) + \sigma_a G_a$
			$d \neq 0, 1, 2,\ n = q$ $k = 0,\ s = \frac{1}{1-d}$ $\mathcal{F} = B$	$X_0 + \nu \hat{F} + \mu \hat{B} + \psi_n \frac{\partial}{\partial u_2}$	$X_0 + \nu \hat{F} + \mu \hat{B}$ $+\psi_n \frac{\partial}{\partial u_2} + \sigma_a G_a$
			$d = 2,\ n = q$ $k = 0,\ s \neq 0$	$X_0 + \mu Y_5 + \psi_n \frac{\partial}{\partial u_2}$	
			$d = 2,\ n = q$ $k = s = 0,\ \mathcal{F} = B$	$X_0 + \nu \hat{B} + \psi_0 \frac{\partial}{\partial u_2}$	$X_0 + \nu \hat{B} + e^{nt} \hat{F}$ $+\psi_0 \frac{\partial}{\partial u_2} + \sigma_a G_a$
			$n = 2\,(q + p)\,,\ d = 2$ $k = s = 0,\ \mathcal{F} = B$	$X_0 + \nu Y_6 + \mu \hat{B} + \psi_0 \frac{\partial}{\partial u_2}$	$X_0 + \nu Y_6 + \mu \hat{B}$ $+\psi_0 \frac{\partial}{\partial u_2} + \sigma_a G_a$
			$d = 2,\ p = -q$ $n = s = 0$	$X_0 + \nu Y_6 + \psi_0 \frac{\partial}{\partial u_2}$ $+\mu \left(\hat{B} - dkt \frac{\partial}{\partial u_2} \right)$	
5	$f^1 = ku_1 \ln u_1 + pu_1$ $f^2 = bu_2 + n \ln u_1 + q$	I	$b = k = q = 0$ $p \neq 0$	$X_0 + \nu Y_7 + \psi_0 \frac{\partial}{\partial u_2}$	
			$q = p = 0$ $b = k$	$X_0 + \nu Y_8 + \psi_b \frac{\partial}{\partial u_2}$	
			$q = p = 0$ $b \neq k,\ b \neq 0$	$X_0 + \nu Y_9 + \psi_b \frac{\partial}{\partial u_2}$	
			$k \neq 0,\ b = p = 0$	$X_0 + \nu Y_9 + \psi_0 \frac{\partial}{\partial u_2}$	
6	$f^1 = pu_1 u_2 + nu_1 \ln u_1$ $f^2 = nu_2 - q \ln u_1$	I	$q = p$	$X_0 + \nu Y_{11} + \mu Y_{12}$	
			$q = 0$	$X_0 + \nu Y_{13} + \mu Y_{14}$	

5 Discussion

Thus we have found all possible versions of systems of diffusion equations which admit a non-trivial Lie symmetry. These results can be used to construct mathematical models with required symmetry properties in, for example, physics, biology, chemistry. On the other hand, our results give ad hoc solution of problems of group analysis of all models using systems of diffusion equations. As an example consider the nonlinear Schrödinger equation in m-dimensional space

$$\left(i\frac{\partial}{\partial t}-\sum_{i=1}^{m}\frac{\partial^2}{\partial x_i^2}\right)\psi=F\left(\psi,\psi^*\right)\psi \tag{5.6}$$

which is also is a particular case of (1.1). If we denote

$$u_1=\frac{1}{2}(\psi+\psi^*),\qquad u_2=\frac{1}{2i}\left(\psi-\psi^*\right) \tag{5.7}$$

then (5.6) reduces to the form (1.1) with $A=i\sigma_2$, and

$$f^1=\frac{1}{2}\left(F^*+F\right)u_2+\frac{1}{2i}\left(F-F^*\right)u_1,$$

$$f^2=\frac{1}{2i}\left(F-F^*\right)u_2-\frac{1}{2}\left(F+F^*\right)u_1.$$

In other words, any solution given in Tables 2, 3 with matrices belonging to Classes I, II give rise to the nonlinearity

$$F=\frac{1}{R^2}\left[\left(u_2f^1-u_1f^2\right)+i\left(u_2f^2+f^1u^1\right)\right]$$

for the nonlinear Schrödinger equation (5.6) which admits a nontrivial Lie symmetry. We see that the number of nonlinearities which guarantee a non-trivial symmetry for the non-linear Schödinger equation is very extended and exceeds one hundred.

We notice that the nonlinear Schrödinger equations with *ad hoc* required symmetry with respect to the (extended) Galilei group where described in [11, 12].

The authors wish to thank the Royal Society fot their support with this research.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 60–67.

Electro-Static Potential Between Two Conducting Cylinders via the Group Method Approach

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The transformation group theoretic approach is applied to present an analysis of distribution of electro-static potential between two eccentric conducting cylindrical surfaces. A conformal mapping is used to map the region between two eccentric circles, in the complex potential-plane onto a region of concentric circles in another complex-plane. The application of one-parameter group reduces the number of independent variables by one, and consequently the Laplace's equation with the boundary conditions to an ordinary differential equation with appropriate corresponding conditions. The obtained differential equation is solved analytically.

1 Introduction

The Laplace equation arises in many branches of physics, from which it attracts a wide band of researchers. Electro-static potential, temperature in the case of steady state heat conduction, velocity potential in the case of steady irrotational flow of an ideal fluid, concentration of a substance that is diffusing through a solid, and the displacements of a two-dimensional membrane in equilibrium state, are counter examples in which the Laplace's equation is satisfied.

The arrangement of two parallel conducting cylinders, each of circular cross section is an important type of transmission line.

Transmission lines are used to transmit electric energy and signals from one point to another. The basic transmission line connects a source to a load. This may be a transmitter and an antenna, a shift register and the memory core in a digital computer, a hydroelectric generating plant and a substation several hundred miles away, a television antenna and a receiver, and one input of the preamplifier, see Hayt [7]. While short transmission line segments (few millimeters in microwave circuits to inches or feet or hundreds of feet in devices at lower frequencies) perform many different functions, within the terminal units of the systems such as: resonant elements, filters and wave-shaping networks, see Chipman [5].

A conformal mapping is used to map the region between two eccentric circles, in the complex potential-plane onto a region of concentric circles in another complex-plane. Then the mathematical technique used in the present analysis is the parameter-group transformation. The group methods, as a class of methods which lead to reduction of the number of independent variables, were first introduced by Birkhoff [4] in 1948, where he made use of one-parameter transformation groups. In 1952, Morgan [9] presented a theory which has led to improvements over earlier similarity methods. The method has been applied intensively by Abd-el-Malek et al. [1–3].

In this work we present a general procedure for applying one-parameter group transformation to the Laplace's equation in a region between two long cylinders with parallel axis. Under the transformation, the partial differential equation with variable boundary conditions, is reduced to an ordinary differential equation with the appropriate corresponding conditions. The equation is solved analytically.

2 Mathematical formulation

Consider the electro-static potential $V(x, y)$ over any cross section Ω (Fig. 1.1 or Fig. 1.2) of a domain between two long cylinders with parallel axis.

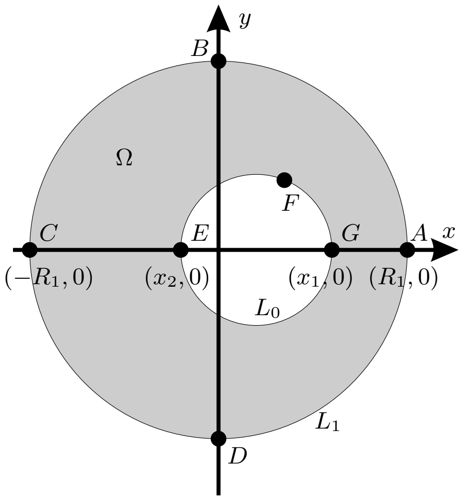

Fig. 1.1. Cross section in two long eccentric cylinders with parallel axis where $-R_1 < x_2 < x_1 < R_1$.

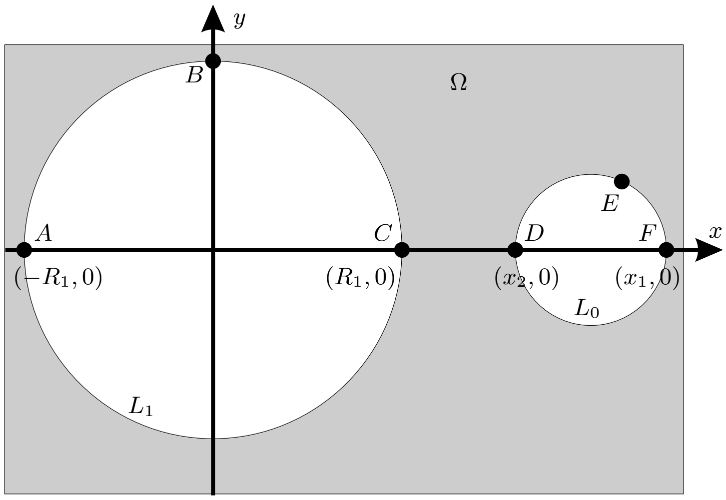

Fig. 1.2. Cross section in two long parallel cylinders where $R_1 < x_2 < x_1$.

Under the assumption that cylinders have variable potentials, the governing equation may be written as

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad (x, y) \in \Omega \tag{1}$$

with the boundary conditions

$$\begin{aligned} V(x, y) &= V_1 q(x, y), \qquad (x, y) \in L_1; \\ V(x, y) &= V_0 q(x, y), \qquad (x, y) \in L_0, \end{aligned} \tag{2}$$

where V_1 and V_0 are constants and $q(x, y)$ is an arbitrary function to be determined later on.

The Möbius or the linear fractional transformation

$$w = \frac{z - aR_1}{az - R_1}, \tag{3}$$

where

$$w = u + iv = re^{i\theta}, \qquad z = x + iy,$$

$$a = \frac{R_1^2 + x_1 x_2 + \sqrt{\left(R_2^1 - x_2^1\right)\left(R_2^1 - x_2^2\right)}}{R_1(x_1 + x_2)}, \tag{4}$$

$$R_0 = \frac{R_1^2 - x_1x_2 + \sqrt{\left(R_1^2 - x_1^2\right)\left(R_1^2 - x_2^2\right)}}{\varepsilon R_1(x_1 - x_2)}, \tag{5}$$

$\varepsilon = 1$ for Ω shown in Fig. (1.1) and $\varepsilon = -1$ for Ω shown in Fig. (1.2), maps the region Ω shown in Fig. 1.1 onto $\bar{\Omega}$ shown in Fig. 2.1 and also maps the region Ω shown in Fig. 1.2 onto $\bar{\Omega}$ shown in Fig. 2.2, see Churchill and Brown [6].

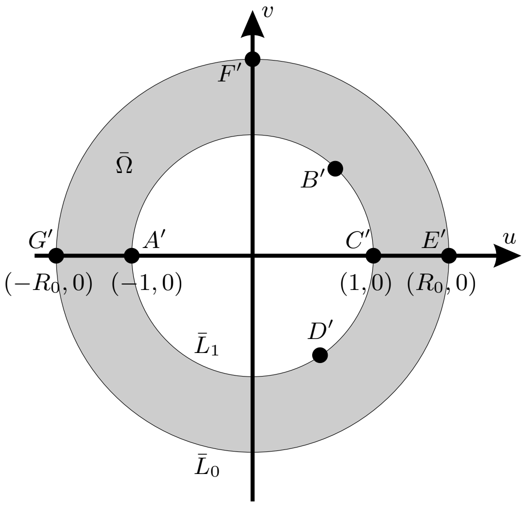

Fig. 2.1. Cross section in mapped two long eccentric cylinders with parallel axis.

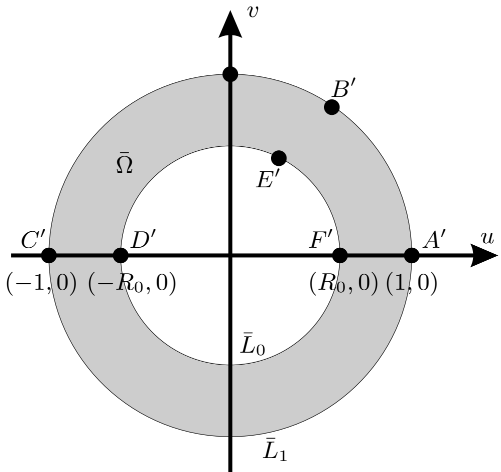

Fig. 2.2. Cross section in mapped two long parallel cylinders.

From (3) we get

$$u = \frac{a\left(x^2 + y^2\right) - \left(a^2 + 1\right)xR_1 + aR_1^2}{a^2\left(x^2 + y^2\right) - 2axR_1 + R_1^2}, \tag{6}$$

$$v = \frac{\left(a^2 - 1\right)yR_1}{a^2\left(x^2 + y^2\right) - 2axR_1 + R_1^2}. \tag{7}$$

Now, the governing equation satisfied in region $\bar{\Omega}$, in polar coordinates, has the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0, \tag{8}$$

with the boundary conditions

$$\begin{aligned} V(1, \theta) &= V_1 q(\theta), \qquad (r, \theta) \in \bar{L}_1, \\ V(R_0, \theta) &= V_0 q(\theta), \qquad (r, \theta) \in \bar{L}_0, \end{aligned} \qquad -\pi < \theta \leq \pi. \tag{9}$$

We restrict θ to an interval $(-\pi, \pi]$. This requires that:

$$V(r, \pi) = V(r, -\pi), \qquad 1 < r < R_0, \tag{10}$$

$$\frac{\partial V}{\partial \theta}(r, \pi) = \frac{\partial V}{\partial \theta}(r, -\pi), \qquad 1 < r < R_0. \tag{11}$$

Write

$$V(r, \theta) = w(r, \theta)q(\theta), \qquad q(\theta) \not\equiv 0 \quad \text{in} \quad \bar{\Omega}. \tag{12}$$

Hence (8) and (9) take the form:

$$q(\theta)\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right]+\frac{1}{r^2}\left[w\frac{d^2 q}{d\theta^2}+2\frac{\partial w}{\partial \theta}\frac{dq}{d\theta}+q\frac{\partial^2 w}{\partial \theta^2}\right]=0 \tag{13}$$

with the boundary conditions:

$$\begin{aligned} &w(1,\theta)=V_1, \qquad (r,\theta)\in \bar{L}_1, \\ &w(R_0,\theta)=V_0, \qquad (r,\theta)\in \bar{L}_0, \end{aligned} \qquad -\pi<\theta<\pi. \tag{14}$$

3 Solution of the problem

The method of solution depends on the application of a one-parameter group transformation to the partial differential equation (13) and the boundary conditions (14). Under this transformation the two independent variables will be reduced by one and the differential equation (13) transforms into an ordinary differential equation in only one independent variable, which is the similarity variable.

3.1 The group systematic formulation

The procedure is initiated with the group G, a class of transformation of one-parameter "b" of the form

$$G:\bar{S}=C^S(b)S+K^S(b), \tag{15}$$

where S stands for r, θ; w, q and the C^S and K^S are real-valued and at least differentiable in the real argument "b".

3.2 The invariance analysis

To transform the differential equation, transformations of the derivatives are obtained from G via chain-rule operations:

$$\bar{S}_{\bar{i}}=\left(\frac{C^S}{C^i}\right)S_i, \qquad \bar{S}_{\bar{i}\bar{j}}=\left(\frac{C^S}{C^iC^j}\right)S_{ij}, \qquad i=r,\theta; \quad j=r,\theta, \tag{16}$$

where S stands for w and q.

Equation (13) is said to be invariantely transformed whenever

$$\begin{aligned} &\bar{q}\left(\bar{r}^2\bar{w}_{\bar{r}\bar{r}}+\bar{r}\bar{w}_{\bar{r}}\right)+\bar{w}\bar{q}_{\bar{\theta}\bar{\theta}}+2\bar{w}_{\bar{\theta}}\bar{q}\bar{w}_{\bar{\theta}\bar{\theta}} \\ &\qquad =H(b)\left[q\left(r^2w_{rr}+rw_r\right)+wq_{\theta\theta}+2w_\theta q_\theta+qw_{\theta\theta}\right] \end{aligned} \tag{17}$$

for some function $H(b)$ which may be a constant.

Substitution from equations (15) into equation (17), using (16), for the independent variables, the functions and their partial derivatives yields

$$\begin{aligned} &q\left(\left[C^qC^w\right]r^2w_{rr}+\left[C^qC^w\right]rw_r\right)+\left[\frac{C^qC^w}{\left(C^\theta\right)^2}\right]wq_{\theta\theta}+2\left[\frac{C^qC^w}{\left(C^\theta\right)^2}\right]w_\theta q_\theta \\ &\qquad +\left[\frac{C^qC^w}{\left(C^\theta\right)^2}\right]qw_{\theta\theta}+\zeta(b)=H(b)\left[q\left(r^2w_{rr}+rw_r\right)+wq_{\theta\theta}+2w_\theta q_\theta+wq_{\theta\theta}\right], \end{aligned} \tag{18}$$

where

$$\begin{aligned}\zeta(b) = K^q \left[(C^r r + K^r)^2 \left(\frac{C^w}{(C^r)^2} \right) w_{rr} + (C^r r + K^r) \left(\frac{C^w}{C^r} \right) w_r \right] \\ + \left[\frac{K^w C^q}{(C^\theta)^2} \right] q_{\theta\theta} + K^q \left[\frac{C^w}{(C^\theta)^2} \right] w_{\theta\theta}.\end{aligned} \tag{19}$$

The invariance of (18) implies $\zeta(b) \equiv 0$. This is satisfied by putting

$$K^q = K^w = 0 \tag{20}$$

and

$$[C^q C^w] = \left[\frac{C^q C^w}{(C^\theta)^2} \right] = H(b), \tag{21}$$

which yields

$$C^\theta = \pm 1. \tag{22}$$

Moreover, the boundary conditions (14) are also invariant in form, imply that $K^r = K^w = 0$ and $C^w = C^r = 1$.

Finally, we get the one-parameter group G which transforms invariantely the differential equation (13) and the boundary conditions (14). The group G is of the form

$$G: \quad \begin{cases} \bar{r} = r, \\ \bar{\theta} = \pm\theta + K^\theta, \\ \bar{w} = w, \\ \bar{q} = C^q q. \end{cases} \tag{23}$$

3.3 The complete set of absolute invariant

Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation (similarity representation) in a single independent variable (similarity variable). Then we have to proceed in our analysis to obtain a complete set of absolute invariant. In addition to the absolute invariant of the independent variable, there are two absolute invariant of the dependent variables w and q.

If $\eta \equiv \eta(r, \theta)$ is an absolute invariant of the independent variables, then

$$g_j(r, \theta; w, q) = F_j[\eta, (r, \theta)], \qquad j = 1, 2 \tag{24}$$

are two absolute invariant corresponding to w and q. The application of a basic theorem in group theory, see Morgan and Gaggioli [8], states that: *a function $g(r, \theta; w, q)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation*

$$\sum_{i=1}^{4} (\alpha_i S_i + \beta_i) \frac{\partial g}{\partial S_i} = 0, \tag{25}$$

where S_i, stands for r, θ, w and q, respectively and

$$\alpha_i = \frac{\partial C^{S_i}}{\partial b} \left(b^0 \right) \qquad \textit{and} \qquad \beta_i = \frac{\partial K^{S_i}}{\partial b} \left(b^0 \right), \qquad i = 1, 2, 3, 4 \tag{26}$$

where b^0 denotes the value of "b" which yields the identity element of the group.

From which we get: $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\beta_1 = \beta_3 = \beta_4 = 0$.

At first, we seek an absolute invariant of the independent variables. Owing to equation (11), $\eta(r, \theta)$ is an absolute invariant if it satisfies the first-order partial differential equation

$$\frac{\partial \eta}{\partial \theta} = 0, \tag{27}$$

which has a solution in the form $\eta(r, \theta) = \Gamma(r)$.

Without loss of generality, take the function Γ as an identity function, hence

$$\eta(r, \theta) = r. \tag{28}$$

The second step is to obtain the absolute invariant of the dependent variables w and q. Applying (25), we get

$$q(\theta) = R(\theta)\Phi(\eta). \tag{29}$$

Since $q(\theta)$ and $R(\theta)$ are independent of η, while Φ is a function of η, then $\phi(\eta)$ must be a constant, say $\Phi(\eta) = 1$, and from which

$$q(\theta) = R(\theta), \tag{30}$$

and the second absolute invariant is:

$$w(r, \theta) = F(\eta). \tag{31}$$

From (30), (31), and (12). the conditions (10) and (11) will be changed to corresponding conditions on $R(\theta)$ as follows

$$R(\pi) = R(-\pi), \tag{32}$$

$$\frac{dR}{d\theta}(\pi) = \frac{dR}{d\theta}(-\pi). \tag{33}$$

4 Reduction to ordinary differential equation

As the general analysis proceeds, the established forms of the dependent and independent absolute invariant are used to obtain ordinary differential equation. Generally, the absolute invariant $\eta(r, \theta)$ has the form given in (28).

Substituting from (28), (30) and (31) into equation (13) yields

$$\frac{d^2F}{d\eta^2} + \frac{1}{\eta}\frac{dF}{d\eta} + \left(\frac{1}{R\eta^2}\frac{d^2R}{d\theta^2}\right) F = 0. \tag{34}$$

For (34) to be reduced to an expression in the single independent invariant η, the coefficients in (34) should be constants or functions of η. Thus take

$$\frac{1}{R}\frac{d^2R}{d\theta^2} = C, \tag{35}$$

where C is an arbitrary constant.

Thus (34) may be written as

$$\eta^2\frac{d^2F}{d\eta^2} + \eta\frac{dF}{d\eta} + CF = 0. \tag{36}$$

Under the similarity variable η, the boundary conditions are:

$$\begin{aligned} F(R_0) &= V_0, \\ F(1) &= V_1. \end{aligned} \tag{37}$$

5 Analytical solution

Case (1): $C = -\alpha^2$, $\alpha \neq 0$. Substituting for C into (35), we get

$$\frac{d^2R}{d\theta^2} + \alpha^2 R = 0. \tag{38}$$

Solution of (38) is:

$$R(\theta) = c_1 \cos\alpha\theta + c_2 \sin\alpha\theta. \tag{39}$$

The function $R(\theta)$ in (39) satisfies boundary conditions (32) and (33) for $\alpha = n$, where $n = \pm1, \pm2, \pm3, \ldots$.

Since $q(\theta)$ is an arbitrary function, then $R(\theta)$ is also an arbitrary function. Thus we can take R(9) in the form

$$R(\theta) = \sin\theta, \qquad -\pi < \theta \leq \pi, \tag{40}$$

and consequently $q(x, y)$ has the form

$$q(x,y) = \sin\left[\tan^{-1}\left(\frac{\left(a^2-1\right) yR_1}{a\left(x^2+y^2\right) - \left(a^2+1\right) xR_1 + aR_1^2}\right)\right].$$

Substituting for C into (36), we get

$$\eta^2 \frac{d^2F}{d\eta^2} + \eta\frac{dF}{d\eta} - n^2 F = 0, \tag{41}$$

which has the solution

$$F(\eta) = k_1\eta^n + \frac{k_2}{\eta^n}, \tag{42}$$

where k_1 and k_2 are constant.

Applying boundary conditions (37), we get

$$F(\eta) = \frac{V_1 - V_0R_0^n}{1-R_0^{2n}}\eta^n + \frac{V_0R_0^n - V_1R_0^{2n}}{\left(1-R_0^{2n}\right)\eta^n}, \qquad n = \pm1, \pm2, \pm3, \ldots. \tag{43}$$

From which we get

$$V(x,y) = \left(\frac{\sin\theta}{1-R_0^{2n}}\right)\left[\left(V_1 - V_0R_0^n\right)\sqrt{u^2+v^2} + \frac{V_0R_0^n - V_1R_0^{2n}}{\sqrt{u^2+v^2}}\right], \quad n = \pm1, \pm2, \pm3, \ldots, \tag{44}$$

where $\theta = \tan^{-1}(v/u)$, u and v are given by (6) and (7), respectively.

To obtain the electrostatic potential in the case of coaxial cylinders, take limit as $a \to \infty$ in (44), and for $n = 1$, we get

$$V(\rho,\Psi) = \frac{\sin\Psi}{\rho_1^2 - \rho_0^2}\left[\left(V_1\rho_0^2 - V_0\rho_0\rho_1\right)\left(\frac{\rho_1}{\rho}\right) + \left(V_0\rho_0 - V_1\rho_1\right)\rho\right], \tag{45}$$

where ρ_0 and ρ_1 are the raduii of the inner and outer cylinders, respectively.

It is noticed that the case of $c = \alpha^2$ is neglected since the obtained form of the function $R(\theta)$ does not satisfy the conditions (32) and (33).

Case (2): $C = 0$. Substituting for C into (35), we get

$$\frac{d^2R}{d\theta^2} = 0. \tag{46}$$

Equation (46) has the solution

$$R(\theta) = c_3\theta + c_4. \tag{47}$$

Since $q(\theta)$ is an arbitrary function, then $R(\theta)$ is also an arbitrary function. Thus we can take, without loss of generality, $R(\theta)$ in the form

$$R(\theta) = 1, \tag{48}$$

which satisfies conditions (32) and (33).

This corresponds to $q(x, y) = 1$, i.e. constant potential along the boundary.

Substituting for C into (36), we get

$$\eta^2\frac{d^2F}{d\eta^2} + \eta\frac{dF}{d\eta} = 0. \tag{49}$$

Equation (49) has the solution

$$F(\eta) = k_3 + k_4 \ln \eta, \tag{50}$$

where k_3 and k_4 are constants.

With the aid of the boundary conditions (37), the solution is

$$F(\eta) = V_1 + \left(\frac{V_0 - V_1}{\ln R_0}\right) \ln \eta, \tag{51}$$

from which we get

$$V(x, y) = V_1 + \left(\frac{V_0 - V_1}{2\ln R_0}\right) \ln\left(u^2 + v^2\right), \tag{52}$$

where u and v are given by (6) and (7) repectively.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 68–72.

New Evolution Completely Integrable System

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We present a detailed algebraic investigation of the evolution system $u_t = u_3 + u_1 v_1 + \delta u_1 + u v_2/2$, $v_t = u^2$ that was obtained in the recent paper by two of the authors. We present the zero curvature representation, the infinite sequences of the conserved densities and Lie–Bäcklund symmetries for the system under consideration. We also found the Noether operator, the Hamiltonian form, and the inverse Noether operator. The one-soliton solution is also obtained.

In our previous paper [1] we presented the classification of evolution systems satisfying the necessary conditions of integrability. This classification was obtained with the help of the conserved canonical densities approach. Here we present more detailed investigation of one of that systems

$$u_t = u_3 + u_1 v_1 - \delta u_1 + \frac{1}{2} u v_2, \qquad v_t = u^2. \tag{1}$$

Here $u_i = (\partial^i u/\partial x^i)$, $u_t = (\partial u/\partial t)$. We found the zero curvature representation, the infinite sequences of the conserved densities and Lie–Bäcklund symmetries for system (1). We also found the Noether operator Θ and the inverse Noether operator J. The operator Θ is implectic and provides the Hamiltonian form of system (1) and the product $\Theta J = \Lambda$ is the recursion operator for system (1). The one-soliton solution was also obtained.

1. To find the linear system realizing the zero curvature representation

$$\Psi_x = U\,\Psi, \qquad \Psi_t = V\,\Psi \tag{2}$$

we assumed that $U = U(u_0, v_0)$. Then we solved the compatibility equation for system (2)

$$U_t - V_x + [\,U,\, V\,] = 0, \tag{3}$$

where $[\,U,\, V\,]$ is the commutator, and obtained the matrices U and V in the following form

$$\begin{aligned} U &= A_1 + A_2\, u + A_3\, v + A_4\, v^2, \\ V &= A_2\, u_2 + 1/2\, A_2\, u\, v_1 + A_5\, u_1 + 1/2\, v\, A_2\, u_1 + 1/8\, v^2 A_2\, u \\ &\quad +1/2\, u^2 A_7 + u\, A_6 + 1/2\, u\, v\, A_5 - \delta\, u\, A_2 + A_8, \end{aligned} \tag{4}$$

where A_i are constant unknown matrices satisfying the following commutation relations:

$$\begin{aligned} &[\,A_4,\, A_7\,] = 0, \quad [\,A_3,\, A_5\,] = 0, \quad [\,A_2,\, A_8\,] + [\,A_1,\, A_6\,] = \delta\, A_5, \quad [\,A_1,\, A_8\,] = 0, \\ &[\,A_2,\, A_7\,] = 0, \quad [\,A_3,\, A_7\,] = -4\, A_4 - A_7, \quad [\,A_3,\, A_8\,] = 0, \qquad [\,A_4,\, A_8\,] = 0, \\ &[\,A_4,\, A_6\,] = -1/8\, A_5, \quad [\,A_3,\, A_6\,] = -1/2\, A_6 + \delta/2\, A_2, \quad [\,A_1,\, A_2\,] = A_5, \\ &[\,A_1,\, A_7\,] + 2\,[\,A_2,\, A_6\,] = -2\, A_3, \quad [\,A_2,\, A_4\,] = 0, \quad [\,A_1,\, A_5\,] = A_6, \\ &[\,A_2,\, A_5\,] = A_7, \quad [\,A_2,\, A_3\,] = -1/2\, A_2, \quad [\,A_3,\, A_5\,] = 0, \quad [\,A_4,\, A_5\,] = -1/8\, A_2. \end{aligned}$$

This table of commutators is obviously not closed, and the first problem is to obtain all commutators $[\,A_i,\, A_j\,]$. Following ideas of Wahlquist and Estabrook [2] we consider the unknown commutators as new elements of Lie algebra. For example, we set $[A_1, A_6] = A_9$ and so on. Then using the Jacobi identity we found some commutation relations for the new elements A_i. But in general case this process is infinite. To make it finite we assume a linear dependence between the elements A_i. It is important that system (1) satisfies sufficiently many conditions of the integrability and representation (3) exists. Therefore if one introduces sufficiently many new elements A_i then the linear constraint provides the closed nontrivial algebra. To close the presented algebra we were forced to consider 19-dimensional Lie algebra. But when we obtained the complete table of the commutators we found a 4-dimensional ideal I. We set the elements of the ideal to be zeros and obtained the 15-dimensional Lie algebra. This new algebra is isomorphic to the factor algebra with respect to ideal I and is simple. We cannot write here the final table of the commutators because it consists of 105 equations.

To construct the representation of the obtained simple algebra we found the Cartan–Weyl basis and the Dynkin diagram for it. It was the diagram of $sl(4)$ algebra. Hence the minimal dimension of a representation of the algebra is 4. The final result takes the following form

$$U = \begin{pmatrix} -\dfrac{v}{2} & \dfrac{1}{2} & 0 & 0 \\ \delta - \dfrac{v^2}{4} & 0 & 0 & \dfrac{1}{2} \\ \dfrac{2\,u}{3} & 0 & 0 & 0 \\ 4\,\mu & \delta - \dfrac{v^2}{4} & -2\,u & \dfrac{v}{2} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & -\dfrac{u}{2} & 0 \\ \dfrac{u^2}{3} & 0 & -u_1 - \dfrac{u\,v}{2} & 0 \\ \dfrac{f}{3} & -\dfrac{u_1}{3} - \dfrac{u\,v}{6} & -\mu & \dfrac{u}{6} \\ 0 & \dfrac{u^2}{3} & -f & 0 \end{pmatrix}, \tag{5}$$

where μ is the spectral parameter and

$$f = 2\,u_2 + u\,v_1 + v\,u_1 + 1/4\,v^2\,u - \delta\,u.$$

2. To check whether the obtained zero curvature representation is nontrivial we constructed from the matrix U the sequence of the conserved densities following to J.M. Alberty, T. Koikawa and R. Sasaki's algorithm [3]. Let c be a constant vector and (c, ψ) be the Euclidean scalar product. Setting $\varphi = \psi/(c, \psi)$ one can obtain from system (2) the following nonlinear system

$$\varphi_x = U\,\varphi - \varphi\,(c\,U\varphi), \qquad \varphi_t = V\varphi - \varphi\,(c\,V\varphi). \tag{6}$$

It is easy to check that the following continuity equation

$$(c\,U\varphi)_t = (c\,V\varphi)_x$$

follows from equation (3). Hence the function

$$\rho = (c\,U\varphi) \tag{7}$$

is the generating function for conserved densities of system (1). Setting

$$c = (0,\, 0,\, 1,\, 0),$$

$$\varphi_1 = \sum_{i=1}^{\infty} f_i\,k^i, \qquad \varphi_2 = \sum_{i=1}^{\infty} g_i\,k^i, \qquad \varphi_4 = \sum_{i=1}^{\infty} h_i\,k^i, \qquad k = 1/(4\,\mu),$$

we obtained from (6) the following recursion formulas

$$
\begin{aligned}
g_i &= 2\,D\,f_i + v\,f_i + \frac{4}{3}\,u\sum_{j=1}^{i-1} f_j\,f_{i-j},\\
h_i &= 2\,D\,g_i + \left(\frac{1}{2}v^2 - 2\,\delta\right)\,f_i + \frac{4}{3}u\sum_{j=1}^{i-1} f_j\,g_{i-j},\\
f_{i+1} &= D\,h_i + \left(\frac{1}{4}\,v^2 - \delta\right)\,g_i - \frac{1}{2}v\,h_i + \frac{2}{3}\,u\sum_{j=1}^{i-1} f_j\,h_{i-j},
\end{aligned}
\tag{8}
$$

where $D = \partial/\partial x,\ f_1 = 2\,u$ and $i > 0$. Formula (7) is reduced now to the form

$$
\rho = \frac{2}{3}\sum_{i=1}^{\infty} \rho_i\,k^i, \qquad \rho_i = u\,f_i,
\tag{9}
$$

and provides an infinite sequence of conserved densities ρ_i. It is obvious that the conserved densities provided by equations (8) are local. It can be easily verified that some first even densities are trivial, but the odd densities are nontrivial. Hence the presented zero curvature representation is nontrivial. The first two nontrivial densities take the following form

$$
\begin{aligned}
\rho_1 &= u^2,\\
\rho_3 &= u_3^2 + u_3\,(u\,v_2 + 2\,u_1\,v_1) + 2\delta\,u_2^2 + \frac{1}{4}\,u^2\,v_2^2\\
&\quad + u\,u_2\,v_1\left(\delta - \frac{1}{2}\,v_1\right) + \left(\delta^2 + \frac{7}{3}\,u^2 + \frac{1}{2}\,v_1^2 - \delta\,v_1\right)u_1^2 + \frac{1}{3}\,u^4\,(\delta - v_1).
\end{aligned}
$$

The subsequent densities are very cumbersome and we do not give them here. Let us note that system (1) possesses other conserved densities, that are not expressed by formula (9). For example, the function

$$
\rho = \frac{1}{4}\,{v_2}^2 - \delta\,{v_1}^2 + \frac{1}{3}\,{v_1}^3 + 3\,{u_1}^2 - 2\,v_1\,u^2
$$

is a conserved density as well.

3. Let us denote by K the vector field that determines system (1), that is, $K = \{u_3 + u_1\,v_1 - \delta\,u_1 + 1/2\,u\,v_2,\ u^2\}$. And let K' be the Fréchet derivative of K and K'^{+} be the adjoint of the operator K'. It is well known (see [4, 5] or [6], for instance) that the equation

$$
(D_t - K')\,\sigma = 0,
$$

is the determining equation for the Lie–Bäcklund symmetries σ of system (1). And the gradients of conserved densities ($\gamma_\alpha = E_\alpha\,\rho \equiv \{\delta\,\rho/\delta\,u,\ \delta\,\rho/\delta\,v\}$) satisfy the equation

$$
(D_t + K'^{+})\,\gamma = 0.
$$

In the papers [7] and [8] two following operators were introduced. An operator Θ satisfying the equation

$$
(D_t - K')\,\Theta = \Theta\,(D_t + K'^{+}),
\tag{10}
$$

maps the set of the gradients of the conserved densities Γ into the set of the Lie–Bäcklund symmetries Σ. It is called the Noether operator. And an operator J satisfying the equation

$$(D_t + K'^{+})\, J = J\,(D_t - K'), \tag{11}$$

provides the inverse map $\Sigma \to \Gamma$. It is called the inverse Noether operator. An elementary computation shows that the operator $\Lambda = \Theta J$ is the recursion one. That is, Λ solves the equation

$$[\, D_t - K',\, \Lambda\,] = 0. \tag{12}$$

We found the operators Θ and J for system (1) in the following form:

$$\Theta = \begin{pmatrix} D^3 + (v_1 + \delta)\, D + \frac{1}{2}\, v_2 & -\frac{4}{3}\, u - \frac{2}{3}\, u_1\, D^{-1} \\ \frac{4}{3}\, u - \frac{2}{3}\, D^{-1}\, u_1 & -\frac{4}{3}\, D - \frac{2}{3}\, (D^{-1}\, v_1 + v_1\, D^{-1}) + \frac{4}{3}\, \delta\, D^{-1} \end{pmatrix},$$

$$J = \begin{pmatrix} 36\, D^3 + w\, D + 18\, v_2 + 32\, u\, D^{-1}\, u & 20\, u\, D^2 + 30\, u_1\, D + 8\, u\, w + 12\, u_2 \\ -20\, u\, D^2 - 10\, u_1\, D - 8\, u\, w - 2\, u_2 & D^5 + 5\, w\, D^3 + \frac{15}{2}\, v_2\, D^2 + p\, D + q \end{pmatrix},$$

where $p = 4\, w^2 + 9/2\, v_3 - 2\, u^2$, $q = 4\, v_2\, w + v_4 - 8\, u\, u_1$, $w = v_1 - \delta$. It may be checked that the operator Θ is implectic . Hence it yields the Hamiltonian form of system (1):

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \Theta\, E\, H, \qquad H = \frac{1}{2}\, u^2 = \frac{1}{2}\, \rho_1,$$

where E is the Euler operator. The Noether operator Θ generates an infinite sequence of Lie–Bäcklund symmetries

$$\sigma_n = \Theta\, E\, \rho_n, \qquad n \geq 0, \qquad \rho_0 = 1; \qquad \sigma_0 = \begin{pmatrix} c_1\, u_1 \\ c_1\, v_1 + c_2 \end{pmatrix}, \qquad \sigma_1 = K, \quad \ldots$$

These symmetries can be constructed by means of the recursion operator Λ. One can easily see that the differential part of Λ has order 6. Therefore $\Lambda\, \sigma_0$ is the 7th-order symmetry. But system (1) possesses the lower order symmetries σ_0, σ_1 and σ_2:

$$\begin{aligned} \sigma_2^u &= u_5 + 5/3\, u_3\, v_1 + 5/2\, u_2\, v_2 + 5/9\, u_0\, v_4 + 35/18\, u_1\, v_3 + 5/9\, v_1\, u_0\, v_2 \\ &\quad + 5/18\, u_0\, \delta\, v_2 + 5/9\, u_1\, v_1^2 + 5/9\, u_1\, \delta\, v_1 - 5/9\, \delta^2\, u_1 + 10/9\, u_0^2\, u_1, \\ \sigma_2^v &= -1/9\, v_5 + 20/9\, u_0\, u_2 + 5/9\, u_1^2 - 5/9\, v_1\, v_3 + 5/9\, \delta\, v_3 + 10/9\, v_1\, u_0^2 \\ &\quad + 5/9\, u_0^2\, \delta - 5/12\, v_2^2 + 5/9\, v_1^2\, \delta - 5/27\, v_1^3. \end{aligned}$$

Hence, we have the triple sequence of symmetries: $\sigma_{3n} = \Lambda^n\, \sigma_0$, $\sigma_{3n+1} = \Lambda^n\, K$, $\sigma_{3n+2} = \Lambda^n \sigma_2$.

So, system (1) possesses the nontrivial zero curvature representation and is exactly solvable. We present the one-soliton solution of system (1):

$$u = \frac{k^2\, \sqrt{3}}{\cosh(k\, x + k^3\, t)}, \qquad v = 3\, k \tanh(k\, x + k^3\, t) + \delta\, x.$$

Here are the plots of this solution for $t = 0$, $k = 2$ and two values of δ:

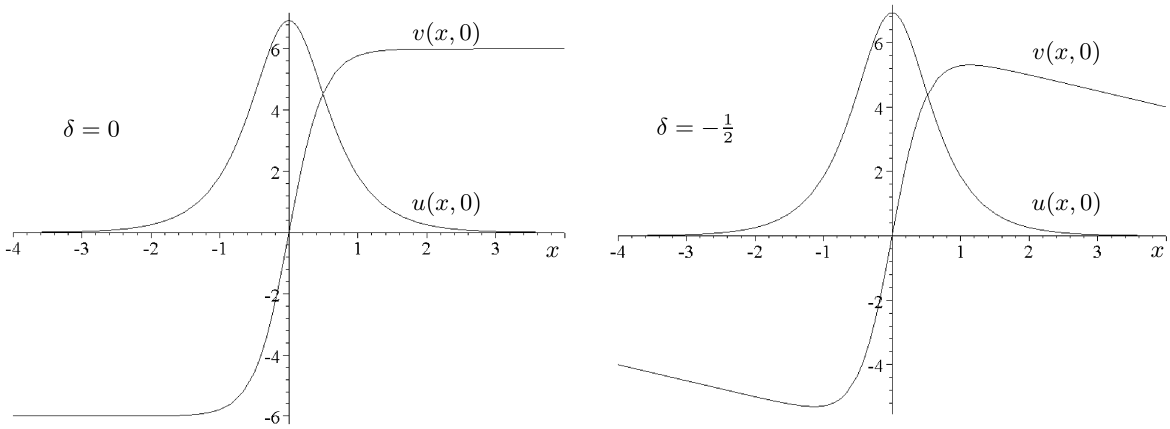

Fig. 1, 2. Soliton solution of system (1).

It is obvious that the u-curve has the typical soliton form and the v-curve has the kink form. The plots of the function v have two asymptotics $v = \delta\, x \pm 3\, k$.

In conclusion we note that system (1) can be reduced to the following single equation

$$v_{tt} = \frac{\partial}{\partial x}\left[v_{txx} - \frac{3}{4}\frac{v_{tx}^2}{v_t} + v_t\, v_x + \delta\, v_t \right]$$

that is integrable of course.

All calculations were performed with the help of an IBM computer and the JET package presented in the separate paper.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 73–82.

Separation of Variables and Construction of Exact Solutions of Nonlinear Wave Equations

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An approach to construction of exact solutions of nonlinear equations on the basis of separated variables is proposed.

1 Introduction

To construct the exact solutions of nonlinear equations in mathematical physics the following ansatz is commonly used

$$u(x) = f(x)\varphi(\omega) + g(x), \tag{1}$$

where $f(x)$, $g(x)$, $\omega = \omega(x, u)$ are certain functions, and functions $\varphi(\omega)$ are undetermined. If the explicit form of variables $\omega = \omega(x, u)$ and functions $f(x)$, $g(x)$ is determined on the basis of subalgebra of invariance algebra of this equation, then ansatz (1) is called as a symmetry or Lie one. Not all ansatzes are symmetry ones.

In [1–4] a definition of conditional invariance of this differential equation was introduced. If the explicit form of new variables $\omega = \omega(x, u)$ and functions $f(x)$, $g(x)$ are determined on the basis of conditional symmetry operators then ansatz (1) is called an arbitrary invariant or non-Lie one. By means of arbitrary invariant ansatzes new classes (types) of exact solutions of many nonlinear equations in mathematical physics were constructed. Let us note an effective algorithm for finding of arbitrary symmetry operators is not found yet.

In this paper an approach to the construction of exact solutions of nonlinear equations is proposed. It is based on the method of separated variables and has a great advantage in view of its simplicity and possibility to be unchanged for construction of exact solutions for many-dimensional equations. We will consider this approach using the Boussinesq equation.

2 Exact solutions of the Boussinesq equation $u_0 = \lambda(\nabla u)^2 + \lambda u \Delta u$

Let us consider the Boussinesq equation

$$u_0 = \lambda(\nabla u)^2 + \lambda u \Delta u, \tag{2}$$

where λ is an arbitrary constant, $u = u(x_0, x_1, \ldots, x_n)$, $u_0 = \dfrac{\partial u}{\partial x_0}$, and

$$(\nabla u)^2 = \left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2, \qquad \Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n}.$$

Certain partial solutions of Eq.(2) for two variables x_0, x have been obtained in [5, 6], and for many variables in [1, 7].

Now let us consider the one-dimensional Boussinesq equation

$$u_0 = \lambda \left(\frac{\partial u}{\partial x_1} \right)^2 + \lambda u \frac{\partial^2 u}{\partial x_1^2}. \tag{3}$$

2.1. We seek for a solution of Eq.(3) in the form $u = a(x_0)b(x_1)$, where functions $a(x_0)$ and $b(x_1)$ are not constants. Substituting into Eq.(3) we have

$$\lambda a^2 bb'' + \lambda a^2 b^{'2} - a'b = 0. \tag{4}$$

It follows from (4) that the functions a^2, a' are linearly dependent. Consequently $a' = \alpha a^2$ for a real number α and Eq.(3) has the form $(\lambda bb'' + \lambda b^{'2})a^2 - \alpha a^2 b = 0$. We find from this equation $\lambda bb'' + \lambda b^{'2} - \alpha b = 0$. Notice that the substitution $a' = \alpha a$ suggests $\alpha = 1$. Thus we will consider the equation

$$\lambda bb'' + \lambda b^{'2} - b = 0. \tag{5}$$

The general solution of Eq.(5) has the form

$$\int \frac{b\,db}{\sqrt{c + b^3}} = \pm \sqrt{\frac{2}{3\lambda}}(x_1 + c_1), \tag{6}$$

where c, c_1 are arbitrary constants. If, for example, $c = 0$, then $b = \frac{1}{6\lambda}(x_1 + c_1)^2$, and we obtain the solution of (3)

$$u = -\frac{(x_1 + c_1)^2}{6\lambda(x_0 + c_2)},$$

which is transformed into

$$u = -\frac{x_1^2}{6\lambda x_0}. \tag{7}$$

The solution (7) is a partial case for

$$u = -\frac{x_1^2}{6\lambda x_0} + f(x_0, x_1).$$

Substituting into Eq.(3), we find

$$f_0 = -\frac{2x_1 f_1}{3x_0} - \frac{x_1^2}{6x_0} f_{11} - \frac{1}{3x_0} f + \lambda f_1^2 + \lambda f f_{11}. \tag{8}$$

The solution of Eq.(8) can be found in the form $f = a(x_0)b(x_1)$ and we have

$$a'b = \frac{a}{x_0} \left(-\frac{2}{3} x_1 b' - \frac{x_1^2}{6} b'' - \frac{1}{3} b \right) + a^2 \left(\lambda b^{'2} + \lambda bb'' \right).$$

Let $a' = \alpha \frac{a}{x_0}$, where α is a real number. Hence, $a = cx_0^\alpha$. To determine the function $b(x_1)$ we find the system of equations:

$$x_1^2 b'' + 4x_1 b' + (2 + 6\alpha)b = 0, \qquad b'^2 + bb'' = 0.$$

Thus, the Boussinesq equation possesses the following solution

$$u = cx_0^{-5/8}x_1^{1/2} - \frac{x_1^2}{6\lambda x_0}.$$

If the function f in (8) depends on x_0 only, then we obtain $f_0 = -\frac{1}{3x_0}f$. Thus, Eq.(3) has a solution

$$u = -\frac{x_1^2}{6\lambda x_0} + cx_0^{-1/3}.$$

2.2. Now let us consider Eq.(2) for the case $n > 1$. We shall look for solution of (2) in the form $u = a(x_0)b(x_1, \ldots, x_k)$, where the functions $a(x_0)$ and $b(x_1, \ldots, x_k)$ are not constant. Substituting this expression into (2) we find

$$\lambda a^2 \left[(\nabla b)^2 + b\Delta b\right] - a'b = 0. \tag{9}$$

It follows from (9) that functions a^2, a' are linearly dependent, thus $a' = \alpha a^2$ and Eq.(9) has a form

$$\left(\lambda b\Delta b + \lambda(\nabla b)^2\right) a^2 - \lambda a^2 b = 0.$$

It can be obtained from this equation that

$$\lambda b\Delta b + \lambda(\nabla b)^2 - \alpha b = 0. \tag{10}$$

The function $b = \varphi(\omega)$, $\omega = x_1^2 + \cdots + x_k^2$, $b \leq n$ satisfies Eq.(10) iff

$$4\lambda\omega\varphi\varphi'' + 2k\lambda\varphi\varphi' + 4\lambda\omega\varphi'^2 - \alpha\varphi = 0. \tag{11}$$

If $\alpha = 2\lambda(k+2)$ then a particular solution of Eq.(11) is the function $\varphi = \omega$. Since the equation $a' = \alpha a^2$ possesses the solution $a = -\frac{1}{\alpha x_0}$, then Eq.(2) has a solution of the form

$$u = -\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0}. \tag{12}$$

The solution (12) is a particular case of

$$u = -\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0} + f(x_0, \ldots, x_k).$$

Substituting this expression into Eq.(2) we obtain

$$\begin{aligned} f_0 = &-\frac{2x_1 f_1}{\lambda(k+2)x_0} - \cdots - \frac{2x_k f_k}{\lambda(k+2)x_0} + \lambda\left(f_1^2 + \cdots + f_k^2 + f_{k+1}^2 + \cdots + f_n^2\right) \\ &+\lambda\left(-\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0} + f\right)(f_{11} + \cdots + f_{nn}) - \frac{k}{(k+2)x_0}f. \end{aligned} \tag{13}$$

Let the function f be independent of $x_1, \ldots, x_k$, then

$$f_0 = \lambda\left(f_{k+1}^2 + \cdots + f_n^2\right) + \lambda\left(-\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0} + f\right)(f_{k+1,k+1} + \cdots + f_{nn}) - \frac{k}{(k+2)x_0}f.$$

Thus,

$$(f_{k+1,k+1} + \cdots + f_{nn}) = 0, \qquad f_0 = \lambda\left(f_{k+1}^2 + \cdots + f_n^2\right) - \frac{k}{(k+2)x_0}f. \tag{14}$$

The solution of (14) can be found in the form

$$f = \mu_{k+1}x_{k+1} + \cdots + \mu_n x_n + \nu,$$

where $\mu_{k+1}, \ldots, \mu_n$, ν are functions dependent on x_0 only. Substituting this expression into the second equation of (14) we have

$$\frac{\partial \mu_{k+1}}{\partial x_0}x_{k+1} + \cdots + \frac{\partial \mu_n}{\partial x_0}x_n + \frac{\partial \nu}{\partial x_0}$$
$$= \lambda\left(\mu_{k+1}^2 + \cdots + \mu_n^2\right) - \frac{k}{(k+2)x_0}\left(\mu_{k+1}x_{k+1} + \cdots + \mu_n x_n + \nu\right).$$

Thus,

$$\begin{aligned}
&\frac{\partial \mu_{k+1}}{\partial x_0} = -\frac{k}{(k+2)x_0}\mu_{k+1},\\
&\dots\dots\dots\dots\dots\\
&\frac{\partial \mu_n}{\partial x_0} = -\frac{k}{(k+2)x_0}\mu_n,\\
&\frac{\partial \nu}{\partial x_0} = \lambda\left(\mu_{k+1}^2 + \cdots + \mu_n^2\right) - \frac{k}{(k+2)x_0}\nu.
\end{aligned} \tag{15}$$

The general solution of (15) has the following form:

$$\mu_{k+1} = c_{k+1}x_0^{-\frac{k}{k+2}}, \quad \ldots, \quad \mu_n = c_n x_0^{-\frac{k}{k+2}},$$
$$\nu = \frac{\lambda(k+2)}{2}\left(c_{k+1}^2 + \cdots + c_n^2\right)x_0^{\frac{-k+2}{k+2}} + c x_0^{-\frac{k}{k+2}},$$

where $c, c_{k+1}, \ldots, c_n$ are arbitrary constants.

Thus, we obtain the multiparameter set of solutions of Eq.(2)

$$\begin{aligned}
u = &-\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0} + \left(c_{k+1}x_{k+1} + \cdots + c_n x_n + c\right)x_0^{-\frac{k}{k+2}}\\
&+\frac{\lambda(k+2)}{2}\left(c_{k+1}^2 + \cdots + c_n^2\right)x_0^{\frac{-k+2}{k+2}}.
\end{aligned} \tag{16}$$

Moreover, if $k = 1$, $n = 3$ then solution (16) takes the form

$$u = -\frac{x_1^2}{6\lambda x_0} + \left(c_2x_2 + c_3x_3\right)x_0^{-1/3} + \frac{3\lambda}{2}\left(c_2^2 + c_3^2\right)x_0^{1/3}.$$

If $k = 2$, $n = 3$ then solution (16) has a form

$$u = -\frac{x_1^2 + x_2^2}{8\lambda x_0} + c_3x_3x_0^{-1/2} + 2\lambda c_3^2.$$

If the function f in (13) does not depend on $x_1, \ldots, x_k$, then we have

$$f_0 = -\frac{k}{(k+2)x_0}f.$$

Thus,

$$f = c x_0^{-\frac{k}{k+2}}.$$

And the Boussinesq equation (2) has also the following solution

$$u = -\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0} + c x_0^{-\frac{k}{k+2}}.$$

If, for example, $k = 2$ then we have

$$u = -\frac{x_1^2 + x_2^2}{8\lambda x_0} + c x_0^{-1/2}.$$

In the case of $k = 3$ we have

$$u = -\frac{x_1^2 + x_2^2 + x_3^2}{10\lambda x_0} + c x_0^{-3/5}.$$

3 Exact solutions of the Boussinesq equation $u_{00} + (\nabla u)^2 + u\Delta u + \Delta(\Delta u) = 0$

Let us consider the Boussinesq equation

$$u_{00} + u u_{11} + u_1^2 + u_{1111} = 0, \tag{17}$$

where

$$u = u(x), \qquad x = (x_0, x_1), \qquad u_1 = \frac{\partial u}{\partial x_1}, \qquad u_{11} = \frac{\partial^2 u}{\partial x_1^2}, \qquad u_{1111} = \frac{\partial^4 u}{\partial x_1^4}.$$

It is invariant with respect to the algebra with operators [8]

$$P_0 = \frac{\partial}{\partial x_0}, \qquad P_1 = \frac{\partial}{\partial x_1}, \qquad D = 2x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} - 2u\frac{\partial}{\partial u}.$$

Operators P_0, P_1 and D give rise to the one-parameter symmetry group of equations:

$$\begin{aligned} &G_0 : (x_0, x_1, u) \to (x_0 + \varepsilon, x_1, u),\\ &G_1 : (x_0, x_1, u) \to (x_0, x_1 + \varepsilon, u),\\ &G_2 : (x_0, x_1, u) \to \left(e^{2\varepsilon}x_0, e^{\varepsilon}x_1, e^{-2\varepsilon}u\right). \end{aligned} \tag{18}$$

Eq.(17) is also invariant under the discrete transformations

$$\begin{aligned} &(x_0, x_1, u) \to (-x_0, x_1, u),\\ &(x_0, x_1, u) \to (x_0, -x_1, u),\\ &(x_0, x_1, u) \to (-x_0, -x_1, u). \end{aligned} \tag{19}$$

One-parameter subgroups (18) and discrete transformations (19) give rise to the group G of Eq.(17). Therefore, the most general solution obtained from $u = f(x_0, x_1)$ by means of the transformations of the group G has the form

$$u = \alpha^2 f\left(\alpha^2 x_0 + \beta_0, \alpha x_1 + \beta_1\right),$$

where α, β_0, β_1 are arbitrary real numbers.

The derivation of exact solutions of Eq.(17) is discussed in [1–4]. A new method of invariant reduction of the Boussinesq equation is proposed in [2]. Exact solutions of Eq.(17) on the basis of the conditional symmetry concept are obtained in [3–4].

3.1. We seek a solution of Eq.(17) in the form $u = a(x_0) + b(x_1)$, where the functions $a(x_0)$ and $b(x_1)$ are not constant. Substituting this expression into Eq.(17) we have

$$a'' + ab'' + \left(bb' + b'^2 + b''''\right) = 0. \tag{20}$$

Since b is independent of x_0, it is clear from Eq.(20) that $a'' = \alpha + \beta a$ for real α and β. Therefore, we obtain from (20) that $a(\beta + b'') + (\alpha + bb'' + b'^2 + b'''') = 0$, i.e.

$$b'' + \beta = 0, \qquad bb'' + b'^2 + b'''' + \alpha = 0. \tag{21}$$

If $\beta = 0$ then the system of Eqs.(21) possesses a solution $b = \gamma x_1 + \delta$, where $\gamma^2 = -\alpha$. The function $b(x_1)$ can be transformed into $2x_1$ by means of a transformation from the group G. Then $\alpha = -4$ and $a = -2x_0^2 + \gamma_1 x_0 + \delta_1$, where γ_1, δ_1 are real numbers. Since a can be rewritten as $a = -2(x_0 - \gamma_1/2)^2 + \delta_1 - \gamma_1^2/4$, this solution can be transformed with the help of the group G to become

$$u = 2\left(x_1 - x_0^2\right). \tag{22}$$

Let us construct another type of solutions to Eq.(17) with partial solution (22) to the Boussinesq equation. A partial solution of Eq.(17) can be found in the form

$$u = 2\left(x_1 - x_0^2\right) + f(x_0, x_1). \tag{23}$$

Ansatz (23) reduces Eq.(17) to the form

$$f_{00} + ff_{11} + f_1^2 + f_{1111} + 2\left(x_1 - x_0^2\right) f_{11} + 4f_1 = 0. \tag{24}$$

Ansatz $\varphi = \varphi(\omega)$, $\omega = x_1 + x_0^2$ reduces Eq.(24) to the ordinary differential equation

$$\varphi\varphi'' + \varphi'^2 + \varphi'''' + 2\omega\varphi'' + 6\varphi' = 0. \tag{25}$$

A partial solution of Eq.(25) we find in the form $\varphi = t\omega^s$, $s \neq 1$. Substituting it into (25) we obtain $s = -2$, $t = -12$. Thus, the function

$$u = 2\left(x_1 - x_0^2\right) - 12\left(x_1 + x_0^2\right)^{-2} \tag{26}$$

is a solution of Eq.(17).

3.2. Now, we look for a solution of Eq.(17) in the form $u = a(x_0)b(x_1)$, where the functions $a(x_0)$ and $b(x_1)$ are not constant. Substituting this expression into (17) we obtain

$$a''b + a^2\left(bb'' + b'^2\right) + ab'''' = 0. \tag{27}$$

In complete analogy with Subsection 3.1 we see that $a'' = \alpha a^2 + \beta a$. Substituting a'' into Eq.(27) and taking into account the functions a and a^2 are linearly independent we obtain the following system to determine the function $b(x_1)$

$$b'''' + \beta b = 0, \qquad bb'' + b'^2 + \alpha b = 0. \tag{28}$$

It may be easily seen from these equations that $\beta = 0$ and $\alpha \neq 0$. We can always set $\alpha = 6$ by multiplying the function a by the number $\alpha/6$ and the function b by $6/\alpha$. Since $\beta = 0$, we see from the first of equations (28) that b is polynomial in x_1 of degree not higher than three. Plugging b in the form of the general polynomial of degree three into the second of equations (28), we see that in fact $b = -x_1^2$. Hence, Eq.(17) possesses the solution

$$u = -x_1^2\mathcal{P}(x_0), \tag{29}$$

$$u = -x_1^2 x_0^{-2}, \tag{30}$$

where $\mathcal{P}(x_0)$ is the Weierstrass function with invariants $g_2 = 0$ and $g_3 = c_1$.

A new class of solutions of Eq.(17) can be constructed using its partial solution (29). We look for these new solutions in the form

$$u = -x_1^2 \mathcal{P}(x_0) + f(x_0, x_1). \tag{31}$$

Ansatz (31) reduces Eq.(17) to

$$\left(f_{00} + f f_{11} + f_1^2 + f_{1111}\right) - \mathcal{P}\left(x_1^2 f_{11} + 4x_1 f_1 + 2f\right) = 0. \tag{32}$$

If the function f is independent of x_1, then we have $f_{00} = 2\mathcal{P}f$. This is the Lamé equation and its solutions are well-known [11]. Thus, the function

$$u = -x_1^2 \mathcal{P}(x_0) + \Lambda(x_0), \qquad \Lambda'' = 2\mathcal{P}\Lambda \tag{33}$$

is a solution of the Boussinesq equation.

If the function f in (32) does not depend on x_0, then we have a system of equations to determine the function f

$$x_1^2 f_{11} + 4x_1 f_1 + 2f = 0, \qquad f f_{11} + f_1^2 + f_{1111} = 0. \tag{34}$$

The first equation of this system is linear and its complementary function is well-known [11]. Hence, $f = -12x_1^{-2}$, and Eq.(17) possesses a solution

$$u = -x_1^2 \mathcal{P}(x_0) - 12x_1^{-2}. \tag{35}$$

We obtain simultaneously that the function

$$u = -12x_1^{-2} \tag{36}$$

is a solution of the Boussinesq equation too.

Then we find a solution of Eq.(32) which is dependent on x_0 and x_1. It can be found in the form $f = a(x_0)b(x_1) + c(x_0)$ where functions $a(x_0)$ and $c(x_0)$ are linearly independent. Substituting into Eq.(17) we obtain

$$c'' + a''b + a^2\left(bb'' + b'^2\right) + acb'' + ab'''' + a\mathcal{P}\left(-x_1^{-2}b'' - 4x_1 b' - 2b\right) - 2\mathcal{P}c = 0. \tag{37}$$

Without going into details let us suppose from the outset that $b'' = 0$. Then $b = \alpha x_1 + \beta$ and consequently $f = \alpha a(x_0)x_1 + (\beta a(x_0) + c(x_0))$. It means that setting $\alpha = 1$, $\beta = 0$ in Eq.(37) we arrive at

$$c'' + \alpha a'' x_1 + a^2 + a\mathcal{P}(-4x_1 - 2x_1) - 2\mathcal{P}c = 0.$$

Thus,

$$a'' - 6\mathcal{P}a = 0, \qquad c'' = -a^2 + 2\mathcal{P}c. \tag{38}$$

The equation $a'' - 6\mathcal{P}a = 0$ is the Lamé equation with a solution $a = \mathcal{P}(x_0)$. Hence the complementary function of the Lamé equation can be written as $a = \gamma_1\mathcal{P}(x_0) + \gamma_2\Lambda(x_0)$, where $\mathcal{P}(x_0)$ and $\Lambda(x_0)$ are linearly independent. The corresponding solution of Eq.(17) has the form

$$u = -\mathcal{P}(x_0)(x_1 - \gamma_1/2)^2 + \gamma_2 x_1 \Lambda(x_0) + \left(c(x_0) + \gamma_1^2/4\mathcal{P}(x_0)\right).$$

Under transformations from the group G it reduces to

$$u = -x_1^2\mathcal{P}(x_0) + \gamma_2 x_1\Lambda(x_0) + d(x_0), \tag{39}$$

where the function $d(x_0)$ is a solution of the following equation

$$d'' = -\gamma_1^2\Lambda^2 + 2\mathcal{P}d.$$

In a similar manner from (30) a new class of the Boussinesq equation solutions can be constructed

$$u = -x_0^2x_1^2 - 12x_1^{-2}, \tag{40}$$

$$u = -x_0^2x_1^2 + c_1x_0^3x_1 - \frac{c_1^2}{54}x_0^8 + c_2x_0^2 + c_3x_0^{-1}. \tag{41}$$

The solution of Eq.(17) is in the form $u = a(x_0)b(x_1) + c(x_0)$, where functions $a(x_0)$ and $c(x_0)$ are linearly independent. By substituting in Eq.(17) we obtain

$$a''b + c'' + a^2\left(b'^2 + bb''\right) + acb'' + ab'''' = 0.$$

If $c'' = \alpha a^2$, $a'' = 0$, then

$$a^2\left(\alpha + b'^2 + bb''\right) + acb'' + ab'''' = 0.$$

It follows from this equation that

$$b'^2 + bb'' + a = 0, \qquad b'' = 0.$$

The solution of this system up to transformations from the group G is a function $b = x_1$ if $\alpha = -1$. Then with the requirement that $c'' = \alpha a^2$, $a'' = 0$, it is possible to obtain $a = x_0$, $c = -\frac{1}{12}x_0^4 + \gamma x_0 + \delta$. Thus the function

$$u = x_0x_1 - \frac{1}{12}x_0^4 + \gamma x_0 + \delta$$

is the Boussinesq equation solution with arbitrary real numbers γ, δ.

3.3. We go now to the construction of exact solutions of the Boussinesq equation for the case $n > 1$. The generalization of Eq.(17) for arbitrary number of variables $x_0, x_1, \ldots, x_n$ is the equation [10]

$$u_{00} + (\nabla u)^2 + u\Delta u + \Delta(\delta u) = 0, \tag{42}$$

where

$$(\nabla u)^2 = \left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2, \qquad \Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$

The solution of (42) can be found in the form $u = a(x_0)b(x_1, \ldots, x_k)$, $k \leq n$. Substituting this expression into (42) we have

$$a''b + a^2\left[b\Delta b + (\nabla b)^2\right] + a\Delta(\Delta b) = 0.$$

Hence $c'' = \alpha a^2 + \beta a$ and as a result we obtain the following system to determine the function $b(x_1, \ldots, x_k)$

$$\Delta(\Delta b) + \beta b = 0, \qquad b\Delta b + (\nabla b)^2 + \alpha b = 0.$$

If $b = 0$, then $\alpha \neq 0$ and it may be considered that $\alpha = 6$. The system has a solution $b = -\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)$ for these values b and α. Therefore, the Boussinesq equation solutions are functions

$$u = -\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)\mathcal{P}(x_0), \tag{43}$$

$$u = -\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)x_0^{-2}. \tag{44}$$

Let us construct another solution of Eq.(42) from (43). We will look for it in the form

$$u = -\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)\mathcal{P}(x_0) + f(x_0, x_1, \ldots, x_k). \tag{45}$$

Ansatz (45) reduces Eq.(42) to

$$\begin{aligned} &f_{00} + (\nabla f)^2 + f\Delta f + \Delta(\Delta f) \\ &\quad -\mathcal{P}\left[\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)\Delta f + (4x_1 f_1 + \cdots + 4x_k f_k) + 2kf\right] = 0. \end{aligned} \tag{46}$$

If f does not depend on variables $x_1, \ldots, x_k$ in Eq.(46) then $f_{00} = \frac{6k}{k+2}f$ and, therefore, the function

$$u = -\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)\mathcal{P}(x_0) + \Lambda(x_0), \qquad \Lambda'' = \frac{6k}{k+2}\mathcal{P}\Lambda \tag{47}$$

is a solution of Eq.(42).

If the function f depends on variables $x_0, x_1, \ldots, x_k$ in (46) then the solution of Eq.(42) can be obtained in the following form

$$u = -\frac{3}{k+2}\left(x_1^2 + \ldots + x_k^2\right)\mathcal{P}(x_0) + \alpha x_1 \Lambda(x_0) + c(x_0), \tag{48}$$

where $\mathcal{P}^{11} = 6\mathcal{P}^2$, $\Lambda'' = (4 + 2k)\mathcal{P}\Lambda$, $c'' = -\alpha^2\Lambda^2 + 2\mathcal{P}c$.

Similarly we find a solution of Eq.(42) from (44):

$$u = -\frac{3}{k+2}\left(x_1^2 + \ldots + x_k^2\right)x_0^{-2} + c_1 x_0^3 x_1 - \frac{k+2}{50k+112}x_0^8 + c_2 x_0^{\frac{1+\sqrt{\frac{25k+2}{k+2}}}{2}} + c_3 x_0^{\frac{1-\sqrt{\frac{25k+2}{k+2}}}{2}},$$

where c_1, c_2, c_3, c_4 are arbitrary real numbers; $k = 1, \ldots, n$.

A new type of solutions of Eq.(42) can be constructed using

$$u = -x_1^2\mathcal{P}(x_0) + f(x_0, x_2, x_3). \tag{49}$$

Substituting anzats (49) into (42) we have

$$f_{00} + (\nabla f)^2 + f(\Delta f) + \Delta(\Delta f) - x_1^2\mathcal{P}(\Delta f) - 2\mathcal{P}f = 0.$$

Since the function f does not depend on x_1, then $\Delta f = 0$ and we obtain the following system of equations to determine the function f:

$$f_{00} + f_2^2 + f_3^2 - 2\mathcal{P}f = 0, \qquad f_{22} + f_{33} = 0. \tag{50}$$

We will seek now a solution of Eqs.(50) in the form $f = a(x_0)x_2 + b(x_0)x_3 + c(x_0)$. Substitution of f into the first equation of (50) gives

$$a''x_2 + b''x_3 + c'' + a^2 + b^2 - 2\mathcal{P}(ax_2 + bx_3 + c) = 0.$$

It follows from this equation that

$$a'' = 2\mathcal{P}a, \qquad b^{11} = 2\mathcal{P}b, \qquad c'' = -a^2 - b^2 + 2\mathcal{P}c. \tag{51}$$

Solving Eq.(51) we find the explicit form of functions $a(x_0)$, $b(x_0)$, $c(x_0)$ and the solution of Eq.(42) too.

If we use the ansatz

$$u = -x_1^2 x_0^{-2} + f(x_0, x_2, x_3),$$

we construct by analogy with the above the following solution of Eq.(42):

$$u = -x_1^2 x_0^{-2} + \left(c_1 x_0^2 + c_4 x_0^{-1}\right) x_2 + \left(c_3 x_0^2 + c_4 x_0^{-1}\right) x_3$$
$$-\frac{c_1^2 + c_3^2}{28} x_0^6 - \frac{c_1 c_2 + c_3 c_4}{2} x_0^3 + \frac{c_2^2 + c_3^2}{28}.$$

And using the ansatz

$$u = -\frac{1}{2}\left(x_1^2 + x_2^2\right)\mathcal{P}(x_0) + f(x_0, x_3)$$

another solution of Eq.(42) can be obtained

$$u = -\frac{1}{2}\left(x_1^2 + x_2^2\right)\mathcal{P}(x_0) + \Lambda(x_0)x_3 + c(x_0),$$

where $\Lambda'' = 2\mathcal{P}\Lambda$, $c'' = -\Lambda^2 + 2\mathcal{P}c$.

Making use of

$$u = -\frac{1}{2}\left(x_1^2 + x_2^2\right) x_0^{-2} + f(x_0, x_3)$$

we find a solution of Eq.(42) in the form

$$u = -\frac{1}{2}\left(x_1^2 + x_2^2\right) x_0^{-2} + \left(c_1 x_0^2 + c_2 x_0^{-1}\right) x_3 + \frac{c_1^2}{28} x_0^6 - \frac{c_1 c_2}{2} x_0^3 + \frac{c_2^2}{2}.$$

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 83–89.

On Exact Solutions of the Nonlinear Heat Conduction Equation with Source

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The symmetry reduction of the equation $u_0 = \nabla\left[u^\mu \nabla u\right] + \delta u$ to ordinary differential equations with respect to all subalgebras of rank three of the invariance algebra of this equation is performed. Some exact solutions of this equation are obtained.

1 Introduction

Symmetry reduction of nonlinear heat conduction equations without a source is investigated in references [1–7]. In this paper, we investigate the equation

$$\frac{\partial u}{\partial x_0} = \nabla\left[u^\mu \nabla u\right] + \delta u, \tag{1}$$

where $u = u(x_0, x_1, x_2, x_3)$, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$; μ, δ are real numbers, $\mu \neq 0$ and $|\delta| = 1$.

The substitution $u = v^{\frac{1}{\mu}}$ transforms equation (1) into the equation

$$\frac{\partial v}{\partial x_0} = v\Delta v + \frac{1}{\mu}(\nabla v)^2 + \delta\mu v. \tag{2}$$

Let L be the maximal invariance algebra of equation (2). If $\mu \neq -\frac{4}{5}$, then L is the direct sum of the extended Euclidean algebras $A\tilde{E}(1) = \langle P_0, D_1\rangle$ and $A\tilde{E}(3) = \langle P_a, J_{ab}, D_2 :\ a, b = 1, 2, 3\rangle$, generated by the vector fields [8]:

$$\begin{gathered} P_0 = e^{-\delta\mu x_0}\left(\frac{\partial}{\partial x_0} + \delta\mu v\frac{\partial}{\partial v}\right), \qquad D_1 = \frac{1}{\delta\mu}\frac{\partial}{\partial x_0}, \qquad P_a = \frac{\partial}{\partial x_a}, \\ J_{ab} = x_a\frac{\partial}{\partial x_b} - x_b\frac{\partial}{\partial x_a}, \qquad D_2 = x_a\frac{\partial}{\partial x_a} + 2v\frac{\partial}{\partial v} \end{gathered} \tag{3}$$

with $a, b = 1, 2, 3$. If $\mu = -\frac{4}{5}$, then L decomposes [8] into the direct sum of $A\tilde{E}(1)$ and the conformal algebra $AC(3) = \langle P_a, K_a, J_{ab}, D_2 : a, b = 1, 2, 3\rangle$, where P_0, P_a, J_{ab}, D_2 are vector fields (3), and

$$K_a = \left(x_1^2 + x_2^2 + x_3^2\right)\frac{\partial}{\partial x_a} - 2x_a D_2, \qquad a = 1, 2, 3.$$

In this paper, the symmetry reduction of equation (2) is performed with respect to all subalgebras of rank three of the algebra L, up to conjugacy with respect to the group $\operatorname{Ad} L$ of inner automorphisms.

Let $u = f(x_1, x_2, x_3)$ be a solution of equation (1). If $\mu + 1 \neq 0$, then $\Delta u^{\mu+1} + \delta(\mu+1)u = 0$, and if $\mu + 1 = 0$, then $\Delta \ln u + \delta u = 0$. Hence, the search for stationary solutions to equation (1) is reduced to a search for relevant solutions of the d'Alembert equation or Liouville equation. Let $u = u(x_0, x_1, x_2, x_3)$ be a solution of equation (1) invariant under P_0. In this case, if $\mu + 1 \neq 0$, then $u = e^{\delta x_0}\varphi(x_1, x_2, x_3)^{\frac{1}{\mu+1}}$, where $\Delta\varphi = 0$. If $\mu + 1 = 0$, then

$$u = e^{\delta x_0 + \psi(x_1,x_2,x_3)},$$

where $\Delta\psi = 0$. In this connection, let us restrict ourselves to those subalgebras of L that do not contain P_0 and D_1. The list of I-maximal subalgebras of rank 3 is obtained in [4, 6, 7].

2 Reduction of equation (2) for an arbitrary μ to ordinary differential equations

Up to the conjugacy under the group of inner automorphisms, the algebra $A\tilde{E}(1) \oplus A\tilde{E}(3)$ has 12 I-maximal subalgebras of rank three, which do not contain P_0 and D_1 [4, 7]:

$$L_1 = \langle P_1, P_2, P_3, J_{12}, J_{13}, J_{23}\rangle;$$

$$L_2 = \langle P_0 + P_1, P_2, P_3, J_{23}\rangle;$$

$$L_3 = \langle P_2, P_3, J_{23}, D_1 + \alpha D_2\rangle \quad (\alpha \in \mathbb{R},\ \alpha \neq 0);$$

$$L_4 = \langle P_0 + P_1, P_3, D_1 + D_2\rangle;$$

$$L_5 = \langle P_3, J_{12}, D_1 + \alpha D_2\rangle\ (\alpha \in \mathbb{R},\ \alpha \neq 0);$$

$$L_6 = \langle P_0 + P_3, J_{12}, D_1 + D_2\rangle;$$

$$L_7 = \langle P_3, J_{12} + \alpha P_0, D_2 + \beta P_0\rangle \quad (\alpha = 1, \beta \in \mathbb{R} \text{ or } \alpha = 0 \text{ and } \beta = 0, \pm 1);$$

$$L_8 = \langle P_2, P_3, J_{23}, D_1 + P_1\rangle;$$

$$L_9 = \langle P_2, P_3, J_{23}, D_2 + \alpha P_0\rangle \quad (\alpha = 0, \pm 1);$$

$$L_{10} = \langle P_3, D_1 + \alpha J_{12}, D_2 + \beta J_{12}\rangle \quad (\alpha, \beta \in \mathbb{R} \text{ and } \alpha > 0);$$

$$L_{11} = \langle J_{12}, J_{13}, J_{23}, D_1 + \alpha D_2\rangle \quad (\alpha \in \mathbb{R},\ \alpha \neq 0);$$

$$L_{12} = \langle J_{12}, J_{13}, J_{23}, D_2 + \alpha P_0\rangle \quad (\alpha = 0, \pm 1).$$

For each of the subalgebras $L_1, \ldots, L_{12}$ we indicate the corresponding ansatz $\omega' = \varphi(\omega)$ solved for v, where ω and ω' are functionally independent invariants of a subalgebra, as well as the reduced equation which is obtained by means of this ansatz. In cases when the reduced equation can be solved, we indicate the corresponding invariant solutions of equation (2).

2.1. $L_1 : v = \varphi(\omega),\ \omega = x_0,\ \dot{\varphi} = \delta\mu\varphi.$

In this case

$$v = Ce^{\delta\mu x_0},$$

where C is an arbitrary constant.

2.2. $L_2 : v = e^{\delta\mu x_0}\varphi(\omega),\ \omega = \dfrac{1}{\delta\mu}e^{\delta\mu x_0} - x_1,\ \varphi\ddot{\varphi} + \dfrac{1}{\mu}\dot{\varphi}^2 - \dot{\varphi} = 0.$

Integrating the reduced equation, we obtain $\varphi = C'$ or

$$\int \frac{d\varphi}{\mu + C|\varphi|^{-\frac{1}{\mu}}} = \omega + C',$$

where C, C' are arbitrary constants and $C \neq 0$. Corresponding invariant solutions to equation (2) are of the form

$$v = C^{-1}e^{-\delta x_0}[1 + \tilde{C}\ \exp(-\delta C e^{-\delta x_0} - Cx_1)],\ \text{if}\ \mu = -1;$$

$$v = Ae^{-\frac{1}{2}\delta x_0}\tan\left(\frac{\delta}{A}e^{-\frac{1}{2}\delta x_0} + \frac{x_1}{2A} + B\right),\ \text{if}\ \mu = -\frac{1}{2},\ C = -\frac{1}{2A^2};$$

$$v = Ae^{-\frac{1}{2}\delta x_0}\tanh\left(\frac{\delta}{A}e^{-\frac{1}{2}\delta x_0} + \frac{x_1}{2A} + B\right)$$

and

$$v = Ae^{-\frac{1}{2}\delta x_0}\coth\left(\frac{\delta}{A}e^{-\frac{1}{2}\delta x_0} + \frac{x_1}{2A} + B\right),\ \text{if}\ \mu = -\frac{1}{2},\ C = \frac{1}{2A^2}.$$

2.3. $L_3 : v = {x_1}^2\varphi(\omega),\ \omega = \alpha\delta\mu x_0 - \ln x_1,$

$$\varphi\ddot{\varphi} + \frac{1}{\mu}\dot{\varphi}^2 - \frac{3\mu+4}{\mu}\varphi\dot{\varphi} - \alpha\delta\mu\dot{\varphi} + \frac{2\mu+4}{\mu}\varphi^2 + \delta\mu\varphi = 0.$$

If $\mu = -2$, $\alpha = -\frac{1}{2}$, then $\varphi = -2\delta\omega + C$ is a solution of the reduced equation. By means of φ we obtain the exact solution

$$v = x_1^2(-2x_0 + 2\delta\ln x_1 + C)$$

of equation (2).

2.4. $L_4 : v = x_2 e^{\delta\mu x_0}\varphi(\omega),\ \omega = \dfrac{\delta\mu x_1 - e^{\delta\mu x_0}}{x_2},$

$$\mu(\mu^2 + \omega)\varphi\ddot{\varphi} + \mu^2\dot{\varphi}^2 + (\varphi - \omega\dot{\varphi})^2 + \delta\mu^2\dot{\varphi} = 0.$$

The function $\varphi = A\omega + B$, where A and B are constants, satisfies this reduced equation if and only if $B^2 = -(\delta A + A^2)\mu^2$. The corresponding invariant solution of equation (2) is of the form

$$v = A\left(\delta\mu x_1 e^{\delta\mu x_0} - e^{2\delta\mu x_0}\right) + Bx_2 e^{\delta\mu x_0}.$$

2.5. $L_5 : v = \left(x_1^2 + x_2^2\right)\varphi(\omega),\ \omega = \alpha\delta\mu x_0 - \dfrac{1}{2}\ln\left(x_1^2 + x_2^2\right),$

$$\varphi\ddot{\varphi} + \frac{\mu+1}{\mu}(2\varphi - \dot{\varphi})^2 - \dot{\varphi}^2 - \alpha\delta\mu\dot{\varphi} + \delta\mu\varphi = 0.$$

2.6. $L_6 : v = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}} e^{\delta\mu x_0}\varphi(\omega),\ \omega = \left(\delta\mu x_3 - e^{\delta\mu x_0}\right)\left(x_1^2 + x_2^2\right)^{-\frac{1}{2}},$

$$\left(\omega^2 + \mu^2\right)\varphi\ddot{\varphi} - \left(1 + \frac{2}{\mu}\right)\omega\varphi\dot{\varphi} + \left(\mu + \frac{\omega^2}{\mu}\right)\dot{\varphi}^2 + \delta\mu\dot{\varphi} + \left(1 + \frac{1}{\mu}\right)\varphi^2 = 0.$$

For $\mu = -1$, this equation has the solution $\varphi = -\delta\omega$. In this case

$$v = \delta e^{-2\delta x_0} + x_3 e^{-\delta x_0}$$

is the corresponding solution of (2).

2.7. $L_7 : v = e^{\delta\mu x_0}\left(x_1^2 + x_2^2\right)\varphi(\omega),\ \omega = \alpha\delta\mu\arctan\dfrac{x_1}{x_2} - \dfrac{\beta\delta\mu}{2}\ln\left(x_1^2 + x_2^2\right) + e^{\delta\mu x_0},$

$$\left(\alpha^2 + \beta^2\right)\mu^2\varphi\ddot{\varphi} + \left(\alpha^2 + \beta^2\right)\mu\dot{\varphi}^2 - 4\beta\delta(1+\mu)\varphi\dot{\varphi} - \delta\mu\dot{\varphi} + \frac{4(\mu+1)}{\mu}\varphi^2 = 0. \qquad (4)$$

For $\alpha = \beta = 0$, $\mu \neq -1$, the reduced equation takes the form

$$-\delta\mu\dot{\varphi} + \frac{4(\mu+1)}{\mu}\varphi^2 = 0.$$

It's general solution is

$$\varphi = \frac{\mu^2}{C - 4\delta(\mu+1)\omega},$$

and the corresponding invariant solution of equation (2) is

$$v = \mu^2 \left(x_1^2 + x_2^2\right) e^{\delta\mu x_0}[C - 4\delta(\mu+1)e^{\delta\mu x_0}]^{-1}.$$

If $\mu = -1$, $\alpha^2 + \beta^2 \neq 0$, then equation (4) takes the form

$$\left(\alpha^2 + \beta^2\right)\varphi\ddot{\varphi} - \left(\alpha^2 + \beta^2\right)\dot{\varphi}^2 + \delta\dot{\varphi} = 0.$$

In this case, $\varphi = C'$ or

$$\varphi = \frac{1}{C}\left[C' \exp\left(\frac{C\omega}{\alpha^2 + \beta^2}\right) - \delta\right],$$

and, therefore,

$$v = C' e^{-\delta x_0}\left(x_1^2 + x_2^2\right)$$

or

$$v = \frac{1}{C}\left(x_1^2 + x_2^2\right) e^{-\delta x_0}\left\{C' \exp\left[\frac{C}{\alpha^2+\beta^2}\left(-\alpha\delta \arctan\frac{x_1}{x_2} + \frac{\beta\delta}{2}\ln\left(x_1^2 + x_2^2\right) + e^{-\delta x_0}\right)\right] - \delta\right\},$$

where C, C' are arbitrary constants and $C \neq 0$.

2.8. $L_8 : v = \varphi(\omega),\ \omega = \delta\mu x_0 - x_1,\ \varphi\ddot{\varphi} + \frac{1}{\mu}\dot{\varphi}^2 + \delta\mu(\varphi - \dot{\varphi}) = 0.$

For $\mu = -1$, the reduced equation has the solution $\varphi = Ce^{\omega}$, where C is an arbitrary constant. The corresponding invariant solution of the equation (2) is of the form $v = C\exp(-\delta x_0 - x_1)$.

2.9. $L_9 : v = x_1^2 e^{\delta\mu x_0}\varphi(\omega),\ \omega = \alpha\ln x_1 - \frac{1}{\delta\mu}e^{\delta\mu x_0},$

$$\alpha^2\varphi\ddot{\varphi} + \frac{\alpha^2}{\mu}\dot{\varphi}^2 + \left(3\alpha + \frac{4\alpha}{\mu}\right)\varphi\dot{\varphi} + \dot{\varphi} + \left(2 + \frac{4}{\mu}\right)\varphi^2 = 0.$$

For $\alpha = 0$, $\mu \neq -2$, we obtain $\varphi = \mu[(2\mu+4)\omega + \tilde{C}]^{-1}$, therefore,

$$v = \frac{\mu^2 x_1^2 e^{\delta\mu x_0}}{C - \delta(2\mu+4)e^{\delta\mu x_0}}.$$

If $\alpha = 0$, $\mu = -2$, then

$$v = Cx_1^2 e^{-2\delta x_0}.$$

For $\alpha \neq 0$, $\mu = -2$, the reduced equation has the solution $\varphi = \frac{1}{\alpha} + Ce^{-2\alpha\omega}$. The corresponding invariant solution of equation (2) is of the form

$$v = \alpha^{-1} x_1^2 \exp(-2\delta x_0) + C \exp[-2\delta x_0 - \alpha\delta \exp(-2\delta x_0)].$$

2.10. $L_{10} : v = \left(x_1^2 + x_2^2\right) \varphi(\omega),\ \omega = \arctan \dfrac{x_1}{x_2} + \alpha\delta\mu x_0 + \dfrac{\beta}{2} \ln\left(x_1^2 + x_2^2\right),$

$$\left(1+\beta^2\right) \varphi\ddot{\varphi} + \frac{1+\beta^2}{\mu} \dot{\varphi}^2 + 4\beta \left(1 + \frac{1}{\mu}\right) \varphi\dot{\varphi} - \alpha\delta\mu\dot{\varphi} + 4\left(1 + \frac{1}{\mu}\right) \varphi^2 + \mu\delta\varphi = 0.$$

2.11. $L_{11} : v = \left(x_1^2 + x_2^2 + x_3^2\right) \varphi(\omega),\ \omega = \alpha\delta\mu x_0 - \dfrac{1}{2} \ln\left(x_1^2 + x_2^2 + x_3^2\right),$

$$\varphi\ddot{\varphi} - \left(5 + \frac{4}{\mu}\right) \varphi\dot{\varphi} + \frac{1}{\mu} \dot{\varphi}^2 - \alpha\delta\mu\dot{\varphi} + \left(6 + \frac{4}{\mu}\right) \varphi^2 + \delta\mu\varphi = 0.$$

For $\alpha = \frac{3}{2}$, $\mu = -\frac{2}{3}$, the reduced equation has the solution $\varphi = \frac{2}{3}\delta\omega + C$, and the corresponding solution of (2) is of the form

$$v = \left(x_1^2 + x_2^2 + x_3^2\right) \left[C - \frac{2}{3} x_0 - \frac{1}{3} \delta \ln\left(x_1^2 + x_2^2 + x_3^2\right)\right].$$

2.12. $L_{12} : v = \left(x_1^2 + x_2^2 + x_3^2\right) e^{\delta\mu x_0} \varphi(\omega),\ \omega = e^{\delta\mu x_0} - \dfrac{\alpha\delta\mu}{2} \ln\left(x_1^2 + x_2^2 + x_3^2\right),$

$$\alpha^2\mu^2 \varphi\ddot{\varphi} - (5\mu + 4)\alpha\delta\varphi\dot{\varphi} + \alpha^2\mu\dot{\varphi}^2 - \delta\mu\dot{\varphi} + \left(6 + \frac{4}{\mu}\right) \varphi^2 = 0.$$

If $\alpha = 0$, then $\varphi = \mu^2[C - (6\mu + 4)\,\delta\omega]^{-1}$, where $C \neq 0$ or $6\mu + 4 \neq 0$. Therefore,

$$v = \frac{\mu^2 \left(x_1^2 + x_2^2 + x_3^2\right) e^{\delta\mu x_0}}{C - (6\mu + 4)\delta e^{\delta\mu x_0}}.$$

For $\mu = -\frac{2}{3}$, $\alpha \neq 0$, the reduced equation has the solution

$$\int \frac{d\varphi}{C\varphi^{\frac{3}{2}} - \frac{3}{\alpha\delta}\varphi + \frac{1}{\alpha^2\delta}} = \omega + C'.$$

If $C = 0$, then

$$\varphi = \frac{1}{3\alpha} + Ae^{-\frac{3}{\alpha\delta}\omega},$$

where A is an arbitrary constant. In this case,

$$v = \frac{1}{3\alpha} \left(x_1^2 + x_2^2 + x_3^2\right) \exp\left(-\frac{2}{3}\delta x_0\right) + A \exp\left[-\frac{2}{3}\delta x_0 - \frac{3}{\alpha\delta} \exp\left(-\frac{2}{3}\delta x_0\right)\right].$$

3 Complementary reduction of equation (2) for $\mu = -\frac{4}{5}$ to ordinary differential equations

Let F be an I-maximal subalgebra of rank three of the algebra $A\tilde{E}(1) \oplus AC(3)$ and $P_0, D_1 \notin F$. If a projection of F onto $AC(3)$ is not conjugate to any subalgebra of the algebra $A\tilde{E}(3)$ under the group $\operatorname{Ad} AC(3)$, then F is conjugate under the group $\operatorname{Ad}(A\tilde{E}(1) \oplus AC(3))$ to one of the following subalgebras [6, 7]:

$F_1 = \langle P_1 + K_1, P_2 + K_2, J_{12}, K_3 - P_3 \rangle$;

$F_2 = \langle P_a + K_a, J_{ab} : a, b = 1, 2, 3 \rangle$;

$F_3 = \langle P_1 + K_1, P_2 + K_2, J_{12}, K_3 - P_3 + \alpha D_1 \rangle$ $(\alpha \in \mathbb{R},\ \alpha > 0)$;

$F_4 = \langle P_1 + K_1, P_2 + K_2, J_{12}, K_3 - P_3 + P_0 \rangle$;

$F_5 = \langle K_a - P_a, J_{ab} : a, b = 1, 2, 3 \rangle$.

3.1. $F_1 : v = \left[\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2\right] \varphi(\omega),\ \omega = x_0,\ \dot{\varphi} = -4\varphi^2 - \dfrac{4}{5}\delta\varphi.$

The general solution of the reduced equation is $\varphi = C\delta \left[e^{\frac{4}{5}\delta\omega} - 5C\right]^{-1}$, where C is an arbitrary constant. The corresponding invariant solution of equation (2) is of the form

$$v = C\delta \left[\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2\right] \left(e^{\frac{4}{5}\delta x_0} - 5C\right)^{-1}.$$

3.2. $F_2 : v = \left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 \varphi(\omega),\ \omega = x_0,\ \dot{\varphi} = -12\varphi^2 - \dfrac{4}{5}\delta\varphi.$

In this case, $\varphi = \frac{C\delta}{15} \left[e^{\frac{4}{5}\delta\omega} - C\right]^{-1}$, therefore,

$$v = \frac{C\delta}{15} \left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 \left(e^{\frac{4}{5}\delta x_0} - C\right)^{-1}.$$

3.3. $F_3 : v = \left[\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2\right] \varphi(\omega),\ \omega = \arctan \dfrac{x_1^2 + x_2^2 + x_3^2 - 1}{2x_3} - \dfrac{8}{5\alpha\delta} x_0,$

$$4\varphi\ddot{\varphi} - 5\dot{\varphi}^2 + \frac{8}{5\alpha\delta}\dot{\varphi} - 4\varphi^2 - \frac{4}{5}\delta\varphi = 0.$$

3.4. $F_4 : v = e^{-\frac{4}{5}\delta x_0} \left[\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2\right] \varphi(\omega),$

$$\omega = \arctan \frac{x_1^2 + x_2^2 + x_3^2 - 1}{2x_3} - \frac{5\delta}{2} e^{-\frac{4}{5}\delta x_0},$$

$$4\varphi\ddot{\varphi} - 5\dot{\varphi}^2 - 2\dot{\varphi} - 4\varphi^2 = 0.$$

3.5. $F_5 : v = \left(x_1^2 + x_2^2 + x_3^2 + 1\right)^2 \varphi(\omega),\ \omega = x_0,\ \dot{\varphi} = 12\varphi^2 - \dfrac{4}{5}\delta\varphi.$

Integrating this equation, we obtain $\varphi = \delta \left[15 - Ce^{\frac{4}{5}\delta\omega}\right]^{-1}$, and, therefore,

$$v = \delta \left(x_1^2 + x_2^2 + x_3^2 + 1\right)^2 \left(15 - Ce^{\frac{4}{5}\delta x_0}\right)^{-1}.$$

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 90–98.

The Exact Linearization of Some Classes of Ordinary Differential Equations for Order $n > 2$

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"When integrating the differential equations the most difficult task is the introduction of suitable variables, which may not be found by the general rule. That's why we have to go in reverse order. After finding a splendid substitution, we should look for such problems, where it might be adopted with success."

Karl Jacobi

The method of exact linearization nonlinear ordinary differential equations (ODE) of order n suggested by one of the authors is demonstrated in [1, 2]. This method is based on the factorization of nonlinear ODE through the first order nonlinear differential the operators, and is also based on using both point and nonpoint, local and nonlocal transformations. Exact linearization of autonomous the third, the fourth and the fifth orders ODE is presented in this paper. For the first time general form of autonomous fourth [3] and fifth order equations, admitting exact linearization with using nonpoint transformation is found by second of the authors. We obtain the formulas in quadratures for finding general and partial solutions of investigated classes of equations. For the realization of transformations and construction of the considered equations we used the computer algebra system MAPLE.

1 Preliminary informations

The following result plays an important role in this paper:

Proposition 1.1 [1]. *The equation*

$$y^{(n)} = f\left(y, y', \ldots, y^{(n-1)}\right), \qquad (') = d/dx \tag{1.1}$$

by means of the invertible transformation

$$y = v(y)z, \qquad dt = u(y)dx, \tag{1.2}$$

where $v(y)$ and $u(y)$ are smooth functions in domain (x, y), reduces to linear autonomous form

$$z^{(n)}(t) + \sum_{k=1}^{n} \binom{n}{k} b_k z^{(n-k)}(t) + c = 0, \qquad b_k, c = \text{const}, \tag{1.3}$$

if and only if (1.1) admits the factorization of the form

$$\prod_{k=1}^{n} \left[\frac{1}{u} D - \frac{v^*}{vu} y' - r_k\right] y + cv = 0, \qquad D = d/dx, \quad (*) = d/dy \tag{1.4}$$

via nonlinear first order differential operators (commutative factorization) or

$$\prod_{k=n}^{1}\left[D-\left(\frac{v^*}{v}+(k-1)\frac{u^*}{u}\right)y'-r_k u\right]y+cu^n v=0 \tag{1.5}$$

(noncommutative factorization), where r_k are distinct roots of the characteristic equation

$$r^n+\sum_{k=1}^{n}\binom{n}{k}b_k r^{n-k}=0. \tag{1.6}$$

In what follows we shall sequentially consider the corresponding class of ODE for the cases $n=3$, $n=4$, and $n=5$. Although first two cases are known [1–3], they are considered here because they form the base of investigation of 5th order ODE.

For finding the transformation (1.2) we use also the proposition about the structure of basis of solutions of linear ODE with variable coefficients.

Proposition 1.2 [2]. *The second order nonlinear nonautonomous equation*

$$v^{**}-\frac{2}{v}v^{*2}+\left(\frac{2}{y}-\frac{n^2-n+2}{2n}\frac{u^*}{u}-f\right)v^*+\left(\frac{n^2-n+2}{2n}\frac{u^*}{u}+f\right)\frac{1}{y}v=0$$

has the solution

$$v(y)=y\left(\alpha+\beta\int u^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)dy\right)^{-1}.$$

2 Linearization of the third order equations

Let us consider the equation

$$y'''=F\left(y,y',y''\right). \tag{2.1}$$

By virtue of Proposition 1.1, it can be reduced to the linear ODE

$$\dddot{z}+3b_1\ddot{z}+3b_2\dot{z}+b_3z+c=0 \tag{2.2}$$

by transformation (1.2), if and only if, it admits the factorization (up to the term cu^3v):

$$\left[D-\left(\frac{v^*}{v}+2\frac{u^*}{u}\right)y'-r_3u\right]\left[D-\left(\frac{v^*}{v}+\frac{u^*}{u}\right)y'-r_2u\right]\left[D-\frac{v^*}{v}y'-r_1u\right]y+cvu^3=0. \tag{2.3}$$

We obtain the differential equation

$$\begin{aligned}
&y'''\left(1-\frac{v^*}{v}y\right)-y''y'\left[3\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(6\frac{v^*}{v}+4\frac{u^*}{u}\right)\right]+y'^3\left[\left(1-\frac{v^*}{v}y\right)\right.\\
&\quad\times\left(6\frac{{v^*}^2}{v^2}+6\frac{u^*v^*}{uv}+3\frac{{u^*}^2}{u^2}-2\frac{v^{**}}{v}-\frac{u^{**}}{u}\right)+4\frac{v^*v^{**}}{v^2}y+3\frac{u^*v^{**}}{uv}y-\frac{v^{**}}{v}-\left.\frac{v^{***}}{v}y\right]\\
&\quad-u(r_1+r_2+r_3)\left\{y''\left(1-\frac{v^*}{v}y\right)-y'^2\left[\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(2\frac{v^*}{v}+\frac{u^*}{u}\right)\right]\right\}\\
&\quad+(r_1r_2+r_1r_3+r_2r_3)u^2y'\left(1-\frac{v^*}{v}y\right)-r_1r_2r_3u^3y+cvu^3=0.
\end{aligned} \tag{2.4}$$

For the sake of determination of the explicit form of transformation (1.2), let us introduce the notation

$$-\left[3\frac{v^{**}}{v}y+\left(6\frac{v^*}{v}+4\frac{u^*}{u}\right)\left(1-\frac{v^*}{v}y\right)\right]=3f(y)\left(1-\frac{v^*}{v}y\right), \tag{2.5}$$

from where we'll obtain the equation for the transformation factor $v(y)$ in (1.2):

$$v^{**}-2\frac{v^{*2}}{v}+\left(\frac{2}{y}-\frac{4}{3}\frac{u^*}{u}-f\right)v^*+\frac{1}{y}\left(\frac{4}{3}\frac{u^*}{u}+f\right)v=0, \qquad f=f(y). \tag{2.6}$$

Proposition 2.1. *Equation (2.6) has general solution, expressed through quadratures*

$$v(y)=\frac{y}{\alpha+\beta\displaystyle\int u^{4/3}\exp\left(\int f dy\right)dy}, \tag{2.7}$$

where α, β are integration constants.

The formulas (2.6) and (2.7) could be obtained from Proposition 1.2 and [2].

Proposition 2.2. *Equation (2.1) can be linearized by means of the transformation of type (1.2), if and only if it has the following form:*

$$\begin{gathered}y'''+3fy'y''+\left(\frac{1}{3}\frac{\varphi^{**}}{\varphi}-\frac{5}{9}\frac{\varphi^{*2}}{\varphi^2}-\frac{1}{3}f\frac{\varphi^*}{\varphi}+f^2+f^*\right)y'^3+3b_1\varphi y''+b_1\varphi\left(3f+\frac{\varphi^*}{\varphi}\right)y'^2\\ +3b_2\varphi^2y'+\varphi^{5/3}\left[b_3\exp\left(-\int f dy\right)\int\varphi^{4/3}\exp\left(\int f dy\right)dy+\frac{c}{\beta}\right]=0,\end{gathered} \tag{2.8}$$

Such an equation can be linearized by means of the transformation

$$z=\beta\int\varphi^{4/3}\exp\left(\int f dy\right)dy, \qquad dt=\varphi dx, \tag{2.9}$$

where β is normalizing factor.

For $c=0$ we shall obtain the one-parameter solutions sets

$$\frac{\displaystyle\int\varphi^{1/3}\exp\left(\int f dy\right)dy}{\displaystyle\int\varphi^{4/3}\exp\left(\int f dy\right)dy}=r_kx+C, \tag{2.10}$$

where r_k are the simple roots of characteristic equation (2.11):

$$r^3+3b_1r^2+3b_2r+b_3=0. \tag{2.11}$$

Transforming (2.3) with the aid of (2.4), (2.7) (where $\alpha=0$), assuming $u=\varphi(y)$ and assuming that $r_k,\ k=1,2,3$ are the distinct roots of the characteristic equation (2.11), we shall arrive at (2.8).

The special case of equation (2.8) is obtained for $\varphi=\exp\left(-\frac{3}{4}\int f dy\right)$.

Remark 1. Thus, the equations of type

$$y'''+f(y)y'y''+\varphi(y)y''+\sum_{k=0}^{3}f_k(y)y'^k=0 \tag{2.12}$$

may be tested by the method of exact linearization.

Example 1. The equation

$$y''' - \frac{y'y''}{y} + 3b_1yy'' + 3b_2y^2y' + \frac{1}{2}b_3y^4 + \frac{c}{2}y^2 = 0 \tag{2.13}$$

by the substitution

$$z = y^2, \qquad dt = ydx \tag{2.14}$$

is reducible to the linear equation of the form (2.2) and has one-parameter set of solutions of the form

$$y = -2/(r_kx + C), \tag{2.15}$$

where r_k are simple roots of the characteristic equation (2.11).

Example 2. The elementary nontrivial system of hydrodynamic type (so-called the triplet). It was shown in [4] that corresponding system can be transformed into the set of Euler equations (the Euler–Poinsot case of a rigid body dynamics), which can be written in in terms of energy variables as

$$\dot u_1 = au_2u_3, \qquad \dot u_2 = bu_3u_1, \qquad \dot u_3 = cu_1u_2, \qquad a + b + c = 0. \tag{2.16}$$

Eliminating the variables of the coupled system (2.16), we obtained decoupled system

$$\begin{aligned} &\dddot u_1 - \frac{1}{u_1}\dot u_1\ddot u_1 - 4bcu_1^2\dot u_1 = 0, \qquad \dddot u_2 - \frac{1}{u_2}\dot u_2\ddot u_2 - 4cau_2^2\dot u_2 = 0,\\ &\dddot u_3 - \frac{1}{u_3}\dot u_3\ddot u_3 - 4bcu_3^2\dot u_3 = 0. \end{aligned} \tag{2.17}$$

The equations (2.17) are factorizables:

$$\left(D_t - \frac{\dot u_i}{u_i} - r_{3i}u_i\right)(D_t - r_{2i}u_i)\left(D_t - \frac{\dot u_i}{u_i} - r_{3i}u_i\right)u_i = 0, \qquad i = \overline{1,3},$$

where r_{ki}, $k = \overline{1,3}$, are roots of corresponding characteristic equations. After the transformations $u_i^2 = z_i$, $d\tau_i = u_idt$ the system (2) reduces to the linear one (see also [1]):

$$z_1'''(\tau_1) - 4bcz_1'(\tau_1) = 0, \qquad z_2'''(\tau_2) - 4acz_2'(\tau_2) = 0, \qquad z_3'''(\tau_3) - 4abz_3'(\tau_3) = 0.$$

3 Linearization of the fourth-order equations

Let us consider the autonomous nonlinear fourth–order differential equation

$$y^{\mathrm{iv}} = F\left(y, y', y'', y'''\right). \tag{3.1}$$

Equation (3.1) could be linearized by the transformation (1.2) to the form

$$\ddddot z + 4b_1 \dddot z + 6b_2\ddot z + 4b_3\dot z + b_4z + c = 0, \tag{3.2}$$

according to Proposition 1.1, if and only if it admits the factorization (up to the term cvu^4)

$$\begin{aligned} &\left[D - \left(\frac{v^*}{v} + 3\frac{u^*}{u}\right)y' - r_4u\right]\left[D - \left(\frac{v^*}{v} + 2\frac{u^*}{u}\right)y' - r_3u\right]\\ &\quad\times\left[D - \left(\frac{v^*}{v} + \frac{u^*}{u}\right)y' - r_2u\right]\left[D - \frac{v^*}{v}y' - r_1u\right]y + cvu^4 = 0. \end{aligned} \tag{3.3}$$

Proposition 3.1. *If equation (3.1) can be linearized by means of (1.2), then it admits the factorization*

$$\begin{aligned}
&\left(1-\frac{v^*}{v}y\right)y^{\rm iv} - y'y'''\left[4\frac{v^{**}}{v}y + \left(1-\frac{v^*}{v}y\right)\left(8\frac{v^*}{v}+7\frac{u^*}{u}\right)\right] - y''^2\left[3\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\right.\\
&\left.\times\left(6\frac{v^*}{v}+4\frac{u^*}{u}\right)\right] + y'^2y''\left[\left(1-\frac{v^*}{v}y\right)\left(36\frac{{v^*}^2}{v^2}-18\frac{v^{**}}{v}+44\frac{u^*v^*}{uv}+25\frac{{u^*}^2}{u^2}+7\frac{u^{**}}{u}\right)\right.\\
&\left.+18\frac{v^{**}v^*}{v^2}y+22\frac{u^*v^{**}}{uv}y-6\frac{v^{***}}{v}y\right] - y'^4\left[\frac{v^{****}}{v}y-\left(2\frac{v^{***}}{v}y-6\frac{v^{**}v^*}{v^2}y\right)\left(2\frac{v^*}{v}+3\frac{u^*}{u}\right)\right.\\
&-6\frac{{v^{**}}^2}{v^2}y+15\frac{v^{**}{u^*}^2}{vu^2}y-4\frac{v^{**}u^{**}}{uv}y+\left(1-\frac{v^*}{v}y\right)\left(4\frac{v^{***}}{v}-24\frac{v^{**}v^*}{v^2}+24\frac{{v^*}^3}{v^3}-18\frac{u^*v^{**}}{uv}\right.\\
&\left.\left.+36\frac{u^*{v^*}^2}{uv^2}+30\frac{{u^*}^2v^*}{u^2v}-8\frac{u^{**}v^*}{uv}-10\frac{u^*u^{**}}{u^2}+15\frac{{u^*}^3}{u^3}+\frac{u^{***}}{u}\right)\right] + 4b_1u\left\{y'''\left(1-\frac{v^*}{v}y\right)\right.\\
&-y'^3\left[\frac{v^{***}}{v}y-3\frac{v^{**}u^*}{uv}y-3\frac{v^{**}v^*}{v^2}y+\left(1-\frac{v^*}{v}y\right)\left(\frac{u^{**}}{u}-6\frac{u^*v^*}{vu}+3\frac{v^{**}}{v}-6\frac{u{v^*}^2}{v^2}-3\frac{{u^*}^2}{u^2}\right)\right]\\
&\left.-y'y''\left[3\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(6\frac{v^*}{v}+4\frac{u^*}{u}\right)\right]\right\}+6b_2u^2\left\{y''\left(1-\frac{v^*}{v}y\right)\right.\\
&\left.-y'^2\left[\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(\frac{u^*}{u}+2\frac{v^*}{v}\right)\right]\right\}+4b_3u^3y'\left(1-\frac{v^*}{v}y\right)+b_4u^4y+cvu^4=0.
\end{aligned}\tag{3.4}$$

Applying the differential operator $\left[D-\left(\frac{v^*}{v}+3\frac{u^*}{u}\right)y'-r_4u\right]$ to (2.4) and adding to the obtained expression term cvu^4, we shall arrive at the formula (3.4), where r_k satisfy the characteristic equation

$$r^4+4b_1r^3+6b_2r^2+4b_3r+b_4=0. \tag{3.5}$$

Introducing the notation $4\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(8\frac{v^*}{v}+7\frac{u^*}{u}\right) = -4f(y)\left(1-\frac{v^*}{v}y\right)$, we shall arrive at the equation

$$v^{**}-\frac{2}{v}{v^*}^2+\left(\frac{2}{y}-\frac{7}{4}\frac{u^*}{u}-f\right)v^*+\left(\frac{7}{4}\frac{u^*}{u}+f\right)\frac{1}{y}v=0, \qquad f=f(y). \tag{3.6}$$

Proposition 3.2. *Equation (3.6) has general solution of the form*

$$v(y)=\frac{y}{\alpha+\beta\int u^{7/4}\exp\left(\int f dy\right)dy}, \tag{3.7}$$

where α, β are arbitrary constants.

The formulas (3.6) and (3.7) could be obtained from Proposition 1.2 and [2].

Proposition 3.3. *The equation (3.1) can be linearized by means of transformation (1.2) if and only if it has the following form:*

$$y^{(\rm iv)}+4f(y)y'y'''+y''^2\left(\frac{5}{4}\frac{\varphi^*}{\varphi}+3f\right)+y'^2y''\left(\frac{7}{2}\frac{\varphi^{**}}{\varphi}-\frac{45}{8}\frac{{\varphi^*}^2}{\varphi^2}-\frac{\varphi^*}{\varphi}f+6\left(f^2+f^*\right)\right)$$

$$+y'^4\left(\frac{195}{64}\frac{\varphi^{*^3}}{\varphi^3}-\frac{57}{16}\frac{\varphi^*\varphi^{**}}{\varphi^2}+\frac{3}{4}\frac{\varphi^{***}}{\varphi}+\left(\frac{5}{4}\frac{\varphi^{**}}{\varphi}-\frac{33}{16}\frac{\varphi^{*^2}}{\varphi^2}\right)f+\frac{21}{4}\frac{\varphi^*}{\varphi}\left(f^2+f^*\right)\right.$$
$$\left.+f^3+3ff^*+f^{**}\right)+4b_1\varphi\left\{y'''+y'y''\left(3f+\frac{5}{4}\frac{\varphi^*}{\varphi}\right)-y'^3\left(\frac{15}{16}\frac{\varphi^{*^2}}{\varphi^2}-\frac{3}{4}\frac{\varphi^{**}}{\varphi}\right.\right.$$
$$\left.\left.-\frac{1}{2}\frac{\varphi^*}{\varphi}f-f^2-f^*\right)\right\}+6b_2\varphi^2\left\{y''+y'^2\left(\frac{3}{4}\frac{\varphi^*}{\varphi}+f\right)\right\}+4b_3\varphi^3y' \tag{3.8}$$
$$+\varphi^{\frac{9}{4}}e^{-\int f dy}\left[b_4\int\varphi^{\frac{7}{4}}e^{\int f dy}dy+\frac{c}{\beta}\right]=0.$$

It can be reduced to (3.2) by means of the transformation

$$z=\beta\int\varphi^{7/4}\exp\left(\int f(y)dy\right)dy,\qquad dt=\varphi(y)dx. \tag{3.9}$$

Transforming (3.3) with the aid of (3.4), (3.6) (where $\alpha=0$, assuming that $\varphi=u(y)$ and that r_k, $k=\overline{1,4}$ are distinct roots of the characteristic equation (3.5), we shall arrive at (3.8).

Equation (3.8) is recovered as the particular case of the above for $\varphi=\exp\left(-\frac{4}{7}\int f(y)dy\right)$.

Remark 2. Thus, the equations of the type

$$y^{\text{iv}}+fy'y'''+\varphi_1y''^2+\varphi_2y'^2y''+f_4y'^4+\varphi y'''+\varphi_3y'y''+\varphi_4y''+f_3y'^3+f_2y'^2+f_1y'+f_0=0$$

may be tested by the method of exact linearization.

Example 3. The equation

$$y^{\text{iv}}-\frac{3}{y}y'y'''-\frac{1}{y}y''^2+\frac{3}{y^2}y'^2y''+4b_1yy'''-4b_1y'y''+6b_2y^2y''+4b_3y^3y'+\frac{1}{2}b_4y^5+\frac{1}{2}cy^4=0$$

is reduced to (3.2) by means of the substitution (2.14) and admits for $c=0$ one-parameter set of solutions (2.15), where r_k are different characteristic roots of the equation (3.5).

4 Linearization of the fifth-order equation

Let us consider the autonomous nonlinear fifth-order differential equation

$$y^{\text{v}}=F\left(y,y',y'',y''',y^{\text{iv}}\right). \tag{4.1}$$

Equation (4.1) can be reduced by means of transformation (1.2) to the linear equation of the form

$$z^{(\text{v})}(t)+5b_1z^{(\text{iv})}(t)+10b_2z'''(t)+10b_3z''(t)+5b_4z'(t)+b_5z(t)+c=0, \tag{4.2}$$

by virtue of Proposition 1.1 if and only if it admits the factorization (up to the term cvu^5)

$$\left[D-\left(\frac{v^*}{v}+4\frac{u^*}{u}\right)y'-r_5u\right]\left[D-\left(\frac{v^*}{v}+3\frac{u^*}{u}\right)y'-r_4u\right]\left[D-\left(\frac{v^*}{v}+2\frac{u^*}{u}\right)y'-r_3u\right]$$
$$\times\left[D-\left(\frac{v^*}{v}+\frac{u^*}{u}\right)y'-r_2u\right]\left[D-\frac{v^*}{v}y'-r_1u\right]y+cvu^5=0. \tag{4.3}$$

Proposition 4.1. *If equation (4.1) can be linearized with the aid of (1.2), then there exists the factorization*

$$
\begin{aligned}
&\Big(1-\frac{v^*}{v}y\Big)y^{\mathrm{v}}-y'y^{\mathrm{iv}}\Big[5\frac{v^{**}}{v}y+\Big(1-\frac{v^*}{v}y\Big)\Big(10\frac{v^*}{v}+11\frac{u^*}{u}\Big)\Big]-5y''y'''\Big[2\frac{v^{**}}{v}y\\
&+\Big(1-\frac{v^*}{v}y\Big)\Big(4\frac{v^*}{v}+3\frac{u^*}{u}\Big)\Big]+y'^2y'''\Big[\Big(1-\frac{v^*}{v}y\Big)\Big(60\frac{{v^*}^2}{v^2}-30\frac{v^{**}}{v}+90\frac{u^*v^*}{uv}+60\frac{{u^*}^2}{u^2}\\
&-14\frac{u^{**}}{u}\Big)+30\frac{v^{**}v^*}{v^2}y+45\frac{u^*v^{**}}{uv}y-10\frac{v^{***}}{v}y\Big]y'y''^2\Big[\Big(1-\frac{v^*}{v}y\Big)\Big(90\frac{{v^*}^2}{v^2}-45\frac{v^{**}}{v}\\
&+120\frac{u^*v^*}{uv}+70\frac{{u^*}^2}{u^2}-18\frac{u^{**}}{u}\Big)+45\frac{v^{**}v^*}{v^2}y+60\frac{u^*v^{**}}{uv}y-15\frac{v^{***}}{v}y\Big]-y'^3y''\Big[10\frac{v^{****}}{v}y\\
&-10\Big(\frac{v^{***}}{v}y-3\frac{v^{**}v^*}{v^2}y\Big)\Big(4\frac{v^*}{v}+7\frac{u^*}{u}\Big)-60\frac{{v^{**}}^2}{v^2}y+195\frac{v^{**}{u^*}^2}{vu^2}y-45\frac{v^{**}u^{**}}{uv}y\\
&+\Big(1-\frac{v^*}{v}y\Big)\Big(40\frac{v^{***}}{v}-240\frac{v^{**}v^*}{v^2}+240\frac{{v^*}^3}{v^3}-210\frac{u^*v^{**}}{uv}+420\frac{u^*{v^*}^2}{uv^2}+390\frac{{u^*}^2v^*}{u^2v}\\
&-90\frac{u^{**}v^*}{uv}-125\frac{u^*u^{**}}{u^2}+210\frac{{u^*}^3}{u^3}+11\frac{u^{***}}{u}\Big)\Big]-y'^5\Big[\frac{v^{*****}}{v}y-5\Big(\frac{v^{****}}{v}y-4\frac{v^{***}v^*}{v^2}y\\
&+12\frac{v^{**}{v^*}^2}{v^3}y-6\frac{v^{**}u^{**}}{uv}y\Big)\Big(\frac{v^*}{v}+2\frac{u^*}{u}\Big)+60\frac{{v^{**}}^2v^*}{v^3}y-20\frac{v^{***}v^{**}}{v^2}y-10\frac{v^{***}u^{**}}{vu}y\\
&-5\frac{v^{**}u^{***}}{vu}y+60\frac{{v^{**}}^2u^*}{v^2u}y+45\frac{v^{***}{u^*}^2}{u^2v}y-135\frac{v^{**}v^*{u^*}^2}{v^2u^2}y-105\frac{v^{**}{u^*}^3}{vu^3}y+\Big(1-\frac{v^*}{v}y\Big)\\
&\times\Big(5\frac{v^{****}}{v}-40\frac{v^{***}v^*}{v^2}-30\frac{{v^{**}}^2}{v^2}+180\frac{v^{**}{v^*}^2}{v^3}-120\frac{{v^*}^4}{v^4}-30\frac{u^{**}v^{**}}{uv}-40\frac{u^*v^{***}}{uv}\\
&+135\frac{v^{**}{u^*}^2}{u^2v}+240\frac{v^{**}v^*u^*}{uv^2}+60\frac{u^{**}{v^*}^2}{uv^2}-270\frac{{u^*}^2{v^*}^2}{u^2v^2}+120\frac{u^*u^{**}v^*}{u^2v}-210\frac{{u^*}^3v^*}{u^3v}\\
&-10\frac{u^{***}v^*}{uv}-240\frac{{v^*}^3u^*}{v^3u}-10\frac{{u^{**}}^2}{u^2}-15\frac{u^*u^{***}}{u^2}+105\frac{{u^*}^2u^{**}}{u^3}-105\frac{{u^*}^4}{u^4}+\frac{u^{****}}{u}\Big)\Big]\\
&+5b_1u\Big\{y^{(\mathrm{iv})}\Big(1-\frac{v^*}{v}y\Big)-y'y'''\Big[4\frac{v^{**}}{v}y+\Big(1-\frac{v^*}{v}y\Big)\Big(8\frac{v^*}{v}+7\frac{u^*}{u}\Big)\Big]+y'^2y''\Big[\Big(1-\frac{v^*}{v}y\Big)\\
&\times\Big(36\frac{{v^*}^2}{v^2}-18\frac{v^{**}}{v}+44\frac{u^*v^*}{uv}+25\frac{{u^*}^2}{u^2}-7\frac{u^{**}}{u}\Big)+18\frac{v^{**}v^*}{v^2}y+22\frac{u^*v^{**}}{uv}y-6\frac{v^{***}}{v}y\Big]\\
&-y''^2\Big[3\frac{v^{**}}{v}y+\Big(1-\frac{v^*}{v}y\Big)\Big(6\frac{v^*}{v}+4\frac{u^*}{u}\Big)\Big]-y'^4\Big[\frac{v^{****}}{v}y-\Big(2\frac{v^{***}}{v}y-6\frac{v^{**}v^*}{v^2}y\Big)\\
&\times\Big(2\frac{v^*}{v}+3\frac{u^*}{u}\Big)-6\frac{{v^{**}}^2}{v^2}y+15\frac{v^{**}{u^*}^2}{vu^2}y-4\frac{v^{**}u^{**}}{uv}y+\Big(1-\frac{v^*}{v}y\Big)\Big(4\frac{v^{***}}{v}-24\frac{v^{**}v^*}{v^2}\\
&+24\frac{{v^*}^3}{v^3}-18\frac{u^*v^{**}}{uv}+36\frac{u^*{v^*}^2}{uv^2}+30\frac{{u^*}^2v^*}{u^2v}-8\frac{u^{**}v^*}{uv}-10\frac{u^*u^{**}}{u^2}+15\frac{{u^*}^3}{u^3}+\frac{u^{***}}{u}\Big)\Big]\Big\}\\
&+10b_2u^2\Big\{y'''\Big(1-\frac{v^*}{v}y\Big)-y'y''\Big[3\frac{v^{**}}{v}y+\Big(1-\frac{v^*}{v}y\Big)\Big(6\frac{v^*}{v}+4\frac{u^*}{u}\Big)\Big]-y'^3\Big[\frac{v^{***}}{v}y-3\frac{v^{**}u^*}{uv}y
\end{aligned}
\tag{4.4}
$$

$$-3\frac{v^{**}v^*}{v^2}y+\left(1-\frac{v^*}{v}y\right)\left(3\frac{v^{**}}{v}-6\frac{v^{*^2}}{v^2}-6\frac{u^*v^*}{uv}-3\frac{u^{*^2}}{u^2}+\frac{u^{**}}{u}\right)\Bigg]\Bigg\}+10b_3u^3\left\{y''\left(1-\frac{v^*}{v}y\right)\right.$$
$$\left.-y'^2\left[\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(2\frac{v^*}{v}+\frac{u^*}{u}\right)\right]\right\}+5b_4u^4y'\left(1-\frac{v^*}{v}y\right)+b_5u^5y+cvu^5=0.$$

Applying the differential operator $\left[D-\left(\frac{v^*}{v}+4\frac{u^*}{u}\right)y'-r_5u\right]$ to (3.3) (up to the term cvu^4) and adding to the obtained expression the term cvu^5, we arrive at (4.4), where r_k satisfy characteristic equation

$$r^5+5b_1r^4+10b_2r^3+10b_3r^2+5b_4r+b_5=0. \tag{4.5}$$

Introducing the notation $\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(10\frac{v^*}{v}+11\frac{u^*}{u}\right)=-5f(y)\left(1-\frac{v^*}{v}y\right)$, we shall arrive at the equation

$$v^{**}-\frac{2}{v}v^{*^2}+\left(\frac{2}{y}-\frac{11}{5}\frac{u^*}{u}-f\right)v^*+\left(\frac{11}{5}\frac{u^*}{u}+f\right)\frac{1}{y}v=0, \qquad f=f(y). \tag{4.6}$$

Proposition 4.2. *Equation (4.6) has general solution of the form*

$$v(y)=\frac{y}{\alpha+\beta\int u^{11/5}\exp\left(\int f\,dy\right)dy}, \tag{4.7}$$

α, β are arbitrary constants.

The formulas (4.6) and (4.7) could be obtained from Proposition 1.2 and [2].

Equation (4.4)is the particular case of the above for $\varphi=\exp\left(-\frac{5}{11}\int f\,dy\right)$.

Proposition 4.3. *Equation (4.1) can be linearized by means of the transformation of the form (1.2) if and only if it has the form:*

$$y^{(\mathrm{v})}+5f(y)y'y^{(\mathrm{iv})}+y''y'''\left(7\frac{\varphi^*}{\varphi}+10f\right)+y'^2y'''\left[8\frac{\varphi^{**}}{\varphi}-\frac{63}{5}\frac{\varphi^{*^2}}{\varphi^2}-\frac{\varphi^*}{\varphi}f+10\left(f^2+f^*\right)\right]$$
$$+y'y''^2\left[15\frac{\varphi^{**}}{\varphi}-\frac{112}{5}\frac{\varphi^{*^2}}{\varphi^2}+6\frac{\varphi^*}{\varphi}f+15\left(f^2+f^*\right)\right]+y'^3y''\left[\frac{987}{25}\frac{\varphi^{*^3}}{\varphi^3}-\frac{244}{5}\frac{\varphi^*\varphi^{**}}{\varphi^2}+11\frac{\varphi^{***}}{\varphi}\right.$$
$$\left.+\left(21\frac{\varphi^{**}}{\varphi}-\frac{169}{5}\frac{\varphi^{*^2}}{\varphi^2}\right)f-4\frac{\varphi^*}{\varphi}\left(f^2+f^*\right)+10\left(f^3+3ff^*+f^{**}\right)\right]-y'^5\left[\frac{8064}{625}\frac{\varphi^{*^4}}{\varphi^4}\right.$$
$$-\frac{2946}{125}\frac{\varphi^{*^2}\varphi^{**}}{\varphi^3}+\frac{102}{25}\frac{\varphi^{**^2}}{\varphi^2}+\frac{186}{25}\frac{\varphi^*\varphi^{***}}{\varphi^2}-\frac{6}{5}\frac{\varphi^{****}}{\varphi}-\left(\frac{1989}{125}\frac{\varphi^{*^3}}{\varphi^3}-\frac{458}{25}\frac{\varphi^*\varphi^{**}}{\varphi^2}+\frac{19}{5}\frac{\varphi^{***}}{\varphi}\right)f \tag{4.8}$$
$$+\left(\frac{129}{25}\frac{\varphi^{*^2}}{\varphi^2}-\frac{16}{5}\frac{\varphi^{**}}{\varphi}\right)\left(f^2+f^*\right)+\frac{6}{5}\frac{\varphi^*}{\varphi}\left(f^3+3ff^*+f^{**}\right)-\left(f^4+6f^2f^*+3f^{*^2}+4ff^{**}\right.$$
$$\left.\left.+f^{***}\right)\right]+5b_1\varphi\left\{y^{(\mathrm{iv})}+y'y'''\left(4f+\frac{9}{5}\frac{\varphi^*}{\varphi}\right)+y'^2y''\left[\frac{31}{5}\frac{\varphi^{**}}{\varphi}-\frac{189}{25}\frac{\varphi^{*^2}}{\varphi^2}+\frac{22}{5}\frac{\varphi^*}{\varphi}f\right.\right.$$
$$\left.+6\left(f^2+f^*\right)\right]+y''^2\left(3f+\frac{13}{5}\frac{\varphi^*}{\varphi}\right)+y'^4\left[\frac{336}{125}\frac{\varphi^{*^3}}{\varphi^3}+\frac{118}{25}\frac{\varphi^*\varphi^{**}}{\varphi^2}+\frac{6}{5}\frac{\varphi^{***}}{\varphi}+\left(\frac{33}{5}\frac{\varphi^{**}}{\varphi}\right.\right.$$

$$-\frac{87}{25}\frac{\varphi^{*2}}{\varphi^2}\bigg) f + \frac{3}{5}\frac{\varphi^*}{\varphi}\left(f^2+f^*\right)+\left(f^3+3ff^*+f^{**}\right)\bigg]\bigg]\bigg\} + 10b_2\varphi^2\left\{y'''+y'y''\left(3f+\frac{13}{5}\frac{\varphi^*}{\varphi}\right)\right.$$

$$\left. +y'^3\left[\frac{6}{5}\frac{\varphi^{**}}{\varphi}-\frac{24}{25}\frac{\varphi^{*2}}{\varphi^2}+\frac{7}{5}\frac{\varphi^*}{\varphi}f+\left(f^2+f^*\right)\right]\right\} + 10b_3\varphi^3\left\{y''+y'^2\left(\frac{6}{5}\frac{\varphi^*}{\varphi}+f\right)\right\}+5b_4\varphi^4y'$$

$$+\varphi^{14/5}\exp\left(-\int f\,dy\right)\left[b_5\int\varphi^{11/5}\exp\left(\int f\,dy\right)dy+\frac{c}{\beta}\right]=0.$$

By means of the transformation

$$z=\beta\int\varphi^{11/5}\exp\left(\int f\,dy\right)dy,\qquad dt=\varphi dx \tag{4.9}$$

it reduces to the linear equation

$$z^v+5b_1z^{iv}+10b_2z'''+10b_3z''+5b_4z'+b_5z+c=0. \tag{4.10}$$

The equation (4.8) admits for $c=0$ one-parameter set of solutions

$$\frac{\displaystyle\int\varphi^{6/5}\exp\left(\int f\,dy\right)dy}{\displaystyle\int\varphi^{11/5}\exp\left(\int f\,dy\right)dy}=r_kx+C, \tag{4.11}$$

where r_k are roots of the characteristic equation

$$r^5+5b_1r^4+10b_2r^3+10b_3r^2+5b_4r+b_5=0. \tag{4.12}$$

Remark 4. Thus, the equations of the type

$$y^{\rm v}+fy'y^{\rm iv}+\varphi_1y''y'''+\varphi_2y'^2y'''+\varphi_3y'y''^2+\varphi_4y'^3y''+\varphi y^{\rm iv}+\varphi_5y'y'''+\varphi_6y''^2$$
$$+\varphi_7y'^2y''+\varphi_8y'''+\varphi_9y'y''+\varphi_{10}y''+\sum_{k=0}^{5}f_ky'^k=0$$

may be tested by the method of exact linearization.

Example 4. Equation

$$\begin{aligned}&y^{\rm v}-\frac{6}{y}y'y^{\rm iv}-\frac{5}{y}y''y'''+\frac{15}{y^2}y'^2y'''+\frac{10}{y^2}y'y''^2-\frac{15}{y^3}y'^3y''+5b_1yy^{\rm iv}-15b_1y'y'''-5b_1y''^2\\&+\frac{15}{y}b_1y'^2y''+10b_2y^2y'''-10b_2yy'y''+10b_3y^3y''+5b_4y^4y'+\frac{1}{2}b_5y^6+\frac{1}{2}cy^5=0\end{aligned} \tag{4.13}$$

by the substitution (2.14) is reduced to (4.2) and admits for $c=0$ one-parameter set of solutions (2.15), where r_k are distinct characteristic roots of the equation (4.5).

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 99–102.

On Galilei Invariance of Continuity Equation

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Classes of the nonlinear Schrödinger-type equations compatible with the Galilei relativity principle are obtained. Solutions of these equations satisfy the continuity equation.

The continuity equation is one of the most fundamental equations of quantum mechanics

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0. \tag{1}$$

Depending on definition of ρ (density) and $\vec{j} = (j^1, \ldots, j^n)$ (current), we can construct essentially different quantum mechanics with different equations of motion, which are distinct from classical linear Schrödinger, Klein–Gordon–Fock, and Dirac equations.

At the beginning we study a symmetry of the continuity equation considering $(\rho, \vec{j})$ as dependent variables related by (1).

Theorem 1 [1]. *The invariance algebra of equation (1) is an infinite-dimensional algebra with basis operators*

$$X = \xi^\mu(x)\frac{\partial}{\partial x_\mu} + \Big(a^{\mu\nu}(x)j^\nu + b^\mu(x)\Big)\frac{\partial}{\partial j^\mu}\ , \tag{2}$$

where $j^0 \equiv \rho$; $\xi^\mu(x)$ *are arbitrary smooth functions;* $x = (x_0 = t, x_1, x_2, \ldots, x_n) \in \mathbf{R}^{n+1}$; $a^{\mu\nu}(x) = \dfrac{\partial \xi^\mu}{\partial x_\nu} - \delta_{\mu\nu}\left(\dfrac{\partial \xi^i}{\partial x_i} + C\right)$; $C = \mathrm{const}$, $\delta_{\mu\nu}$ *is the Kronecker delta;* $\mu, \nu, i = 0, 1, \ldots, n$, $\Big(b^0(x), b^1(x), \ldots, b^n(x)\Big)$ *is an arbitrary solution of equation (1).*

Here and below we imply summation over repeated indices.

An infinite-dimensional algebra with basis operators (2) contains as subalgebras the generalized Galilei algebra

$$AG_2(1,n) = \langle P_\mu, J_{ab}, G_a, D^{(1)}, A\rangle \tag{3}$$

and the conformal algebra

$$AP_2(1,n) = AC(1,n) = \langle P_\mu, J_{ab}, J_{0a}, D^{(2)}, K_\mu\rangle. \tag{4}$$

We use the following designations in (3) and (4)

$$P_\mu = \partial_\mu, \qquad J_{ab} = x_a\partial_b - x_b\partial_a + j^a\partial_{j^b} - j^b\partial_{j^a} \quad (a < b),$$

$$G_a = x_0\partial_a + \rho\partial_{j^a}, \qquad J_{0a} = x_a\partial_0 + x_0\partial_a + j^a\partial_\rho + \rho\partial_{j^a},$$

$$D^{(1)} = 2x_0\partial_0 + x_a\partial_a - n\rho\partial_\rho - (n+1)j^a\partial_{j^a}, \qquad D^{(2)} = x_\mu\partial_\mu - n\rho\partial_\rho - nj^a\partial_{j^a},$$

$$A = x_0^2\partial_0 + x_0x_a\partial_a - nx_0\rho\partial_\rho + (x_a\rho - (n+1)x_0j^a)\partial_{j^a},$$

$$K_\mu = 2x_\mu D^{(2)} - x_\nu x^\nu g_{\mu i}\partial_i - 2x^\nu S_{\mu\nu}, \quad S_{\mu\nu} = g_{\mu i}j^\nu\partial_{j^i} - g_{\nu i}j^\mu\partial_{j^i},$$

$$g_{\mu\nu} = \begin{cases} 1, & \mu = \nu = 0 \\ -1, & \mu = \nu \neq 0 \\ 0, & \mu \neq \nu, \end{cases} \qquad \mu, \nu, i = 0, 1 \ldots, n; \quad a, b = 1, 2, \ldots, n.$$

Thus, the continuity equation satisfies the Galilei relativity principle as well as the Lorentz–Poincare–Einstein relativity principle and, depending on the definition of ρ and $\vec{j}$, we will come to different quantum mechanics.

Let us consider the scalar complex-valued wave functions and define ρ and $\vec{j}$ in the following way

$$\rho = f(uu^*), \qquad j^k = -\frac{1}{2} i g(uu^*) \left(\frac{\partial u}{\partial x_k} u^* - u \frac{\partial u^*}{\partial x_k} \right) + \frac{\partial \varphi(uu^*)}{\partial x_k}, \quad k = 1, 2, \ldots, n, \tag{5}$$

where f, g, φ are arbitrary smooth functions, $f \neq \text{const}$, $g \neq 0$. Without loss of generality, we assume that $f \equiv uu^*$.

Let us describe all functions $g(uu^*)$, $\varphi(uu^*)$ for continuity equation (1), (5) to be compatible with the Galilei relativity principle, defined by the following transformations:

$$t \to t' = t, \qquad x_a \to x'_a = x_a + v_a t.$$

Here we do not fix transformation rules for the wave function u.

If ρ and $\vec{j}$ are defined according to formula (5), then the continuity equation (1) is Galilei-invariant iff

$$\rho = uu^*, \qquad j^k = -\frac{1}{2} i \left(\frac{\partial u}{\partial x_k} u^* - u \frac{\partial u^*}{\partial x_k} \right) + \frac{\partial \varphi(uu^*)}{\partial x_k}, \qquad k = 1, 2, \ldots, n. \tag{6}$$

The corresponding generators of Galilei transformations have the form

$$G_a = x_0 \partial_a + i x_a \left(u \partial_u - u^* \partial_{u^*} \right), \quad a = 1, 2, \ldots, n.$$

If in (6)

$$\varphi = \lambda uu^*, \qquad \lambda = \text{const}, \tag{7}$$

then the continuity equation (1), (6), (7) coincides with the Fokker–Planck equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} + \lambda \Delta \rho = 0, \tag{8}$$

where

$$\rho = uu^*, \qquad j^k = -\frac{1}{2} i \left(\frac{\partial u}{\partial x_k} u^* - u \frac{\partial u^*}{\partial x_k} \right), \quad k = 1, 2, \ldots, n. \tag{9}$$

The continuity equation (1), (6), (7) was considered in [3, 5].

In [1] we investigated the symmetry properties of the nonlinear Schrödinger equation the following form

$$i u_0 + \frac{1}{2} \Delta u + i \frac{\Delta \varphi(uu^*)}{2uu^*} u = F\left(uu^*, (\vec{\nabla}(uu^*))^2, \Delta(uu^*) \right) u, \tag{10}$$

where F is an arbitrary real smooth function.

For the solutions of equation (10), equation (1), (6) is satisfied and therefore this equation is compatible with the Galilei relativity principle.

In terms of the phase and amplitude $\left(u = R \exp(i\Theta)\right)$, equation (10) has the form

$$\begin{aligned} & R_0 + R_k \Theta_k + \frac{1}{2} R \Delta \Theta + \frac{1}{2R} \Delta \varphi = 0, \\ & \Theta_0 + \frac{1}{2} \Theta_k^2 - \frac{1}{2R} \Delta R + F\left(R^2, \left(\vec{\nabla}\left(R^2\right) \right)^2, \Delta R^2 \right) = 0. \end{aligned} \tag{11}$$

Theorem 2 [1]. *The maximal invariance algebras for system (11) if $F = 0$ are the following:*

$$1. \quad \langle P_\mu, J_{ab}, Q, G_a, D\rangle \tag{12}$$

when φ is an arbitrary function;

$$2. \quad \langle P_\mu, J_{ab}, Q, G_a, D, I, A\rangle \tag{13}$$

when $\varphi = \lambda R^2$, $\lambda = \mathrm{const}$.

In (12) and (13) we use the following designations:

$$\begin{aligned}
&P_\mu = \partial_\mu, \qquad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a}, \qquad a < b,\\
&G_a = x_0\partial_{x_a} + ix_a\partial_\Theta, \quad Q = \partial_\Theta, \quad D = 2x_0\partial_{x_0} + x_a\partial_{x_a}, \quad I = R\partial_R,\\
&A = x_0^2\partial_{x_0} + x_0x_a\partial_{x_a} - \frac{n}{2}x_0R\partial_R + \frac{1}{2}x_a^2\partial_\Theta,\\
&\mu = 0, 1, \ldots, n; \qquad a, b = 1, 2, \ldots, n.
\end{aligned} \tag{14}$$

Algebra (13) coincides with the invariance algebra of the linear Schrödinger equation.

Corolarry. *System (11), (7) is invariant with respect to algebra (13) if*

$$F = R^{-1}\Delta R\, N\left(\frac{R\Delta R}{(\vec{\nabla}R)^2}\right),$$

where N is an arbitrary real smooth function.

Let us consider a more general system than (10)

$$iu_0 + \frac{1}{2}\Delta u = (F_1 + iF_2)u, \tag{15}$$

where F_1, F_2 are arbitrary real smooth functions,

$$F_m = F_m\left(uu^*, (\vec{\nabla}(uu^*))^2, \Delta(uu^*)\right)u, \qquad m = 1, 2. \tag{16}$$

The structure of functions F_1, F_2 may be described in form (16) by virtue of conditions for system (15) to be Galilei-invariant.

In terms of the phase and amplitude, equation (15) has the form

$$\begin{aligned}
&R_0 + R_k\Theta_k + \frac{1}{2}R\Delta\Theta - RF_2 = 0,\\
&\Theta_0 + \frac{1}{2}\Theta_k^2 - \frac{1}{2R}\Delta R + F_1 = 0,
\end{aligned} \tag{17}$$

where $F_m = F_m\left(R^2, \left(\vec{\nabla}\left(R^2\right)\right)^2, \Delta R^2\right)$, $\quad m = 1, 2$.

Theorem 3. *System (17) is invariant with respect to algebra (13) if it has the form*

$$\begin{aligned}
&R_0 + R_k\Theta_k + \frac{1}{2}R\Delta\Theta - \Delta R\, M\left(\frac{R\Delta R}{(\vec{\nabla}R)^2}\right) = 0,\\
&\Theta_0 + \frac{1}{2}\Theta_k^2 - \frac{1}{2R}\Delta R + \frac{\Delta R}{R}N\left(\frac{R\Delta R}{(\vec{\nabla}R)^2}\right) = 0,
\end{aligned} \tag{18}$$

where N, M are arbitrary real smooth functions.

System (18) written in terms of the wave function has the form

$$iu_0 + \frac{1}{2}\Delta u = \frac{\Delta|u|}{|u|}\left(N\left(\frac{|u|\Delta|u|}{(\vec{\nabla}|u|)^2}\right) + iM\left(\frac{|u|\Delta|u|}{(\vec{\nabla}|u|)^2}\right)\right)u. \tag{19}$$

Thus, equation (19) admits an invariance algebra which coincides with the invariance algebra of the linear Schrödinger equation with arbitrary functions M, N.

With certain particular M and N the symmetry of system (18) can be essentially extended. If in (18) $N = \frac{1}{2}$, then the second equation of the system (equation for the phase) will be the Hamilton–Jacobi equation [4].

Let us consider some forms of the continuity equation (1) for equation (19).

Case 1. If $M = 0$, then for solutions of equation (18) equation (1) holds true, where the density and current can be defined in the classical way (9).

Case 2. If $M\Delta R = -\lambda\Big(\Delta R + \frac{(\vec{\nabla}R)^2}{R}\Big)$, then for solutions of equation (19), the continuity equation (1), (6), (7) (or the Fokker–Planck equation (8), (9)) is valid.

Case 3. If M is arbitrary then for solutions of equation (19), the continuity equation is valid, where the density and current can be defined by the conditions

$$\rho = uu^*, \qquad \vec{\nabla}\cdot\vec{j} = \frac{\partial}{\partial x_k}\left(-\frac{1}{2}i\left(\frac{\partial u}{\partial x_k}u^* - u\frac{\partial u^*}{\partial x_k}\right)\right) - 2|u|\Delta|u|\; M\left(\frac{|u|\Delta|u|}{(\vec{\nabla}|u|)^2}\right).$$

Thus, we constructed wide classes of the nonlinear Schrödinger-type equations which are invariant with respect to algebra (13) (maximal invariance algebra of the linear Schrödinger equation) and for whose solutions the continuity equation (1) is valid.

The necessary and sufficient condition for the Lorentz invariance of the continuity equation for the electromagnetic field, where energy density and Poiting vectors depend on the vector fields $\vec{E}$, $\vec{H}$ has been obtained in [6].

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On Symmetry Reduction of the Five-Dimensional Dirac Equation

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Using the subgroup structure of the generalized Poincaré group $P(1,4)$, the symmetry reduction of the five-dimensional Dirac equation to systems of differential equations with fewer independent variables is done.

The Dirac equation in spaces of various dimensions has many applications (see, for example, [1–9]).

Let us consider the equation

$$\left(\gamma_k P^k - m\right)\psi(x) = 0, \qquad (1)$$

where $x = (x_0, x_1, x_2, x_3, x_4)$, $P_k = i\frac{\partial}{\partial x_k}$, $k = 0,1,2,3,4$; γ_k are 4×4 Dirac matrices. Equation (1) is invariant under the generalized Poincaré group $P(1,4)$. Continuous subgroups of the group $P(1,4)$ were in [10–14]. For all continuous subgroups of the group $P(1,4)$ invariants in five-dimensional Minkowski space are constructed. The majority of these invariants has been presented in [15, 16].

Following [17–23] and using the subgroup structure of the group $P(1,4)$ ansatzes which reduce equation (1) to systems of differential equations with fewer independent variables were constructed and the corresponding symmetry reduction has been obtained. Some of these results were presented in [24].

In the present paper we give some new results which are obtained on the basis of subgroup structure of the group $P(1,4)$ and invariants of its nonconjugate subgroups.

First we present some ansatzes which reduce the equation (1) to systems of ordinary differential equations and we give the corresponding systems of reduced equations.

1. $$\psi(x) = \exp\left[-\frac{1}{2}\gamma_4\gamma_0\ln(x_0+x_4) + \frac{1}{2}\gamma_2\gamma_1\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right]\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2};$$

$$i\left[\gamma_2\varphi' + \frac{1}{2}\left(\gamma_0+\gamma_4+\frac{1}{\omega}\gamma_2\right)\varphi\right] - m\varphi = 0.$$

2. $$\psi(x) = \exp\left[\frac{1}{2}(\gamma_2\gamma_1 + d\gamma_4\gamma_0)\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right]\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2}, \quad d>0;$$

$$i\left[\gamma_2\varphi' + \frac{1}{2\omega}(\gamma_2 + d\gamma_0\gamma_1\gamma_4)\varphi\right] - m\varphi = 0.$$

3. $$\psi(x) = \exp\left\{\frac{1}{2}\left[\gamma_2\gamma_1 + \varepsilon(\gamma_0+\gamma_4)\gamma_3\right]\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right\}\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2}, \ \varepsilon = \pm 1;$$

$$i\left[\gamma_2\varphi' + \frac{1}{2\omega}\left(\gamma_2 + \varepsilon\gamma_1(\gamma_0+\gamma_4)\gamma_3\right)\varphi\right] - m\varphi = 0.$$

4. $$\psi(x) = \exp\left[\frac{1}{2}\gamma_4\gamma_0\left(-\frac{1}{a}x_3 - \frac{d}{a}\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right) + \frac{1}{2}\gamma_2\gamma_1\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right]\varphi(\omega),$$

$$\omega = \left(x_1^2+x_2^2\right)^{1/2}, \quad a \neq 0, \quad d \neq 0;$$

$$i\left\{\gamma_2\varphi' - \frac{1}{2}\left[\frac{1}{\omega}\left(\frac{d}{a}\gamma_0\gamma_1\gamma_4 - \gamma_2\right) + \frac{1}{a}\gamma_0\gamma_3\gamma_4\right]\varphi\right\} - m\varphi = 0.$$

5. $$\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}} + \frac{1}{2}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2};$$

$$i\left(\gamma_2\varphi' + \frac{1}{2\omega}\gamma_2\varphi\right) - m\varphi = 0.$$

6. $$\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}} - \frac{1}{2}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2};$$

$$i\left(\gamma_2\varphi' + \frac{1}{2\omega}\gamma_2\varphi\right) - m\varphi = 0.$$

7. $$\psi(x) = \exp\left[-\frac{1}{2a}\gamma_4\gamma_0x_3 + \frac{1}{2}\gamma_2\gamma_1\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right]\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2}, \quad a \neq 0;$$

$$i\left\{\gamma_2\varphi' + \frac{1}{2}\left[\frac{1}{\omega}\gamma_2 - \frac{1}{a}\gamma_0\gamma_3\gamma_4\right]\varphi\right\} - m\varphi = 0.$$

8. $$\psi(x) = \exp\left[-\frac{1}{2e}(\gamma_2\gamma_1 + e\gamma_4\gamma_0)\ln(x_0+x_4)\right]\varphi(\omega), \quad \omega = \left(x_0^2-x_4^2\right)^{1/2}, \quad e > 0;$$

$$\frac{i}{2}\left\{\left[\omega(\gamma_0+\gamma_4) + \frac{1}{\omega}(\gamma_0-\gamma_4)\right]\varphi' + (\gamma_0+\gamma_4)(1 - \frac{1}{e}\gamma_2\gamma_1)\varphi\right\} - m\varphi = 0.$$

9. $$\psi(x) = \exp\left\{-\frac{1}{2}\gamma_4\gamma_0\ln(x_0+x_4) + \frac{1}{2\mu}\gamma_2\gamma_1[\alpha\ln(x_0+x_4) - x_3]\right\}\varphi(\omega),$$

$$\omega = \left(x_0^2-x_4^2\right)^{1/2}, \quad \alpha \neq 0, \quad \mu \neq 0;$$

$$\frac{i}{2}\left\{\left[\omega(\gamma_0+\gamma_4) + \frac{1}{\omega}(\gamma_0-\gamma_4)\right]\varphi' + \left[(\gamma_0+\gamma_4)\left(1 + \frac{\alpha}{\mu}\gamma_2\gamma_1\right) + \frac{1}{\mu}\gamma_1\gamma_2\gamma_3\right]\varphi\right\} - m\varphi = 0.$$

10. $$\psi(x) = \exp\left[-\frac{1}{2}\gamma_4\gamma_0\ln(x_0+x_4) - \frac{1}{2\alpha}\gamma_2\gamma_1x_3\right]\varphi(\omega), \quad \omega = \left(x_0^2-x_4^2\right)^{1/2}, \quad \alpha \neq 0;$$

$$\frac{i}{2}\left\{\left[\omega(\gamma_0+\gamma_4) + \frac{1}{\omega}(\gamma_0-\gamma_4)\right]\varphi' + \left[\gamma_0+\gamma_4 + \frac{1}{\alpha}\gamma_1\gamma_2\gamma_3\right]\varphi\right\} - m\varphi = 0.$$

11. $$\psi(x) = \exp\left(\frac{1}{2}\gamma_4\gamma_3\arcsin\frac{x_3}{\sqrt{x_3^2+x_4^2}} + \frac{1}{2d}\gamma_2\gamma_1x_0\right)\varphi(\omega), \quad \omega = \left(x_3^2+x_4^2\right)^{1/2}, \quad d \neq 0;$$

$$i\left[\gamma_4\varphi' + \frac{1}{2}\left(\frac{1}{d}\gamma_0\gamma_2\gamma_1 + \frac{1}{\omega}\gamma_4\right)\varphi\right] - m\varphi = 0.$$

12. $\psi(x) = \exp\left[\frac{1}{2}\left(\gamma_4\gamma_3 + \frac{e}{2}\gamma_2\gamma_1\right)\arcsin\frac{x_3}{\sqrt{x_3^2+x_4^2}}\right]\varphi(\omega), \quad \omega = \left(x_3^2+x_4^2\right)^{1/2}, \quad e \neq 0;$

$$i\left[\gamma_4\varphi' + \frac{1}{2\omega}\left(\gamma_4 + \frac{e}{2}\gamma_3\gamma_2\gamma_1\right)\varphi\right] - m\varphi = 0.$$

13. $\psi(x) = \exp\left(-\frac{1}{2\alpha}\gamma_2\gamma_1 x_3\right)\varphi(\omega), \quad \omega = x_0, \quad \alpha \neq 0;$

$$i\left(\gamma_0\varphi' + \frac{1}{2\alpha}\gamma_1\gamma_2\gamma_3\varphi\right) - m\varphi = 0.$$

14. $\psi(x) = \exp\left[-\frac{1}{2}(\gamma_0+\gamma_4)\gamma_3 x_1\right]\varphi(\omega), \quad \omega = x_2;$

$$i\left[\gamma_2\varphi' - \frac{1}{2}\gamma_1(\gamma_0+\gamma_4)\gamma_3\varphi\right] - m\varphi = 0.$$

15. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = x_2;$

$$i\gamma_2\varphi' - m\varphi = 0.$$

16. $\psi(x) = \exp\left[-\frac{1}{2}\gamma_4\gamma_0\ln(x_0+x_4)\right]\varphi(\omega), \quad \omega = x_3;$

$$i\left[\gamma_3\varphi' + \frac{1}{2}(\gamma_0+\gamma_4)\varphi\right] - m\varphi = 0.$$

17. $\psi(x) = \exp\left(-\frac{1}{2\tilde{a}_2}\gamma_4\gamma_0 x_2\right)\varphi(\omega), \quad \omega = x_3, \quad \tilde{a}_2 > 0;$

$$i\left(\gamma_3\varphi' - \frac{1}{2\tilde{a}_2}\gamma_0\gamma_2\gamma_4\varphi\right) - m\varphi = 0.$$

18. $\psi(x) = \exp\left(\frac{1}{d}\gamma_2\gamma_1 x_0\right)\varphi(\omega), \quad \omega = x_3, \quad d > 0;$

$$i\left(\gamma_3\varphi' - \frac{1}{d}\gamma_0\gamma_1\gamma_2\varphi\right) - m\varphi = 0.$$

19. $\psi(x) = \exp\left[-\frac{1}{2e}(\gamma_2\gamma_1 + e\gamma_4\gamma_0)\ln(x_0+x_4)\right]\varphi(\omega), \quad \omega = x_3, \quad e > 0;$

$$i\left[\gamma_3\varphi' + \frac{1}{2}(\gamma_0+\gamma_4)\left(1 - \frac{1}{e}\gamma_2\gamma_1\right)\varphi\right] - m\varphi = 0.$$

20. $\psi(x) = \exp\left(-\frac{1}{2d_3}\gamma_2\gamma_1 x_3\right)\varphi(\omega), \quad \omega = x_4, \quad d_3 \neq 0;$

$$i\left(\gamma_4\varphi' + \frac{1}{2d_3}\gamma_1\gamma_2\gamma_3\varphi\right) - m\varphi = 0.$$

21. $\psi(x) = \exp\left[-\frac{1}{2}\gamma_4\gamma_0\ln(x_0+x_4)\right]\varphi(\omega), \quad \omega = \ln(x_0+x_4) - \frac{x_3}{a_3}, \quad a_3 > 0;$

$$i\left[\left(\gamma_0+\gamma_4-\frac{1}{a_3}\gamma_3\right)\varphi' + \frac{1}{2}(\gamma_0+\gamma_4)\varphi\right] - m\varphi = 0.$$

22. $\psi(x) = \exp\left[\frac{1}{2}\gamma_4\gamma_0\ln(x_0-x_4)\right]\varphi(\omega), \quad \omega = \ln(x_0-x_4) + \frac{x_3}{a_3}, \quad a_3 > 0;$

$$i\left[\left(\gamma_0-\gamma_4+\frac{1}{a_3}\gamma_3\right)\varphi' + \frac{1}{2}(\gamma_0-\gamma_4)\varphi\right] - m\varphi = 0.$$

23. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = x_1 - \frac{1}{b}x_3 + \frac{1}{2b}(x_0+x_4)^2, \quad b \neq 0;$

$$i\gamma_1\varphi' - m\varphi = 0.$$

24. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\frac{\gamma_1x_1+(x_2-x_3)\gamma_2}{x_0+x_4}\right]\varphi(\omega), \quad \omega = x_0+x_4;$

$$i\left\{(\gamma_0+\gamma_4)\varphi' + \frac{1}{\omega}(\gamma_0+\gamma_4)\left[1-\frac{1}{2}\gamma_2\gamma_3\right]\varphi\right\} - m\varphi = 0.$$

25. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\gamma_3\frac{x_3-bx_1}{x_0+x_4}\right]\varphi(\omega), \quad \omega = x_0+x_4, \quad b \neq 0;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2\omega}(\gamma_0+\gamma_4)(1+b\gamma_1\gamma_3)\varphi\right] - m\varphi = 0.$$

26. $\psi(x) = \exp\left[\frac{1}{2\varepsilon}\left(\gamma_2\gamma_1+\varepsilon(\gamma_0+\gamma_4)\gamma_3\right)\frac{x_3}{x_0+x_4}\right]\varphi(\omega), \quad \omega = x_0+x_4, \quad \varepsilon = \pm 1;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2\varepsilon\omega}\left(\varepsilon(\gamma_0+\gamma_4)-\gamma_1\gamma_2\gamma_3\right)\varphi\right] - m\varphi = 0.$$

27. $\psi(x) = \exp\left[-\frac{1}{2}(\gamma_0+\gamma_4)\gamma_3x_2\right]\varphi(\omega), \quad \omega = x_0+x_4;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2}(\gamma_0+\gamma_4)\gamma_2\gamma_3\varphi\right] - m\varphi = 0.$$

28. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\left(x_0-x_4-\frac{x_3^2}{x_0+x_4}\right) + \frac{1}{2}(\gamma_0+\gamma_4)\gamma_3\frac{x_3}{x_0+x_4}\right]\varphi(\omega), \quad \omega = x_0+x_4;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2}\left(\frac{1}{\omega}(\gamma_0+\gamma_4)+\gamma_0-\gamma_4\right)\varphi\right] - m\varphi = 0.$$

29. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\left(x_4-x_0+\frac{x_3^2}{x_0+x_4}\right) + \frac{1}{2}(\gamma_0+\gamma_4)\gamma_3\frac{x_3}{x_0+x_4}\right]\varphi(\omega), \quad \omega = x_0+x_4;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2}\left(\frac{1}{\omega}(\gamma_0+\gamma_4)-\gamma_0+\gamma_4\right)\varphi\right] - m\varphi = 0.$$

30. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1(x_4 - x_0)\right]\varphi(\omega), \quad \omega = x_0 + x_4;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2}\gamma_2\gamma_1(\gamma_4-\gamma_0)\varphi\right] - m\varphi = 0.$$

31. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1(x_0 - x_4)\right]\varphi(\omega), \quad \omega = x_0 + x_4;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2}\gamma_2\gamma_1(\gamma_0-\gamma_4)\varphi\right] - m\varphi = 0.$$

32. $\psi(x) = \exp\left[\frac{1}{2\delta}(\gamma_0+\gamma_4)\left((\delta\gamma_2-\gamma_1)\frac{x_2}{x_0+x_4} - \frac{x_3}{2}\gamma_1\right)\right]\varphi(\omega), \quad \omega = x_0 + x_4, \quad \delta > 0;$

$$i(\gamma_0+\gamma_4)\left[\varphi' - \frac{1}{2\delta\omega}\left(\gamma_1(\gamma_2+\omega\gamma_3)-\delta\right)\varphi\right] - m\varphi = 0.$$

33. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\left(\frac{x_2}{x_0+x_4}\gamma_2 - x_3\gamma_1\right)\right]\varphi(\omega), \quad \omega = x_0 + x_4;$

$$i(\gamma_0+\gamma_4)\left[\varphi' + \frac{1}{2}\left(\frac{1}{\omega} - \gamma_1\gamma_3\right)\varphi\right] - m\varphi = 0.$$

34. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\left(\frac{\mu x_1 - x_3}{1+\mu(x_0+x_4)}\gamma_1 + \frac{x_2\gamma_2}{x_0+x_4}\right)\right]\varphi(\omega), \quad \omega = x_0 + x_4, \quad \mu > 0;$

$$i(\gamma_0+\gamma_4)\left\{\varphi' + \frac{1}{2}\left[\frac{1}{\mu\omega+1}(\mu - \gamma_1\gamma_3) + \frac{1}{\omega}\right]\varphi\right\} - m\varphi = 0.$$

35. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\left(\frac{2}{3\alpha^2}(x_0+x_4)^3 - \frac{2}{\alpha}x_3(x_0+x_4) + x_0 - x_4\right) + \right.$

$$\left. + \frac{1}{2\alpha}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = \alpha x_3 - \frac{1}{2}(x_0+x_4)^2, \quad \alpha \neq 0;$$

$$i\left\{\alpha\gamma_3\varphi' + \left[\left(\frac{\omega}{\alpha^2}+\frac{1}{2}\right)\gamma_0 + \left(\frac{\omega}{\alpha^2}-\frac{1}{2}\right)\gamma_4\right]\gamma_2\gamma_1\varphi\right\} - m\varphi = 0.$$

36. $ds\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\left(-\frac{2}{3\alpha^2}(x_0+x_4)^3 + \frac{2}{\alpha}x_3(x_0+x_4) - x_0 + x_4\right) + \right.$

$$\left. + \frac{1}{2\alpha}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = \alpha x_3 - \frac{1}{2}(x_0+x_4)^2, \quad \alpha \neq 0;$$

$$i\left\{\alpha\gamma_3\varphi' + \left[\left(\frac{\omega}{\alpha^2}-\frac{1}{2}\right)\gamma_0 + \left(\frac{\omega}{\alpha^2}+\frac{1}{2}\right)\gamma_4\right]\gamma_2\gamma_1\varphi\right\} - m\varphi = 0.$$

Let us note that the ansatzes (1)–(36) are obtained with the help of four-dimensional non-Abelian subalgebras of the Lie algebra of the group $P(1,4)$. The basis elements of these subalgebras commute if they belong to the Lie algebra of the group $SO(1,4)$.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 109–115.

On Symmetry Reduction and Some Exact Solutions of the Multidimensional Born–Infeld Equation

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Using the subgroup structure of the extended generalized Poincaré group $\tilde{P}(1,4)$, symmetry reduction of the multidimensional Born–Infeld equation to differential equations with fewer independent variables is made. Some classes of exact solutions of the equation under investigation are constructed.

The Born–Infeld equation in spaces of various dimensions has many applications (see, for example, [1–8]).

The symmetry properties of the multidimensional Born–Infeld equation were studied in [9–11]. In these works, multiparameter families of exact solutions were constructed using special ansatzes.

Let us consider the equation

$$\Box u\,(1-u_\nu u^\nu)+u^\mu u^\nu u_{\mu\nu}=0, \tag{1}$$

where $u=u(x)$, $x=(x_0,x_1,x_2,x_3)\in M(1,3)$, $u_\mu\equiv\frac{\partial u}{\partial x^\mu}$, $u_{\mu\nu}\equiv\frac{\partial^2 u}{\partial x^\mu\partial x^\nu}$, $u^\mu=g^{\mu\nu}u_\nu$, $g_{\mu\nu}=(1,-1,-1,-1)\delta_{\mu\nu}$, $\mu,\nu=0,1,2,3$, $\Box$ is the d'Alembert operator.

The symmetry group of equation (1) is the group $\tilde{P}(1,4)$ [9–11].

On the basis of the subgroup structure of the group $\tilde{P}(1,4)$ and the invariants of its subgroups [12], the symmetry reduction of the investigated equation to differential equations with fewer independent variables was done. In many cases the reduced equations are linear ODEs. Taking into account solutions of the reduced equations, we found multiparameter families of exact solutions of the equation under consideration. Below we only present ansatzes which reduce equation (1) to ordinary differential equations, and we list the ODEs obtained as well as some exact solutions of the Born–Infeld equation

1. $u^2=-x_2^2\varphi^2(\omega)+x_0^2,\quad \omega=\dfrac{x_3}{x_2},$

   $$\left(\varphi^2-\omega^2-1\right)\varphi\varphi''-\left(\varphi^2-\omega^2-1\right)\varphi'^2-2\varphi'^2-2\omega\varphi\varphi'=1;$$

2. $u^2=-\left(x_1^2+x_2^2\right)\varphi^2(\omega)+x_0^2,\quad \omega=\dfrac{x_3}{\left(x_1^2+x_2^2\right)^{1/2}};$

   $$\left(\varphi^2-\omega^2-1\right)\varphi\varphi''+\left(\left(3\varphi^2+1\right)\left(\omega^2+1\right)-\varphi\right)\varphi'^2-3\omega\left(\varphi^2+1\right)\varphi\varphi'+\varphi^4=1;$$

3. $u=-x_2^{\frac{\alpha+1}{\alpha}}\left(\varphi(\omega)\right)^{\frac{\alpha+1}{\alpha}}+x_0,\quad \omega=\dfrac{x_3}{x_2},\ \alpha\neq 0;$

   $$\varphi''=0;\qquad u=x_0-\left(c_1x_2+c_2x_3\right)^{\frac{\alpha+1}{\alpha}};$$

4. $u=-\varphi(\omega)-2\ln x_2+x_0,\quad \omega=\dfrac{x_3}{x_2};$

   $$2\varphi''+\varphi'^2=0;\qquad u=2c_1e^{-\frac{x_3}{2x_2}}-2\ln x_2+x_0-c_2;$$

5. $u = -\varphi(\omega) + x_0 - \ln\left(x_1^2 + x_2^2\right) + 2\alpha \operatorname{arctg} \dfrac{x_2}{x_1}, \quad \omega = \dfrac{x_3}{\left(x_1^2 + x_2^2\right)^{1/2}}, \ \alpha \geq 0;$

$4\left(\alpha^2\omega^2 + \alpha^2 + 1\right)\varphi'' - \omega\left(\omega^2 + 1\right)\varphi'^3 + 2\varphi'^2 = 0;$

6. $u = -\varphi(\omega) + x_0 + 2 \operatorname{arctg} \dfrac{x_2}{x_1}, \quad \omega = \dfrac{x_3}{\left(x_1^2 + x_2^2\right)^{1/2}};$

$4\left(\omega^2 + 1\right)\varphi'' - \omega\left(\omega^2 + 1\right)\varphi'^3 + 4\omega\varphi' = 0;$

7. $u^2 = x_0^2\varphi^2(\omega) - x_3^2, \quad \omega = \dfrac{\left(x_1^2 + x_2^2\right)^{1/2}}{x_0};$

$\omega\left(\varphi^2 + \omega^2 - 1\right)\varphi\varphi'' + \left(\omega^2 - 1\right)\varphi\varphi'^3 - \omega\left(2\varphi^2 + \omega^2 - 1\right)\varphi^2 +$
$+ \left(\varphi^2 + \omega\left(2\omega - 1\right)\right)\varphi\varphi' - \omega\left(\varphi^2 - 1\right) = 0;$

8. $u^2 = -\left(x_1^2 + x_2^2\right)\varphi^2(\omega) + x_0^2 - x_3^2, \quad \omega = \ln \dfrac{(x_0 - u)^2}{x_1^2 + x_2^2} - 2c \operatorname{arctg} \dfrac{x_2}{x_1}, \quad c > 0;$

$4\left(c^2\varphi^4 - \left(c^2 - 1\right)\varphi^2 - 1\right)\varphi\varphi'' - 8\left(c^2 + 1\right)\varphi^3\varphi'^3 + \left(\varphi^4 + 8c^2\varphi^2 - 4\right)\varphi'^4 -$
$- \left(6\varphi^4 + 4\varphi^2 - 10\right)\varphi\varphi' + \varphi^2\left(\varphi^4 + \varphi^2 - 2\right) = 0;$

9. $u^2 = -\left(x_1^2 + x_2^2\right)\varphi^2(\omega) + x_0^2 - x_3^2,$

$\omega = (1 + \alpha)\ln\left(x_0^2 - x_3^2 - u^2\right) - 2\alpha\ln\left(x_0 - u\right) - 2\beta \operatorname{arctg} \dfrac{x_2}{x_1}, \quad \alpha \neq 0, \quad \beta \geq 0;$

$4\left(\alpha^2 + \left(\beta^2 - \alpha^2 + 1\right)\varphi^2 - \beta^2\varphi^4\right)\varphi^2\varphi'' + 8\left(\alpha^2 + 1\right)\left(\beta^2\varphi^2 + \alpha^2 + \alpha - 2\right)\varphi'^3 -$
$-4\left(\left(2\beta^2 - \alpha^2 + 1\right)\varphi^2 + 3\alpha^2 - 2\alpha - 6\right)\varphi\varphi'^2 +$
$+2\left(\varphi^2 - \alpha - 6\right)\varphi^2\varphi' - \varphi^3\left(\varphi^4 + \varphi^2 - 2\right) = 0;$

10. $u^2 = -\left(x_1^2 + x_2^2\right)\varphi^2(\omega) + x_0^2 - x_3^2, \quad \omega = \ln\left(x_0^2 - x_3^2 - u^2\right) - 2\alpha \operatorname{arctg} \dfrac{x_2}{x_1}, \quad \alpha > 0;$

$4\left(\alpha^2 + 1 - \alpha^2\varphi^2\right)\varphi^4\varphi'' + 8\left(\alpha^2\varphi^2 - 2\right)\varphi'^3 - 4\left(\left(2\alpha^2 + 1\right)\varphi^2 - 6\right)\varphi\varphi'^2 +$
$+2\left(\varphi^2 - 6\right)\varphi^2\varphi' - \varphi^3\left(\varphi^4 + \varphi^2 - 2\right) = 0;$

11. $u^2 = -\left(x_1^2 + x_2^2\right)\varphi^2(\omega) + x_0^2 - x_3^2, \quad \omega = 2 \operatorname{arctg} \dfrac{x_2}{x_1} - \ln\left(x_1^2 + x_2^2\right);$

$4\varphi\left(\alpha^2\left(\varphi^2 - 1\right) - 1\right)\varphi'' - 8\left(\alpha^2 + 1\right)\varphi\varphi'^3 + 4\left(3\varphi^2 + 2\left(\alpha^2 + 1\right)\right)\varphi'^2 -$
$-2\left(3\varphi^2 + 2\right)\varphi\varphi' + \varphi^2\left(\varphi^2 + 1\right) = 2.$

The ansatzes (1)–(11) can be written in the following form: $h(u) = f(x)\varphi(\omega) + g(x)$, where $h(u)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function. $\omega = \omega(x, u)$ are one-dimensional invariants of subgroups of the group $\tilde{P}(1, 4)$.

12. $x_2\omega = x_3\varphi(\omega), \quad \omega = \dfrac{x_3}{(x_0 - u)},$

$\varphi'' = 0, \qquad u = x_0 + c_1x_2 + c_2x_3;$

13. $x_3\omega = \left(x_1^2 + x_2^2\right)^{1/2}\varphi(\omega), \quad \omega = \dfrac{\left(x_1^2 + x_2^2\right)^{1/2}}{u};$

$\omega\left(\varphi^2 + \omega^2 + 1\right)\varphi'' + \left(\omega^2 + 1\right)\varphi'^3 - 2\omega\varphi\varphi'^2 + \left(\varphi^2 + 1\right)\varphi' = 0;$

14. $x_3\omega = \left(x_1^2 + x_2^2\right)^{1/2} \varphi(\omega), \quad \omega = \dfrac{\left(x_1^2 + x_2^2\right)^{1/2}}{x_0 - u};$

$$\omega\varphi'' + \varphi'^3 + \varphi' = 0,$$

$$\frac{x_3}{x_0 - u} = \ln\left(\frac{2\sqrt{x_1^2 + x_2^2}}{x_0 - u} + 2\sqrt{\frac{x_1^2 + x_2^2}{(x_0 - u)^2} - 1}\right) + c;$$

15. $\dfrac{x_3}{\left(x_3\left(2x_0\omega - x_3\right) - x_1^2\omega^2\right)^{1/2}} = \varphi(\omega), \quad \omega = \dfrac{x_3}{x_0 - u},$

$$\left(\varphi^2\varphi^2 - 1\right)\varphi\varphi'' + \left(4 - 3\omega^2\varphi^2\right)\varphi'^2 + 2\omega\varphi\varphi^3\varphi' + \varphi^4 = 0;$$

16. $\omega\left(\dfrac{x_3}{\omega}\left(2x_0 - \dfrac{x_3}{\omega}\right) - x_1^2 - x_2^2\right)^{1/2} = x_3\varphi(\omega), \quad \omega = \dfrac{x_3}{x_0 - u};$

$$\left(\omega^2 - \varphi^2\right)\varphi^2\varphi'' + \left(\omega^2 - 3\varphi^2\right)\varphi'^2 + 4\omega\varphi\varphi' - 2\varphi^2 = 0;$$

17. $\left(x_0^2 - x_1^2 - x_2^2\right)^{1/2}\omega = x_3\varphi(\omega), \quad \omega = \dfrac{x_3}{u},$

$$\left(\varphi^2 - \omega^2 - 1\right)\varphi''\varphi + 2\left(\omega^2 + 1\right)\varphi'^2 - 4\omega\varphi\varphi' + 2\varphi^2 - 2 = 0;$$

18. $\ln\dfrac{x_3}{\omega} + \dfrac{x_1\omega}{x_3} = \varphi(\omega), \quad \omega = \dfrac{x_3}{x_0 - u};$

$$\varphi'' = 0; \qquad \frac{c_1x_3 - x_1}{x_0 - u} - \ln(x_0 - u) + c_2 = 0;$$

19. $\ln x_3^2\omega - \dfrac{x_3^2}{\omega} = \varphi(\omega) - 2x_0, \quad \omega = \dfrac{x_3^2}{x_0 - u};$

$$4\omega(\omega + 1)\varphi'' - 2\omega\varphi'^2 + 2(\omega + 1)\varphi' = -1;$$

20. $\dfrac{x_3}{\omega x_2 - x_1} = \varphi(\omega), \quad \omega = x_0 - u;$

$$\varphi = 0; \qquad u = x_0 - \frac{x_1 + x_3}{x_2};$$

21. $\ln\dfrac{x_3^2}{\omega} - \dfrac{x_1^2\omega}{x_3^2} - \dfrac{x_3^2}{\omega} = \varphi(\omega) - 2x_0, \quad \omega = \dfrac{x_3^2}{x_0 - u};$

$$\omega(\omega + 4)\varphi'' - 4\omega\varphi'^2 + 2(2\omega + 1)\varphi' = -3;$$

22. $\dfrac{x_3^2}{(\omega + 1)\left(\omega(2x_0 - \omega) - x_1^2\right) - \omega x_2^2} = \varphi(\omega), \quad \omega = x_0 - u;$

$$(\omega(\omega + 1))^2\left(\varphi\varphi'' - 2\varphi'^2\right) + 4\omega(\omega + 1)(2\omega + 1)\varphi\varphi' + 2\omega(\omega + 1)\varphi' -$$
$$-2\left(7\omega^2 + 7\omega + 2\right)\varphi^2 + 6(2\omega + 1)\varphi = 0;$$

23. $\ln\dfrac{x_3^2}{\omega} - \dfrac{\left(x_1^2 + x_2^2\right)\omega}{x_3^2} - \dfrac{x_3^2}{\omega} = \varphi(\omega) - 2x_0, \quad \omega = \dfrac{x_3^2}{x_0 - u};$

$$\omega(\omega + 4)\varphi'' - 6\omega\varphi'^2 + 2(3\omega + 1)\varphi' = -5;$$

24. $x_3\omega = \left(x_1^2+x_2^2\right)^{1/2}\varphi(\omega) - \left(x_1^2+x_2^2\right)^{1/2}\operatorname{arctg}\dfrac{x_2}{x_1}, \quad \omega = \dfrac{\left(x_1^2+x_2^2\right)^{1/2}}{x_0-u};$

$\omega\left(\omega^2+1\right)\varphi'' + \omega^2\varphi'^3 + \left(\omega^2+2\right)\varphi' = 0;$

$$\frac{x_3}{x_0-u} + \operatorname{arctg}\frac{x_2}{x_1} = i\left(\ln\frac{\sqrt{x_1^2+x_2^2}}{\sqrt{x_1^2+x_2^2+(x_0-u)^2}+x_0-u} + \frac{\sqrt{x_1^2+x_2^2+(x_0-u)^2}}{x_0-u} + c\right);$$

25. $\left(2x_0\omega\left(x_1^2+x_2^2\right)^{1/2} - x_3^2\omega^2 - x_1^2 - x_2^2\right)^{1/2} = \left(x_1^2+x_2^2\right)^{1/2}\varphi(\omega), \quad \omega = \dfrac{\left(x_1^2+x_2^2\right)^{1/2}}{x_0-u};$

$\omega\left(\varphi^3-2\varphi^2+\omega^2\right)\varphi'' - \left(\varphi\varphi'\right)^3 + \omega^3\varphi'^2 - \left(\varphi^2-2\omega^2\right)\varphi\varphi' - 5\omega\varphi^2 = 0;$

26. $x_3\omega + \sqrt{x_1^2+x_2^2}\ln\dfrac{\sqrt{x_1^2+x_2^2}}{\omega} = \left(x_1^2+x_2^2\right)^{1/2}\varphi(\omega) - \alpha\left(x_1^2+x_2^2\right)^{1/2}\operatorname{arctg}\dfrac{x_2}{x_1},$

$\omega = \dfrac{\left(x_1^2+x_2^2\right)^{1/2}}{x_0-u}, \quad \alpha \geq 0;$

$\omega\left(\omega^2+\alpha^2\right)\varphi'' + \omega^2\varphi'^3 + \left(\omega^2+2\alpha^2\right)\varphi' = 0;$

$$\frac{x_3}{x_0-u} + \ln(x_0-u) + \alpha\operatorname{arctg}\frac{x_2}{x_1} =$$

$$= i\left(\alpha^2\ln\frac{\sqrt{x_1^2+x_2^2}}{\sqrt{x_1^2+x_2^2+\alpha^2(x_0-u)^2}+\alpha^2(x_0-u)} + \frac{\sqrt{x_1^2+x_2^2+\alpha^2(x_0-u)^2}}{x_0-u} + c\right);$$

27. $3\ln\left(\omega\left(x_1^2+x_2^2\right)^{1/2}\right) - 2\ln\left(12x_0 - \left(\omega\left(x_1^2+x_2^2\right)^{1/2} + 4x_3\right)^{1/2} \times\right.$

$\left.\times\left(6+2x_3-\omega\left(x_1^2+x_2^2\right)^{1/2}\right)\right) = \varphi(\omega), \quad \omega = \dfrac{(x_0-u)^2-4x_3}{\left(x_1^2+x_2^2\right)^{1/2}};$

$144\left(\left(\omega^2+16\right)e^{\varphi}-\omega^2\right)\omega\varphi'' + \omega^4\left(\omega^2+16\right)\varphi'^3 -$

$-24\omega\left(3\left(\omega^2+16\right)e^{\varphi}+2\omega^2\right)\varphi'^2 + 144\left(\left(40+\omega^2\right)e^{\varphi}-\omega^2\right)\varphi' = 0;$

28. $\ln\left(\omega\left(x_1^2+x_2^2\right)\right) - \omega\left(x_1^2+x_2^2\right) - \dfrac{x_3^2}{\omega\left(x_1^2+x_2^2\right)} = \varphi(\omega) - 2\alpha\operatorname{arctg}\dfrac{x_2}{x_1} - 2x_0,$

$\omega = \dfrac{x_0-u}{x_1^2+x_2^2}, \quad \alpha \geq 0;$

$\omega^2\left(4\omega\left(1-\alpha^2\omega\right)+1\right)\varphi'' + 2\omega^5\varphi'^3 - 2\omega^3\varphi'^2 - 2\left(\alpha^2\omega^2-2\omega+1\right)\varphi'\omega - 2\alpha^2\omega + 3 = 0;$

29. $\omega\left(x_1^2+x_2^2\right) + \dfrac{x_3^2}{\omega\left(x_1^2+x_2^2\right)} = \varphi(\omega) + 2\operatorname{arctg}\dfrac{x_2}{x_1} + 2x_0, \quad \omega = \dfrac{x_0-u}{x_1^2+x_2^2},$

$\omega^2\left(1-4\omega^2\right)\varphi'' + 2\omega^5\varphi'^3 - 2\omega^3\varphi'^2 - 2\omega\left(\omega^2+1\right)\varphi' - 2\omega + 1 = 0;$

30. $\ln\left((\omega+\alpha)\left(\omega\left(2x_0-\omega\right)-x_1^2\right)-x_2^2\omega\right) - 2\ln\left(\dfrac{\beta x_2}{\omega+\alpha} + \dfrac{x_1}{\omega} - x_3\right) = \varphi(\omega),$

$\omega = x_0-u, \quad \alpha > 0, \quad \beta \geq 0;$

$$\omega^4(\omega+\alpha)^4\varphi''+\omega^4(\omega+\alpha)^4\varphi'^2-4\omega^3(\omega+\alpha)^3(2\omega+\alpha)\varphi'+2\omega(\omega+\alpha)\beta^2\omega^2+(\omega+\alpha)^2+$$
$$+\left(\omega(\omega+\alpha)^2\right)e^{\varphi}\varphi'-2e^{\varphi}\left(\beta^2\omega^2(7\omega+3\alpha)+3(2\omega+\alpha)\left(\omega+\alpha^2\right)^2+\right.$$
$$\left.+(4\omega+\alpha)(\omega+\alpha)^2\omega^2\right)=0;$$

31. $\ln\left(\omega\left(2x_0-\omega\right)-x_1^2-x_2^2\right)-2\ln\left(x_3\omega-x_1\right)=\varphi(\omega), \quad \omega=x_0-u;$
$$\omega^2\varphi''+\omega^2\varphi'^2+2\omega\left(e^{\varphi}\left(\omega^2+1\right)-1\right)\varphi'-2\left(\omega^2+2\right)e^{\varphi}=4;$$

32. $\ln\left(\omega\left(2x_0-\omega\right)-x_1^2-\dfrac{\omega}{\omega+1}x_2^2\right)-2\ln\left(\dfrac{\alpha x_2}{\omega+1}+\dfrac{x_1}{\omega}-x_3\right)=\varphi(\omega),$
$\omega=x_0-u, \quad \alpha>0;$
$$\omega^4\left(\omega+1\right)^3\left(\varphi''+\varphi'^2\right)+2\left(\omega+1\right)\omega\left(e^{\varphi}\left(\omega^2+\left(\omega^2+1\right)^3\right)-2\omega^2\left(\omega+1\right)^2-\omega^3\right)\varphi'+$$
$$+2e^{\varphi}\left(\left(\omega+1\right)^2\left(\omega^4-3\omega^3-2\omega^2-4\omega-4\right)-2\alpha^2\left(2\omega+1\right)\omega^2\right)+$$
$$+2\left(3\omega+2\right)\left(\omega+1\right)^2\omega^2=0.$$

The ansatzes (12)–(32) can be written in the following form: $h(x,\omega)=f(x)\varphi(\omega)+g(x)$, where $h(x,\omega)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function. $\omega=\omega(x,u)$ are one-dimensional invariants of subgroups of the group $\tilde{P}(1,4)$.

33. $2\ln\left(x_0-u\right)=\varphi(\omega)+\ln\left(x_1^2+x_2^2\right)+2c\,\mathrm{arctg}\,\dfrac{x_2}{x_1}, \quad \omega=\dfrac{x_3}{\left(x_1^2+x_2^2\right)^{1/2}};$
$$4\left(c^2\left(\omega^2+1\right)+1\right)\varphi''-\omega\left(\omega^2+1\right)\varphi'^3+2\left(3\omega^2+2\right)\varphi'^2-12\omega\varphi'+8\left(c^2+1\right)=0;$$

34. $\dfrac{x_0}{\left(x_1^2+x_2^2+u^2\right)^{1/2}}=\varphi(\omega), \quad \omega=\dfrac{x_3}{x_0};$
$$\left(\varphi^2-\omega^2-1\right)\varphi^2\varphi''+2\omega\varphi'^3-4\varphi\varphi'^2-2\omega\varphi^4\varphi'+2\varphi^7=0;$$

35. $\dfrac{\left(u-x_0\right)^{\alpha-1}}{\left(u+x_0\right)^{\alpha+1}}=\varphi(\omega), \quad \omega=\dfrac{x_3}{\left(x_0^2-u^2\right)^{1/2}}, \quad \alpha>0;$
$$\left(\alpha^2-1-\omega^2\right)\varphi^2\varphi''+\omega\left(1-\omega^2\right)\varphi'^3+\left(4\alpha^2-8+\left(4\alpha^2+6\right)\omega^2\right)\varphi\varphi'^2-$$
$$-2\left(\alpha^2-2\alpha+7\right)\omega\varphi'\varphi^2-8\left(\alpha^2-1\right)\varphi^3=0;$$

36. $2\alpha\ln\left(x_0-u\right)=\varphi(\omega)+(1+\alpha)\ln\left(x_1^2+x_2^2\right)-2\beta\,\mathrm{arctg}\,\dfrac{x_2}{x_1},$
$\omega=\dfrac{x_3}{\sqrt{x_1^2+x_2^2}}, \quad \beta\geq 0, \quad \alpha^2+\beta^2\neq 0;$
$$4\alpha\left(\left(\alpha+1\right)^2+\beta^2\left(\omega^2+1\right)\right)\varphi''-\alpha\omega\left(\omega^2+1\right)\varphi'^3+$$
$$+2\left(3\alpha(\alpha+1)\omega^2+\alpha^2-\beta^2-1\right)\varphi'^2+4\left(2\beta\left(\alpha+1-\beta^2\right)-3\alpha(\alpha+1)^2\right)\omega\varphi'+$$
$$+4\alpha(\alpha+1)\left(2(\alpha+1)^2+2\beta^2+1\right)+2\left(\beta^2-\alpha^2+1\right)=0;$$

37. $\dfrac{x_3}{\sqrt{x_0^2-x_1^2-u^2}}=\varphi(\omega), \quad \omega=2\gamma\ln\left(x_0-u\right)-\left(1+\gamma\right)\ln\left(x_0^2-x_1^2-u^2\right), \quad \gamma\neq 0;$
$$2\left(\gamma^2\varphi^2+(\gamma+1)(3\gamma+1)\right)\varphi''-4(\gamma+1)\left(\gamma^2+\gamma-2\right)\varphi'^3+$$
$$+2\left(\gamma^2+\gamma-2\right)\varphi\varphi'^2+\left((\gamma+6)\varphi^2-3\gamma-1\right)\varphi'-\varphi^3+\varphi=0;$$

38. $\dfrac{x_3}{\sqrt{x_0^2 - x_1^2 - x_2^2 - u^2}} = \varphi(\omega),$

$\omega = 2\alpha \ln (x_0 - u) - (1+\alpha) \ln \left(x_0^2 - x_1^2 - x_2^2 - u^2\right), \ \alpha \neq 0;$

$4\left(\alpha^2\varphi^2 + (\alpha+1)(3\alpha+1)\right)\varphi'' - 8(\alpha+1)\left(\alpha^2+\alpha-3\right)\varphi'^3 +$
$+4\left(2\alpha^2 - 6\alpha - 9\right)\varphi\varphi'^2 + 2\left((2\alpha+9)\varphi^2 - 2\alpha - 4\right)\varphi' - 3\left(\varphi^3 - \varphi\right) = 0;$

39. $\dfrac{x_3}{(x_0-u)^2 - 4x_1} = \varphi(\omega),$

$\omega = 3\ln\left((x_0-u)^2 - 4x_1\right) - 2\ln\left(6\,(x_0+u) - 6x_1\,(x_0-u) + (x_0-u)^3\right);$

$144\left(\left(16\varphi^2+1\right)e^\omega - 1\right)\varphi'' - 2592e^\omega\varphi'^3 + 432e^\omega\varphi\varphi'^2 + 9\left(\left(16\varphi^2+1\right)e^\omega + 2\right)\varphi' - 8\varphi = 0;$

40. $(x_0-u)^2 = x_2\varphi(\omega) + 4x_1, \quad \omega = \dfrac{x_3}{x_2}; \quad \left(\varphi^2 + 16\omega^2 + 16\right)\varphi'' = 0;$

$u = x_0 - (4x_1 + c_1x_2 + c_2x_3)^{1/2}, \quad u = x_0 - 2\sqrt{x_1 + i\left(x_2^2 + x_3^2\right)^{1/2}};$

41. $\left(\dfrac{x_1^2 + x_2^2}{x_3^2 + u}\right)^{1/2} = \varphi(\omega), \quad \omega = c\,\mathrm{arctg}\,\dfrac{x_2}{x_1} - \mathrm{arctg}\,\dfrac{u}{x_3}, \quad 0 < c \leq 1;$

$\left(\varphi^2+1\right)\left(\varphi^2+c^2\right)\varphi\varphi'' - \left(\varphi^2 + c^2\left(\varphi+2\right)\right)\varphi'^2 + \varphi^2\left(\varphi^4 - 1\right) = 0;$

42. $\left(\dfrac{x_1^2 + x_2^2}{x_3^2 + u}\right)^{1/2} = \varphi(\omega), \quad \omega = 2\alpha\,\mathrm{arctg}\,\dfrac{x_2}{x_1} + 2\beta\,\mathrm{arctg}\,\dfrac{u}{x_3} - \ln\left(x_1^2 + x_2^2\right), \quad \alpha > 0, \quad \beta \geq 0;$

$4\varphi\left(\varphi^4 + \left(\alpha^2+\beta^2+1\right)\varphi^2 + \alpha^2\right)\varphi'' - 4\left(\varphi^2\left(\alpha^2+\beta^2+1\right) + 2\alpha + 1\right)\varphi'^2 -$
$-4\varphi\varphi' + \varphi^2\left(\varphi^4 - 1\right) = 0;$

43. $\dfrac{\left(x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2\right)^{1/2}}{x_0 - u} = \varphi(\omega), \quad \omega = \dfrac{x_1}{x_0 - u} + \ln\left(x_0 - u\right);$

$\varphi\left(\varphi^2 - 1\right)\varphi'' + \left(3\varphi^2 - 1\right)\varphi'^2 + 5\varphi\varphi' + 3\varphi^2 = 0;$

44. $(x_0-u)^2 = \left(x_1^2 + x_2^2\right)^{1/2}\varphi(\omega) + 4x_3, \quad \omega = 4c\,\mathrm{arctg}\,\dfrac{x_2}{x_1} + \ln\left(x_1^2 + x_2^2\right), \quad c > 0;$

$16\left(c^2\varphi^2 + 4\left(4c^2+1\right)\right)\varphi'' + 8\left(4c^2+1\right)\varphi'^3 + 12\varphi\varphi'^2 + 2\left(32 + 3\varphi^2\right)\varphi' + \varphi^3 + 16\varphi = 0;$

45. $\dfrac{x_1 + x_2 + x_3^2}{x_0 - u} = \varphi(\omega), \quad \omega = x_0 + u + \ln\left(x_0 - u\right);$

$\varphi\left(4+\varphi\right)\varphi'' - \varphi'^3 - 2\left(4+\varphi\right)\varphi'^2 - 4\varphi\varphi' + 6\varphi = 0.$

The ansatzes (33)–(45) can be written in the following form: $h(x,u) = f(x)\varphi(\omega) + g(x)$, where $h(x,u)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function, $\omega = \omega(x,u)$ are one-dimensional invariants of subgroups of the group $\tilde{P}(1,4)$.

Let us note that the equation (1) was also studied with the help of the subgroup structure of the group $P(1,4)$ as well as invariants of its nonconjugate subgroups. Some of the results we obtained were published in [13, 14].

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 116–119.

Nonlinear Integrable Models with Higher d'Alembertian Operator in Any Dimension

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We consider a nonlinear $\mathbf{C}\mathrm{P}^1$-model on Minkowski space of any dimension. To solve its equations of motion is in general not easy, so we study not the full equations but subequations. It is well known that they have rich solutions and an infinite number of conserved currents.

We extend the subequations and show that extended ones also have rich solutions and an infinite number of conserved currents.

1 Introduction

Nonlinear sigma models play an important role in field theory. They are very interesting objects (toy models) to study not only at the classical but also the quantum level, [1, 2].

The $\mathbf{C}\mathrm{P}^1$-model in (1+1)-dimensions is particularly well understood (Belavin–Polyakov et al). The one in $(1+2)$-dimensions is also relatively well studied. But the $\mathbf{C}\mathrm{P}^1$-model in any higher dimensions has not been studied sufficiently because of difficulties arising from higher dimensionality.

We consider the $\mathbf{C}\mathrm{P}^1$-model in $(1+n)$-dimensions. But it is not easy to solve its equations of motion directly, so we change our strategy. We decompose the full equations into subequations (those determine a submodel in the terminology of [3]). By the way these equations have a long history since [4, 5]. Smirnov and Sobolev have constructed (maybe) general solutions for them. At the same time we can construct an infinite number of conserved currents for them. In this sense the submodel is integrable. The construction by Smirnov and Sobolev (S–S construction in our terminology) is clear and suggestive. Getting a hint from S–S construction we extend the submodel stated above, [9]. For our extended submodel we can construct

(A) (maybe) general solutions,

(B) an infinite number of conserved currents

similarly to the case of submodel. That is, our extended system is also integrable.

In this talk I will discuss (A) and (B) in some detail.

2 $\mathbf{CP}^1$-Models in Any Dimension and Submodels

Let M^{1+n} be a $(1+n)$-dimensional Minkowski space and $\eta = (\eta_{\mu\nu}) = \mathrm{diag}(1, -1, \ldots, -1)$ its metric. For a function

$$u : M^{1+n} \to \mathbf{C} \tag{2.1}$$

an action $\mathcal{A}(u)$ is defined as

$$\mathcal{A}(u) \equiv \int d^{1+n}x \, \frac{\partial^\mu u \, \partial_\mu \bar{u}}{(1+|u|^2)^2}. \tag{2.2}$$

This action is invariant under the transformation

$$u \to \frac{1}{u}. \tag{2.3}$$

Therefore u can be lifted from $\mathbf{C}$ to $\mathbf{C}\mathrm{P}^1$, [6]. The equations of motion of (2.2) read

$$\left(1+|u|^2\right)\partial^\mu\partial_\mu u - 2\bar{u}\partial^\mu u\partial_\mu u = 0. \tag{2.4}$$

We want to solve (2.4) completely, but it is not easy. When $n = 1$ many solutions were constructed by Belavin and Polyakov, [1]. For $n \geq 2$ much is not known about the construction of solutions as far as we know.

Here changing the way of thinking, we try to solve not the full equations (2.4) but

$$\partial^\mu\partial_\mu u = 0 \quad \text{and} \quad \partial^\mu u\partial_\mu u = 0. \tag{2.5}$$

Of course if u is a solution of (2.5), then u satisfies (2.4). We call (2.5) subequations of (2.4) (or submodel of $\mathbf{C}\mathrm{P}^1$-model in the terminology of [3]). (2.5) is much milder than (2.4).

Now our aim in the following is

(A) to write down all solutions of (2.5),

(B) to write down all conserved currents of (2.5).

Here a conserved current is a vector $(V_\mu)_{\mu=0,\ldots,n}$ satisfying

$$\partial^\mu V_\mu(u,\bar{u}) = 0. \tag{2.6}$$

3 Construction of Solutions and Conserved Currents

(A) has a long history [4, 5]. Now we make a short review of Smirnov–Sobolev's construction (S–S construction as abbreviated). For u let $a_0(u), a_1(u), \ldots, a_n(u)$ and $b(u)$ be functions given and we set

$$\delta \equiv \sum_{\mu=0}^{n} a_\mu(u)x_\mu - b(u) \tag{3.1}$$

and

$$a^\mu(u)a_\mu(u) \equiv a_0^2(u) - \sum_{j=1}^{n} a_j^2(u) = 0. \tag{3.2}$$

Putting $\delta = 0$, we solve as

$$\delta = 0 \quad \Rightarrow \quad u = u(x_0, x_1, \ldots, x_n) \tag{3.3}$$

by the inverse function theorem. Then

Proposition 3.1 ([4, 5]). *u is a solution of (2.5).*

Next we turn to (B). In [3, 6, 7] an infinite number of conserved currents was constructed by the representation theory of Lie algebras ($su(2)$ or $su(1,1)$). But their results are extended further.

Let f be a function of C^2-class on $\mathbf{C}$. For $\tilde{f}(x_\mu) \equiv f(u(x_\mu), \bar{u}(x_\mu))$ we set

$$V_\mu(\tilde{f}) \equiv \partial_\mu u\frac{\partial f}{\partial u} - \partial_\mu \bar{u}\frac{\partial f}{\partial \bar{u}}. \tag{3.4}$$

Then

Proposition 3.2 ([9]). *$V_\mu(\tilde{f})$ is a conserved current of (2.5).*

From this proposition we find that (2.5) has uncountably many conserved currents (all C^2-class functions on $\mathbf{C}$).

Remark 3.1. (A) and (B) seem at first sight unrelated. But the existence of an infinite number of conserved currents implies the infinite number of symmetries, therefore they give a strict restriction on ansatz of construction of solutions. As a consequence we have only Smirnov–Sobolev's construction. This is our story (conjecture). We want to prove this at any cost.

4 New Models with Higher Order Derivatives

Let us extend the results in Section 2 and Section 3. For that we look over the S–S construction again.

$$\delta \equiv \sum_{\mu=0}^{n} a_\mu(u)x_\mu - b(u), \tag{4.1}$$

$$a_0^2(u) - \sum_{j=1}^{n} a_j^2(u) = 0. \tag{4.2}$$

Here we try to change the power in (4.2) from 2 to an arbitrary integer p ($p \geq 2$)

$$a_0^p(u) - \sum_{j=1}^{n} a_j^p(u) = 0. \tag{4.3}$$

Under this condition we solve (4.1) as

$$\delta = 0 \quad \Rightarrow \quad u = u(x_0, x_1, \ldots, x_n). \tag{4.4}$$

We call this an extended S–S construction.

Problem. What are differential equations which u in (4.4) satisfies?

We are considering the converse of Section 2 and Section 3. That is, first of all a "solution" is given and next we look for a system of equations which u satisfies. But it is not easy to extend subequations (2.5) in this fashin. Trying to transform (2.5) in an equivalent manner we reach

Lemma 4.1. *(2.5) is equivalent to*

$$\Box_2 u \equiv \partial^\mu \partial_\mu u = 0 \quad \textit{and} \quad \Box_2(u^2) = 0. \tag{4.5}$$

This form is very clear and suggestive. We can extend (4.5) to obtain

Definition 4.1. $$\Box_p(u^k) \equiv \left(\frac{\partial^p}{\partial x_0^p} - \sum_{j=1}^{n} \frac{\partial^p}{\partial x_j^p}\right)(u^k) = 0 \quad \text{for} \quad 1 \leq k \leq p. \tag{4.6}$$

Next let F_n be a Bell polynomial (see [9, 11] for details) and we set $F_{n,\mu}$ as

$$F_{n,\mu} \equiv: F_n\left(\partial_\mu u \frac{\partial}{\partial u}, \partial_\mu^2 u \frac{\partial}{\partial u}, \cdots, \partial_\mu^n u \frac{\partial}{\partial u}\right) : \tag{4.7}$$

Here : : is the normal ordering (moving differentials to right end). Let $\bar{F}_{n,\mu}$ be the complex conjugate of $F_{n,\mu}$ ($u \to \bar{u}$). For f any C^p-class function on $\mathbf{C}$ we set

$$V_{p,\mu}(\tilde{f}) \equiv \sum_{k=0}^{p-1} (-1)^k : F_{p-1-k,\mu}\bar{F}_{k,\mu} : (f). \tag{4.8}$$

For examples when $p = 2$ and 3

$$V_{2,\mu}(\tilde{f}) \equiv F_{1,\mu}(f) - \bar{F}_{1,\mu}(f) = \partial_\mu u \frac{\partial f}{\partial u} - \partial_\mu \bar{u} \frac{\partial f}{\partial \bar{u}}, \tag{4.9}$$

$$\begin{aligned} V_{3,\mu}(\tilde{f}) &\equiv F_{2,\mu}(f) - : F_{1,\mu}\bar{F}_{1,\mu} : (f) + \bar{F}_{2,\mu}(f) \\ &= \partial_\mu^2 u \frac{\partial f}{\partial u} + (\partial_\mu u)^2 \frac{\partial^2 f}{\partial u^2} - \partial_\mu u \partial_\mu \bar{u} \frac{\partial^2 f}{\partial u \partial \bar{u}} + \partial_\mu^2 \bar{u} \frac{\partial f}{\partial \bar{u}} + (\partial_\mu \bar{u})^2 \frac{\partial^2 f}{\partial \bar{u}^2}. \end{aligned} \tag{4.10}$$

Under the preparations mentioned above

Theorem 4.1 ([9, 10]). *We have the following:*

(A) u in (4.4) is a solution of (4.6),

(B) $V_{p,\mu}(\tilde{f})$ in (4.8) is a conserved current of (4.6).

We could extend the results corresponding to $p = 2$ in Section 3 to ones corresponding to any p in a complete manner.

Remark 4.1. Our extended S–S construction may give general solutions. And moreover the statement in the comment in Section 3 may hold even in this case.

Acknowledgements

I would like to thank my collaborators Yasushi Homma and Tatsuo Suzuki at Waseda University, and Prof. Ryu Sasaki at Kyoto University for useful comments.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 120–126.

Nonlocal Symmetries of Nonlinear Integrable Models

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In this paper, nonlocal symmetries are considered for some integrable equations including the first equation of the AKNS hierarchy, the so-called breaking soliton equation, the Boussinesq equation and the Toda equation. Besides, using invariant transformations of corresponding spectral problems, more nonlocal symmetries can be produced from one seed symmetry.

1 Introduction

Symmetries and conservation laws for differential equations are the central themes of perpetual interest in mathematical physics. During past thirty years, the study of symmetries has been connected with the development of soliton theory and, in fact, it constitutes an indispensable and important part of soliton theory.

Let us begin with the celebrated Korteweg de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

where the subscripts represent derivatives. A symmetry of the KdV equation (1) is defined as a solution of its linearized equation

$$\sigma_t + 6(u\sigma)_x + \sigma_{xxx} = 0. \tag{2}$$

It is well known that x-translation and t-translation invariance of (1) leads to the following symmetries: u_x, u_t of the KdV equation (1). In order to find more generalized symmetries, the concepts of recursion operators or strong symmetries, and hereditary symmetries were introduced by Olver and Fuchssteiner and used to find these symmetries [1, 2]. Furthermore, Galilean invariance of the KdV equation (1) leads to symmetry $tu_x - \frac{1}{6}$, which may be viewed as the origin of active research on the time-dependent symmetries and the corresponding Lie algebraic structures for nonlinear equations; and these time-dependent symmetries are connected with nonisospectral problems (see, e.g. [3–6]). Apart from the symmetries mentioned above, there exist so-called nonlocal symmetries expressed by spectral functions, e.g., $\sigma = (\phi^2)_x$ is a symmetry of the KdV equation (1), where ϕ is a spectral function of Lax pair

$$\phi_{xx} + u\phi = \lambda\phi, \tag{3}$$

$$\phi_t = u_x\phi - (2u + 4\lambda)\phi_x. \tag{4}$$

To search for nonlocal symmetries is an interesting topic. On one hand, these nonlocal symmetries enlarge class of symmetries. Besides, nonlocal symmetries are connected with integrable models. Such an example is the nonlocal symmetry $\sigma = (\phi^2)_x$ generates well-known sinh-Gordon equation and Liouville equation [7]. Further more examples can be found in [8–10]. A natural problem arises now: how to find nonlocal symmetries? An effective method to find nonlocal symmetries seems to find inverse of the corresponding recursion operators (see [11]). However, to find inverse of recursion operators is a difficult problem by itself. Recently one of authors (Lou) re-obtained the nonlocal symmetry $\sigma = (\phi^2)_x$ from the conformal invariance of the Schwartz form of the KdV equation (1) [12]. It is an interesting result. As explained below, this nonlocal symmetry is basic one of the KdV equation, from which all the known nonlocal symmetries can be obtained. In fact, we know from [12] that $\frac{d^n}{d\lambda^n}(\phi^2)_x$ is also a symmetry and inverse recursion operator of the KdV equation (1) appears naturally when $\frac{d^n}{d\lambda^n}(\phi^2)_x$ is rewritten as a single multiplication form. Secondly, the other two nonlocal seed symmetries $\partial_x\phi^2\partial_x^{-1}\phi^{-2}$ and $\partial_x\phi^2\partial_x^{-1}\phi^{-2}\partial_x^{-1}\phi^{-2}$ are easily obtained from $(\phi^2)_x$ by considering the fact that Lax pair (3), (4) of the KdV equation is invariant under transformation $\phi \longrightarrow \phi\partial_x^{-1}\phi^{-2}$ and (3), (4) are linear differential equations with respect to ϕ. That means all the known nonlocal symmetries of the KdV equation in literature can be obtained from one seed symmetry $(\phi^2)_x$.

In this paper, we intend to search for nonlocal seed symmetries of some integrable models. It is noticed that recently there have been active research on nonlinearization of spectral problems and generation of finite dimensional integrable systems (see, e.g. [13]). In literature, there are two cases to be considered: Bargmann and Neumann constraints. For the KdV equation, it is obvious that Bargmann constraint is equivalent to symmetry constraint $u_x = (\phi^2)_x$. With this observation in mind, we are going to derive nonlocal symmetries along this line. Besides, using invariance of spectral problem, more nonlocal symmetries can be produced from one seed symmetry.

2 The AKNS case

The AKNS hierarchy is

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n K_0 = L^n \begin{pmatrix} -iq \\ ir \end{pmatrix}, \tag{5}$$

where

$$L = \begin{pmatrix} -D + 2qD^{-1}r & 2qD^{-1}q \\ -2rD^{-1}r & D - 2rD^{-1}q \end{pmatrix}$$

with $D = \frac{\partial}{\partial x}$, $D^{-1} = \int_{-\infty}^{x} dx$. In what follows, we only consider $n = 2$ case for the sake of convenience in calculation. In this case, (5) becomes

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = i \begin{pmatrix} -q_{xx} + 2q^2 r \\ r_{xx} - 2r^2 q \end{pmatrix}, \qquad i = \sqrt{-1}. \tag{6}$$

Its Lax pair is [14]

$$\begin{pmatrix} \phi_{1x} \\ \phi_{2x} \end{pmatrix} = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{7}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = \begin{pmatrix} 2i\lambda^2 + iqr & -2q\lambda - iq_x \\ -2r\lambda + ir_x & -2i\lambda^2 - iqr \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{8}$$

It is known that the Bargmann constraint is [15]

$$\begin{pmatrix} q \\ r \end{pmatrix} = c_0 \begin{pmatrix} \phi_1^2 \\ -\phi_2^2 \end{pmatrix}$$

and $K_0 = \begin{pmatrix} -iq \\ ir \end{pmatrix}$ is a symmetry of (6). Thus $\sigma = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix}$ is possible to become a symmetry of (6). Indeed, a direct calculation shows that $\sigma = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix}$ is a symmetry (see also [16]). In order to obtain more seed symmetries, we now consider the invariance property of (7) and (8). To this end, we have

Proposition 1. *Lax pair (7) and (8) is invariant under transformation*

$$\phi_1 \longrightarrow F(t)\phi_1 + (\alpha - 1)\frac{1}{\phi_2} + \phi_1 \left(\alpha \int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha - 1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right),$$

$$\phi_2 \longrightarrow F(t)\phi_2 + \alpha\frac{1}{\phi_1} + \phi_2 \left(\alpha \int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha - 1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right),$$

where α is an arbitrary constant and $F(t)$ is a function of t defined by

$$F(t) = -\int^{t} \left[\alpha \frac{iq_x + 2\lambda q}{\phi_1^2} + (1 - \alpha)\frac{ir_x - 2\lambda r}{\phi_2^2} \right]_{x = x_0} dt.$$

Proof: direct calculation.

Proposition 2. *Suppose that* $\begin{pmatrix} \phi_1^{(i)} \\ \phi_2^{(i)} \end{pmatrix}$ *$(i = 1, 2)$ is a solution of (7) and (8). Then* $\begin{pmatrix} \phi_1^{(1)}\phi_1^{(2)} \\ \phi_2^{(1)}\phi_2^{(2)} \end{pmatrix}$ *is a symmetry of (6).*

Using Proposition 1 and 2, we know that

$$\begin{pmatrix} F(t)\phi_1^2 + (\alpha - 1)\dfrac{\phi_1}{\phi_2} + \phi_1^2 \left(\alpha \displaystyle\int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha - 1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right) \\ F(t)\phi_2^2 + \alpha\dfrac{\phi_2}{\phi_1} + \phi_2^2 \left(\alpha \displaystyle\int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha - 1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right) \end{pmatrix}$$

and

$$\begin{pmatrix} \left[F(t)\phi_1 + (\alpha - 1)\dfrac{1}{\phi_2} + \phi_1 \left(\alpha \displaystyle\int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha - 1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right) \right]^2 \\ \left[F(t)\phi_2 + \alpha\dfrac{1}{\phi_1} + \phi_2 \left(\alpha \displaystyle\int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha - 1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right) \right]^2 \end{pmatrix}$$

are symmetries of (6). Furthermore, in [17], the inverse of recursion operator L was obtained. Thus more symmetries can be obtained from seed symmetries and inverse recursion operator L^{-1}.

3 The breaking soliton equation

The breaking soliton equation under consideration is

$$u_{xt} = 4u_x u_{xy} + 2u_y u_{xx} - u_{xxxy} \tag{9}$$

which was first introduced by Calogero and Degasperis [18]. Set $v = u_x$, then (9) can be written as

$$v_t = 4vv_y + 2(\partial_x^{-1} v_y)v_x - v_{xxy}. \tag{10}$$

Its bi-Hamiltonian structure and the Lax pair equations with non-isospectral problem have been discussed in [19]. In [10], one of authors (Lou) found a nonlocal symmetry of (9)

$$\sigma = 2\phi_x \phi(1 + \partial_x^{-1}\phi^{-3}\phi_y) + \phi^{-1}\phi_y$$

with

$$-\phi_{xx} + v\phi = 0, \tag{11}$$

$$\phi_t = -v_y\phi + 2\phi_x \partial_x^{-1} v_y. \tag{12}$$

It is easily verified that Lax pair (11), (12) is invariant under the transformation $\phi \longrightarrow \phi\partial_x^{-1}\frac{1}{\phi^2}$. Besides, we have

Proposition 3. *Suppose ϕ_1 and ϕ_2 are two solutions of (11), (12). Then*

$$\begin{aligned}\sigma(\epsilon,\delta) = {} & 2(\epsilon\phi_1 + \delta\phi_2)_x(\epsilon\phi_1 + \delta\phi_2)\left(1 + \partial_x^{-1}(\epsilon\phi_1 + \delta\phi_2)^{-3}(\epsilon\phi_1 + \delta\phi_2)_y\right) \\ & + (\epsilon\phi_1 + \delta\phi_2)^{-1}(\epsilon\phi_1 + \delta\phi_2)_y\end{aligned}$$

and $\frac{\partial^{m+n}}{\partial\epsilon^m\partial\delta^n}\sigma(\epsilon,\delta)$ are symmetries of (10) (here ϵ, δ are arbitrary constants).

Using these results, many nonlocal symmetries can be obtained.

4 The Boussinesq equation

The Boussinesq equation under consideration is [20]

$$u_{tt} + \left(u^2\right)_{xx} + \frac{1}{3}u_{xxxx} = 0. \tag{13}$$

The corresponding Lax pair is

$$\phi_{xxx} + \frac{3}{2}u\phi_x + \left(\frac{3}{4}u_x - \frac{3}{4}\partial_x^{-1}u_t\right)\phi = 0, \tag{14}$$

$$\phi_t = -\phi_{xx} - u\phi \tag{15}$$

and its adjoint version is

$$\phi^*_{xxx} + \frac{3}{2}u\phi^*_x + \left(\frac{3}{4}u_x + \frac{3}{4}\partial_x^{-1}u_t\right)\phi^* = 0, \tag{16}$$

$$\phi^*_t = \phi^*_{xx} + u\phi^*. \tag{17}$$

Just as the KP case [12, 21], it is easily verified that $(\phi\phi^*)_x$ is a symmetry of (13). In the following, we want to give more symmetries of (13) by considering invariance property of (14), (15) and (16), (17). To this end, we obtain

Proposition 4. *Suppose ϕ_1 and ϕ_2 are two linearly independent spectral functions of (14), (15) (or (16), (17)) corresponding to u. Then*

$$\Phi = \psi_1(t)\phi_1 - \psi_2(t)\phi_2 + \phi_1 \int_{x_0}^{x} \frac{\phi_2}{W^2(\phi_1, \phi_2)}\, dx - \phi_2 \int_{x_0}^{x} \frac{\phi_1}{W^2(\phi_1, \phi_2)}\, dx$$

is also a spectral function of (14), (15) (or (16), (17)) corresponding to u, where

$$W(\phi_1, \phi_2) \equiv \phi_{1x}\phi_2 - \phi_1\phi_{2x}, \tag{18}$$

$$\psi_1(t) = \int^t \left[\frac{\phi_{2x}}{W^2(\phi_1, \phi_2)}\right]_{x=x_0} dt, \tag{19}$$

$$\psi_2(t) = \int^t \left[\frac{\phi_{1x}}{W^2(\phi_1, \phi_2)}\right]_{x=x_0} dt. \tag{20}$$

Proof: direct calculation.

From Proposition 4, we know

$$\begin{aligned}\sigma = \Bigg[\Bigg(& c_0\phi_1 + c_1\phi_2 + c_2\psi_1(t)\phi_1 - c_2\psi_2(t)\phi_2 + c_2\phi_1 \int_{x_0}^{x} \frac{\phi_2}{W^2(\phi_1, \phi_2)}\, dx \\ & -c_2\phi_2 \int_{x_0}^{x} \frac{\phi_1}{W^2(\phi_1, \phi_2)}\, dx\Bigg) \big(c_3\phi_1^* + c_4\phi_2^* + c_5\psi_1^*(t)\phi_1^* - c_5\psi_2^*(t)\phi_2^* \\ & +c_5\phi_1^* \int_{x_0^*}^{x} \frac{\phi_2^*}{W^2(\phi_1^*, \phi_2^*)}\, dx - c_5\phi_2^* \int_{x_0^*}^{x} \frac{\phi_1^*}{W^2(\phi_1^*, \phi_2^*)}\, dx\Bigg)\Bigg]_x\end{aligned} \tag{21}$$

is also a symmetry of (13), where ϕ_i and ϕ_i^* $(i = 1, 2)$ are spectral functions of (14), (15) and (16), (17) respectively, $\psi_i(t)$ $(i = 1, 2)$ is defined by (19), (20) and

$$\psi_1^*(t) = \int^t \left[\frac{\phi_{2x}^*}{W^2(\phi_1^*, \phi_2^*)}\right]_{x=x_0^*} dt, \tag{22}$$

$$\psi_2^*(t) = \int^t \left[\frac{\phi_{1x}^*}{W^2(\phi_1^*, \phi_2^*)}\right]_{x=x_0^*} dt. \tag{23}$$

5 The Toda equation

The Toda equation under consideration is [22]

$$\frac{d^2}{dt^2} \ln v(n) = v(n-1) - 2v(n) + v(n+1) \tag{24}$$

or equivalently

$$\frac{dv(n)}{dt} = v(n)\partial_t^{-1}[v(n-1) - 2v(n) + v(n+1)] \tag{25}$$

which may be rewritten in a coupled form

$$\frac{dp(n)}{dt} = v(n) - v(n+1), \tag{26}$$

$$\frac{dv(n)}{dt} = v(n)(p(n-1) - p(n)). \tag{27}$$

It is known that (25) or (26), (27) has a Lax pair

$$y_{n+1} + p(n)y_n + v(n)y_{n-1} = \lambda y_n, \tag{28}$$

$$y_{nt} = v(n)y_{n-1} - \frac{1}{2}\lambda y_n \tag{29}$$

and its adjoint version is

$$y^*_{n-1} + p(n)y^*_n + v(n+1)y^*_{n+1} = \lambda y^*_n, \tag{30}$$

$$-y^*_{nt} = v(n+1)y^*_{n+1} - \frac{1}{2}\lambda y^*_n. \tag{31}$$

Here the adjoint operator of a difference operator is defined by

$$\left(a(n)e^{k\partial_n}\right)^* = e^{-k\partial_n}a(n).$$

A symmetry of the Toda equation (25) is defined as a solution of its linearized equation

$$\frac{d\sigma(n)}{dt} = \sigma(n)\partial_t^{-1}[v(n-1) - 2v(n) + v(n+1)] + v(n)\partial_t^{-1}[\sigma(n-1) - 2\sigma(n) + \sigma(n+1)]. \tag{32}$$

Just as the two-dimensional Toda equation [23], it is easily verified that $\sigma(n) = (y_n y^*_{n-1})_t$ is a symmetry of the Toda equation (25). To obtain more seed symmetries,we now consider invariance property of (28), (29) and (30), (31). We obtain

Proposition 5. *Suppose y_n is a spectral function of (28), (29) and* $\lim\limits_{n\to-\infty} p(n) = 0$. *Then*

$$\bar{y}_n = y_n \sum_{k=-\infty}^{n} \frac{\prod\limits_{i=-\infty}^{k-1} v(i)}{y_k y_{k-1}}$$

is also a spectral function of (28), (29).

Proof: direct calculation.

Similarly, we have

Proposition 6. *Suppose y^*_n is a spectral function of (30), (31) and* $\lim\limits_{n\to\infty} p(n) = 0$. *Then*

$$\bar{y}^*_n = y^*_n \sum_{k=n}^{\infty} \frac{\prod\limits_{i=k+2}^{\infty} v(i)}{y^*_k y^*_{k+1}}$$

is also a spectral function of (30), (31).

From Proposition 5 and 6, we know

$$\left[\left(c_1 y_n + c_2 y_n \sum_{k=-\infty}^{n} \frac{\prod\limits_{i=-\infty}^{k-1} v(i)}{y_k y_{k-1}}\right)\left(c_3 y^*_{n-1} + c_4 y^*_{n-1} \sum_{k=n-1}^{\infty} \frac{\prod\limits_{i=k+2}^{\infty} v(i)}{y^*_k y^*_{k+1}}\right)\right]_t$$

is also a symmetry of (25), where y_n and y^*_n are spectral functions of (28), (29) and (30), (31) respectively and c_i is an arbitrary constant ($i = 1, 2, 3, 4$).

6 Summary

In this paper, nonlocal symmetries are considered for four integrable equations as examples which include the first equation of the AKNS hierarchy, the so-called breaking soliton equation, the Boussinesq equation and the Toda equation. Besides, using invariance properties of corresponding spectral problems under suitable transformations, more nonlocal symmetries can be produced from one seed symmetry.

Acknowledgements

The work was supported by National Natural Science Foundation of China. One of the authors (XBH) would like to express his thanks to Prof. A. Nikitin for his invitation to attend Third International Conference Symmetry in Nonlinear Mathematical Physics.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 127–130.

Oscillation of Solutions of Ordinary Differential Equations Systems Generated by Finite-Dimensional Group Algebra

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The class of nonlinear second order systems having oscillation solutions has been described. Let us take note that periodic solution is particular case of oscillation solution. The algorithm of construction of a reducible transformation transforming initial system to system generated by finite-dimensional group algebra has been developed. It is important that initial systems can be essentially nonlinear. The class of relaxation oscillations has been reduced to the considered case.

Among the variety of second order systems is of interest to select systems in which periodic or "similar" to it change of system state almost periodic, recurrent or oscillatory can take place.

Consider a nonlinear autooscillatory system of differential equations with one degree of freedom

$$\begin{cases} \dot{x} = f_1(x, y), \\ \dot{y} = f_2(x, y), \end{cases} \tag{1}$$

where $(x, y) \in \mathrm{R}^2$, the overdot in (1) means derivative "d/dt" with respect to $t \in [0, +\infty)$, functions $f_i(x, y)$, $i = 1, 2$ are arbitrary analytical functions in some open domain D of plane (x, y) that satisfy Lipschitz condition in any bounded closed region that is subset of D.

The mathematical model of autooscillatory system is essentially nonlinear. Restriction of amplitude of autooscillations takes place in autooscillatory system due to its nonlinearity. The form of them can be diverse including nonusual. Among similar class of nonlinear systems, the relaxation systems, are of special class

$$\begin{cases} \varepsilon\dot{x} = f_1(x, y), \\ \dot{y} = f_2(x, y), \end{cases} \qquad 0 < \varepsilon \ll 1,$$

where oscillations are very far from harmonic. The construction of approximative analytical expressions for them cannot be obtained within the limits of classical methods of perturbation theory and requires special methods. This problem was solved for indicated class of systems.

Suppose that system (1) has integral in the form of curve

$$F(x, y) = C, \qquad 0 \leq C \leq C^*, \tag{2}$$

that satisfies the following conditions:

1. Curve (2) is sectionally smooth;
2. The curve surrounding a system's state of equilibrium (x_0, y_0) is closed;
3. Curve (2) restricts some simply connected domain D^* that is subset of determination region of system (1) $D^* \subseteq D$;

4. Curve (2) does not have points of self-intersection that means that for given implicit function $F(x,y)=C$ the following condition takes place:

$$\Delta = \left(\frac{\partial^2 F(x,y)}{\partial x^2}\right)_{(0,0)} \cdot \left(\frac{\partial^2 F(x,y)}{\partial y^2}\right)_{(0,0)} - \left(\frac{\partial^2 F(x,y)}{\partial x \partial y}\right)^2_{(0,0)} \geq 0. \qquad (3)$$

Theorem 1. *If an autooscillatory nonlinear system with one power of freedom (1) has an integral (2) in the form of sectionally smooth closed curve for which the conditions 1–4 are satisfied then system (1) has a restricted and oscillatory solution*

$$u = \sum_{i=1}^{m} A_i \sum_{\substack{j=2p \\ p=0,1,\ldots,[\frac{i}{2}]}} C_i^j \cdot (-1)^{\frac{j}{2}} \cdot \cos^{i-j}\varphi \cdot \sin^j \varphi,$$

$$v = \sum_{i=1}^{m} A_i \sum_{\substack{j=2p+1 \\ p=0,1,\ldots,[\frac{i-1}{2}]}} C_i^j \cdot (-1)^{\frac{j-1}{2}} \cdot \cos^{i-j}\varphi \cdot \sin^j \varphi.$$

And, vice versa, a restricted and oscillatory solution of system (1) corresponds to a phase trajectory in the form of sectionally smooth closed curve $F(x,y)=C$ that does not have the points of self-intersection.

Represent system (1) in complex plane by means of the change of variables

$$\begin{cases} x = \dfrac{1}{2}(w+\overline{w}), \\ y = -\dfrac{i}{2}(w-\overline{w}), \end{cases} \qquad (4)$$

where $w=u+iv$, $\overline{w}=u-iv$.

In view of special properties of the change the system (1) may be rewriten as

$$\begin{cases} \dot{u} = f_1(u,v), \\ \dot{v} = f_2(u,v), \end{cases} \qquad (5)$$

where $u+iv=w$ and integral (2) in complex variables shall respectively have the form of

$$F(u,v)=C, \qquad u+iv=w, \qquad (6)$$

where sectionally smooth closed curve (6) will restrict respective one-connected domain D_w of complex plane w.

Theorem 2 (approximative transformation of nonlinear system). *Suppose that autooscillatory nonlinear system (5) where functions $f_i(u,v)$, $i=1,2$ are analytical in some domain D of complex plane w has an integral (6) in the form of sectionally smooth closed curve that satisfies conditions 1–4.*

By the method of trigonometric interpolation we construct the power function mapping unit circle $|W|=1$ on curve the (6) to some fixed value of parameter C

$$w = \sum_{n=1}^{m} A_n W^n, \qquad w=u+iv, \quad W=U+iV, \quad m\to\infty. \qquad (7)$$

The inverse function

$$W = U + iV = G(w) = \sqrt{c}\sum_{n=1}^{m} B_n w^n \tag{8}$$

transforms integral (6) to the canonical form

$$W\overline{W} = C, \qquad U^2 + V^2 = C. \tag{9}$$

Then transformation (8) reduces system (5) to a system generated by finite-dimensional group algebra $so(2)$

$$\begin{cases} \dot{U} = -\alpha(U,V)V, \\ \dot{V} = \alpha(U,V)U. \end{cases} \tag{10}$$

Remark 1. The system (10) has an oscillation solution

$$\begin{cases} U = \rho\cos\varphi(t), \\ V = \rho\sin\varphi(t), \end{cases} \tag{11}$$

where the functions $\varphi(t)$ and $\alpha(U,V)$ satisfy differential equation for phase $\varphi(t)$ and amplitude ρ of oscillations

$$\frac{d\varphi}{dt} = \alpha(\rho,\varphi). \tag{12}$$

Thus we determine a transformation of coordinates in form of power series for receiving phase trajectory of system (1) in the form of a family of concentric circles with centre in the origin. But a representative point moves along one of phase trajectory with variable angular velocity $\frac{d\varphi}{dt} = \text{var}$. Remark that in particular case with $\alpha = 1$ the point moves on the circle uniformly. In this case we have a harmonic solution

$$\begin{cases} U = A\cos(t+\phi), \\ V = A\sin(t+\phi), \end{cases}$$

where A is the amplitude of oscillations and ϕ is the phase of oscillations.

Remark 2. The solution (11) is periodic if function $\varphi(t)$ is periodic or $\varphi(t) = t$.

The Riemannian theorem, the theorem of conformity of domain boundaries at one-to-one conformal mapping of domains and Christoffel–Schwarz integral [5] realizing mapping of unit circle $|W| \le 1$ on internal region of polygon are theoretical base for transformation (7). The constants of integral will be unit circle points which correspond to vertices of a polygon when mapping.

For numerical solution we use of stated problem Filchakov method of trigonometric interpolation of conformal mapping of domains. This method allows to obtain with help of simple formulas any given accuracy of construction of function mapping unit circle on internal region of any previously given simply connected and one-sheet domain D_w restricted by curve (6).

It is of great importance that the method of trigonometric interpolation does not give any restrictions on the manner of setting of contour what means that curve (6) can be given analytically, graphically or tabular, only by a discrete series of points.

Remark 3. Taking into consideration that in power series (8) the coefficients are imaginary

$$B_n = B_n^{(1)} + iB_n^{(2)}, \qquad w = u + iv,$$

and using the Newton binomial formula it is possible to determine real and imaginary parts for the transformation $W = G(w)$:

$$U = \sqrt{c}\sum_{n=1}^{m} B_n^{(1)} \sum_{\substack{k=2l \\ l=0,1,\ldots,\left[\frac{n}{2}\right]}} C_n^k (-1)^{\frac{k}{2}} u^{n-k} v^k - \sqrt{c}\sum_{n=1}^{m} B_n^{(2)} \sum_{\substack{k=2l+1 \\ l=0,1,\ldots,\left[\frac{n-1}{2}\right]}} C_n^k (-1)^{\frac{k-1}{2}} u^{n-k} v^k.$$

$$V = \sqrt{c}\sum_{n=1}^{m} B_n^{(1)} \sum_{\substack{k=2l+1 \\ l=0,1,\ldots,\left[\frac{n-1}{2}\right]}} C_n^k (-1)^{\frac{k-1}{2}} u^{n-k} v^k + \sqrt{c}\sum_{n=1}^{m} B_n^{(2)} \sum_{\substack{k=2l \\ l=0,1,\ldots,\left[\frac{n}{2}\right]}} C_n^k (-1)^{\frac{k}{2}} u^{n-k} v^k.$$

Similar problem for analysis of autonomous second order systems that are closed to nonlinear conservative is solved in [4]. The main result of this paper is a considerable extension of the class of studied systems was without essential restrictions for the functions $f_1(x, y)$, $f_2(x, y)$. Moreover there is a possibility of generalization of theory in case $n > 2$.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 131–136.

About Symmetries of Exterior Differential Equations, Appropriated to a System of Quasilinear Differential Equations of the First Order

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The symmetries for a quasilinear system of first order partial differential equations are determined. The transformation to a system of exterior differential equations is used. Is shown that the use of this method allows to simplify a problem of defining equations determination.

1 Introduction

Most calculations of symmetries of differential equations are done with the classical L.V. Ovsyannikov method [1]. In 1965 K.P. Surovikhin published a paper [2], in which the differential forms were applied for searching symmetries. In the K.P. Surovikhin paper the system of hyperbolic type equations was considered, and the canonization method for finding symmetries of a exterior differential equations system was applied. In 1971 F.B. Estabrook and B.K. Harrison published a paper [3] in which the Lie derivatives were used for finding symmetries of the exterior differential equations system. This method is easier and universal compared to a canonization method. This method has also certain advantages compared to the L.V. Ovsyannikov method.

However, as it was noted by B.K. Harrison [4], this method was not used widely in the literature. The author also developed a method for finding the symmetries of exterior differential equations with use of the Lie derivatives [5] (the author did not know about the F.B. Estabrook and B.K. Harrison method). In the present paper this method is applied to finding symmetries of quasilinear partial differential equations of the first order. The advantages of this method on a comparison with the L.V. Ovsyannikov method are considered.

2 System of exterior differential equations

A quasilinear system of the first order partial differential equations is considered as a submainfold (surface) Σ in 1-jets space $J^1(\pi)$ of a bundle $\pi : E \longrightarrow M$ local cuts [6]. This submainfold is determined by the system of equations

$$F^k\left(x^i, u^j, p_i^j\right) = 0, \tag{1}$$

where $x^i \in M \subset R^n$, $u^j \in U \subset R^m$, $p_i^j \in J^1(\pi)$, $E = M \times U$. Thus in the space $J^1(\pi)$ there is a Cartan distribution C, defined by Cartan 1-forms

$$\Omega^j = du^j - \sum_{i=1}^{n} p_i^j dx^i. \tag{2}$$

The surface Σ is integral variety Cartan's distribution. Therefore together with (1) should be fulfilled

$$\Omega^j = 0. \tag{3}$$

Thus, a cut $u : M \longrightarrow E$ is a solution of the system (1) if the relations (1), (3) are fulfilled. We shall designate system of relations (1), (3) as $C\Sigma$.

For the quasilinear system of equations (1), we have

$$F^k = c^k_{ji}(x, u)p^j_i + c^k_0(x, u), \tag{4}$$

where $c^k_{ji}(x, u)$, $c^k_0(x, u)$ are continuous functions.

We can obtained now a system of exterior differential equations. For this purpose we shall multiply each equation of the system (1) and the base volume M

$$\omega^k_F = F^k dx^1 \wedge \ldots \wedge dx^n. \tag{5}$$

From Cartan 1-forms we can obtain the n-forms

$$\Omega^j_i = \Omega^j \wedge (dx^1 \wedge \ldots \wedge dx^n)_{\bar{i}} = du^j \wedge (dx^1 \wedge \ldots \wedge dx^n)_{\bar{i}} + p^j_i(-1)^i dx^1 \wedge \ldots \wedge dx^n, \tag{6}$$

where $(dx^1 \wedge \ldots \wedge dx^n)_{\bar{i}} = dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n$.

The system of exterior differential equations $\Lambda(\Sigma)$ is obtained by the following method

$$\omega^k = \omega^k_F - (-1)^i c^k_{ji}\Omega^j_i = 0. \tag{7}$$

After substitution (5), (6) we have

$$\omega^k = -(-1)^i c^k_{ji}(x, u)du^j \wedge (dx^1 \wedge \ldots \wedge dx^n)_{\bar{i}} + c^k_0(x, u)dx^1 \wedge \ldots \wedge dx^n. \tag{8}$$

Let us note, that the system $\Lambda(\Sigma)$ on the space E is determined.

Thus initial system of equations $C\Sigma$, defined as the surface Σ with Cartan distribution C, appropriated by the system of the exterior differential equations $\Lambda(C\Sigma)$

$$\Omega^j = 0, \qquad \omega^k = 0. \tag{9}$$

From $\Omega^j = 0$ and $\omega^k = 0$ it follows that $F^k = 0$. Hence the systems $C\Sigma$ and $\Lambda(C\Sigma)$ are equivalent and any integrated variety $C\Sigma$ is an integrated variety for $\Lambda(C\Sigma)$ and vice versa.

3 About symmetries for $C\Sigma$ and $\Lambda(C\Sigma)$

Let us consider now a problem of symmetries for $C\Sigma$ and $\Lambda(C\Sigma)$.

According to [1] and [6], the classical infinitesimal symmetry of the equations $C\Sigma$ is Lie vector field $\overline{X}$, such that

$$\overline{X}(\Omega^k) = \lambda^j \Omega^j, \tag{10}$$

$$\overline{X}(F^k) = \alpha^j F^j. \tag{11}$$

Here λ^j, α^j are some functions, and $\overline{X}(\Omega^k)$ is determined by the Lie derivative

$$\overline{X}(\Omega^k) = d(\overline{X} \rfloor \Omega^k) + \overline{X} \rfloor d(\Omega^k),$$

where $\rfloor$ is an interior product. The Lie vector field $\overline{X}$ belongs to a space, tangent to $J^1(\pi)$, and $\overline{X}$ is a lift of a vector field X, tangent to a space of a bundle E

$$\overline{X} = X + X^{(1)}, \tag{12}$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \phi^j(x,u)\frac{\partial}{\partial u^j}, \qquad X^{(1)} = \zeta_i^j(x,u,p)\frac{\partial}{\partial p_i^j}, \tag{13}$$

ξ^i, ϕ^j, ζ_i^j are some functions. We shall designate the Lie algebra of vector fields X as $\mathrm{sym}(\Sigma)$.

Theorem. *For point infinitesimal symmetries of systems $C\Sigma$ and $\Lambda(C\Sigma)$ the relation is fulfilled*

$$\mathrm{sym}(\Sigma) = \mathrm{cosym}(\Sigma).$$

Proof. The vector field X of a point symmetry is uniquely determined by the Lie vector field $\overline{X}$. Therefore it is enough to show that any symmetry of a system $C\Sigma$ is a symmetry of a system $\Lambda(C\Sigma)$ and vice versa.

Let at first $\overline{X}$ is infinitesimal symmetry $C\Sigma$, i.e.,

$$\overline{X}(F^k)\mid_{C\Sigma} = (X + X^{(1)})(F^k)\mid_{C\Sigma} = 0. \tag{14}$$

Let us show, that

$$X(\omega^k)\mid_{\Lambda(C\Sigma)} = X(\omega^k)\mid_{\Lambda(\Sigma)} = 0. \tag{15}$$

Taking into account (6) and (10), we have

$$\overline{X}(\Omega_i^k)\mid_{\Lambda(C\Sigma)} = [\lambda^j\Omega^j \wedge (dx^1 \wedge \ldots \wedge dx^n)_{\bar{i}} + \gamma^l\Omega^k \wedge (dx^1 \wedge \ldots \wedge dx^n)_{\bar{l}}]\mid_{\Lambda(C\Sigma)} = 0, \tag{16}$$

where γ is some function. Taking into account (5) and (11), we have

$$\begin{aligned}\overline{X}(\omega_F^k) &= \overline{X}(F^k)(dx^1 \wedge \ldots \wedge dx^n) + F^k\overline{X}(dx^1 \wedge \ldots \wedge dx^n)\\ &= \alpha^j F^j(dx^1 \wedge \ldots \wedge dx^n) + \beta F^k(dx^1 \wedge \ldots \wedge dx^n) = (\alpha^j\omega_F^j + \beta\omega_F^k),\end{aligned}$$

and, thus

$$\overline{X}(\omega_F^k) = \mu^j\omega_F^j,$$

where μ^j, β are some functions. Hence, taking into account (7) and (17)

$$\overline{X}(\omega^k)\mid_{\Lambda(C\Sigma)} = [\overline{X}(\omega_F^k) - (-1)^i\overline{X}(c_{ji}^k\Omega_i^j)]\mid_{\Lambda(C\Sigma)} = \mu^j\omega_F^j\mid_{\Lambda(C\Sigma)} = 0.$$

As the forms ω^k are defined in coordinates of space E, then $X^{(1)}(\omega^k) = 0$. Therefore

$$X(\omega^k)\mid_{\Lambda(C\Sigma)} = 0.$$

We can write the latter equality as

$$X(\omega^k) = \rho_j^k\omega^j + \sigma_j^k\Omega^j,$$

where ρ_j^k, σ_j^k are some functions. As the forms ω^j and vector field X are defined on a space of a bundle E, then, $\sigma_j^k \equiv 0$ and, therefore, (16) is fulfilled.

Let now X be an infinitesimal symmetry of $\Lambda(\Sigma)$, i.e., (16) is fulfilled. Let us define the vector field $\overline{X} = X + X^{(1)}$ so, that it is a Lie field (saves the Cartan distribution). Let us show that (15) also is fulfilled.

Taking into account (17) we have

$$X(\omega^k)\mid_{\Lambda(\Sigma)}=X(\omega^k)\mid_{\Lambda(C\Sigma)}=X(\omega^k)\mid_{C\Sigma}= \\ =\overline{X}(\omega^k)\mid_{C\Sigma}=[\overline{X}(\omega^k_F)-(-1)^i\overline{X}(c^k_{ji}\Omega^j_i)]\mid_{C\Sigma}=[\overline{X}(\omega^k_F)]\mid_{C\Sigma}=0$$

and we can write

$$\overline{X}(\omega^k_F)=\mu^j\omega^j_F.$$

From here follows, that

$$\overline{X}(F^k)=\alpha^jF^j$$

and consequently (15) is fulfilled. The theorem is proved.

So the problem of searching infinitesimal symmetries for the given class of equations system $C\Sigma$ is equivalent to a problem of searching infinitesimal symmetries for the system $\Lambda(C\Sigma)$, to be exact systems $\Lambda(\Sigma)$. Thus infinitesimal symmetries are vector fields, tangents to base of bundle E, and not to a space of a bundle $J^1(\pi)$. As it is shown below, such lowering of a vector field space dimensionality reduces to some decreasing of difficulty in construction of the defining equations system.

4 Example

As example, let us consider a system of two equations

$$\frac{\partial u^k}{\partial x^1}+c^k_j\left(x^1,x^2,u^1,u^2\right)\frac{\partial u^j}{\partial x^2}+c^k_0\left(x^1,x^2,u^1,u^2\right)=0, \tag{17}$$

where $i,j,k=1,2$.

The system of the exterior differential equations $\Lambda(\Sigma)$ will look like

$$\omega^k=du^k\wedge dx^2-c^k_jdu^j\wedge dx^1+c^k_0dx^1\wedge dx^2=0. \tag{18}$$

Infinitesimal symmetry of a system $\Lambda(\Sigma)$ will be a vector field

$$X=\xi^1\left(x^1,x^2,u^1,u^2\right)\frac{\partial}{\partial x^1}+\xi^2\left(x^1,x^2,u^1,u^2\right)\frac{\partial}{\partial x^2} \\ +\phi^1\left(x^1,x^2,u^1,u^2\right)\frac{\partial}{\partial u^1}+\phi^2\left(x^1,x^2,u^1,u^2\right)\frac{\partial}{\partial u^2}.$$

The defining equations for $\mathrm{cosym}(\Sigma)$ are obtained from a condition (16). We have

$$\begin{aligned} X(\omega^k)&=d\phi^k\wedge dx^2+du^k\wedge d\xi^2-c^k_j\left(d\phi^j\wedge dx^1+du^j\wedge d\xi^1\right)\\ &-X\rfloor d(c^k_j)du^j\wedge dx^1+c^k_0\left(d\xi^1\wedge dx^2+dx^1\wedge d\xi^2\right)+X\rfloor d(c^k_0)dx^1\wedge dx^2. \end{aligned} \tag{19}$$

The system of defining equations for functions ξ^i, ϕ^j is obtained by a substitution (20) to (16). The decomposition is carried on under the forms: $du^1\wedge du^2$, $du^1\wedge dx^1$, $du^2\wedge dx^1$, $dx^1\wedge dx^2$.

We have after decomposition from the first equation of the system (19)

$$\frac{\partial\xi^2}{\partial u^2}-c^1_1\frac{\partial\xi^1}{\partial u^2}+c^1_2\frac{\partial\xi^1}{\partial u^1}=0,$$

$$c_2^1\frac{\partial\phi^2}{\partial u^1}-c_1^2\frac{\partial\phi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial x^1}+c_1^1\left(\frac{\partial\xi^1}{\partial x^1}-\frac{\partial\xi^2}{\partial x^2}\right)+\xi^i\frac{\partial c_1^1}{\partial x^i}+\phi^j\frac{\partial c_1^1}{\partial u^j}$$
$$+\left[\left(c_1^1\right)^2+c_2^1c_1^2\right]\frac{\partial\xi^1}{\partial x^2}+c_0^1\left(c_1^1\frac{\partial\xi^1}{\partial u^1}+c_1^2\frac{\partial\xi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial u^1}\right)=0,$$

$$c_2^1\left(\frac{\partial\phi^2}{\partial u^2}-\frac{\partial\phi^1}{\partial u^1}\right)-\left(c_2^2-c_1^1\right)\frac{\partial\phi^1}{\partial u^2}+c_2^1\left(\frac{\partial\xi^1}{\partial x^1}-\frac{\partial\xi^2}{\partial x^2}\right)+c_2^1\left(c_1^1+c_2^2\right)\frac{\partial\xi^1}{\partial x^2}$$
$$+\xi^i\frac{\partial c_2^1}{\partial x^i}+\phi^j\frac{\partial c_2^1}{\partial u^j}++c_0^1\left(c_2^1\frac{\partial\xi^1}{\partial u^1}+c_2^2\frac{\partial\xi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial u^2}\right)=0,$$

$$\frac{\partial\phi^1}{\partial x^1}+c_j^1\frac{\partial\phi^j}{\partial x^2}+\xi^i\frac{\partial c_0^1}{\partial x^i}+\phi^j\frac{\partial c_0^1}{\partial u^j}+c_0^1\left(\frac{\partial\xi^1}{\partial x^1}+\frac{\partial\xi^2}{\partial x^2}-c_0^1\frac{\partial\xi^1}{\partial u^1}-c_0^2\frac{\partial\xi^1}{\partial u^2}\right)=0.$$

From the second equation we have

$$\frac{\partial\xi^2}{\partial u^1}-c_1^2\frac{\partial\xi^1}{\partial u^2}+c_2^2\frac{\partial\xi^1}{\partial u^1}=0,$$

$$c_1^2\left(\frac{\partial\phi^2}{\partial u^2}-\frac{\partial\phi^1}{\partial u^1}\right)-\left(c_1^1-c_2^2\right)\frac{\partial\phi^2}{\partial u^1}+c_1^2\left(\frac{\partial\xi^1}{\partial x^1}-\frac{\partial\xi^2}{\partial x^2}\right)+c_1^2\left(c_1^1+c_2^2\right)\frac{\partial\xi^1}{\partial x^2}$$
$$+\xi^i\frac{\partial c_1^2}{\partial x^i}+\phi^j\frac{\partial c_1^2}{\partial u^j}+c_0^2\left(c_1^1\frac{\partial\xi^1}{\partial u^1}+c_1^2\frac{\partial\xi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial u^1}\right)=0,$$

$$-c_2^1\frac{\partial\phi^2}{\partial u^1}+c_1^2\frac{\partial\phi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial x^1}+c_2^2\left(\frac{\partial\xi^1}{\partial x^1}-\frac{\partial\xi^2}{\partial x^2}\right)+\xi^i\frac{\partial c_2^2}{\partial x^i}+\phi^j\frac{\partial c_2^2}{\partial u^j}$$
$$+\left[\left(c_2^2\right)^2+c_2^1c_1^2\right]\frac{\partial\xi^1}{\partial x^2}+c_0^2\left(c_2^1\frac{\partial\xi^1}{\partial u^1}+c_2^2\frac{\partial\xi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial u^2}\right)=0,$$

$$\frac{\partial\phi^2}{\partial x^1}+c_j^2\frac{\partial\phi^j}{\partial x^2}+\xi^i\frac{\partial c_0^2}{\partial x^i}+\phi^j\frac{\partial c_0^2}{\partial u^j}+c_0^2\left(\frac{\partial\xi^1}{\partial x^1}+\frac{\partial\xi^2}{\partial x^2}-c_0^1\frac{\partial\xi^1}{\partial u^1}-c_0^2\frac{\partial\xi^1}{\partial u^2}\right)=0.$$

Thus we have obtained a system of defining equations for determination of symmetries of the system (19). The system of defining equations is over-determined. The number N_d of the defining system equations is determined by expression $N_d = mN_c - N_l$. Here $m = 2$ is the number of the initial system equations, N_c is number of decomposition conditions, N_l is number of linearly dependent equations for the defining equations system. For the considered system $N_c = 4$ and $N_l = 0$ (all equations of a defining system are linearly independent). Therefore we have $N_d = 8$.

If the system of defining equations obtained by L.V. Ovsjannikov's technique [2], then the number of decomposition conditions $N_c = 6$ (decomposition under p_1^1, p_1^2, $\left(p_1^1\right)^2$, $\left(p_1^2\right)^2$, $p_1^1p_1^2$ and under absolute terms). Therefore $2N_c = 12$. Thus the general number of the equations will be also eight, as $N_l = 4$ (four equations will linearly depend on other equations).

5 About number of decomposition conditions

In more common case the number of decomposition conditions N_c for the exterior differential equations system corresponding quasilinear first order system is determined by expression

$$N_c = C_{m+n}^m - m,$$

where m and n are numbers of dependent and independent variables, C^m_{m+n} is the number of combinations from $n+m$ elements under n,

$$C^n_{m+n} = \frac{(n+m)!}{n!m!}.$$

Thus, the transformation to a system of the exterior differential equations (for a considered class of the equations) allows to lower the number of decomposition conditions for searching of symmetries and to eliminate from consideration linearly dependent equations of a defining system. In some cases it reduces complexity at deriving of the defining equations system.

Acknowledgments

The author expresses thanks to Professors K.G. Garaev and V.G. Pavlov for attention to work. Author also expresses thanks to Professor K.B. Harrison for discussions on the differential forms use during the Third International Conference “Symmetry in Nonlinear Mathematical Physics”.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 137–140.

Construction of Invariants for a System of Differential Equations in the $(n + 2m)$-Dimensional Space

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In this paper an algorithm of construction of infinitesimal operator and invariants for $(n + 2m)$-dimensional space is represented.

Let us consider the system of differential equations of the following form:

$$\begin{aligned}
&\frac{dx}{dt} = -\lambda y + X(x, y, z_1, \ldots, z_n),\\
&\frac{dy}{dt} = \lambda x + Y(x, y, z_1, \ldots, z_n),\\
&\frac{dz_j}{dt} = \sum_{i=1}^{n} r_{ji} z_i + Z_j(x, y, z_1, \ldots, z_n), \qquad j = 1, \ldots, n,
\end{aligned} \tag{1}$$

where $x, y \in \mathbb{R}^m$. Hence, the system of differential equations (1) is in $(n + 2m)$-dimensional space.

V.I. Zubov [1] suggested the following theorem:

Theorem. *A necessary and sufficient condition for the family (1) to have a family of limited solutions in a neighborhood of point $(0, \ldots, 0, z_1^o, \ldots, z_n^o)$, is the existence of m holomorphic integrals of (1) the form:*

$$c_s^2 = x_s^2 + y_s^2 + \Psi_s(x, y, z_1, \ldots, z_n).$$

By substitution $y_s = \rho_s \cos\varphi_s$, $x_s = \rho_s \sin\varphi_s$, $s = 1, \ldots, m$ we transform (1) to the form:

$$\begin{aligned}
&\frac{d\rho_s}{dt} = R_s,\\
&\frac{d\varphi_s}{dt} = \lambda_s + \Theta_s, \qquad s = 1, \ldots, m,\\
&\frac{dz_j}{dt} = \sum_{i=1}^{n} r_{ji} z_i + P_j(\rho_1 \ldots, \rho_m, \varphi_1, \ldots, \varphi_m, z_1, \ldots, z_n), \qquad j = 1, \ldots, n,
\end{aligned} \tag{2}$$

where

$$R_s = \cos\varphi_s X_s + \sin\varphi_s Y_s, \qquad \Theta_s = \frac{\cos\varphi_s Y_s - \sin\varphi_s X_s}{\rho_s},$$

$$\begin{aligned}
&P_j(\rho_1 \ldots, \rho_m, \varphi_1, \ldots, \varphi_m, z_1, \ldots, z_n)\\
&\quad = Z_j(\rho_1 \cos\varphi_1, \ldots, \rho_m \cos\varphi_m, \rho_1 \sin\varphi_1, \ldots, \rho_m \sin\varphi_m, z_1, \ldots, z_n).
\end{aligned}$$

We will seek a solution in the following form of rows:

$$\rho_s = c_s + \sum_{k=2}^{\infty} r_s^{(k)}(\varphi_1, \ldots, \varphi_m, c_1, \ldots, c_m),$$

$$z_j = \sum_{k=2}^{\infty} z_s^{(k)}(\varphi_1, \ldots, \varphi_m, c_1, \ldots, c_m).$$

By making an appropriate substitution into (2) and stating the coefficients at equal degrees as equal, we receive functions $r_s^{(k)}$, $z_j^{(k)}$. If all such functions are periodical with respect to $\varphi_1, \ldots, \varphi_m$ and at sufficiently small $\|c_s\|$, we obtain the following family of solutions:

$$\rho_s = c_s + F_s(z_1, \ldots, z_n, \varphi_1, \ldots, \varphi_m, c_1, \ldots, c_m), \qquad s = 1, \ldots, m.$$

However, in order to find ρ_s, an infinite number of differential equation needs to be solved.

To solve this problem, we will use invariants with respect to transformations of $SO(2)$. If we obtain the whole system of invariants, their number will define the number of equations, which have to be solved in order to find a solution to the original problem. In a particular case, when (1) is in the form:

$$\frac{dx_s}{dt} = -\lambda_s y_s + X_s(x_s, y_s, z_1, \ldots, z_n),$$

$$\frac{dy_s}{dt} = \lambda_s x_s + Y_s(x_s, y_s, z_1, \ldots, z_n), \qquad s = 1, \ldots, m,$$

$$\frac{dz_j}{dt} = \sum_{i=1}^{n} r_{ji} z_i + Z_j(x_1, \ldots, x_m, y_1, \ldots, y_m, z_1, \ldots, z_n), \qquad j = 1, \ldots, n$$

the quantity of invariants for each pair of imaginary numbers is obtained in [2, 3].

Let us build an infinitesimal operator for finding invariants of (1).

To do this, we will consider one pair of imaginary numbers and corresponding equations:

$$\begin{aligned} \frac{dx_s}{dt} &= -\lambda_s y_s + X_s(x_1, \ldots, x_m, y_1, \ldots, y_m, z_1, \ldots, z_n), \\ \frac{dy_s}{dt} &= \lambda_s x_s + Y_s(x_1, \ldots, x_m, y_1, \ldots, y_m, z_1, \ldots, z_n), \qquad s = 1, \ldots, m. \end{aligned} \tag{3}$$

If X_s and Y_s are viewed as polynomials of the variables $z_1, \ldots, z_n$, let us consider X_s and Y_s at random monomial $z_1, \ldots, z_n$. Suppose that

$$X_s = \sum_{i_1+i_2+\ldots+i_{2m}=\{l\}} c_{i_1\ldots i_{2m}} x_1^{i_1} \ldots x_m^{i_m} y_1^{i_1} \ldots y_m^{i_m},$$

$$Y_s = \sum_{i_1+i_2+\ldots+i_{2m}=\{l\}} b_{i_1\ldots i_{2m}} x_1^{i_1} \ldots x_m^{i_m} y_1^{i_1} \ldots y_m^{i_m}.$$

For building an infinitesimal operator, we use the same method, with is used in case $m = 1$ [3]. The right part of the system of equations is written in a matrix form G_{sl}. Then, after a transformation at the $SO(2)$ (a rotation by $\delta = (\delta_1, \ldots, \delta_m)$), the variables will change

$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} = \Delta_s \begin{pmatrix} \bar{x}_s \\ \bar{y}_s \end{pmatrix},$$

where

$$\Delta_s = \begin{pmatrix} \cos\delta_s & -\sin\delta_s \\ \sin\delta_s & \cos\delta_s \end{pmatrix}.$$

Accordingly, the matrix G_{sl} will also change. The new matrix will be in the form:

$$\bar{G}_{sl}(\delta) = \Delta_s^{-1} G_{sl} D_s(\delta).$$

Respectively, each element of $\bar{G}_{sl}$ is a linear transformation of the elements of G_{sl}

$$\bar{g}_{sl}^{(ij)} = B(\delta) g_{sl}.$$

In $\delta = 0$, we seek the differential $\bar{g}_{sl}^{(ij)}$

$$d\bar{g}_{sl}^{(ij)} = \frac{\partial B(\delta)}{\partial \delta} G_{sl} \,|_{\delta=0}\, d\delta = k_{sl}^{(ij)} d\delta.$$

Therefore, the infitesimal operator of the $SO(2)$ group for the right part of (3) with homogeneous polynomials of degree l, may be written in the following way:

$$U_s = \frac{\partial}{\partial \delta} + \sum_{i,j} k_{sl}^{(ij)} \frac{\partial}{\partial g_{sl}^{(ij)}},$$

where $\sum\limits_{i,j} k_{sl}^{(ij)} \frac{\partial}{\partial g_{sl}^{(ij)}}$ is a linear transformation of the elements $k_{sl}^{(ij)} \frac{\partial}{\partial g_{sl}^{(ij)}}$.

If the right hand part represents a sum of homogeneous spaces $L_s = L_{\otimes p_1} + L_{\otimes p_2} + \cdots + L_{\otimes p_s}$, then the infinitesimal operator we seek will be expressed as a sum of infinitesimal operators (as shown in [3]) from respective spaces $L_{\otimes l}$

$$U_s = \frac{\partial}{\partial \delta} + \sum_{l=p_1}^{p_s} \sum_{i,j} k_{sl}^{(ij)} \frac{\partial}{\partial g_{sl}^{(ij)}}, \qquad s = 1, \dots m. \tag{4}$$

Now we assume that all previous conditions, valid for the pair of imaginary solutions, are preserved. Let us consider a general infinitesimal operator for the whole system. Then, for each pair of imaginary solutions $i\lambda$, $s = 1, \dots, m$, the infinitesimal operator will be obtained using the same method and will have the form (4). Thus, the general infinitesimal operator will look:

$$U = \sum_{s=1}^{m} U_s. \tag{5}$$

Theorem. *Let the right hand side of (1) be fixed with respect to the variables $z_1, \dots, z_n$. We will consider a part of the system, which is a system of degree $2m$. Then the invariants of the $SO(2)$ group in the space of coefficients are solutions for the differential equation $Uf = 0$, where U is expressed in (5).*

Example. Consider the system

$$\frac{dx_1}{dt} = -y_1 + a_{11}x_1 + a_{12}y_1 + a_{13}x_2 + a_{14}y_2,$$
$$\frac{dy_1}{dt} = x_1 + a_{21}x_1 + a_{22}y_1 + a_{23}x_2 + a_{24}y_2,$$
$$\frac{dx_2}{dt} = -y_2 + a_{31}x_1 + a_{32}y_1 + a_{33}x_2 + a_{34}y_2,$$
$$\frac{dy_2}{dt} = x_2 + a_{41}x_1 + a_{42}y_1 + a_{43}x_2 + a_{44}y_2.$$

By substitution

$$x_i = \frac{1}{2}(\bar{w}_i + w_i), \qquad y_i = \frac{i}{2}(\bar{w} - w)$$

we transform the system to the form

$$\begin{aligned}\frac{dw_1}{dt} &= \frac{1}{2}\bar{w}_1(a_{11} + ia_{12} + ia_{21} - a_{22}) + \frac{1}{2}w_1(a_{11} - i_{12} + ia_{21} + a_{22})\\ &+ \frac{1}{2}\bar{w}_2(a_{13} + ia_{14} + ia_{23} - a_{24}) + \frac{1}{2}w_2(a_{13} - ia_{14} + ia_{23} + a_{24}),\\ \frac{dw_2}{dt} &= \frac{1}{2}\bar{w}_1(a_{31} + ia_{32} + ia_{41} - a_{42}) + \frac{1}{2}w_1(a_{31} - i_{32} + ia_{41} + a_{42})\\ &+ \frac{1}{2}\bar{w}_2(a_{33} + ia_{34} + ia_{43} - a_{44}) + \frac{1}{2}w_2(a_{33} - ia_{34} + ia_{43} + a_{44}).\end{aligned}$$

After substitution

$$w_j = w'_j \exp i\varphi_j, \qquad j = 1, 2$$

we receive the following form:

$$\begin{aligned}\frac{dw_1}{dt} &= \frac{1}{2}\bar{w}_1 e^{-2i\varphi_1} z_{11} + \frac{1}{2}w_1 z_{12} + \frac{1}{2}\bar{w}_2 e^{-i(\varphi_1+\varphi_2)} z_{13} + \frac{1}{2}w_2 e^{i(\varphi_2-\varphi_1)} z_{14},\\ \frac{dw_2}{dt} &= \frac{1}{2}\bar{w}_1 e^{-i(\varphi_1+\varphi_2)} z_{21} + \frac{1}{2}w_1 e^{i(\varphi_1-\varphi_2)} z_{22} + \frac{1}{2}\bar{w}_2 e^{-2i\varphi_2} z_{23} + \frac{1}{2}w_2 z_{24}.\end{aligned}$$

Find differentials

$$\begin{aligned}&dz_{11} = -2ie^{-2i\varphi_1} z_{11} d\varphi_1, && dz_{13} = -iz_{13}e^{-i(\varphi_1+\varphi_2)}(d\varphi_1 + d\varphi_2),\\ &dz_{14} = ie^{i(\varphi_2-\varphi_1)} z_{14}(d\varphi_2 - d\varphi_1), && dz_{21} = -ie^{-i(\varphi_1+\varphi_2)} z_{21}(d\varphi_1 + d\varphi_2),\\ &dz_{22} = ie^{i(\varphi_1-\varphi_2)} z_{22}(d\varphi_2 - d\varphi_2), && dz_{23} = -2ie^{-2i\varphi_2} z_{23} d\varphi_2.\end{aligned}$$

Build infitesimal operator

$$\begin{aligned}U = \frac{\partial}{\partial\varphi} &- 2ie^{-2i\varphi_1} z_{11}\frac{\partial}{\partial z_{11}} - ie^{-i(\varphi_1+\varphi_2)} z_{13}\frac{\partial}{\partial z_{13}} + ie^{i(\varphi_2-\varphi_1)} z_{14}\frac{\partial}{\partial z_{14}}\\ &-ie^{-i(\varphi_1+\varphi_2)} z_{21}\frac{\partial}{\partial z_{21}} + ie^{i(\varphi_1-\varphi_2)} z_{22}\frac{\partial}{\partial z_{22}} + dz_{23} = -2ie^{-2i\varphi_2} z_{23}\frac{\partial}{\partial z_{23}}.\end{aligned}$$

The invariants

$$z_{12},\ z_{24},\ z_{14}z_{22},\ z_{11}\bar{z}_{11},\ z_{23}\bar{z}_{23},\ z_{13}\bar{z}_{21}$$

are solutions of equations $Uz = 0$.

Hence, for the existence of family of limited solutions in a neighborhood of point $(0, 0, 0, 0)$, we have only 6 conditions.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 141–145.

The Ovsjannikov's Theorem on Group Classification of a Linear Hyperbolic Type Partial Differential Equation Revisited

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The group classification of linear hyperbolic partial differential equation is carried out with the use of the new approach to solving group classification problems suggested recently by Zhdanov and Lahno (**J. Phys. A: Math. Gen., V.32, 7405 (1999)**).

1. Consider a partial differential equation of the hyperbolic type

$$u_{tx} + A(t,x)u_t + B(t,x)u_x + C(t,x)u = 0, \tag{1}$$

where $u = u(t,x)$, $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{tx} = \frac{\partial^2 u}{\partial t \partial x}$. Group classification of equations (1) admitting non-trivial (finite-parameter) symmetry group has been performed by L.V. Ovsjannikov [1, 2]. His classification scheme is based on using the Laplace invariants

$$h = A_t + AB - c, \qquad k = B_x + AB - C.$$

The results obtained can be formulated as follows.

Theorem 1 (Ovsjannikov [1, 2]). *Equation (1) admits a Lie algebra of the dimension higher than 1 if and only if the functions*

$$p = \frac{k}{h}, \qquad q = \frac{\partial_x \partial_y (\ln h)}{h}$$

are constant. If p and q are constant, then equation (1) is equivalent either to the Euler–Poisson equation $(q \neq 0)$

$$u_{tx} - \frac{2u_t}{q(t+x)} - \frac{2pu_x}{q(t+x)} + \frac{4pu}{q^2(t+x)^2} = 0 \tag{2}$$

or to equation $(q = 0)$

$$u_{tx} + tu_t + pxu_x + ptxu = 0 \tag{3}$$

and its symmetry algebra is a three-dimensional Lie algebra L^3.

What is more, Ovsjannikov has proved that the basis of the Lie algebra L^3 is formed by the operators

$$\partial_t - \partial_x, \qquad t\partial_t + x\partial_x, \qquad t^2\partial_t - x^2\partial_x + \frac{2}{q}(pt - x)u\partial_u$$

for equation (2), and by the operators

$$t\partial_t - x\partial_x, \qquad \partial_t - xu\partial_u, \qquad \partial_x - ptu\partial_u$$

for equation (3).

In this paper we perform group classification of equation (1) by using an alternative approach suggested in [3].

2. Using the infinitesimal Lie method we obtain that equation (1) is invariant under infinite-dimensional transformation group, which is generated by the operator

$$X_\infty = \omega(t,x)\partial_u, \qquad \omega_{tx} + A\omega_t + B\omega_x + C\omega = 0,$$

and under the one-parameter transformation group, whose infinitesimal operators reads as

$$X = f(t)\partial_t + g(x)\partial_x + h(t,x)u\partial_u, \tag{4}$$

where

$$\begin{aligned} &h_t + B\dot{f} + fB_t + gB_x = 0,\\ &h_x + Ag' + gA_x + fA_t = 0,\\ &h_{tx} + C\dot{f} + fC_t + Cg' + gC_x + Ah_t + Bh_x = 0. \end{aligned} \tag{5}$$

In (5) the following notations are used, $\dot{f} = \frac{df}{dt}$, $g' = \frac{dg}{dx}$.

Furthermore, as the direct calculations show, the equivalence group of the equation (1) is a superposition of the following transformations:

$$\begin{aligned} &(a) \quad \tau = \alpha(t), \quad \xi = \beta(x), \quad v = \theta(t,x)u + \rho(t,x),\\ &(b) \quad \tau = \alpha(x), \quad \xi = \beta(t), \quad v = \theta(t,x)u + \rho(t,x), \end{aligned} \tag{6}$$

where α and β are arbitrary smooth functions, $\theta \neq 0$ and θ, ρ satisfy the condition

$$\theta_t\rho_x + \rho_t\theta_x - \theta\rho_{tx} + \rho\theta_{tx} - 2\rho\theta^{-1}\theta_t\theta_x - C\theta\rho = 0.$$

In order to perform group classification of equation (1), we start with studying realizations of real Lie algebras within the class of operators (4) up to the equivalence relation determined by transformations (6). As a next step, we select those realizations, that form bases of invariance algebras of equations (1).

Remark 1. We use the known classification of non-isomorphic real Lie algebras (see, for example, [4, 5]).

Remark 2. Equation

$$u_{tx} = 0 \tag{7}$$

is invariant under infinite-dimensional transformation group, which is generated by the operator

$$X_\infty = f(t)\partial_t + g(x)\partial_x + \lambda u\partial_u,$$

where f and g are arbitrary smooth functions and $\lambda = \text{const}$. What is more, its general solution reads as

$$u = \varphi(t) + \psi(x)$$

with arbitrary smooth functions φ, ψ. Furthermore, the equation

$$u_{tx} + B(x)u_x = 0, \qquad B \neq 0, \tag{8}$$

has the following general solution:

$$u = \int \varphi(x)e^{-tB(x)}dx + \psi(t),$$

where φ, ψ are arbitrary smooth functions.

Therefore, we consider only equations of the form (1), which are inequivalent to (7) and (8).

It is well-known, that a linear partial differential equation of the form (1) is invariant under the operator $u\partial_u$ and this operator satisfies the following commutation relation:

$$[X, u\partial_u] = 0,$$

where X has the form (4).

Consequently, the function $h(t,x)$ in operator (3) is determined up to a constant summand.

The list of non-isomorphic two-dimensional real Lie algebras is exhausted by the following two algebras:

$$A_{2.1} = \langle e_1, e_2\rangle, \qquad [e_2, e_2] = 0;$$
$$A_{2.2} = \langle e_1, e_2\rangle, \qquad [e_2, e_2] = e_2.$$

If these algebras are maximal invariance algebras of equation (1), then one of their basis operators must coincide with the operator $u\partial_u$. Consequently, we have to consider realizations of the algebra $A_{2.1}$ only.

Proposition 1. *Let the algebra $A_{2.1}$ be invariance algebra of equation (1). The set of inequivalent realizations of this algebra is exhausted by the following two realizations:*

$$A_{2.1} = \langle u\partial_u, \partial_t\rangle;$$
$$A_{2.2} = \langle u\partial_u, \partial_t + \partial_x\rangle.$$

The corresponding invariant equations can be taken in the following form:

$$A^1_{2.1}: \; u_{tx} + B(x)u_x + u = 0; \tag{9}$$

$$A^2_{2.2}: \; u_{tx} + B(z)u_x + C(z)u = 0, \qquad z = t - x, \quad C \neq 0. \tag{10}$$

Proof. First of all we note that the operator $u\partial_u$ is invariant under action of the changes of variables (6). Choose $e_1 = u\partial_u$ as the first basis operator of the Let in the algebra $A_{2.1}$ and let the second basis operator e_2 have the general form (4).

If $f \cdot g \neq 0$ in the operator e_2, then making the change of variables (6), where α, β, θ, ρ are solutions of the system of differental equations

$$\dot\alpha f = 1, \qquad \beta' g = 1, \qquad f\theta_t + g\theta_x + h\theta = 0, \qquad \theta \neq 0, \qquad f\rho_t + g\rho_x = 0,$$

reduces this operator to the operator

$$e_2' = \partial_\tau + \partial_\xi.$$

If $f \neq 0$, $g = 0$ in the operator e_2, then performing the change of variables (6), where $\beta = \beta(x)$, $\rho = \rho(x)$ and functions α, θ are solutions of system of differential equations

$$\dot\alpha f = 1, \qquad f\theta_t + h\theta = 0, \qquad \theta \neq 0,$$

reduces this operator to become

$$e_2' = \partial_\tau.$$

If $f = 0$, $g \neq 0$ in the operator e_2, then making another change of variables (6) ($t \to x$, $x \to t$) reduces this case to the previous one.

If, finally, $f = g = 0$ in the operator e_2, then $h \neq 0$ and

$$e_2 = h(t,x)u\partial_u, \qquad h \neq \text{const}.$$

Thus, we obtain three inequivalent realizations of the algebra $A_{2.1}$ within the class of operators (3):

$$A^1_{2.1} = \langle u\partial_u, \partial_t\rangle,$$
$$A^2_{2.1} = \langle u\partial_u, \partial_t + \partial_x\rangle,$$
$$A^3_{2.1} = \langle u\partial_u, h(t,x)u\partial_u\rangle, \qquad h \neq \text{const}.$$

The direct verification of conditions (5) for the obtained realizations yields the following results:

- Invariant equations for the first and second realizations have the form:

$$A^1_{2.1}: \ u_{tx} + A(x)u_t + B(x)u_x + C(x)u = 0, \qquad C \neq 0; \tag{11}$$

$$A^2_{2.1}: \ u_{tx} + A(z)u_t + B(z)u_x + C(z)u = 0, \qquad z = t - x. \tag{12}$$

- If the realization $A^3_{2.1}$ is invariance algebra of an equation of the form (1), then $h = \text{const}$.

Furthermore, it is not difficult to verify that the realization $A^1_{2.1}$ is invariant with respect to the change of variables

$$\tau = t + \lambda, \qquad \lambda = \text{const}, \qquad \xi = \beta(x), \qquad v = \theta(x)u + \rho(x), \qquad \theta \neq 0. \tag{13}$$

If in (13) β, θ and ρ are solutions of the system of differential equations

$$\theta_x = \theta A, \qquad \beta' = C - \theta^{-1}\partial_x B, \qquad B\theta^{-1}\theta_x\rho - B\rho_x - C\rho = 0,$$

then this change of variables reduces equation (11) to equation of the form (9).

Using analogous reasonings, it is not difficult to show that equation (12) is equivalent to (10).

Proposition 1 is proved.

Thus obtained classification of equations (1), which are invariant under two-dimensional Lie algebras, permits realizing further group classification of equation (1) by the method suggested in [3].

The system of determining equations (5) for equation (9) reads as

$$h_t + B\dot{f} + gB_x = 0, \qquad h_x = 0, \qquad \dot{f} + g' = 0. \tag{14}$$

The second and third equations from (14) imply that $h = h(t)$, $f = \lambda_1 t + \lambda_2$, $g = -\lambda_1 x + \lambda_3$, where $\lambda_1, \lambda_2, \lambda_3 = \text{const}$.

Consequently, extension of the symmetry of equation (9) is only possible, if the function B in first equation (14) within the equivalence relation has the form

$$B = mx, \qquad m = \text{const}, \qquad m \neq 0,$$

which means that equation (9) reads as

$$u_{tx} + mxu_x + u = 0, \qquad m = \text{const}. \tag{15}$$

Its invariance algebra is the four-dimensional Lie algebra

$$\langle u\partial_u, \partial_t, t\partial_t - x\partial_x, \partial_x - mtu\partial_u\rangle.$$

Analogously, we verify that extension of symmetry of equation (10) is only possible, if it has the form

$$u_{tx} + \frac{m}{z}u_x + \frac{k}{z^2}u = 0, \qquad m, k = \mathrm{const}, \quad k \neq 0, \quad z = t - x. \tag{16}$$

The invariance algebra of equation (16) is the four-dimensional Lie algebra

$$\langle u\partial_u, \partial_t + \partial_x, t\partial_t + x\partial_x + \frac{1}{2}mu\partial_u, t^2\partial_t + x^2\partial_x + mtu\partial_u\rangle.$$

Cosequently, the following assertion holds true:

Proposition 2. *Equation (1) admits a Lie algebra of infinitesimal operators (4), whose dimension is higher than two, if it is either equivalent to equation (15) or to (16), its invariance algebra being necessarily four-dimensional.*

It is straightforward to verify that the results obtained in Proposition 2 are equivalent to results obtained by Ovsjannikov.

Acknowledgements

Authors thank Renat Zhdanov for his critical remarks and interest to the paper.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 146–150.

Towards a Classification of Realizations of the Euclid Algebra $e(3)$

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We classify realizations of the Lie algebras of the rotation $O(3)$ and Euclid $E(3)$ groups within the class of first-order differential operators in arbitrary finite dimensions. It is established that there are only two distinct realizations of the Lie algebra of the group $O(3)$ which are inequivalent within the action of a diffeomorphism group. Using this result we describe a special subclass of realizations of the Euclid algebra which are called covariant.

1. In the present paper we study realizations of the Lie algebra of the Euclid group $E(3)$ (which will be called in the sequel the Euclid algebra $e(3)$) within the class of Lie vector fields on the space $V = X \otimes U$ of independent and dependent variables. In the case under study X is the three-dimensional Euclid space having the coordinates $x = (x_1, x_2, x_3)$; U is the space of real-valued scalar functions $u(x) = (u_1(x), u_2(x), \dots, u_n(x))$, and Lie vector fields are first-order differential operators of the form

$$Q = \xi_a(x,u)\partial_{x_a} + \eta_i(x,u)\partial_{u_i}, \tag{1}$$

where ξ_a, η_i ($a = 1, 2, 3$; $i = 1, \dots, n$) are some sufficiently smooth real-valued functions defined on the space V, $\partial_{x_a} = \frac{\partial}{\partial x_a}$, $\partial_{u_i} = \frac{\partial}{\partial u_i}$. Hereafter, we use the summation convention for the repeated indices.

We say that the operators P_a, J_b ($a, b = 1, 2, 3$) belonging to class (1) form a basis of the realization of the Euclid algebra $e(3)$ if (a) they are linearly independent, and (b) they satisfy the following commutation relations:

$$[P_a, P_b] = 0, \tag{2}$$

$$[J_a, P_b] = \varepsilon_{abc} P_c, \tag{3}$$

$$[J_a, J_b] = \varepsilon_{abc} J_c, \tag{4}$$

where

$$\varepsilon_{abc} = \begin{cases} 1, & (abc) = \text{cycle}\,(123), \\ -1, & (abc) = \text{cycle}\,(213), \\ 0, & \text{in the remaining cases.} \end{cases}$$

The realization of the Euclid algebra $e(3)$ within the class of Lie vector fields (1) is called covariant if coefficients of the basis elements

$$P_a = \xi^{(1)}_{ab}(x,u)\partial_{x_b} + \eta^{(1)}_{ai}(x,u)\partial_{u_i} \qquad (a, b = 1, 2, 3;\ i = 1, \dots, n) \tag{5}$$

satisfy the following condition:

$$\mathrm{rank}\left\|\begin{matrix} \xi_{11}^{(1)} & \xi_{12}^{(1)} & \xi_{13}^{(1)} & \eta_{11}^{(1)} & \dots & \eta_{1n}^{(1)} \\ \xi_{21}^{(1)} & \xi_{22}^{(1)} & \xi_{23}^{(1)} & \eta_{21}^{(1)} & \dots & \eta_{2n}^{(1)} \\ \xi_{31}^{(1)} & \xi_{32}^{(1)} & \xi_{33}^{(1)} & \eta_{31}^{(1)} & \dots & \eta_{3n}^{(1)} \end{matrix}\right\| = 3. \tag{6}$$

It is easy to check that the relations (2)–(4) are invariant with respect to an arbitrary invertible transformation of variables x, u

$$y_a = f_a(x,u), \quad a = 1,2,3; \qquad v_i = g_i(x,u) \quad i = 1,\dots,n, \tag{7}$$

where f_a, g_i are sufficiently smooth functions defined on the space V. That is why we can introduce on the set of realizations of the Euclid algebra $e(3)$ the following relation: two realizations of the algebra $e(3)$ are called equivalent if they are transformed one into another by means of an invertible transformation (7). As invertible transformations of the form (7) form a group (called diffeomorphism group), this relation is the equivalence relation. It divides the set of all realizations of the Euclid algebra into equivalence classes $A_1,\dots,A_r$. Consequently, to describe all possible realizations of $e(3)$ it suffices to construct one representative of each equivalence class A_j, $j = 1,\dots,r$.

2. As it follows from commutation relations (2)–(4) of the algebra $e(3)$, the latter is the semi-direct sum of the commutative ideal $t^3 = \langle P_1, P_2, P_3\rangle$ and of the simple algebra $so(3) = \langle J_1, J_2, J_3\rangle$. That is why we start investigation of covariant realizations of the algebra $e(3)$ by studying realizations of the translation generators P_a $(a = 1,2,3)$ within the class of operators (1). To this end we will make use of the following lemma.

Lemma 1. *Let the operators P_a $(a = 1,2,3)$ of the form (5) satisfy relation (6). Then there exists a transformation of the form (7) reducing the operators P_a to become $P'_a = \partial_{y_a}$, $a = 1,2,3$.*

Proof. In view of (6) $P_a \neq 0$ for all $a = 1,2,3$. It is well-known [1] that a non-zero operator

$$P_1 = \xi_{1b}^{(1)}(x,u)\partial_{x_b} + \eta_{1i}^{(1)}(x,u)\partial_{u_i}$$

can be always reduced to the form $P'_1 = \partial_{y_1}$ by transformation (7). If we denote by P'_2, P'_3 the operators P_2, P_3 written in the new variables y, v, then owing to commutation relations (2) they commute with the operator $P'_1 = \partial_{y_1}$. Hence, we conclude that their coefficients are independent of y_1.

Furthermore, due to the condition (6) at least one of the coefficients $\xi_{22}^{\prime(1)}$, $\xi_{23}^{\prime(1)}$, $\eta_{21}^{\prime(1)}$, …, $\eta_{2n}^{\prime(1)}$ of the operator P'_2 is not equal to zero.

Summing up, we conclude that the operator P'_2 is of the form

$$P'_2 = \xi_{2b}^{\prime(1)}(y_2,y_3,v)\partial_{y_b} + \eta_{2i}^{\prime(1)}(y_2,y_3,v)\partial_{v_i},$$

not all the functions $\xi_{22}^{\prime(1)}$, $\xi_{23}^{\prime(1)}$, $\eta_{21}^{\prime(1)},\dots,\eta_{2n}^{\prime(1)}$ being identically equal to zero.

Making a transformation

$$\begin{aligned} &z_1 = y_1 + F(y_2,y_3,v), \qquad z_2 = G(y_2,y_3,v), \\ &z_3 = \omega_0(y_2,y_3,v), \qquad \omega_i = \omega_i(y_2,y_3,v), \quad i = 1,\dots,n, \end{aligned} \tag{8}$$

where the functions F, G are particular solutions of differential equations

$$\begin{aligned} &\xi_{22}^{\prime(1)}(y_2,y_3,v)F_{y_2} + \xi_{23}^{\prime(1)}(y_2,y_3,v)F_{y_2} + \eta_{2i}^{\prime(1)}(y_2,y_3,v)F_{u_i} + \xi_{21}^{\prime(1)}(y_2,y_3,v) = 0, \\ &\xi_{22}^{\prime(1)}(y_2,y_3,v)G_{y_2} + \xi_{23}^{\prime(1)}(y_2,y_3,v)G_{y_3} + \eta_{2i}^{\prime(1)}(y_2,y_3,v)G_{u_i} = 1 \end{aligned}$$

and $\omega_0, \omega_1, \dots, \omega_n$ are functionally-independent first integrals of the Euler–Lagrange system

$$\frac{dy_2}{\xi'^{(1)}_{22}} = \frac{dy_3}{\xi'^{(1)}_{23}} = \frac{dv_1}{\eta'^{(1)}_{21}} = \cdots = \frac{dv_n}{\eta'^{(1)}_{2n}},$$

which has exactly $n+1$ functionally-independent integrals, we reduce the operator P'_2 to the form $P''_2 = \partial_{z_2}$. It is easy to check that transformation (8) does not alter the form of the operator P'_1. Being rewritten in the new variables z, ω it reads as $P''_1 = \partial_{z_1}$.

As the right-hand sides of (8) are functionally-independent by construction, transformation (8) is invertible. Consequently, operators P_a are equivalent to operators P''_a, where $P''_1 = \partial_{z_1}$, $P''_2 = \partial_{z_2}$ and

$$P''_3 = \xi''^{(1)}_{3b}(z_3, \omega)\partial_{z_b} + \eta''^{(1)}_{3i}(z, \omega)\partial_{\omega_i} \neq 0$$

(coefficients of the above operator are independent of z_1, z_2 because of the fact that it commutes with the operators P''_1, P''_2). And what is more, due to (6) at least one of the coefficients $\xi''^{(1)}_{33}, \eta''^{(1)}_{31}, \dots, \eta''^{(1)}_{3n}$ of the operator P''_3 is not identically equal to zero.

It is not difficult to verify that there exists the invertible transformation

$$\begin{aligned} &Z_1 = z_1 + F(z_3, \omega), \qquad Z_2 = z_2 + G(z_3, \omega), \\ &Z_3 = H(z_3, \omega), \qquad W_i = \Omega_i(z_3, \omega), \quad i = 1, \dots, n, \end{aligned}$$

which reduces the operators P''_a, $a = 1, 2, 3$ to the form $P'''_a = \partial_{z_a}$, $a = 1, 2, 3$.

Lemma is proved.

Due to Lemma 1 the operators P_a can be reduced to the form $P_a = \partial_{x_a}$ by means of a properly chosen transformation (7). Inserting the operators

$$P_a = \partial_{x_a}, \qquad J_a = \xi_{ab}(x, u)\partial_{x_b} + \eta_{ai}(x, u)\partial_{u_i}, \qquad a, b = 1, 2, 3; \quad i = 1, \dots, n,$$

into commutation relations (3) and equating the coefficients of the linearly-independent operators ∂_{x_a}, ∂_{u_i} ($a = 1, 2, 3$; $i = 1, \dots, n$) we arrive at the system of partial differential equations for the functions $\xi_{ab}(x, u)$, $\eta_{ai}(x, u)$

$$\xi_{acx_b} = -\varepsilon_{abc}, \qquad \eta_{aix_b} = 0, \qquad a, b, c = 1, 2, 3, \quad i = 1, \dots, n.$$

Integrating the above system we conclude that the operators J_a have the form

$$J_a = -\varepsilon_{abc}x_b\partial_{x_c} + j_{ab}(u)\partial_{x_b} + \tilde{\eta}_{ai}(u)\partial_{u_i}, \qquad a, b = 1, 2, 3, \quad i = 1, \dots, n, \tag{9}$$

where j_{ab}, $\tilde{\eta}_{ab}$ are arbitrary smooth functions.

Inserting (9) into the commutation relations (4) and equating the coefficients of ∂_{u_i} ($i = 1, \dots, n$) show that the operators $\mathcal{J}_a = \tilde{\eta}_{ai}\partial_{u_i}$, ($a = 1, 2, 3$) have to fulfill (4) with $J_a \to \mathcal{J}_a$.

Lemma 2. *Let first-order differential operators*

$$\mathcal{J}_a = \eta_{ai}(u)\partial_{u_i}, \qquad a = 1, 2, 3, \quad i = 1, \dots, n, \tag{10}$$

satisfy commutation relations (4) of the Lie algebra so(3). Then either all of them are equal to zero, i.e.

$$\mathcal{J}_a = 0, \qquad a = 1, 2, 3, \tag{11}$$

or there exists a transformation

$$v_i = F_i(u), \qquad i = 1, \dots, n,$$

reducing these operators to one of the following forms:

$$\begin{aligned} 1.\quad & \mathcal{J}_1 = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}, \\ & \mathcal{J}_2 = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}, \\ & \mathcal{J}_3 = \partial_{u_1}; \end{aligned} \tag{12}$$

$$\begin{aligned} 2.\quad & \mathcal{J}_1 = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} + \sin u_1 \sec u_2 \partial_{u_3}, \\ & \mathcal{J}_2 = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} + \cos u_1 \sec u_2 \partial_{u_3}, \\ & \mathcal{J}_3 = \partial_{u_1}. \end{aligned} \tag{13}$$

The proof of Lemma 2 requires long cumbersome calculations which are omitted here.

Notice that the set of inequivalent realizations of the Lie algebra $so(3)$ within the class of first-order differential operators (10) is exhausted by the realizations given in (12), (13).

Hence, taking into account Lemma 2 we conclude that any covariant realization of the algebra $e(3)$ is equivalent to the following one:

$$P_a = \partial_{x_a}, \qquad J_a = -\varepsilon_{abc} x_b \partial_{x_c} + j_{ab}(u)\partial_{x_b} + \mathcal{J}_a, \qquad a,b,c = 1,2,3, \tag{14}$$

operators $\mathcal{J}_a$ being given by one of formulae (11)–(13).

Making a transformation

$$y_a = x_a + F_a(u), \qquad v_i = u_i, \qquad a = 1,2,3, \quad i = 1,\ldots,n,$$

we reduce operators J_a from (14) to become

$$\begin{aligned} & J_1 = -y_2\partial_{y_3} + y_3\partial_{y_2} + A\partial_{y_1} + B\partial_{y_2} + C\partial_{y_3} + \mathcal{J}_1, \\ & J_2 = -y_3\partial_{y_1} + y_1\partial_{y_3} + F\partial_{y_2} + G\partial_{y_3} + \mathcal{J}_2, \\ & J_3 = -y_1\partial_{y_2} + y_2\partial_{y_1} + H\partial_{y_3} + \mathcal{J}_3, \end{aligned} \tag{15}$$

where A, B, C, F, G, H are arbitrary smooth functions of $v_1,\ldots,v_n$.

Substituting the operators (15) into (4) and equating the coefficients of linearly-independent operators ∂_{y_1}, ∂_{y_2}, ∂_{y_3}, ∂_{v_i} $(i = 1,\ldots,n)$ result in the following system of partial differential equations:

$$\begin{aligned} & \mathcal{J}_2 A = -C, \qquad \mathcal{J}_3 C - \mathcal{J}_1 H = G, \qquad \mathcal{J}_3 F = -B, \\ & \mathcal{J}_1 G - \mathcal{J}_2 C = H - A - F, \qquad \mathcal{J}_3 A = B, \qquad \mathcal{J}_3 B = F - A - H, \\ & \mathcal{J}_1 F - \mathcal{J}_2 B = G, \qquad A - F - H = 0, \qquad \mathcal{J}_2 H - \mathcal{J}_3 G = C. \end{aligned} \tag{16}$$

Analyzing system (16) we arrive at the following assertion.

Theorem 1. *Any covariant realizations of the algebra $e(3)$ within the class of first-order differential operators is equivalent to one of the following realizations:*

$$1.\quad P_a = \partial_{x_a}, \qquad J_a = -\varepsilon_{abc} x_b \partial_{x_c}, \qquad a,b,c = 1,2,3;$$

$$\begin{aligned} 2.\quad & P_a = \partial_{x_a}, \qquad a = 1,2,3, \\ & J_1 = -x_2\partial_{x_3} + x_3\partial_{x_2} + f\partial_{x_1} - f_{u_2}\sin u_1 \partial_{x_2} - \sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}, \\ & J_2 = -x_3\partial_{x_1} + x_1\partial_{x_3} + f\partial_{x_2} - f_{u_2}\cos u_2 \partial_{x_3} - \cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}, \\ & J_3 = -x_1\partial_{x_2} + x_2\partial_{x_1} + \partial_{u_1}; \end{aligned}$$

3. $P_a = \partial_{x_a}, \qquad a = 1, 2, 3,$
$$J_1 = -x_2\partial_{x_3} + x_3\partial_{x_2} + g\partial_{x_1} - (\sin u_1 g_{u_2} + \cos u_1 \sec u_2 g_{u_3})\partial_{x_3} - \sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} + \sin u_1 \sec u_2 \partial_{u_3},$$
$$J_2 = -x_3\partial_{x_1} + x_1\partial_{x_3} + g\partial_{x_2} - (\cos u_1 g_{u_2} - \sin u_1 \sec u_2 g_{u_3})\partial_{x_3} - \cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} + \cos u_1 \sec u_2 \partial_{u_3},$$
$$J_3 = -x_1\partial_{x_2} + x_2\partial_{x_1} + \partial_{u_1}.$$

Here $f = f(u_2, \ldots, u_n)$ is given by the formula
$$f = \alpha \sin u_2 + \beta \left(\sin u_2 \ln \frac{\sin u_2 + 1}{\cos u_2} - 1 \right),$$
α, β are arbitrary smooth functions of $u_3, \ldots, u_n$ and $g = g(u_2, \ldots, u_n)$ is a solution of the following linear partial differential equation:
$$\cos^2 u_2 g_{u_2u_2} + g_{u_3u_3} - \sin u_2 \cos u_2 g_{u_2} + 2\cos^2 u_2 g = 0.$$

3. Summarizing the results obtained in the previous section yields the following structure of realizations of the Lie algebra $so(3)$ by Lie vector fields in n variables.

1. If $n = 1$, then there are no non-zero realizations.
2. As there is no realization of $so(3)$ by real non-zero 2×2 matrices, the only non-zero realizations is given by (12).
3. In the case $n = 3$ there are two more inequivalent realizations (12) and (13).
4. Provided $n > 3$, there is no new realizations of $so(3)$ and, furthermore. any realization can be reduced to a linear one.

Notice that a complete description of covariant realizations of the conformal algebra $c(n, m)$ in the space of $n+m$ independent and one dependent variables was obtained in [2, 3]. Some new realizations of the Galilei algebra $g(1, 3)$ were suggested in [4]. Yehorchenko [5], and Fushchych, Tsyfra and Boyko [6] have constructed new (nonlinear) realizations of the Poincaré algebras $p(1, 2)$ and $p(1, 3)$ correspondingly. Complete description of realizations of the Galilei algebra $g_2(1, 1)$ in the space of two dependent and two independent variables was obtained in [7, 8].

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 151–156.

Realizations of the Euclidean Algebra within the Class of Complex Lie Vector Fields

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We obtained a complete description of inequivalent realizations of the Euclidean algebra in the class of Lie vector fields with three independent and n dependent variables. In particular, principally new nonlinear realizations of the above algebra are constructed. We also construct functional bases of differential invariants for one realization of the Euclidean algebra and one realization of the extended Euclidean algebra.

As is well known, the problem of classifying linear and nonlinear partial differential equations (PDEs) admitting some Lie transformation group G is closely connected to that of describing inequivalent realizations of its Lie algebra AG in the class of differential operators of the first order or Lie vector fields (LVFs) [1–3]. Having realizations of the Lie algebra AG, we can to construct all PDEs admitting the group G by means of the Lie infinitesimal method [1, 2]. Fushchych and Yehorchenko found the complette set of first- and second-order differential invariants for the standard realizations of the Poincaré group $P(1,n)$, Euclidean group $E(n)$ [4] and for nonlinear realization of $P(1,n)$ [10, 11].

Rideau, Winternitz [5] and Zhdanov, Fushchych [6] have done complete description of inequivalent realizations of the Galilean group and its natural extensions in the class of LVFs with two independent and two dependent variables. Results are used in constructing of the general evolution equation of the second order

$$\Psi_t + F(t, x, \Psi, \Psi^*, \Psi_x, \Psi^*_x, \Psi_{xx}, \Psi^*_{xx}) = 0,$$

invariant under the Galilean, Galilean-similitude, and Schrödinger groups. All second-order PDEs, invariant under the Poincaré group, extended Poincaré group and conformal group in a two-dimensional space are constructed in [7, 8].

In this paper we study realizations of the Lie algebra $AE(3)$ of the Euclidean group $E(3)$ on the space $X \otimes U$ of complex variables $x = (x_1, x_2, x_3)$ and $u = (u_1, \ldots, u_n)$.

1. Consider a problem of constructing realizations of the Lie algebra $AE(3)$ in the class of Lie vector fields (LVF realizations) of the form

$$Q_a = \xi^{ab}(x, u)\partial_{x_b} + \eta^{aj}(x, u)\partial_{u_j}, \tag{1}$$

Here ξ^{ab}, η^{aj} are some sufficiently smooth complex functions on the space $X \otimes U$. We use the notation $\partial_{x_b} = \frac{\partial}{\partial x_b}$, $\partial_{u_j} = \frac{\partial}{\partial u_j}$ and we sum over repeated indices ($a, b = 1, 2, 3$, $j = 1, 2, \ldots, n$).

Definition 1. *We say that operators P_a, J_b $(a, b, c = 1, 2, 3)$ of the form (1) compose a realization of the Euclidean algebra $AE(3)$ in the class of Lie vector fields if*

- *they are linearly independent,*
- *they satisfy the following commutation relations:*

$$[P_a, P_b] = 0, \qquad [P_a, J_b] = i\varepsilon_{abc}P_c, \tag{2}$$

$$[J_a, J_b] = i\varepsilon_{abc}J_c. \tag{3}$$

In the above formulae $[Q_1, Q_2] \equiv Q_1Q_2 - Q_2Q_1$ is the commutator; $a, b, c = 1, 2, 3$; ε_{abc} is third order antisymmetic tensor with $\varepsilon_{123} = 1$; i is imaginary unit: $i^2 = -1$.

Let us note that linearly independent differential operators J_b satisfying commutation relations (3), compose a realization of the Lie algebra $AO(3)$ of the rotations group.

Algebra $AE(3)$ is a semi-direct sum of the Lie algebra of the rotation group $O(3)$ and the commutative ideal $I = \langle P_1, P_2, P_3\rangle$.

Here we study LVF realizations of the Euclidean algebra $AE(3)$, where translation generators P_a are of the form

$$P_a = i\partial_{x_a}, \qquad a = 1, 2, 3. \tag{4}$$

Precisely these LVF realizations of the Euclidean algebra $AE(3)$ are important in different problems of theoretical and mathematical physics (see, e.g., [9]).

Therefore, the problem of studying all LVF realizations of the Euclidean algebra $AE(3)$ is reduced to solving of relations (2) and (3) within the class of linear first-order differential operators, where P_a are of the form (4) and J_b are of the form (1). It is known [2] that commutation relations do not change after an arbitrary nondegenerate change of variables x, u

$$\tilde{x}_\alpha = f_\alpha(x, u), \qquad \alpha = 1, 2, 3, \tag{5}$$

$$\tilde{u}_\beta = g_\beta(x, u), \qquad \beta = 1, \ldots, n, \tag{6}$$

where f_α, g_β are sufficiently smooth complex functions defined on the space $X \otimes U$. Invertible transformations (5), (6) form a transformation group (a group of diffeomorphisms) and determine a natural equivalence relations of LVF realizations of the algebra $AE(3)$. Two realizations of the Euclidean algebra are called equivalent if the corresponding basis operators can be transformed one into another by a change of variables (5) and (6).

Let P_a, J_b $(a, b = 1, 2, 3)$ be differential operator of the form (4) and (1) respectively. From the commutation relations (2) we find that

$$J_a = -i\varepsilon_{abc}x_b\partial_{x_c} + \zeta_{ab}(u)\partial_{x_b} + A_a, \qquad a = 1, 2, 3, \tag{7}$$

where A_a are operators of the form

$$A_a = \tilde{\eta}_{aj}(u)\partial_{u_j}, \tag{8}$$

which are satisfing the commutation relations

$$[A_a, A_b] = i\varepsilon_{abc}A_c, \qquad a, b, c = 1, 2, 3. \tag{9}$$

In (7), (8) ζ_{ab} and $\tilde{\eta}_{aj}$ are some smooth functions.

Therefore, we begin the classification of LVF realizations of Euclidean algebra from construction of inequivalent realizations of the Lie algebra of the rotation group in the class of operators (8).

Theorem 1. *Let differential operators A_a $(a = 1, 2, 3)$ of the form (8) satisfy commutation relations (9). Then there exist changes of variables (6), reducing these operators to one of the following triplets of operators:*

$$A_a = 0, \qquad a = 1, 2, 3; \tag{10}$$

$$A_1 = \sin u_1 \partial_{u_1}, \qquad A_2 = \cos u_1 \partial_{u_1}, \qquad A_3 = i\partial_{u_1}; \tag{11}$$

$$\begin{aligned} A_1 &= -\sin u_1 \coth u_2 \partial_{u_1} + \cos u_1 \partial_{u_2} + \varepsilon \frac{\sin u_1}{\sinh u_2}\partial_{u_3}, \\ A_2 &= -\cos u_1 \coth u_2 \partial_{u_1} - \sin u_1 \partial_{u_2} + \varepsilon \frac{\cos u_1}{\sinh u_2}\partial_{u_3}, \\ A_3 &= i\partial_{u_1}, \qquad \varepsilon = 0, 1; \end{aligned} \tag{12}$$

$$\begin{aligned} A_1 &= \sin u_1 \partial_{u_1} + \cos u_1 \partial_{u_2}, \\ A_2 &= \cos u_1 \partial_{u_1} - \sin u_1 \partial_{u_2}, \\ A_3 &= i\partial_{u_1}; \end{aligned} \tag{13}$$

$$\begin{aligned} A_1 &= \sin u_1 \partial_{u_1} + u_2 \cos u_1 \partial_{u_2} + u_2 \sin u_1 \partial_{u_3}, \\ A_2 &= \cos u_1 \partial_{u_1} - u_2 \sin u_1 \partial_{u_2} + u_2 \cos u_1 \partial_{u_3}, \\ A_3 &= i\partial_{u_1}. \end{aligned} \tag{14}$$

It follows from the theorem 1 and definition of LVF realization of Euclidean algebra that the following statement is valid

Corollary. *The algebra $AO(3)$ possesses five nonequivalent LVF realizations presented by formulae (11)–(14).*

2. Now, using the results of Theorem 1, we shall construct inequivalent LVF realizations of the Euclidean algebra where P_a $(a = 1, 2, 3)$ are of the form (4) and J_b $(b = 1, 2, 3)$ are of the form (7). We will call these realizations of the Lie algebra $AE(3)$ *covariant* realizations.

Theorem 2. *Any covariant LVF realization of the Euclidean algebra $AE(3)$ is equivalent to one of the following realizations:*

1. $$P_a = i\partial_{x_a}, \qquad J_a = -i\varepsilon_{abc} x_b \partial_{x_c}, \qquad a, b, c, = 1, 2, 3; \tag{15}$$

2. $$\begin{aligned} &P_a = i\partial_{x_a}, \qquad a = 1, 2, 3, \\ &J_1 = i(x_3\partial_{x_2} - x_2\partial_{x_3}) + \sin u_1 \partial_{u_1}, \\ &J_2 = i(x_1\partial_{x_3} - x_3\partial_{x_1}) + \cos u_1 \partial_{u_1}, \\ &J_3 = i(x_2\partial_{x_1} - x_1\partial_{x_2}) + i\partial_{u_1}; \end{aligned} \tag{16}$$

3. $$\begin{aligned} &P_a = i\partial_{x_a}, \qquad a = 1, 2, 3, \\ &J_1 = i(x_3\partial_{x_2} - x_2\partial_{x_3}) + f\partial_{x_1} - i\sin u_1 \frac{\partial f}{\partial u_2}\partial_{x_3} - \sin u_1 \coth u_2 \partial_{u_1} + \cos u_1 \partial_{u_2}, \\ &J_2 = i(x_1\partial_{x_3} - x_3\partial_{x_1}) + f\partial_{x_2} - i\cos u_1 \frac{\partial f}{\partial u_2}\partial_{x_3} - \cos u_1 \coth u_2 \partial_{u_1} - \sin u_1 \partial_{u_2}, \\ &J_3 = i(x_2\partial_{x_1} - x_1\partial_{x_2}) + i\partial_{u_1}; \end{aligned} \tag{17}$$

4. $$\begin{aligned} &P_a = i\partial_{x_a}, \qquad a = 1, 2, 3, \\ &J_1 = i(x_3\partial_{x_2} - x_2\partial_{x_3}) + g\partial_{x_1} - i\left(\sin u_1 \frac{\partial g}{\partial u_2} - \frac{\cos u_1}{\sinh u_2}\frac{\partial g}{\partial u_3}\right)\partial_{x_3} \\ &\qquad - \sin u_1 \coth u_2 \partial_{u_1} + \cos u_1 \partial_{u_2} + \frac{\sin u_1}{\sinh u_2}\partial_{u_3}, \end{aligned} \tag{18}$$

$$J_2 = i(x_1\partial_{x_3} - x_3\partial_{x_1}) + g\partial_{x_2} - i\left(\cos u_1 \frac{\partial g}{\partial u_2} + \frac{\sin u_1}{\sinh u_2}\frac{\partial g}{\partial u_3}\right)\partial_{x_3}$$
$$- \cos u_1 \coth u_2 \partial_{u_1} - \sin u_1 \partial_{u_2} + \frac{\cos u_1}{\sinh u_2}\partial_{u_3},$$
$$J_3 = i(x_2\partial_{x_1} - x_1\partial_{x_2}) + i\partial_{u_1};$$

5. $$P_a = i\partial_{x_a}, \qquad a = 1,2,3,$$
$$J_1 = i(x_3\partial_{x_2} - x_2\partial_{x_3}) + h\partial_{x_1} - i\sin u_1 \frac{\partial h}{\partial u_2}\partial_{x_3} + \sin u_1\partial_{u_1} + \cos u_1 \partial_{u_2},$$
$$J_2 = i(x_1\partial_{x_3} - x_3\partial_{x_1}) + h\partial_{x_2} - i\cos u_1 \frac{\partial h}{\partial u_2}\partial_{x_3} + \cos u_1\partial_{u_1} - \sin u_1 \partial_{u_2}, \tag{19}$$
$$J_3 = i(x_2\partial_{x_1} - x_1\partial_{x_2}) + i\partial_{u_1};$$

6. $$P_a = i\partial_{x_a}, \qquad a = 1,2,3,$$
$$J_1 = i(x_3\partial_{x_2} - x_2\partial_{x_3}) + r\partial_{x_1} - iu_2\left(\cos u_1 \frac{\partial r}{\partial u_2} + \sin u_1 \frac{\partial r}{\partial u_3}\right)\partial_{x_3}$$
$$+ \sin u_1 \partial_{u_1} + u_2 \cos u_1 \partial_{u_2} + u_2 \sin u_1 \partial_{u_3},$$
$$J_2 = i(x_1\partial_{x_3} - x_3\partial_{x_1}) + r\partial_{x_2} - iu_2\left(\sin u_1 \frac{\partial r}{\partial u_2} - \cos u_1 \frac{\partial r}{\partial u_3}\right)\partial_{x_3} \tag{20}$$
$$+ \cos u_1 \partial_{u_1} - u_2 \sin u_1 \partial_{u_2} + u_2 \cos u_1 \partial_{u_3},$$
$$J_3 = i(x_2\partial_{x_1} - x_1\partial_{x_2}) + i\partial_{u_1}.$$

Here $f = f(u_2, \ldots, u_n)$ and $h = h(u_2, \ldots, u_n)$ are given by the formulae

$$f = f_1 \cosh u_2 + f_2\left(\cosh u_2 \ln\left|\tanh \frac{u_2}{2}\right| - 1\right),$$

and

$$h = h_1 e^{-u_2} + h_2 e^{2u_2}$$

respectively, where f_1, f_2, h_1, h_2 are arbitrary function of $u_3, \ldots, u_n$; $g = g(u_2, \ldots, u_n)$ is a solution of differential equation

$$\sinh^{-2} u_2 g_{u_3u_3} + g_{u_2u_2} + \coth u_2 g_{u_2} - 2g = 0$$

and $r = r(u_2, \ldots, u_n)$ is a solution of the equation

$$u_2^2\left(r_{u_3u_3} + r_{u_2u_2}\right) - 2r = 0.$$

3. Now, we use the obtained realizations to construct PDEs, invariant under the Euclidean group.

Let X_a ($a = 1, 2, \ldots, 6$) be basis operators of Lie algebra $AE(3)$ of the Euclidean group in the space of $X \otimes U$. A differential equation

$$F(x, u, \underset{1}{u}) = 0,$$

where $\underset{1}{u}$ is a set of the first derivatives of u, is invariant under group $E(3)$ if the function F satisfies the following relations [1, 2]

$$\mathrm{pr}^{(1)} X_a F\Big|_{F=0} = 0, \qquad a = 1, 2, \ldots, 6. \tag{21}$$

Here $\mathrm{pr}^{(1)} X_a$ are first prolongations of the operators X_a.

Solving the system (21) we obtain the complete set of elementary differential invariants

$$I_r(x, u, \underset{1}{u}), \qquad r = 1, 2, \ldots, 4n-3,$$

and the invariant equation has the form

$$\Phi(I_1, I_2, \ldots, I_{4n-3}) = 0. \tag{22}$$

Hence to describe the general form of PDEs admitting Euclidean group, we must find a complete set of elementary differential invariants.

Let the basis operators of Lie algebra of Euclidean group be of the form (16). The prolongations of translation operators are equal to the P_a of (4) and prolongations of rotation operators read

$$\begin{aligned}\mathrm{pr}^{(1)} J_1 = {} & i(x_3\partial_{x_2} - x_2\partial_{x_3}) + \sin u^1 \partial_{u^1} + u^1_1 \cos u^1 \partial_{u^1_1} + (u^1_2 \cos u^1 + iu^1_3)\partial_{u^1_2} \\ & + (u^1_3 \cos u^1 - iu^1_2)\partial_{u^1_3} + i(u^k_3 \partial_{u^k_2} - u^k_2 \partial_{u^k_3}),\end{aligned} \tag{23}$$

$$\begin{aligned}\mathrm{pr}^{(1)} J_2 = {} & i(x_1\partial_{x_3} - x_3\partial_{x_1}) + \cos u^1 \partial_{u^1} - (u^1_1 \sin u^1 + iu^1_3)\partial_{u^1_1} \\ & - u^1_2 \sin u^1 \partial_{u^1_2} + (iu^1_1 - u^1_3 \sin u^1)\partial_{u^1_3} + i(u^k_1 \partial_{u^k_3} - u^k_3 \partial_{u^k_1}),\end{aligned} \tag{24}$$

$$\mathrm{pr}^{(1)} J_3 = i\{(x_2\partial_{x_1} - x_1\partial_{x_2}) + \partial_{u^1} + u^1_2 \partial_{u^1_1} - u^1_1 \partial_{u^1_2} + u^k_2 \partial_{u^k_1} - u^k_1 \partial_{u^k_2}\}, \quad k = 2, 3, \ldots, n. \tag{25}$$

Here and below we use the notation

$$u^a = u_a, \qquad u^a_b = \frac{\partial u^a}{\partial x_b}, \qquad \partial_{u^a_b} = \frac{\partial}{\partial u^a_b}.$$

Functions I_r are invariant under operators $P_a = i\partial_{x_a}$, $(a = 1, 2, 3)$, consequently they do not depend on x explictly.

Next, solving the system (21) for operator (23)–(25) we obtain the following elementary invariants:

$$I_1 = \frac{i\left(u^1_1 \sin u^1 + u^1_2 \cos u^1\right) - u^1_3}{\left(u^k_1\right)^2 + \left(u^k_2\right)^2 + \left(u^k_3\right)^2}, \qquad I^k_2 = \frac{i\left(u^k_1 \sin u^1 + u^k_2 \cos u^1\right) - u^k_3}{\sqrt{\left(u^k_1\right)^2 + \left(u^k_2\right)^2 + \left(u^k_3\right)^2}},$$

$$I^k_3 = \frac{u^k_3\left(u^1_2 \sin u^1 - u^1_1 \cos u^1\right) + u^1_3\left(u^k_1 \cos u^1 - u^k_2 \sin u^1\right) + i\left(u^1_1 u^k_2 - u^k_1 u^1_2\right)}{\left(u^k_1\right)^2 + \left(u^k_2\right)^2 + \left(u^k_3\right)^2},$$

$$I^k_4 = u^k, \qquad I^k_5 = \left(u^k_1\right)^2 + \left(u^k_2\right)^2 + \left(u^k_3\right)^2, \qquad k = 2, 3, \ldots, n.$$

The general form of the first order differential equation admitting the Euclidean group is given by (22).

4. The results of Theorem 2 can also be used for construction of inequivalent realization of the Lie algebra of the extended Euclidean group $\tilde{E}(3)$.

Definition 2. *We say that operators P_a, J_b, D $(a, b = 1, 2, 3)$ of the form (1) compose a realization of the extended Euclidean algebra $A\tilde{E}(3)$ in the class of Lie vector fields if*

- *they are linearly independent,*
- *they satisfy the commutation relations (2), (3) and the following ones*

  $$[P_a, D] = P_a, \qquad [J_b, D] = 0.$$

It is not difficult to make sure that operators (16) and the dilatation operator

$$D = i(x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}) + \varepsilon\partial_v, \tag{26}$$

where $\varepsilon = 0$ or $\varepsilon = 1$, compose realization of algebra $A\tilde{E}(3)$.

Let us consider the problem of construction of the general form of PDEs, admitting this realization of the extended Euclidean group.

The prolongation of operators P_a and J_b $(a, b = 1, 2, 3)$ are of the form (4) and (23)–(25) respectively. Prolongation of operators D reads

$$D = i(x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}) + \varepsilon\partial_v - i\left(u_1^j\partial_{u_1^j} + u_2^j\partial_{u_2^j} + u_3^j\partial_{u_3^j}\right), \qquad j = 1, \ldots, n.$$

Obviously, differential invariants of algebra $A\tilde{E}(3)$ are invariant with respect to the corresponding algebra $AE(3)$, hence, they are functions of I_1, I_2^k, I_3^k, I_4^k, I_5^k.

Taking into account all the above, instead of the operator D we can consider an operator

$$D' = \varepsilon\frac{\partial}{\partial I_4^k} + iI_1\frac{\partial}{\partial I_1} - 2iI_5\frac{\partial}{\partial I_5}.$$

Solving system (21), corresponding to above operator we construct the following differential invariant of extended Euclidean group

$$I_4^k + i\varepsilon\ln I_1, \quad I_2^k, \quad I_3^k, \quad (I_1)^2 I_5^k, \qquad k = 2, 3, \ldots, n.$$

The general form of invariant equations reads

$$\Phi\left(I_4^k + i\varepsilon\ln I_1, I_2^k, I_3^k, (I_1)^2 I_5^k\right) = 0.$$

Also we use results of Theorem 1 to construct inequivalent LVF realizations of the Lie algebra of the Poincaré group $P(1,3)$.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 157–164.

The Integrability of Some Underdetermined Systems

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The problem of integrability of special nonholonomic systems with single-functional arbitrariness of solutions is studied. The algorithm and exact formulas are obtained. As an example the problem of "Integrating Wheel" motion is considered, and symmetry algebra for flat control system of n-th order are calculated.

1 Introduction

Mathematical models of various problems of science may be described by the systems of ordinary differential equations

$$F_j(t, x, \dot{x}, \ldots, x^{(n)}) = 0, \quad j = \overline{1, r}, \qquad x \subset X, \qquad \dim X = m, \quad m > r, \tag{1}$$

which contain more unknown functions (m) than equations (r). Similar systems are considered, for example, in geometry problems [1], problems of mathematical physics [2], nonholonomic mechanics [3], control theory [4]. Following [5], we will define such systems as "underdetermined systems".

In the present paper we will consider only underdetermined systems of the form

$$\begin{aligned} &\frac{d\xi^i}{du} - f^i(u)\frac{d\tau}{du} = 0, \qquad (\text{or} \quad \omega^i = d\xi^i - f^i(u)d\tau = 0),\\ &\tau = \tau(u), \qquad \xi^i = \xi^i(u), \qquad i = \overline{1, n} \end{aligned} \tag{2}$$

containing n equations and $(n + 1)$ unknown functions.

The aim of this research is to get exact formulas for the general solution of system (2). It is well known that sets of solutions of ordinary differential equations of n-th order are defined by n arbitrary constants. On the contrary the general solution of underdetermined systems may depends on arbitrary functions (not only constants). Let us consider a well-known example [3, 6, 7]. The motion of mechanical system with coordinates (x, y, z) is described by equation

$$\frac{dy}{dt} - z\frac{dx}{dt} = 0, \tag{3}$$

or in terms of differential forms

$$\omega = dy - zdx = 0, \qquad (\partial_z, \ \partial_x + z\partial_y). \tag{4}$$

The integrability conditions for this system are not fulfilled:

$$d\omega \wedge \omega = -dz \wedge dx \wedge dy \neq 0. \tag{5}$$

Therefore there does not exist any two-dimensional solutions of the form $\Phi(x, y, z) = C$. After H. Hertz such systems are known as "nonholonomic systems" [8]. But there exist one-dimensional solutions admitted by equation (3). For example, in [7] we can find a solution $\{x = t^2,\ y = t^4,\ z = 2t^2\}$. It is easy to construct the solution $\{x = \cos t,\ y = t\cos t - \sin t,\ z = t\}$. The question is: Can we construct a formula, which includes all one-dimensional solutions? We may find the positive answer in Pars' book [6]. His solution for (4) is

$$y = f(x), \qquad z = f'(x). \tag{6}$$

But this solution is only guess and we do not know what can we do in a more difficult situation. The algorithm for the general case is given by M. Gromov in his book [1]. Let us illustrate his algorithm for solving of the system

$$\begin{cases} \dfrac{d\xi^1}{du} = -u^2\dfrac{d\tau}{du}, \\ \dfrac{d\xi^2}{du} = u\dfrac{d\tau}{du}. \end{cases} \tag{7}$$

Following Gromov, rewrite system (7) in the form

$$\mathbf{B}\mathbf{x}' = 0, \tag{8}$$

where

$$\mathbf{B} = \begin{pmatrix} -u^2 & -1 & 0 \\ u & 0 & -1 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} \tau \\ \xi^1 \\ \xi^2 \end{pmatrix} \tag{9}$$

and take the solution in the form:

$$\mathbf{B}\mathbf{x} = \sigma, \qquad \sigma = \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix}, \tag{10}$$

where $\sigma^i(u)$ are arbitrary functions. Let us differentiate (10) taking into account (8). We obtain

$$\mathbf{B}'\mathbf{x} = \sigma'. \tag{11}$$

System (10), (11) is algebraic with respect to (τ, ξ^1, ξ^2)

$$\begin{cases} -u^2\tau - \xi^1 = \sigma^1, \\ u\tau - \xi^2 = \sigma^2, \\ -2u\tau = \dfrac{d\sigma^1}{du}, \\ \tau = \dfrac{d\sigma^2}{du}, \end{cases} \tag{12}$$

and has nontrivial solutions iff the condition

$$\frac{d\sigma^1}{du} + 2u\frac{d\sigma^2}{du} = 0. \tag{13}$$

takes place. Equation (13) is also underdetermined but it contains only 2 unknown functions. Proceeding in the similar way and making in (13) the substitution

$$\sigma^1 + 2u\sigma^2 = h, \tag{14}$$

where $h = h(u)$ is an arbitrary function we have

$$\sigma^2 = \frac{h'(u)}{2}, \tag{15}$$

and accordingly, from (14)

$$\sigma^1 = h - uh'(u). \tag{16}$$

At the last step we substitute (σ^1, σ^2) in (12) and finally obtain the solution in the form

$$\begin{cases} \tau = \dfrac{h''}{2}, \\ \xi^1 = -u^2\dfrac{h''}{2} + uh' - h, \\ \xi^2 = u\dfrac{h''}{2} - \dfrac{h'}{2}. \end{cases} \tag{17}$$

Thus the Gromov anzats reduces underdetermined system also to underdetermined system with dimension is less than of the initial system. Therefore for solving system (2) we have to input consecutively $(n-1)$ times $n, (n-1), \ldots, 1$ new functions. But the role of these new functions is intermediate while the solution of initial problem may be defined by only one arbitrary function. Our goal is to exclude these intermediate calculations. With respect to this at first we have to calculate the number of arbitrary functions defining the general solution and then to get exact formulas.

2 General solution

In the general case we use the definition of "width of solution" which was introduced by E. Cartan in [9]. We will consider only nonholonomic systems (2), so $(n+1)$-dimensional integral manifolds are absent.

$$\Phi(\tau, \xi^i) = C. \tag{18}$$

At the following step we have to obtain for system (2) Cartan's characteristics (s_i). Direct calculations give

$$s = n, \qquad s_1 = 1. \tag{19}$$

Therefore, the general solution of system (2) depends on one arbitrary function $\sigma(u)$.

The next step is based on the following. The general solutions of simple cases show that the final formulas are linearly dependent on $\sigma(u)$ and its derivatives up to n-th order $(\sigma', \sigma'', \ldots, \sigma^{(n)})$. Hence we can try to find the general solution of (2) in the form

$$\begin{aligned} \tau &= \sum_{k=0}^{n} A_k \sigma^{(k)}, \qquad \left(\sigma^{(k)} = U^k \sigma\right), \\ \xi^i &= \sum_{k=0}^{n} B_k^i \sigma^{(k)}, \qquad U = \frac{d}{du}, \end{aligned} \tag{20}$$

with undefined coefficients (A^i, B_k^i). The substitution (20) into (2) leads us (after decomposition by $\sigma^{(k)}$) to the system

$$B_n^i = f^i A_n, \tag{21}$$

$$UB_k^i + B_{k-1}^i = f^i U A_k + f^i A_{k-1}, \qquad k = \overline{1,n}, \tag{22}$$

$$UB_0^i = f^i U A_0. \tag{23}$$

The substitution (21) into (22) gives us

$$\begin{aligned} B_{n-1}^i &= f^i A_{n-1} - A_n U f^i, \\ B_{n-2}^i &= f^i A_{n-2} - A_{n-1} U f^i + A_n U^2 f^i + U A_n U f^i, \\ B_{n-3}^i &= f^i A_{n-3} - A_{n-2} U f^i + U\left(A_{n-1} U f^i\right) - U^2\left(A_n U f^i\right), \end{aligned} \tag{24}$$

and we may assume that

$$B_{n-k}^i = f^i A_{n-k} + \sum_{m=0}^{k-1} (-1)^{m+1} U^m \left(A_{n-k+m+1} U f^i\right), \tag{25}$$

or, redefining the subscript $(n-k \longrightarrow k)$,

$$B_k^i = f^i A_k + \sum_{m=0}^{n-k-1} (-1)^{m+1} U^m \left(A_{m+k+1} U f^i\right), \qquad k = \overline{1, n-1}. \tag{26}$$

In fact, we get identity via substitution (B_k^i) into (22). From (26) with respect to $k = 0$ we have

$$B_0^i = f^i A_0 + \sum_{m=0}^{n-1} (-1)^{m+1} U^m \left(A_{m+1} U f^i\right). \tag{27}$$

The substitution (27) into (23) gives us A_i:

$$\sum_{m=0}^{n} (-1)^m U^m \left(A_m U f^i\right) = 0. \tag{28}$$

Let us make the following transformations at (28):

1) rewrite according to Leibnitz formula (see, for example, [10]) the expression

$$U^m(A_m U f^i) = \sum_{s=0}^{m} \binom{m}{s} (U^{m-s} A_m) U^{m+1} f^i; \tag{29}$$

2) define

$$D_s = \sum_{m=s}^{n} (-1)^m \binom{m}{s} U^{m-s} A_m. \tag{30}$$

Then (28) takes the form

$$\sum_{s=0}^{n} D_s U^{s+1} f^i = 0, \qquad i = \overline{1,n}. \tag{31}$$

This system is linear algebraic one with respect to $(n+1)$ unknown variables D_s. The existence of solutions of the latter system is connected with the rank of functional $n \times (n+1)$ matrix

$$W(u) = a_j^i, \qquad a_j^i = \frac{d^{j+1} f^i}{du^{j+1}}. \tag{32}$$

The determinant of square matrix $\hat{W}(u)$ (which is $W(u)$ without last column) is a Wronskian for functions $\frac{df^i}{du}$ (see, for example, [11]). If rank of the matrix $W(u)$ is equal to n, then system (31) has a solution. We may get $D_n(u)$ as arbitrary function.

$$D_n(u) = h(u). \tag{33}$$

The remaining coefficients are defined from the system

$$\sum_{s=0}^{n-1} D_s U^{s+1} f^i = (-1) U^{n+1} f^i h(u). \tag{34}$$

In the particular case $U^{n+1} f^i = 0$ ($\forall\ i$) we have $D_s = 0$, $s = \overline{0, n-1}$. It is easy to show that function $h(u)$ is not essential, because of for any $h(u)$ the substitution $\hat{\sigma} = h\sigma$ leave only one arbitrary function in general solution. Therefore we may assume without loss of generality that

$$D_n(u) = h(u) = (-1)^n. \tag{35}$$

In this case we have from (28) $A_n = 1$, and from (21) $B_n^i = f^i$. By inverting formula (34) we can calculate the coefficients A_i

$$A_i = \sum_{m=i}^{n-1} (-1)^m \binom{m}{i} U^{m-i} D_m, \tag{36}$$

and according to (26) we can obtain B_i^k. As a result we may formulate the following theorem:

Theorem 1. *If Wronskian of functions $\varphi^i = \frac{df^i}{du}$ in system (2) is not equal to zero ($W(\varphi^i) \neq 0$), then the general solution of system (2) is given by the formulas*

$$\begin{cases} \tau = \sigma^{(n)} + \displaystyle\sum_{k=0}^{n-1} A_k \sigma^{(k)}, \\ \xi^i = f^i \sigma^{(n)} + \displaystyle\sum_{k=0}^{n-1} B_k^i \sigma^{(k)}, \end{cases} \tag{37}$$

where $\sigma = \sigma(u)$ is an arbitrary function and for calculating coefficients (A^i, B_k^i) one needs to follow the following algorithm:

1) solve the linear system

$$\sum_{s=0}^{n-1} (U^{s+1} f^i) D_s = (-1)^{n+1} U^{n+1} f^i \tag{38}$$

with respect to D_s;

2) calculate A_i from (36);

3) calculate B_k^i from recursion relations (21), (22) or from formulas (26).

In an important particular case $U^{n+1} f^i = 0$ ($\forall\ i$) (system (38) is homogeneous) the latter formulas simplify to

$$A_n = 1, \qquad A_i = 0, \qquad B_n = f^i, \qquad B_k^i = (-U)^{n-k} f^i, \qquad k = \overline{0, n-1}. \tag{39}$$

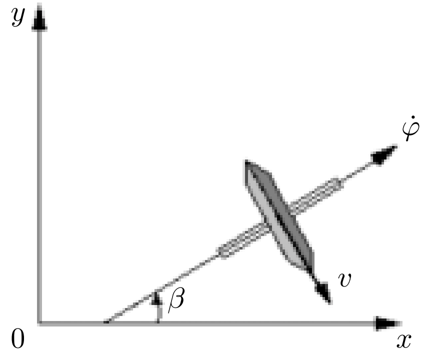

Figure 1. “Integrating Wheel”

3 Examples

Example 1 [3, p.28]. Let us consider the motion of a nonholonomic system (“Integrating Wheel”) on the plane OXY (see Fig. 1).

During the rotation of wheel around its axis the coordinates x and y are bounded by following equations

$$\dot{x} = R\dot{\varphi}\sin\beta, \qquad \dot{y} = R\dot{\varphi}\cos\beta, \tag{40}$$

where $\dot{\varphi}$ is angular velocity, R is radius, β is angle of orientation of the wheel on the plane. Denoting by

$$\xi^1 = \frac{x}{R}, \qquad \xi^2 = \frac{y}{R}, \qquad \tau = \varphi, \qquad \beta = u, \tag{41}$$

we get the system

$$\begin{cases} \dfrac{d\xi^1}{du} = \sin u \dfrac{d\tau}{du}, \\ \dfrac{d\xi^2}{du} = \cos u \dfrac{d\tau}{du}. \end{cases} \tag{42}$$

According to Theorem 1 the result will be following. System (38) has the form

$$\begin{pmatrix} \cos u & \sin u \\ -\sin u & -\cos u \end{pmatrix} \begin{pmatrix} D_0 \\ D_1 \end{pmatrix} = \begin{pmatrix} \cos u \\ -\sin u \end{pmatrix}, \tag{43}$$

and its solution is

$$D_0 = 1, \qquad D_1 = 0. \tag{44}$$

By formulas (36) one can obtain

$$A_1 = -D_1 = 0, \qquad A_0 = D_0 - UD_1 = 1. \tag{45}$$

Finally, by using (26) we have

$$B_1^1 = -\cos u, \qquad B_0^1 = 0, \qquad B_1^2 = \sin u, \qquad B_0^2 = 0, \tag{46}$$

and the general solution takes the form

$$\begin{cases} \tau = \sigma_{uu} + \sigma, \\ \xi^1 = \sin u\sigma_{uu} - \cos u\sigma_u, \\ \xi^2 = \cos u\sigma_{uu} + \sin u\sigma_u, \end{cases} \tag{47}$$

where $\sigma = \sigma(u)$ is arbitrary function from $\mathbf{C}^3$.

Example 2. Let us calculate the symmetry algebra of a control system

$$x^{(n)} = u, \qquad x^{(n)} = \frac{d^n x}{dt^n}. \tag{48}$$

Rewrite system (48) in Cauchy form

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = x^3, \quad \ldots, \quad \dot{x}^n = u, \tag{49}$$

where $x^1 = x$, $x^2 = \dot{x}$, …, $x^n = x^{(n-1)}$. Now with system (49) we can associate the differential operator

$$X_0 = \hat{X}_0 + u\partial_{x^n}, \qquad \hat{X}_0 = \partial_t + x^{i+1}\partial_{x^i}, \qquad i = \overline{1, n-1}. \tag{50}$$

The symmetry operator is

$$X = \tau(t,x,u)\partial_t + \xi^j(t,x,u)\partial_{x^j} + \varphi(t,x,u)\partial_u, \qquad j = \overline{1,n}. \tag{51}$$

Symmetry conditions give us the following determining equations

$$Xf^j + f^j X_0\tau - X_0\xi^j = 0, \tag{52}$$

$$f^j U\tau - U\xi^j = 0, \qquad j = \overline{1,n}. \tag{53}$$

The last equation is the same as (2). From (53) we have

$$\xi^i_u - x^{i+1}\tau_u = 0, \tag{54}$$

$$\xi^n_u - u\tau_u = 0. \tag{55}$$

According to Theorem 1 the solution of (55) has the form

$$\tau = \sigma_u, \qquad \xi^n = u\sigma_u - \sigma, \tag{56}$$

where $\sigma = \sigma(t,x,u)$ is arbitrary function. Now for (54) we have

$$\xi^i = x^{i+1}\sigma_u + g^i, \tag{57}$$

where $g^i = g^i(t,x)$. Omitting the intermediate calculations, we can formulate the general result as following.

Theorem 2. *The maximal invariance algebra for control flat system (49) is infinite-dimensional and its infinitesimal operator is*

$$\begin{aligned} X = {} & -\frac{\partial}{\partial x^n}\left(\hat{X}_0^{n-2}g\right)\partial_t + \left(-x^{i+1}\frac{\partial}{\partial x^n}\left(\hat{X}_0^{n-2}g\right) + \hat{X}_0^{i-1}g\right)\partial_{x^i} \\ & + \left(\hat{X}_0^{n-1}g\right)\partial_{x^n} + \left(X_0^2\hat{X}_0^{n-2}\right)g\partial_u, \qquad i = \overline{1,n-1}. \end{aligned} \tag{58}$$

where $g = g(t, x^1, \ldots, x^{n-1})$.

Important details and other examples the reader may find in [12].

4 Conclusion

The calculation of the general solution of system (2) according to Theorem 1 consists of only regular actions (solving of linear system and differentiation), so these procedures are easy realized in the analytical system **REDUCE**. Besides, exact formulas are very useful in supervising of control systems (see the details in [12]).

Acknowledgments

The author is grateful to Professor Kent Harrison for constant attention to this work and useful discussions.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 165–169.

The Group Classification of Nonlinear Wave Equations Invariant under Two-Dimensional Lie Algebras

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The group classification of the one class of the nonlinear wave equations which are invariant under one- and two-dimensional Lie algebras is obtained.

The problem of group classification of differential equations is one of the central problems of the modern symmetry analysis of the differential equations. In this raper we consider the problem of the group classification of the equations of form

$$u_{tt} = u_{xx} + F(t, x, u, u_x), \tag{1}$$

where $u = u(t, x)$, F is an arbitrary nonlinear differentiable function of its variables. In (1) $F_{u_x} \neq 0$ is an arbitrary nonlinear smooth function, which depends on variables u or u_x. Also we denoted $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $F_{u_x} = \frac{\partial F}{\partial u_x}$, $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$.

We note that the problem of group classification of nonlinear hyperbolic equations was studied in the works [1–5]. We describe the equations of the form (1), which are invariant under one- and two-dimensional Lie algebras.

At first we give a form of an infinitesimal operator of the symmetry group of the equation (1), and the group of equivalence transformation of this equation.

Theorem 1. *The infinitesimal operator of the symmetry group of the equation (1) has following form:*

$$X = (\lambda t + \lambda_1)\partial_t + (\lambda x + \lambda_2)\partial_x + (h(x)u + r(t, x))\partial_u, \tag{2}$$

where λ, λ_1, λ_2 are the arbitrary real constants, $h(x)$, $r(t, x)$ are the arbitrary functions, which satisfy the following condition:

$$\begin{aligned} & r_{tt} - \frac{d^2 h}{dx^2} - r_{xx} + (h - 2\lambda)F - (\lambda t + \lambda_1)F_t - (\lambda x + \lambda_2)F_x \\ & \quad -(hu + r)F_u - 2u_x\frac{dh}{dx} - u_x(h - \lambda)F_{u_x} - \frac{dh}{dx}uF_{u_x} - r_xF_{u_x} = 0. \end{aligned} \tag{3}$$

The proof of the theorem is done by the Lie method [6]. Then by the way of direct calculations it is not difficult to show, that the group of the equvalence of the equation (1) is determined by the transformation:

$$\bar{t} = \gamma t + \gamma_1, \qquad \bar{x} = \epsilon\gamma x + \gamma_2, \qquad v = \rho(x)u + \theta(t, x), \tag{4}$$

$\gamma \neq 0$, $\rho \neq 0$, $\epsilon = \pm 1$.

The first step of the group classification is the study of nonequivalent realizations of one-dimensional Lie algebra in class of operators (2).

Theorem 2. *There are transformations (4) which reduce operator (2) to one of the following operators:*

1) $X = \lambda(t\partial_t + x\partial_x), \quad \lambda \neq 0, \quad \lambda = \text{const};$
2) $X = \partial_t + \beta\partial_x, \quad \beta > 0;$
3) $X = \partial_x;$
4) $X = \partial_t;$
5) $X = \partial_t + h(x)u\partial_u, \quad h(x) \neq 0;$
6) $X = h(x)u\partial_u, \quad h(x) \neq 0;$
7) $X = r(t,x)\partial_u.$

Proof. We distinguish two cases.

Case 1. $\lambda \neq 0$.

$$X = \gamma[\lambda t + \lambda_1]\partial_{\bar{t}} + (\lambda x + \lambda_2)\epsilon\gamma\partial_{\bar{x}} + [\theta_t(\lambda t + \lambda_1) + (\lambda x + \lambda_2)(\rho' u + \theta_x) + (hu + r)\rho]\partial_v. \quad (5)$$

Check that operator (5) which by means of transformations (4) one be reduced to

$$X = \lambda\bar{t}\partial_{\bar{t}} + \lambda\bar{x}\partial_{\bar{x}}. \quad (6)$$

According to (5) and (6) we have

$$\begin{aligned} &\gamma(\lambda t + \lambda_1) = \lambda(\gamma t + \gamma_1), \\ &\epsilon\gamma(\lambda x + \lambda_2) = \lambda(\epsilon\gamma x + \gamma_1), \\ &\theta_t(\lambda t + \lambda_2) + (\lambda x + \lambda_2)\theta_x + r\rho = 0, \\ &(\lambda x + \lambda_2)\rho' + h\rho = 0. \end{aligned} \quad (7)$$

We have $\gamma_1 = \lambda_1\gamma\lambda^{-1}$, $\gamma_2 = \lambda_2\epsilon\gamma\lambda^{-1}$, ρ, θ are solutions of the system (7).

With the help of trasformations (4) the operator (2) reduces to form

$$X = \lambda(t\partial_t + x\partial_x).$$

Case 2. $\lambda = 0$ is treated the same way. ■

In accordance with Theorem 2 there are seven nonequivalent one-dimensional algebras:

$$\begin{aligned} &A_1^1 = \langle t\partial_t + x\partial_x \rangle; \\ &A_1^2 = \langle \partial_t + \beta\partial_x \rangle, \quad \beta > 0; \\ &A_1^3 = \langle \partial_x \rangle; \\ &A_1^4 = \langle \partial_t \rangle; \\ &A_1^5 = \langle \partial_t + h(x)u\partial_u \rangle, \quad h(x) \neq 0; \\ &A_1^6 = \langle h(x)u\partial_u \rangle, \quad h(x) \neq 0; \\ &A_1^7 = \langle r(t,x)\partial_u \rangle. \end{aligned}$$

Below we give the list of corresponding values of function F in the equation (1), when those one-dimensional algebras will be the algebras of invariance.

1) $A_1^1: \quad F = u_x^2 G(u, \omega_1, \omega_2), \quad \omega_1 = tx^{-1}, \quad \omega_2 = xu_x;$

2) $A_1^2: \quad F = G(\omega, u, u_x), \quad \omega = x - \beta t;$

3) $A_1^3: \quad F = G(t, u, u_x);$

4) $A_1^4: \quad F = G(x, u, u_x);$

5) $A_1^5: \quad F = h^{-1}h'' - 2h'h^{-1}u_x \ln|u| + (h'h^{-1})^2 u \ln^2|u| + uG(x, \omega_1, \omega_2),$
$\omega_1 = ue^{-ht}, \quad \omega_2 = u^{-1}u_x - h^{-1}h' \ln|u|;$

6) $A_1^6: \quad F = h^{-1}h'' - 2h'h^{-1}u_x \ln|u| + (h'h^{-1})^2 u \ln^2|u| + uG(t, x, \omega),$
$\omega = u^{-1}u_x - h'h^{-1} \ln|u|;$

7) $A_1^7: \quad F = r^{-1}(r_{tt} - r_{xx})u + G(t, x, \omega), \quad \omega = r_x u - r u_x.$

There are two real one- and two-dimensional Lie algebras:

1) $A_{2.1} = A_1 \oplus A_1 = \langle e_1, e_2 \rangle, \quad [e_1, e_2] = 0;$

2) $A_{2.2} = \langle e_1, e_2 \rangle, \quad [e_1, e_2] = e_2.$

Studying their realizations in the class of the operators (2), on the base of the results of the Theorem 2 we can put one of the basis operators of one- and two-dimensional Lie algebras equal to one of operators, which are given in Theorem 2.

After the next steps, we have 19 realizations of the algebra $A_{2.1}$ and 15 realizations $A_{2.2}$, which are the algebras of invariance of equation (1). We give the realizations of the algebra $A_{2.1}$ and the corresponding values of the functions F in the equation (1).

$A_{2.1}^1 = \langle ku\partial_u,\ t\partial_t + x\partial_x \rangle, \quad k \neq 0,$
$F = u^{-1}u_x^2 G(\omega, v), \quad \omega = tx^{-1}, \quad v = xu^{-1}u_x;$

$A_{2.1}^2 = \langle r(\zeta)\partial_u,\ t\partial_t + x\partial_x \rangle, \quad \zeta = tx^{-1},$
if $r = \text{const}, \quad F = u_x^2 G(\omega, v), \quad \omega = \zeta, \quad v = xu_x,$
if $r_\zeta \neq 0, \quad F = x^{-1}u_x(\zeta^2 - 1)r_{\zeta\zeta}(\zeta r_\zeta)^{-1} + x^{-2}G(\zeta, \omega), \quad \omega = \zeta r_\zeta u + rxu_x;$

$A_{2.1}^3 = \langle \partial_t + \beta\partial_x,\ ku\partial_u \rangle, \quad \beta > 0, \quad k \neq 0,$
$F = uG(v, \omega), \quad v = x - \beta t, \quad \omega = u^{-1}u_x;$

$A_{2.1}^4 = \langle \partial_t + \beta\partial_x,\ r(\zeta)\partial_u \rangle, \quad \zeta = x - \beta t, \quad \beta > 0,$
$F = r^{-1}r_{\zeta\zeta}(1 - \beta^2)u + G(\zeta, \omega), \quad \omega = r_\zeta u + ru_x;$

$A_{2.1}^5 = \langle \partial_t + \beta\partial_x,\ \partial_t + ku\partial_u \rangle, \quad k \neq 0,$
$F = uG(v, \omega), \quad v = \beta u + k\zeta, \quad \omega = u^{-1}u_x, \quad \zeta = x - \beta t;$

$A_{2.1}^6 = \langle \partial_t,\ \partial_x \rangle,$
$F = G(u, u_x);$

$$A^7_{2.1} = \langle \partial_x,\ ku\partial_u \rangle, \quad k \neq 0,$$
$$F = uG(t, u^{-1}u_x);$$

$$A^8_{2.1} = \langle \partial_x,\ r(t)\partial_u \rangle, \quad r \neq 0,$$
$$F = r^{-1}\ddot{r}u + G(t, u_x), \quad \ddot{r} = \frac{d^2r}{dt^2};$$

$$A^9_{2.1} = \langle \partial_x,\ \partial_t + \lambda u\partial_u \rangle, \quad \lambda \neq 0,$$
$$F = uG(e^{-\lambda t}u, e^{-\lambda t}u_x);$$

$$A^{10}_{2.1} = \langle h(x)u\partial_u,\ \partial_t \rangle,$$
$$F = -2\frac{h'u_x}{h}\ln|u| + \frac{(h')^2}{h^2}u\ln^2|u| + \frac{h''}{h} + uG(x,\omega), \quad \omega = \frac{u_x}{u} - \frac{h'}{h}\ln|u|,$$

$$A^{11}_{2.1} = \langle \partial_u,\ \partial_t \rangle,$$
$$F = g(x, u_x);$$

$$A^{12}_{2.1} = \langle f(x)u\partial_u,\ \partial_t + ku\partial_u \rangle, \quad k \neq 0,$$
$$F = f^{-1}f''u\ln|u| - u^{-1}u_x^2 + uG(x,v), \quad v = kf^{-1}f't + u^{-1}u_x - f^{-1}f'\ln|u|;$$

$$A^{13}_{2.1} = \left\langle e^{\lambda t}\partial_u,\ \partial_t + \lambda\partial_u \right\rangle, \quad \lambda \neq 0,$$
$$F = \lambda^2 u + u_x G(x,\omega), \quad \omega = u_x e^{-\lambda t};$$

$$A^{14}_{2.1} = \langle f(x)u\partial_u,\ \partial_t + h(x)u\partial_u \rangle, \quad h' \neq 0,$$
$$F = -\omega^2 + G(x,V), \quad V = AT + x, \quad A = hf'f^{-1} - h', \quad A \neq 0,$$
$$\omega = u^{-1}u_x - f'f^{-1}\ln|u|, \quad h'' = hf^{-1}f'', \quad \frac{f'}{f} \neq \frac{h'}{h}, \quad \frac{h''}{h} = \frac{f''}{f};$$

$$A^{15}_{2.1} = \left\langle e^{h(x)t}\partial_u,\ \partial_t + f(x)u\partial_u \right\rangle, \quad f' \neq 0,$$
$$F = f^2 + 2f'f^{-1} - 2(f'f^{-1})^2 + 2f'f^{-1}\omega\ln|u| + (f'f^{-1})^2\ln^2|v|$$
$$+ [f^{-1}f'' - 2f'f^{-1} + 2(f'f^{-1})^2]\ln|v| + G(x,w), \quad w = \omega v + f^{-1}f'v\ln|v|,$$
$$v = ue^{-f(x)t}, \quad \omega = u^{-1}u_x + f^{-1}f'\ln|u|;$$

$$A^{16}_{2.1} = \langle \lambda xu\partial_u,\ ku\partial_u \rangle, \quad \lambda, k \neq 0,$$
$$F = -u^{-1}u_x^2 + uG(t,x);$$

$$A^{17}_{2.1} = \langle f(x)u\partial_u,\ h(x)u\partial_u \rangle, \quad h' \neq 0, \quad f' \neq 0,$$
$$F = -u^{-1}u_x^2 + uG(t,x), \quad f''f^{-1} = h''h^{-1};$$

$$A^{18}_{2.1} = \langle \frac{\varphi(t)}{\dot{\varphi}(t)}\partial_u,\ \frac{1}{\sqrt{\dot{\varphi}(t)}}\partial_u \rangle, \quad \dot{\varphi} \neq 0,$$
$$F = \frac{1}{4}\dot{\varphi}^{-2}[3(\ddot{\varphi})^2 - 2\dot{\varphi}\dddot{\varphi}]u + G(t, x, \dot{\varphi}^{-1/2}u_x);$$

$A_{2.1}^{19} = \langle \lambda(t\partial_t + x\partial_x,\ r(\zeta)\partial_u \rangle\,, \quad \zeta = tx^{-1},$

$1)\ r = k = \mathrm{const}, \quad r \neq 0, \quad k \neq 0, \quad F = G(\zeta, xu_x),$

$2)\ r_\zeta \neq 0, \quad F = \zeta^{-1} r_\zeta^{-1} v^{-1}[(1-\zeta^2)r_{\zeta\zeta} - 2\zeta r_\zeta] + \tilde{G}(\zeta,\omega), \quad v = xu_x,$

$\omega = \zeta r_\zeta u + rv.$

On base of the results of the Theorem 2 we proved the following theorem.

Theorem 3. *In the class of operators (2) there are no realizations of the algebras* $so(3)$ *and* $sl(2,R)$*.*

From this theorem we have the following results:

- in class of operators (2) there are no realizations of real semi-simple Lie algebras;
- there are not such equations (1) which have algebras of invariance, which isomorphic by real semi-simple algebras, or contain those algebras as subalgebras.

Thus, we must study of existence of realizations of only real solvable Lie algebras in the class of operators (2) for the complete group classification equation (1).

The author is gratefull to R.Z. Zhdanov and V.I. Lahno for formulation of the problem and for the help in the research.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 170–176.

Hamiltonian Formulation and Order Reduction for Nonlinear Splines in the Euclidean 3-Space

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The authors use the procedure developed in [9] to develop a Hamiltonian structure into the variational problem given by the integral of the squared curvature on the spatial curves. The solutions of that problem are the elasticae or nonlinear splines. The symmetry of the problem under rigid motions is then used to reduce the Euler–Lagrange equations to a first-order dynamical system.

1 Introduction

Elasticae, or nonlinear spline curves, are the extremal curves of the second order variational problem given by the functional $\int \kappa^2 \mathrm{d}s$ (*cf.* [2, 4, 5]), where κ is the geodesic curvature of the path, and ds is the arc-length element. This is one of the simplest examples of a general type of variational problems called parameter-invariant problems (see [6]), as the action integral is invariant under arbitrary changes of parameter. As all the problems in this class, our problem is *singular*, that is, its Hessian vanishes. As a consequence, the standard Hamiltonian formalism (momenta, Hamilton equations) cannot be directly applied. Hence it is not clear how to reduce the order of the equations by using Noether invariants attached to the symmetries of the problem.

In [9], the authors devised a general procedure called *parameter elimination*, in order to pass from parameter-invariant Lagrangian to a nonparametric version of it. Roughly speaking, the parameter elimination consists in taking the parameter "time" apart and using one of the "spatial" coordinates as the new parameter. So, the Lagrangian projects onto a "deparametrized" Lagrangian, whose extremals, parametrized arbitrarily, are the extremals of the original problem. Furthermore, the projected Lagrangian gives rise in some cases (which can be suitably characterized) to a regular problem, and hence Hamiltonian formalism can be applied.

In this paper this general procedure is applied to the particular case of the nonlinear splines in the 3-dimensional space, thus introducing a natural Hamiltonian formulation to the problem. Within this setting, the Noether invariants associated to the rigid motions of the space are found (the rigid motions – translations and rotations – are symmetries of the variational problem, as both the curvature and the arc-length are invariant under isometries). This invariants are then used to reduce the order of the equations of extremals, from a fourth order system to a nonlinear system of the form

$$\begin{cases} \dfrac{\mathrm{d}y'}{\mathrm{d}x} = F(y', z'), \\ \dfrac{\mathrm{d}z'}{\mathrm{d}x} = G(y', z'), \end{cases}$$

where $y' = \mathrm{d}y/\mathrm{d}x$ and $z' = \mathrm{d}z/\mathrm{d}x$.

The term *nonlinear splines* is used to distinguish the extremal curves of the squared curvature functional of the "usual" linear splines (or just splines), or piecewise cubic polynomials. The latter are the extremals of the linear approximation to the variational problem under consideration. Of course, we are just interested in the *exact* extremals, not in the approximations.

2 Hamiltonian formulation for elasticae

In the standard coordinates of $\mathbb{R}^3$, the expression of the nonlinear splines Lagrangian is (see [12]):

$$\mathcal{L} = \frac{(\dot{y}\ddot{z} - \ddot{y}\dot{z})^2 + (\dot{x}\ddot{z} - \ddot{x}\dot{z})^2 + (\dot{x}\ddot{y} - \ddot{x}\dot{y})^2}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{5/2}}.$$

The variational problem defined by this Lagrangian is not regular. In fact, as a simple computation shows, we have $\det\left(\partial^2\mathcal{L}/\partial\ddot{x}_i\partial\ddot{x}_j\right) \equiv 0$. Hence, Hamiltonian formalism cannot be applied. Nevertheless, this variational problem is parameter-invariant (see [6]); *i.e.*,

$$\int_a^b \mathcal{L}\left(j_t^2(\sigma\circ u)\right)\mathrm{d}t = \int_\alpha^\beta \mathcal{L}\left(j_u^2(\sigma)\right)\mathrm{d}u,$$

for every orientation-preserving diffeomorphism $u\colon [a,b]\to[\alpha,\beta]$. This fact can easily be checked, either geometrically, or making use of the Zermelo conditions (see [6]).

Using the procedure described in [9], we pass from the Lagrangian $\mathcal{L}$ given above to the deparametrized version $\bar{\mathcal{L}}$. Roughly speaking, this is done by eliminating the variable t, using x as the new independent variable, and passing from the "dots" (which represent derivatives with respect to t), to the "primes" (standing for derivatives with respect to x). The following relations are used:

$$y' = \frac{\dot{y}}{\dot{x}}, \qquad z' = \frac{\dot{z}}{\dot{x}}, \qquad y'' = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}, \qquad z'' = \frac{\dot{x}\ddot{z} - \dot{z}\ddot{x}}{\dot{x}^3}. \tag{1}$$

It is important to notice that the process is applied to the Lagrangian density $\mathcal{L}\,\mathrm{d}t$ to convert it into a density $\bar{\mathcal{L}}\,\mathrm{d}x$ modulo a contact form $\mathrm{d}x - \dot{x}\,\mathrm{d}t$.

The following are some of the main results given in [9] (here $\mathcal{L}$ stands for a generic second-order Lagrangian):

(i) $\bar{\mathcal{L}}$ is regular if and only if the Hessian matrix of $\mathcal{L}$ has rank 2.

(ii) The extremals of $\mathcal{L}\,\mathrm{d}t$ are the extremals of $\bar{\mathcal{L}}\,\mathrm{d}x$, endowed with an arbitrary parametrization.

In the case of the squared curvature functional, the rank of the Hessian matrix is 2, so the non-parametric Lagrangian, which can be easily computed by using the formulas (1),

$$\bar{\mathcal{L}} = \frac{(y'z'' - y''z')^2 + y''^2 + z''^2}{(1 + y'^2 + z'^2)^{5/2}}, \tag{2}$$

is regular, and we can apply the Hamiltonian formalism to it.

2.1 Hamilton equations for elasticae

Using the standard Hamiltonian formalism for second order problems (see, [1, 3, 4, 7, 11]) we can write the Jacobi–Ostrogradski momenta associated to the non-parametric Lagrangian $\bar{\mathcal{L}}$,

$$p = v^{-5}\left[-2\left(z'^2+1\right)y''' + 2y'z'z'''\right] + v^{-7}\left[5y'\left(z'^2+1\right)y''^2 + 2z'\left(-2y'^2+3z'^2+3\right)y''z'' - y'\left(y'^2+6z'^2+1\right)z''^2\right],$$

$$
\begin{aligned}
q &= v^{-5}\left[2y'z'y''' - 2\left(y'^2+1\right)z'''\right] + v^{-7}\left[-z'\left(6y'^2+z'^2+1\right)y''^2\right.\\
&\quad\left. + 2y'\left(3y'^2-2z'^2+3\right)y''z'' + 5z'\left(y'^2+1\right)z''^2\right],\\
p' &= v^{-5}\left[2\left(z'^2+1\right)y'' - 2y'z'z''\right],\\
q' &= v^{-5}\left[-2y'z'y'' + 2\left(y'^2+1\right)z''\right],\\
v &= \left(1+y'^2+z'^2\right)^{1/2},
\end{aligned}
$$

and also the Hamiltonian $H = py' + qz' + \left(v^3/4\right)\left[(p'y'+q'z')^2 + (p')^2 + (q')^2\right]$, the Poincaré–Cartan form

$$
\begin{aligned}
\bar{\Theta} &= \bar{\mathcal{L}}\,\mathrm{d}x + p(\mathrm{d}y - y'\,\mathrm{d}x) + q(\mathrm{d}z - z'\,\mathrm{d}x) + p'(\mathrm{d}y' - y''\,\mathrm{d}x) + q'(\mathrm{d}z' - z''\,\mathrm{d}x)\\
&= -H\,\mathrm{d}x + p\,\mathrm{d}y + q\,\mathrm{d}z + p'\,\mathrm{d}y' + q'\,\mathrm{d}z',
\end{aligned}
\tag{3}
$$

and, finally, the Hamilton equations:

$$
\begin{aligned}
&\frac{\mathrm{d}y}{\mathrm{d}x} = y', \qquad \frac{\mathrm{d}z}{\mathrm{d}x} = z', \qquad \frac{\mathrm{d}p}{\mathrm{d}x} = 0, \qquad \frac{\mathrm{d}q}{\mathrm{d}x} = 0,\\
&\frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{1}{2}v^3\left[\left(y'^2+1\right)p' + y'z'q'\right],\\
&\frac{\mathrm{d}z'}{\mathrm{d}x} = \frac{1}{2}v^3\left[y'z'p' + \left(z'^2+1\right)q'\right],\\
&\frac{\mathrm{d}p'}{\mathrm{d}x} = -p - \frac{1}{4}v\left[y'\left(5y'^2+2z'^2+5\right)(p')^2 + 2z'\left(4y'^2+z'^2+1\right)p'q' + 3y'\left(z'^2+1\right)(q')^2\right],\\
&\frac{\mathrm{d}q'}{\mathrm{d}x} = -q - \frac{1}{4}v\left[3z'\left(y'^2+1\right)(p')^2 + 2y'\left(y'^2+4z'^2+1\right)p'q' + z'\left(2y'^2+5z'^2+5\right)(q')^2\right].
\end{aligned}
\tag{4}
$$

Our aim shall be to reduce this system to a first-order ordinary differential system in the variables y', z'. Also note that $p' = q' = 0$ if and only if the spline curve is a geodesic; *i.e.*, $y = \alpha_1 x + \beta_1$, $z = \alpha_2 x + \beta_2$, $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$.

3 Generalized symmetries and reduction to first order

It is straightforward to see that the variational problem defined by the squared curvature functional is invariant under isometries, as the curvature and arc-length element are themselves invariant under isometries. Thus, the infinitesimal generators of the rigid motions of $\mathbb{R}^3$ are infinitesimal symmetries of $\mathcal{L}\,\mathrm{d}t$, that is, if $X \in \mathfrak{X}(\mathbb{R}^3)$ is the infinitesimal generator of a rigid motion, $\mathrm{L}_{X_{(2)}}(\mathcal{L}\,\mathrm{d}t) = 0$, where $X_{(r)}$ is the prolongation of the vector field $(0, X) \in \mathfrak{X}(\mathbb{R}\times\mathbb{R}^3)$ to $J^r(\mathbb{R}, \mathbb{R}^3)$ by means of infinitesimal contact transformations (*cf.* [8, 9]). For infinitesimal symmetries, Noether's Theorem (see [7]) states that the function $f_X = i_{X_{(3)}}\Theta(\mathcal{L}\,\mathrm{d}t)\colon J^{2r-1}(\mathbb{R}, M) \to \mathbb{R}$ is constant along each extremal of the variational problem defined by $\mathcal{L}\,\mathrm{d}t$. The function f_X is called the Noether invariant associated to X.

Nevertheless, as it was stated in [9], if $(0, X) \in \mathfrak{X}(\mathbb{R}\times\mathbb{R}^3)$ is an infinitesimal symmetry of $\mathcal{L}\,\mathrm{d}t$, it does not need to be $X \in \mathfrak{X}(\mathbb{R}\times\mathbb{R}^2)$ an infinitesimal symmetry of $\bar{\mathcal{L}}\,\mathrm{d}t$ (in fact, it can even be not projectable onto $\mathbb{R}^2$). But it is a generalized infinitesimal symmetry of $\bar{\mathcal{L}}\,\mathrm{d}t$, *i.e.*, $\mathrm{L}_{X_{[2]}}(\bar{\mathcal{L}}\,\mathrm{d}t)$

is a contact form, that is, it vanishes on every 2-jet of curve on M (*cf.* [10, Definition 5.25]). Here, $X_{[r]}$ denotes the prolongation of $X \in \mathfrak{X}(\mathbb{R}^3)$ to $\mathfrak{X}(J^r(\mathbb{R},\mathbb{R}^2))$ by means of infinitesimal contact transformations.

It can be shown that if $X \in \mathfrak{X}(\mathbb{R}\times M)$ is a generalized infinitesimal symmetry of $\bar{\mathcal{L}}\,\mathrm{d}x$, then f_X is constant on the extremals of $\bar{\mathcal{L}}\,\mathrm{d}x$; *i.e.*, generalized symmetries also produce Noether invariants.

Now we shall use the (generalized) symmetries of the deparametrized squared curvature Lagrangian to reduce (by means of the associated Noether invariants) the order of the Hamilton equations for the nonlinear splines on $\mathbb{R}^3$. More precisely, in the general case we shall reduce the equations (4) to a system of the form:

$$\frac{\mathrm{d}z'}{\mathrm{d}y'} = F\left(y',z'\right), \qquad \frac{\mathrm{d}y'}{\mathrm{d}x} = G\left(y',z'\right).$$

Hence the problem is reduced to two ordinary differential equations in the plane as we can first solve z' in terms of y' from the first equation above and then substitute the obtained expression for z' into the second equation thus leading us to an equation of the form $\mathrm{d}y'/\mathrm{d}x = \bar{G}(\mu,x,y')$, where μ is a constant.

As we have already said, the infinitesimal generators of the isometries of $\mathbb{R}^3$ are infinitesimal symmetries of the Lagrangian density $\mathcal{L}\,\mathrm{d}t$, and hence infinitesimal generalized symmetries of $\bar{\mathcal{L}}\,\mathrm{d}x$. The isometries of $\mathbb{R}^3$ are generated by the translations along the three axes, and the rotations around these axes. The infinitesimal generators of the translations are $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$, which are their own prolongations by infinitesimal contact transformations. As the Poincaré–Cartan form has the expression given in (3), the Noether invariants associated to the translations are $H = -\bar{\Theta}(\partial/\partial x)$, $p = \bar{\Theta}(\partial/\partial y)$ and $q = \bar{\Theta}(\partial/\partial z)$. The infinitesimal generators of the three rotations are

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \qquad Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

We obtain their prolongations $X_{[3]}, Y_{[3]}, Z_{[3]} \in \mathfrak{X}(J^3(\mathbb{R},\mathbb{R}^2))$ by imposing that $X_{[3]}$, $Y_{[3]}$ and $Z_{[3]}$ project onto X, Y and Z respectively, and leave invariant the differential system spanned by the contact forms $\mathrm{d}y - y'\,\mathrm{d}x$, $\mathrm{d}z - z'\,\mathrm{d}x$, $\mathrm{d}y' - y''\,\mathrm{d}x$, $\mathrm{d}z' - z''\,\mathrm{d}x$, $\mathrm{d}y'' - y'''\,\mathrm{d}x$ and $\mathrm{d}z'' - z'''\,\mathrm{d}x$:

$$\begin{aligned}
X_{[3]} &= X + z'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial z'} + z''\frac{\partial}{\partial y''} - y''\frac{\partial}{\partial z''} + z'''\frac{\partial}{\partial y'''} - y'''\frac{\partial}{\partial z'''},\\
Y_{[3]} &= Y + y'z'\frac{\partial}{\partial y'} + \left(z'^2+1\right)\frac{\partial}{\partial z'} + \left(y'z''+2y''z'\right)\frac{\partial}{\partial y''} + 3z'z''\frac{\partial}{\partial z''}\\
&\quad + \left(y'z'''+3y''z''+3y'''z'\right)\frac{\partial}{\partial y'''} + \left(4z'z'''+3z''^2\right)\frac{\partial}{\partial z'''},\\
Z_{[3]} &= Z + (y'^2+1)\frac{\partial}{\partial y'} + y'z'\frac{\partial}{\partial z'} + 3y'y''\frac{\partial}{\partial y''} + \left(2y'z''+y''z'\right)\frac{\partial}{\partial z''}\\
&\quad + \left(4y'y'''+3y''^2\right)\frac{\partial}{\partial y'''} + \left(y'''z'+3y''z'''+3y'z'''\right)\frac{\partial}{\partial z'''}.
\end{aligned}$$

The associated Noether invariants are:

$$\begin{aligned}
C_x &= pz - qy + p'z' - q'y',\\
C_y &= Hz + qx + y'z'p' + \left(z'^2+1\right)q',\\
C_z &= Hy + px + \left(y'^2+1\right)p' + y'z'q'.
\end{aligned}$$

3.1 The level $H = 0$

The Hamiltonian H has a unique critical point at the point $y' = z' = p = q = p' = q' = 0$, which is non-degenerate of signature $(4, 2)$ as follows from Morse's lemma taking into account that $H = py' + qz' + P^2 + Q^2$, where

$$P = \frac{1}{2}v^{3/2}\left(\left(y'^2+1\right)^{1/2}p' + y'z'\left(y'^2+1\right)^{-1/2}q'\right),$$

$$Q = \frac{1}{2}v^{5/2}\left(y'^2+1\right)^{-1/2}q'.$$

If the Hamiltonian vanishes, then we can substitute the expressions of C_y and C_z into the third and fourth Hamilton equations (4) to obtain the following system of first-order ordinary differential equations:

$$\frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{1}{2}v^3(C_z - px), \qquad \frac{\mathrm{d}z'}{\mathrm{d}x} = \frac{1}{2}v^3(C_y - qx).$$

3.2 $H \neq 0$

If H is not zero, we use the formulas of C_y and C_z to eliminate y and z, and the third and fourth Hamilton equations (4) to eliminate p' and q'. Substitution in the expressions of H and C_x then yields, after defining C as

$$2C = C_x - \frac{pC_y - qC_z}{H},$$

the following system of ordinary differential equations of the first order:

$$Cv^3 = \left(\frac{q}{H} + z'\right)\frac{\mathrm{d}y'}{\mathrm{d}x} - \left(\frac{p}{H} + y'\right)\frac{\mathrm{d}z'}{\mathrm{d}x}, \tag{5}$$

$$(H - py' - qz')v^5 = \left(z'^2+1\right)\left(\frac{\mathrm{d}y'}{\mathrm{d}x}\right)^2 - 2y'z'\frac{\mathrm{d}y'}{\mathrm{d}x}\frac{\mathrm{d}z'}{\mathrm{d}x} + \left(y'^2+1\right)\left(\frac{\mathrm{d}z'}{\mathrm{d}x}\right)^2. \tag{6}$$

3.2.1 The case $C = 0$

If C vanishes, then the equation (5) is just

$$(Hz' + q)\frac{\mathrm{d}y'}{\mathrm{d}x} - (Hy' + p)\frac{\mathrm{d}z'}{\mathrm{d}x} = 0,$$

or, equivalently,

$$\begin{pmatrix} \mathrm{d}y'/\mathrm{d}x \\ \mathrm{d}z'/\mathrm{d}x \end{pmatrix}^{\perp} = \begin{pmatrix} \mathrm{d}z'/\mathrm{d}x \\ -\mathrm{d}y'/\mathrm{d}x \end{pmatrix} \perp \left(H\begin{pmatrix} y' \\ z' \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}\right).$$

Hence, for every x there exists a $\lambda(x) \in \mathbb{R}$ such that

$$\frac{\mathrm{d}y'}{\mathrm{d}x} = \lambda\left(Hy' + p\right), \qquad \frac{\mathrm{d}z'}{\mathrm{d}x} = \lambda\left(Hz' + q\right).$$

Solving this pair of differential equations, we obtain

$$y'(x) = K_y e^{H\Lambda} - \frac{p}{H}, \qquad z'(x) = K_z e^{H\Lambda} - \frac{q}{H},$$

where $\Lambda(x) = \int \lambda(x)\mathrm{d}x$ and $K_y, K_z \in \mathbb{R}$. Substitution in equation (6) then yields

$$\left(H - (pK_y + qK_z)e^{H\Lambda} - \frac{p^2+q^2}{H}\right)\left(\left(K_y^2 + K_z^2\right)e^{2H\Lambda} - \frac{pK_y + qK_z}{H}e^{H\Lambda} + \frac{H^2+p^2+q^2}{H^2}\right)^{5/2}$$
$$= (\Lambda')^2\left(H^2(K_y^2+K_z^2) + (qK_y - pK_z)^2\right)e^{2H\Lambda},$$

which is a first-order ordinary differential equation on Λ.

3.2.2 The general case

Let us now assume that $H \neq 0$ and $C \neq 0$. From there, and (5), we know that $p + Hy'$ and $q + Hz'$ do not vanish simultaneously. Let us suppose that $p + Hy' \neq 0$. Then, from equation (5) we can write the derivative of z' in terms of the derivative of y' as

$$\frac{\mathrm{d}z'}{\mathrm{d}x} = \frac{q + Hz'}{p + Hy'}\frac{\mathrm{d}y'}{\mathrm{d}x} - \frac{CHv^3}{p + Hy'}, \tag{7}$$

and then we can write this equation as a differential equation for $z'(y')$:

$$\frac{\mathrm{d}z'}{\mathrm{d}y'} = \frac{q + Hz'}{p + Hy'} - \frac{CHv^3}{(\mathrm{d}y'/\mathrm{d}x)(p + Hy')}. \tag{8}$$

Substitution of (7) in (6) then yields

$$\left(\frac{\mathrm{d}y'}{\mathrm{d}x}\right)^2\left[(pz' - qy')^2 + (Hz' + q)^2 + (hy' + p)^2\right] + 2\frac{\mathrm{d}y'}{\mathrm{d}x}CHv^3\left[y'(pz' - qy') - (Hz' + q)\right]$$
$$+ C^2H^2v^6({y'}^2 + 1) - (H - py' - qz')(Hy' + p)v^5 = 0,$$

whose solutions are

$$\frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{v^3\left(y'(pz' - qy') - (Hz' + q)\right) \pm \Delta^{1/2}}{A}, \tag{9}$$

where

$$\Delta = -C^2H^2v^8(Hy' + p)^2 + v^5(Hy' + p)(H - py' - qz')A,$$
$$A = (qy' - pz')^2 + (Hy' + p)^2 + (Hz' + q)^2.$$

We have thus reduced the Hamilton equations to a pair of first-order ordinary differential equations, (8) and (9). Also note that for $C \neq 0$ and $H \neq 0$, the above system is non-singular where it is defined so that the singularities of the system can only arise in the particular cases previously studied.

We remark that in order to find out where the solutions of (9) are defined, we have to study the behaviour of the discriminant Δ. It is clear that where $H - py' - qz' < 0$ the discriminant is negative and hence no solution is defined in that region. On the half-plane $H - py' - qz' > 0$, Δ is positive if and only if

$$(H - py' - qz')^2\left((qy' - pz')^2 + (Hy' + p)^2 + (Hz' + q)^2\right)^2 > C^4H^4\left(1 + {y'}^2 + {z'}^2\right)^3.$$

The analysis of the sign of this expression has turned too long to be stated here, and we leave it for a future work.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 177–179.

A Class of Non-Lie Solutions for a Non-linear d'Alembert Equation

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New class of exact solutions of multidimensional complex nonlinear d'Alembert equation is constructed. These solutions in principle can not be obtained within the frame work of the traditional Lie approach.

In paper [1] the symmetry reduction of multidimensional complex non-linear d'Alembert equation was carried out

$$\Box u = F(|u|)u \tag{1}$$

to ordinary differential equations. In (1) $\Box = \partial^2/\partial x_0^2 - \Delta$ is the d'Alembert operator, $u = u(x_0, x_1, x_2, x_3)$ is a complex twice continuously differentiable function, $F(|u|)$ is a continuous arbitrary function.

Amongst the obtained reduced equations there appear not only ordinary differential equations, but also purely algebraical ones.

For example, if we take the Ansatz

$$u(x) = \exp\left(-i(\sigma x_1 + \gamma x_2 + \sigma x_2(x_0 + x_3))\right) \varphi(x_0 + x_3),$$

where σ, γ are real constants, so that $|\gamma| + |\sigma| \neq 0$.

Substituting this Ansatz into (1), we obtain

$$\left(\sigma^2 + \gamma^2 + 2\gamma\sigma\omega + \sigma\omega^2\right)\varphi = F(|\varphi|)\varphi.$$

Hence, we find that $|\varphi|$ is given implicitly:

$$F(|\varphi|) = \sigma^2 + \gamma^2 + 2\gamma\sigma\omega + \sigma\omega^2.$$

So, this procedure gives a class of solutions of equation (1) with the only real function $\Phi(x_0 + x_3)$:

$$\varphi = F^{-1}\left(\sigma^2 + \gamma^2 + 2\gamma\sigma\omega + \sigma\omega^2\right)\exp(i\Phi(x_0 + x_3)).$$

Our aim is to describe all possible Ansätze of the type:

$$u(x) = \exp(ia(x))\varphi(\omega(x)), \tag{2}$$

which reduce equation (1) to algebraic one.

The full solution of this task is given by the following theorem.

Theorem. *Ansatz (2) reduces equation (1) to algebraic one if and only if*

$$\begin{aligned} 1)\quad & A_\mu(\omega)x^\mu + B(\omega) = 0, \qquad A_\mu(\omega)A^\mu(\omega) = 0, \qquad a_{x_\mu}\omega_{x^\mu} = 0, \\ & \Box a = 0, \qquad a_{x_\mu}a_{x^\mu} = -w_1^2(\omega), \qquad a(x) = \frac{w_1(\omega)}{\left(-\dot{A}_\nu \dot{A}^\nu\right)}\dot{A}_\mu x^\mu + w_2(\omega), \end{aligned} \tag{3}$$

where $\nu = 0,1,2,3$*;* $A_\mu(\omega)$*,* $B(\omega)$*,* $w_1(\omega)$*,* $w_2(\omega)$ *are arbitrary functions; the point above the symbol means derivative by* ω*; by the repeated indexes ment summing up (raising and lowering of the index is carried out with the help of the metrical tensor of the Minkowski space* $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$*).*

$$2)\quad \omega(x) = w_0\left(\theta_\mu x^\mu\right), \quad a(x) = w_1\left(\theta_\mu x^\mu\right) a_\nu x^\nu + w_2\left(\theta_\mu x^\mu\right) b_\nu x^\nu + w_3\left(\theta_\mu x^\mu\right),$$

where w_0*,* w_1*,* w_2*,* w_3 *are arbitrary functions of their arguments;* θ_μ*,* a_μ*,* b_μ *are arbitrary real parameters, which satisfy the following orthogonal relations*

$$a_\mu a^\mu = b_\mu b^\mu = -1, \qquad a_\mu b^\mu = a_\mu \theta^\mu = b_\mu \theta^\mu = \theta_\mu \theta^\mu = 0.$$

We omit the proof, which bases on the results of papers [2–3].

The result of substituting Ansatz (2), where $a(x)$, $\omega(x)$ are given by formulas (3), into equation (1) will give the following algebraic equation:

$$w_1^2(\omega)\varphi = F(|\varphi|)\varphi.$$

Hence, we obtain the following class of solutions of the complex non-linear d'Alembert equation (1)

$$u(x) = \rho(\omega)\exp\{ia(x)\},$$

where $a(x)$, $\omega(x)$ are given by the formulas (3), and $\rho(\omega) > 0$ is determined in implicit way:

$$F(\rho(\omega)) = w_1^2(\omega).$$

Let us mention that the obtained class of solutions is non-Lie and thus can not be derived with the help of the method of symmetry reduction.

Consider the example, where the exact solution of d'Alembert equation with the cubical non-linearity

$$\Box u = \lambda |u|^2 u \tag{4}$$

is constructed in explicit way. Substituting $A_0 = 1$, $A_1 = \omega$, $A_2 = \sqrt{1-\omega^2}$, $A_3 = B = 0$ into (3) we obtain that

$$\omega = \left(x_1^2 + x_2^2\right)^{-1}\left(x_0 x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}\right).$$

Hence, we find

$$a(x) = iw_1(\omega)x_3 + iw_2(\omega).$$

So, we obtain the following class of solutions

$$u(x) = \frac{1}{\sqrt{\lambda}} w_1(\omega)\exp\{iw_1(\omega)x_3 + iw_2(\omega)\}$$

of the equation (4). We emphasize that this solution has singularity at the point $\lambda = 0$, so it can not be obtained by the methods of the perturbation theory by a small parameter λ.

Acknowledgments

I would like to express my gratitude to Professor R.Z. Zhdanov for permanent assistance and formulation of the problem.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 180–183.

On $SO(3)$-Partially Invariant Solutions of the Euler Equations

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$SO(3)$-partially invariant solutions having minimal defect are constructed for the Euler equations describing flows of an ideal incompressible fluid.

The concept of partially invariant solutions was introduced by Ovsiannikov [1] as a generalization of invariant solutions, which is possible for systems of partial differential equations (PDEs). The algorithm for finding partially invariant solutions is very difficult to apply. For this reason it is used more rarely than the classical Lie algorithm for constructing invariant solutions.

The Euler equations (EEs) describing flows of an ideal incompressible fluid have the following form:

$$\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p = \vec{0}, \qquad \operatorname{div}\vec{u} = 0. \tag{1}$$

It is well known [2, 3] that the maximal Lie invariance algebra of EEs is the infinite dimensional algebra $A(E)$, generated by the following basis elements:

$$\begin{aligned}
&\partial_t, \quad J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a} \quad (a<b),\\
&D^t = t\partial_t - u^a\partial_{u^a} - 2p\partial_p, \quad D^x = x_a\partial_a + u^a\partial_{u^a} + 2p\partial_p,\\
&R(\vec{m}) = R(\vec{m}(t)) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p,\\
&Z(\chi) = Z(\chi(t)) = \chi(t)\partial_p.
\end{aligned} \tag{2}$$

In the following $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity of the fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian, $m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0,t_1),\mathbb{R})$). The fluid density is set equal to unity. Summation over repeated indices is implied, and we have $a,b = 1,2,3$. Subscripts of functions denote differentiation with respect to the corresponding variables.

Let us note that the algebra $so(3)$ generated by the operators J_{ab} is a subalgebra of $A(E)$.

Invariant solutions of (1) have been already constructed. For example, in [4, 5] EEs are reduced to partial differential equations in two and three independent variables by means of the Lie algorithm. In this paper we obtain $SO(3)$-partially invariant solutions of the minimal defect that is equal to 1 for the given representation of $so(3)$.

A complete set of functionally independent invariants of the group $SO(3)$ in the space of the variables $(t,\vec{x},\vec{u},p)$ is exhausted by the functions t, $|\vec{x}|$, $\vec{x}\cdot\vec{u}$, $|\vec{u}|$, p, so $SO(3)$-partially invariant solution of the minimal defect has the form

$$\begin{aligned}
u^R &= v(t,R),\\
u^\theta &= w(t,R)\sin\psi(t,R,\theta,\varphi),\\
u^\varphi &= w(t,R)\cos\psi(t,R,\theta,\varphi),\\
p &= p(t,R).
\end{aligned} \tag{3}$$

Hereafter for convenience the spherical coordinates are used:

$$R = |\vec{x}|, \qquad \varphi = \arctan\frac{x_2}{x_1}, \qquad \theta = \arccos\frac{x_3}{|\vec{x}|},$$

$$u^R = \frac{\vec{x}\cdot\vec{u}}{|\vec{x}|}, \qquad u^\varphi = \frac{(\vec{e}_3\times\vec{x})\cdot\vec{u}}{|(\vec{e}_3\times\vec{x})|}, \qquad u^\theta = \frac{((\vec{e}_3\times\vec{x})\times\vec{x})\cdot\vec{u}}{|((\vec{e}_3\times\vec{x})\times\vec{x})|}, \qquad \vec{e}_3 := (0,0,1).$$

Substituting (3) into EEs (1), we obtain the system of PDEs for the functions v, w, ψ, p :

$$\begin{aligned}
&v_t + vv_R - R^{-1}w^2 + p_r = 0,\\
&w_t + vw_R + R^{-1}vw = 0,\\
&w(\psi_t + v\psi_R + R^{-1}w\psi_\theta\sin\psi + R^{-1}w\cos\psi(\sin\theta)^{-1}(\psi_\varphi - \cos\theta)) = 0,\\
&Rv_r + 2v + w\psi_\theta\cos\psi - (\sin\theta)^{-1}w\sin\psi(\psi_\varphi - \cos\theta) = 0.
\end{aligned}\tag{4}$$

It follows from (4) if $w = 0$ that $v = \eta R^{-2}$, $p = \eta_t R^{-1} - \frac{1}{2}\eta^2 R^{-4} + \chi$, where η and χ are arbitrary smooth functions of t. The corresponding solution of EEs

$$u^R = \frac{\eta}{R^2}, \qquad u^\theta = u^\varphi = 0, \qquad p = \frac{\eta_t}{R} - \frac{\eta^2}{2R^4} + \chi \tag{5}$$

is invariant with respect to $SO(3)$. Note that flow (5) is a solution of the Navier-Stokes equations too, and it is the unique $SO(3)$-partially invariant solutions of the minimal defect for the Navier-Stokes equations.

Below $w \neq 0$. Then two last equations of (4) form an overdetermined system in the function ψ. This system can be rewritten as follows

$$\begin{aligned}
&\psi_\theta + Rw^{-1}\sin\psi(\psi_t + v\psi_R) = -G\cos\psi,\\
&\psi_\varphi + Rw^{-1}\cos\psi(\psi_t + v\psi_R)\sin\theta = G\sin\psi\sin\theta + \cos\theta,
\end{aligned}\tag{6}$$

where $G = G(t,R) := w^{-1}(Rv_R + 2v)$. The Frobenius theorem gives the compatibility condition of (6):

$$G_t + vG_R = R^{-1}w(1 + G^2). \tag{7}$$

If condition (7) holds, system (6) can be integrated implicitely. Namely, its general solution has the form

$$F(\Omega_1, \Omega_2, \Omega_3) = 0, \tag{8}$$

where F is an arbitrary function of Ω_1, Ω_2, and Ω_3,

$$\Omega_1 = \frac{\sin\theta\sin\psi - G\cos\theta}{\sqrt{1+G^2}}, \qquad \Omega_2 = \varphi + \arctan\frac{\cos\psi}{\cos\theta\sin\psi + G\sin\theta}, \qquad \Omega_3 = h(t,r),$$

$h = h(t,R)$ is a fixed solution of the equation $h_t + vh_R = 0$ such that $(h_t, h_R) \neq (0,0)$. Equation (8) can be solved with respect to ψ in a number of cases, for example, if either $F_{\Omega_1} = 0$ or $F_{\Omega_2} = 0$.

Equation (7) and two first equations of (4) form the "reduced" system for the invariant functions v, w, and p. It can be represented as the union of the system

$$\begin{aligned}
&R^2 f_{tR} + ff_{RR} - (f_R)^2 = g, \qquad && f := R^2 v\\
&R^2 g_t + fg_R = 0, && g := (Rw)^2,
\end{aligned}\tag{9}$$

for the functions v and w (this system can be also considered a system for the functions f and g) and the equation

$$p_R = -v_t - vv_R - R^{-1}w^2 \tag{10}$$

which is one for the function p if v and w are known. Therefore, to construct solutions for EEs, we are to carry out the following chain of actions: 1) to solve the system (9); 2) to integrate (10) with respect to p; 3) to find the function ψ from (8).

Theorem. *The maximal Lie invariance algebra of (9) is the algebra*

$$\mathcal{A} = \langle\, \partial_t,\ D^R = R\partial_R + v\partial_v + w\partial_w,\ D^t = t\partial_t - v\partial_v - w\partial_w \,\rangle.$$

A complete set of $\mathcal{A}$-inequivalent one-dimensional subalgebras of $\mathcal{A}$ is exhausted by four algebras. Let us enumerate these algebras and the corresponding ansatzes for the functions v and w as well as the reduced systems arising after substituting the ansatzes into (9).

1. $\langle\, \partial_t \,\rangle:$ $\quad v = R^{-2}\varphi^1(\omega),\ w = R^{-1}\varphi^2(\omega),\ \omega = R, \quad \varphi^2 \neq 0;$
   $\varphi^1\varphi^1_{\omega\omega} - (\varphi^1_\omega)^2 = (\varphi^2)^2, \quad \varphi^1\varphi^2_\omega = 0.$
2. $\langle\, D^R \,\rangle:$ $\quad v = R\varphi^1(\omega),\ w = R/\varphi^2(\omega),\ \omega = t, \quad \varphi^1\varphi^2 \neq 0;$
   $3\varphi^1_\omega = 3(\varphi^1)^2 + (\varphi^2)^{-2}, \quad \varphi^2_\omega = 2\varphi^1\varphi^2.$
3. $\langle\, \partial_t + D^R \,\rangle:$ $\quad v = R\varphi^1(\omega),\ w = R\varphi^2(\omega),\ \omega = \ln R - t, \quad \varphi^1_\omega\varphi^2_\omega \neq 0;$
   $(\varphi^1 - 1)\varphi^1_{\omega\omega} - (\varphi^1_\omega)^2 - \varphi^1_\omega(\varphi^1 + 3) - 3(\varphi^1)^2 = (\varphi^2)^2,$
   $(\varphi^1 - 1)\varphi^2_\omega + 2\varphi^1\varphi^2 = 0.$
4. $\langle\, D^t + \kappa D^R \,\rangle:$ $\quad v = Rt^{-1}\varphi^1(\omega),\ w = Rt^{-1}\varphi^2(\omega),\ \omega = \ln R - \kappa\ln|t|, \quad \varphi^1\varphi^2 \neq 0;$
   $(\varphi^1 - \kappa)\varphi^1_{\omega\omega} - (\varphi^1_\omega)^2 - \varphi^1_\omega(\varphi^1 + 3\kappa + 1) - 3(\varphi^1)^2 - 3\varphi^1 = (\varphi^2)^2,$
   $(\varphi^1 - \kappa)\varphi^2_\omega + (2\varphi^1 - 1)\varphi^2 = 0.$

Two first reduced systems can be integrated completely. As a result we obtain the following expressions for the functions v, w, and p:

1. $v = \dfrac{C_3}{2R^2}\Big(e^{C_2R} + C_1^2e^{-C_2R}\Big), \quad w = \dfrac{C_1C_2C_3}{R}, \quad C_1, C_2, C_3 = \mathrm{const}, \quad C_1C_2C_3 \neq 0,$

   $$\Longrightarrow \quad p = -\frac{C_3^2}{8R^4}\Big(e^{2C_2R} + 2C_1^2 + C_1^4e^{-2C_2R}\Big) - \frac{C_1^2C_2^2C_3^2}{2R^2} + \chi(t),$$

   $$G = \frac{1}{2C_1}\Big(e^{C_2R} - C_1^2e^{-C_2R}\Big), \quad h = t - \int\frac{dR}{v(R)}.$$

2. $v = \dfrac{\wp_t}{2\wp}R, \quad w = \dfrac{3C}{2\wp}R, \quad \Longrightarrow \quad p = C^2\Big(\dfrac{1}{\wp^2} - \wp\Big)R^2 + \chi(t), \quad G = \dfrac{\wp_t}{C}, \quad h = \dfrac{R^2}{\wp}.$

Here $C = \mathrm{const}$, $C \neq 0$, $\wp = \wp(Ct, 0, 1)$ is the Weierstrass function, χ is an arbitrary smooth function of t.

System (9) has solutions for which f and g are polynomial with respect to R. Thus, if $\deg(f, R) = 1$ and, therefore, $\deg(g, R) \leqslant 2$, then $f = C^2tR$, $g = C^2R^2 - C^4t^2$, where $C = \mathrm{const}$, $C \neq 0$, i.e.

$$v = \frac{C^2t}{R}, \quad w = \frac{C}{R}\sqrt{R^2 - C^2t^2} \quad \Longrightarrow \quad p = \chi(t), \quad G = \frac{Ct}{\sqrt{R^2 - C^2t^2}}, \quad h = \sqrt{R^2 - C^2t^2}.$$

The solution of system (9), given above, is invariant with respect to the algebra $\langle D^t + D^R \rangle$.

If $\deg(f, R) = 3$ and, therefore, $\deg(g, R) \leqslant 4$, we have two families of solutions:

a) $f = -\dfrac{R^3}{t} + C_1^2(t^3 + C_2)R, \quad g = 3C_1^2 t^2 R^2 - C_1^4(t^3 + C_2)^2, \quad C_1, C_2 = \mathrm{const},\ C_1 \neq 0,$ i.e.

$$v = -\frac{R}{t} + C_1^2\frac{t^3 + C_2}{R}, \quad w = \frac{C_1}{R}\sqrt{3t^2R^2 - C_1^2(t^3 + C_2)^2}, \quad \Longrightarrow \quad p = -\frac{R^2}{t^2} + \chi(t),$$

$$G = \frac{-3R^2 + C_1^2 t(t^3 + C_2)^2}{C_1 t\sqrt{3t^2R^2 - C_1^2(t^3 + C_2)^2}}, \quad h = \sqrt{3t^2R^2 - C_1^2(t^3 + C_2)^2}.$$

b) $f = \dfrac{\wp_t}{2\wp}R^3 + (C_1\cos\alpha + C_2\sin\alpha)R, \quad g = \left(\dfrac{3C_0}{2\wp}R^2 - C_1\sin\alpha + C_2\cos\alpha\right)^2 - C_1^2 - C_2^2,$

$$v = \frac{f}{R^2}, \quad w = \frac{\sqrt{g}}{R} \quad \Longrightarrow \quad p = C_0^2\Big(\frac{1}{\wp^2} - \wp\Big)R^2 + \chi(t), \quad G = \frac{f_R}{\sqrt{g}}, \quad h = \sqrt{g}.$$

Here $C_0, C_1, C_2 = \mathrm{const}$, $C_0 \neq 0$, $\wp = \wp(C_0 t, 0, 1)$ is the Weierstrass function, $\alpha = \int 3C_0\wp^{-1}dt$, χ is an arbitrary smooth function of t. The last solution is a generalization of the invariant solution with respect to the algebra $\langle D^R \rangle$.

The solutions given above exhaust all the solutions of system (9), for which f and g are polynomial with respect to R.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 184–189.

Equivalence of Q-Conditional Symmetries under Group of Local Transformation

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The definition of Q-conditional symmetry for one PDE is correctly generalized to a special case of systems of PDEs and involutive families of operators. The notion of equivalence of Q-conditional symmetries under a group of local transformation is introduced. Using this notion, all possible single Q-conditional symmetry operators are classified for the n-dimensional ($n \geqslant 2$) linear heat equation and for the Euler equations describing the motion of an incompressible ideal fluid.

The concept of Q-conditional symmetry called also nonclassical symmetry was introduced by Bluman and Cole in 1969. This year is the year of the 30^{th} anniversary of appearance of their pioneering paper [1]. Although the concept of Q-conditional symmetry exists for a long time and has various applications many problem of its theory are not solved so far.

Before 1986 the nonclassical symmetry was only mentioned in few papers, e.g., in [2]. The intensive application of Q-conditional symmetries to finding exact solutions of partial differential equations (PDEs) and the parallel search for their foundations was begun after publication of the papers of Olver and Rosenau in 1986 and 1987 [3, 4] as well as the paper of Fushchych and Tsyfra in 1987 [5].

The first correct definition of a Q-conditional symmetry operator for one PDE was given in [5]. Later it was generalized to involutive families of operators [6–8]. We stress that it can be directly extended only to some special cases of Q-conditional invariance for systems of PDEs. For all the other cases this definition must be essentially modified and is much more complicated.

In this paper we correctly generalize the definition of Q-conditional symmetry [6–8] to a special case of systems of PDEs and involutive families of operators. Further, we introduce the notion of equivalence of Q-conditional symmetries under a group of local transformation. Using this notion, we can, first, classify all the possible Q-conditional symmetries and, correspondingly, all the possible reductions of systems of PDEs [7, 8] and, secondly, essentially simplify the procedure of finding Q-conditional symmetries in some cases when the Lie symmetry group is sufficiently wide.

Consider a system of k PDEs of the order r for m unknown functions $u = (u^1, \ldots, u^m)$ depending on n independent variables $x = (x_1, \ldots, x_n)$ of the form

$$L(x, u_{(r)}(x)) = 0, \qquad L = (L^1, \ldots, L^k). \tag{1}$$

Here the order of a system is the order of the major partial derivative appearing in the system. The symbol $u_{(r)}$ denotes for the set of partial derivatives of the functions u of the orders from 0 to r. Within the local approach system (1) is treated as a system of algebraic equations in the jet space $J^{(r)}$ of the order r.

Consider also an involutive family Q of l differential operators

$$Q^s = \xi^{si}(x,u)\partial_{x_i} + \eta^{sa}(x,u)\partial_{u_a}, \qquad \text{where} \qquad l \leqslant n, \quad \operatorname{rank} ||\xi^{si}(x,u)|| = l. \tag{2}$$

The requirement of involution means for the family Q that the commutator of any pair of operators from Q belongs to the span of Q over the ring of smooth functions of the variables x and u, i.e.

$$\forall\, s,p \quad \exists\, \zeta^{sps'} = \zeta^{sps'}(x,u)\colon \quad [Q^s, Q^p] = \zeta^{sps'} Q^{s'}. \tag{3}$$

Here and below the indices a and b run from 1 to m, the indices i and j run from 1 to n, the indices s and p run from 1 to l, and the indices μ and ν run from 1 to $n-l$. The sumation is imposed over the repeated indices. Subscripts of functions denote differentiation with respect to the corresponding variables.

If operators (2) form an involutive family, then the family $\widetilde{Q}$ of differential operators

$$\widetilde{Q}^s = \lambda^{sp} Q^p, \qquad \text{where} \qquad \lambda^{sp} = \lambda^{sp}(x,u), \quad \det ||\lambda^{sp}|| \neq 0, \tag{4}$$

is also involutive. And family (4) is called *equivalent* to family (2) [6–8].

Notation: $\widetilde{Q} = \{\widetilde{Q}^s\} \sim Q = \{Q^s\}$.

By the Frobenius theorem, condition (3) is sufficient for the system of PDEs

$$Q^s[u^a] := \eta^{sa}(x,u) - \xi^{si}(x,u)\frac{\partial u^a}{\partial x_i} = 0 \tag{5}$$

to be compatible.

Denote the manifold defined by the system of algebraic equations $L = 0$ in $J^{(r)}$ by $\mathcal{L}$ and the manifold corresponding to the set of all the differential consequences of the system of PDEs (5) in $J^{(r)}$ by $\mathcal{M}$:

$$\begin{aligned}\mathcal{L} &= \{(x, u_{(r)}) \in J^{(r)} \,|\, L(x, u_{(r)}) = 0\},\\ \mathcal{M} &= \{(x, u_{(r)}) \in J^{(r)} \,|\, D_1^{\alpha_1} \dots D_n^{\alpha_n} Q^s[u^a] = 0, \ \ \alpha_i \in \mathbb{N}\cup\{0\}, \ \ |\alpha|\colon = \alpha_1 + \dots + \alpha_n < r\},\end{aligned}$$

where $D_i = \partial_{x_i} + \sum\limits_{\alpha} u^a_{\alpha,i}\partial_{u^a_\alpha}$ is the operator of total differentiation with respect to the variable x_i, u^a_α and $u^a_{\alpha,i}$ denote the variables in $J^{(r)}$, corresponding to derivatives $\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$ and $\frac{\partial^{|\alpha|+1} u}{\partial x_1^{\alpha_1}\dots\partial x_{i-1}^{\alpha_{i-1}}\partial x_i^{\alpha_i+1}\partial x_{i+1}^{\alpha_{i+1}}\dots\partial x_n^{\alpha_n}}$.

Let the system $L|_{\mathcal{M}} = 0$ do not includes equations which are differential consequences of other its equations. Moreover, let all the differential consequences of the system $L|_{\mathcal{M}} = 0$, the orders of which (as equations) are less than or equal to its order, vanish on $\mathcal{L}\cap\mathcal{M}$.

Definition 1. *System of smaller PDEs (1) is called* Q-conditional invariant *with respect to involutive family of differential operators (2) if the relation*

$$\left(Q^s_{(r)}L\right)\Big|_{\mathcal{M}\cap\mathcal{L}} = 0 \tag{6}$$

holds true. Here the symbol $Q^s_{(r)}$ *denotes the* r*th prolongation of the operator* Q^s*:*

$$Q^s_{(r)} = Q^s + \sum_{|\alpha|\leqslant r} \eta^{sa\alpha}\partial_{u^a_\alpha}, \qquad \eta^{sa\alpha} = D_1^{\alpha_1}\dots D_n^{\alpha_n}(\eta^{sa} - \xi^{si}u^a_i) + \xi^{si}u^a_{\alpha,i}.$$

Denote the set of involutive families of l operators of Q-conditional symmetry of system (1) as $\mathcal{B}(\mathcal{L}, l)$:

$$\mathcal{B}(\mathcal{L}, l) = \left\{ Q = \{Q^1,\dots,Q^l\} \;\middle|\; \begin{array}{l}\text{the system } L = 0 \text{ is } Q\text{-conditionally}\\ \text{invariant with respect to } Q\end{array} \right\}$$

Lemma [6–8]. *Let system of PDEs (1) be Q-conditionally invariant with respect to involutive family of operators (2). Then, it is Q-conditionally invariant with respect to an arbitrary family of the form (4), i.e.*

$$Q \in \mathcal{B}(\mathcal{L}, l),\ \widetilde{Q} \sim Q \quad \Longrightarrow \quad \widetilde{Q} \in \mathcal{B}(\mathcal{L}, l).$$

An important consequence of the lemma is that we can study Q-conditionally invariance up to equivalence relation (4) which is defined on the set of involutive families of l operators as well as in $\mathcal{B}(\mathcal{L}, l)$. Then it is possible for an arbitrary family of operators (2) to choose the functions $\lambda^{sp}(x,u)$ and, if it is necessary, to change enumeration of the variables $x_1, \ldots, x_n$ in such a way that operators (4) take the following form: $\widehat{Q}^s = \partial_{x_s} + \hat{\xi}^{s,l+\nu}\partial_{x_{l+\nu}} + \hat{\eta}^{sa}\partial_{u^a}$. Operators $\widehat{Q}^s$ generate a commutative Lie algebra.

Let $A(\mathcal{L})$ and $G(\mathcal{L})$ denote the maximal Lie invariance algebra of system (1) and its maximal local symmetry correspondingly. Now we strengthen the equivalence relation in $\mathcal{B}(\mathcal{L}, l)$, given by formula (4), by means of generalizing equivalence of l-dimensional subalgebras of the algebra $A(\mathcal{L})$ under the adjoint representation of the group $G(\mathcal{L})$ in $A(\mathcal{L})$.

We use the following lemma for this generalization.

Lemma. *Let g be an arbitrary local transformation from $G(\mathcal{L})$. Then the adjoint action of g in the set of differential operators generate a one-to-one mapping from $\mathcal{B}(\mathcal{L}, l)$ into itself.*

Let $Q = \{Q^s\}$ and $\widetilde{Q} = \{\widetilde{Q}^s\}$ be involutive families of differential operators.

Definition. *The families Q and $\widetilde{Q}$ are called equivalent with respect to a group G of local transformations if there exists a local transformation g from G for which the families Q and $\mathrm{Ad}(g)\widetilde{Q}$ are equivalent.*

Notation: $Q \sim \widetilde{Q} \bmod G$.

Definition. *The families Q and $\widetilde{Q}$ are called equivalent with respect to a Lie algebra A of differential operators if they are equivalent with respect to the one-parametric group generated by an operator from A.*

Notation: $Q \sim \widetilde{Q} \bmod A$.

Therefore,

$$Q \sim \widetilde{Q} \bmod G \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists\, g \in G\colon Q \sim \mathrm{Ad}(g)\widetilde{Q}. \tag{7}$$

$$Q \sim \widetilde{Q} \bmod A \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists\, V \in A\colon Q \sim \widetilde{Q} \bmod \{e^{\varepsilon V},\ \varepsilon \in U(0,\delta) \subset \mathbb{R}\} \tag{8}$$

Lemma. *Formulas (7) and (8) define equivalence relations in the set of involutive families of l differential operators. Moreover, if G is a subgroup of $G(\mathcal{L})$ and A is a subalgebra of $A(\mathcal{L})$ then formulas (7) and (8) define equivalence relations in $\mathcal{B}(\mathcal{L}, l)$.*

Comsider two examples.

Example 1. Investigate Q-conditional invariance of the linear n-dimensional heat equation

$$u_t = u_{aa}, \quad \text{where} \quad u = u(t, \vec{x}), \quad t = x_0, \quad \vec{x} = (x_1, \ldots, x_n), \tag{9}$$

with respect to a single operator ($l = 1$).

It is just the problem with $n = 1$ for which Bluman and Cole introduced the concept of nonclassical symmetry. In the one-dimensional case the problem was completely solved in [9]. That is why we pay our attention to the multidimensional problem.

Lie symmetry of equation (9) is well known. In the one-dimensional case it was investigated by Lie. The maximal Lie invariance algebra $A(\mathrm{LHE})$ of (9) is generated by the following operators:

$$\begin{aligned} &\partial_t = \partial/\partial t, \quad \partial_a = \partial/\partial x_a, \quad D = 2t\partial_t + x_a\partial_a, \quad G_a = t\partial_a - \tfrac{1}{2}x_a u\partial_u, \quad I = u\partial_u, \\ &J_{ab} = x_a\partial_b - x_b\partial_a \ (a < b), \quad \Pi = 4t^2 + 4tx_a\partial_a - (x_a x_a + 2t)u\partial_u, \quad f(t,\vec{x})\partial_u, \end{aligned} \tag{10}$$

where $f = f(t,\vec{x})$ is an arbitrary solution of (9).

Theorem 1. *For any operator Q of Q-conditional symmetry of equation (9) one of three following conditions holds:*

1. $Q \sim \widetilde{Q}^0$, *where* $\widetilde{Q}^0 \in A(\mathrm{LHE})$;
2. $Q \sim \widetilde{Q}^1 = \partial_n + g_n g^{-1} u\partial_u \bmod ASO(n) + A^\infty(\mathrm{LHE})$, *where $g = g(t,x_n)$ ($g_n \neq 0$) is a solution of the one-dimensional heat equation, that is, $g_t = g_{nn}$;*
3. $Q \sim \widetilde{Q}^2 = J_{12} + \varphi(\theta)u\partial_u \bmod AG(1,n) + A^\infty(\mathrm{LHE})$, *where $\varphi = \varphi(\theta)$ is a solution of the equation $\varphi_{\theta\theta} + 2\varphi\varphi_\theta = 0$, $\varphi_\theta \neq 0$, θ is the polar angle in the plane OX_1X_2.*

Here A^∞(LHE) $= \langle f(t,\vec{x})\partial_u | f = f(t,\vec{x})\colon f_t = f_{aa}\rangle$, $AG(1,n) = \langle \partial_t, \partial_a, G_a, J_{ab}\rangle$, $ASO(n) = \langle J_{ab}\rangle$.

It follows from Theorem 1 that there exist only three classes of the possible reductions on one independent variable for the linear multidimensional heat equation.

The first class is formed by Lie reductions.

The second class involves reductions which are similar to separation of variables in the Cartesian coordinates:

$$u = g(t,x_n)v(\omega_0,\ldots,\omega_{n-1}), \quad \text{where} \quad \omega_0 = t,\ \omega_i = x_i; \qquad (9) \quad \Longrightarrow \quad v_0 = v_{ii}.$$

The third class is formed by reductions which are similar to separation of variables in the cylindrical coordinates:

$$\begin{aligned} &u = \exp\left(\textstyle\int \varphi(\theta)d\theta\right)v(\omega_0,\ldots,\omega_{n-1}), \quad \text{where} \quad \omega_0 = t,\ \omega_1 = r,\ \omega_s = x_{s+1},\ s = \overline{2,n-1}; \\ &(9) \quad \Longrightarrow \quad v_0 = v_{11} + \omega_1^{-1}v_1 - \lambda\omega_1^{-2}v + v_{ss}. \end{aligned}$$

Here $\lambda = -\varphi_\theta - \varphi^2 = \mathrm{const}$, (r,θ) are the polar coordinates in the plane OX_1X_2. As the equation $\varphi_{\theta\theta} + 2\varphi\varphi_\theta = 0$ has four essentially different (under translations with respect to θ) families of solutions with $\varphi_\theta \neq 0$, there are four inequivalent cases for the third class of reductions ($\varkappa \neq 0$):

$$\begin{aligned} &\text{a)}\ \varphi = -\varkappa\tan\varkappa\theta: && u = v(\omega_0,\ldots,\omega_{n-1})\cos\varkappa\theta, && \lambda = \varkappa^2; \\ &\text{b)}\ \varphi = \varkappa\tanh\varkappa\theta: && u = v(\omega_0,\ldots,\omega_{n-1})\cosh\varkappa\theta, && \lambda = -\varkappa^2; \\ &\text{c)}\ \varphi = \varkappa\coth\varkappa\theta: && u = v(\omega_0,\ldots,\omega_{n-1})\sinh\varkappa\theta, && \lambda = -\varkappa^2; \\ &\text{d)}\ \varphi = \theta^{-1}: && u = v(\omega_0,\ldots,\omega_{n-1})\theta, && \lambda = 0. \end{aligned}$$

Example 2. Consider the Euler equations

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \vec{0}, \qquad \mathrm{div}\,\vec{u} = 0 \tag{11}$$

describing the motion of an incompressible ideal fluid. In the following $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity of the fluid, $p = p(t,\vec{x})$ denotes the pressure, $n = 3$, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian. The fluid density is set equal to unity.

Lie symmetry of system (11) was investigated by Buchnev [10, 11]. The maximal Lie invariance algebra $A(\mathrm{E})$ of (11) is infinite dimensional and generated by the following operators:

$$
\begin{aligned}
&\partial_t, \quad J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a} \quad (a<b),\\
&D^t = t\partial_t - u^a\partial_{u^a} - 2p\partial_p, \quad D^x = x_a\partial_a + u^a\partial_{u^a} + 2p\partial_p,\\
&R(\vec m) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p, \quad Z(\chi) = \chi(t)\partial_p,
\end{aligned}
\tag{12}
$$

where $m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0,t_1),\mathbb{R})$). Let us investigate Q-conditional symmetry of (11) with respect to alone differential operator $Q = \xi^0(t,\vec x,\vec u,p)\partial_t + \xi^a(t,\vec x,\vec u,p)\partial_a + \eta^a(t,\vec x,\vec u,p)\partial_{u^a} + \eta^0(t,\vec x,\vec u,p)\partial_p$.

Theorem 2. *Any operator Q of Q-conditional symmetry of the Euler equations (11) either is equivalent to a Lie symmetry operator of (11) or is equivalent ($\mathrm{mod}\,A(E)$) to the operator*

$$
\widetilde{Q} = \partial_3 + \zeta\left(t,x_3,u^3\right)\partial_{u^3} + \chi(t)x_3\partial_p,
\tag{13}
$$

where $\zeta_{u^3}\neq 0$, $\zeta_3 + \zeta\zeta_{u^3} = 0$, $\zeta_t + \left(u^3\zeta + \chi x_3\right)\zeta_{u^3} + (\zeta)^2 + \chi = 0$.

It follows from Theorem 2 that there exist two classes of the possible reductions w.r.t. independent variable for the Euler equations, namely, the Lie reductions and the reductions corresponding to operators of form (13).

Lie reductions of the Euler equations (11) are investigated in [12–14].

An ansatz constructed with the operator $\widetilde{Q}$ has the following form:

$$
u^1 = v^1, \quad u^2 = v^2, \quad u^3 = x_3v^3 + \psi\left(t,v^3\right), \quad p = q + \tfrac{1}{2}\chi(t)x_3^2,
$$

where $v^a = v^a(t,x_1,x_2)$, $q = q(t,x_1,x_2)$, $\psi = \psi(t,v^3)$ is a solution of the equation

$$
\psi_t - \left(\left(v^3\right)^2 + \chi\right)\psi_{v^3} + v^3\psi = 0.
$$

Substituting this ansatz into (11), we obtain the corresponding reduced system ($i,j = 1,2$):

$$
v^i_t + v^jv^i_j + q_i = 0, \quad v^3_t + v^jv^3_j + \left(v^3\right)^2 + \chi = 0, \quad v^j_j + v^3 = 0.
$$

The analogous problem for the Navier–Stokes equations

$$
\vec u_t + (\vec u\cdot\nabla)\vec u + \nabla p - \nu\triangle\vec u = \vec 0, \qquad \operatorname{div}\vec u = 0 \qquad (\nu\neq 0)
\tag{14}
$$

describing the motion of an incompressible viscous fluid was solved by Ludlow, Clarkson, and Bassom in [15]. Their result can be reformulated as follows.

Theorem 3. *Any (real) operator Q of Q-conditional symmetry of the Navier–Stokes equations (14) is equivalent to a Lie symmetry operator of (14).*

Therefore, all the possible reductions of the Navier–Stokes equations w.r.t. independent variable are exhausted by the Lie reductions. Lie symmetry of system (14) was studied by Danilov [16, 17]. The maximal Lie invariance algebra of the Navier-Stokes equations (14 is similar to one of the Euler equations (see (12)):

$$
A(\mathrm{NS}) = \langle \partial_t,\ J_{ab},\ D^t + \tfrac{1}{2}D^x,\ R(\vec m(t)),\ Z(\zeta(t))\rangle.
$$

The Lie reductions of the Navier–Stokes equations were completely described in [18].

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 190–195.

Lie Submodels of Rank 1 for MHD Equations

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The MHD equations describing flows of a viscous homogeneous incompressible fluid of finite electrical conductivity are reduced by means of Lie symmetries to partial differential equations in three independent variables. Symmetry properties of the reduced systems are investigated.

1. Introduction. The MHD equations (the MHDEs) describing flows of a viscous homogeneous incompressible fluid of finite electrical conductivity have the following form:

$$\begin{aligned} &\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \triangle\vec{u} + \vec{\nabla}p + \vec{H}\times\operatorname{rot}\vec{H} = \vec{0}, \qquad \operatorname{div}\vec{u} = 0,\\ &\vec{H}_t - \operatorname{rot}(\vec{u}\times\vec{H}) - \nu_m\triangle\vec{H} = \vec{0}, \qquad \operatorname{div}\vec{H} = 0. \end{aligned} \tag{1}$$

System (1) is very complicated and construction of its new exact solutions is a difficult problem. In [1, 2] the MHDEs (1) are reduced to ordinary differential equations and to partial differential equations in two independent variables. Following [3], in this paper we reduce the MHDEs (1) to partial differential equations in three independent variables by means of one-dimensional subalgebras of the maximal Lie invariance algebra of the MHDEs.

In (1) and below, $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity field of a fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{H} = \{H^a(t,\vec{x})\}$ denotes the magnetic intensity, ν_m is the coefficient of magnetic viscosity, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian. The kinematic coefficient of viscosity and fluid density are set equal to unity, permeability is done $(4\pi)^{-1}$. Subscripts of functions denote differentiation with respect to the corresponding variables.

The maximal Lie invariance algebra of the MHDEs (1) is an infinite-dimensional algebra $A(\mathrm{MHD})$ with the basis elements (see [4])

$$\begin{aligned} &\partial_t, \quad D = t\partial_t + \tfrac{1}{2}x_a\partial_a - \tfrac{1}{2}u^a\partial_{u^a} - \tfrac{1}{2}H^a\partial_{H^a} - p\partial_p,\\ &J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a} + H^a\partial_{H^b} - H^b\partial_{H^a}, \ a<b,\\ &R(\vec{m}) = m^a\partial_a + m^a_t\partial_{u^a} - m^a_{tt}x_a\partial_p, \quad Z(\chi) = \chi\partial_p, \end{aligned} \tag{2}$$

where $m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0,t_1),\mathbb{R})$). Summation is understood over repeated indices. The indices a, b take values $1,2,3$ and i, j takes respectivily values $1,2$. The algebra $A(\mathrm{MHD})$ is isomorphic to the maximal Lie invariance algebra $A(\mathrm{NS})$ of the Navier–Stokes equations [5, 6, 7].

In addition to continuous transformations generated by operators (2), the MHDEs admit discrete transformations I_b of the form

$$\begin{aligned} &\tilde{t} = t, \qquad x_b = -x_b, \qquad \tilde{x}_a = x_a,\\ &\tilde{p} = p, \qquad \tilde{u}^b = -u^b, \qquad \tilde{H}^b = -H^b, \qquad \tilde{u}^a = u^a, \qquad \tilde{H}^a = H^a, \quad a \neq b, \end{aligned}$$

where b is fixed.

2. Inequivalent one-dimensional subalgebras of $A(\mathrm{MHD})$**.** A complete set of $A(\mathrm{MHD})$-inequivalent one-dimensional subalgebras of $A(\mathrm{MHD})$ is exhausted by the following algebras:

1. $A^1_1(\varkappa) = \langle D + \varkappa J_{12}\rangle$, where $\varkappa \geq 0$.

2. $A^1_2(\varkappa) = \langle \partial_t + \varkappa J_{12} \rangle$, where $\varkappa \in \{0;1\}$.

3. $A^1_3(\eta,\chi) = \langle J_{12} + R(0,0,\eta(t)) + Z(\chi(t)) \rangle$ with smooth functions η and χ. Algebras $A^1_3(\eta,\chi)$ and $A^1_3(\tilde\eta,\tilde\chi)$ are equivalent if $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\,\varepsilon_1,\varepsilon_2\in\{-1;1\}$, $\exists\,\lambda\in C^\infty((t_0,t_1),\mathbb{R})$:

$$\tilde\eta(\tilde t) = \varepsilon_1 e^{-\varepsilon}\eta(t), \quad \tilde\chi(\tilde t) = \varepsilon_2 e^{2\varepsilon}(\chi(t) + \lambda_{tt}(t)\eta(t) - \lambda(t)\eta_{tt}(t)), \tag{3}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

4. $A^1_4(\vec m,\chi) = \langle R(\vec m(t)) + Z(\chi(t)) \rangle$ with smooth functions $\vec m$ and χ: $(\vec m,\chi) \not\equiv (\vec 0, 0)$. Algebras $A^1_4(\vec m,\chi)$ and $A^1_4(\tilde{\vec m},\tilde\chi)$ are equivalent if $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\,\varepsilon_1\in\{-1;1\}$, $\exists\, C\neq 0$, $\exists\, B\in O(3)$, $\exists\,\vec l\in C^\infty((t_0,t_1),\mathbb{R}^3)$:

$$\tilde{\vec m}(\tilde t) = Ce^{-\varepsilon}B\vec m(t), \quad \tilde\chi(\tilde t) = C\varepsilon_1 e^{2\varepsilon}(\chi(t) + \vec l_{tt}(t)\cdot\vec m(t) - \vec m_{tt}(t)\cdot\vec l(t)), \tag{4}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

3. Lie ansatzes of codimension one for the MHD field. Using of the algebras $A^1_1 - A^1_4$ (in the case when additional restrictions for parameters are satisfied), we can construct ansatzes of codimension one for the MHD field. Let us list these ansatzes.

1. $$\begin{aligned} &\vec u = |t|^{-1/2}O(\tau)\vec v + \tfrac12 t^{-1}\vec x + \varkappa t^{-1}\vec e_3\times\vec x,\\ &\vec H = |t|^{-1/2}O(\tau)\vec G,\\ &p = |t|^{-1}q + \tfrac18 t^{-2}|\vec x|^2 + \tfrac12\varkappa t^{-2}r^2, \end{aligned} \tag{5}$$

where $\vec y = |t|^{-1/2}O^T(\tau)\vec x$, $\tau = \varkappa\ln|t|$. Here and below

$$v^a = v^a(y_1,y_2,y_3), \quad G^a = G^a(y_1,y_2,y_3), \quad q = q(y_1,y_2,y_3),$$

$$O(\tau) = \begin{pmatrix} \cos\tau & -\sin\tau & 0\\ \sin\tau & \cos\tau & 0\\ 0 & 0 & 1\end{pmatrix}, \quad r = (x_1^2 + x_2^2)^{1/2}, \quad \vec e_3 = (0,0,1).$$

2. $$\vec u = O(\tau)\vec v + \varkappa\vec e_3\times\vec x, \quad \vec H = O(\tau)\vec G, \quad p = q + \tfrac12\varkappa r^2, \tag{6}$$

where $\vec y = O^T(\tau)\vec x$, $\tau = \varkappa t$.

3. $$\begin{aligned} &u^1 = x_1r^{-1}v^1 - x_2r^{-2}v^2, \quad u^2 = x_2r^{-1}v^1 + x_1r^{-2}v^2,\\ &u^3 = v^3 + \eta(t)r^{-2}v^2 + \eta_t(t)\arctan x_2/x_1,\\ &H^1 = x_1r^{-1}G^1 - x_2r^{-2}G^2, \quad H^2 = x_2r^{-1}G^1 + x_1r^{-2}G^2, \quad H^3 = G^3 + \eta(t)r^{-2}G^2,\\ &p = q - \tfrac12\eta_{tt}(t)(\eta(t))^{-1}x_3^2 + \chi(t)\arctan x_2/x_1, \end{aligned} \tag{7}$$

where $y_1 = r$, $y_2 = x_3 - \eta(t)\arctan x_2/x_1$, $y_3 := \tau = t$.

Notion. The expression for the pressure p from ansatz (7) is indeterminate in the points $t\in(t_0,t_1)$ where $\eta(t) = 0$. If there are such points t, we will consider ansatz (7) on the intervals (t_0^n, t_1^n) that are contained in the interval (t_0,t_1) and that satisfy one of the conditions:

a) $\eta(t)\neq 0 \quad \forall\, t\in(t_0^n,t_1^n)$;

b) $\eta(t) = 0 \quad \forall\, t\in(t_0^n,t_1^n)$.

In the latter case we consider $\eta_{tt}/\eta := 0$.

With the algebra $A_4^1(\vec m,\chi)$, an ansatz can be constructed only for such t wherefore $\vec m(t)\neq\vec 0$. If this condition is satisfied, it follows from (4) that the algebra $A_4^1(\vec m,\chi)$ is equivalent to the algebra $A_5^1(\vec m,0)$. An ansatz constructed with the algebra $A_4^1(\vec m,0)$ has the following form:

$$\begin{aligned}
4.\quad & \vec u = v^i\vec n^i + v^3|\vec m|^{-2}\vec m + (\vec m\cdot\vec x)|\vec m|^{-2}\vec m_t - y_i|\vec m|^{-1}\vec n^i_t,\\
& \vec H = G^i\vec n^i + G^3|\vec m|^{-2}\vec m,\\
& p = |\vec m|q - \tfrac12|\vec H|^2 - |\vec m|^{-2}(\vec m_{tt}\cdot\vec x)(\vec m\cdot\vec x) + \tfrac12(\vec m_{tt}\cdot\vec m)|\vec m|^{-4}(\vec m\cdot\vec x)^2 -\\
& \quad - \tfrac32|\vec m|^{-4}\big((\vec m_t\cdot\vec n^i)y_i\big)^2 + \big(\tfrac14|\vec m|_{tt}|\vec m|^{-2} - \tfrac38(|\vec m|_t)^2|\vec m|^{-3}\big)y_iy_i,
\end{aligned}\tag{8}$$

where $y_i=\vec n^i\cdot\vec x$, $y_3=\tau:=\int|\vec m|dt$, $\vec n^i$ are smooth vector-functions such that

$$\vec n^i\cdot\vec m=\vec n^1\cdot\vec n^2=\vec n^1_t\cdot\vec n^2=0,\quad |\vec n^i|=|\vec m|^{1/2}.\tag{9}$$

Notion. There exist vector-functions $\vec n^i$ which satisfy conditions (9). They can be constructed in the following way [3]: let us fix smooth vector-functions $\vec k^i=\vec k^i(t)$ such that $\vec k^i\cdot\vec m=\vec k^1\cdot\vec k^2=0$, $|\vec k^i|=|\vec m|^{1/2}$, and set

$$\vec n^1=\vec k^1\cos\psi(t)-\vec k^2\sin\psi(t),\qquad \vec n^2=\vec k^1\sin\psi(t)+\vec k^2\cos\psi(t).\tag{10}$$

Then $\vec n^1_t\cdot\vec n^2=\vec k^1_t\cdot\vec k^2-\psi_t=0$ if $\psi=\int(\vec k^1_t\cdot\vec k^2)dt$.

4. Reduced systems in three independent variables. Substituting ansatzes (5) and (6) into the MHDEs (1), we obtain reduced systems of PDEs with the same general form

$$\begin{aligned}
&(\vec v\cdot\nabla)\vec v-\triangle\vec v+\nabla q+\vec G\times\operatorname{rot}\vec G+\gamma_1\vec e_3\times\vec v=\vec 0,\\
&(\vec v\cdot\nabla)\vec G-(\vec G\cdot\nabla)\vec v-\nu_m\triangle\vec G+\gamma_2\vec G=\vec 0,\\
&\operatorname{div}\vec v=\tfrac32\gamma_2,\quad \operatorname{div}\vec G=0.
\end{aligned}\tag{11}$$

Hereafter the functions v^a, G^a, and q are differentiated with respect to the variables y_1, y_2, and y_3. The constants γ_a take the values

1. $\gamma_1=2\varkappa\operatorname{sign}t,\quad \gamma_2=-\operatorname{sign}t;$
2. $\gamma_1=2\varkappa,\quad \gamma_2=0.$

For ansatzes (7) and (8) the reduced equations have the form

$$\begin{aligned}
3.\quad & \mathcal M^1+q_1+y_1^{-3}\big((G^3)^2-(v^3)^2-2\eta v^3_2\big)-v^1_1y_1^{-1}+v^1y_1^{-2}=0,\\
& \mathcal M^2+(1+\eta^2y_1^{-2})q_2+2\eta y_1^{-3}(G^1G^3-v^1v^3+v^3_1-\eta v^1_2-\\
& \quad -2v^3y_1^{-1})-y_1^{-1}v^2_1+2\eta_ty_1^{-2}v^3-\eta_{tt}\eta^{-1}y_2-\eta\chi y_1^{-2}=0,\\
& \mathcal M^3-\eta q_2+v^3_1y_1^{-1}+2\eta y_1^{-1}v^1_2+\chi=0,\\
& \mathcal N^1+\nu_m(-2\eta y_1^{-3}G^3_2+y_1^{-2}G^1-y_1^{-1}G^1_1)=0,\\
& \mathcal N^2+\nu_m(-2\eta^2y_1^{-3}G^1_2+2\eta y_1^{-3}G^3_1-y_1^{-1}G^2_1-4\eta G^3y_1^{-4})=0,\\
& \mathcal N^3+2y_1^{-1}(v^3G^1-v^1G^3)+2\nu_m\eta y_1^{-1}G^1_2+\nu_mG^3y_1^{-1}=0,\\
& v^i_i+v^1y_1^{-1}=0,\quad G^i_i+G^1y_1^{-1}=0,
\end{aligned}\tag{12}$$

where $\mathcal M^a=v^a_\tau+v^jv^a_j-G^jG^a_j-v^a_{11}-(1+\eta^2y_1^{-2})v^a_{22},$

$\mathcal N^a=G^a_t+v^iG^a_i-G^iv^a_i-\nu_mG^a_{11}-\nu_m(1+\eta^2y_1^{-2})G^a_{22}.$

$$
\begin{aligned}
4.\quad & v^i_\tau + v^j v^i_j - G^j G^i_j - v^i_{jj} + q_i + 2\beta^i\alpha^{-3}v^3 = 0,\\
& v^3_\tau + v^j v^3_j - G^j G^3_j - v^3_{jj} = 0,\\
& G^i_\tau + v^j G^i_j - G^j v^i_j - \nu_m G^i_{jj} + \alpha_\tau\alpha^{-1}G^i = 0,\\
& G^3_\tau + v^j G^3_j - G^j v^3_j - \nu_m G^3_{jj} - 2\beta^j G^j - 2\alpha_\tau\alpha^{-1}G^3 = 0,\\
& v^i_i = 0, \quad G^i_i = 0,
\end{aligned}
\tag{13}
$$

where $\alpha = \alpha(\tau) = |\vec{m}|$, $\beta^i = \beta^i(\tau) = (\vec{m}_\tau \cdot \vec{n}^i)$.

5. Symmetry of reduced systems. Let us study symmetry properties of systems (11), (12), and (13). All results of this subsection are obtained by means of the standard Lie algorithm [9, 8].

Symmetry properties of the systems (11). The maximal Lie invariance algebra of system (11) is the algebra

a) $\langle \partial_a,\ \partial_q,\ J^1_{12}\rangle$ if $\gamma_1 \neq 0$;

b) $\langle \partial_a,\ \partial_q,\ J^1_{ab}\rangle$ if $\gamma_1 = 0$, $\gamma_2 \neq 0$;

c) $\langle \partial_a,\ \partial_q,\ J^1_{ab},\ D^1_1\rangle$ if $\gamma_1 = \gamma_2 = 0$.

Here $J^1_{ab} = y_a\partial_{y_b} - y_b\partial_{y_a} + v^a\partial_{v^b} - v^b\partial_{v^a} + G^a\partial_{G^b} - G^b\partial_{G^a}$,

$D^1_1 = y_a\partial_{y_a} - v^a\partial_{v^a} - G^a\partial_{G^a} - 2q\partial_q$.

Note. All Lie symmetry operators of (11) are induced by operators from $A(\mathrm{MHD})$: The operators J^1_{ab} and D^1_1 are induced by J_{ab} and D. The operators $c_a\partial_a$ $(c_a = \mathrm{const})$ and ∂_q are induced by either

$$R(|t|^{1/2}(c_1\cos\tau - c_2\sin\tau, c_1\sin\tau + c_2\cos\tau, c_3)),\quad Z(|t|^{-1}),$$

where $\tau = \varkappa\ln|t|$, for ansatz (5) or

$$R(c_1\cos\varkappa t - c_2\sin\varkappa t, c_1\sin\varkappa t + c_2\cos\varkappa t, c_3),\quad Z(1)$$

for ansatz (6), respectively. Therefore, Lie reduction of system (11) gives only solutions that can be obtained by reducing the MHDEs with two- and three-dimensional subalgebras of $A(\mathrm{MHD})$.

Symmetry properties of the systems (12). Let $A^{\max}$ be the maximal Lie invariance algebra of system (12). Studying symmetry properties of (12), one has to consider the following cases:

A. $\eta, \chi \equiv 0$. Then

$$A^{\max} = \langle \partial_\tau,\ D^1_2,\ R_1(\zeta(\tau)),\ Z^1(\lambda(\tau))\rangle,$$

where $D^1_2 = 2\tau\partial_\tau + y_i\partial_{y_i} - v^i\partial_{v^i} - 2v^3\partial_{v^3} - G^i\partial_{G^i} - 2G^3\partial_{G^3} - 2q\partial_q$,

$R_1(\zeta(\tau)) = \zeta\partial_2 + \zeta_\tau\partial_{v^2} - \zeta_{\tau\tau}y_2\partial_q$, $\quad Z^1(\lambda(\tau)) = \lambda(\tau)\partial_q$.

Here and below $\zeta = \zeta(\tau)$ and $\lambda = \lambda(\tau)$ are arbitrary smooth functions of $\tau = t$.

B. $\eta \equiv 0$, $\chi \not\equiv 0$. In this case an extension of $A^{\max}$ exists for $\chi = (C_1\tau + C_2)^{-1}$, where $C_1, C_2 = \mathrm{const}$. Let $C_1 \neq 0$. We can make C_2 vanish by means of equivalence transformation (3), i.e., $\chi = C\tau^{-1}$, where $C = \mathrm{const}$. Then

$$A^{\max} = \langle D^1_2,\ R_1(\zeta(\tau)),\ Z^1(\lambda(\tau))\rangle.$$

If $C_1 = 0$, $\chi = C = \mathrm{const}$ and $A^{\max} = \langle \partial_\tau,\ R_1(\zeta(\tau)),\ Z^1(\lambda(\tau))\rangle$.

For other values of χ, i.e., when $\chi_{\tau\tau}\chi \neq \chi_\tau\chi_\tau$, $A^{\max} = \langle R_1(\zeta(\tau)),\ Z^1(\lambda(\tau))\rangle$.

C. $\eta \neq 0$. Using equivalence transformation (3) we always can make $\chi = 0$. In this case an extension of $A^{\max}$ exists for $\eta = \pm|C_1\tau + C_2|^{1/2}$, where $C_1, C_2 = \mathrm{const}$. Let $C_1 \neq 0$. We can annihilate C_2 by means of equivalence transformation (3), i.e., $\eta = C|\tau|^{1/2}$, where $C = \mathrm{const}$. Then

$$A^{\max} = \langle D_2^1,\ R_2(|\tau|^{1/2}),\ R_2(|\tau|^{1/2}\ln|\tau|),\ Z^1(\lambda(\tau))\rangle,$$

where $R_2(\zeta(\tau)) = \zeta\partial_{y_2} + \zeta_\tau\partial_{v^2}$. If $C_1 = 0$, i.e., $\eta = C = \mathrm{const}$,

$$A^{\max} = \langle \partial_\tau,\ \partial_{y_2},\ \tau\partial_{y_2} + \partial_{v^2},\ Z^1(\lambda(\tau))\rangle.$$

For other values of η, i.e., when $(\eta^2)_{\tau\tau} \neq 0$,

$$A^{\max} = \langle R_2(\eta(\tau)),\ R_2(\eta(\tau)\textstyle\int(\eta(\tau))^{-2}d\tau), Z^1(\lambda(\tau))\rangle.$$

Note. In all cases considered above the Lie symmetry operators of (12) are induced by operators from $A(\mathrm{MHD})$: The operators ∂_τ, D_2^1, and $Z^1(\lambda(\tau))$ are induced by ∂_t, D, and $Z(\lambda(t))$, respectively. The operator $R(0,0,\zeta(t))$ induces the operator $R_1(\zeta(\tau))$ for $\eta \equiv 0$ and the operator $R_2(\zeta(\tau))$ (if $\zeta_{\tau\tau}\eta - \zeta\eta_{\tau\tau} = 0$) for $\eta \neq 0$. Therefore, the Lie reduction of system (12) gives only the solutions that can be obtained by reducing the MHDEs with two- and three-dimensional subalgebras of $A(\mathrm{MHD})$.

Symmetry properties of the systems (13). Let us introduce the notations

$$S^1 = \partial_{v^3} - 2\beta^i\alpha^{-3}y_i\partial_q, \quad S^2 = (\alpha)^2\partial_{G^3}, \quad \tilde{Z}(\lambda(\tau)) = \lambda\partial_q,$$
$$\tilde{R}(\bar{\psi}(\tau)) = \psi^i\partial_{y_i} + \psi^i_\tau\partial_{v^i} - \psi^i_{\tau\tau}y_i\partial_q, \quad \bar{\psi} = (\psi^1, \psi^2),$$
$$\tilde{D} = \tau\partial_\tau + \tfrac{1}{2}y_i\partial_{y_i} - \tfrac{1}{2}v^i\partial_{v^i} - \tfrac{1}{2}G^i\partial_{G^i} - q\partial_q,\ \tilde{I} = v^3\partial_{v^3} + G^3\partial_{G^3},$$
$$\tilde{J}_{12} = y_1\partial_{y_2} - y_2\partial_{y_1} + v^1\partial_{v^2} - v^2\partial_{v^1} + G^1\partial_{G^2} - G^2\partial_{G^1}.$$

For arbitrary values of the parameter-functions α and β^i, the system (13) is invariant under the algebra

$$A^{\mathrm{all}} = \langle \tilde{R}(\bar{\psi}),\ S^1,\ S^2,\ \tilde{Z}(\lambda)\rangle.$$

Extensions of the maximal Lie invariance algebra of system (13) exist in the following cases (for each extension we write down its basis operators):

1. $\beta^i = 0$, $\alpha_\tau = 0$, $\nu_m = 1$: $\tilde{D},\ \partial_\tau,\ \tilde{J}_{12},\ I,\ G^3\partial_{v^3} + v^3\partial_{G^3}$.
2. $\beta^i = 0$, $\alpha_\tau = 0$, $\nu_m \neq 1$: $\tilde{D},\ \partial_\tau,\ \tilde{J}_{12},\ I$.
3. $\beta^i = 0$, $\alpha = a_2|\tau + a_0|^{a_1}$, $a_1a_2 \neq 0$: $\tilde{D} + a_0\partial_\tau,\ \tilde{J}_{12},\ I$.
4. $\beta^i = 0$, $\alpha = a_2e^{a_1\tau}$, $a_1a_2 \neq 0$: $\partial_\tau,\ \tilde{J}_{12},\ I$.
5. $\beta^i = 0$, $\alpha\alpha_\tau\alpha_{\tau\tau\tau} + (\alpha_\tau)^2\alpha_{\tau\tau} - 2\alpha(\alpha_{\tau\tau})^2 \neq 0$: $\tilde{J}_{12},\ I$.
6. $\beta^i \neq 0$, $\beta^i_\tau = 0$, $\alpha_\tau = 0$: $\tilde{D} - \frac{3}{2}I,\ \partial_\tau$.
7. $\beta^1 = \rho\cos\theta$, $\beta^2 = \rho\sin\theta$, $\alpha = a_2|\tau + a_0|^{a_1}$, where
   $\rho = b_1|\tau + a_0|^{a_1/2-1}$, $\theta = b_2\ln|\tau + a_0| + b_3$, and $(a_1b_1, b_2) \neq (0,0)$:
   $\tilde{D} + a_0\partial_\tau - b_2\tilde{J}_{12} + \frac{1}{2}(a_1 - 1)I$.
8. $\beta^1 = \rho\cos\theta$, $\beta^2 = \rho\sin\theta$, $\alpha = a_2e^{a_1\tau}$, where
   $\rho = b_1e^{3a_1\tau/2}$, $\theta = b_2\tau + b_3$, and $(a_1b_1, b_2) \neq (0,0)$:
   $\partial_\tau - b_2\tilde{J}_{12} + \frac{3}{2}a_1I$.

Note. The vector-functions $\vec{n}^i$ from Ansatz 4 are determined up to the transformation

$$\vec{n}^1 \longrightarrow \vec{n}^1\cos\delta - \vec{n}^2\sin\delta, \qquad \vec{n}^2 \longrightarrow \vec{n}^1\sin\delta + \vec{n}^2\cos\delta,$$

where $\delta = \mathrm{const}$. Therefore, δ can be chosen such that $b_3 = 0$.

Note. The operators $R(\bar{\psi}(t)) + C_1 S^1$, $\tilde{Z}(\lambda(\tau))$ from A^{all} are induced by the operators $R(\vec{l}(t))$, $Z(\chi(t))$, respectively. Here

$$\chi(t) = \lambda(\tau(t)), \quad \vec{l}(t) = \psi^i(\tau(t))\vec{n}^i(t) + \varphi(t)\vec{m}(t),$$
$$\text{where} \quad 2\psi^i(\tau(t))(\vec{n}^i_t(t)\cdot\vec{m}(t)) + \varphi(t)|\vec{m}(t)|^2 = C_1.$$

The operator S^2 is not induced by operators from $A(\mathrm{MHD})$. Therefore, Lie reduction of system (13) can give solutions that can not be obtained by reducing the MHDEs with two- and three-dimensional subalgebras of $A(\mathrm{MHD})$.

Consider inducing the operators from extension of A^{all}. The operators I and $G^3\partial_{v^3} + v^3\partial_{G^3}$ are not induced by operators from $A(\mathrm{MHD})$.

The operator $\tilde{J}_{12}$ belongs to the maximal Lie invariance algebra of the system (13) if $\beta^i = 0$. In this case $\vec{m} = |\vec{m}|\vec{e}$, where $\vec{e} = \mathrm{const}$ and $|\vec{e}| = 1$. Then, the operator $\tilde{J}_{12}$ is induced by $e_1 J_{23} + e_2 J_{31} + e_3 J_{12}$.

For $\vec{m} = e^{\sigma t}(c_2\cos\theta,\, c_2\sin\theta,\, c_1)$ with $c_1, c_2, \sigma, \varkappa, \delta = \mathrm{const}$ and $\theta = \varkappa t + \delta$, where $c_1^2 + c_2^2 = 1$, the operator $\partial_t + \varkappa J_{12}$ induces the operator $\partial_\tau - c_1\varkappa\tilde{J}_{12} + \sigma I$ if the following vector-functions $\vec{n}^i$ are chosen:

$$\vec{n}^1 = \vec{k}^1\cos c_1\theta + \vec{k}^2\sin c_1\theta, \quad \vec{n}^2 = -\vec{k}^1\sin c_1\theta + \vec{k}^2\cos c_1\theta, \tag{14}$$

where $\vec{k}^1 = (-\sin\theta, \cos\theta, 0)$ and $\vec{k}^2 = (c_1\cos\theta, c_1\sin\theta, -c_2)$.

For $\vec{m} = |t+\tilde{\delta}|^{\sigma+1/2}(c_2\cos\theta,\, c_2\sin\theta,\, c_1)$ with $\theta = \varkappa\ln|t+\tilde{\delta}| + \delta$ and $c_1, c_2, \sigma, \varkappa, \delta, \tilde{\delta} = \mathrm{const}$, where $c_1^2 + c_2^2 = 1$, the operator $D + \tilde{\delta}\partial_t + \varkappa J_{12}$ induces the operator $\tilde{D} + \tilde{\delta}\partial_\tau - c_1\varkappa\tilde{J}_{12} + \sigma I$, if the vector-functions $\vec{n}^i$ are chosen in form (14).

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 196–203.

On Local Time-Dependent Symmetries of Integrable Evolution Equations

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We consider scalar $(1+1)$-dimensional evolution equation of order $n \geq 2$, which possesses time-independent formal symmetry (i.e. it is integrable in the sense of symmetry approach), shared by all local generalized time-independent symmetries of this equation. We show that if such equation possesses the nontrivial canonical conserved density ρ_m, $m \in \{-1, 1, 2, \ldots\}$, then it has no polynomial in time local generalized symmetries (except time-independent ones) of order higher than $n+m+1$. Some generalizations of this result and related results are also presented. Using them, we have found all local generalized time-dependent symmetries of Harry Dym and mKdV equations.

Introduction

The scalar $(1+1)$-dimensional evolution equation, having the time-independent formal symmetry, is either linearizable or integrable via inverse scattering transform (see e.g. [1, 2, 3, 4] for the survey of known results and [5] for the generalization to $(2+1)$ dimensions).

It is natural to ask whether such equation may have local generalized *time-dependent* symmetries, different from time-independent ones, forming the integrable hierarchy of the equation considered, and how to find all of them (cf. Ch. V of [4]). To the best of our knowledge, there were no attempts to find general answer to this question, although long ago all local generalized symmetries of KdV [6, 7] and Burgers [6] equations were found.

In this paper we present the results, enabling one to answer this question for a large class of evolution equations. In particular, we prove that if $\rho_{-1} = (\partial F/\partial u_n)^{-1/n} \notin \operatorname{Im} D$, but $\nabla_F(\rho_{-1}) \in \operatorname{Im} D$, then the equation $u_t = F(x, u, \ldots, u_n)$, $n \geq 2$, has no local generalized time-dependent symmetries of order higher than n (see Section 1 and Theorem 1 in Section 2 for details).

Next, for the majority of nonlinear evolution equations one can prove the *polynomiality* in time of *all* their local generalized symmetries, using scaling or other arguments, so it is interesting to consider the conditions of existence of polynomial in time symmetries, especially for the equations with $\rho_{-1} \in \operatorname{Im} D$. To this end one can apply our Theorem 2, stating that if canonical conserved density $\rho_m \notin \operatorname{Im} D$ for some $m \in \mathbb{N}$, then the equation $u_t = F(x, u, \ldots, u_n)$, $n \geq 2$, possessing time-independent formal symmetry, has no polynomial in time local generalized symmetries (except time-independent ones) of order higher than the number $p_F = p_F(m)$, given by (20).

Finally, on the basis of Theorems 1 and 2 we suggest the scheme of finding all local generalized time-dependent symmetries of a given integrable evolution equation and apply it to Harry Dym and modified KdV equations. We also discuss in brief the generalization of our results to the systems of evolution equations.

1 Basic definitions and known results

We consider the scalar $(1+1)$-dimensional evolution equation

$$\partial u/\partial t = F(x,u,u_1,\dots,u_n), \qquad n\geq 2, \quad \partial F/\partial u_n \neq 0, \tag{1}$$

where $u_l = \partial^l u/\partial x^l$, $l=0,1,2,\dots$, $u_0\equiv u$, and the local generalized symmetries of this equation, i.e. the right hand sides G of evolution equations

$$\partial u/\partial \tau = G(x,t,u,u_1,\dots,u_k), \tag{2}$$

compatible with equation (1) (following [3, 6] we identify the symmetries with their characteristics).

For any function $H=H(x,t,u,u_1,\dots,u_q)$ the greatest number m such that $\partial H/\partial u_m\neq 0$ is called its order and is denoted as $m=\operatorname{ord} H$. For $H=H(x,t)$ we assume that $\operatorname{ord} H=0$. We shall call the function f of $x,t,u,u_1,\dots$ *local* [4], if it has finite order.

We shall denote by $S_F^{(k)}$ the space of local generalized symmetries of order not higher than k of Eq.(1). Let also $S_F=\bigcup\limits_{j=0}^{\infty} S_F^{(j)}$, $\Theta_F=\{H(x,t)|H(x,t)\in S_F\}$, $S_{F,k}=S_F^{(k)}/S_F^{(k-1)}$ for $k=1,2,\dots$, $S_{F,0}=S_F^{(0)}/\Theta_F$.

Finally, let Ann_F be the set of all local *time-independent* generalized symmetries of Eq.(1). In what follows we shall always consider time-dependent local generalized symmetries of Eq.(1) as the elements of quotient space S_F/Ann_F. In other words, we shall consider time-dependent symmetries modulo time-independent ones (i.e. up to the addition of linear combinations of time-independent symmetries).

S_F is Lie algebra with respect to the so-called Lie bracket [2, 4]

$$\{h,r\} = r_*(h)-h_*(r) = \nabla_h(r)-\nabla_r(h),$$

where for any sufficiently smooth function f of $x,t,u,u_1,\dots,u_s$ we have introduced the notation

$$f_* = \sum_{i=0}^{s}\partial h/\partial u_i D^i, \qquad \nabla_f = \sum_{i=0}^{\infty} D^i(f)\partial/\partial u_i.$$

Here $D=\partial/\partial x+\sum\limits_{i=0}^{\infty}u_{i+1}\partial/\partial u_i$ is the total derivative with respect to x. We shall denote by $\operatorname{Im} D$ the image of the space of local functions under the action of the operator D.

G is symmetry of Eq.(1) if and only if [4]

$$\partial G/\partial t = -\{F,G\}. \tag{3}$$

Let us note without proof (cf. Lemma 5.21 from [4]) that for any $G\in S_F$, $\operatorname{ord} G=k\geq n_0$, we have

$$\partial G/\partial u_k = c_k(t)\Phi^{k/n}, \tag{4}$$

where $c_k(t)$ is a function of t, $\Phi=\partial F/\partial u_n$,

$$n_0 = \begin{cases} \max(1-j,0), \ \text{if } F \text{ is such that } \partial F/\partial u_{n-i}=\phi_i(x), \quad i=0,\dots,j, \\ 2 \ \text{otherwise.}\end{cases} \tag{5}$$

In what follows we shall assume without loss of generality that any symmetry $G\in S_{F,k}$, $k\geq n_0$ vanishes, provided the relevant function $c_k(t)$ is identically equal to zero.

We make also a blanket assumption that all the functions that appear below in this paper (the function F, symmetries G, etc.) are locally analytical functions of their arguments.

For any local functions P, Q the relation $R = \{P, Q\}$ implies [2]

$$R_* = \nabla_P(Q_*) - \nabla_Q(P_*) + [Q_*, P_*], \tag{6}$$

$$[\nabla_P, \nabla_Q] = \nabla_R. \tag{7}$$

Here $\nabla_P(Q_*) \equiv \sum\limits_{i,j=0}^{\infty} D^j(P) \frac{\partial^2 Q}{\partial u_j \partial u_i} D^i$ and likewise for $\nabla_Q(P_*)$; $[\cdot,\cdot]$ stands for the usual commutator of linear differential operators.

For $P = G$, $Q = F$, using Eq.(3), we obtain

$$\partial G_*/\partial t \equiv (\partial G/\partial t)_* = \nabla_G(F_*) - \nabla_F(G_*) + [F_*, G_*]. \tag{8}$$

Now let us remind some facts concerning the formal series in powers of D (see e.g. [1, 3, 5] for more information; in contrast with these references we let the coefficients of the series depend explicitly on time, but this obviously doesn't alter the results, listed below), i.e. the expressions of the form

$$\mathrm{H} = \sum_{j=-\infty}^{m} h_j(x, t, u, u_1, \ldots) D^j. \tag{9}$$

The greatest integer m such that $h_m \neq 0$ is called the degree of formal series H and is denoted by $\deg \mathrm{H}$. For any formal series H of degree $m \neq 0$ there exists unique [5] (up to the multiplication by m-th root of unity) formal series $\mathrm{H}^{1/m}$ of degree 1 (or -1 for $m < 0$) such that $(\mathrm{H}^{1/m})^m = \mathrm{H}$. The fractional powers of H are defined as $\mathrm{H}^{l/m} = (\mathrm{H}^{1/m})^l$ for all integer l.

Let us also define [3] the residue of the formal series H as the coefficient at D^{-1}, i.e. $\operatorname{res} \mathrm{H} = h_{-1}$, and the logarithmic residue as $\operatorname{res} \ln \mathrm{H} = h_{m-1}/h_m$.

The formal series R is called the *formal symmetry* (of infinite rank) of Eq.(1), if it satisfies the relation (cf. [3])

$$\partial \mathrm{R}/\partial t + \nabla_F(\mathrm{R}) - [F_*, \mathrm{R}] = 0. \tag{10}$$

Finally, let us introduce the important notion of master symmetry [8] for the particular case of local functions: the local function $B(x, u, u_1, \ldots)$ is called (time-independent local) *master symmetry* of Eq.(1), if for any $P \in \mathrm{Ann}_F$ we have $\{B, P\} \in \mathrm{Ann}_F$. If in addition $\{B, F\} \neq 0$, we shall call B *strong master symmetry.* Like for the time-dependent symmetries, we shall always consider master symmetries up to the addition of the terms, being the linear combinations of time-independent symmetries.

2 The no-go theorem

By Theorem 1 from [10] for any symmetry G of Eq.(1) of order $k > n + n_0 - 2$ we have

$$G_* = \sum_{j=k-n+1}^{k} c_j(t) F_*^{j/n} + \left(\frac{1}{n}\dot{c}_k(t) D^{-1}(\Phi^{-1/n}) - \frac{k}{n} c_k(t) D^{-1}(\nabla_F(\Phi^{-1/n}))\right) F_*^{\frac{k-n+1}{n}} + \mathrm{N}, \tag{11}$$

where $c_j(t)$ are some functions of t and N is some formal series, $\deg \mathrm{N} < k - n + 1$.

Analyzing the terms, standing under D^{-1}, we conclude that if $\Phi^{-1/n} \notin \operatorname{Im} D$, while $\nabla_F(\Phi^{-1/n}) \in \operatorname{Im} D$, then G_* (and hence G itself) becomes nonlocal, and nonlocal terms vanish only if $\dot{c}_k(t) = 0$.

Lemma 1 *If $G \in S_{F,k}$, $k \geq n_0$, and there exists a linear differential operator $L = \sum\limits_{j=0}^{q} a_j \partial^j/\partial t^j$, $a_j \in \mathbb{C}$, such that $L(c_k(t)) = 0$, then $L(G) = 0$.*

Proof of the lemma. Let us assume that the statement of the lemma is wrong, i.e. $L(G) = \tilde{G} \neq 0$. It is obvious that $\tilde{G} \in S_F^{(k-1)}$ ($\tilde{G} \in \Theta_F$ for $k = 0$) and that the determining equations (3) for $\tilde{G}$ contain neither $c_k(t)$ nor its time derivatives. Since by assumption $G \in S_{F,k}$, G must vanish if $c_k(t)$ vanishes. On the other hand, this is impossible, because $\tilde{G}$ is independent of $c_k(t)$ and its derivatives and $\tilde{G} \neq 0$. This contradiction may be avoided only if $\tilde{G} = 0$, what proves the lemma.

Using Lemma 1 with $L = \partial/\partial t$, we conclude from the above that if $\Phi^{-1/n} \notin \operatorname{Im} D$ and $\nabla_F(\Phi^{-1/n}) \in \operatorname{Im} D$, then all the elements of $S_{F,q}$ for $q > n + n_0 - 2$ are time-independent, and hence, since we consider time-dependent symmetries modulo time-independent ones, Eq.(1) has no local time-dependent generalized symmetries of order higher than $n + n_0 - 2$.

Finally, from the definition (5) of n_0 it is clear that for $n_0 = 0, 1$ $\Phi^{-1/n} = \tilde{\phi}(x) \in \operatorname{Im} D$ and hence the case $\Phi^{-1/n} \notin \operatorname{Im} D$ is possible only for $n_0 = 2$. Thus, we have proved

Theorem 1 *If $(\partial F/\partial u_n)^{-1/n} \notin \operatorname{Im} D$, while $\nabla_F((\partial F/\partial u_n)^{-1/n}) \in \operatorname{Im} D$, then Eq.(1) has no local time-dependent generalized symmetries of order higher than n.*

Let B be strong master symmetry of Eq.(1) and $\operatorname{ord} B > n$. Then $Q = B + t\{B, F\}$ obviously is time-dependent symmetry of Eq.(1) of order higher than n, what contradicts to Theorem 1. This contradiction proves the following

Corollary 1 *If the conditions of Theorem 1 hold for Eq.(1), then it has no local time-independent strong master symmetries of order higher than n.*

As an example, let us consider Harry Dym equation

$$u_t = u^3 u_3.$$

It is straightforward to check that we have $(\partial F/\partial u_3)^{-1/3} = u^{-1} \notin \operatorname{Im} D$, but $\nabla_F(u^{-1}) \in \operatorname{Im} D$. Hence, the equation in question has no time-dependent symmetries of order higher than 3. Further computation of symmetries of orders $0, \ldots, 3$ shows that, apart from the infinite hierarchy of time-independent symmetries, Harry Dym equation has only two local time-dependent generalized symmetries: $u + 3tu^3u_3$ and $xu_1 + 3tu^3u_3$, and both of them are equivalent to Lie point symmetries.

3 Structure of linear in time symmetries

Consider polynomial in time t symmetries of Eq.(1) from the space $S_{F,q}$. Using Lemma 1 with $L = \partial^s/\partial t^s$, one may easily check that in order to possess polynomial in time symmetry from $S_{F,q}$ Eq.(1) must possess (at least one) linear in t symmetry $Q = K + tH \in S_{F,q}$, $\partial K/\partial t = \partial H/\partial t = 0$, $H \in S_{F,q}$. It is obvious that

$$\{F, H\} = 0. \tag{12}$$

Since $Q \in S_{F,q}$, it is clear that $k \equiv \operatorname{ord} K \leq q$. The substitution of $G = Q$ and $P = F$ into (3) and (8) yields

$$\{F, K\} = -H, \tag{13}$$

$$\nabla_K(F_*) - \nabla_F(K_*) + [F_*, K_*] = H_*. \tag{14}$$

Since for arbitrary F and K $\operatorname{ord}\{F,K\} \le k+n-1$, Eq.(13) implies that $k+n-1 \ge q$ and hence $k \ge q-n+1$.

Plugging the symmetry Q into (11) and setting $t=0$, we immediately obtain the following representation for K_*, provided $q > n+n_0-2$:

$$K_* = \sum_{j=q-n+1}^{k} \kappa_j F_*^{j/n} + \left(\frac{\gamma}{n} D^{-1}(\Phi^{-1/n}) - \delta_{k,q}\frac{k}{n}\kappa_k D^{-1}(\nabla_F(\Phi^{-1/n}))\right) F_*^{\frac{q-n+1}{n}} + \mathrm{N}, \qquad (15)$$

where $\kappa_j \in \mathbb{C}$, $\gamma = \Phi^{-q/n}\partial H/\partial u_q \in \mathbb{C}$, $\gamma \ne 0$; N is some formal series with time-independent coefficients, $\deg \mathrm{N} < q-n+1$; $\delta_{k,q}=1$ if $k=q$ and 0 otherwise.

Let us mention that if $k=q$, $\kappa_k \ne 0$, we may consider the symmetry $Q' = Q - (\kappa_q/\gamma)H = tH + K' \in S_{F,q}$ instead of Q, and for Q' we have $\operatorname{ord} K' < q$. Thus, we can always assume that $\operatorname{ord} K < q$ and hence $\kappa_k\delta_{k,q} = 0$.

4 Polynomial in time symmetries of evolution equation having formal symmetry

From now on we shall consider the evolution equation (1), possessing a time-independent ($\partial\mathrm{L}/\partial t = 0$) formal symmetry L of nonzero degree p. By definition, L satisfies the equation

$$[\nabla_F - F_*, \mathrm{L}] = 0. \qquad (16)$$

It is clear that for any integer q $c\mathrm{L}^{q/p}$, $c=\mathrm{const}$ also is formal symmetry of Eq.(1) [3]. Therefore, without loss of generality we may assume in what follows that $\deg \mathrm{L} = 1$ and $\mathrm{L} = (\partial F/\partial u_n)^{1/n} D + \cdots$ [3].

It is known [2] that there exists at most one (up to the addition of linear combination of local generalized *time-independent* symmetries $Z = Z(x,u,u_1,\ldots)$ of Eq.(1), satisfying $[\nabla_Z - Z_*, \mathrm{L}] = 0$) such local generalized time-independent symmetry Y of Eq.(1) that[1]

$$[\nabla_Y - Y_*, \mathrm{L}] \ne 0.$$

Let us choose Y to be of minimal possible order r, adding to it, if necessary, the appropriate linear combination of the symmetries $Z \in \mathrm{Ann}_F$, which satisfy the condition $[\nabla_Z - Z_*, \mathrm{L}] = 0$.

From now on we shall assume (it is clear that this does not lead to the loss of generality) that for any local generalized *time-independent* symmetry $P = P(x,u,u_1,\ldots) \in S_F/S_F^{(r)}$

$$[\nabla_P - P_*, \mathrm{L}] = 0. \qquad (17)$$

Finally, let us assume that $(\partial F/\partial u_n)^{-1/n} \in \operatorname{Im} D$, i.e. the necessary condition of existence of time-dependent symmetries of order higher than n, given in Theorem 1, is satisfied.

Now let us consider again the symmetry $Q = K + tH \in S_{F,q}$, $\operatorname{ord} Q = q > \max(r, n+n_0-2)$. Using Jacobi identity, Eq.(17) for $P=H$, Eqs. (13), (16), and Eqs.(6), (7) for $P=F$, $Q=K$, we obtain that for all integer s

$$[\nabla_F - F_*, [\mathrm{L}^s, \nabla_K - K_*]] = -[\nabla_K - K_*, [\nabla_F - F_*, \mathrm{L}^s]] - [\mathrm{L}^s, [\nabla_K - K_*, \nabla_F - F_*]] =$$
$$-[\mathrm{L}^s, [\nabla_K - K_*, \nabla_F - F_*]] = [\mathrm{L}^s, \nabla_H - H_*] = 0.$$

[1] If, e.g., as in the majority of cases, all time-independent symmetries are generated by hereditary recursion operator R from one seed symmetry, leaving R invariant, the symmetry Y with such property does not exist.

Hence, by Lemma 8 from [2] $[\mathrm{L}^s, \nabla_K - K_*] = \sum\limits_{j=-\infty}^{k_s} c_{j,s}\mathrm{L}^j, c_{j,s} \in \mathbb{C}$. Straightforward but lengthy check, which we omit here, shows that in fact $k_s = s + q - n$, and thus

$$[\mathrm{L}^s, \nabla_K - K_*] = \sum_{j=-\infty}^{s+q-n} c_{j,s}\mathrm{L}^j, c_{j,s} \in \mathbb{C}, c_{s+q-n,s} \neq 0. \tag{18}$$

Since $\mathrm{res}\,\nabla_G(\mathrm{L}^s) = 0$ for $s \leq -2$ and $\mathrm{res}\,\mathrm{L}^j = 0$ for $j < -1$, Eq.(18) for $s \leq -2$ yields

$$\mathrm{res}\,[\mathrm{L}^s, K_*] = -\sum_{j=-\infty}^{s+q-n} c_{j,s}\mathrm{res}\,\mathrm{L}^j = -\sum_{j=-1}^{s+q-n} c_{j,s}\mathrm{res}\,\mathrm{L}^j. \tag{19}$$

But the residue of the commutator of two formal series always lies in $\mathrm{Im}\,D$ [3]. On the other hand, $\rho_j = \mathrm{res}\,\mathrm{L}^j$, $j = -1, 1, 2, 3, \dots$ (and $\rho_0 = \mathrm{res}\ln\mathrm{L}$) are nothing but the so-called *canonical conserved densities* for Eq.(1) [3], and hence $\nabla_F(\rho_j) \in \mathrm{Im}\,D$. The density ρ_j is called *nontrivial*, if $\rho_j \notin \mathrm{Im}\,D$, and *trivial* otherwise.

If the density ρ_{s+q-n} is nontrivial, while ρ_j, $j = -1, 1, \dots, s+q-n-1$ are trivial, then Eq.(19) contains a contradiction. Namely, its l.h.s. lies in $\mathrm{Im}\,D$, while the nonzero term $c_{s+q-n,s}\rho_{s+q-n}$ on its r.h.s. does not belong to $\mathrm{Im}\,D$.

Since the density $\rho_{-1} = (\partial F/\partial u_n)^{-1/n}$ is trivial by assumption, let us restrict ourselves to the case $s + q - n \geq 1$. This inequality is compatible with the condition $s \leq -2$ for the non-empty range of values of s if and only if $q > n+2$. Therefore, the range of values of s, for which Eq.(19) may contain the contradiction, is $n - q + 1, \dots, -2$.

Thus, if for $q > \max(n + 2, r)$ at least one of the densities $\rho_1, \dots, \rho_{q-n-2}$ is nontrivial, then Eq.(18) (and hence Eq.(13) with $H \in S_{F,q}$ as well) has no local time-independent solutions K.

Let

$$p_F = \begin{cases} m + n + 1, & \text{if (17) is satisfied for all } P \in \mathrm{Ann}_F, \\ \max(r, m + n + 1) & \text{otherwise}, \end{cases} \tag{20}$$

where $m \in \mathbb{N}$ is the smallest number such that $\rho_m \notin \mathrm{Im}\,D$, while for $j = -1, 1, \dots, m-1$, $j \neq 0$, $\rho_j \in \mathrm{Im}\,D$ ($\rho_{-1} \in \mathrm{Im}\,D$ by assumption).

It is clear from the above that Eq.(1) has no polynomial in time symmetries from $S_F/S_F^{(p_F)}$ (except time-independent ones), and we obtain (cf. Theorem 1 and Corollary 1)

Theorem 2 *If Eq.(1) has time-independent formal symmetry* L, $\deg\mathrm{L} \neq 0$, *of infinite rank, and for some* $m \in \mathbb{N}$ $\rho_m \notin \mathrm{Im}\,D$, *while for* $j = -1, 1, \dots, m-1$, $j \neq 0$, $\rho_j \in \mathrm{Im}\,D$, *then Eq.(1) has no polynomial in time*[2] *local generalized symmetries from* $S_F/S_F^{(p_F)}$.

Corollary 2 *If the conditions of Theorem 2 hold for Eq.(1), then it has no local time-independent strong master symmetries of order higher than* p_F.

Let us mention that provided one can prove that all the symmetries from $S_F/S_F^{(p_F)}$ are polynomial in time, Theorem 2, exactly like Theorem 1, implies the absence of *any* time-dependent local generalized symmetries of order higher than p_F of Eq.(1).

It is also important to stress that the application of Theorem 2 does *not* require the check of triviality of the density ρ_0, as shows the well known example of Burgers equation

$$u_t = u_2 + uu_1.$$

[2]Of course, except time-independent ones.

This equation has time-independent formal symmetry of degree 1 and local generalized polynomial in time symmetries of all orders [6], and its canonical densities $\rho_{-1}, \rho_1, \rho_2, \ldots$ are trivial, while ρ_0 is nontrivial.

Let us mention without proof that our results may be partially generalized to the case of systems of evolution equations of the form (1), where u is s-component vector, provided $s \times s$ matrix $\Phi = \partial F/\partial u_n$ is nondegenerate ($\det \Phi \neq 0$), may be diagonalized by means of some similarity transformation $\Phi \to \Phi' = \Omega\Phi\Omega^{-1}$ and has s distinct eigenvalues λ_i. We shall call the systems (1) with such properties *nondegenerate weakly diagonalizable.* For such systems we have the following analogs of Theorems 1 and 2:

Theorem 3 *If for all eigenvalues λ_i of Φ we have $\lambda_i^{-1/n} \notin \operatorname{Im} D$, but $\nabla_F(\lambda_i^{-1/n}) \in \operatorname{Im} D$, then nondegenerate weakly diagonalizable system (1) has no local time-dependent generalized symmetries of order higher than n.*

Theorem 4 *If nondegenerate weakly diagonalizable system (1) has time-independent formal symmetry $\mathrm{L} = \eta D^q + \cdots$ of infinite rank, with $\det \eta \neq 0$ and $q = \deg \mathrm{L} \neq 0$, (17) is satisfied for all time-independent symmetries P of (1), and[3] for some $m \in \mathbb{N}$ $\rho_m^l \notin \operatorname{Im} D$ for all $l = 1, \ldots, s$, while for $j = -1, 1, \ldots, m-1$, $j \neq 0$, $\rho_j^a \in \operatorname{Im} D$ for all $a = 1, \ldots, s$, then (1) has no polynomial in time local generalized symmetries (except time-independent ones) from $S_F/S_F^{(m+n+1)}$.*

The modification of Corollaries 1 and 2 for the case of nondegenerate weakly diagonalizable systems is obvious, so we leave it to the reader.

Let us note that the requirements of Theorem 4 may be relaxed. Namely, if there exist nontrivial densities ρ_j^a with $j < m$, $j \neq 0$, but only for $j = m$ *all* the densities ρ_m^l, $l = 1, \ldots, s$, are nontrivial, and the nontrivial densities ρ_j^a with $j < m$, $j \neq 0$, are linearly independent of ρ_m^a with the same value of index a, then the statement of Theorem 4 remains true.

Thus, Theorems 1 – 4 reveal interesting duality between time-dependent symmetries and canonical conserved densities of integrable evolution equations, which is completely different from the one coming e.g. from the famous Noether's theorem. Namely, as one can conclude from Theorems 1 – 4, the *nontriviality* of these densities (except ρ_0) turns out to be an obstacle to existence of polynomial in time (or even any time-dependent) local generalized symmetries of sufficiently high order of such equations, provided they possess time-independent formal symmetry. This result appears to be rather unexpected in view of the well known fact that the existence of canonical conserved densities is the necessary condition of existence of high order time-independent symmetries of the evolution equations, see e.g. [1]. However, the apparent contradiction between these two results vanishes, if we consider nonlocal symmetries. Indeed, integrable evolution systems usually possess the infinite number of *nonlocal* polynomial in time symmetries, which form the so-called hereditary algebra, see e.g. [9], and the nonlocal variables that these symmetries depend on turn out to be nothing but the integrals of nontrivial conserved densities.

5 Applications

It is well known that the straightforward finding of all time-dependent local generalized symmetries of a given integrable evolution equation, and especially the proof of completeness of the obtained set of symmetries, is a highly nontrivial task (see e.g. [7] for the case of KdV equation), in contrast with time-independent symmetries, all of which usually can be obtained by the repeated application of the recursion operator to one seed symmetry.

[3] See [3] for the definition of densities ρ_k^l.

Fortunately, our results allow to suggest a very simple and efficient way to find *all* local generalized time-dependent symmetries of a given integrable evolution equation.

First of all, one should find the smallest $m \in \{-1, 1, 2, \ldots\}$ such that the canonical conserved density ρ_m is nontrivial. If $m \neq -1$, then one should evaluate the number p_F and check the polynomiality in time of all local generalized symmetries from the space $S_F/S_F^{(p_F)}$, using scaling arguments or e.g. the results of [11]. If $m = -1$ or the polynomiality really takes place, then by Theorem 1 or 2 there exist no time-dependent symmetries (of course, modulo the infinite hierarchy of time-independent ones) of order higher than n or p_F respectively. Finally, all time-dependent symmetries of orders $0, \ldots, n$ or $0, \ldots, p_F$ can be found by straightforward computation, using e.g. computer algebra.

The similar scheme, this time based on Theorems 3 and 4, works for integrable nondegenerate weakly diagonalizable systems of evolution equations as well, provided all $P \in \mathrm{Ann}_F$ satisfy (17) (for $m = -1$ this is not required).

Our method fails, if ρ_j are trivial for all $j = -1, 1, 2, \ldots$ or it is impossible (for $m \neq -1$) to prove that all the elements of the space $S_F/S_F^{(p_F)}$ are polynomial in time. However, such situations are typical for linearizable equations, while for genuinely nonlinear integrable equations one usually encounters no difficulties in the application of the above scheme.

Let us consider for instance the modified Korteweg-de Vries (mKdV) equation

$$u_t = u_3 + u^2 u_1.$$

It has the recursion operator (see e.g. [4]) $\mathrm{R} = D^2 + (2/3)u^2 - (2/3)u_1 D^{-1} u$ and $\mathrm{L} = \mathrm{R}^{1/2}$ is the formal symmetry of degree 1, which satisfies (17) for all $P \in \mathrm{Ann}_{\mathrm{mKdV}}$, since the operator R is hereditary. The density $\rho_1 = u^2$ is nontrivial, while $\rho_{-1} = 1 \in \mathrm{Im}\, D$, so we have $p_{\mathrm{mKdV}} = 5$. All local time-dependent generalized symmetries of mKdV equation are polynomial in time [11], so by Theorem 2 it has no local generalized time-dependent symmetries of order greater than 5. The computation of symmetries of orders $0, \ldots, 5$ shows that the only generalized symmetry of mKdV equation, that doesn't belong to the infinite hierarchy of time-independent symmetries, is the dilatation $xu_1 + u + 3t(u_3 + u^2 u_1)$, which is equivalent to Lie point symmetry.

Acknowledgements

I am sincerely grateful to Profs. M. Błaszak, B. Kupershmidt and W.X. Ma for stimulating discussions.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 204–209.

One-Dimensional Fokker–Planck Equation Invariant under Four- and Six-Parametrical Group

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Symmetry properties of the one-dimensional Fokker–Planck equations with arbitrary coefficients of drift and diffusion are investigated. It is proved that the group symmetry of these equations can be one-, two-, four- or six-parametric and corresponding criteria are obtained. The changes of the variables reducing Fokker–Planck equations to the heat and Schrödinger equations with certain potential are determined.

1 Introduction

Fokker–Planck equation (FPE) is a basic equation in the theory of continuous Markovian processes. In an one-dimensional case FPE has the form [1, 2]

$$L = \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[A(t,x)u] - \frac{1}{2}\frac{\partial^2}{\partial x^2}[B(t,x)u] = 0, \tag{1}$$

where $u = u(t,x)$ is the probability density, $A(t,x)$ and $B(t,x)$ are differentiable functions meaning coefficients of drift and diffusion correspondingly.

We investigated symmetry properties of the equation (1) under the infinitesimal basis operators [3–5]

$$X = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^1(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2}$$

The symmetry operators are defined from the invariance condition

$$\underset{2}{\hat{X}} L\Big|_{L=0} = 0, \tag{3}$$

where $\underset{2}{\hat{X}}$ is the second prolongation of the operator X, which is constructed according to the formulae [3–5]. From the condition of invariance (3), equating coefficients by a function u and its derivatives u_x, u_{tt}, u_{tx}, u_{xx} (u_t can be expressed from equation (1)) to zero it is possible to determine the following system of equations on functions ξ^0, ξ^1, η:

$$\begin{gathered}
\xi^0 = \xi^0(t), \qquad \xi^1 = \xi^1(t,x), \qquad \eta = \chi(t,x)u, \qquad 2\xi^1_x B - \xi^0_t B - \xi^1 B_x - \xi^0 B_t = 0, \\
\xi^0_t(A - B_x)\xi^1_t + \xi^0(A_t - B_{tx}) + \xi^1(A_x - B_{xx}) - \xi^1_x(A - B_x) + \frac{1}{2}B\xi^1_{xx} = B\chi_x, \\
\chi_t + \xi^0_t\left(A_x - \frac{1}{2}B_{xx}\right) + \xi^0\left(A_{tx} - \frac{1}{2}B_{txx}\right) \\
+\xi^1\left(A_{xx} - \frac{1}{2}B_{xxx}\right) + \chi_x(A - B_x) - \frac{1}{2}B\chi_{xx} = 0.
\end{gathered} \tag{4}$$

Here lower indexes t, x mean differentiation on corresponding variables. Let us also introduce the following notations $\frac{\partial}{\partial t} = \partial_t$, $\frac{\partial}{\partial x} = \partial_x$, $\frac{\partial}{\partial u} = \partial_u$.

2 Criterion of invariance FPE under four- and six-parametrical group of symmetry

In [6] following Theorem was proved:

Theorem 1. *If there is a symmetry operator (2) $Q \neq u\partial_u$ for FPE (1) then there exists a transformation of a form*

$$\tilde{t} = T(t), \qquad \tilde{x} = X(t,x), \qquad u = v(t,x)\tilde{u},$$

which reduces it to equation (1) with coefficients of drift and diffusion $\tilde{A} = A(\tilde{x})$, $\tilde{B} = B(\tilde{x})$. And, if $\xi^0 \not\equiv 0$ then

$$\tilde{t} = T(t), \qquad \tilde{x} = \omega, \qquad u = v(t,x)\tilde{u}, \tag{5}$$

where $T(t) = \displaystyle\int \frac{dt}{\xi^0(t)}$, and the functions $\omega = \omega(t,x)$, $v(t,x)$ satisfy the equations:

$$\xi^0\omega_t + \xi^1\omega_x = 0, \qquad \xi^0 v_t + \xi^1 v_x = \chi v, \tag{6}$$

where $\omega \neq \mathrm{const}$ is further meant as any fixed solution of the equation (6).

The consequence of this theorem is

Theorem 2. *The dimension of an invariance algebra of FPE (1) can be equal to 1, 2, 4, 6.*

Proof. If dimension of algebra more than 1 then equation (1) is reduced to the equation with $\tilde{A} = \tilde{A}(\tilde{x})$, $\tilde{B} = \tilde{B}(\tilde{x})$, but classification of such equations is known: dimension of their invariance algebra is either 2 or 4 or 6 [7].

In work [8] it is shown that any diffusion process with coefficient of drift $A(t,x)$ and diffusion $B(t,x)$ can be reduced to a process with appropriate coefficienct $\tilde{A}(t,x) = A(t,x)/B(t,x)$ and $\tilde{B}(t,x) = 1$ through random replacement of time $\tau(t)$. Using result of the theorem 1 we carry out symmetry classification of FPE for the coefficient $B(t,x) = 1$ and any $A(t,x)$ just as it was made in [7] for a case $A = A(x)$ (homogeneous process). So puting in the equations (4) $B = 1$ it is easy to show that

$$\begin{gathered}
\xi^0 = \tau(t), \qquad \xi^1 = \frac{1}{2}x\tau' + \varphi(t), \\
\frac{3}{2}\tau' M + \tau M_t + (\frac{1}{2}\tau' x + \varphi)M_x = \frac{1}{2}\tau' x + \varphi'', \\
\chi = \frac{1}{2}\tau' x A(t,x) - \frac{1}{4}x^2\tau'' - \varphi' x + \varphi A(t,x) + \tau\int_{x_0}^{x} \frac{\partial A(t,\xi)}{\partial t}d\xi + \theta(t),
\end{gathered} \tag{7}$$

where $M = A_t + \frac{1}{2}A_{xx} + AA_x$, x_0 and $\theta(t)$ are arbitrary point and function correspondently. Let us find a condition on M under wich there existes at least two the linearly independent solutions $\tau(t)$ of the equations (7). In this case from the Theorem 2 it is followed that there exists either 3 or 5 operators of symmetry (besides trivial $u\partial_u$). Let's assume that $M_{xx} \neq 0$. After differentiating twice on x both parts (7) we have:

$$\frac{5}{2}\tau' M_{xx} + \tau M_{txx} + \left(\frac{1}{2}\tau' x + \varphi\right) M_{xxx} = 0. \tag{8}$$

Now if we assume that $M_{xxx} = 0$, i.e. $M_{xx} = F(t)$, then the following condition takes place:

$$\frac{5}{2}\tau' F + \tau F' = 0. \tag{9}$$

For this equation there is only one linearly independent solution, therefore $M_{xxx} \neq 0$. Then from (8):

$$-\varphi(t) = \frac{5M_{xx} + xM_{xxx}}{2M_{xxx}}\tau' + \frac{M_{txx}}{M_{xxx}}\tau = h(t,x)\tau' + r(t,x)\tau.$$

So if (τ_1, φ_1), (τ_2, φ_2) are linearly independent then τ_1, τ_2 are linearly independent, and also $h_x\tau' + r_x\tau = 0$. Thus

$$h_x\tau_1' + r_x\tau_1 = 0, \qquad h_x\tau_1' + r_x\tau_1 = 0.$$

As Wronskian $\begin{vmatrix} \tau_1' & \tau_1 \\ \tau_2' & \tau_2 \end{vmatrix} \neq 0$, then from this system it is followed that $h_x \equiv 0$, $r_x \equiv 0$, i.e.

$$\frac{5M_{xx} + xM_{xxx}}{2M_{xxx}} = h(t), \qquad \frac{M_{xxt}}{M_{xxx}} = r(t). \tag{10}$$

From conditions (10) it is easy to deduce that

$$M = \lambda(x - H(t))^{-3} + F(t)x + G(t), \tag{11}$$

where $\lambda = \mathrm{const} \neq 0$, H, F, G are arbitrary functions. Now notice that if $M_{xx} = 0$, M has form (11) with $\lambda = 0$. Thus the condition (11) is necessary for the invariance algebra to have dimension either 4 or 6. Substituting (11) in (8) and equating zero factors at $x - H$, $(x - H)^{-4}$ and 1 we obtain the following conditions:

$$\begin{gathered} 2\tau' F + \tau F' = \frac{1}{2}\tau''', \qquad \lambda\left(\tau H' - \frac{1}{2}\tau' H - \varphi\right) = 0, \\ \frac{3}{2}\tau'(FH + G) + \tau(F'H + G') + F\left(\frac{1}{2}\tau' H + \varphi\right) = \frac{1}{2}\tau''' H + \varphi'''. \end{gathered} \tag{12}$$

1) Let $\lambda \neq 0$. Then expressing from the second equation $\varphi(t) = \tau H' - \frac{1}{2}\tau' H$ and substituting it in the third equation we have

$$\frac{3}{2}\tau'(FH + G - H'') + \tau(FH + G - H'')' = 0.$$

Condition of existence of at least 2 independent solutions τ_1, τ_2 results in the equation $FH + G - H'' = 0$. In this case the number of the fundamental solutions of system (12) is three. Really, there are three linear independent solutions τ_1, τ_2, τ_3 of the first equation (12). From the second equation (12) φ_i is expressed through τ_i, $i = 1, 2, 3$.

2) If $\lambda = 0$ the system of the equations (12) has 5 linearly independent solution (τ_i, φ_i), $i = \overline{1, 5}$.

So the following theorem is proved.

Theorem 3. *1) The class FPE (1) with* $B = 1$ *admitting four-dimentional algebra of invariance is described by the condition*

$$A_t + \frac{1}{2}A_{xx} + AA_x = \lambda(x - H(t))^{-3} + F(t)x + G(t), \tag{13}$$

where $\lambda = \mathrm{const} \neq 0$, *$G$ satisfies the condition*

$$G = H'' - FH, \tag{14}$$

$F(t)$, $H(t)$ are arbitrary functions.

2) The class FPE (1) with $B = 1$ admitting six-dimensional invariance algebra invariance is described by condition (13) in which $\lambda = 0$, F, G are arbitrary functions.

Remark. In particular, if the coefficient $A(t, x)$ satisfies the Burgers equation then FPE (1) is reduced to the heat equation (see [9]).

3 Transformation of the Fokker–Planck equations to homogeneous equations

1) It turns out that FPE (1) ($B = 1$), (13) at $\lambda = 0$ is reduced to the heat equation [9]. We find the appropriate transformation (5), (6). Let τ be any solution of system (12) and $\tau > 0$ (evidently that it is always possible to choose a solution $\tau(t) > 0$ on some interval). From the formulae (6), (7) it is easy to prove that $\omega(t, x) = \tau^{1/2}x - \int\limits_{t_0}^{t} \varphi(\xi)\tau^{-3/2}(\xi)dt$, where t_0 is arbitrary fixed point. Let us consider the transformation:

$$\begin{aligned} &\tilde{t} = \frac{1}{2}\int \frac{dt}{\tau}, \\ &\tilde{x} = \omega(t, x) = \tau^{-1/2}x - \int\limits_{t_0}^{t} \varphi(\xi)\tau^{-3/2}(\xi)d\xi, \\ &u(t, x) = v(t, x)\tilde{u}(\tilde{t}, \tilde{x}). \end{aligned} \tag{15}$$

Having substitued into (1), (13) the replacement variable (15) we come to the equation:

$$\begin{aligned} \tilde{u}_{\tilde{t}} = {}& -2\tau\left(\frac{v_t}{v} + A_x + A\frac{v_x}{v} - \frac{1}{2}\frac{v_{xx}}{v}\right)\tilde{u} \\ & -2\left(-\frac{1}{2}\tau^{1/2}\tau' x - \varphi\tau^{-1/2} + A\tau^{1/2} - \frac{v_x}{v}\tau^{1/2}\right)\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}}. \end{aligned} \tag{16}$$

Equating zero factor at $\tilde{u}_{\tilde{x}}$, we shall get:

$$v = \exp\left(-\frac{1}{4}\tau^{-1}\tau' x^2 - \tau^1\varphi x + \int\limits_{x_0}^{x} A(t, \xi)d\xi + h(t)\right), \tag{17}$$

where $h(t)$ is an arbitrary function, x_0 is some fixed point. Substituting (17) into the expression $\frac{v_t}{v} + A_x + A\frac{v_x}{v} - \frac{1}{2}\frac{v_{xx}}{v}$ (factor at $\tilde{u}$ in (16)) and equating its to zero we get:

$$h'(t) = \frac{1}{2}\left[\tau^{-2}\varphi^2 - \frac{1}{2}\tau^{-1}\tau^1 - A_x(t, x_0) - A^2(t, x_0)\right], \tag{18}$$

$$\frac{1}{2}\tau^{-1}\tau'' - \frac{1}{4}\tau^2(\tau')^2 = F, \qquad \tau^{-1}\varphi' - \frac{1}{2}\tau^2\tau'\varphi = G. \tag{19}$$

It is easy to prove that if ($\tau \neq 0$, φ) is some solution of system (19) then it satisfies to system (12) ($\lambda = 0$, $M = 0$). Then we have the transformation (15), where funcitons $v(t,x)$, $\tau(t)$, $\varphi(t)$ can be found from (17)–(19), resulting FPE (1), (13) ($\lambda = 0$) to the heat equation

$$\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}}. \tag{20}$$

Let us notice, that the system (19) is reduced to following:

$$2y' + y^2 = 4F, \qquad y = \frac{\tau'}{\tau}, \qquad \varphi = \tau^{1/2}\int_{t_0}^{t} \tau^{1/2} G dt. \tag{21}$$

2) We consider now FPE (1), (13) with $\lambda \neq 0$. As in the case of 1) the transformation (15) reduces this equation to the equation (16). The conditions for (16) to be FPE are the following:

$$\begin{gathered}\tilde{A} = \tilde{A}(\omega) = -\tau^{-1/2}\tau' x - 2\varphi\tau^{-1/2} + 2A\tau^{1/2} - 2\tau^{1/2}\frac{v_x}{v},\\ \tilde{A}_\omega = 2\tau\left(\frac{v_t}{v} + A_x + A\frac{v_x}{v} - \frac{1}{2}\frac{v_{xx}}{v}\right),\end{gathered} \tag{22}$$

where ω is given in (15). The first condition is equivalent to the equation

$$\partial_{\tilde{t}}\tilde{A} = \left[\tau\partial_t + \left(\frac{1}{2}\tau' x + \varphi\right)\partial_x\right]\left(-\tau^{-1/2}\tau' x - 2\varphi\tau^{1/2} + 2A\tau^{1/2} - 2\tau^{1/2}\frac{v_x}{v}\right) = 0. \tag{23}$$

Omitting intermediate calculations we give the general solution $v(t,x)$ of equation (23):

$$v(t,x) = \exp\left[\int_{x_0}^{x} A(t,\xi)d\xi - \frac{1}{4}\tau^{-1}\tau' x^2 - \tau^{-1}\varphi x + k(\omega)\right], \tag{24}$$

where $k(\omega)$ is an arbitrary function, x_0 is some fixed point. Substituting (24) into the first equation (22) one can prove that $\tilde{A} = -k'(\omega)$ $\left(k'(\omega) = \frac{dk(\omega)}{d\omega}\right)$. Let us substitute $\tilde{A}(\omega) = -k'(\omega)$, $v(t,x)$ (24) in the second equation (22). Under chosen conditions

$$\tau^{1/2}\int_{t_0}^{t} \varphi\tau^{-3/2} dt = H, \qquad \frac{1}{2}\tau^{-1}\tau'' - \frac{1}{4}\tau^{-2}\tau'^2 = F, \qquad \tau^{-1}\varphi' - \frac{1}{2}\tau^{-2}\tau'\varphi = G, \tag{25}$$

$$k'' - k'^2 = \lambda\omega^{-2}, \tag{26}$$

the second equaiton (22) is satisfied. It is possible to choose the condition (25) because, as it is easy to prove, any solution $\tau \neq 0$, φ of the given system is a particular solution of th system equations (12), (14) that it is enough for construction of the transformation (15). System (25) (taking into account (14)) is equivalent to:

$$2y' + y^2 = 4F, \qquad y = \frac{\tau'}{\tau}, \qquad \varphi = \tau^{3/2}(\tau^{-1/2}H)'. \tag{27}$$

Thus we have proved

Theorem 4. *FP equation (1), (13), (14) with $\lambda \neq 0$, invariant under four-parameter algebra of invariance, through transformations*

$$\tilde{t} = T(t), \qquad \tilde{x} = \tau^{-1/2}x - \tau^{-1/2}H(t), \qquad u = v(t,x)\tilde{u}(\tilde{t},\tilde{x}),$$

where $T = \dfrac{1}{2}\displaystyle\int \dfrac{dt}{\tau(t)}$, $v(t,x)$ *has the form (24),* $\tau \neq 0$ *is any solution of the first equation (27),* $k(\omega)$ *is a solution of the equation (26), is reduced to the equation*

$$\tilde{u}_{\tilde{t}} = 2k''(\omega)\tilde{u} + 2k'(\omega)\tilde{u}_\omega + \tilde{u}_{\omega\omega}.$$

Remark. Making the replacement in last equation

$$\bar{t} = \tilde{t}, \qquad \bar{x} = \omega, \qquad \tilde{u} = \exp(k(\omega))\bar{u},$$

and taking into account the condition (26), we can reduce this equation to the following Schrödinger equation:

$$\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \frac{\lambda}{\bar{x}^2}\bar{u}.$$

Thus in the case FPE with four-parametrical group of symmetry there exists an “initial” equation, to which they are reduced; though it is not FPE as it is in the case of the six-parametrical group.

Acknowlegments

Authors is grateful to Prof. R.Z. Zhdanov for valuable discusions.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 210–217.

Periodic Soliton Solutions to the Davey–Stewartson Equation

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The periodic soliton resonances and recurrent wave solutions to the Davey–Stewartson equation are presented. The solutions that described the interaction between a y-periodic soliton and a line soliton are analyzed to show the existence of the soliton resonances. The various recurrent solutions (The growing-and-decaying mode, breather and rational growing-and-decaying mode solutions) are presented. The y-periodic soliton and breather solutions can be constructed as the imbricate series of algebraic soliton solutions and rational growing-and-decaying mode solutions, respectively.

1 Introduction

It is well known that spin and statistics in quantum mechanics come from symmetries of transformation. The soliton solutions to some soliton equations show Fermion-like behavior. We could not obtain the solutions from the initial value problem which forgive the coexistence of completely same solitons in the wave field to some soliton equations. It is very interesting to know that what symmetries are hidden in soliton equations related to this problem. Before studying the symmetries to the soliton equations from the point of view, we will show some propagation properties of solitons to the Davey–Stewartson (DS) equation which is the two-dimensional generalization of the nonlinear Schrödinger equation [1].

The higher-dimensional nonlinear wave fields have richer phenomena than one-dimensional ones since various localized solitons may be considered in higher-dimensional space. The DS equation has four kinds of soliton solutions: the conventional line, algebraic, periodic and lattice solitons. The conventional line soliton has an essentially one-dimensional structure. On the other hand, the algebraic, periodic and lattice solitons have a two-dimensional localized structure.

The solutions to the DS equation have been studied previously in various aspects [2–9]. The existence of solitons having the structures peculiar to a higher-dimensionality may contribute to the variety of the dynamics of nonlinear waves. To clarify the dynamics, we must investigate various interactions between two different kinds of solitons. In the previous papers [10, 11], the various interactions between two y-periodic solitons, line and periodic and periodic and algebraic solitons were investigated. And we found the periodic resonant interactions which are qualitatively different from the interaction between two line solitons. We expect that the periodic soliton resonances play fundamental role in the nonlinear development of higher-dimensional wave field as the existence of the periodic soliton resonances may be related to the instability of the solitons, accompanied by their decay and merging.

The governing equations for the description of the long time evolution of unstable wave train have been studied by many authors. The extension to the two-dimensional case was examined by Zakharov [12], Benny and Roskes [13] and Davey and Stewartson [1]. The time evolution of the solution of the 1D-NLS equation with periodic boundary condition and with a Benjamin–Feir unstable initial condition was studied numerically by Lake *et al.* [15]. They found that a modulated unstable wave train achieves a state of maximum modulation and returns to an unmodulated initial state, which is well known as the Fermi–Pasta–Ulam (FPU) recurrence. One of the important feature of the solutions of the NLS equations in one- and two-dimensions is the recurrence of the unstable wavetrain to its initial state.

The purposes of this study are (i) to review periodic soliton solutions and recurrent solutions and (ii) to show that these solutions can be constructed by imbricate series of rational soliton solutions or rational growing-and-decaying mode solutions.

2 Periodic soliton resonances

The Davey–Stewartson equation may be written as

$$\begin{cases} \mathrm{i}u_t + pu_{xx} + u_{yy} + r|u|^2u - 2uv = 0, \\ pv_{xx} - v_{yy} - pr(|u|^2)_{xx} = 0, \end{cases} \tag{1}$$

where $p = \pm 1$, r is constant, eq. (1) with $p = 1$ and $p = -1$ are called the DS I and DS II equations, respectively. In this section, we study the resonant interactions between y-periodic soliton and line soliton mutually parallel propagating to the x-direction of the DS I equation with $r > 0$. The solution that describes the interaction between a y-periodic soliton and a line soliton is written as [11]

$$u = u_0 e^{\mathrm{i}(kx+ly-\omega t)}\frac{g}{f}, \qquad v = -2p(\log f)_{xx} \qquad (p = 1) \tag{2}$$

with

$$\begin{aligned} f = 1 &- \frac{1}{\alpha^2}\exp(\xi_1)\cos\eta + \frac{M}{4\alpha^4}\exp(2\xi_1) \\ &+ \exp(\xi_2)\left\{1 - \frac{N}{\alpha^2}\exp(\xi_1)\cos\eta + \frac{MN^2}{4\alpha^4}\exp(2\xi_1)\right\}, \end{aligned} \tag{3}$$

$$\begin{aligned} g = 1 &- \frac{1}{\alpha^2}\exp(\xi_1 + \mathrm{i}\phi)\cos\eta + \frac{M}{4\alpha^4}\exp 2(\xi_1 + \mathrm{i}\phi) \\ &+ \exp(\xi_2 + \mathrm{i}\psi)\left\{1 - \frac{N}{\alpha^2}\exp(\xi_1 + \mathrm{i}\phi)\cos\eta + \frac{MN^2}{4\alpha^4}\exp 2(\xi_1 + \mathrm{i}\phi)\right\}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\xi_1 = \alpha x - \Omega_P t + \sigma_1, \qquad \xi_2 = \beta x - \Omega_L t + \sigma_2, \qquad \eta = \delta y - \gamma t + \kappa, \\ &\sin^2\frac{\phi}{2} = \frac{\alpha^2 + \delta^2}{2ru_0^2}, \qquad \sin^2\frac{\psi}{2} = \frac{\beta^2}{2ru_0^2}, \\ &\Omega_P = 2k\alpha - (\alpha^2 - \delta^2)\cot\frac{\phi}{2}, \qquad \Omega_L = \beta\left(2k - \beta\cot\frac{\psi}{2}\right), \qquad \gamma = 2l\delta, \\ &M = 1\bigg/\left[1 - \frac{(\alpha^2 + \delta^2)^2}{2\delta^2 ru_0^2}\right], \qquad N = \frac{2ru_0^2\sin\frac{\phi}{2}\sin\frac{\psi}{2}\cos\frac{\phi-\psi}{2} - \alpha\beta}{2ru_0^2\sin\frac{\phi}{2}\sin\frac{\psi}{2}\cos\frac{\phi+\psi}{2} - \alpha\beta}. \end{aligned} \tag{5}$$

We can investigate the phase shifts after the collision between y-periodic soliton and line soliton by using the solution. The condition $|N| = \infty$, corresponds to the phase shift in the propagation direction becomes infinite for the case $\alpha\beta > 0$. This means that the period of the intermediate state, where the periodic soliton propagates together with the line soliton, persist infinitely. This is thought as a resonance between the y-periodic soliton and the line soliton. By equating the denominator of N to zero, this condition is given by

$$2ru_0^2 \sin\frac{\phi}{2}\sin\frac{\psi}{2}\cos\frac{\phi+\psi}{2} - \alpha\beta = 0. \tag{6}$$

The condition $N = 0$ corresponds to the phase shift in the propagating direction becomes negative infinity for $\alpha\beta > 0$. This means that two solitons can interact infinitely apart each other. This is thought as extremely repulsive or long range interaction between the y-periodic soliton and the line soliton. The explicit expression of the condition is obtained by equating the numerator of N with zero as

$$2ru_0^2 \sin\frac{\phi}{2}\sin\frac{\psi}{2}\cos\frac{\phi-\psi}{2} - \alpha\beta = 0. \tag{7}$$

We can also show the existence of periodic soliton resonances in the interactions of periodic soliton-periodic soliton and periodic soliton-algebraic soliton [10, 11].

3 Recurrent solutions

One of the important feature of the solution to the DS equation is the recurrence of the unstable wavetrain to its initial state. Three kinds of recurrent solutions, growing-and-decaying mode, breather and rational growing-and-decaying mode solutions are shown in this section, which can be constructed from the two-soliton solution [15]. The two-soliton solution may be written as [3]

$$u = u_0 e^{\mathrm{i}(kx+ly-\omega t)}\frac{g}{f}, \qquad v = -2p(\log f)_{xx}, \tag{8}$$

with

$$f = 1 + e^{\eta_1} + e^{\eta_2} + De^{\eta_1+\eta_2}, \qquad g = 1 + e^{\eta_1+\mathrm{i}\phi_1} + e^{\eta_2+\mathrm{i}\phi_2} + De^{\eta_1+\eta_2+\mathrm{i}(\phi_1+\phi_2)},$$

where

$$\begin{aligned} &\eta_j = K_j x + L_j y - \Omega_j t + \eta_j^0, \qquad \sin^2\frac{\phi_j}{2} = \frac{pK_j^2 - L_j^2}{2ru_0^2},\\ &\Omega_j = 2pkK_j + 2lL_j - (pK_j^2 + L_j^2)\cot\frac{\phi_j}{2} \qquad (j = 1, 2). \end{aligned} \tag{9}$$

(i) growing-and-decaying mode solution. Taking wave numbers and frequencies pure imaginary and complex, respectively,

$$K_1 = K_2^* = \mathrm{i}\beta, \quad L_1 = L_2^* = \mathrm{i}\delta, \qquad \Omega_1 = \Omega_2^* = \Omega + \mathrm{i}\gamma,$$

$$\phi_1 = \phi_2 = \phi \ \text{: real}, \qquad \eta_1^0 = \eta_2^{0*}, \qquad e^{\eta_1^0} = e^{\eta_2^{0*}} = -(1/2)e^{-\tilde\sigma+\mathrm{i}\theta},$$

we have the following dispersion relation and D

$$\sin^2\frac{\phi}{2} = \frac{\delta^2 - p\beta^2}{2ru_0^2}, \qquad \Omega = -\left(\delta^2 + p\beta^2\right)\cot\frac{\phi}{2},$$

$$\gamma = 2pk\beta + 2l\delta, \qquad D = \frac{2}{1+\cos\phi} > 1.$$

Then, the solution is given by

$$u = u_0 e^{\mathrm{i}(kx+ly-\omega t+\phi)} \left[\sqrt{D}\cosh(\Omega t+\sigma-\mathrm{i}\phi)-\cos(\beta x+\delta y-\gamma t+\theta)\right] \times \left[\sqrt{D}\cosh(\Omega t+\sigma)-\cos(\beta x+\delta y-\gamma t+\theta)\right]^{-1}, \tag{10}$$

$$v = 2p\beta^2 \frac{\sqrt{D}\cosh(\Omega t+\sigma)\cos(\beta x+\delta y-\gamma t+\theta)-1}{\left[\sqrt{D}\cosh(\Omega t+\sigma)-\cos(\beta x+\delta y-\gamma t+\theta)\right]^2}, \tag{11}$$

where $\sigma = \tilde{\sigma} + \log \frac{2}{\sqrt{D}}$. The existence condition for the non-singular solution is given by $D > 1$, which is satisfied for $\delta^2 - p\beta^2 > 0$. This solution grows exponentially at the initial stage and the growth rate is given by Ω, which is in agreement with the growth rate of the Benjamin–Feir instability. Therefore, we can regard as this growing-and-decaying mode solution as one described the nonlinear evolution of the unstable mode.

(ii) Breather solution. To obtain analytical expression for the breathing wave solution, we set $K_1 = K_2 = a$, $L_1 = L_2 = b$ and $\phi_1 = -\phi_2 = \mathrm{i}\Phi$ in eq. (9), where a and b are real. Then, frequencies Ω_1 and Ω_2 are complex and are complex conjugate with each other and the solution is given by,

$$u = u_0 e^{\mathrm{i}(kx+ly-\omega t)} \frac{\sqrt{D}\cosh\xi - \cosh\Phi\cos(\gamma t+\theta)+\mathrm{i}\sinh\Phi\sin(\gamma t+\theta)}{\sqrt{D}\cosh\xi-\cos(\gamma t+\theta)}, \tag{12}$$

$$v = -2pa^2 D \frac{1-\frac{1}{\sqrt{D}}\cosh\xi\cos(\gamma t+\theta)}{\left[\sqrt{D}\cosh\xi-\cos(\gamma t+\theta)\right]^2}, \tag{13}$$

where

$$\begin{gathered} \xi = ax+by-\Omega t+\sigma, \qquad \sinh^2\frac{\Phi}{2} = \frac{b^2-pa^2}{2ru_0^2} > 0, \\ \Omega = 2(pka+lb), \qquad \gamma = \left(b^2+pa^2\right)\sqrt{\frac{2ru_0^2}{b^2-pa^2}+1}, \qquad D = 1+\frac{b^2-pa^2}{2ru_0^2}, \end{gathered} \tag{14}$$

where σ and θ are arbitrary phase constants.

(iii) Rational growing-and-decaying mode solution. We consider the long wave limit of the growing-and-decaying mode solution. Putting $K_1 = K_2^* = \mathrm{i}\varepsilon c$, $L_1 = L_2^* = \mathrm{i}\varepsilon d$, $\eta_1^0 = \eta_2^{0^*} = \varepsilon(\mathrm{i}\tilde{\theta}' - \tilde{\sigma}') + \mathrm{i}\pi$, and taking the limit as $\varepsilon \to 0$, we have

$$u = u_0 e^{\mathrm{i}(kx+ly-\omega t)} \left\{1 - \frac{4\alpha(\alpha \pm \mathrm{i}\eta)}{\alpha^2+\eta^2+\xi^2}\right\}, \tag{15}$$

$$v = -4pc^2 \frac{\alpha^2+\eta^2-\xi^2}{(\alpha^2+\eta^2+\xi^2)^2}, \tag{16}$$

where

$$\begin{gathered} \xi = cx+dy-\gamma t+\theta', \qquad \eta = \Omega t + \sigma', \\ \Omega = \pm\left(d^2+pc^2\right)\sqrt{\frac{2ru_0^2}{d^2-pc^2}}, \qquad \gamma = 2pkc+2ld, \qquad \alpha^2 = \frac{d^2-pc^2}{2ru_0^2}. \end{gathered} \tag{17}$$

4 Periodic soliton and recurrent solutions as imbricate series of rational solutions

Zaitsev has succeeded in obtaining a periodic soliton solution by the imbricate series of algebraic soliton solutions for the Kadomtsev–Petviashvili (KP) equation with positive dispersion [16]. It is known that the lattice soliton solution to the KP equation with positive dispersion that have doubly periodic array of the localized structure in the x-y plane was constructed as doubly imbricate series of algebraic soliton solutions, which was expressed by using Weierstrass's $\wp$ function or the Riemann theta functions [17]. In this section, we show that the y-periodic soliton and breather solutions can be constructed as the imbricate series of algebraic soliton solutions and rational growing-and-decaying mode solutions, respectively.

(i) $\boldsymbol{Y}$-periodic soliton solution as imbricate series of algebraic soliton solutions. It is interesting to note that the algebraic soliton solutions is given as the following form:

$$u = u_0 e^{i\zeta}\left[1 + \frac{2iB}{\xi + i\sqrt{\eta^2 + A^2}}\right]\left[1 + \frac{2iB}{\xi - i\sqrt{\eta^2 + A^2}}\right], \tag{18}$$

$$v = 2p\left[\frac{1}{(\xi + i\sqrt{\eta^2 + A^2})^2} + \frac{1}{(\xi - i\sqrt{\eta^2 + A^2})^2}\right]. \tag{19}$$

where

$$\begin{aligned} &\zeta = kx + ly - \omega t, \qquad \xi = x - \left(2pk - \frac{p - R^2}{B}\right)t + \xi^0, \qquad \eta = R(y - 2lt) + \eta^0, \\ &B = \sqrt{\frac{p + R^2}{2ru_0^2}}, \qquad A^2 = 4B^2 \Bigg/ \left(2B^2 - \frac{p - R^2}{ru_0^2}\right), \end{aligned} \tag{20}$$

The y-periodic soliton solution can be obtained from two-soliton solution of Satsuma and Ablowitz as follows:

$$\begin{aligned} u = u_0 e^{i\zeta}(1 - \tan^2\frac{\phi}{2})\cos^2\frac{\phi}{2}\Bigg[1 - 2\frac{\tan\frac{\phi}{2}}{1 - \tan^2\frac{\phi}{2}} \\ \times \frac{\frac{1}{\sqrt{D}}\tan\frac{\phi}{2}\cos(\delta y - \gamma t + \theta) - i\sinh(\alpha x - \Omega t + \sigma)}{\cosh(\alpha x - \Omega t + \sigma) - \frac{1}{\sqrt{D}}\cos(\delta y - \gamma t + \theta)}\Bigg], \end{aligned} \tag{21}$$

$$v = -2p\alpha^2\frac{1 - \frac{1}{\sqrt{D}}\cosh(\alpha x - \Omega t + \sigma)\cos(\delta y - \gamma t + \theta)}{\left[\cosh(\alpha x - \Omega t + \sigma) - \frac{1}{\sqrt{D}}\cos(\delta y - \gamma t + \theta)\right]^2}, \tag{22}$$

where

$$\begin{aligned} &D = \left[1 - \frac{(\delta^2 + p\alpha^2)^2}{2ru_0^2\delta^2}\right]^{-1} > 1, \qquad \sin^2\frac{\phi}{2} = \frac{\delta^2 + p\alpha^2}{2ru_0^2}, \\ &\Omega = 2pk\alpha - (p\alpha^2 - \delta^2)\cot\frac{\phi}{2}, \qquad \gamma = 2l\delta. \end{aligned} \tag{23}$$

On the basis of eqs. (18) and (19), we assume the form of the y-periodic soliton solution as follows:

$$u = \hat{u}_0 e^{i\zeta}\left[1 + \sum_m \frac{ib}{\xi' + i\nu(\eta) + im\pi}\right]\left[1 + \sum_m \frac{ib}{\xi' - i\nu(\eta) + im\pi}\right], \tag{24}$$

$$v = \frac{p\alpha^2}{2}\sum_m\left[\frac{1}{(\xi' + i\nu(\eta) + im\pi)^2} + \frac{1}{(\xi' - i\nu(\eta) + im\pi)^2}\right], \tag{25}$$

where the summation $\sum_m$ means $\lim_{N\to\infty}\sum_{m=-N}^{N}$, $\nu(\eta)$ is a function of η to be determined afterward. The function $\sqrt{\eta^2 + A}$ is deformed to $\nu(\eta)$ by nonlinear effects. Equations (24) and (25) are rewritten as follows

$$u = \hat{u}_0 e^{i\zeta}(1 - b^2)\left[1 - \frac{2b}{1 - b^2}\frac{b\cos 2\nu(\eta) + i\sinh 2\xi'}{\cosh 2\xi' - \cos 2\nu(\eta)}\right], \tag{26}$$

$$v = -2p\alpha^2\frac{1 - \cos 2\nu(\eta)\cosh 2\xi'}{[\cosh 2\xi' - \cos 2\nu(\eta)]^2}. \tag{27}$$

Comparing these eqs. (26) and (27) with eqs. (21) and (22), respectively, we find

$$\begin{aligned} &\hat{u}_0 = u_0\cos^2\frac{\phi}{2}, \qquad b = -\tan\frac{\phi}{2}, \qquad \xi' = \frac{1}{2}(\alpha x - \Omega t + \sigma'), \\ &\nu(\eta) = \tfrac{1}{2}\cos^{-1}\left[\tfrac{1}{\sqrt{D}}\cos(\delta y - \gamma t + \theta)\right]. \end{aligned} \tag{28}$$

The substitution of eq.(28) into eqs.(24) and (25) gives the y-periodic soliton solution as an imbricate series of algebraic solitons. Taking the $\lim\phi \to 0$ ($\alpha \to 0$, $\delta \to 0$ but $\delta/\alpha = R$ is finite), we have

$$\xi' = \frac{\alpha}{2}\xi, \qquad \nu(\eta) = \frac{\alpha}{2}\sqrt{\eta^2 + A^2}, \qquad b = \alpha B. \tag{29}$$

This means that the solutions (21) and (22) are simple summations of algebraic soliton solutions for very small ϕ.

Recently, the lattice soliton solution to the DS equation was constructed as doubly imbricate series of algebraic soliton solutions which was expressed by using Weierstrass's $\wp$ function or the Riemann theta functions [18].

(ii) Breather solution as imbricate series of rational growing-and-decaying mode solutions. At first, we have to note that the rational growing-and-decaying mode solution is rewritten as following form,

$$u = u_0 e^{i\zeta}\left[1 \mp \frac{2i\alpha}{\eta + i\sqrt{\xi^2 + \alpha^2}}\right]\left[1 \mp \frac{2i\alpha}{\eta - i\sqrt{\xi^2 + \alpha^2}}\right], \tag{30}$$

$$\begin{aligned} v = -2pc^2&\left[\left\{\frac{1}{(\eta + i\sqrt{\xi^2 + \alpha^2})^2} + \frac{1}{(\eta - i\sqrt{\xi^2 + \alpha^2})^2}\right\}\right. \\ &\left.+\frac{2\alpha}{(\eta + i\sqrt{\xi^2 + \alpha^2})^2}\frac{2\alpha}{(\eta - i\sqrt{\xi^2 + \alpha^2})^2}\right]. \end{aligned} \tag{31}$$

On the basis of eqs. (30) and (31), we assume the form of the breather solution as follows,

$$u = \bar{u}_0 e^{i\zeta} \left\{ 1 + ib \sum_m \frac{1}{\eta' + i\nu(\xi) + m\pi} \right\} \left\{ 1 + ib \sum_m \frac{1}{\eta' - i\nu(\xi) + m\pi} \right\}, \tag{32}$$

$$\begin{aligned} v = 4A\alpha^2 \Bigg[& \sum_m \frac{1}{(\eta' + i\nu(\xi) + m\pi)^2} + \sum_m \frac{1}{(\eta' - i\nu(\xi) + m\pi)^2} \\ & + 4\alpha^2 \left\{ \sum_m \frac{1}{(\eta' + i\nu(\xi) + m\pi)^2} \right\} \left\{ \sum_m \frac{1}{(\eta' - i\nu(\xi) + m\pi)^2} \right\} \Bigg], \end{aligned} \tag{33}$$

Equations (32) and (33) are rewritten as follows,

$$u = \bar{u}_0 (1 - b^2) e^{i\zeta} \left[1 - \frac{2b}{1 - b^2} \frac{b \cos 2\eta' - i \sin 2\eta'}{\cosh 2\nu(\xi) - \cos 2\eta'} \right], \tag{34}$$

$$v = 16A\alpha^2 \left[\frac{4\alpha^2 + 1 - \cos 2\nu(\xi) \cos 2\eta'}{(\cosh 2\nu(\xi) - \cos 2\eta')^2} \right]. \tag{35}$$

Comparing these eqs. (34) and (35) with eqs. (12) and (13), respectively, we find

$$\begin{aligned} & \eta' = \frac{1}{2}\eta = \frac{1}{2}(\Omega t + \sigma), \qquad b = \left[\frac{2ru_0^2}{b^2 - pa^2} + 1 \right]^{-\frac{1}{2}} = \tanh \frac{\Psi}{2}, \qquad A = -\frac{pa^2 ru_0^2}{b^2 - pa^2}, \\ & \bar{u}_0 = \frac{u_0}{1 - b^2}, \qquad \nu(\xi) = \frac{1}{2} \ln \left(\sqrt{D} \cosh \xi + \sqrt{D \cosh^2 \xi - 1} \right), \end{aligned} \tag{36}$$

where

$$D = 1 + \frac{b^2 - pa^2}{2ru_0^2}. \tag{37}$$

Substituting eq. (36) into eqs. (32) and (33), we have the imbricate series constructing breather solution.

5 Conclusion

The DS equation has four kinds of soliton solutions and three kinds of recurrent wave solutions. We have investigated the interaction between y-periodic soliton and line soliton. There are two-types of singular interactions, namely the resonant interaction and the long range interaction. In the long range interaction, the line soliton receives a small transverse disturbance of the same wave number as approaching y-periodic soliton. The disturbance on the line soliton develops into the same y-periodic soliton as approaching soliton. The line soliton emits the y-periodic soliton forward and changes into the messenger line soliton. Then, we observe that the same y-periodic solitons coexist in the wave field when the messenger line soliton is propagating between them. It was also shown that the periodic soliton solutions and the recurrent wave solutions can be constructed as imbricate series of algebraic soliton solutions and rational growing-and-decaying mode solutions, respectively. If we can regard the y-periodic soliton as a sequence of infinite algebraic solitons, we see that same algebraic soliton can not coexist, but infinite algebraic solitons can coexist in the wave field, which is a kind of condensation. We would like to go on to investigate on the symmetries related to these phenomena.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 218–222.

Symmetries and Reductions of Partial Differential Equations

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We consider different types of symmetries of partial differential equations. Using symmetry operators we construct corresponding ansatzes, reducing initial equations to the system with fewer independent variables.

It is well known that invariance of system of partial differential equations with respect to a Lie group of point transformations of independent and dependent variables is a sufficient condition of reduction of the system under study to a system of equations with fewer number of independent variables with help of a corresponding ansatz. This property is sucessfully exploited in constructing of exact solutions for many linear and nonlinear equations of mathematical physics [1]. By using the results of [2] we construct an ansatz for $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$, which reduces the nonlinear Maxwell equations

$$\begin{aligned} &\frac{\partial \vec{D}}{\partial t} = \mathrm{rot}\vec{H}, \qquad && \frac{\partial \vec{B}}{\partial t} = -\mathrm{rot}\vec{E}, \\ &\mathrm{div}\vec{D} = 0, && \mathrm{div}\vec{B} = 0, \end{aligned} \tag{1}$$

$$\vec{D} = M(I_1, I_2)\vec{E} + N(I_1, I_2)\vec{B}, \qquad \vec{H} = M(I_1, I_2)\vec{B} - N(I_1, I_2)\vec{E}, \tag{2}$$

where $M, N \in C^1(R^2, R^1)$, to the system of ordinary differential equations.

The ansatz invariant with respect to the 3-dimensional subalgebra $\langle -J_{01} - J_{13}, J_{03}, P_2 \rangle$ of the Poincaré algebra has the form

$$\begin{aligned} E^1 &= \frac{1}{2}\left(\frac{1}{\xi} + \xi\right)\widetilde{E}_1 + \frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\widetilde{B}_2 - \frac{x_1}{\xi}\widetilde{E}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{B}_2 - \widetilde{E}_1\right), \\ E^2 &= \frac{1}{2}\left(\frac{1}{\xi} + \xi\right)\widetilde{E}_2 - \frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\widetilde{B}_1 + \frac{x_1}{\xi}\widetilde{B}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{E}_2 - \widetilde{B}_1\right), \\ E^3 &= \widetilde{E}_3 - x_1\left(\widetilde{B}_2 - \widetilde{E}_1\right), \end{aligned} \tag{3}$$

$$\begin{aligned} B^1 &= \frac{1}{2}\left(\frac{1}{\xi} + \xi\right)\widetilde{B}_1 - \frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\widetilde{E}_2 - \frac{x_1}{\xi}\widetilde{B}_3 - \frac{x_1^2}{2\xi}\left(\widetilde{E}_2 + \widetilde{B}_1\right), \\ B^2 &= \frac{1}{2}\left(\frac{1}{\xi} + \xi\right)\widetilde{B}_2 + \frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\widetilde{E}_1 - \frac{x_1}{\xi}\widetilde{E}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{B}_2 - \widetilde{E}_1\right), \\ B^3 &= \widetilde{B}_3 - x_1\left(\widetilde{E}_2 + \widetilde{B}_1\right), \end{aligned} \tag{4}$$

$$\begin{aligned}
&D^1 = \frac{1}{2}\left(\frac{1}{\xi}+\xi\right)\widetilde{D}_1 + \frac{1}{2}\left(\frac{1}{\xi}-\xi\right)\widetilde{H}_2 - \frac{x_1}{\xi}\widetilde{D}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{H}_2 - \widetilde{D}_1\right),\\
&D^2 = \frac{1}{2}\left(\frac{1}{\xi}+\xi\right)\widetilde{D}_2 - \frac{1}{2}\left(\frac{1}{\xi}-\xi\right)\widetilde{H}_1 + \frac{x_1}{\xi}\widetilde{H}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{D}_2 - \widetilde{H}_1\right),\\
&D^3 = \widetilde{D}_3 - x_1\left(\widetilde{H}_2 - \widetilde{D}_1\right),
\end{aligned} \tag{5}$$

$$\begin{aligned}
&H^1 = \frac{1}{2}\left(\frac{1}{\xi}+\xi\right)\widetilde{H}_1 - \frac{1}{2}\left(\frac{1}{\xi}-\xi\right)\widetilde{D}_2 - \frac{x_1}{\xi}\widetilde{H}_3 - \frac{x_1^2}{2\xi}\left(\widetilde{D}_2 + \widetilde{H}_1\right),\\
&H^2 = \frac{1}{2}\left(\frac{1}{\xi}+\xi\right)\widetilde{H}_2 + \frac{1}{2}\left(\frac{1}{\xi}-\xi\right)\widetilde{D}_1 - \frac{x_1}{\xi}\widetilde{D}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{H}_2 - \widetilde{D}_1\right),\\
&H^3 = \widetilde{H}_3 - x_1\left(\widetilde{D}_2 + \widetilde{H}_1\right),
\end{aligned} \tag{6}$$

where $\widetilde{E}_a$, $\widetilde{B}_a$, $\widetilde{D}_a$, $\widetilde{H}_a$ are unknown functions of the variable $\omega = x_0^2 - x_1^2 - x_3^2$, $\xi = x_0 - x_3$.
Substituting (3)–(6) in (1) we obtain the reduced system

$$\begin{aligned}
&\left(\widetilde{B}_1' + \widetilde{E}_2'\right)\omega + \widetilde{B}_1' - \widetilde{E}_2' + \widetilde{B}_1 + \widetilde{E}_2 = 0,\\
&\left(\widetilde{B}_2' - \widetilde{E}_1'\right)\omega + \widetilde{B}_2' + \widetilde{E}_1' + 2\left(\widetilde{B}_2 - \widetilde{E}_1\right) = 0,\\
&\widetilde{B}_3' = 0, \qquad \widetilde{B}_3 = 0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\left(\widetilde{H}_1' + \widetilde{D}_2'\right)\omega - \widetilde{H}_1' + \widetilde{D}_2' + 2\left(\widetilde{H}_1 + \widetilde{D}_2\right) = 0,\\
&\left(\widetilde{H}_2' - \widetilde{D}_1'\right)\omega - \left(\widetilde{H}_2' + \widetilde{D}_1'\right) + \widetilde{H}_2 - \widetilde{D}_1 = 0,\\
&\widetilde{D}_3' = 0, \qquad \widetilde{D}_3 = 0,
\end{aligned} \tag{8}$$

$$\vec{\widetilde{D}} = M\vec{\widetilde{E}} + N\vec{\widetilde{B}}, \qquad \vec{\widetilde{H}} = M\vec{\widetilde{B}} - N\vec{\widetilde{E}}, \tag{9}$$

where M, N are functions of $I_1 = \vec{\widetilde{E}}^2 - \vec{\widetilde{B}}^2$, $I_2 = \vec{\widetilde{B}}\vec{\widetilde{E}}$, “$'$” designates differentiation.

To construct invariant solutions it is necessary to the integrate nonlinear system of differential equations (7)–(9). We obtained a partial solution of the system, when $N = 0$, $M = M(I_1)$ in the form

$$\widetilde{E}_1 = C_1\left(\omega^{-1/2} - \omega^{-3/2}\right), \qquad \widetilde{E}_2 = C_1\left(\omega^{-1/2} + \omega^{-3/2}\right), \qquad \widetilde{E}_3 = 0, \tag{10}$$

$$\widetilde{B}_1 = -C_1\left(\omega^{-1/2} - \omega^{-3/2}\right), \qquad \widetilde{B}_2 = C_1\left(\omega^{-1/2} + \omega^{-3/2}\right), \qquad \widetilde{B}_3 = 0, \tag{11}$$

$$\widetilde{D}_1 = mC_1\left(\omega^{-1/2} - \omega^{-3/2}\right), \qquad \widetilde{D}_2 = mC_1\left(\omega^{-1/2} + \omega^{-3/2}\right), \qquad \widetilde{D}_3 = 0, \tag{12}$$

$$\widetilde{H}_1 = -mC_1\left(\omega^{-1/2} - \omega^{-3/2}\right), \qquad \widetilde{H}_2 = mC_1\left(\omega^{-1/2} + \omega^{-3/2}\right), \qquad \widetilde{H}_3 = 0, \tag{13}$$

where $m = M(0)$.

Substituting the solution in (3)–(6), we obtain an exact solution of the nonlinear Maxwell equations

$$E^1 = \frac{2C_1x_3}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad E^2 = \frac{2C_1x_0}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad E^3 = -\frac{2C_1x_1}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}},$$

$$B^1 = -\frac{2C_1x_3}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad B^2 = \frac{2C_1x_0}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad B^3 = \frac{2C_1x_1}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}},$$

$$D^1 = \frac{2C_1mx_3}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad D^2 = \frac{2C_1mx_0}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad D^3 = -\frac{2C_1mx_1}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}},$$

$$H^1 = -\frac{2C_1mx_3}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad H^2 = \frac{2C_1mx_0}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad H^3 = \frac{2C_1mx_1}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}.$$

In analogous way we construct solutions invariant under the following subalgebras of Poincaré algebra: $\langle J_{03}, P_1, P_2\rangle$, $\langle J_{12}+\alpha P_0, P_1, P_2\rangle$, $\langle J_{03}+\alpha J_{12}, P_1, P_2\rangle$, $\langle P_0 - J_{01} - J_{13}, P_0 + P_3, P_2\rangle$, where $\alpha = \mathrm{const}$.

The existence of the operator of the classical symmetry is not a necessary condition for reduction of partial differential equations, as it is shown in [3, 4, 5]. It was proved in [6] that the conditional symmetry under involutive set of operators is the necessary and sufficient condition for reduction of partial differential equations by means of a corresponding ansatz.

Operators of nonpoint symmetry can be used to reduction of differential equations too. For simplicity we consider a second order equation of the form

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1x_1}, u_{x_1x_2}, u_{x_2x_2}) = 0. \tag{14}$$

We search for a solution of (14) as a solution of system

$$\frac{\partial u}{\partial x_1} = v^1(x_1, x_2, u), \qquad \frac{\partial u}{\partial x_2} = v^2(x_1, x_2, u). \tag{15}$$

Denote $u \equiv x_3$ and consider v^1, v^2 as a functions of variables x_1, x_2, x_3; $v^1, v^2 \in C^1\left(R^3, R^1\right)$. Then the compability condition of the system (15) takes the form

$$v_2^1 + v_3^1v^2 = v_1^2 + v_3^2v^1. \tag{16}$$

Any solution of (15) satisfies (14), if the following equality holds

$$F\left(x_1, x_2, x_3, v_1^1 + v_3^1v^1, v_2^1 + v_3^1v^2, v_2^2 + v_3^2v^2\right) = 0. \tag{17}$$

Thus the problem of construction of an ansatz of type (15) is reduced to the problem of finding of operators of classical and conditional symmetry of the system (16), (17).

Let us consider the infinitesimal operator of one–parametrical group of transfomations of independent and dependent variables

$$Q = \xi^1(x_1, x_2, u)\partial_{x_1} + \xi^2(x_1, x_2, u)\partial_{x_2} + \eta(x_1, x_2, u)\partial_u \tag{18}$$

and first prolongation of Q

$$\begin{aligned} \underset{1}{Q} = Q &+ \left(\eta_1 + \eta_u u_1 - u_1\left(\xi_1^1 + \xi_u^1u_1\right) - u_2\left(\xi_1^2 + \xi_u^2u_1\right)\right)\partial_{u_1} \\ &+ \left(\eta_2 + \eta_u u_2 - u_1\left(\xi_2^1 + \xi_u^1u_2\right) - u_2\left(\xi_2^2 + \xi_u^2u_2\right)\right)\partial_{u_2}, \end{aligned} \tag{19}$$

where lower indices designate differentiation of ξ^p, η $(p = 1, 2)$ with respect to the corresponding variables. We associate the operator Q'

$$\begin{aligned} Q' &= \xi^1(x_1, x_2, x_3)\partial_{x_1} + \xi^2(x_1, x_2, x_3)\partial_{x_2} + \xi^3(x_1, x_2, x_3)\partial_{x_3} \\ &\quad + \left(\xi^3_1 + \xi^3_3 v^1 - v^1\left(\xi^1_1 + \xi^1_3 v^1\right) - v^2\left(\xi^2_1 + \xi^2_3 v^1\right)\right)\partial_{v^1} \\ &\quad + \left(\xi^3_2 + \xi^3_3 v^2 - v^1\left(\xi^1_2 + \xi^1_3 v^2\right) - v^2\left(\xi^2_2 + \xi^2_3 v^2\right)\right)\partial_{v^2}, \\ \eta(x_1, x_2, x_3) &= \xi^3(x_1, x_2, x_3) \end{aligned} \tag{20}$$

with operator Q.

Theorem. *Let equation (14) be invariant with respect to one-parameter group generator Q (18). Then the operator Q' belongs to the invariance algebra of system (16), (17).*

Proof. Acting by the operator $\underset{1}{Q'}$ on the manifold (16), we obtain

$$\begin{aligned} &\xi^3_{12} + \xi^3_{32}v^1 - v^1\left(\xi^1_{12} + \xi^1_{32}v^1\right) - v^2\left(\xi^2_{12} + \xi^2_{32}v^1\right) + v^1_2\left(\xi^3_3 - \xi^1_1 - \xi^2_3 v^2 - 2v^1\xi^1_3\right) \\ &\quad - v^2_2\left(\xi^2_1 + \xi^2_3 v^1\right) - \xi^1_2 v^1_1 - \xi^2_2 v^1_2 - v^1_3\xi^3_2 + v^2\left(\xi^3_{13} + \xi^3_{33}v^1 - v^1\left(\xi^1_{13} + \xi^1_{33}v^1\right)\right. \\ &\quad \left. - v^2\left(\xi^2_{13} + \xi^2_{33}v^1\right) + v^1_3\left(\xi^3_3 - \xi^1_1 - 2v^1\xi^1_3 - \xi^2_3 v^2\right) - v^2_3\left(\xi^2_1 + \xi^2_3 v^1\right)\right. \\ &\quad \left. - v^1_1\xi^1_3 - v^1_2\xi^2_3 - v^1_3\xi^3_3\right) + \left(\xi^3_2 + \xi^3_3 v^2 - v^1\left(\xi^1_2 + \xi^1_3 v^2\right) - v^2\left(\xi^2_2 + \xi^2_3 v^2\right)\right)v^1_3 \\ &= \xi^3_{21} + \xi^3_{31}v^2 - v^1\left(\xi^1_{21} + \xi^1_{31}v^2\right) - v^2\left(\xi^2_{21} + \xi^2_{31}v^2\right) + v^2_1\left(\xi^3_3 - \xi^2_2 - \xi^1_3 v^1 - 2v^2\xi^2_3\right) \\ &\quad - v^1_1\left(\xi^1_2 + \xi^1_3 v^2\right) - \xi^1_1 v^2_1 - \xi^2_1 v^2_2 - v^2_3\xi^3_1 + v^1\left(\xi^3_{23} + \xi^3_{33}v^2 - v^1\left(\xi^1_{23} + \xi^1_{33}v^2\right)\right. \\ &\quad \left. - v^2\left(\xi^2_{23} + \xi^2_{33}v^2\right) + v^2_3\left(\xi^3_3 - \xi^2_2 - 2v^2\xi^2_3 - \xi^1_3 v^1\right) - v^1_3\left(\xi^1_2 + \xi^1_3 v^2\right) - v^2_1\xi^1_3\right. \\ &\quad \left. - v^2_2\xi^2_3 - v^2_3\xi^3_3\right) + \left(\xi^3_1 + \xi^3_3 v^1 - v^1\left(\xi^1_1 + \xi^1_3 v^1\right) - v^2\left(\xi^2_1 + \xi^2_3 v^1\right)\right)v^2_3. \end{aligned}$$

It is easy to verify, that this equality is fulfilled identically on the manifold (16). Thus we obtain

$$\underset{1}{Q'}\left(v^1_2 + v^1_3 v^2 - v^2_1 - v^2_3 v^1\right)\Big|_{v^1_2 + v^1_3 v^2 = v^2_1 + v^2_3 v^1} \equiv 0. \tag{21}$$

It is necessary to prove that the equation (17) admits operator Q' to prove the theorem. One property of coordinates of prolonged operators $\underset{2}{Q}$, $\underset{1}{Q'}$ is used for this purpose, where

$$\underset{2}{Q} = \underset{1}{Q} + \varepsilon^{u_{11}}\partial_{u_{11}} + \varepsilon^{u_{12}}\partial_{u_{12}} + \varepsilon^{u_{22}}\partial_{u_{22}}, \tag{22}$$

$$\begin{aligned} \varepsilon^{u_{ab}} &= \eta_{ab} + u_b\eta_{au} + u_a\eta_{bu} + u_a u_b\eta_{uu} + u_{ab}\eta_u - u_c\left(\xi^c_{ab} + u_b\xi^c_{au}\right) \\ &\quad - u_a u_c\left(\xi^c_{bu} + u_b\xi^c_{uu}\right) - u_{ac}\left(\xi^c_b + u_b\xi^c_u\right) - u_{cb}\left(\xi^c_a + u_a\xi^c_u\right) - u_{ab}u_c\xi^c_u, \end{aligned} \tag{23}$$

$a, b, c = 1, 2$, we mean summation over the index c.

Making the substitution $u = x_3$, $u_1 = v^1$, $u_2 = v^2$, $u_{11} = v^1_1 + v^1_3 v^1$, $u_{12} = v^1_2 + v^1_3 v^2$, $u_{22} = v^2_2 + v^2_3 v^2$ in (23), we obtain coefficients $\varepsilon'^{u_{12}}$, $\varepsilon'^{u_{22}}$ associated with $\varepsilon^{u_{11}}$, $\varepsilon^{u_{12}}$, $\varepsilon^{u_{22}}$. Then the following equalities

$$\varepsilon'^{u_{11}} = \underset{1}{Q'}\left(v^1_1 + v^1_3 v^1\right), \qquad \varepsilon'^{u_{12}} = \underset{1}{Q'}\left(v^1_2 + v^1_3 v^2\right), \qquad \varepsilon'^{u_{22}} = \underset{1}{Q'}\left(v^2_2 + v^2_3 v^2\right) \tag{24}$$

are fulfilled. The correctness of (24) is verified by direct calculations. Thus, derivatives u_{11}, u_{12}, u_{22} are transformed in the same way as the combinations $v^1_1 + v^1_3 v^1$, $v^1_2 + v^1_3 v^2$, $v^2_2 + v^2_3 v^2$ under the group transformations. From this it follows that the system (16), (17) is invariant under the group G'_1 provided G_1 is the invariance group of equation (14). ■

Thus, we conclude that the group of transformations admissible by equation (14) is not wider than the symmetry group of the system (16), (17). In the general case the symmetry group of system (16), (17) contains the invariance group of (14) as a subgroup. There is a possibility of expansion of this group by studying the symmetry properties of the system, as it is shown in [7, 8]. To obtain new solutions it is necessary to use the symmetry operators of system (16), (17), which are not prolongated operators of point symmetry of equation (14), as well as the operators of conditional symmetry of the system. By using this approach we constructed ansatzes reducing nonlinear equations to the system of ordinary differential equations. Integrating the reduced system we obtained new solutions of nonlinear heat and wave equations.

The method of conditional symmetry is generalised to the Lie–Bäcklund operators (see for example [9])

$$X = \eta(x, u, \ldots, \underset{r}{u})\partial_u. \tag{25}$$

Let us consider differential equations

$$U(x, u, \underset{1}{u}, \ldots, \underset{k}{u}) = 0, \tag{26}$$

where $u \in C^k\left(R^n, R^1\right)$, $x \in R^n$.

Definition. *Equation (26) is conditionaly invariant with respect to operator (25), if the following condition is satisfied*

$$\underset{k}{X}U\Big|_{[\eta=0]_\infty} = M, \tag{27}$$

where $M \neq 0$, $M\Big|_{[U=0]_r} = 0$, $[\eta = 0]_\infty$ *is a set of all differential consequences of the equation* $\eta = 0$, $[U = 0]_r$ *is a set of all differential consequences of* r*-th order of the equation* $U = 0$.

The corresponding ansatz, which is a solution of the equation

$$\eta(x, u, \ldots, \underset{r}{u}) = 0,$$

reduces equation (26) to the system of equations with smaller number of independent variables. Using this property we can construct exact solutions of partial differential equations.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 223–230.

Symmetry Groups and Conservation Laws in Structural Mechanics

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Recent results concerning the application of Lie transformation group methods to structural mechanics are presented. Focus is placed on the point Lie symmetries and conservation laws inherent to the Bernoulli–Euler and Timoshenko beam theories as well as to the Marguerre-von Kármán equations describing the large deflection of thin elastic shallow shells within the framework of the nonlinear Donnell–Mushtari–Vlasov theory.

1 Introduction

The present paper is concerned with the invariance properties (point Lie symmetries) of three classes of self-adjoint partial differential equations arising in structural mechanics – the dynamic beam equations of Bernouli–Euler and Timoshenko type governing vibration of beams on a variable elastic foundation and dynamic stability of fluid conveying pipes, and Marguerre-von Kármán equations describing the large deflection of thin isotropic elastic shallow shells subjected to an external transverse load and a nonuniform heating.

Once the invariance properties of a given differential equation are established, several important applications are available. First, it is possible to obtain classes of group-invariant solutions. For a self-adjoint equation another application of its symmetries arises since it is the Euler–Lagrange equation of a certain functional. If a symmetry group of such an equation turned out to be its variational symmetry as well, that is a symmetry of the associated functional, then Noether's theorem guarantees the existence of a conservation law for the solutions of this equation. Needless to recall the fundamental role of the conserved quantities and conservation laws (or the corresponding balance laws) for the natural sciences, however it is worthy to point out that the available conservation (balance) laws should not be overlooked in the examination of discontinuous solutions (acceleration waves, shock waves, etc.) or in the numerical analysis (when constructing finite difference schemes or verifying numerical results, for instance) of any system of differential equations of physical interest. It should be remarked also that the path-independent integrals (such as the well known J-, L- and M-integrals) related to the conservation laws are basic tools in fracture analysis of solids and structures.

Throughout this paper: Greek (Latin) indices have the range 1, 2 (1, 2, 3), unless explicitly stated otherwise, and the usual summation convention over a repeated index is employed. The k-th order partial derivatives of a dependent variable, say w, that is $\partial^k w/\partial x^{\alpha_1}\partial x^{\alpha_2}\ldots\partial x^{\alpha_k}$ $(k,\alpha_1,\alpha_2,\ldots,\alpha_k = 1,2,\ldots)$, are denoted either by $w_{\alpha_1\alpha_2\ldots\alpha_k}$ or $w_{x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_k}}$, where $x^1, x^2, \ldots$ are the independent variables. A similar notation is used for the partial derivatives of any other function, say f, of the independent variables but, in this case, the indices indicating the differentiation are preceded by a coma. D_α $(\alpha = 1,2,\ldots)$ denote the total derivative operators. For the basic notions and statements used in the group analysis of differential equations and variational problems see [1] or [2].

2 Symmetries and conservation laws of beam equations

Bernoulli–Euler beams. Consider the class of self-adjoint partial differential equations

$$\gamma w_{1111} + \chi^{\alpha\beta} w_{\alpha\beta} + \kappa(x) w = 0, \tag{1}$$

in two independent variables $x = (x^1, x^2)$ and one dependent variable $w(x)$, where $\gamma = \text{const} \neq 0$, $\chi^{\alpha\beta}$ are arbitrary constants, and $\kappa(x)$ is an arbitrary smooth function. Equations of this special type are used to study problems concerning dynamics and stability of both elastic beams resting on elastic foundations and pipes conveying fluid. In these cases, x^1 is associated with the spatial variable along the rod axis, x^2 – with the time, and w represents the transversal displacement field.

In [3] the point Lie symmetries of (1) are examined and the solution of the corresponding group-classification problem with respect to the arbitrary element $\{\gamma, \chi^{\alpha\beta}, \kappa(x)\}$ is given. Evidently, each equation of form (1) is invariant under the point Lie groups generated by the vector fields $X_0 = w\partial/\partial w$ and $X_u = u(x)\,\partial/\partial w$, where $u(x)$ is any smooth solution of the respective equation. The results of the group-classification are summarized in Table 1 below, where the equations invariant under larger groups are given through their coefficients together with the generators of the associated symmetry groups.

Table 1

#	Coefficients	Generators
1	$\kappa(x) = f(\beta^2 x^1 - \beta^1 x^2)$	$\beta^1 X_1 + \beta^2 X_2$
2	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = \left(\beta^2 + x^2\right)^{-2} f(y)$, $y = \left(\beta^2 + x^2\right)^{-1/2} \left\{\beta^1 + x^1 - \chi x^2\right\}$	$\{\beta^1 + 2\chi\beta^2\}X_1$ $+2\beta^2 X_2 + X_3$
3	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = \left(\beta^2 + x^2\right)^{-4/3} f(y)$, $y = \left(\beta^2 + x^2\right)^{-1/3} \left\{\beta^1 + 2x^1 - \left(\chi^{11}/\chi^{12}\right) x^2\right\}$	$\{\beta^1 + 3(\chi^{11}/\chi^{12})\beta^2\}X_1$ $+6\beta^2 X_2 + 2\tilde{X}_3$
4	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = \kappa_0 \left(\beta + x^2\right)^{-2}$,	X_1, $2\beta X_2 + X_3$
5	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = \kappa_0 \left(\beta + x^2\right)^{-4/3}$	X_1, $3\beta X_2 + \tilde{X}_3$
6	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0$, $\kappa(x) = \kappa_0 \left(\beta + x^1 - \chi x^2\right)^{-4}$	$\beta X_1 + X_3$, $\chi X_1 + X_2$
7	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0$, $\kappa(x) = \kappa_0 \left(\beta + 2x^1 - \left(\chi^{11}/\chi^{12}\right) x^2\right)^{-4}$	$\beta X_1 + 2\tilde{X}_3$, $(\chi^{11}/\chi^{12}) X_1 + 2X_2$
8	$\chi^{22} \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = \text{const}$	X_1, X_2
9	$\chi^{22} \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = \text{const} \neq 0$	X_1, X_2
10	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = 0$	X_1, X_2, X_3
11	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = 0$	X_1, X_2, $\tilde{X}_3$

Here f is an arbitrary function, β, β^1, β^2 are arbitrary real constants, $\chi = \chi^{12}/\chi^{22}$ and $X_\alpha = \partial/\partial x^\alpha$, $X_3 = \left(x^1 + \chi x^2\right) \partial/\partial x^1 + 2x^2 \partial/\partial x^2$, $\tilde{X}_3 = \left(x^1 + \chi x^2\right) \partial/\partial x^1 + 3x^2 \partial/\partial x^2$.

It is found [3] that all vector fields quoted under numbers 1, 3, 5, 7, 8, 9 and 11 generate variational symmetries of the respective equations of form (1), while in case # 2 variational symmetries are associated with $\{\beta^1 + 2\chi\beta^2\}X_1 + 2\beta^2 X_2 + X_3 + (1/2)X_0$, in case # 4 – with X_1

and $2\beta X_2 + X_3 + (1/2)X_0$, in case # 6 – with $\chi X_1 + X_2$ and $\beta X_1 + X_3 + (1/2)X_0$, and in case # 10 – with X_1, X_2 and $X_3 + (1/2)X_0$.

Once the variational symmetries are identified, we derive the corresponding conservation laws. They are listed in Table 2 in the same order as in Table 1 using the notation:

$$
\begin{aligned}
B^1_{(1)} &= -(1/2)\left\{\gamma(2w_1w_{111} - w_{11}^2) + \chi^{11}w_1^2 - \chi^{22}w_2^2 + \kappa w^2\right\} - (1/2)(\chi^{2\mu}ww_\mu)_{,2}\,,\\
B^2_{(1)} &= -\chi^{2\mu}w_1w_\mu + (1/2)(\chi^{2\mu}ww_\mu)_{,1}\,,\\
B^1_{(2)} &= -\chi^{1\mu}w_2w_\mu + \gamma(w_2w_{111} - w_{11}w_{12}) - (1/2)(\gamma w_1w_{11} - \chi^{1\mu}ww_\mu)_{,2}\,,\\
B^2_{(2)} &= -(1/2)\left\{\gamma w_{11}^2 + \chi^{22}w_2^2 - \chi^{11}w_1^2 + \kappa w^2\right\} + (1/2)(\gamma w_1w_{11} - \chi^{1\mu}ww_\mu)_{,1}\,,\\
B^\alpha_{(3)} &= \left\{x^1 + \chi x^2\right\}B^\alpha_{(1)} + 2x^2B^\alpha_{(2)} + \chi^{\alpha\mu}ww_\mu + (1/2)\gamma\delta^{1\alpha}(ww_{111} - w_1w_{11}),\\
\tilde{B}^\alpha_{(3)} &= \left\{x^1 + (\chi^{11}/\chi^{12})x^2\right\}B^\alpha_{(1)} + 3x^2B^\alpha_{(2)}\\
&\quad + (1/2)\left\{\chi^{\alpha\mu}ww_\mu + \delta^{1\alpha}(\chi^{11}ww_1 + 2\chi^{12}ww_2 - \gamma w_1w_{11})\right\}.
\end{aligned}
$$

Table 2

#	Conservation laws
1	$D_\alpha\left\{\beta^1B^\alpha_{(1)} + \beta^2B^\alpha_{(2)}\right\} = 0$
2	$D_\alpha\left\{\left(\beta^1 + 2\chi\beta^2\right)B^\alpha_{(1)} + 2\beta^2B^\alpha_{(2)} + B^\alpha_{(3)}\right\} = 0$
3	$D_\alpha\left\{\left(\beta^1 + 3\chi\beta^2\right)B^\alpha_{(1)} + 6\beta^2B^\alpha_{(2)} + 2\tilde{B}^\alpha_{(3)}\right\} = 0$
4	$D_\alpha B^\alpha_{(1)} = 0, \quad D_\alpha\left\{2\beta B^\alpha_{(2)} + B^\alpha_{(3)}\right\} = 0$
5	$D_\alpha B^\alpha_{(1)} = 0, \quad D_\alpha\left\{3\beta B^\alpha_{(2)} + \tilde{B}^\alpha_{(3)}\right\} = 0$
6	$D_\alpha\left\{\beta B^\alpha_{(1)} + B^\alpha_{(3)}\right\} = 0, \quad D_\alpha\left\{\chi B^\alpha_{(1)} + B^\alpha_{(2)}\right\} = 0$
7	$D_\alpha\left\{\beta B^\alpha_{(1)} + 2\tilde{B}^\alpha_{(3)}\right\} = 0, \quad D_\alpha\left\{(\chi^{11}/\chi^{12})B^\alpha_{(1)} + 2B^\alpha_{(2)}\right\} = 0$
8	$D_\alpha B^\alpha_{(1)} = 0, \quad D_\alpha B^\alpha_{(2)} = 0$
9	$D_\alpha B^\alpha_{(1)} = 0, \quad D_\alpha B^\alpha_{(2)} = 0$
10	$D_\alpha B^\alpha_{(1)} = 0, \quad D_\alpha B^\alpha_{(2)} = 0, \quad D_\alpha B^\alpha_{(3)} = 0$
11	$D_\alpha B^\alpha_{(1)} = 0, \quad D_\alpha B^\alpha_{(2)} = 0, \quad D_\alpha\tilde{B}^\alpha_{(3)} = 0$

In addition, each equation (1) admits conservation laws of form

$$D_\alpha\{\chi^{\alpha\mu}(uw_\mu - u_{,\mu}\,w) + \delta^{1\alpha}\gamma(uw_{111} + u_{,11}\,w_1 - u_{,111}\,w - u_{,1}\,w_{11})\} = 0,$$

$u(x)$ being any solution of the equation considered.

Timoshenko beams. The Timoshenko beam equations

$$\varrho J\varphi_{tt} = EJ\varphi_{xx} + nGA(w_x - \varphi), \qquad \varrho Aw_{tt} = nGA(w_{xx} - \varphi_x), \tag{2}$$

describe the motion of beams accounting for the buckling of the beam cross-section. They are two coupled second order linear partial differential equations in two independent variables – the time t and the coordinate along the beam axis x, the dependent variables being $w(x,t)$ and

$\varphi(x,t)$, associated with the transversal displacement of the beam axis and the rotation angle, respectively. In these equations ϱ is the density of the beam material, E – the modulus of elasticity, G – the shear modulus, J and A – the moment of inertia and the area of the beam cross-section, n – a coefficient related to the buckling of the cross-section.

The generator X_H of each one-parameter group H, admitted by (2), has the form

$$X_H = C_1 X_1 + C_2 X_2 + C_3 X_3 + X_S$$

(see [4, 5]), where C_i $(i = 1, 2, 3)$ are real constants, $(\tilde{w}, \tilde{\varphi})$ is a solution of (2), and

$$X_1 = \partial/\partial x, \quad X_2 = \partial/\partial t, \quad X_3 = w\partial/\partial w + \varphi\partial/\partial\varphi, \quad X_S = \tilde{w}(x,t)\partial/\partial w + \tilde{\varphi}(x,t)\partial/\partial\varphi.$$

Denoting $r_* = \operatorname{rank}(C_1, C_2, C_3 w + \tilde{w}(x,t),\ C_3\varphi + \tilde{\varphi}(x,t))$, $r_{**} = \operatorname{rank}(C_1, C_2)$, where rank$(\cdot)$ is the rank of the matrix in parentheses, the necessary conditions for existence of solutions to Timoshenko beam equations invariant under the transformations of H (i.e. H-invariant solutions) are of the form

$$r_* \le 2, \qquad r_{**} = r_*. \tag{3}$$

The inequality (3) holds for every choice of C_i, $\tilde{w}(x,t)$ and $\tilde{\varphi}(x,t)$, because r_* is either 1 or 0. There exist only two opportunities to satisfy the equality (3). They are $C_1^2 + C_2^2 > 0$, if $r_* = 1$ or $C_1^2 + C_2^2 = 0$, if $r_* = 0$. The only interesting alternative here is the first one, because if $r_* = 0$, the group H consists of the identity only. Thus, we proved the following.

Proposition 1 [4, 5]. *H-invariant solutions of the Timoshenko beam equations exist only if the group generator X_H incorporates at least one of the vector fields X_1 or X_2 associated with the translations along the independent variables.*

The invariant of the group H with generator X_H could be obtained, seeking for solutions to the equation $X_H(f) = 0$. Examining the cases $C_1 \ne 0$ and $C_1 = 0$, we found the most general form of the H-invariant solutions of (2) to be

$$w(x,t) = [\overline{w}(y) + W(x,t)]\Sigma, \qquad \varphi(x,t) = [\overline{\varphi}(y) + \Phi(x,t)]\Sigma, \tag{4}$$

where $y = C_2 x - C_1 t$ and the functions $W(x,t)$ and $\Phi(x,t)$ are solutions of the equations

$$C_1 W_x + C_2 W_t = \tilde{w}(x,t)\Sigma^{-1}, \qquad C_1 \Phi_x + C_2 \Phi_t = \tilde{\varphi}(x,t)\Sigma^{-1}. \tag{5}$$

In (4) and (5) we denote $\Sigma = \exp(C_3 x/C_1)$ if $C_1 \ne 0$, otherwise $\Sigma = \exp(C_3 t/C_2)$. Equations (5) are first-order linear partial differential equations, so it is a simple matter to obtain their solutions once C_i, $\tilde{w}(x,t)$ and $\tilde{\varphi}(x,t)$ are specified.

The following basic conservation laws of densities A^t and fluxes A^x, that is

$$\partial A^t/\partial t + \partial A^x/\partial x = 0,$$

are found to hold on the smooth solutions of the Timoshenko beam equations [4, 5].

Table 3

w - translations		**transversal linear momentum**
$X_w = \frac{\partial}{\partial w}$	$A^t_w = \rho A w_t,\ A^x_w = nGA(w_x - \varphi)$	
x - translations		**wave momentum**
$X_1 = \frac{\partial}{\partial x}$	$A^t_1 = \rho A w_x w_t + \rho J \varphi_x \varphi_t,\ A^x_1 = -\mathcal{E} - nGA(w_x - \varphi)\varphi$	
time - translations		**energy**
$X_2 = \frac{\partial}{\partial t}$	$A^t_2 = \mathcal{E} = (1/2)\left\{EJ\varphi_x^2 + nGA(w_x - \varphi)^2 + \rho A w_t^2 + \rho J \varphi_t^2\right\}$ $A^x_2 = -nGA(w_x - \varphi)w_t - EJ\varphi_x\varphi_t$	
		reciprocity relation
$X_S = \tilde{w}\frac{\partial}{\partial w} + \tilde{\varphi}\frac{\partial}{\partial\varphi}$	$\tilde{A}^t = \rho A(w\tilde{w}_t - w_t\tilde{w}) + \rho J(\varphi\tilde{\varphi}_t - \varphi_t\tilde{\varphi})$ $\tilde{A}^x = EJ(\varphi_x\tilde{\varphi} - \varphi\tilde{\varphi}_x) + nGA\{(w_x - \varphi)\tilde{w} - w(\tilde{w}_x - \tilde{\varphi})\}$	

3 Marguerre-von Kármán equations

Marguerre-von Kármán (MvK) equations (see e.g. [6, 7, 8]) describe the large deflection of thin isotropic elastic shallow shells. They can be written in the form [7, 8]:

$$\begin{aligned}
&D\Delta^2 W - \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} W_{\alpha\beta}\varPhi_{\mu\nu} = P,\\
&\frac{1}{Eh}\Delta^2\varPhi + \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} W_{\alpha\beta}W_{\mu\nu} = Q.
\end{aligned} \tag{6}$$

Here, the independent variables are the coordinates $x = \left(x^1, x^2\right)$ on the shell middle-surface F supposed to be given by the equation $z = f\left(x^1, x^2\right)$, $\left(x^1, x^2\right) \in \Omega \subset \mathbf{R}^2$, where $\left(x^1, x^2, z\right)$ is a fixed right-handed rectangular Cartesian coordinate system in the 3-dimensional Euclidean space in which the middle-surface F of the shell is embedded, and $f : \mathbf{R}^2 \to \mathbf{R}$ is a smooth function on a certain domain of interest Ω. The dependent variables are Airy's stress function $\varPhi$, and $W = w + f$, where w is the transversal displacement function. In (6): $\varepsilon^{\alpha\beta}$ is the alternating tensor of F; E, h and $D = Eh^3/\left[12\left(1-\nu^2\right)\right]$ are Young's modulus, thickness and bending rigidity of the shell, respectively, ν being Poisson's ratio; Δ is the Laplace–Beltrami operator on F;

$$P = D\delta^{\alpha\beta}\delta^{\mu\nu} f_{,\alpha\beta\mu\nu} + p, \qquad Q = \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} f_{,\alpha\beta}f_{,\mu\nu} + q$$

(functions p and q appear when the shell is subjected to an external transversal load and nonuniform heating). System (6) includes as a special case, with $f_{,\alpha\beta} = 0$, the well-known von Kármán equations for large deflection of plates.

Actually (6) describe the state of equilibrium of the shell, but introducing, according to d'Alembert principle, the inertia force $-\rho w_{33} = -\rho W_{33}$ in the right-hand side of the first MvK equation, w_{33} being the second derivative of the displacement field with respect to the time $t \equiv x^3$ and ρ – the mass per unit area of the shell middle-surface, one can extend (6) to describe the dynamic behaviour of shells. In this case we will speak about the time-dependent MvK equations, otherwise (6) will be referred to as the time-independent MvK equations. In both cases, the moment tensor $M^{\alpha\beta}$, membrane stress tensor $N^{\alpha\beta}$, and shear-force vector Q^{α} are given in terms of W and $\varPhi$ by the expressions

$$\begin{aligned}
&M^{\alpha\beta} = D\left\{(1-\nu)\delta^{\alpha\mu}\delta^{\beta\nu} + \nu\delta^{\alpha\beta}\delta^{\mu\nu}\right\}\left\{W_{\mu\nu} - f_{,\mu\nu}\right\},\\
&N^{\alpha\beta} = \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\varPhi_{\mu\nu}, \qquad Q^{\alpha} = M^{\alpha\mu}_{,\mu} + N^{\alpha\mu}\left\{W_{\mu} - f_{,\mu}\right\}.
\end{aligned}$$

Symmetry groups. The following is known [11] for the symmetry groups of the homogeneous time-independent and time-dependent MvK equations.

Proposition 2. *The homogeneous time-independent MvK equations admit the group $G_{(S)}$ generated by the basic vector fields (operators):*

$$\begin{aligned}
&Y_1 = \partial/\partial W,\ Y_2 = \partial/\partial x^1,\ Y_3 = \partial/\partial x^2,\ Y_4 = x^2\partial/\partial x^1 - x^1\partial/\partial x^2,\ Y_5 = x^1\partial/\partial\Phi,\\
&Y_6 = x^2\partial/\partial\varPhi,\ Y_7 = \partial/\partial\varPhi,\ Y_8 = x^1\partial/\partial W,\ Y_9 = x^2\partial/\partial W,\ Y_{10} = x^1\partial/\partial x^1 + x^2\partial/\partial x^2.
\end{aligned}$$

Proposition 3. *The homogeneous time-dependent MvK equations admit the group $G_{(D)}$ generated by the basic vector fields:*

$$\begin{aligned}
&X_1 = Y_1,\ X_2 = Y_2,\ X_3 = Y_3,\ X_4 = \partial/\partial x^3,\ X_5 = x^1\partial/\partial x^1 + x^2\partial/\partial x^2 + 2x^3\partial/\partial x^3,\\
&X_6 = Y_4,\ X_7 = x^1\partial/\partial W,\ X_8 = x^2\partial/\partial W,\ X_9 = x^3\partial/\partial W,\ X_{10} = x^1x^3\partial/\partial W,\\
&X_{11} = x^2x^3\partial/\partial W,\ X_{12} = x^1 f(x^3)\partial/\partial\varPhi,\ X_{13} = x^2 g(x^3)\partial/\partial\varPhi,\ X_{14} = h(x^3)\partial/\partial\varPhi,
\end{aligned}$$

where f, g and h are arbitrary functions depending on the time only.

As for the symmetries of the nonhomogeneous MvK equations, we proved that:

Proposition 4. *A nonhomogeneous time-independent MvK system is invariant under a vector field Y iff $Y = c^j Y_j$ $(j = 1, \ldots, 10)$, where c^j are real constants, and*

$$2P\xi^{\mu}_{,\mu} + \xi^{\mu} P_{,\mu} = 0, \qquad 2Q\xi^{\mu}_{,\mu} + \xi^{\mu} Q_{,\mu} = 0, \qquad \xi^{\alpha} = Y(x^{\alpha}), \tag{7}$$

Y being regarded as an operator acting on the functions $\zeta : \Omega \to \mathbf{R}$, $\Omega \subset \mathbf{R}^2$.

Proposition 5. *A nonhomogeneous time-dependent MvK system is invariant under a vector field X iff $X = C^j X_j$ $(j = 1, \ldots, 14)$, where C^j are real constants, and*

$$P\xi^{i}_{,i} + \xi^{i} P_{,i} = 0, \qquad Q\xi^{i}_{,i} + \xi^{i} Q_{,i} = 0, \qquad \xi^{i} = X(x^{i}), \tag{8}$$

X being regarded as an operator acting on the functions $\chi : \Omega \times T \to \mathbf{R}$, $\Omega \subset \mathbf{R}^2$, $T \subset \mathbf{R}$.

The above Propositions imply the following group classification results.

Proposition 6. *The time-independent MvK equations admit a group G iff G is generated by a vector field $Y = c^j Y_j$ $(j = 1, \ldots, 10)$ and the right-hand sides P and Q are invariants of G (when $c^{10} = 0$) or eigenfunctions (when $c^{10} \neq 0$) of its generator Y.*

Proposition 7. *The time-dependent MvK equations admit a group G iff G is generated by a vector field $X = C^j X_j$ $(j = 1, \ldots, 14)$ and the right-hand sides P and Q are invariants of G (when $C^5 = 0$) or eigenfunctions (when $C^5 \neq 0$) of its generator X.*

Conservation laws. Both the time-independent and time-dependent MvK equations constitute self-adjoint systems and are the Euler–Lagrange equations for the functionals

$$I^{(S)}[W, \Phi] = \int\int\int \Pi \, dx^1 dx^2 \quad \text{and} \quad I^{(D)}[W, \Phi] = \int\int\int (\mathrm{T} - \Pi) \, dx^1 dx^2 dx^3,$$

$$\Pi = \frac{D}{2}\left\{(\Delta W)^2 - (1-\nu)\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} W_{\alpha\beta} W_{\mu\nu}\right\} + \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\Phi_{\alpha\beta} W_{\mu} W_{\nu}$$
$$-\frac{1}{2Eh}\left\{(\Delta\Phi)^2 - (1+\nu)\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\Phi_{\alpha\beta}\Phi_{\mu\nu}\right\} - PW - Q\Phi,$$
$$\mathrm{T} = \frac{\rho}{2}(W_3)^2,$$

Π and T being the strain and kinetic energies per unit area of the shell middle-surface.

In [10], the variational symmetries of the above functionals with $P = Q = 0$ are established and the associated conservation laws admitted by the smooth solutions of the homogeneous MvK equations are presented (see Appendices A and B). In particular, each such conservation law for the time-dependent MvK equations is a linear combination of the basic linearly independent conservation laws

$$\partial\Psi_{(j)}/\partial x^3 + \partial P^{\mu}_{(j)}/\partial x^{\mu} = 0 \qquad (j = 1, 2, \ldots, 14)$$

whose densities $\Psi_{(j)}$ and fluxes $P^{\mu}_{(j)}$ are presented (together with the generators of the respective symmetries) on the Table 4 below in terms of Q^{α}, $M^{\alpha\beta}$, $G^{\alpha\beta}$ and F^{α},

$$G^{\alpha\beta} = \frac{1}{Eh}\left\{(1+\nu)\delta^{\alpha\mu}\delta^{\beta\nu} - \nu\delta^{\alpha\beta}\delta^{\mu\nu}\right\}\Phi_{\mu\nu} - \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} w_{\mu} w_{\nu}, \qquad F^{\alpha} = G^{\alpha\nu}_{,\nu}.$$

Table 4

w - translations	**transversal linear momentum** (first MvK equation)
$X_1 = \frac{\partial}{\partial w}$	$P^\alpha_{(1)} = -Q^\alpha,\ \Psi_{(1)} = \rho\, w_3$
Φ - translations	**compatibility condition** (second MvK equation)
$X_{14} = \frac{\partial}{\partial \Phi}$	$P^\alpha_{(14)} = F^\alpha,\ \Psi_{(14)} = 0$
time - translations	**energy**
$X_4 = \frac{\partial}{\partial x^3}$	$P^\alpha_{(4)} = -w_3 Q^\alpha - \Phi_3 F^\alpha + w_{3\beta} M^{\alpha\beta} + \Phi_{3\beta} G^{\alpha\beta}$ $\Psi_{(4)} = \mathrm{T} + \Pi$
x^1 & x^2- translations	**wave momentum**
$X_2 = \frac{\partial}{\partial x^1}$	$P^\alpha_{(2)} = \delta^{\alpha 1}(\mathrm{T} - \Pi) + w_1 Q^\alpha + \Phi_1 F^\alpha - w_{1\beta} M^{\alpha\beta} - \Phi_{1\beta} G^{\alpha\beta}$ $\Psi_{(2)} = -\rho\, w_1 w_3$
$X_3 = \frac{\partial}{\partial x^2}$	$P^\alpha_{(3)} = \delta^{\alpha 2}(\mathrm{T} - \Pi) + w_2 Q^\alpha + \Phi_2 F^\alpha - w_{2\beta} M^{\alpha\beta} - \Phi_{2\beta} G^{\alpha\beta}$ $\Psi_{(3)} = -\rho\, w_2 w_3$
rotations	**moment of the wave momentum**
$X_6 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$	$P^\alpha_{(6)} = x^2 P^\alpha_{(2)} - x^1 P^\alpha_{(3)} + \varepsilon^{\ \mu}_{\nu} w_\mu M^{\alpha\nu} + \varepsilon^{\ \mu}_{\nu} \Phi_\mu G^{\alpha\nu}$ $\Psi_{(6)} = x^2\, \Psi_{(2)} - x^1\, \Psi_{(3)}$
rigid body rotations	**angular momentum**
$X_7 = x^1 \frac{\partial}{\partial w}$	$P^\alpha_{(7)} = M^{\alpha 1} - x^1 Q^\alpha + w\varepsilon^{\alpha\nu}\, \Phi_{\nu 2},\ \Psi_{(7)} = \rho x^1 w_3$
$X_8 = x^2 \frac{\partial}{\partial w}$	$P^\alpha_{(8)} = M^{\alpha 2} - x^2 Q^\alpha + w\varepsilon^{\nu\alpha}\, \Phi_{\nu 1},\ \Psi_{(8)} = \rho x^2 w_3$
scaling	
$X_5 = x^\mu \frac{\partial}{\partial x^\mu} + 2x^3 \frac{\partial}{\partial x^3}$	$P^\alpha_{(5)} = x^1 P^\alpha_{(2)} + x^2 P^\alpha_{(3)} - 2x^3 P^\alpha_{(4)} - w_\beta M^{\alpha\beta} - \Phi_\beta G^{\alpha\beta}$ $\Psi_{(5)} = x^1\, \Psi_{(2)} + x^2\, \Psi_{(3)} - 2x^3\, \Psi_{(4)}$
Galilean boost	**center-of-mass theorem**
$X_9 = x^3 \frac{\partial}{\partial w}$	$P^\alpha_{(9)} = -x^3 Q^\alpha,\ \Psi_{(9)} = \rho\,(x^3 w_3 - w)$
$X_{10} = x^1 x^3 \frac{\partial}{\partial w}$	$P^\alpha_{(10)} = x^3 P^\alpha_{(7)},\ \Psi_{(10)} = x^1\, \Psi_{(9)}$
$X_{11} = x^2 x^3 \frac{\partial}{\partial w}$	$P^\alpha_{(11)} = x^3 P^\alpha_{(8)},\ \Psi_{(11)} = x^2\, \Psi_{(9)}$
$X_{12} = x^1 \frac{\partial}{\partial \Phi}$	$P^\alpha_{(12)} = x^1 F^\alpha - G^{\alpha 1},\ \Psi_{(12)} = 0$
$X_{13} = x^2 \frac{\partial}{\partial \Phi}$	$P^\alpha_{(13)} = x^2 F^\alpha - G^{\alpha 2},\ \Psi_{(13)} = 0$

The following statements [6] hold for the nonhomogeneous MvK equations.

Proposition 8. *A conservation law of flux* $A^\alpha_{(j)}$ *and characteristic* $\Lambda^\alpha_{(j)}$ $(j = 1, \dots, 9)$ *admitted by the smooth solutions of the homogeneous time-independent MvK equations takes the form*

$$A^\mu_{(j),\mu} + S_{(j)} = 0, \qquad S_{(j)} = -\Lambda^1_{(j)} P - \Lambda^2_{(j)} Q, \tag{9}$$

on the smooth solutions of the non-homogeneous time-independent MvK equations;

$$S_{(j)} = \widetilde{A}^\mu_{(j),\mu},$$

iff (7) hold, and then (9) can be written as a divergence free expression (i.e. it becomes a proper conservation law), otherwise it has supply (production) $S_{(j)}$.

Proposition 9. *Each conservation law of density $\Psi_{(i)}$, flux $P^{\alpha}_{(i)}$ and characteristic $\Lambda^{\alpha}_{(i)}$ ($i = 1, \ldots, 14$) admitted by the smooth solutions of the homogeneous time-dependent MvK equations takes the form*

$$\Psi_{(i),3} + P^{\mu}_{(i),\mu} + S_{(i)} = 0, \qquad S_{(i)} = -\Lambda^{1}_{(i)} P - \Lambda^{2}_{(i)} Q, \tag{10}$$

on the smooth solutions of the non-homogeneous time-dependent MvK equations;

$$S_{(i)} = \widetilde{\Psi}_{(i),3} + \widetilde{P}^{\mu}_{(i),\mu},$$

iff (8) hold, and hence (10) becomes a proper conservation law, otherwise it has supply (production) $S_{(i)}$.

Acknowledgments

This research was partly supported by Contract No. TH 564/95 with the NSF, Bulgaria.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 231–238.

Modelling System for Relaxing Media. Symmetry, Restrictions and Attractive Features of Invariant Solutions

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A model describing non-equilibrium processes in relaxing media is considered. Restrictions arising from the symmetry principles and the second law of thermodynamics are stated. System of ODE describing a set of travelling wave solutions is obtained via group theory reduction. Bifurcation analysis of this system reveals the existence of periodic invariant solutions as well as limiting to them solitary wave solutions. These families play the role of intermediate asymptotics for a wide set of Cauchy and boundary value problems

1 Introduction

Analysis of experimental studies of multi-component media subjected to shock loading [1] enables one to conclude that some internal state variations are possible at constant values of the "external" parameters (temperature T, pressure p, mass velocity u, etc.). Phenomena arising then as an afteraction of relaxing processes might be formally described by introducion of an internal variable λ, expressing deviation of the system from the state of complete thermodynamic equilibrium and formally obeying the chemical kinetics equation with unknown affinity A of the relaxing process. Connection between the "internal" and "external" variables is stated by the second law of thermodynamics, written in the Gibbs form [2]:

$$TdS = dE - p\rho^{-2}d\rho + Ad\lambda. \tag{1}$$

To describe the long nonlinear waves propagation in such media, the following system may be proposed [3]:

$$\begin{gathered}\rho\left(\frac{\partial u^i}{\partial t} + u^j\frac{\partial u^i}{\partial x^j}\right) + \frac{\partial p}{\partial x^i} = 0, \qquad \frac{\partial \rho}{\partial t} + u^i\frac{\partial \rho}{\partial x^i} + \rho\frac{\partial u^i}{\partial x^i} = 0,\\ \frac{\partial p}{\partial t} + u^i\frac{\partial p}{\partial x^i} + M\frac{\partial u^i}{\partial x^i} = N, \qquad \frac{\partial \lambda}{\partial t} + u^j\frac{\partial \lambda}{\partial x^j} = Q \equiv aA,\end{gathered} \tag{2}$$

where M, and N are functions connected with internal energy E and affinity of the relaxing processes $A = a^{-1}Q$ by means of the relations

$$M = (p - \rho^2 E_\rho)/(\rho E_p), \qquad N = -E_\lambda Q/E_p. \tag{3}$$

Here and henceforth lower indices mean partial derivatives with corresponding to subsequent variables.

The aim of this work is to show that arbitrainess in the choice of functions E and A may be reduced to the great extent if we impose restrictions arising from symmetry principles and the second law of thermodynamics. Another goal is to state the conditions leading to the invariant autowave solutions appearance as well as to study their attractive features.

2 Group theory classification of system (2)

Let us study the symmetry of system (2), that contains three unknown functions, linked together by means of equations (3). We look for infinitesimal operators (IFO), having the following form:

$$X = \xi^{\alpha}\frac{\partial}{\partial x^{\alpha}} + \eta^{i}\frac{\partial}{\partial u^{i}} + \tau\frac{\partial}{\partial p} + \theta\frac{\partial}{\partial \rho} + \gamma\frac{\partial}{\partial \lambda}, \tag{4}$$

where ξ^{α}, η^{i}, τ, θ, γ depend on x^{α}, u^{i}, p, ρ, λ, $\alpha = 0,\ldots,n$, $i = 1,\ldots,n$ (we identify variable x^0 with t). The procedure of "splitting" of a PDE system, arising from action of the first extension of the operator (4) on system (2), is very similar to that described in [4]. It is possible to select a subsystem defining coordinates ξ^{μ}, η^{i}, τ and θ:

$$\begin{gathered} \tau = p[m-(2+n)gt] + f(t), \qquad \theta = \rho(\alpha - ngt), \qquad \xi^0 = gt^2 + bt + h, \\ \xi^i = (gx^i + l^i)t + cx^i + \Sigma a^i_j x^j + q^i, \qquad \eta^i = gx^i + l^i + u^i(c-b-gt) + \Sigma a^i_j u^j, \end{gathered} \tag{5}$$

where g, b, h, l^i, m, c are arbitrary constants, $a^i_j = -a^j_i$, $\alpha = 2(b-c)+m$, $f(t)$ is an arbitrary function. Besides, it may be shown that $\gamma^j_{x^i} = \gamma^j_{u^k} = 0$. The remaining part of the system, containing the unknown functions M, N and Q, is presented in the following form:

$$\hat{Z}\begin{pmatrix} M \\ N \\ Q \end{pmatrix} \equiv \left(\theta\frac{\partial}{\partial\rho} + \tau\frac{\partial}{\partial p} + \gamma\frac{\partial}{\partial\lambda}\right)\begin{pmatrix} M \\ N \\ Q \end{pmatrix} = \begin{bmatrix} \tau_p M \\ \tau_0 + (\tau_p - \xi^0_0)N + ngM \\ \gamma_0 + \gamma_p N + \gamma_\lambda Q - \xi^0_0 Q \end{bmatrix}. \tag{6}$$

Note that system (6) does not contain the parameters h, l^i, q^i, a^i_j . Therefore, for arbitrary functions M, N and Q system (2) admits operators $\hat{P}_0$, $\hat{P}_i$, $\hat{J}_{ab}$ and $\hat{G}_a$, $i, a, b = 1,\ldots,n$, forming the standard representation of the Galilei algebra $AG(n)$ [4]. We did not succeed in obtaining the general solution of system (6), but knowledge of particular one enables to prove that symmetry algebra is infinite-dimensional.

Theorem 1. *Let $E = p/[(\nu-1)\rho] + q\lambda$, $\nu = (n+2)/n$, $Q = \rho^{\sigma\nu-1}\psi(\omega)$, where ψ is arbitrary function of $\omega = p/\rho^{\nu}$. Then system (2), in addition to $AG(n)$, admits the following operators:*

$$\begin{aligned} K_1 &= t^2\frac{\partial}{\partial t} + tx^i\frac{\partial}{\partial x^i} + (x^i - tu^i)\frac{\partial}{\partial u^i} - nt\rho\frac{\partial}{\partial\rho} \\ &\quad - (2+n)tp\frac{\partial}{\partial p} + \left\{\int\frac{\rho f(\rho) - 2p}{\rho^{\sigma\nu+1}\psi(\omega)}dp + t[2p/\rho - f(\rho)]\right\}\frac{\partial}{\partial\lambda}, \\ \hat{D} &= 2t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i} - u^i\frac{\partial}{\partial u^i} - (n+2)p\frac{\partial}{\partial p} - n\rho\frac{\partial}{\partial\rho} - \frac{2p}{\rho}\frac{\partial}{\partial\lambda}, \\ L &= \Gamma(\rho, \lambda + p/\rho)\frac{\partial}{\partial\lambda}, \end{aligned} \tag{7}$$

where $i, a, b = 1,\ldots,n$. Note that the latter two expressions contain arbitrary functions $f(\rho)$ and $\Gamma(\rho, \lambda + p/\rho)$.

If to decline the requirement of maximal symmetry existence, then the problem of group theory classification of system (2) may be effectively solved. The results obtained are presented in Table 1 where the following notation is used:

$$\hat{D}_1 = t\frac{\partial}{\partial t} - u^i\frac{\partial}{\partial u^i} + 2\rho\frac{\partial}{\partial\rho}, \qquad \hat{D}_2 = x_i\frac{\partial}{\partial x^i} + u^i\frac{\partial}{\partial u^i} - 2\rho\frac{\partial}{\partial\rho},$$

$$\hat{D}_3 = \rho\frac{\partial}{\partial\rho} + p\frac{\partial}{\partial p}, \qquad \hat{L}_1 = \partial/\partial\rho, \qquad \hat{L}_2 = \partial/\partial p, \qquad \hat{L}_3 = \partial/\partial\lambda.$$

It is seen from the analysis of Table 1 that symmetry extension takes place in many cases including those for which E, Q are arbitrary functions of the invariants of subsequent ISO and this gives way for effective use of qualitative methods in the relaxing media models investigations.

Table 1

E, Q	IFO
$E = p\rho^{-1}f(\omega)$, $Q = \lambda g(\omega)$, $\omega = p\lambda^{\kappa}$	$\hat{Z}_1 = \lambda\hat{L}_3 - \kappa\hat{D}_3$, $\hat{Z}_2 = \hat{D}_2$
$E = \rho^{-1}[f(\omega) - p]$, $Q = e^{-p}g(\omega)$, $\omega = \lambda - \sigma\ln\rho$,	$\hat{Z}_1 = \hat{D}_1 + \hat{D}_2 + \hat{L}_2$, $\hat{Z}_2 = \hat{D}_2 - 2\sigma\hat{L}_3$
$E = \rho^{-1}[f(\omega) - p]$, $Q = \rho^{\nu}e^{p}g(\omega)$, $\omega = \lambda/\rho^{\sigma}$	$\hat{Z}_1 = \hat{D}_1 + \hat{D}_2 - \hat{L}_2$, $\hat{Z}_2 = \hat{D}_2 + 2(\sigma - \nu)\hat{L}_2 + 2\sigma\lambda\hat{L}_3$
$E = p\rho^{-1}f(\omega)$, $\omega = \lambda - \tau\ln\rho$, $Q = p^{\mu}\rho^{\nu}g(\omega)$,	$\hat{Z}_1 = \nu\hat{D}_1 + (\nu + 1/2)\hat{D}_2 - \tau\hat{L}_3$, $\hat{Z}_2 = (\mu + \nu)\hat{D}_2 + 2\nu\hat{D}_3 - 2\mu\tau L_3$
$E = \rho^{\xi-1}F(\omega_1, \omega_2)$, $Q = \rho^{\sigma-\beta}G(\omega_1, \omega_2)$ $\omega_1 = p/\rho^{\xi}$, $\omega_2 = \lambda/\rho^{\sigma}$	$\hat{Z} = 2\beta\hat{D}_1 + (2\beta + \xi - 1)\hat{D}_2 + 2\xi\hat{D}_3 + 2\sigma\lambda\hat{L}_3$
$E = \rho^{-1}[F(\omega_1, \omega_2) - \tau\ln\rho]$, $Q = \rho^{\sigma-\beta}G(\omega_1, \omega_2)$, $\omega_1 = \rho^{\tau}e^{-p}$, $\omega_2 = \lambda\rho^{-\sigma}$,	$\hat{Z} = 2\beta\hat{D}_1 + 2\sigma\lambda\hat{L}_3 + (2\beta - 1)\hat{D}_2 + 2\tau\hat{L}_2$
$E = \rho^{\xi-1}F(\omega_1, \omega_2)$, $Q = \rho^{-\beta(1+\xi)}G(\omega_1, \omega_2)$, $\omega_1 = p/\rho^{\xi}$, $\omega_2 = \lambda - \delta\ln\rho$	$\hat{Z} = 2\beta(1 + \xi)\hat{D}_1 + 2\xi\hat{D}_3 + 2\delta\hat{L}_3 +$ $+ [2\beta(1 + \xi) + (\xi - 1)]\hat{D}_2$
$E = pF(\rho, \omega)$, $Q = p^{-\beta}G(\rho, \omega)$, $\omega = \lambda - \tau\ln p$	$\hat{Z} = \beta\hat{D}_1 + (\beta + 1/2)\hat{D}_2 + \hat{D}_3 + \tau\hat{L}_3$
$E = F(\rho, \omega) - p/\rho$, $Q = e^{-p}G(\rho, \omega)$, $\omega = \lambda - \nu p$	$\hat{Z} = \hat{D}_1 + \hat{D}_2 + \hat{L}_2 + \nu\hat{L}_3$
$E = F(\rho, \omega) - p/\rho$, $Q = G(\rho, \omega)$, $\omega = \lambda - \nu p$	$\hat{Z} = \hat{L}_2 + \nu\hat{L}_3$
$E = \rho^{\tau-1}F(\lambda, \omega)$, $Q = p^{-\beta}G(\lambda, \omega)$, $\omega = p/\rho^{\tau}$,	$\hat{Z} = 2\tau\beta(\hat{D}_1 + \hat{D}_2) - (1 - \tau)\hat{D}_2 + 2\tau\hat{D}_3$
$E = \rho^{\tau-1}F(\lambda, \omega)$, $Q = G(\lambda, \omega)$, $\omega = p/\rho^{\tau}$,	$\hat{Z} = (1 - \tau)\hat{D}_2 - 2\tau\hat{D}_3$
$E = \rho^{-1}[F(\lambda, \omega) - \tau(1 + \ln\rho)]$, $Q = e^{-p}G(\lambda, \omega)$, $\omega = \rho^{\tau}e^{-p}$	$\hat{Z} = 2\tau(\hat{D}_1 + \hat{L}_2) + (2\tau - 1)\hat{D}_2$
$E = pF(\rho, \lambda)$, $Q = p^{-\beta}G(\rho, \lambda)$	$Z = \hat{D}_3 + \beta\hat{D}_1 + (\beta + 1/2)\hat{D}_2$
$E = \rho^{-1}F(p, \lambda)$, $Q = \rho^{\tau}G(p, \lambda)$	$\hat{Z} = 2\tau\hat{D}_1 + (2\tau + 1)\hat{D}_2$

3 Restrictions imposed by the second law of thermodynamics

Let us consider the governing functions, defining in the following form, widely used in applications [5]:

$$E = \frac{p}{(\sigma - 1)\rho} - h(\lambda), \qquad Q \equiv aA = a\, g(\lambda)\phi(p, \rho). \tag{8}$$

Employing the consequences of the second law of thermodynamics (1), we may obtain some restrictions on functions $g(\lambda)$, $h(\lambda)$ and $\phi(p, \rho)$.

Equating partial derivative of the entropy function S with the corresponding terms standing at the RHS of the formula (1), we obtain:

$$(S_p)_{V,\lambda} = T^{-1}\,(E_p)_{V,\lambda}\,, \tag{9}$$

$$(S_V)_{p,\lambda} = T^{-1}[(E_V)_{p,\lambda} + p], \tag{10}$$

$$(S_\lambda)_{V,p} = T^{-1}[(E_\lambda)_{V,p} + A], \tag{11}$$

where $V = \rho^{-1}$. Comparison of the mixed partial derivatives $S_{p,V}$, $S_{p,\lambda}$ and $S_{V,\lambda}$, calculated from (9)–(11), gives the following expressions for T and S:

$$T = V^{-1/\Gamma}\Phi(\Omega), \qquad \Omega = p\,V^\sigma, \tag{12}$$

$$S = S_1(\lambda) + \Gamma \int \frac{d\Omega}{\Phi(\Omega,\lambda)}, \tag{13}$$

where $\Gamma = (\sigma - 1)^{-1}$. Functions $\Phi(\Omega,\lambda)$ and $S_1(\lambda)$ are connected with functions defining E and Q by means of the equation

$$V^{-1/\Gamma}\Phi(\Omega,\lambda)\left(\dot{S}_1(\lambda) - \Gamma\int \frac{d\Omega}{\Phi(\Omega,\lambda)}\Phi_\lambda\right) = g(\lambda)\phi(p,\rho) - \dot{h}(\lambda). \tag{14}$$

Assuming that $g(\lambda) = \dot{h}(\lambda)/m$, $\phi(p,\rho) = m + \rho^{1/\Gamma}\theta(\Omega)$ and $\Phi(\Omega,\lambda) = f(\lambda)R(\Omega)$ we obtain the solution

$$A = g(\lambda)\left\{m + \rho^{1/\Gamma}R(\Omega)\left(1 + \Gamma\int\frac{d\Omega}{R(\Omega)}\right)\right\}, \qquad E = \Gamma\rho^{1/\Gamma}\Omega - h(\lambda), \tag{15}$$

where $f = C\exp[g(\lambda)]$, $g(\lambda) = \dot{h}(\lambda)/m$. Note that at $R = C_1\exp(\Omega/r)$, $r = \mathrm{const}$ function A describes kinetics of the Ahrrenius type [5].

Assuming that $R(\Omega) = \kappa^{-1}\Omega$, $h(\lambda) = -q\,(\lambda - \lambda_0)$ we obtain the governing equations

$$A = q\kappa^{-1}\left(\frac{p}{\rho} - \kappa\right), \tag{16}$$

$$E = \frac{p}{\rho(\sigma-1)} + q\,(\lambda - \lambda_0)\,, \tag{17}$$

at which system (2) admits scaling symmetry group, generated by the operator $\hat{D}_3$. So the symmetry requirements together with the restrictions arising from the second law of thermodynamics completely remove the arbitrariness in the choice of the functions E and A.

If the processes under consideration are not far from the state of complete thermodynamic equilibrium, which is the case, e.g., when the long nonlinear wave propagation is studied, then we may substitute the energy balance equation for the the finite-differenc equation

$$p - p_0 = +\sigma\kappa V_0^{-2}\,(V - V_0) + q(\sigma - 1)V_0^{-1}\,(\lambda - \lambda_0) = 0, \tag{18}$$

where p_0, V_0 and λ_0 denote the values of the subsequent parameters in the state of complete thermodynamic equilibrium. Expressing λ from this equation, differentiating (18) with respect to temporal variable and next employing (16) with RHS expanded near the equilibrium, it is possible to express the governing equation merely in terms of “external” variables. Restricting our consideration to one-dimensional case and going to the Lagrangian representation

$$t_l = t, \qquad x_l = \int \rho\, dx \tag{19}$$

we obtain a closed system which is more simple than (2):

$$\frac{\partial u}{\partial t_l} + \frac{\partial p}{\partial x_l} = \Im, \qquad \frac{\partial V}{\partial t_l} - \frac{\partial u}{\partial x_l} = 0, \qquad \tau\left[\frac{\partial p}{\partial t_l} + \frac{\chi}{\tau V^2}\frac{\partial u}{\partial x_l}\right] = \frac{\kappa}{V} - p, \tag{20}$$

where $\Im$ is an external force, $V \equiv \rho^{-1}$ is the specific volume,

$$\tau^{-1} = -a\,(A_\lambda)_V = a(\sigma-1)q^2/\kappa, \qquad \chi/\tau = (\partial p/\partial\rho)_\lambda = \kappa\sigma, \qquad \kappa = (\partial p/\partial\rho)_{A=0}\,.$$

(we will drop index "l" in the forthcoming formulae). So, instead of unknown functions, system (20) contains three parameters that completely define dynamical features of relaxing media near the equilibrium [3].

4 On attractive features of invariant solutions of system (20)

A wide employment of the group theory methods in non-linear mathematical physics is justified to the great extent by the fact that invariant solutions of evolution systems very often play the role of intermediate asymptotics for sufficiently large class of Cauchy and boundary value problems. It is shown below that attractive features are inherent to periodic invariant solutions of system (22) as well as reducible to them solitary wave solutions.

When $\Im = \gamma = \text{const}$ then ansatz

$$u = U(\omega), \qquad V = \frac{R(\omega)}{x_0 - x}, \qquad p = (x_0 - x)\,\Pi(\omega), \qquad \omega = t\xi + \ln\frac{x_0}{x_0 - x} \tag{21}$$

leads to an ODE system. Substituting (21) into the second equation of systems (20) we find that $\dot U = \xi \dot R$. Variables R, Π satisfy the following system of ODE:

$$\begin{aligned} &\xi\left[\tau\,(\xi R)^2 - \chi\right]\dot R = -R\left[(1+\tau\xi)R\Pi - \kappa + \tau\gamma\xi R\right],\\ &\xi\left[\tau\,(\xi R)^2 - \chi\right]\dot \Pi = \xi\left[\xi R(R\Pi - \kappa) + \xi(\Pi + \gamma)\right]. \end{aligned} \tag{22}$$

It is easy to see that the singular point $\mathbf{A}(R_1, \Pi_1)$, where $\Pi_1 = -\gamma > 0$, $R_1 = \kappa/\Pi_1$, corresponds to the invariant stationary solution of system (20), belonging to the set (21):

$$u_0 = 0, \qquad p_0 = \gamma(x - x_0), \qquad V_0 = \kappa/[\gamma(x - x_0)]. \tag{23}$$

For this special case we are able to express transition to the Eulerian co-ordinate x_e in explicit form:

$$x_e = (\kappa/\gamma)\ln\left[(x_0 - x_l)/x_0\right]. \tag{24}$$

So, according to the formula (24), the Lagrangian co-ordinate $x_l = x_0$ corresponds to the point on infinity in the Eulerian reference frame.

We are going to formulate conditions assuring existence of periodic solutions in vicinity of the singular point $\mathbf{A}(R_1, \Pi_1)$. For this purpose we rewrite the linear part of system (22) in co-ordinates $x = R - R_1$, $y = \Pi - \Pi_1$:

$$\xi\Delta\begin{pmatrix} x \\ y\end{pmatrix}' = \begin{bmatrix} -\kappa, & -R_1^2\sigma \\ \kappa\xi^2, & (\xi R_1)^2 + \chi\xi \end{bmatrix}\begin{pmatrix} x \\ y\end{pmatrix} + O(|x|, |y|). \tag{25}$$

Periodic solutions appearance would take place when the eigenvalues of matrix $\hat{\mathbf{M}}$ standing at the RHS of equation (25) intersect imaginary axis [6], and this is so when the following relations hold:

$$\xi = \xi_{cr} = -\left(\chi + \sqrt{\chi^2 + 4\kappa R_1^2}\right)/(2R_1^2), \qquad 0 < R_1 < \sqrt{\chi/(\tau\xi^2)}. \tag{26}$$

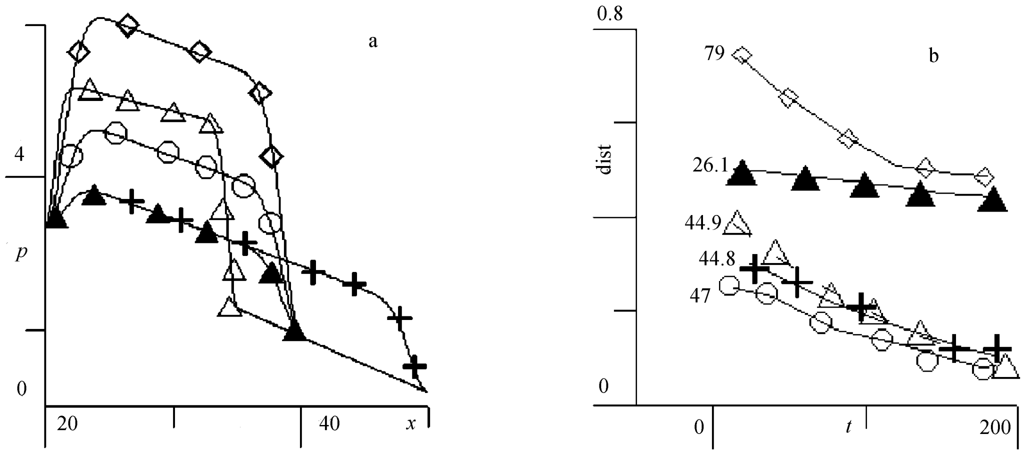

Fig. 1. Perturbations used in numerical experiments (a) and temporal dependence of distances between the wave packs and the solitary wave invariant solution (b). Numbers near the graphs show the energies of the initial perturbations with the same marks.

Numerical analysis of system (22) gives such changes of regimes in a vicinity of the singular point $\mathbf{A}\,(R_1, \Pi_1)$. When value of the parameter ξ is a little less than ξ_{cr}, then the singular point is a stable focus. Above this value a stable limiting cycle appears in a soft manner. Its radius grows with further increase of the parameter ξ until the homoclinic bifurcation takes place. After that the singular point becomes unstable focus.

We performed numerical simulation of system (20) based on the Godunov numerical scheme [7]. The values of the parameters were choosen in accordance with the requirements posed by (26). In numerical experiments we observed that solutions of the Cauchy problems evolved in self-similar modes when invariant periodic solutions belonging to set (21) as well as limiting to them homoclinic solutions were taken as Cauchy data.

Numerical simulations have also shown that wave packs created by sufficiently large class of perturbations of the initial inhomogeneous state (23) tend to the solution associated with the homoclinic loop. Whether or not the wave pack would tend to the homoclinic solution depends on the energy of initial perturbation, more preciesly, on that part of the total energy that is travelled with the pack moving “downward” i.e. towards the domain with decreasing p.

Using the equation (18), we can express the energy of perturbation in the following form:

$$E^{\rm tot} = \int_0^x \left\{ \frac{u^2}{2} - p_0\,(V - V_0) + \int_0^{x'} \left[\frac{\partial p}{\partial x} - \gamma \right] V\,dx'' \right\} dx'.$$

Numerical simulation shows that energy estimation of the initial perturbation well enough characterizes convergency to the invariant soliton-like solution. For $\chi = 1.5$, $\tau = 0.07$, $\kappa = 10$ convergency is observed when $E^{\rm tot}$ is close to 45. Fig. 1a shows variety of perturbation used whereas Fig. 1b – temporal dependence of minimal distances between the wave packs created by perturbations and the family of solutions associated with the homoclinic loop. On the left side of Fig. 2 initial perturbations are shown together with invariant homoclinic solution marked by the dotted line, whereas on the right side the homoclinic trajectories and the wave packs created by the subsequent perturbations are shown at large distances from the origin. Case b corresponds to the initial perturbations having the energy close to 45, cases a and c – to the initial perturbations having sufficiently different energies.

We also interested in attracting features of periodic invariant solutions. Our experiments showed that it is impossible to obtain convergency to a periodic invariant solution when Cauchy data are chosen among monotonic functions. But if we solve the boundary value problem with periodically initiated impulses then convergency may be attained. Here again the energy criterion works, besides, perturbations should be separated by proper temporal intervals.We solved numerically a piston problem taking again as Cauchy data stationary invariant solutions (23), associated with the critical point $A\,(R_1, \Pi_1)$. It was observed that the convergency takes place when the energy of a single perturbation created by the piston that works in a pulse regime is close to 17 and the temporal intervals between the pulses lies near 22. Fig. 3 shows typical patterns obtained. An invariant periodic solution envelopes succession of wave perturbations, that are essentialy different from the autowave mode near the origin (Fig. 3a), but approaches it in the long run (Fig. 3b).

So both autowave invariant solutions and solitary waves play roles of intermediate asymptotics for sufficiently large classes of solutions of system (20).

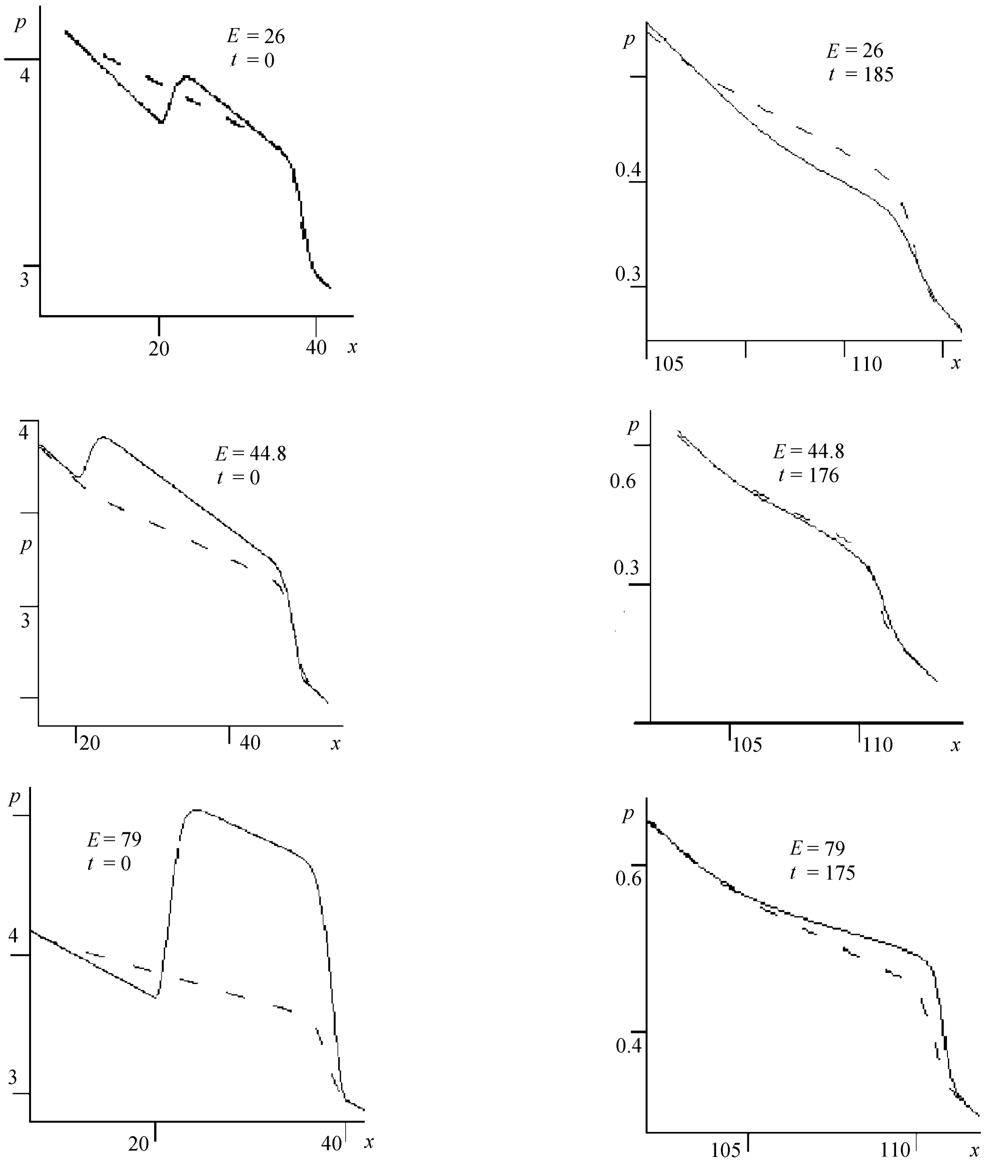

Fig. 2. Perturbations of the stationary inhomogeneous solution (23) (left) and wave packs created by these perturbations (right) on the background of the invariant solitary wave solution indicated by dotted lines.

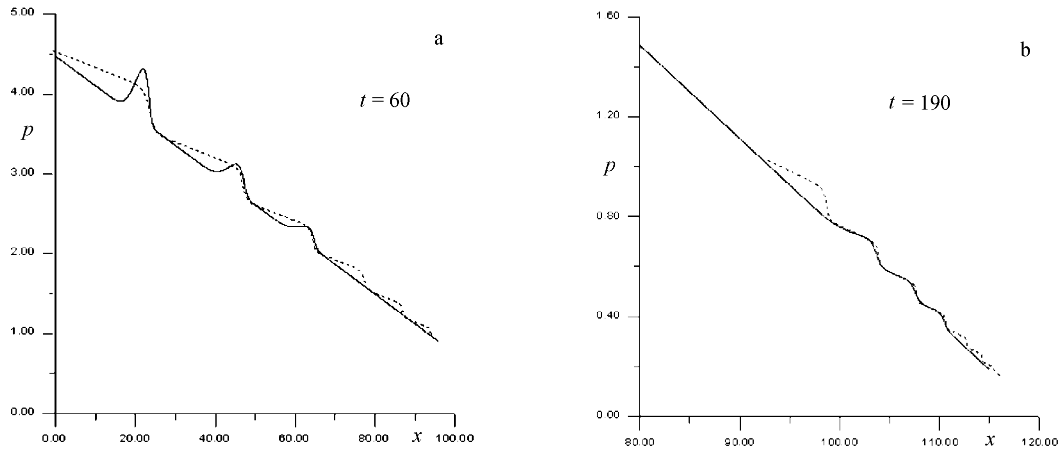

Fig. 3. Wave packs initiated by a piston moving periodically in the pulse regime on the background of a periodic invariant solution, indicated by the dotted line: patterns near the source (a) and at large distance from source (b).

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On Invariant Wave Patterns in Non-Local Model of Structured Media

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A set of invariant solutions of a modelling system describing long nonlinear waves propagation in medium with internal structure is considered. Using the well known symmetry reduction method we perform transition from the initial system of PDE to a third-order dynamical system. Employing the qualitative theory method as well as direct numerical simulation, we study forms of multiperiodic, quasiperiodic, chaotic and soliton-like invariant solutions. Parametric portraits are presented and the structure of a set of parameters corresponding to the soliton-like solutions is analyzed. Within the method applied it manifests fractal features.

It is well known, that most of the earth materials possess internal structure. Such are rocks, soils, layered media and lithosphere itself. Such are a lot of artificial substances – concrete, air-liquid mixtures, polymers and so on. When studying high-speed high-intense loading afteractions in structured media the problem of their adequate description arises, since continual approach in such circumstances is not valid and integro-differential relations must be used [1]. Such relations are not easy to deal with and, besides, they contain, as a rule, unknown kernels of relaxations [1] that must be determined in every particular case.

It turns out that knowledge of details of the relaxing mechanisms is almost unnecessary if the processes to be described are weakly non-equilibrium, which is the case when we restrict to the consideration of the long waves propagation. Analysis performed within the asymptotic approach in paper [2] shows that the balance equations for mass and momentum in the long wave limit do not depend on structure, retaining their classical form. So all the information about the structure in this approximation should be concentrated in the dynamical equation of state (DES) which may be obtained using the methods of phenomenological thermodynamics of non-equilibrium processes. Here we employ the DES, which describes relaxing effects, as well as purely spatial non-locality [3]. Together with balance equations for mass and momentum taken in the hydrodynamic approximation, it forms a closed system of the following form:

$$
\begin{gathered}
\frac{d\rho}{dt} + \rho\frac{\partial u}{\partial x} = 0, \qquad \rho\frac{du}{dt} + \frac{\partial p}{\partial x} = \gamma\rho, \\
\tau\left(\frac{dp}{dt} - \chi\frac{d\rho}{dt}\right) = \kappa\rho - p + \sigma\left\{\frac{\partial^2 p}{\partial x^2} + \frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial x} - \chi\left(\frac{\partial^2 \rho}{\partial x^2} - \frac{1}{\rho}\left(\frac{\partial \rho}{\partial x}\right)^2\right)\right\},
\end{gathered} \tag{1}
$$

where p is pressure, ρ is density, u is the mass velocity, $d(\cdot)/dt = \partial(\cdot)/\partial t + u\partial(\cdot)/\partial x$ is substantial derivative with respect to time, $\rho\gamma$ is the mass force, κ is equal to the square of equilibrium (low-frequency) sound velocity, τ is the time of relaxation, χ is equal to the square of frozen (high-frequency) sound velocity, σ defines the effective range of non-local effects. System (1) describes long non-linear waves evolution in structured media. In this work we present some results concerning the features of a set of travelling wave solutions of this system.

It is easy to prove that system (1) admits a one-parameter group generated by the operator

$$\hat{X} = \frac{\partial}{\partial t} + D\frac{\partial}{\partial x} + \xi\left(\rho\frac{\partial}{\partial \rho} + p\frac{\partial}{\partial p}\right). \tag{2}$$

Using invariants of this operator, obtained with the help of standard technique [4], one is able to construct the ansatz

$$\begin{aligned} &u = U(\omega) + D, \qquad \omega = x - Dt,\\ &\rho = \exp[\xi t + S(\omega)], \qquad p = \rho Z(\omega), \end{aligned} \tag{3}$$

describing a travelling wave moving with constant velocity D (the meaning of the parameter ξ will be explained later on). We are going to show that the set (3) contains periodic, quasiperiodic, multiperiodic, stochastic and soliton-like solutions. The last regime is analyzed in more detail, in particular, a set of parameter space $\left(D^2, \kappa\right)$, for which solitary wave solutions do exist is studied.

Inserting (3) into (1) we obtain an ODE cyclic with respect to the variable S. Functions U, Z and $W = dU/d\omega \equiv \dot{U}$ satisfy the following dynamical system:

$$\begin{aligned} &U\frac{dU}{d\omega} = UW, \qquad U\frac{dZ}{d\omega} = \gamma U + \xi Z + W(Z - U^2),\\ &U\frac{dW}{d\omega} = \left\{U^2\left(\gamma U + \xi Z - WU^2 + \chi\tau W + Z - \kappa\right)\right.\\ &\qquad\left. + \sigma\left\{\left[(\xi + W)(2U(\gamma - UW) + \chi W) + (UW)^2\right]\right\}\right\}\left[\sigma\left(\chi - U^2\right)\right]^{-1}. \end{aligned} \tag{4}$$

Analysis shows that system (4) possesses only one critical point belonging to the physical parameter's range. If $\gamma = \xi Z_0/D$ then this point, having the coordinates

$$U_0 = -D, \qquad Z_0 = \frac{\kappa}{[1 - 2\sigma(\xi/D)^2]}, \qquad W_0 = 0, \tag{5}$$

defines a stationary solution

$$u = 0, \qquad \rho = \rho_0 \exp\left[\frac{\xi x}{D}\right], \qquad p = Z_0\rho,$$

belonging to the set (3). One immediately concludes from the above formula that ξ defines an inclination of inhomogeneity of this solution if the rest of the parameters are fixed.

We begin our study of patterns formation from some analytical estimations, enabling to state the conditions leading to the limiting cycle appearance in vicinity of critical point (5). It is more convenient to work in the coordinate system $X = U + D$, $Y = Z - Z_0$, W, having the origin at the critical point (5), so we rewrite the linear part of system (4) in this reference frame:

$$U\frac{d}{d\omega}\begin{pmatrix} X\\ Y\\ W\end{pmatrix} = \hat{M}\begin{pmatrix} X\\ Y\\ W\end{pmatrix} + o\left(|X, Y, W|\right), \tag{6}$$

where

$$\hat{M} = \begin{pmatrix} 0 & 0 & -D\\ \gamma & \xi & \Delta\\ A & B & C\end{pmatrix}, \qquad A = \frac{\kappa\xi D(2\xi\sigma - \tau D^2)}{Q\sigma(2\sigma\xi^2 - D^2)}, \qquad B = \frac{D^2(1 + \xi\tau)}{Q},$$

$$Q = \sigma(\chi - D^2), \qquad C = Q^{-1}\left\{\xi\sigma\left(\chi - 2D^2\right) - \frac{2D^2\xi\sigma\kappa}{D^2 - 2\sigma\xi^2} + \tau D^2\left(\chi - D^2\right)\right\}.$$

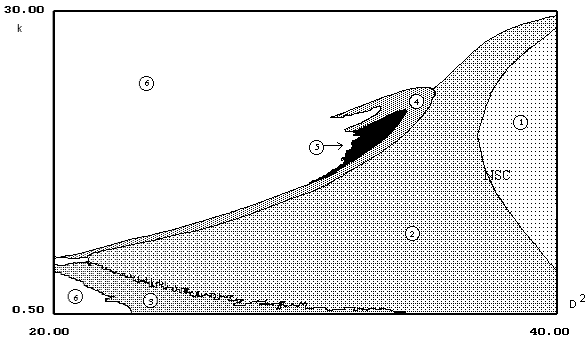

Fig 1. Bifurcation diagram of system (4) in parametric space (D^2, κ): 1 – stable focus; 2 – 1T-cycle; 3 – torus; 4 – multiperiodic attractor; 5 – chaotic attractor; 6 – lose of stability.

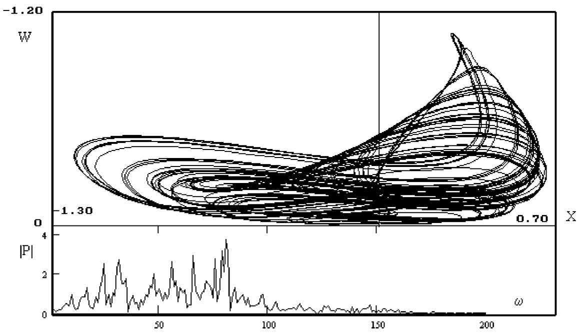

Fig 2. Phase portrait (above) and Fourier spectrum (below) of chaotic solution of system (4) obtained at $\kappa = 17$ and $D^2 = 32.3$.

Let us formulate conditions leading to the appearence of periodic solutions of system (4). According the Hopf theorem [5], limiting cycle is created when a pair of complex conjugate eigenvalues of the matrix $\hat{M}$ crosses the imaginary axis and the third eigenvalue is strictly negative. This is so if the following relations hold:

$$\alpha = \xi + C > 0, \tag{7}$$

$$\Omega^2 = AD - B\Delta + \xi C > 0, \tag{8}$$

$$\alpha\Omega^2 = \xi(AD - Z_0 B). \tag{9}$$

The first two of them take on the form of inequalities, imposing some restrictions on the parameters, while the third one determines the neutral stability curve (NSC) in the plane (κ, D^2), providing that the rest of parameters are fixed. If $\sigma = 0.76$, $\tau = 0.1$, $\chi = 50$, $\xi = 1.8$ then the NSC looks like a parabola with branches directed from right to left. It is presented on Fig. 1 as a bold line.

Numerical investigations illustrated by Fig. 1 show that inside the parabola the critical point $A(-D, Z_0, 0)$ is a stable focus. When we cross the neutral stability curve (9) from right to left, then the stable periodic solution softly appears. The radius of the limiting cycle grows as one goes away from the curve, until it remains stable. Further evolution of periodic regime depends strongly on the values of parameter κ. For $\kappa > 20$ the amplitude of oscilations grows up as the parameter D^2 decreases, till the domain is attained where the regimes become completely unstable.

If the κ lies betwen 10 and 20, then scenario of the development of oscilations is as follows: limiting cycle $\rightarrow$ finite period doubling cascade $\rightarrow \cdots \rightarrow$ chaotic attractor $\rightarrow$ global lose of stability or falling onto the separated regime existing simultaneously with the main cascade. A typical phase portrait of the regime arising as a result of period doubling cascade is presented on the Fig. 2. Fourier spectrum of X-coordinate, shown on the lower part of Fig. 2, looks like a continious function, so we really deal with the chaotic solutions in this domain of parameters' values.

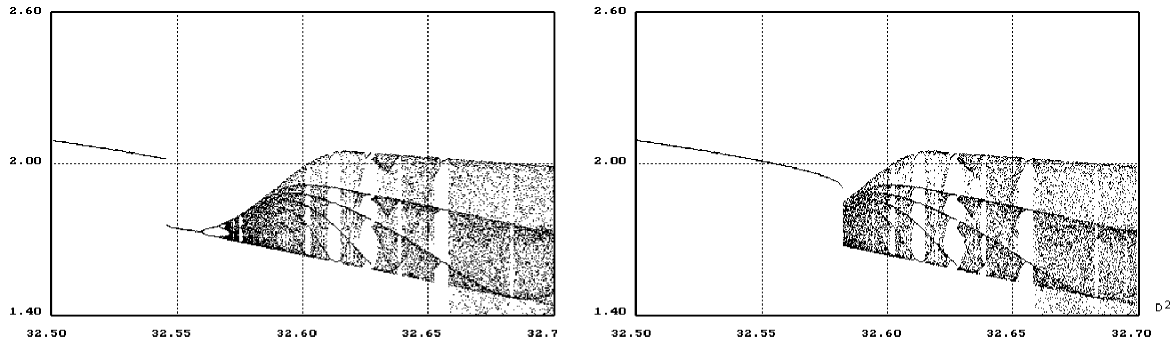

Fig 3. Poincaré sections obtained for $\kappa = 17.82$ when D^2 is decreasing (left) and when D^2 is growing up (right).

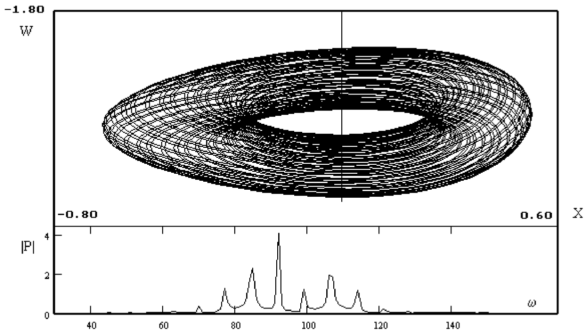

Fig 4. Phase portrait (above) and Fourier spectrum (below) of toroidal attractor of system (4), obtained at $\kappa = 1$ and $D^2 = 24.2$.

In order to study a fine structure of the strange attractor the Poincare sections technique was used [5, 6]. A section plain transversal to the phase traejectories was defined by equation $W = 0$. The bifurcation diagrams shown on Fig. 3 were formed in the following way: we took X coordinates of the points of intersection of the phase trajectories with the plane $W = 0$ and set them on the vertical axis, whereas the corresponding values of the bifurcation parameter D^2 on the horizontal one. Complete list of bifurcation diagrams obtained this way is published in [7]. The most interesting features of system (4) seen on Fig. 3 are the phenomenon of hysteresis and coexistence of different regimes in certain domains of parameters' values.

When $\kappa < 10$, the scenario of the oscillationg regimes development is provided with quasiperiodic solutions and spiral attractors of Šilnikov type [8]. Creation of spiral attractors takes place at the point of intersection of NSC with the horizontal axis, where the linearization matrix $\hat{M}$ has one zero and two pure imaginary eigenvalues. This may be shown explicitely by studying canonical Poincaré form [5] of system (4):

$$\dot{r} = a_1 r y, \qquad \dot{y} = b_1 r^2, \tag{10}$$

where $a_1 = -\frac{D^3\tau}{Q}$, $b_1 = \frac{(\omega^2+\xi^2)\sigma\xi\chi}{DQ\omega^2}$.

A simple analysis shows that system (10) has a center and there arises a stable focus, corresponding to a spiral attractor of system (4), in the domain $r > 0$ when κ is small and positive. The spiral attractor transforms into the isolated regime, existing simultaneously with the main cascade and visible on some of the bifurcation diagrams presented in [7], as the parameter κ sufficiently grows, but in proximity of the horizontal axis it turns into the stable torus when D^2 goes sufficiently far away of the point of NSC intersection with axis. A typical phase portrait of the toroidal regime is shown on Fig. 4. The corresponding Fourier spectrum, shown below the phase portrait, evidently contains maxima defining main frequencies of this quasiperiodic regime.

As it was allready mentioned, system (4) possesses selected regimes, coexisting with the oscillating solutions from the main bifurcation cascade. For certain values of the parameters the

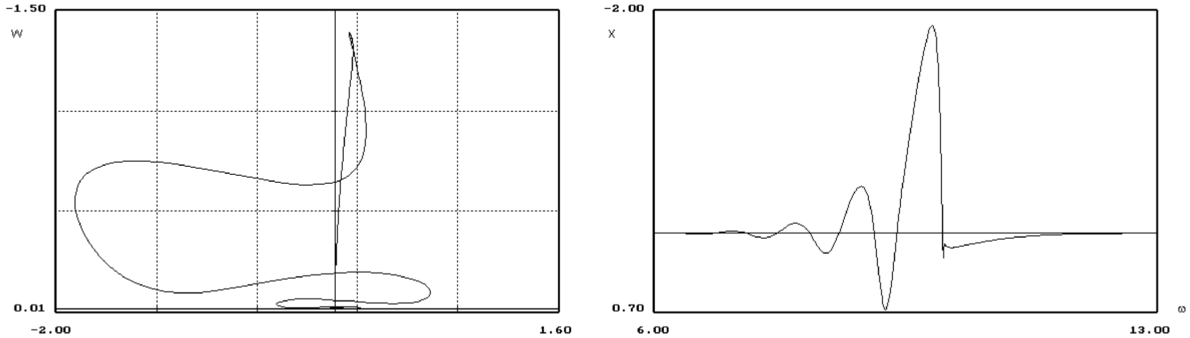

Fig 5. Phase portrit of the soliton-like solution (left) and coordinate U versus ω (right).

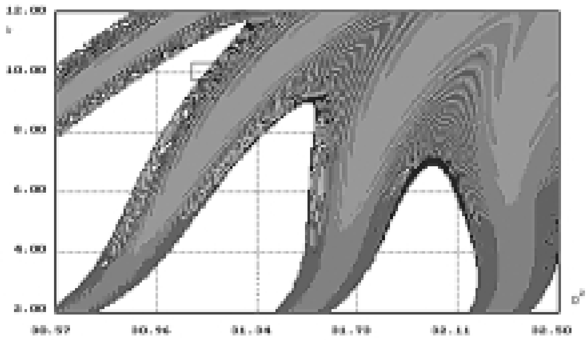

Fig 6. A portrait of subsets of parameter space (D^2, κ), corresponding to different intervals of the function $f^{\Gamma}_{\min}(\kappa, D^2)$ values: $f^{\Gamma}_{\min} > 1.2$ for white colour; $0.6 < f^{\Gamma}_{\min} \leq 1.2$ for light grey; $0.3 < f^{\Gamma}_{\min} \leq 0.6$ for grey; $0.01 < f^{\Gamma}_{\min} \leq 0.3$ for deep grey; $f^{\Gamma}_{\min} \leq 0.01$ for black colour.

isolated regime forms the homoclinic loop, which may be calculated numerically by means of the special technique (Fig. 5). Existence of homoclinic loops among the solutions of system (4) is a very important fact, because these regimes correspond to solitary-wave solutions of the initial PDE system. Therefore we investigate a set of points of paremeter space $\left(D^2, \kappa\right)$, for which trajectories going out of the origin along the one-dimensional unstable invariant manifold W^u return to the origin along the two-dimensional stable invariant manifold W^s. In practice, for a given values of the parameters κ, D^2, we define numerically a distance between the origin and the point $\left(X^{\Gamma}(\omega), Y^{\Gamma}(\omega), W^{\Gamma}(\omega)\right)$ of the phase trajectory $\Gamma\left(\cdot; \kappa, D^2\right)$:

$$f^{\Gamma}\left(\kappa, D^2; \omega\right) = \sqrt{\left[X^{\Gamma}(\omega)\right]^2 + \left[Y^{\Gamma}(\omega)\right]^2 + \left[W^{\Gamma}(\omega)\right]^2}, \tag{11}$$

starting from the fixed Cauchy data. Next we determine minimum $f^{\Gamma}_{\min}(\kappa, D^2)$ of the function (11) for that part of the trajectory that lies beyond the point at which the distance gains its first local maximum, providing that it still lies inside the ball centered at the origin and having a fixed (sufficiently large) radius. The results are presented on Figs. 6–8. First of them is of the most rough scale among this series. Here white colour marks values of the parameters κ, D^2 for which $f^{\Gamma}_{\min} > 1.2$, light gray corresponds to the cases when $0.9 < f^{\Gamma}_{\min} < 1.2$ and so on (further explanations are given in the subsequent captions). The black coloured patches correspond to the case when $f^{\Gamma}_{\min} < 0.03$. Fig. 7 presents the enlargement of the rectangle shown in Fig. 6, whereas Fig. 8 – the enlargement of the rectangle from the Fig. 7. Note, that colours in the last one are re-scaled so e.g. the black colour corresponds to the points at which $f^{\Gamma}_{\min} < 0.003$.

Conclusion

Thus, the modeling system (1), describing media with memory and spatial non-locality possesses a set of complicated invariant solutions, which are effectively investigated with the help of qualitative methods as well as numerical simulation. It is stated the existence of tori and spiral

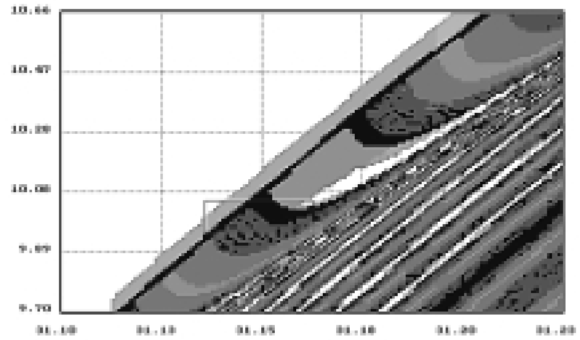

Fig 7. Enlargement of the part of Fig. 6, lying inside the rectangle.

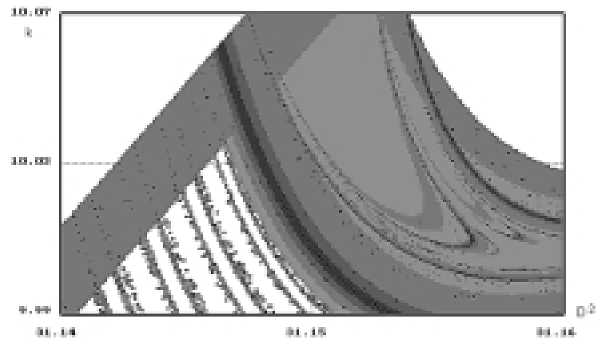

Fig 8. Enlargement of the part of Fig. 7, lying inside the rectangle: $f^{\Gamma}_{\min} > 0.011$ for white colour; $0.007 < f^{\Gamma}_{\min} \leq 0.011$ for light grey; $0.005 < f^{\Gamma}_{\min} \leq 0.007$ for grey; $0.003 < f^{\Gamma}_{\min} \leq 0.005$ for deep grey; $f^{\Gamma}_{\min} \leq 0.003$ for black colour.

attractors of Šilnikov type appearing at the point of parametric space $\left(D^2, \kappa\right)$, corresponding to the doubly degeneracy of the linearized system. The last regime coexists with the main bifurcation cascade (including multiperiodic and chaotic oscillations) and causes hysteretic features of the system. Besides, it gives rise to saddle loops, corresponding to soliton-like solutions of system (1). Numerical investigations show that there are domains of parameter space $\left(D^2, \kappa\right)$ where the soliton-like solutions are not observed and there are domains where the points $\left(D^2, \kappa\right)$, corresponding to the homoclinic loops form dense sets. Within the numerical algorithm applied, the set of points corresponding to the homoclinic solutions manifests fractal features.

It is worth noting that soliton-like solutions of system (1) do not have the classical bell shape. They possess, as a rule, many humps and oscillating tails. Compact wave perturbations of this sort are rather typical to the media with internal structure. One such pulse was created during the Great Chilean Earthquake as it was shown in paper [9]. Another examples may be seen in papers dealing with the models of block geophysical media [10].

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 245–248.

On Symmetry of a Class of First Order PDEs Equations

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The problem of group classiffication for the class of first-order scalar PDEs invariant under the Euclid algebra $E(n)$ in considered. We found new nonlinear equations of the form $u_a u_a = F(u_t)$ with wide symmetry properties.

In this paper we study group classification of a class of nonlinear first-order multidimensional equations

$$u_t = \Phi(u, u_a u_a). \tag{1}$$

$u_a u_a$ is a designation for the sum

$$\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \ldots + \left(\frac{\partial u}{\partial x_n}\right)^2, \qquad u_t = \frac{\partial u}{\partial t}.$$

u is a scalar function of time t and n spatial variables $(x_1, x_2, \ldots, x_n)$. The class (1) includes many well-known equations with wide symmetry properties.

We will not consider cases when $n < 3$. It will be more convenient to investigate the class (1) in the form

$$u_a u_a = F(u, u_t). \tag{2}$$

The function F is assumed to be sufficiently smooth.

Why is it interesting to study symmetries for this particular class of equations? First, it is the general class of first order PDEs, that includes many physically interesting equations. It is interesting to find new equations invariant under known symmetry algebras and new symmetry algebras. Invariant first-order equations can be used for study of conditional symmetry of higher order PDEs. First order PDEs may also have interesting generalizations.

The class of equations (2) includes such well-known equations with wide symmetries as the eikonal equation, the Hamilton–Jacobi and the Hamilton equations.

The Hamilton–Jacobi equation

$$u_t + u_a u_a = 0 \tag{3}$$

is invariant under the Galilei group. Its maximal Lie invariance algebra was studied in [4] and can be described by the following basis operators:

$$P_0 = \partial_t, \qquad P_a = \partial_a, \qquad P_u = \partial_u, \qquad J_{ab} = x_a\partial_b - x_b\partial_a, \qquad G_a^{(1)} = t\partial_a + \frac{1}{2}x_a\partial_u,$$

$$D^{(1)} = t\partial_0 + \frac{1}{2}x_a\partial_a, \qquad A^{(1)} = t^2\partial_0 + tx_a\partial_a + \frac{1}{4}x_a x_a\partial_u, \qquad G_a^{(2)} = u\partial_a + \frac{x_a}{2}\partial_t,$$

$$D_a^{(2)} = u\partial_u + \frac{1}{2}x_a\partial_a, \qquad A^{(2)} = u^2\partial_u + ux_a\partial_a + \frac{1}{4}x^2\partial_t,$$
$$K_a = 2x_a\left(D^{(1)} + D^{(2)}\right) + \left(\frac{1}{4}tu - x^2\right)\partial_a \qquad \left(x^2 \equiv x_a x_a\right).$$

The equation (3) is also invariant under a discrete transformation $u \to t$, $t \to u$.

Symmetry of the relativistic Hamilton equation

$$u_\alpha u_\alpha = 1 \tag{4}$$

was studied in [1, 5]. Here

$$u_\alpha u_\alpha \equiv u_0^2 - u_1^2 - \ldots - u_n^2,$$

$u_0 \equiv u_t$.

The maximal Lie invariance group of the equation (4) is the conformal group $C(1, n+1)$.

Basis elements for the corresponding Lie algebra can be written as follows:

$$\partial_A = ig_{AB}\frac{\partial}{\partial x_B}, \qquad g_{AB} = \mathrm{diag}\,(1, -1, \ldots, -1),$$
$$J_{AB} = x_A\partial_B - x_B\partial_A, \qquad D = x_A\partial_A, \qquad K_A = 2x_A D - x_B x_B\partial_A,$$

where $A, B = 0, 1, 2, \ldots, n+1$; $x_{n+1} \equiv u$, summation over the repeated indices is as follows:

$$x_A x_A = x_0^2 - x_1^2 - x_2^2 - \cdots - x_{n+1}^2.$$

The eikonal equation

$$u_\alpha u_\alpha = 0, \tag{5}$$

$\alpha = 0, 1, \ldots, n$; is invariant [1, 5] under an infinite-dimensional algebra, defined by operators

$$X = (b^{\mu\nu}x_\nu + a^\mu)\partial_\mu + \eta\partial_u,$$

where $b^{\mu\nu} = -b^{\nu\mu}$, a^μ, η are arbitrary differentiable functions on u;

$$\partial_\alpha = ig_{\alpha\beta}\frac{\partial}{\partial x_B}, \qquad g_{\alpha\beta} = \mathrm{diag}\,(1, -1, \ldots, -1).$$

The class of equations we consider will be a natural generalization of equations (3)–(5).

We look for a Lie symmetry operator of the equation (2) in the form

$$X = \xi^t(t, x_a, u)\partial_t + \xi^a(t, x_b, u)\partial_{xa} + \eta(t, x_a, u)\partial_u. \tag{6}$$

The general Lie invariance condition is

$$\overset{1}{X}(u_a u_a - F(u, u_t))\Big|_{u_a u_a = F(u, u_t)} = 0, \tag{7}$$

where $\overset{1}{X}$ is the first Lie prolongation for the operator X.

The condition (7) gives the the following determining equations for operators of invarianse algebra of the equation (2):

$$\xi_b^a + \xi_a^b = 0, \quad b \neq a; \qquad \xi_a^a = \xi_b^b \tag{8}$$

(we will designate $\xi^a_a = d(x_a, t, u)$);

$$2(\eta_a - \xi^t_a u_t - \xi^a_u F) + F_{u_t}(\xi^a_t + \xi^a_u u_t) = 0; \tag{9}$$

$$2F(\eta_u - d - \xi^t_u u_t) = \eta F_u + F_{u_t}(\eta_t + (\eta_u - \xi^t_t)u_t - \xi^t_u u_t^2). \tag{10}$$

Lower indices always designate corresponding derivatives.

Determining equations (8) are fulfilled for all equations from the class (2). From (8) we get the following form for coefficients ξ^a of the operator X (6):

$$\xi^a = c_a + \tilde{d}x_a + \lambda_{ab}x_b + 2k_b x_b x_a - k_a x_b x_b, \tag{11}$$

where $\lambda_{ab} = -\lambda_{ba}$, c_a, $\tilde{d}$, k_a are functions on u and t.

The following operators are symmetry operators for all equations from the class (2) irrespective of the form of the function $F(u, u_t)$:

$$P_t = \partial_t, \qquad P_a = \partial_a, \qquad J_{ab} = x_a\partial_b - x_b\partial_a, \tag{12}$$

that form the basis of the Euclid algebra $E(n)$ in the space of n variables $x_1, \ldots, x_n$, plus the translation operator by time variable.

Now we look for equations from the class (2) admitting wider symmetry than the algebra (12). We need to find functions F for which the conditions (9), (10) are fulfilled with some coefficients being non-zero.

From the determining equation (9) we conclude that there are two options:

$$I.\ \eta_a = \xi^t_a = \xi^a_u = \xi^a_t = 0, \tag{13}$$

and $F = F(u, u_t)$ is determined by the equation (10).

$$II.\ F = r(u)u_t^2 + s(u)u_t + q(u). \tag{14}$$

The class (2) with F having the form (14) includes all well-known equations (3)–(5).

Let us consider the first option in detail. It follows from the conditions (13) that

$$\eta = \eta(t, u), \qquad \xi^t = \xi^t(t, u), \qquad \xi^a = \xi^a(x_1, \ldots, x_n).$$

The equation (10) takes the form

$$2F(\eta_u - d - \xi^t_u u_t) = \eta F_u + F_{u_t}\left(\eta_t + \eta_u u_t - \xi^t_t u_t - \xi^t_u u_t^2\right). \tag{15}$$

As $d_u = 0$, we conclude from (15) that $d = \text{const}$, and the expression for the coefficients ξ^a can only take the form

$$\xi^a = c^a + dx_a + \lambda_{ab}x_b,$$

where $\lambda_{ab} = -\lambda_{ba}$, d, c_a are constants.

There will be no conformal or projective symmetry operators in this case.

We adduce some new equations with additional symmetry to (12). For example, if we put $\eta_u = \xi^t_t$, then in the case $\xi^t_u \cdot \eta_t < 0$ we get the function F of the form

$$F = \left(1 + u_t^2\right)\exp(\lambda \operatorname{arctg} u_t), \qquad \lambda = \text{const}. \tag{16}$$

In the case $\xi^t_u \cdot \eta_t > 0$ we get $F = (a + bu_t)^2$ (a, b are constants) from the class (14).

The equation

$$u_a u_a = \left(1 + u_t^2\right)\exp(\lambda \operatorname{arctg} u_t) \tag{17}$$

has three additional symmetry operators of the form

$$\partial_u, \qquad -u\partial_t + t\partial_u - \frac{\lambda}{2}x_a\partial_a, \qquad u\partial_u + t\partial_t + x_a\partial_a.$$

It is interesting to note that the change $u \to t$, $t \to u$ leaves the equation (17) invariant up to the change of λ. In this aspect this equation is similar to the Hamilton–Jacobi equation (3).

There are other examples of equations of the form

$$u_a u_a = F(u_t)$$

with additional to (12) symmetry operators:

1. $u_a u_a = u_t^k$.

If $k \neq 0$, $k \neq 1$, $k \neq 2$, we get three additional operators:

$$t\partial_t + u\partial_u + x_a\partial_a,\ \partial_u, \qquad kx_a\partial_a + 2t\partial_t.$$

2. $u_a u_a = \exp u_t$.

We get two additional symmetry operator

$$\partial_u, \qquad 2t\partial_u - x_a\partial_a.$$

Summary. We studied the problem of the group classification for the equation (2). Determining equations for the function F were found, and some partial solutions for these equations constructed. Further research will be required for description of all nonequivalent equations of the form (2) that have additional invariance operators compared to space rotations and space and time translations. Other research opportunities in this respect include investigation of higher order PDEs invariant under the some algebras, of conditional symmetry of second-order PDEs with new equations as additional conditions.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 249–254.

On Some New Classes of Separable Fokker–Planck Equations

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We communicate some recent results on variable separation in the (1+3)-dimensional Fokker–Planck equations with a constant diagonal diffusion matrix.

The principal object of the study is a problem of separation of variables in the Fokker–Planck equation (FPE) with a constant diagonal diffusion matrix

$$u_t + \Delta u + (B_a(\vec{x})u)_{x_a} = 0, \tag{1}$$

where $\vec{B}(\vec{x}) = (B_1(\vec{x}), B_2(\vec{x}), B_3(\vec{x}))$ is the drift velocity vector. Here $u = u(t, \vec{x})$ and $B_i(\vec{x})$, $i = 1, 2, 3$ are smooth real-valued functions. Hereafter, summation over the repeated Latin indices from 1 to 3 is understood.

FPE (1) is a basic equation in the theory of continuous Markov processes. Therefore, it is widely used in different fields of physics, chemistry and biology [1], where stochastic methods are utilized.

We solve the problem of variable separation in FPE (1) into second-order ordinary differential equations in a sense that we obtain possible forms of the drift coefficients $B_1(\vec{x})$, $B_2(\vec{x})$, $B_3(\vec{x})$ providing separability of (1). Furthermore, we construct inequivalent coordinate systems enabling to separate variables in the corresponding FPEs.

Our analysis is based on the direct approach to variable separation in linear PDEs suggested in [3, 4]. It has been successfully applied to solving variable separation problem the Schrödinger equations [3, 4, 5] with variable coefficients.

For an alternative (symmetry) approach to separation of variables in FPE, see [2].

We say that FPE (1) is *separable* in a coordinate system t, $\omega_a = \omega_a(t, \vec{x})$, $a = 1, 2, 3$ if the separation Ansatz

$$u(t, \vec{x}) = \varphi_0(t) \prod_{a=1}^{3} \varphi_a\left(\omega_a(t, \vec{x}), \vec{\lambda}\right) \tag{2}$$

reduces PDE (1) to four ordinary differential equations for the functions φ_μ, $(\mu = 0, 1, 2, 3)$

$$\varphi_0' = U_0\left(t, \varphi_0; \vec{\lambda}\right), \qquad \varphi_a'' = U_a\left(\omega_a, \varphi_a, \varphi_a'; \vec{\lambda}\right). \tag{3}$$

Here $U_0, \ldots, U_3$ are some smooth functions of the indicated variables, $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda = \{$an open domain in $\mathbf{R}^3\}$ are separation constants (spectral parameters, eigenvalues) and, what is more,

$$\operatorname{rank} \left\| \frac{\partial U_\mu}{\partial \lambda_a} \right\|_{\mu=0}^{3} {}_{a=1}^{3} = 3. \tag{4}$$

For more details, see our paper [5].

Next, we introduce an equivalence relation $\mathcal{E}$ on the set of all coordinate systems providing separability of FPE. We say that two coordinate systems t, ω_1, ω_2, ω_3 and $\tilde{t}$, $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$ are *equivalent* if the corresponding Ansatzes (2) are transformed one into another by the invertible transformations of the form

$$t \to \tilde{t} = f_0(t), \qquad \omega_i \to \tilde{\omega}_i = f_i(\omega_i), \tag{5}$$

where $f_0, \ldots, f_3$ are some smooth functions and $i = 1, 2, 3$. These equivalent coordinate systems give rise to the same solution with separated variables, therefore we shall not distinguish between them. The equivalence relation (5) splits the set of all possible coordinate systems into equivalence classes. In a sequel, when presenting the lists of coordinate systems enabling us to separate variables in FPE we will give only one representative for each equivalence class.

Following [5] we choose the reduced equations (3) to be

$$\varphi_0' = (T_0(t) - T_i(t)\lambda_i)\,\varphi_0, \qquad \varphi_a'' = (F_{a0}(\omega_a) + F_{ai}(\omega_a)\lambda_i)\,\varphi_a, \tag{6}$$

where T_0, T_i, F_{a0}, F_{ai} are some smooth functions of the indicated variables, $a = 1, 2, 3$. With this remark the system of nonlinear PDEs for unknown functions ω_1, ω_2, ω_3 takes the form

$$\frac{\partial \omega_i}{\partial x_a}\frac{\partial \omega_j}{\partial x_a} = 0, \qquad i \neq j, \quad i, j = 1, 2, 3; \tag{7}$$

$$\sum_{i=1}^{3} F_{ia}(\omega_i)\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} = T_a(t), \qquad a = 1, 2, 3; \tag{8}$$

$$B_j\frac{\partial \omega_a}{\partial x_j} + \frac{\partial \omega_a}{\partial t} + \Delta\omega_a = 0, \qquad a = 1, 2, 3; \tag{9}$$

$$\sum_{i=1}^{3} F_{i0}(\omega_i)\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} + T_0(t) + \frac{\partial B_a}{\partial x_a} = 0. \tag{10}$$

The system of equations (7), (8) has been integrated in [5]. Its general solution $\vec{\omega} = \vec{\omega}(t, \vec{x})$ is given implicitly by the following formulae:

$$\vec{x} = \mathcal{T}(t)H(t)\,\vec{z}(\vec{\omega}) + \vec{w}(t). \tag{11}$$

Here $\mathcal{T}(t)$ is the time-dependent 3×3 orthogonal matrix:

$$\mathcal{T}(t) = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\gamma & -\cos\alpha\sin\beta - \sin\alpha\cos\beta\cos\gamma & \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta\cos\gamma & -\sin\alpha\sin\beta + \cos\alpha\cos\beta\cos\gamma & -\cos\alpha\sin\gamma \\ \sin\beta\sin\gamma & \cos\beta\sin\gamma & \cos\gamma \end{pmatrix}, \tag{12}$$

α, β, γ being arbitrary smooth functions of t; $\vec{z} = \vec{z}(\vec{\omega})$ is given by one of the eleven formulae

1. Cartesian coordinate system,

$$z_1 = \omega_1, \qquad z_2 = \omega_2, \qquad z_3 = \omega_3, \qquad \omega_1, \omega_2, \omega_3 \in \mathbf{R}.$$

2. Cylindrical coordinate system,

$$z_1 = e^{\omega_1}\cos\omega_2, \qquad z_2 = e^{\omega_1}\sin\omega_2, \qquad z_3 = \omega_3, \qquad 0 \leq \omega_2 < 2\pi, \qquad \omega_1, \omega_3 \in \mathbf{R}.$$

3. Parabolic cylindrical coordinate system,

$$z_1 = (\omega_1^2 - \omega_2^2)/2, \qquad z_2 = \omega_1\omega_2, \qquad z_3 = \omega_3, \qquad \omega_1 > 0, \quad \omega_2, \omega_3 \in \mathbf{R}.$$

4. Elliptic cylindrical coordinate system,

$$z_1 = a\cosh\omega_1\cos\omega_2, \qquad z_2 = a\sinh\omega_1\sin\omega_2, \qquad z_3 = \omega_3,$$
$$\omega_1 > 0, \qquad -\pi < \omega_2 \le \pi, \qquad \omega_3 \in \mathbf{R}, \qquad a > 0.$$

5. Spherical coordinate system,

$$z_1 = \omega_1^{-1}\operatorname{sech}\omega_2\cos\omega_3, \qquad z_2 = \omega_1^{-1}\operatorname{sech}\omega_2\sin\omega_3, \qquad z_3 = \omega_1^{-1}\tanh\omega_2,$$
$$\omega_1 > 0, \qquad \omega_2 \in \mathbf{R}, \qquad 0 \le \omega_3 < 2\pi.$$

6. Prolate spheroidal coordinate system,

$$z_1 = a\operatorname{csch}\omega_1\operatorname{sech}\omega_2\cos\omega_3, \quad a > 0, \qquad z_2 = a\operatorname{csch}\omega_1\operatorname{sech}\omega_2\sin\omega_3,$$
$$z_3 = a\coth\omega_1\tanh\omega_2, \qquad \omega_1 > 0, \qquad \omega_2 \in \mathbf{R}, \qquad 0 \le \omega_3 < 2\pi. \tag{13}$$

7. Oblate spheroidal coordinate system,

$$z_1 = a\csc\omega_1\operatorname{sech}\omega_2\cos\omega_3, \quad a > 0, \qquad z_2 = a\csc\omega_1\operatorname{sech}\omega_2\sin\omega_3,$$
$$z_3 = a\cot\omega_1\tanh\omega_2, \qquad 0 < \omega_1 < \pi/2, \qquad \omega_2 \in \mathbf{R}, \qquad 0 \le \omega_3 < 2\pi.$$

8. Parabolic coordinate system,

$$z_1 = e^{\omega_1+\omega_2}\cos\omega_3, \qquad z_2 = e^{\omega_1+\omega_2}\sin\omega_3, \qquad z_3 = (e^{2\omega_1} - e^{2\omega_2})/2,$$
$$\omega_1, \omega_2 \in \mathbf{R}, \qquad 0 \le \omega_3 \le 2\pi.$$

9. Paraboloidal coordinate system,

$$z_1 = 2a\cosh\omega_1\cos\omega_2\sinh\omega_3, \quad a > 0, \qquad z_2 = 2a\sinh\omega_1\sin\omega_2\cosh\omega_3,$$
$$z_3 = a(\cosh 2\omega_1 + \cos 2\omega_2 - \cosh 2\omega_3)/2, \qquad \omega_1, \omega_3 \in \mathbf{R}, \qquad 0 \le \omega_2 < \pi.$$

10. Ellipsoidal coordinate system,

$$z_1 = a\,\frac{1}{\operatorname{sn}(\omega_1,\, k)}\operatorname{dn}(\omega_2,\, k')\operatorname{sn}(\omega_3,\, k), \quad a > 0, \quad k^2 + k'^2 = 1,$$
$$z_2 = a\,\frac{\operatorname{dn}(\omega_1,\, k)}{\operatorname{sn}(\omega_1,\, k)}\operatorname{cn}(\omega_2,\, k')\operatorname{cn}(\omega_3,\, k), \quad 0 < k, k' < 1,$$
$$z_3 = a\,\frac{\operatorname{cn}(\omega_1,\, k)}{\operatorname{sn}(\omega_1,\, k)}\operatorname{sn}(\omega_2,\, k')\operatorname{dn}(\omega_3,\, k),$$
$$0 < \omega_1 < K, \qquad -K' \le \omega_2 \le K', \qquad 0 \le \omega_3 \le 4K.$$

11. Conical coordinate system,

$$z_1 = \omega_1^{-1}\operatorname{dn}(\omega_2,\, k')\operatorname{sn}(\omega_3,\, k), \quad k^2 + k'^2 = 1, \quad 0 < k, k' < 1,$$
$$z_2 = \omega_1^{-1}\operatorname{cn}(\omega_2,\, k')\operatorname{cn}(\omega_3,\, k), \qquad z_3 = \omega_1^{-1}\operatorname{sn}(\omega_2,\, k')\operatorname{dn}(\omega_3,\, k),$$
$$\omega_1 > 0, \qquad -K' \le \omega_2 \le K', \qquad 0 \le \omega_3 \le 4K.$$

$H(t)$ is the 3×3 diagonal matrix

$$H(t) = \begin{pmatrix} h_1(t) & 0 & 0 \\ 0 & h_2(t) & 0 \\ 0 & 0 & h_3(t) \end{pmatrix}, \tag{14}$$

where

(a) $h_1(t)$, $h_2(t)$, $h_3(t)$ are arbitrary smooth functions for the completely split coordinate system (case 1 from (13)),

(b) $h_1(t) = h_2(t)$, $h_1(t)$, $h_3(t)$ being arbitrary smooth functions, for the partially split coordinate systems (cases 2–4 from (13)),

(c) $h_1(t) = h_2(t) = h_3(t)$, $h_1(t)$ being an arbitrary smooth function, for non-split coordinate systems (cases 5–11 from (13))

and $\vec{w}(t)$ stands for the vector-column whose entries $w_1(t)$, $w_2(t)$, $w_3(t)$ are arbitrary smooth functions of t.

Note that we have chosen the coordinate systems ω_1, ω_2, ω_3 with the use of the equivalence relation $\mathcal{E}$ (5) in such a way that the relations

$$\Delta\omega_a = 0, \qquad a = 1, 2, 3 \tag{15}$$

hold for all the cases 1–11 in (13). Solving (9) with respect to $B_j(\vec{x})$, $i = 1, 2, 3$ we get (see, also [5])

$$\vec{B}(\vec{x}) = \mathcal{M}(t)(\vec{x} - \vec{w}) + \dot{\vec{w}}. \tag{16}$$

Here we use the designation

$$\mathcal{M}(t) = \dot{\mathcal{T}}(t)\mathcal{T}^{-1}(t) + \mathcal{T}(t)\dot{H}(t)H^{-1}(t)\mathcal{T}^{-1}(t), \tag{17}$$

where $\mathcal{T}(t)$, $H(t)$ are variable 3×3 matrices defined by formulae (12) and (14), correspondingly, $\vec{w} = (w_1(t), w_2(t), w_3(t))^T$ and the dot over a symbol means differentiation with respect to t.

As the functions B_1, B_2, B_3 are independent of t, it follows from (16) that

$$\vec{B}(\vec{x}) = \mathcal{M}\vec{x} + \vec{v}, \qquad \vec{v} = \text{const}, \tag{18}$$

$$\mathcal{M} = \text{const}, \tag{19}$$

$$\dot{\vec{w}} = \mathcal{M}\vec{w} + \vec{v}. \tag{20}$$

Taking into account that $\dot{\mathcal{T}}\mathcal{T}^{-1}$ is antisymmetric and $\mathcal{T}\dot{H}H^{-1}\mathcal{T}^{-1}$ is symmetric part of $\mathcal{M}$ (17), correspondingly, we get from (19)

$$\dot{\mathcal{T}}(t)\mathcal{T}^{-1}(t) = \text{const}, \tag{21}$$

$$\mathcal{T}(t)\dot{H}(t)H^{-1}(t)\mathcal{T}^{-1}(t) = \text{const}. \tag{22}$$

Relation (21) yields the system of three ordinary differential equations for the functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$

$$\begin{aligned} &\dot{\alpha} + \dot{\beta}\cos\gamma = C_1, \\ &\dot{\beta}\cos\alpha\sin\gamma - \dot{\gamma}\sin\alpha = C_2, \\ &\dot{\beta}\sin\alpha\sin\gamma + \dot{\gamma}\cos\alpha = C_3, \end{aligned} \tag{23}$$

where C_1, C_2, C_3 are arbitrary real constants. Integrating the above system we obtain the following form of the matrix $\mathcal{T}(t)$:

$$\mathcal{T}(t) = \mathcal{C}_1\tilde{\mathcal{T}}\mathcal{C}_2, \tag{24}$$

where $\mathcal{C}_1$, $\mathcal{C}_2$ are arbitrary constant 3×3 orthogonal matrices and

$$\tilde{\mathcal{T}} = \begin{pmatrix} -\cos s\cos bt & \sin s & \cos s\sin bt \\ \sin bt & 0 & \cos bt \\ \sin s\cos bt & \cos s & -\sin s\sin bt \end{pmatrix} \tag{25}$$

with arbitrary constants b and s.

The substitution of equality (24) into (22) with subsequent differentiation of the obtained equation with respect to t yields

$$\mathcal{C}_2^{-1}\tilde{\mathcal{T}}^{-1}\dot{\tilde{\mathcal{T}}}\,\mathcal{C}_2 L + \dot{L} + L\,\mathcal{C}_2^{-1}(\dot{\tilde{\mathcal{T}}^{-1}})\tilde{\mathcal{T}}\,\mathcal{C}_2 = 0, \tag{26}$$

where $L = \dot{H}H^{-1}$, i.e. $l_i = \dot{h}_i/h_i$, $i = 1, 2, 3$. From (26) we have

$$\begin{aligned}
&l_i = \text{const}, \qquad i = 1, 2, 3;\\
&b\,(l_1 - l_2)\cos\alpha_2\,\sin\gamma_2 = 0,\\
&b\,(l_1 - l_3)(-\sin\alpha_2\,\sin\beta_2 + \cos\alpha_2\,\cos\beta_2\cos\gamma_2) = 0,\\
&b\,(l_2 - l_3)(\sin\alpha_2\,\cos\beta_2 + \cos\alpha_2\,\sin\beta_2\cos\gamma_2) = 0,
\end{aligned} \tag{27}$$

where α_2, β_2, γ_2 are the Euler angles for the orthogonal matrix $\mathcal{C}_2$. Thus we obtain the following forms of h_i:

$$h_i = c_i\exp(l_i t), \qquad c_i = \text{const}, \quad l_i = \text{const}, \quad i = 1, 2, 3. \tag{28}$$

From (27) we get the possible forms of b, l_i and $\mathcal{C}_2$:

$$\begin{aligned}
&(i) \quad b = 0, \quad l_1, l_2, l_3 \text{ are arbitrary constants,}\\
&\qquad \mathcal{C}_2 \text{ is an arbitrary constant orthogonal matrix;}\\
&(ii) \quad b \neq 0, \quad l_1 = l_2 = l_3,\\
&\qquad \mathcal{C}_2 \text{ is an arbitrary constant orthogonal matrix;}\\
&(iii) \quad b \neq 0, \quad l_1 = l_2 \neq l_3, \quad \mathcal{C}_2 = \begin{pmatrix} \varepsilon_1\cos\theta & -\varepsilon_1\sin\theta & 0\\ 0 & 0 & -\varepsilon_1\varepsilon_2\\ \varepsilon_2\sin\theta & \varepsilon_2\cos\theta & 0 \end{pmatrix},
\end{aligned} \tag{29}$$

where ε_1, $\varepsilon_2 = \pm 1$, and θ is arbitrary constant. We do not adduce cases $b \neq 0$, $l_1 \neq l_2 = l_3$ and $b \neq 0$, $l_2 \neq l_1 = l_3$ because they are equivalent to case (iii).

Finally, we give a list of the drift velocity vectors $\vec{B}(\vec{x})$ providing separability of the corresponding FPEs. They have the following form:

$$\vec{B}(\vec{x}) = \mathcal{M}\vec{x} + \vec{v},$$

where $\vec{v}$ is arbitrary constant vector and $\mathcal{M}$ is constant matrix given by one of the following formulae:

1. $\mathcal{M} = \mathcal{T}L\mathcal{T}^{-1}$, where $L = \begin{pmatrix} l_1 & 0 & 0\\ 0 & l_2 & 0\\ 0 & 0 & l_3 \end{pmatrix}$, l_1, l_2, l_3 are constants and $\mathcal{T}$ is an arbitrary constant 3×3 orthogonal matrix, i.e. $\mathcal{M}$ is a real symmetric matrix with eigenvalues l_1, l_2, l_3.

   (a) l_1, l_2, l_3 are all distinct. The new coordinates ω_1, ω_2, ω_3 are given implicitly by formula

   $$\vec{x} = \mathcal{T}H(t)\,\vec{z}(\vec{\omega}) + \vec{w}(t), \tag{30}$$

   where $\vec{z}(\vec{\omega})$ is given by formula 1 from (13), $\vec{w}(t)$ is solution of system of ordinary differential equations (20) and

   $$H(t) = \begin{pmatrix} c_1 e^{l_1 t} & 0 & 0\\ 0 & c_2 e^{l_2 t} & 0\\ 0 & 0 & c_3 e^{l_3 t} \end{pmatrix} \tag{31}$$

   with arbitrary constants c_1, c_2, c_3.

(b) $l_1 = l_2 \neq l_3$. The new coordinates ω_1, ω_2, ω_3 are given implicitly by (30), where $\vec{z}(\vec{\omega})$ is given by one of the formulae 1–4 from (13) and $H(t)$ is given by (31) with arbitrary constant c_1, c_2, c_3 satisfying the condition $c_1 = c_2$ for the partially split coordinates 2–4 from (13).

(c) $l_1 = l_2 = l_3$, i.e. $M = l_1 I$, where I is unit matrix. The new coordinates ω_1, ω_2, ω_3 are given implicitly by formula (30), where $\vec{z}(\vec{\omega})$ is given by one of the eleven formulae (13) and $H(t)$ is given by (31) with arbitrary constants c_1, c_2, c_3 satisfying the condition $c_1 = c_2$ for the partially split coordinates 2–4 from (13) and the condition $c_1 = c_2 = c_3$ for the non-split coordinates 5–11 from (13).

2. $M = b\,\mathcal{C}_1 \begin{pmatrix} 0 & \cos s & 0 \\ -\cos s & 0 & \sin s \\ 0 & -\sin s & 0 \end{pmatrix} \mathcal{C}_1^{-1} + l_1 I$, where I is the unit matrix and $\mathcal{C}_1$ is an arbitrary constant 3×3 orthogonal matrix, b, s, l_1 are arbitrary constants and $b \neq 0$. The new coordinates ω_1, ω_2, ω_3 are given implicitly by formula (11), where $\vec{z}(\vec{\omega})$ is given by one of the eleven formulae (13), $\mathcal{T}(t)$ is given by (24)–(25), $\vec{w}(t)$ is solution of system of ordinary differential equations (20) and $H(t) = \exp(l_1 t) \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$ with arbitrary constants c_1, c_2, c_3 satisfying the condition $c_1 = c_2$ for the partially split coordinates 2–4 from (13) and the condition $c_1 = c_2 = c_3$ for non-split coordinates 5–11 from (13).

3. $M = \mathcal{C}_1 \begin{pmatrix} \frac{1}{2}(l_1 + l_3 + (l_1 - l_3)\cos 2s) & b\cos s & \frac{1}{2}(l_3 - l_1)\sin 2s \\ -b\cos s & l_1 & b\sin s \\ \frac{1}{2}(l_3 - l_1)\sin 2s & -b\sin s & \frac{1}{2}(l_1 + l_3 - (l_1 - l_3)\cos 2s) \end{pmatrix} \mathcal{C}_1^{-1}$, where $\mathcal{C}_1$ is an arbitrary constant 3×3 orthogonal matrix, b, s, l_1, l_2 are arbitrary constants, $l_1 \neq l_3$ and $b \neq 0$. The new coordinates ω_1, ω_2, ω_3 are given implicitly by formula (11), where $\vec{z}(\vec{\omega})$ is given by one of the formulae 1–4 from (13), $\mathcal{T}(t)$ is given by (24), (25) and (iii) from (29), $\vec{w}(t)$ is solution of system of ordinary differential equations (20) and

$$H(t) = \begin{pmatrix} c_1 e^{l_1 t} & 0 & 0 \\ 0 & c_2 e^{l_1 t} & 0 \\ 0 & 0 & c_3 e^{l_3 t} \end{pmatrix}$$

with arbitrary constants c_1, c_2, c_3 satisfying the condition $c_1 = c_2$ for the partially split coordinates 2–4 from (13).

Acknowledgments

I would like to thank Renat Zhdanov for his time, suggestions, and encouragement.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 255–263.

Higher Conditional Symmetries and Reduction of Initial Value Problems for Nonlinear Evolution Equations

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We prove that the presence of higher conditional symmetry is the necessary and sufficient condition for reduction of an arbitrary evolution equation in two variables to a system of ordinary differential equations. Furthermore, we give the sufficient condition for an initial value problem for an evolution equation to be reducible to a Cauchy problem for a system of ordinary differential equations, provided it possesses higher conditional symmetry.

1 Introduction

Consider a nonlinear evolution type partial differential equation (PDE) in two independent variables t, x

$$u_t = F(t, x, u, u_1, u_2, \ldots, u_n), \tag{1}$$

where $u \in C^n(\mathbf{R^2}, \mathbf{R^1})$, $u_k = \partial^k u/\partial x^k$, $1 \le k \le n$.

As is well known, a possibility of reduction of (1) to a single ordinary differential equation (ODE) is intimately connected to its Lie symmetry under a group of point transformations (see, e.g., [1–3]). It has been recently established that a reducibility of any PDE in two variables to a single ODE is in one-to-one correspondence with its Q-conditional (non-classical) symmetry [4] (see, also [5–10]). Furthermore, integrability of equations of the form (1) by the method of the inverse scattering transform is a consequence of its invariance with respect to a non-point group of infinitesimal transformations

$$u' = u + \varepsilon\, \eta(t, x, u, u_1, \ldots, u_N),$$
$$u'_x = u_x + \varepsilon\, D_x\, \eta(t, x, u, u_1, \ldots, u_N), \quad \ldots$$

generated by the Lie–Bäcklund vector field (LBVF)

$$Q = \sum_{k=0}^{\infty} \left(D_x^k\, \eta\right) \frac{\partial}{\partial u_k} \equiv \eta \frac{\partial}{\partial_u} + (D_x\, \eta) \frac{\partial}{\partial_{u_1}} + \left(D_x^2\, \eta\right) \frac{\partial}{\partial_{u_2}} + \cdots. \tag{2}$$

In the above formulae we denote by the symbols D_x the total differentiation operator with respect to the variable x, i.e.

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1} \frac{\partial}{\partial u_k}.$$

Note that if the function η has the structure

$$\eta = \tilde{\eta}(t, x, u) - \xi_0(t, x, u) u_t - \xi_1(t, x, u) u_x, \tag{3}$$

then LBVF (2) is equivalent to the usual Lie vector field and can be represented in the standard form [11]:

$$Q = \xi_0(t,x,u)\frac{\partial}{\partial t} + \xi_1(t,x,u)\frac{\partial}{\partial x} + \tilde{\eta}(t,x,u)\frac{\partial}{\partial u}.$$

It was noted by Galaktionov [12] that a number of nonlinear PDEs, that were non-integrable within the framework of the method of the inverse scattering transform, possessed a remarkable property, namely, they could be reduced to *systems* of ordinary differential equations with the help of appropriate Ansätze. A natural question arises, which symmetry is responsible for this kind of reduction? Evidently, this symmetry cannot be Q-conditional symmetry since the latter gives rise to reduction of PDE under study to a single ODE. It has been conjectured in [13, 14] (see, also [15, 16]) that it is higher conditional symmetry that provides this type of reduction. This conjecture has been proved in [17]. In the present paper, we show that the property of reducibility of evolution type equations (1) to several ODEs is in one-to-one correspondence with their higher conditional symmetry. Next, we give the sufficient condition for the initial value problem for PDE (1) to be reducible to the Cauchy problem for some system of ODEs.

2 Reduction criterion

Let us first introduce the necessary definitions.

Definition 1. *We say that PDE (1) is invariant under the LBVF (2) if the condition*

$$Q(u_t - F)\Big|_M = 0 \tag{4}$$

holds. In (4) M is a set of all differential consequences of the equation $u_t - F = 0$.

Definition 2. *We say that PDE (1) is conditionally-invariant under LBVF (2) if the following condition*

$$Q(u_t - F)\Big|_{M\cap L_x} = 0 \tag{5}$$

holds. Here the symbol L_x denotes the set of all differential consequences of the equation $\eta = 0$ with respect to the variable x.

Evidently, condition (4) is nothing else than the usual invariance criterion for equation (1) under LBVF (2) written in a canonical form (see, e.g. [11]). The most of the “soliton equations” admit infinitely many LBVFs which can be obtained by repeatedly applying the recursion operator to some initial LBVF.

Clearly, if PDE (1) is invariant under LBVF (2), then it is conditionally-invariant under it; however, the inverse assertion is not true. This means, in particular, that Definition 2 is a generalization of the standard definition of invariance of partial differential equation with respect to LBVF. Provided (2) is a Lie vector field, Definition 2 coincides with the one of Q-conditional invariance under the Lie vector field.

If we consider the nonlinear PDE

$$\eta(t,x,u,u_1,\ldots,u_N) = 0 \tag{6}$$

as the N-th order ODE with respect to variable x, then its general integral can be (locally) represented in the form

$$u(t,x) = U\left(t,x,\varphi_1(t),\varphi_2(t),\ldots,\varphi_N(t)\right), \tag{7}$$

where $\varphi_j(t)$, $(j = 1, \dots N)$ are arbitrary smooth functions. In a sequel, we call expression (7) the Ansatz invariant under LBVF (2).

Theorem 1. *Let equation (1) with $F \in C^{N+1}(\mathcal{D})$, where $\mathcal{D}$ is an open domain in $\mathbf{R}^{n+3}$, be conditionally-invariant under LBVF (2) with $\eta \in C^2(\mathcal{D}')$, where $\mathcal{D}'$ is an open domain in $\mathbf{R}^{N+3}$ and, furthermore, $\partial\eta/\partial u_N \neq 0$ on $\mathcal{D}'$. Then Ansatz (7) invariant under LBVF (2) reduces PDE (1) to a system of N ODEs for the functions $\varphi_j(t)$, $(j = 1, \dots, N)$*

$$\dot{\varphi}_j = F_j(t, \varphi_1, \dots, \varphi_N), \qquad j = 1, \dots, N. \tag{8}$$

Suppose now the inverse, namely, that Ansatz (7), where the function U and its derivatives $\partial U^{k+1}/\partial\varphi_j\partial x^k$, $(j = 1, \dots, N,\ k = 0, \dots, N)$ exist and are continuous on an open domain $\mathcal{D}_1$ in $\mathbf{R}^{N+2}$, reduces (1) to system of ODEs (8) with $F_i \in C^1(\mathcal{D}'_1)$, where $\mathcal{D}'_1$ is an open domain in $\mathbf{R}^{N+2}$. Then, there exists such LBVF (2) that equation (1) is conditionally-invariant with respect to it.

The proof of the first part of the theorem (i.e., of the assertion *conditional symmetry* $\rightarrow$ *reduction*) is given in our paper [17]. That is why, we give the proof of the second part of the theorem, namely, we prove the implication *reduction* $\rightarrow$ *conditional symmetry.* As the functions F_j, $(j = 1, \dots, N)$ satisfy the conditions of the theorem on existence and uniqueness of a solution of a Cauchy problem for system of ODEs (8), there exists an open domain $\mathcal{T} \times \mathcal{D}_2 \subset \mathbf{R}^{N+1}$ such that for any $t_0 \in \mathcal{T}$, $(C_1, \dots, C_N) \in \mathcal{D}_2$ there is a solution of (8) such that

$$\varphi_j(t_0) = C_j, \quad j = 1, \dots, N.$$

Thus we have the N-parameter family of exact solutions of equation (1)

$$u(t, x) = u_0(t, x; C_1, \dots, C_N), \qquad (C_1, \dots, C_N) \in \mathcal{D}_2. \tag{9}$$

Consider now the system of equations

$$\begin{aligned}
&u = U\left(t, x, \varphi_1(t), \varphi_2(t), \dots, \varphi_N(t)\right),\\
&u_1 = D_x\, U\left(t, x, \varphi_1(t), \varphi_2(t), \dots, \varphi_N(t)\right), \dots,\\
&u_{N-1} = D_x^{N-1}\, U\left(t, x, \varphi_1(t), \varphi_2(t), \dots, \varphi_N(t)\right).
\end{aligned}$$

Given the conditions of the theorem, we can solve (locally) the above system with respect to $\varphi_j(t)$, $(j = 1, \dots, N)$ and get

$$\varphi_j(t) = \Phi_j\left(t, x, u, u_1, \dots, u_{N-1}\right), \qquad j = 1, \dots, N.$$

Differentiating any of the above equations (say, the first one) with respect to x yields an Nth order ODE

$$\tilde{\eta}(t, x, u, u_1, \dots, u_N) = 0$$

such that (7) is its general integral. Consequently, the system of partial differential equations

$$u_t = F(t, x, u, u_1, u_2, \dots, u_n), \qquad \tilde{\eta}(t, x, u, u_1, \dots, u_N) = 0 \tag{10}$$

has (locally) a solution (9) that depends on N arbitrary constants $(C_1, \dots, C_N) \in \mathcal{D}_2$. Whence, using the Cartan's criterion we conclude that over-determined system (10) is in involution. It has been proved in [17] that system of PDEs (10) is in involution if and only if condition (5) with

$$Q = \sum_{k=0}^{\infty} \left(D_x^k\, \tilde{\eta}\right) \frac{\partial}{\partial u_k}$$

holds true. Whence we conclude that (1) is conditionally-invariant with respect to so constructed LBVF Q, which is the same as what was to be proved.

We will finish this section by giving the two examples of reduction of nonlinear evolution equations with the use of higher symmetries.

Example 1. Consider the KdV equation

$$u_t = u_3 + uu_1. \tag{11}$$

It is a common knowledge (see, e.g., [11]) that the KdV equation (11) possesses infinitely many higher symmetries within the class of LBVFs (2). In particular, it admits symmetry operator (2) with

$$\eta = u_5 + \frac{5}{3}\,uu_3 + \frac{10}{3}\,u_1u_2 + \frac{5}{6}\,u^2u_1. \tag{12}$$

Though we have not succeeded in integrating ODE $\eta = 0$ and, consequently, have not constructed the explicit form of Ansatz (7), it proves to be possible to derive the form of system of ODEs (8) in the case under consideration. To this end we choose in (7) the functions $\varphi_j(t)$, $(j = 1, \ldots, 5)$ in the following way:

$$\varphi_j(t) = \left(D^{j-1}u(t,x)\right)\big|_{x=x_0}, \qquad j = 1, 2, \ldots, N \tag{13}$$

with some constant x_0. Then the first equation of system (8) is obtained by putting in (11) $x = x_0$ and using (13). The second equation is obtained by differentiating (11) with respect to x with subsequent putting $x = x_0$ and using (13) and so on. This procedure will end when we will take the fifth derivative of (11), since due to invariance of the equation under study under LBVF (2) with η of the form (12) thus obtained relation turns out to be the identity. So that the system of ODEs for unknown functions $\varphi_j(t)$, $(j = 1, \ldots, 5)$ reads as

$$\dot\varphi_j = D_x^{j-1}\left(u_3 + uu_1\right)$$

or

$$\begin{aligned}
&\dot\varphi_1 = \varphi_1\varphi_2 + \varphi_4,\\
&\dot\varphi_2 = \varphi_2^2 + \varphi_1\varphi_3 + \varphi_5,\\
&\dot\varphi_3 = -\frac{5}{6}\varphi_1^2\varphi_2 - \frac{1}{3}\varphi_2\varphi_3 - \frac{2}{3}\varphi_1\varphi_4,\\
&\dot\varphi_4 = -\frac{5}{3}\varphi_1\varphi_2^2 - \frac{5}{6}\varphi_1^2\varphi_3 - \frac{1}{3}\varphi_3^2 - \varphi_2\varphi_4 - \frac{2}{3}\varphi_1\varphi_5,\\
&\dot\varphi_5 = \frac{5}{9}\varphi_1^3\varphi_2 - \frac{5}{6}\varphi_2^3 - \frac{25}{9}\varphi_1\varphi_2\varphi_3 + \frac{5}{18}\varphi_1^2\varphi_4 - \frac{5}{3}\varphi_3\varphi_4 - \frac{5}{3}\varphi_2\varphi_5.
\end{aligned} \tag{14}$$

Thus it is possible to use efficiently higher symmetries of solitonic equations in order to study the dynamics of solitons which is described by the system of ODEs of the form (14). Needless to say, that the above described method for obtaining systems of ODEs (8) without direct integration of equation (6) can be applied to any evolution type PDE (1).

Example 2. As the direct check shows, the nonlinear PDE

$$u_t = uu_2 - \frac{3}{4}u_1^2 + k^2u^2, \qquad k = \text{const}, \quad k \neq 0 \tag{15}$$

is conditionally-invariant with respect to LBVF (2) under

$$\eta = u_5 + 5k^2u_3 + 4k^4u_1. \tag{16}$$

Note that equation (15) admits no higher symmetries and is non-integrable by the method of the inverse scattering transform.

Integrating the equation $\eta = 0$ yields the Ansatz for $u(t,x)$

$$u(t,x) = \varphi_1(t) + \varphi_2(t)\cos(kx) + \varphi_3(t)\sin(kx) + \varphi_4(t)\cos(2kx) + \varphi_5(t)\sin(2kx) \tag{17}$$

that reduces PDE (15) to the system of five ODEs for unknown functions $\varphi_j(t)$, $(j = 1, \ldots, 5)$

$$\begin{aligned}
\dot\varphi_1 &= -3k^2(\varphi_4^2 + \varphi_5^2) - \frac{3k^2}{8}(\varphi_2^2 + \varphi_3^2) + k^2\varphi_1^2,\\
\dot\varphi_2 &= -3k^2(\varphi_2\varphi_4 + \varphi_3\varphi_5) + k^2\varphi_1\varphi_2,\\
\dot\varphi_3 &= -3k^2(\varphi_2\varphi_5 - \varphi_3\varphi_4) + k^2\varphi_1\varphi_3,\\
\dot\varphi_4 &= \frac{3k^2}{8}(\varphi_1^2 - \varphi_2^2) - 2k^2\varphi_1\varphi_4,\\
\dot\varphi_5 &= -2k^2\varphi_1\varphi_5 + \frac{3k^2}{4}\varphi_3\varphi_5.
\end{aligned} \tag{18}$$

3 Reduction of initial value problems

Consider an initial value problem for an evolution type PDE (1)

$$\begin{cases} u_t = F(t,x,u,u_1,u_2,\ldots,u_n),\\ (\alpha(x)u_1 + \beta(x)u)|_{t=0} = \gamma(x), \end{cases} \tag{19}$$

where $\alpha(x)$, $\beta(x)$, $\gamma(x)$ are some smooth functions.

There is a technique that enables using Lie (first and higher order) symmetry in order to carry our the dimensional reduction of problem (19). So there arises a natural question, whether higher conditional symmetry can be used in this respect. It is natural to expect that, provided PDE (1) admits higher order conditional symmetry, there exist such functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$ that the initial value problem (19) reduces by virtue of the Ansatz (7) to the Cauchy problem for the functions $\varphi_j(t)$, $(j = 1, \ldots, N)$. This means that PDE (1) should reduce to a system of ODEs (8) and the initial condition given in (19) should reduce to algebraic relations prescribing the values of the functions $\varphi_j(t)$, $(j = 1, \ldots, N)$ under $t = 0$. Saying it another way, we have to answer the two fundamental questions:

- Is the above described reduction of the initial value problem (19) possible?
- Which constraints should be imposed on the functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$ in order to provide dimensional reduction of the problem (18)?

The answer to the first question is positive, which is quite predictable in view of the fact that higher conditional symmetry is a generalization of a usual higher Lie symmetry. What is more, we will give without proof a simple assertion that provides us with an efficient way of describing initial conditions enabling dimensional reduction of the initial value problem for an evolution type PDE that admits higher conditional symmetry. To this end we need the notion of a regular compatibility to be introduced below.

Consider the following system of two PDEs

$$\eta(t,x,u,u_1,\ldots,u_N) = 0, \qquad a(t,x)u_1 + b(t,x)u - c(t,x) = 0 \tag{20}$$

and suppose that the PDE $\eta = 0$ is conditionally invariant with respect to a one-parameter group having the generator

$$X = a(t,x)\frac{\partial}{\partial x} - (b(t,x)u - c(t,x))\frac{\partial}{\partial u}. \tag{21}$$

Integrating the second PDE from (20) yields the Ansatz for the function $u(t,x)$

$$u(t,x) = f(t,x)\varphi(t) + g(t,x)$$

with some fixed functions f, g and an arbitrary smooth function φ. As the equation $\eta = 0$ is conditionally invariant with respect to the operator Q, inserting the above Ansatz into the first PDE from (20) yields an equation of the form $F(t,\varphi(t)) = 0$. We say that system (20) is regularly compatible if the solution of the equation $F = 0$ exists, and furthermore, inserting it into the Ansatz for $u(t,x)$ yields a non-singular solution of the equation $\eta = 0$ considered as ODE with respect to x.

Theorem 2. *Let equation (1) be conditionally invariant with respect to LBVF*

$$Q = \sum_{k=0}^{\infty} \left(D_x^k \eta\right) \frac{\partial}{\partial u_k}, \qquad \eta = \eta(t,x,u,u_1,\ldots,u_N)$$

and PDE $\eta = 0$ be conditionally invariant with respect to operator (21). Furthermore, we suppose that system (20) is regularly compatible. Then Ansatz (7) invariant under LBVF Q reduces (19) with $\alpha(x) = a(0,x), \beta(x) = b(0,x), \gamma(x) = c(0,x)$ to a Cauchy problem for the functions $\varphi_j(t)$, $(j = 1,\ldots,N)$.

Since usual and higher Lie symmetries as well as Q-conditional (non-classical) symmetry are particular cases of higher conditional symmetry, it follows from the above theorem that the enumerated symmetries can also be applied to reduce the initial value problem (19).

As an illustration to Theorem 2, we give the following two examples.

Example 3. Consider the initial value problem for PDE (15)

$$\begin{cases} u_t = uu_2 - \dfrac{3}{4}u_1^2 + k^2u^2, \\ (\alpha(x)u_1 + \beta(x)u)|_{t=0} = \gamma(x). \end{cases} \tag{22}$$

As we have mentioned in the previous section, PDE (15) is conditionally-invariant with respect to LBVF (2), (16). Using the standard Lie method (see, e.g., [1]–[3]) one can prove that PDE $u_5 + 5k^2u_3 + 4k^2u_1 = 0$ is invariant with respect to the group having the infinitesimal generator (21), where

$$\begin{aligned} a(t,x) &= C_1\cos(kx) + C_2\sin(kx) + C_3, \\ b(t,x) &= 2k(C_1\sin(kx) - C_2\cos(kx)) + C_0, \\ c(t,x) &= C_4\cos(2kx) + C_5\sin(2kx) + C_6\cos(kx) + C_7\sin(kx) + C_8, \end{aligned} \tag{23}$$

$C_0, C_1, \ldots, C_8$ being arbitrary constants.

Ansatz (17) reduces PDE (15) to system of ODEs (18). Next, inserting (17) into the initial condition from (22) under (23) yields the system of algebraic relations

$$A\vec{\varphi}(0) = \vec{B}, \tag{24}$$

where $\vec{\varphi}(0) = (\varphi_1(0),\ldots,\varphi_5(0))$, $\vec{B} = (C_4, C_8, C_7, C_6, C_5)$ and

$$A = \begin{pmatrix} C_0 & -\frac{3k}{2}C_2 & \frac{3k}{2}C_1 & 0 & 0 \\ 0 & \frac{k}{2}C_1 & -\frac{k}{2}C_2 & -2kC_3 & C_0 \\ 0 & -\frac{k}{2}C_2 & -\frac{k}{2}C_1 & C_0 & -2kC_3 \\ 2kC_1 & -kC_3 & C_0 & -2kC_1 & -2kC_2 \\ -2kC_2 & C_0 & kC_3 & -2kC_2 & 2kC_1 \end{pmatrix}.$$

Note that the determinant of the matrix A equals to zero only in the following three cases,

$$(1)\quad C_0 = 0;$$

$$(2)\quad k = \frac{C_0}{2}\left(C_1^2 + C_2^2 - C_3^2\right)^{-1/2}, \qquad C_0 \neq 0;$$

$$(2)\quad k = C_0\left(C_1^2 + C_2^2 - C_3^2\right)^{-1/2}, \qquad C_0 \neq 0.$$

Provided none of the above relations holds true, the matrix A is non-singular and we can resolve (24) with respect to $\vec{\varphi}(0)$ thus getting the initial Cauchy data for the system of ODEs (18)

$$\vec{\varphi}(0) = A^{-1}\vec{B}.$$

Example 4. Let us apply the results of the previous example for constructing the (unique) solution of the following initial value problem:

$$\begin{cases} u_t = uu_2 - \dfrac{1}{4}\left(3u_1^2 + u^2\right), \\ u(0, x) = \sin x. \end{cases} \tag{25}$$

Evidently, the above problem is a particular case of the initial value problem (22), (23) under $k = 1/2$, $C_0 = C_1 = C_2 = C_3 = C_4 = 0$, $C_5 = 1$, $C_6 = C_7 = C_8 = 0$. That is why we can use the Ansatz (17) with $k = 1/2$ in order to reduce the problem (25). The initial condition reduces to the following Cauchy data:

$$\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = \varphi_4(0) = 0, \qquad \varphi_5(0) = 1.$$

Taking into account this fact we put in (18)

$$\varphi_2(t) \equiv 0, \qquad \varphi_3(t) \equiv 0, \qquad \varphi_4(t) \equiv 0,$$

the remaining functions $\varphi_1(t)$, $\varphi_5(t)$ satisfying the system of ODEs

$$\dot{\varphi}_1 = \frac{1}{4}\varphi_1^2 - \frac{3}{4}\varphi_5^2, \qquad \dot{\varphi}_5 = -\frac{1}{2}\varphi_1\varphi_5$$

under the following initial conditions

$$\varphi_1(0) = 0, \qquad \varphi_5(0) = 1.$$

The above system of ODEs is integrated in a closed form. Imposing the initial Cauchy data we arrive at the following solution:

$$\varphi_1(t) = -\sqrt{(w(t))^2 - (w(t))^{-1}}, \qquad \varphi_5(t) = w(t),$$

where $w(t)$ is the Jacobi elliptic function

$$\int_1^{w(t)} \frac{dy}{\sqrt{y^4 - y}} = \frac{1}{2}t.$$

Inserting the obtained expressions for the functions $\varphi_j(t)$ into the Ansatz (17) with $k = 1/2$ yields the final form of the (unique) solution of the initial value problem (25) for the nonlinear heat conductivity equation (15)

$$u(t,x) = -\sqrt{(w(t))^2 - (w(t))^{-1}} + w(t)\sin x.$$

4 Some conclusions

Thus it is higher conditional symmetry which is responsible for a phenomena of “anti-reduction” or “nonlinear separation of variables” in evolution PDEs. It is one of the principal results of the paper that the one-to-one correspondence **reduction to a single ODE ↔ conditional (non-classical) symmetry** is extended to the following one: **reduction to a system of ODEs ↔ higher conditional symmetry**.

Another important conclusion is that higher conditional symmetries play the same role in the theory of PDEs admitting “nonlinear separation of variables” as second order Lie symmetries in the theory of variable separation in linear PDEs (see, e.g., [18, 19]). This intriguing analogy makes one suspicious that there is a possibility to exploit second-order conditional symmetries in order to get new coordinate systems providing separability of classical linear equations of mathematical physics.

Recently, a number of papers devoted to application of higher Lie symmetries to analysis of boundary problems for PDEs in two dimensions, that admit higher Lie symmetries, have been published (see the paper [20] and the references therein). We believe that higher conditional symmetries can also be efficiently applied to reduction of boundary problems.

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Наукове видання

Праці
Третьої міжнародної конференції

Симетрія
в нелінійній математичній фізиці

Частина 1

Редактори: *А.Г. Нікітін, В.М. Бойко*

Комп'ютерна верстка: *В.М. Бойко, Р.О. Попович*

Художнє оформлення: *Р.О. Попович, Г.В. Попович*

Коректори: *І.А. Єгорченко, А.Г. Сергєєв*

Підп. до друку 21.12.1999. Формат 60x90/8. Папір офсетний. Друк цифровий.
Обл.-вид. арк. 23,6. Ум. друк. арк. 30,8. Тираж 200 пр.

Підготовано до друку в Інституті математики НАН України
01601 Київ-4, МСП, вул. Терещенківська, 3
тел.: (044) 224–63–22, E-mail: appmath@imath.kiev.ua, Web-page: www.imath.kiev.ua/~appmath

Надруковано в Міжрегіональній Академії управління персоналом (МАУП)
03039 Київ-39, вул. Фрометівська, 2
тел.: (044) 264–57–94, E-mail: iapm@carrier.kiev.ua

Proceedings

of the Third International Conference

SYMMETRY

in Nonlinear Mathematical Physics

KYIV 1999

Mathematics and its applications

Institute of Mathematics
National Academy of Sciences of Ukraine
Kyiv, Ukraine

Proceedings of Institute of Mathematics of the National Academy of Sciences of Ukraine **Volume 30**

SERIES Mathematical Physics

Proceedings

of the Third International Conference

SYMMETRY

in Nonlinear Mathematical Physics

Kyiv, Ukraine *12–18 July 1999*

Part 2

Institute of Mathematics of NAS of Ukraine

Kyiv • 2000

УДК 517.95:517.958:512.81(06)

Симетрія у нелінійній математичній фізиці // Праці Інституту математики НАН України. — Т. 30. — Ч. 2. — Київ: Інститут математики НАН України, 2000 / Ред.: А.Г. Нікітін, В.М. Бойко. — 288 с.

Цей том "Праць Інституту математики НАН України" є збірником статей учасників Третьої міжнародної конференції "Симетрія у нелінійній математичній фізиці". Збірник складається з двох частин, кожна з яких видана окремою книгою.

Дане видання є другою частиною і включає оригінальні праці, присвячені застосуванню теоретико-групових методів у сучасній математичній та теоретичній фізиці, а також інших природничих науках. Коло досліджуваних проблем включає квантові групи, зображення груп і алгебр Лі та деякі прикладні задачі, що піддаються дослідженню з використанням симетрійного аналізу.

Розраховано на наукових працівників, аспірантів, які цікавляться сучасними методами теорії груп і узагальненими симетріями математичних моделей.

Symmetry in Nonlinear Mathematical Physics // Proceedings of Institute of Mathematics of NAS of Ukraine. — V. 30. — Part 2. — Kyiv: Institute of Mathematics of NAS of Ukraine, 2000 / Eds.: A.G. Nikitin, V.M.Boyko. — 288 p.

This volume of the Proceedings of Institute of Mathematics of NAS of Ukraine includes papers of participants of the Third International Conference "Symmetry in Nonlinear Mathematical Physics". The collection consists of two parts which are published as separate issues.

This issue is the second part which is devoted to the applications of group theoretical methods in modern mathematical and theoretical physics and other natural sciences. The main topics covered are quantum groups, representations of Lie groups and Lie algebras, and some applied problems which can be investigated using symmetry methods.

The book may be useful for researchers and post graduate students who are interested in modern problems of group theory and generalized symmetries of mathematical models.

Затверджено до друку вченою радою Інституту математики НАН України

Редактори: *А.Г. Нікітін, В.М. Бойко*
Editors: *A.G. Nikitin, V.M. Boyko*

Рецензенти: *В.А. Даниленко, І.О. Луковський*
Referees: *V.A. Danylenko, I.O. Lukovsky*

ISBN 966–02–1444–8

ISBN 966–02–1402–2 (Part 2)

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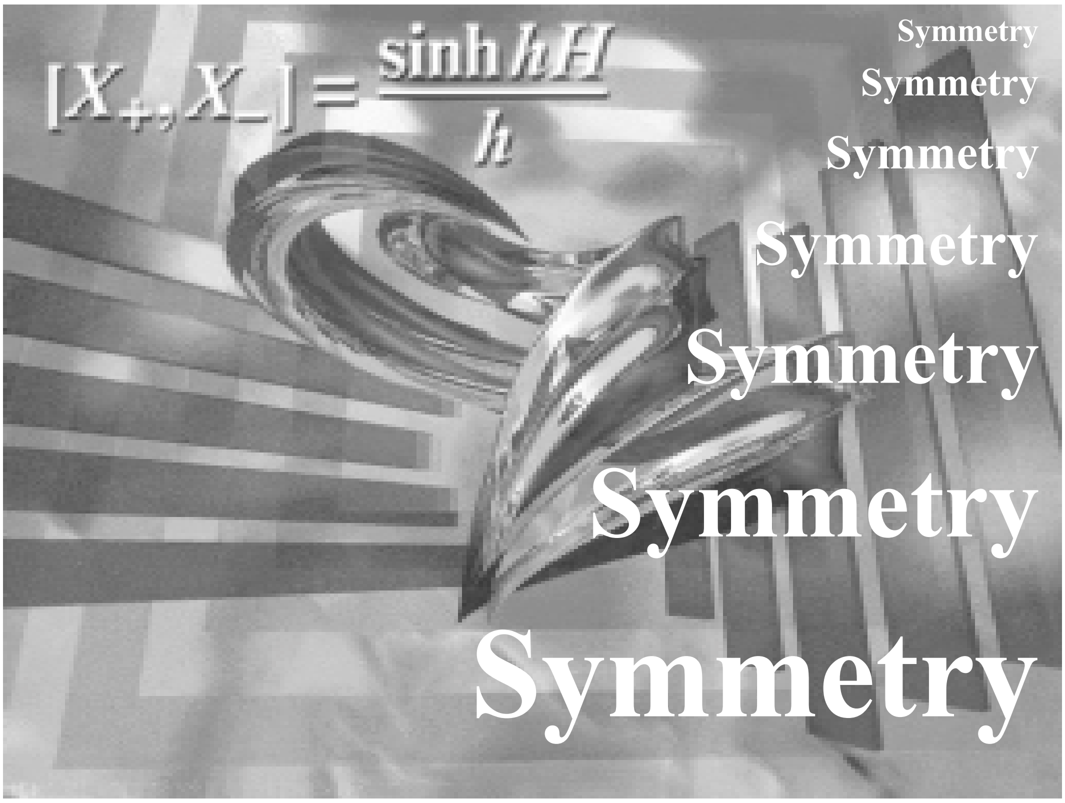

Symmetry in Nonlinear Mathematical Physics

Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 275–279.

On Nonlinear Deformations of Lie Algebras and their Applications in Quantum Physics

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The $sl(2,R)$-Lie algebra is the one of the simplest Lie algebras dealing with particularly important concepts in *quantum* physics, i.e. the angular momentum theory. Taken as an example, we then study some of its specific *polynomial* deformations leading to quadratic and cubic nonlinearities appearing inside symmetry algebras of recent interest in conformal field theory and quantum optics. The determination of their finite-dimensional representations in terms of *differential* operators is then discussed and their interest in connection with multi-boson Hamiltonians is pointed out.

1 Introduction

Already introduced in the proceedings of the *second* conference [1], the role of the linear simple $sl(2,R)$-Lie algebra is very well understood by physicists and mathematicians due mainly to its interest in connection with the famous angular momentum theory [2, 3] when *quantum* aspects of physics are considered. There, I have reported on some new results already published elsewhere [4–7] obtained in the characterization of irreducible representations of finite dimensions *but* for the so-called "*nonlinear*" $sl(2,R)$-algebras with a particular emphasis on the Higgs algebra [8, 9] which is frequently mentioned as a *cubic* deformation of $sl(2,R)$.

Here I also want to insist on another approach of such finite-dimensional irreducible representations characterizing these "*nonlinear*" $sl(2,R)$-algebras by coming on already published [10] and not yet published [11] results dealing more particularly with *differential* realizations of the generators. These polynomial deformations of $sl(2,R)$, in prolongation of well known results obtained in the linear context by Turbiner [12, 13] or (and) Ushveridze [12, 14] in particular, will be of special interest for the study of multi-boson Hamiltonians introduced in quantum *optical* models [15], for example. In fact, these nonlinear structures can play the role of "spectrum generating algebras" for such Hamiltonian descriptions and their irreducible representations can give us a lot of nice and meaningful contexts.

In *Section 2*, we recall a few interesting relations and information on well known results but go relatively quickly to *Section 3* for characterizing the differential forms of special interest for the generators of the structures we are visiting. In *Section 4*, the connection with optical models is proposed and the discussion of the multi-boson Hamiltonians is considered: it finally leads to conclusions on constructive developments associated with the Higgs algebra. Some considerations on supersymmetric properties are also pointed out by taking care of Witten's proposal [16] of supersymmetric quantum mechanics when two supercharges enter the game.

2 A brief survey of the "nonlinear" context

Our *"nonlinear"* $sl(2,R)$-algebras [4] are characterized by the typical commutation relation

$$[J_+, J_-] = f(J_3) = \sum_{p=0}^{N} \beta_p (2J_3)^{2p+1} \tag{1}$$

instead of the following one

$$[J_+, J_-] = 2J_3 \tag{2}$$

referring to the *linear* context, each of these relations being evidently supplemented by the usual commutators

$$[J_3, J_\pm] = \pm J_\pm. \tag{3}$$

In the latest context, the raising (J_+), lowering (J_-) and diagonal (J_3) operators act on vectors belonging to the well known orthogonal basis $\{\mid j,m\rangle\}$ [2, 3] in the following way

$$J_\pm \mid j,m\rangle = \sqrt{(j \mp m)(j \pm m+1)} \mid j, m \pm 1\rangle, \tag{4}$$

$$J_3 \mid j,m\rangle = m \mid j,m\rangle, \tag{5}$$

where j refers to the Casimir eigenvalues

$$C \mid j,m\rangle \equiv \left(\frac{1}{2}(J_+J_- + J_-J_+) + J_3^2\right) \mid j,m\rangle = j(j+1) \mid j,m\rangle \tag{6}$$

and takes the values $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ while m, in eq.(5), runs from $-j$ to j giving the dimensions $(2j+1)$ to the irreducible representations of the linear $sl(2,R)$-context.

If the relation (1) is substituted to eq.(2), we then get [4, 5] the irreducible representations characterized by the following relations

$$J_+ \mid j,m\rangle = \sqrt{g(m)} \mid j, m+c\rangle, \tag{7}$$

$$J_- \mid j,m\rangle = \sqrt{g(m-c)} \mid j, m-c\rangle, \tag{8}$$

$$J_3 \mid j,m\rangle = \left(\frac{m}{c} + \gamma\right) \mid j,m\rangle, \tag{9}$$

where c is a nonnegative and nonvanishing integer, γ is a real scalar parameter while the function g is γ- and c-dependent [4, 5].

A "nonlinear" typical context is the one corresponding to the (cubic) Higgs algebra [8] given in (1) by $N = 1$, $p = 0, 1$, $\beta_0 = 1$ and $\beta_1 = 8\beta$, β being a real continuous parameter so that we have

$$f(J_3) = 2J_3 + 8\beta J_3^3. \tag{10}$$

All its finite-dimensional irreducible representations can be obtained by exploiting the corresponding actions (7), (8) and (9). In that way, we recover old well known results [8, 9, 17] but also find new ones in these angular momentum basis developments [4–7].

3 On polynomial deformations and differential realizations

If we search for *finite-dimensional* representations, the operators J_+, J_- and J_3 have to act, for example, on the $(n+1)$-dimensional vector spaces $P(n) \equiv \{1, x, x^2, \ldots, x^n\}$ when *differential* realizations are prescribed. Such a point of view has already been adopted in the *linear* context [12–14] since the late eighties. More recently Fradkin [18] has proposed a nice way for discussing such purposes and we have extended his method to the *nonlinear* context [10].

By coming back on the example of the Higgs algebra characterized by the structure relations (1) and (3) but with the expression (10), we can realize the generators in the following way:

$$J_+ = x^N F(D), \qquad J_- = G(D)\frac{d^N}{dx^N}, \qquad J_0 = \frac{1}{N}\,(D+\alpha), \qquad N = 1, 2, 3, \ldots, \tag{11}$$

where α is a constant and

$$D \equiv x\frac{d}{dx}$$

is the dilatation operator which, due to the Heisenberg commutation relation

$$\left[\frac{d}{dx}, x\right] = 1$$

satisfies

$$\left[D, x^N\right] = N x^N, \qquad \left[\frac{d^N}{dx^N}, D\right] = N\frac{d^N}{dx^N}.$$

Let us introduce also the Fradkin notations [18]

$$\frac{d^N}{dx^N}x^N = \prod_{k=1}^{N}(D+k) = \frac{(D+N)!}{D!}, \qquad x^N\frac{d^N}{dx^N} = \prod_{k=0}^{N-1}(D-k) = \frac{D!}{(D-N)!} \tag{12}$$

and notice that the relations (1), (10) and (11) imply the constraint

$$F(D-N)G(D-N)\frac{D!}{(D-N)!} - F(D)G(D)\frac{(D+N)!}{D!} = \frac{2}{N}(D+\alpha) + \frac{8\beta}{N^3}(D+\alpha)^3.$$

With the simplifying choice $G(D) = 1$, we get in the *cubic* context

$$F(D) = -f\frac{D!}{(D+N)!}(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4),$$

where $f = 2\beta N^{-4}$ and where the four λ's have to satisfy the system

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4\alpha + 2N,$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = N^2 + 6\alpha N + 6\alpha^2 + \frac{N^2}{2\beta},$$

$$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 = 2\alpha N^2 + 6\alpha^2 N + 4\alpha^3 + \frac{\alpha N^2}{\beta} + \frac{N^3}{2\beta}.$$

Nonsingular realizations (look at the definitions (12)) only appear when $N = 1, 2, 3, 4$ and finite-dimensional representations are obtained only for the $N =$ 1- and 2-cases. These results are in perfect agreement with those obtained in previous developments [5, 6] but in the angular momentum basis rather than, here, in the $P(n)$-basis. In particular, when $N = 2$ and $\alpha = -\frac{n}{2}$ we recover specific families already quoted elsewhere [5–7].

4 Differential realizations and quantum optical Hamiltonians

Lie algebras being strongly related to (kinematical as well as dynamical) symmetries as everybody knows, it is interesting to learn about new symmetries from "nonlinear" Lie algebras dealing with physical models. This is the aim of this section by visiting more particularly *quantum optical models* subtended by typical multi-photon Hamiltonians already put in evidence for describing some scattering processes. We refer more particularly to Karassiov–Klimov proposals [15] which, in 2-dimensional flat spaces, considered the superposition of two harmonic oscillators. By taking care of $\omega_1 = \omega_2 = \omega$ at the level of their angular frequencies and of a real coupling constant g, the corresponding Hamiltonian can be written on the form with integers m and n $(0 \leq m \leq n)$:

$$H = \omega\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) + g\left(a_1^\dagger\right)^n a_2^m + \left(a_2^\dagger\right)^m a_1^n, \tag{13}$$

where the characteristics of the two harmonic oscillators are immediately fixed through the commutation relations

$$[a_j, a_k^\dagger] = \delta_{jk} I, \qquad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0, \qquad j,k = 1,2.$$

An interesting result due to Debergh [5] is that the Higgs algebra can play the role of the *"spectrum generating algebra"* for the quantum optical model subtended by the Hamiltonian (13) iff $m = n = 2$, the raising and lowering operators being second powers of the linear ones and the diagonal J_3 being half of the linear one. In such a context, the deformation parameter is fixed by

$$\beta = -\frac{2}{2j^2 + 2j - 1}, \qquad j = 0, \frac{1}{2}, 1, \dots$$

and the specific actions of the generators J_+, J_- and J_3 become, in correspondence with eqs. (7)–(9) when the angular momentum basis is considered:

$$J_+ \mid j,m\rangle = ((j-m)(j+m+1)(j-m-1)(j+m+2))^{\frac{1}{2}} \mid j, m+2\rangle,$$

$$J_- \mid j,m\rangle = ((j+m)(j+m-1)(j-m+1)(j-m+2))^{\frac{1}{2}} \mid j, m-2\rangle,$$

$$J_3 \mid j,m\rangle = \frac{m}{2} \mid j,m\rangle.$$

For the whole set of j-values, we thus have the ($c = 2$ and $\gamma = 0$)-family of representations pointed out by Debergh [5] and simply related to meaningful physical models. Let us also mention that another interesting result, once again due to Debergh [6], is the *twofold* degeneracy of all the energy eigenvalues of the Hamiltonian (13) inside a Schrödinger-type (stationary) equation with the above characteristics of the Higgs algebra seen as the spectrum generating algebra of a quantum optical model. These degeneracies have moreover been interpreted as a property of supersymmetry in quantum mechanics [16] as it can be shown [6] through the construction of (two) supercharges generating with the Hamiltonian the *graded* Lie algebra $sqm(2)$.

In order to show such an interesting supersymmetric property, we have also considered [11] the *differential* realizations of the generators $J_\pm$ and J_3 and their introduction in the Hamiltonian operator. So, coming back to the $(n+1)$-dimensional vector spaces $P(n)$ of polynomials of degree at most n in the variable x, the Hamiltonian with arbitrary N is found on the form

$$H_n^{(N)} = \omega n + g\left(\frac{d^N}{dx^N} + x^N (D-n)(D-n+1)\dots(D-n+N-1)\right).$$

In our previous $N = 2$-context this gives

$$H_n^{(2)} = \omega n + g\left(\left(1 + x^4\right)\frac{d^2}{dx^2} + 2(1-n)x^3\frac{d}{dx} + n(n-1)x^2\right).$$

It is easy to see that these Hamiltonians preserve the spaces $P(n)$ and act invariantly on the subspaces $\epsilon(n) \equiv \{e_a(x)\}$ and $O(n) \equiv \{o_a(x)\}$ of even (e_a) and odd (o_a) polynomials of $P(n) = \epsilon(n) \oplus O(n)$.

It is remarkable that we get the following properties

$$H_n^{(2)} e_a(x) = E_a e_a(x) \qquad \text{and} \qquad H_n^{(2)} o_a(x) = E_a o_a(x)$$

with positive eigenvalues $E_a = \lambda_a^2$ pointing out immediately the double degeneracies. The existence of two supercharges Q and $\bar{Q}$ becomes evident if we require

$$Qe_a = 0, \qquad Qo_a = \lambda_a e_a \qquad \text{and} \qquad \bar{Q}e_a = \lambda_a o_a, \qquad \bar{Q}o_a = 0.$$

Specific realizations of such supercharges have been proposed elsewhere [11] as well as some contexts for different even values of N. Supersymmetry is always present in these applications so that we have some hope that, as in *nuclear physics* [19] or in *atomic physics* [20], supersymmetry can also reveal its presence in some models of *quantum optics*.

Acknowledgements

Cordial thanks are due to my coauthors Y. Brihaye and N. Debergh who have both played an important role more particularly in the "differential" part of the program.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 280–287.

Representations of the q-Deformed Algebra $\mathrm{so}_q(2,1)$

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We give a classification theorem for irreducible weight representations of the q-deformed algebra $U_q(\mathrm{so}_{2,1})$ which is a real form of the nonstandard deformation $U_q(\mathrm{so}_3)$ of the Lie algebra $\mathrm{so}(3,\mathbf{C})$. The algebra $U_q(\mathrm{so}_3)$ is generated by the elements I_1, I_2 and I_3 satisfying the relations $[I_1,I_2]_q := q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 = I_3$, $[I_2,I_3]_q = I_1$ and $[I_3,I_1]_q = I_2$. The real form $U_q(\mathrm{so}_{2,1})$ is determined for real q by the $*$-involution $I_1^* = -I_1$ and $I_2^* = I_2$. Weight representations of $U_q(\mathrm{so}_{2,1})$ are defined as representations T for which the operator $T(I_1)$ can be diagonalized and has a discrete spectrum. A part of the irreducible representations of $U_q(\mathrm{so}_{2,1})$ turn into irreducible representations of the Lie algebra $\mathrm{so}_{2,1}$ when $q \to 1$. Representations of the other part have no classical analogue.

1 The algebras $U_q(\mathrm{so}_3)$ and $U_q(\mathrm{so}_{2,1})$

The algebra $U_q(\mathrm{so}_3)$ is obtained by a q-deformation of the standard commutation relations $[I_1,I_2] = I_3$, $[I_2,I_3] = I_1$, $[I_3,I_1] = I_2$ of the Lie algebra $\mathrm{so}(3,\mathbf{C})$ and is defined as the complex associative algebra (with a unit element) generated by the elements I_1, I_2, I_3 satisfying the defining relations

$$[I_1,I_2]_q := q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 = I_3, \tag{1}$$

$$[I_2,I_3]_q := q^{1/2}I_2I_3 - q^{-1/2}I_3I_2 = I_1, \tag{2}$$

$$[I_3,I_1]_q := q^{1/2}I_3I_1 - q^{-1/2}I_1I_3 = I_2. \tag{3}$$

A Hopf algebra structure is not known on $U_q(\mathrm{so}_3)$. However, it can be embedded into the Hopf algebra $U_q(\mathrm{sl}_3)$ as a Hopf coideal (see [1]). This embedding is very important for the possible application in spectroscopy.

It follows from the relations (1)–(3) that for the algebra $U_q(\mathrm{so}_3)$ the Poincaré–Birkhoff–Witt theorem is true and this theorem can be formulated as: *The elements* $I_3^k I_2^m I_1^n$, $k,m,n = 0,1,2,\ldots$, *form a basis of the linear space* $U_q(\mathrm{so}_3)$. This theorem is proved by using the diamond lemma [2] (or its special case from Subsect. 4.1.5 in [3]).

By (1) the element I_3 is not independent: it is determined by the elements I_1 and I_2. Thus, the algebra $U_q(\mathrm{so}_3)$ is generated by I_1 and I_2, but now instead of quadratic relations (1)–(3) we must take the relations

$$I_1I_2^2 - \left(q+q^{-1}\right) I_2I_1I_2 + I_2^2I_1 = -I_1, \qquad I_2I_1^2 - \left(q+q^{-1}\right) I_1I_2I_1 + I_1^2I_2 = -I_2, \tag{4}$$

which are obtained if we substitute the expression (1) for I_3 into (2) and (3). The equation $I_3 = q^{1/2}I_1I_2 - q^{-1/2}I_2I_1$ and the relations (4) restore the relations (1)–(3).

Up to now we did not introduce $*$-involutions on $U_q(\mathrm{so}_3)$ determining real forms of this algebra. The $*$-involution $I_1 = -I_1$, $I_2 = -I_2$ determines the real form of $U_q(\mathrm{so}_3)$ which can be called a compact real form of $U_q(\mathrm{so}_3)$. The $*$-involution uniquely determined by the relations

$$I_1^* = -I_1, \qquad I_2^* = I_2 \tag{5}$$

gives a noncompact real form of $U_q(\mathrm{so}_3)$ which is denoted by $U_q(\mathrm{so}_{2,1})$. It is a q-analogue of the real form $\mathrm{so}_{2,1}$ of the complex Lie algebra $\mathrm{so}(3,\mathbf{C})$.

Note that for real q the equations $I_1^* = -I_1$ and $I_2^* = I_2$ do not mean that $I_3^* = I_3$ or $I_3^* = -I_3$:

$$I_3^* = \left(q^{1/2}I_1I_2 - q^{-1/2}I_2I_1\right)^* = q^{1/2}I_2^*I_1^* - q^{-1/2}I_1^*I_2^* = -q^{1/2}I_2I_1 + q^{-1/2}I_1I_2 \neq \pm I_3.$$

However, if $|q| = 1$ then $I_3^* = I_3$. Really,

$$I_3^* = \left(q^{1/2}I_1I_2 - q^{-1/2}I_2I_1\right)^* = q^{-1/2}I_2^*I_1^* - q^{1/2}I_1^*I_2^* = -q^{-1/2}I_2I_1 + q^{1/2}I_1I_2 = I_3.$$

In this paper we are interested in irreducible infinite dimensional representations of the algebras $U_q(\mathrm{so}_{2,1})$. Infinite dimensional irreducible representations of $U_q(\mathrm{so}_{2,1})$ are important for physical applications. For example, irreducible $*$-representations of the so called strange series (these representations were defined in [4]) are related to a certain type of Schrödinger equation [5]. Infinite dimensional representations of $U_q(\mathrm{so}_{2,1})$ appear in the theory of quantum gravity [6].

Infinite dimensional representations of $U_q(\mathrm{so}_{2,1})$ were studied in [4]. However, not all such representations were found there. Note that $*$-representations of real forms of $U_q(\mathrm{so}_3)$ different from $U_q(\mathrm{so}_{2,1})$ were studied in [7] and [8]. Irreducible representations of $U_q(\mathrm{so}_3)$ (including the case when q is a root of unity) are studied in [9–11].

2 Definition of weight representations of $U_q(\mathrm{so}_{2,1})$

From this point we assume that q is not a root of unity.

Definition 1. *By a weight representation T of $U_q(\mathrm{so}_{2,1})$ we mean a homomorphism of $U_q(\mathrm{so}_{2,1})$ into the algebra of linear operators (bounded or unbounded) on a Hilbert space $\mathcal{H}$, defined on an everywhere dense invariant subspace $\mathcal{D}$, such that the operator $T(I_1)$ can be diagonalized, has a discrete spectrum (with finite multiplicities of spectral points if T is irreducible), and its eigenvectors belong to $\mathcal{D}$. Two weight representations T and T' of $U_q(\mathrm{so}_{2,1})$ on spaces $\mathcal{H}$ and $\mathcal{H}'$, respectively, are called (algebraically) equivalent if there exist everywhere dense invariant subspaces $V \subset \mathcal{H}$ and $V' \subset \mathcal{H}'$ and a one-to-one linear operator $A : V \to V'$ such that $AT(a)v = T'(a)Av$ for all $a \in U_q(\mathrm{so}_{2,1})$ and $v \in V$.*

Remark. Note that the element $I_1 \in U_q(\mathrm{so}_{2,1})$ corresponds to the compact part of the group $SO(2,1)$. Therefore, as in the classical case, it is natural to demand in the definition of representations of $U_q(\mathrm{so}_{2,1})$ that the operator $T(I)$ has a discrete spectrum (with finite multiplicities of spectral points for irreducible representations T). Such representations correspond to Harish–Chandra modules of Lie algebras. Note that the algebra $U_q(\mathrm{so}_{2,1})$ has irreducible representations T for which the operator $T(I_1)$ can be diagonalized and has a continuous spectrum (this follows from the results of Section 4 in [12]). We do not consider such representations in this paper.

Since we shall consider only weight representations, below speaking about weight representations we shall omit the word “weight”.

Definition 2. *By a $*$-representation T of $U_q(\mathrm{so}_{2,1})$ we mean a representation of $U_q(\mathrm{so}_{2,1})$ in a sense of Definition 1 such that the equations $T(I_1)^* = -T(I_1)$ and $T(I_2)^* = T(I_2)$ are fulfilled on the domain $\mathcal{D}$.*

Definition 1 does not use the $*$-structure of $U_q(\mathrm{so}_{2,1})$. This means that representations of Definition 1 are in fact representations of $U_q(\mathrm{so}_3)$.

3 Representations of the principal series

Let us study irreducible infinite dimensional representations of the algebra $U_q(\mathrm{so}_{2,1})$ which were constructed in [4] and [11].

Let $q = e^\tau$ and ϵ be a fixed complex number such that $0 \le \mathrm{Re}\,\epsilon < 1$ and $\epsilon \ne \pm \mathrm{i}\pi/2\tau$. Let $\mathcal{H}_\epsilon$ be a complex Hilbert space with the orthonormal basis

$$|m\rangle, \qquad m = n + \epsilon, \quad n = 0, \pm 1, \pm 2, \ldots.$$

To every complex number a there corresponds the representation $R_{a\epsilon}$ of $U_q(\mathrm{so}_{2,1})$ on the Hilbert space $\mathcal{H}_\epsilon$ defined by the formulas

$$R_{a\epsilon}(I_1)|m\rangle = \mathrm{i}[m]|m\rangle, \tag{6}$$

$$R_{a\epsilon}(I_2)|m\rangle = \frac{1}{q^m + q^{-m}} \left\{ [a-m]|m+1\rangle - [a+m]|m-1\rangle \right\}, \tag{7}$$

$$R_{a\epsilon}(I_3)|m\rangle = \frac{\mathrm{i}q^{1/2}}{q^m + q^{-m}} \left\{ q^m[a-m]|m+1\rangle + q^{-m}[a+m]|m-1\rangle \right\}. \tag{8}$$

(Everywhere below, under considering representations of $U_q(\mathrm{so}_{2,1})$, we do not give the operator corresponding to I_3 since it can be easily calculated by using formula (3).)

Note that we excluded the cases $\epsilon = \pm\mathrm{i}\pi/2\tau$ since for these ϵ the coefficients in (7) and (8) are singular.

If $\epsilon = -\mathrm{i}\pi/2\tau + \sigma$ and $q^\sigma = \lambda$, then the representation $R_{a\epsilon}$ can be reduced to the following form:

$$R_{a\epsilon}(I_1)|n\rangle = \frac{\lambda q^n + \lambda^{-1}q^{-n}}{q - q^{-1}}|n\rangle,$$

$$R_{a\epsilon}(I_2)|n\rangle = \frac{-1}{\lambda q^n - \lambda^{-1}q^{-n}} \left(\frac{\lambda q^{n-a} + \lambda^{-1}q^{-n+a}}{q-q^{-1}}|n{+}1\rangle + \frac{\lambda q^{n+a} + \lambda^{-1}q^{-n-a}}{q-q^{-1}}|n{-}1\rangle \right),$$

where the basis elements $|n+\epsilon\rangle$ are denoted by $|n\rangle$, $n = 0, \pm 1, \ldots$. In particular, if $a = \pm\mathrm{i}\pi/2\tau$ and $q^\sigma = \lambda$, $0 \le \mathrm{Re}\,\sigma < 1$, then after rescaling the basis vectors the representation $R_{a\epsilon}$ (we denote it in this case as Q^+_λ) takes the form

$$Q^+_\lambda(I_1)|m\rangle = \frac{\lambda q^m + \lambda^{-1}q^{-m}}{q-q^{-1}}\,|m\rangle, \qquad Q^+_\lambda(I_2)|m\rangle = \frac{1}{q-q^{-1}}\,|m+1\rangle + \frac{1}{q-q^{-1}}\,|m-1\rangle.$$

If $a = \pm\mathrm{i}\pi/2\tau$ and $q^\sigma = -\lambda$, $0 \le \mathrm{Re}\,\sigma < 1$, then we obtain the representation $R_{a\epsilon}$ (we denote it in this case as Q^-_λ) in the form

$$Q^-_\lambda(I_1)|m\rangle = -\frac{\lambda q^m + \lambda^{-1}q^{-m}}{q-q^{-1}}\,|m\rangle, \qquad Q^-_\lambda(I_2) = Q^+_\lambda(I_2).$$

Since the representations $R_{a\epsilon}$ are determined for $\epsilon \neq \pm\mathrm{i}\pi/2\tau$, then the representations $Q_\lambda^\pm$ are determined for $\lambda \neq 1$. However, the operators $Q_\lambda^\pm(I_j)$, $j=1,2,3$, are well defined also for $\lambda = \pm 1$ and satisfy the defining relations (1)–(3). Thus, the representations $Q_\lambda^\pm$ are determined for all complex values of λ.

Theorem 1. *The representation $R_{a\epsilon}$ is irreducible if and only if $a \not\equiv \pm\epsilon \,(\mathrm{mod}\,\mathbf{Z})$ or if $\epsilon \not\equiv \pm\mathrm{i}\pi/2\tau + 1/2$ or if (a,ϵ) does not coincide with one of four couples $(\pm\mathrm{i}\pi/2\tau, \pm\mathrm{i}\pi/2\tau + 1/2)$. The representation $Q_\lambda^\pm$ is irreducible if and only if $\lambda \neq \pm 1, \pm q^{1/2}$.*

This theorem follows from Theorem 1 in [4] and the results of Section 7 in [11].

There exist equivalence relations between irreducible representations $R_{a\epsilon}$. They are completely described in [4].

In the excluded cases of Theorem 1, representations $R_{a\epsilon}$ and $Q_\lambda^\pm$ are reducible. In particular, the representations $Q_\lambda^\pm$, $\lambda = \pm 1, \pm q^{1/2}$, are reducible (see [11]) and leads to the irreducible representations which are described as follows.

Let V_1 and V_2 be the vector spaces with the bases

$$|m\rangle', \quad m = 0,1,2,\ldots, \qquad \text{and} \qquad |m\rangle'', \quad m = 1,2,3,\ldots,$$

respectively. Then the operators $Q_1^{1,\pm}(I_1)$, $Q_1^{1,\pm}(I_2)$, $Q_1^{2,\pm}(I_1)$, $Q_1^{2,\pm}(I_2)$ given by the formulas

$$Q_1^{1,\pm}(I_1)|m\rangle' = \pm\frac{q^m + q^{-m}}{q-q^{-1}}\,|m\rangle', \qquad Q_1^{2,\pm}(I_1)|m\rangle'' = \pm\frac{q^m + q^{-m}}{q-q^{-1}}\,|m\rangle'',$$

$$Q_1^{1,\pm}(I_2)|0\rangle = \frac{\sqrt{2}}{q-q^{-1}}\,|1\rangle', \qquad Q_1^{2,\pm}(I_2)|1\rangle'' = \frac{1}{q-q^{-1}}\,|2\rangle'',$$

$$Q_1^{1,\pm}(I_2)|1\rangle' = \frac{\sqrt{2}}{q-q^{-1}}\,|0\rangle' + \frac{1}{q-q^{-1}}\,|2\rangle', \qquad Q_1^{2,\pm}(I_2)|2\rangle' = \frac{1}{q-q^{-1}}\,|1\rangle' + \frac{1}{q-q^{-1}}\,|3\rangle',$$

$$Q_1^{1,\pm}(I_2)|m\rangle' = \frac{1}{q-q^{-1}}\,|m+1\rangle' + \frac{1}{q-q^{-1}}\,|m-1\rangle', \qquad m > 1,$$

$$Q_1^{2,\pm}(I_2)|m\rangle'' = \frac{1}{q-q^{-1}}\,|m+1\rangle'' + \frac{1}{q-q^{-1}}\,|m-1\rangle'', \qquad m > 2,$$

determine irreducible representations of $U_q(\mathrm{so}_{2,1})$ which are denoted by $Q_1^{1,\pm}$ and $Q_1^{2,\pm}$, respectively.

Let W_1 and W_2 be the vector spaces spanned by the basis vectors

$$|m+\frac{1}{2}\rangle', \quad m = 0,1,2,\ldots, \qquad \text{and} \qquad |m+\frac{1}{2}\rangle'', \quad m = 0,1,2,\ldots,$$

respectively. Then the operators $Q_{\sqrt{q}}^{1,\pm}(I_1)$, $Q_{\sqrt{q}}^{1,\pm}(I_2)$, $Q_{\sqrt{q}}^{2,\pm}(I_1)$, $Q_{\sqrt{q}}^{2,\pm}(I_2)$ given by the formulas

$$Q_{\sqrt{q}}^{1,\pm}(I_1)|m+\tfrac{1}{2}\rangle' = \pm\frac{q^{m+1/2} + q^{-m-1/2}}{q-q^{-1}}\,|m+\tfrac{1}{2}\rangle',$$

$$Q_{\sqrt{q}}^{2,\pm}(I_1)|m+\tfrac{1}{2}\rangle'' = \pm\frac{q^{m+1/2} + q^{-m-1/2}}{q-q^{-1}}\,|m+\tfrac{1}{2}\rangle''$$

and

$$Q_{\sqrt{q}}^{1,\pm}(I_2)|\tfrac{1}{2}\rangle' = -\frac{1}{q-q^{-1}}\,|\tfrac{1}{2}\rangle' + \frac{1}{q-q^{-1}}\,|\tfrac{3}{2}\rangle',$$

$$Q^{1,\pm}_{\sqrt{q}}(I_2)|m+\tfrac12\rangle' = \frac{1}{q-q^{-1}}\,|m+\tfrac32\rangle' + \frac{1}{q-q^{-1}}\,|m-\tfrac12\rangle', \qquad m>0,$$

$$Q^{2,\pm}_{\sqrt{q}}(I_2)|\tfrac12\rangle'' = \frac{1}{q-q^{-1}}\,|\tfrac12\rangle'' + \frac{1}{q-q^{-1}}\,|\tfrac32\rangle'',$$

$$Q^{2,\pm}_{\sqrt{q}}(I_2)|m+\tfrac12\rangle'' = \frac{1}{q-q^{-1}}\,|m+\tfrac32\rangle'' + \frac{1}{q-q^{-1}}\,|m-\tfrac12\rangle'', \qquad m>0,$$

determine irreducible representations of $U_q(\mathrm{so}_{2,1})$ which are denoted by $Q^{1,\pm}_{\sqrt{q}}$ and $Q^{2,\pm}_{\sqrt{q}}$, respectively. We have

$$Q_1^{\pm} = Q_1^{1,\pm} \oplus Q_1^{2,\pm}, \qquad Q^{\pm}_{\sqrt{q}} = Q^{1,\pm}_{\sqrt{q}} \oplus Q^{2,\pm}_{\sqrt{q}}.$$

The representations $R_{a\epsilon}$ with $\epsilon = \pm\mathrm{i}\pi/2\tau + \frac12$ are also reducible. They lead to the following irreducible representations. For any complex number a we define the representations $R_a^{(\mathrm{i},\pm)}$ and $R_a^{(-\mathrm{i},\pm)}$ of $U_q(\mathrm{so}_{2,1})$ acting on the Hilbert space $\mathcal{H}$ with the orthonormal basis $|n\rangle$, $n=1,2,3,\ldots$, by the formulas

$$R_a^{(\mathrm{i},\pm)}(I_1)|k\rangle = -\frac{q^{k-1/2}+q^{-k+1/2}}{q-q^{-1}}|k\rangle,$$

$$R_a^{(\mathrm{i},\pm)}(I_2)|1\rangle = \pm\frac{[a]}{q^{1/2}-q^{-1/2}}|1\rangle + \mathrm{i}\frac{[a-1]}{q^{1/2}-q^{-1/2}}|2\rangle,$$

$$R_a^{(\mathrm{i},\pm)}(I_2)|k\rangle = \mathrm{i}\frac{[a-k]}{q^{k-1/2}-q^{-k+1/2}}|k+1\rangle + \mathrm{i}\frac{[a+k-1]}{q^{k-1/2}-q^{-k+1/2}}|k-1\rangle, \qquad k\neq 1,$$

and by the formulas

$$R_a^{(-\mathrm{i},\pm)}(I_1)|k\rangle = \frac{q^{k-1/2}+q^{-k+1/2}}{q-q^{-1}}|k\rangle, \qquad R_a^{(-\mathrm{i},\pm)}(I_2) = -R_a^{(\mathrm{i},\pm)}(I_2).$$

For $\epsilon = \pm\mathrm{i}\pi/2\tau + \frac12$ we have

$$R_{a,\pm\mathrm{i}\pi/2\tau+1/2} = R_a^{(\mathrm{i},\pm)} \oplus R_a^{(-\mathrm{i},\pm)}.$$

Note that for $a=1/2$ the representations $R_a^{(\pm\mathrm{i},\pm)}$ are equivalent to the corresponding representations $Q^{1,\pm}_{\sqrt{q}}$ and $Q^{2,\pm}_{\sqrt{q}}$.

The algebra $U_q(\mathrm{so}_{2,1})$ has also irreducible infinite dimensional representations with highest weights or with lowest weights which are classified in the paper [4]. They are subrepresentations of the corresponding representations $R_{a\epsilon}$. We give a list of these representations.

Let $l=\frac12,1,\frac32,2,\ldots$. We denote by R_l^+ the representation of $U_q(\mathrm{so}_3)$ acting on the Hilbert space $\mathcal{H}_l$ with the orthonormal basis $|m\rangle$, $m=l,l+1,l+2,\ldots$, and given by formulas (6)–(8) with $a=-l$. By R_l^- we denote the representation of $U_q(\mathrm{so}_3)$ acting on the Hilbert space $\hat{\mathcal{H}}_l$ with the orthonormal basis $|m\rangle$, $m=-l,-l-1,-l-2,\ldots$, and given by formulas (6)–(8) with $a=l$.

Now let $a\neq 0 \pmod{\mathbf{Z}}$ and $a\neq \frac12 \pmod{\mathbf{Z}}$. We denote by $\mathcal{H}_a$ the Hilbert space with the orthonormal basis $|m\rangle$, $m=-a,-a+1,-a+2,\ldots$. On this space the representation R_a^+ acts which is given by formulas (6)–(8). On the Hilbert space $\hat{\mathcal{H}}_a$ with the orthonormal basis $|m\rangle$, $m=a,a-1,a-2,\ldots$, the representation R_a^- acts which is given by (6)–(8).

4 Other infinite dimensional representations of $U_q(\mathrm{so}_{2,1})$

Let us construct additional two series of infinite dimensional irreducible representations of $U_q(\mathrm{so}_{2,1})$ which cannot be obtained from the representations $R_{a\epsilon}$. Let $\mathcal{H}$ be the complex Hilbert space with the basis $|m\rangle$, $m = 0, \pm 1, \pm 2, \ldots$. Let a and b be complex numbers such that $a^2 + b^2 = 1$, $a \neq 0$, $b \neq 0$ and $a \neq b$. We define on the operators $\hat{Q}^{\pm}_{ab}(I_1)$ and $\hat{Q}^{\pm}_{ab}(I_2)$ determined by the formulas

$$\hat{Q}^{\pm}_{ab}(I_1)|m\rangle = \pm \frac{q^m + q^{-m}}{q - q^{-1}}\, |m\rangle,$$

$$\hat{Q}^{\pm}_{ab}(I_2)|m\rangle = \frac{1}{q - q^{-1}}\, |m-1\rangle + \frac{1}{q - q^{-1}}\, |m+1\rangle, \qquad m \neq 0, \pm 1,$$

$$\hat{Q}^{\pm}_{ab}(I_2)|0\rangle = \frac{b\sqrt{2}}{q - q^{-1}}\, |1\rangle + \frac{a\sqrt{2}}{q - q^{-1}}\, |-1\rangle,$$

$$\hat{Q}^{\pm}_{ab}(I_2)|1\rangle = \frac{b\sqrt{2}}{q - q^{-1}}\, |0\rangle + \frac{1}{q - q^{-1}}\, |2\rangle,$$

$$\hat{Q}^{\pm}_{ab}(I_2)|-1\rangle = \frac{a\sqrt{2}}{q - q^{-1}}\, |0\rangle + \frac{1}{q - q^{-1}}\, |-2\rangle.$$

A direct computation shows that these operators satisfy the determining relations (1)–(3) and therefore determine a representation of $U_q(\mathrm{so}_{2,1})$ which is denoted by $\hat{Q}^{\pm}_{ab}$.

Let now $\mathcal{H}'$ be the complex Hilbert space with the basis $|k\rangle$, $k = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$. Let a and b be complex numbers such that $a^2 + b^2 = 1$, $a \neq 0$, $b \neq 0$. We define on the space $\mathcal{H}'$ the operators $\breve{Q}^{\pm}_{ab}(I_1)$ and $\breve{Q}^{\pm}_{ab}(I_2)$ determined by the formulas

$$\breve{Q}^{\pm}_{ab}(I_1)|k\rangle = \frac{q^k + q^{-k}}{q - q^{-1}}\, |k\rangle,$$

$$\breve{Q}^{\pm}_{ab}(I_2)|k\rangle = \frac{1}{q - q^{-1}}\, |k-1\rangle + \frac{1}{q - q^{-1}}\, |k+1\rangle, \qquad k \neq \pm\frac{1}{2},$$

$$\breve{Q}^{\pm}_{ab}(I_2)|\tfrac{1}{2}\rangle = \frac{a}{q - q^{-1}}\, |\tfrac{1}{2}\rangle + \frac{1}{q - q^{-1}}\, |\tfrac{3}{2}\rangle + \frac{b}{q - q^{-1}}\, |-\tfrac{1}{2}\rangle,$$

$$\breve{Q}^{\pm}_{ab}(I_2)|-\tfrac{1}{2}\rangle = -\frac{a}{q - q^{-1}}\, |-\tfrac{1}{2}\rangle + \frac{b}{q - q^{-1}}\, |\tfrac{1}{2}\rangle + \frac{1}{q - q^{-1}}\, |-\tfrac{3}{2}\rangle.$$

A direct computation shows that these operators also determine representations of $U_q(\mathrm{so}_{2,1})$ which are denoted by $\breve{Q}^{\pm}_{ab}$.

Thus, we have constructed the following classes of irreducible infinite dimensional representations of the algebra $U_q(\mathrm{so}_{2,1})$:

(a) The representations $R_{a\epsilon}$ with the exclusions of Theorem 1.
(b) The representations $R^{\pm \mathrm{i},\pm}_a$, $a \in \mathbf{C}$.
(c) The representations $R^{\pm}_l$, $l = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and $R^{\pm}_a$, $a \neq 0 \,(\mathrm{mod}\, \mathbf{Z})$, $a \neq \frac{1}{2} \,(\mathrm{mod}\, \mathbf{Z})$.
(d) The representations $Q^{1,\pm}_1$ and $Q^{2,\pm}_1$.
(e) The representations $Q^{1,\pm}_{\sqrt{q}}$ and $Q^{2,\pm}_{\sqrt{q}}$.
(f) The representations $\hat{Q}^{\pm}_{ab}$ and $\breve{Q}^{\pm}_{ab}$, $a^2 + b^2 = 1$, $a \neq 0$, $b \neq 0$, and $a \neq b$ for $\hat{Q}^{\pm}_{ab}$.

Theorem 2. *Every irreducible infinite dimensional weight representation of the algebra* $U_q(\mathrm{so}_{2,1})$ *is equivalent to one of the representations of classes* (a)–(f) *describe above.*

A proof of this theorem is long and will be given in a separate paper. In particular, the proof uses the following proposition:

Proposition. *Let* $|q| \neq 1$. *If* $b \neq \frac{1}{2}$ *and* $b \neq 1$, *then the set*

$$\frac{q^{b+m} + q^{-b-m}}{q - q^{-1}}, \qquad m \in \mathbf{Z},$$

has no coinciding numbers. If $b = \frac{1}{2}$, *then this set consists only of pairs of coinciding numbers. If* $b = 1$, *then this set consists of the point 0 and pairs of coinciding numbers.*

This proposition show for which representations the operator $R(I_1)$ has multiple eigenvalues.

5 $*$-representations of $U_q(\mathrm{so}_{2,1})$

In the previous section we described all irreducible infinite dimensional representations of $U_q(\mathrm{so}_{2,1})$. The aim of this section is to separate $*$-representations of $U_q(\mathrm{so}_{2,1})$ from the set of the representations (a)–(f).

Note that $*$-representations of the universal enveloping algebra $U(\mathrm{so}_{2,1})$ correspond to unitary representations of the Lie group $SO(2,1)$. Irreducible $*$-representations of $U_q(\mathrm{so}_{2,1})$ can be found by using the method described, for example, in Section 6.4 of [13]. The same method is used for separation of $*$-representations in the set of the representations (a)–(f). Let us give the result of this separation.

Theorem 3. *Let* $q = e^h$, $h \in \mathbf{R}$. *Then the following representations from the set (a)–(f) are $*$-representations or equivalent to $*$-representations:*

(a) *the representations* $R_{a\epsilon}$, $a = \mathrm{i}\rho - 1/2$, $\rho \in \mathbf{R}$, $\epsilon = c + \mathrm{i}n\pi/h$, $0 \le c < 1$, $n = 0, 1$ *(the principal series);*

(b) *the representations* $R_{a\epsilon}$, $a \in \mathbf{R}$, $\epsilon = c + \mathrm{i}n\pi/h$, $0 \le c < 1$, $n = 0, 1$, *such that* $-c < a < c - 1$ *for* $c > 1/2$ *and* $c - 1 < a < -c$ *for* $c < 1/2$ *(the supplementary series);*

(c) *the representations* $R_{a\epsilon}$, $\mathrm{Im}\, a = \pi/2h$, $\epsilon = c + \mathrm{i}n\pi/h$, $0 \le c < 1$, $n = 0, 1$ *(the strange series);*

(d) *all the representations* R_a^+, $a \ge -1/2$, *and* R_a^-, $a \le 1/2$ *(the discrete series).*

This list of irreducible $*$-representations of $U_q(\mathrm{so}_{2,1})$ coincides with that of [4].

Theorem 4. *Let* $q = e^{\mathrm{i}\varphi}$, $0 < \varphi \le 2\pi$. *We suppose that q is not a root of unity. The following representations from the set (a)–(f) are $*$-representations or equivalent to $*$-representations:*

(a) *the representations* $R_{a\epsilon}$, $a = \mathrm{i}\rho - 1/2$, $\rho \in \mathbf{R}$, $0 \le \epsilon < 1$, *if*

$$\cos(\epsilon + n)\varphi \cdot \cos(\epsilon + n + 1)\varphi > 0 \qquad \text{for all} \quad n \in \mathbf{Z};$$

(b) *the representations* $R_{a\epsilon}$, $\mathrm{Re}\, a = \pi/2\varphi$, $0 \le \epsilon < 1$, *if*

$$\sin(\epsilon + n - a)\varphi \cdot \sin(\epsilon + n + a + 1)\varphi \cdot \cos(\epsilon + n)\varphi \cdot \cos(\epsilon + n + 1)\varphi > 0 \qquad \text{for all} \quad n \in \mathbf{Z};$$

(c) *the representations* $R_a^{\pm\mathrm{i},\pm}$ *if*

$$\sin(a - n)\varphi \cdot \sin(a + n)\varphi \cdot \sin(n - 1/2)\varphi \cdot \sin(n + 1/2)\varphi < 0 \qquad \text{for} \quad n = 1, 2, 3, \ldots.$$

Acknowledgements

The second co-author (A.U.K.) was supported by Ukrainian State Foundation of Fundamental Researchs Grant No. 1.4/206 and by INTAS Grant INTAS–93–1038EXT. This co-author would like to thank Department of Mathematics of FNSPE, Czech Technical University, for the hospitality when part of this work was done.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 288–297.

C_λ-Extended Oscillator Algebras: Theory and Applications to (Variants of) Supersymmetric Quantum Mechanics

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C_λ-extended oscillator algebras, where C_λ is the cyclic group of order λ, are introduced and realized as generalized deformed oscillator algebras. For $\lambda = 2$, they reduce to the well-known Calogero–Vasiliev algebra. For higher λ values, they are shown to provide in their bosonic Fock space representation some interesting applications to supersymmetric quantum mechanics and some variants thereof: an algebraic realization of supersymmetric quantum mechanics for cyclic shape invariant potentials of period λ, a bosonization of parasupersymmetric quantum mechanics of order $p = \lambda - 1$, and, for $\lambda = 3$, a bosonization of pseudosupersymmetric quantum mechanics and orthosupersymmetric quantum mechanics of order two.

1 Introduction

Deformations and extensions of the oscillator algebra have found a lot of applications to physical problems, such as the description of systems with non-standard statistics, the construction of integrable lattice models, the investigation of nonlinearities in quantum optics, as well as the algebraic treatment of quantum exactly solvable models and of n-particle integrable systems.

The generalized deformed oscillator algebras (GDOAs) (see e.g. Ref. [1] and references quoted therein) arose from successive generalizations of the Arik–Coon [2] and Biedenharn–Macfarlane [3] q-oscillators. Such algebras, denoted by $\mathcal{A}_q(G(N))$, are generated by the unit, creation, annihilation, and number operators I, $a^\dagger$, a, N, satisfying the Hermiticity conditions $\left(a^\dagger\right)^\dagger = a$, $N^\dagger = N$, and the commutation relations

$$\left[N, a^\dagger\right] = a^\dagger, \qquad [N, a] = -a, \qquad \left[a, a^\dagger\right]_q \equiv aa^\dagger - qa^\dagger a = G(N), \tag{1.1}$$

where q is some real number and $G(N)$ is some Hermitian, analytic function.

On the other hand, $\mathcal{G}$-extended oscillator algebras, where $\mathcal{G}$ is some finite group, appeared in connection with n-particle integrable models. For the Calogero model [4], for instance, $\mathcal{G}$ is the symmetric group S_n [5].

For two particles, the S_2-extended oscillator algebra $\mathcal{A}^{(2)}_\kappa$, where $S_2 = \{\, I, K \mid K^2 = I \,\}$, is generated by the operators I, $a^\dagger$, a, N, K, subject to the Hermiticity conditions $\left(a^\dagger\right)^\dagger = a$, $N^\dagger = N$, $K^\dagger = K^{-1}$, and the relations

$$\begin{aligned} &\left[N, a^\dagger\right] = a^\dagger, \qquad [N, K] = 0, \qquad K^2 = I, \\ &\left[a, a^\dagger\right] = I + \kappa K \qquad (\kappa \in \mathrm{R}), \qquad a^\dagger K = -K a^\dagger, \end{aligned} \tag{1.2}$$

together with their Hermitian conjugates.

When the S_2 generator K is realized in terms of the Klein operator $(-1)^N$, $\mathcal{A}_\kappa^{(2)}$ becomes a GDOA characterized by $q = 1$ and $G(N) = I + \kappa(-1)^N$, and known as the Calogero–Vasiliev [6] or modified [7] oscillator algebra.

The operator K may be alternatively considered as the generator of the cyclic group C_2 of order two, since the latter is isomorphic to S_2. By replacing C_2 by the cyclic group of order λ, $C_\lambda = \{\, I, T, T^2, \ldots, T^{\lambda-1} \mid T^\lambda = I \,\}$, one then gets a new class of $\mathcal{G}$-extended oscillator algebras [8], generalizing that describing the two-particle Calogero model. In the present communication, we will define the C_λ-extended oscillator algebras, study some of their properties, and show that they have some interesting applications to supersymmetric quantum mechanics (SSQM) [9] and some of its variants.

2 Definition and properties of C_λ-extended oscillator algebras

Let us consider the algebras generated by the operators I, $a^\dagger$, a, N, T, satisfying the Hermiticity conditions $\left(a^\dagger\right)^\dagger = a$, $N^\dagger = N$, $T^\dagger = T^{-1}$, and the relations

$$\begin{aligned} &\left[N, a^\dagger\right] = a^\dagger, \qquad [N, T] = 0, \qquad T^\lambda = I, \\ &\left[a, a^\dagger\right] = I + \sum_{\mu=1}^{\lambda-1} \kappa_\mu T^\mu, \qquad a^\dagger T = e^{-\mathrm{i}2\pi/\lambda}\, T a^\dagger, \end{aligned} \tag{2.1}$$

together with their Hermitian conjugates [8]. Here T is the generator of (a unitary representation of) the cyclic group C_λ (where $\lambda \in \{2, 3, 4, \ldots\}$), and κ_μ, $\mu = 1, 2, \ldots, \lambda - 1$, are some complex parameters restricted by the conditions $\kappa_\mu^* = \kappa_{\lambda-\mu}$ (so that there remain altogether $\lambda - 1$ independent real parameters).

C_λ has λ inequivalent, one-dimensional matrix unitary irreducible representations (unirreps) Γ^μ, $\mu = 0, 1, \ldots, \lambda - 1$, which are such that $\Gamma^\mu\left(T^\nu\right) = \exp(\mathrm{i}2\pi\mu\nu/\lambda)$ for any $\nu = 0, 1, \ldots, \lambda - 1$. The projection operator on the carrier space of Γ^μ may be written as

$$P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{-\mathrm{i}2\pi\mu\nu/\lambda}\, T^\nu, \tag{2.2}$$

and conversely T^ν, $\nu = 0, 1, \ldots, \lambda - 1$, may be expressed in terms of the P_μ's as

$$T^\nu = \sum_{\mu=0}^{\lambda-1} e^{\mathrm{i}2\pi\mu\nu/\lambda} P_\mu. \tag{2.3}$$

The algebra defining relations (2.1) may therefore be rewritten in terms of I, $a^\dagger$, a, N, and $P_\mu = P_\mu^\dagger$, $\mu = 0, 1, \ldots, \lambda - 1$, as

$$\begin{aligned} &\left[N, a^\dagger\right] = a^\dagger, \qquad [N, P_\mu] = 0, \qquad \sum_{\mu=0}^{\lambda-1} P_\mu = I, \\ &\left[a, a^\dagger\right] = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu, \qquad a^\dagger P_\mu = P_{\mu+1}\, a^\dagger, \qquad P_\mu P_\nu = \delta_{\mu,\nu} P_\mu, \end{aligned} \tag{2.4}$$

where we use the convention $P_{\mu'} = P_\mu$ if $\mu' - \mu = 0 \bmod \lambda$ (and similarly for other operators or parameters indexed by μ, μ'). Equation (2.4) depends upon λ real parameters

$\alpha_\mu = \sum_{\nu=1}^{\lambda-1} \exp(\mathrm{i}2\pi\mu\nu/\lambda)\kappa_\nu$, $\mu = 0, 1, \ldots, \lambda-1$, restricted by the condition $\sum_{\mu=0}^{\lambda-1} \alpha_\mu = 0$. Hence, we may eliminate one of them, for instance $\alpha_{\lambda-1}$, and denote C_λ-extended oscillator algebras by $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$.

The cyclic group generator T and the projection operators P_μ can be realized in terms of N as

$$T = e^{\mathrm{i}2\pi N/\lambda}, \qquad P_\mu = \frac{1}{\lambda}\sum_{\nu=0}^{\lambda-1} e^{\mathrm{i}2\pi\nu(N-\mu)/\lambda}, \qquad \mu = 0, 1, \ldots, \lambda-1, \tag{2.5}$$

respectively. With such a choice, $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$ becomes a GDOA, $\mathcal{A}^{(\lambda)}(G(N))$, characterized by $q = 1$ and $G(N) = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu$, where P_μ is given in Eq.(2.5).

For any GDOA $\mathcal{A}_q(G(N))$, one may define a so-called structure function $F(N)$, which is the solution of the difference equation $F(N+1) - qF(N) = G(N)$, such that $F(0) = 0$ [1]. For $\mathcal{A}^{(\lambda)}(G(N))$, we find

$$F(N) = N + \sum_{\mu=0}^{\lambda-1} \beta_\mu P_\mu, \qquad \beta_0 \equiv 0, \qquad \beta_\mu \equiv \sum_{\nu=0}^{\mu-1} \alpha_\nu \quad (\mu = 1, 2, \ldots, \lambda-1). \tag{2.6}$$

At this point, it is worth noting that for $\lambda = 2$, we obtain $T = K$, $P_0 = (I+K)/2$, $P_1 = (I-K)/2$, and $\kappa_1 = \kappa_1^* = \alpha_0 = -\alpha_1 = \kappa$, so that $\mathcal{A}^{(2)}_{\alpha_0}$ coincides with the S_2-extended oscillator algebra $\mathcal{A}^{(2)}_\kappa$ and $\mathcal{A}^{(2)}(G(N))$ with the Calogero–Vasiliev algebra.

In Ref. [10], we showed that $\mathcal{A}^{(\lambda)}(G(N))$ (and more generally $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$) has only two different types of unirreps: infinite-dimensional bounded from below unirreps and finite-dimensional ones. Among the former, there is the so-called bosonic Fock space representation, wherein $a^\dagger a = F(N)$ and $aa^\dagger = F(N+1)$. Its carrier space $\mathcal{F}$ is spanned by the eigenvectors $|n\rangle$ of the number operator N, corresponding to the eigenvalues $n = 0, 1, 2, \ldots$, where $|0\rangle$ is a vacuum state, i.e., $a|0\rangle = N|0\rangle = 0$ and $P_\mu|0\rangle = \delta_{\mu,0}|0\rangle$. The eigenvectors can be written as

$$|n\rangle = \mathcal{N}_n^{-1/2}\left(a^\dagger\right)^n|0\rangle, \qquad n = 0, 1, 2, \ldots, \tag{2.7}$$

where $\mathcal{N}_n = \prod_{i=1}^{n} F(i)$. The creation and annihilation operators act upon $|n\rangle$ in the usual way, i.e.,

$$a^\dagger|n\rangle = \sqrt{F(n+1)}\,|n+1\rangle, \qquad a|n\rangle = \sqrt{F(n)}\,|n-1\rangle, \tag{2.8}$$

while P_μ projects on the μth component $\mathcal{F}_\mu \equiv \{\,|k\lambda+\mu\rangle \mid k = 0, 1, 2, \ldots\}$ of the Z_λ-graded Fock space $\mathcal{F} = \sum_{\mu=0}^{\lambda-1} \oplus\mathcal{F}_\mu$. It is obvious that such a bosonic Fock space representation exists if and only if $F(\mu) > 0$ for $\mu = 1, 2, \ldots, \lambda-1$. This gives the following restrictions on the algebra parameters α_μ,

$$\sum_{\nu=0}^{\mu-1} \alpha_\nu > -\mu, \qquad \mu = 1, 2, \ldots, \lambda-1. \tag{2.9}$$

In the bosonic Fock space representation, we may consider the bosonic oscillator Hamiltonian, defined as usual by

$$H_0 \equiv \frac{1}{2}\left\{a, a^\dagger\right\}. \tag{2.10}$$

It can be rewritten as

$$H_0 = a^\dagger a + \frac{1}{2}\left(I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu\right) = N + \frac{1}{2}I + \sum_{\mu=0}^{\lambda-1} \gamma_\mu P_\mu, \tag{2.11}$$

where $\gamma_0 \equiv \frac{1}{2}\alpha_0$ and $\gamma_\mu \equiv \sum\limits_{\nu=0}^{\mu-1} \alpha_\nu + \frac{1}{2}\alpha_\mu$ for $\mu = 1, 2, \ldots, \lambda - 1$.

The eigenvectors of H_0 are the states $|n\rangle = |k\lambda+\mu\rangle$, defined in Eq.(2.7), and their eigenvalues are given by

$$E_{k\lambda+\mu} = k\lambda + \mu + \gamma_\mu + \frac{1}{2}, \qquad k = 0, 1, 2, \ldots, \qquad \mu = 0, 1, \ldots, \lambda - 1. \tag{2.12}$$

In each $\mathcal{F}_\mu$ subspace of the Z_λ-graded Fock space $\mathcal{F}$, the spectrum of H_0 is therefore harmonic, but the λ infinite sets of equally spaced energy levels, corresponding to $\mu = 0, 1, \ldots, \lambda - 1$, may be shifted with respect to each other by some amounts depending upon the algebra parameters $\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-2}$, through their linear combinations γ_μ, $\mu = 0, 1, \ldots, \lambda - 1$.

For the Calogero–Vasiliev oscillator, i.e., for $\lambda = 2$, the relation $\gamma_0 = \gamma_1 = \kappa/2$ implies that the spectrum is very simple and coincides with that of a shifted harmonic oscillator. For $\lambda \geq 3$, however, it has a much richer structure. According to the parameter values, it may be nondegenerate, or may exhibit some $(\nu + 1)$-fold degeneracies above some energy eigenvalue, where ν may take any value in the set $\{1, 2, \ldots, \lambda - 1\}$. In Ref. [11], we obtained for $\lambda = 3$ the complete classification of nondegenerate, twofold and threefold degenerate spectra in terms of α_0 and α_1.

In the remaining part of this communication, we will show that the bosonic Fock space representation of $\mathcal{A}^{(\lambda)}(G(N))$ and the corresponding bosonic oscillator Hamiltonian H_0 have some useful applications to SSQM and some of its variants.

3 Application to supersymmetric quantum mechanics with cyclic shape invariant potentials

In SSQM with two supercharges, the supersymmetric Hamiltonian $\mathcal{H}$ and the supercharges $Q^\dagger$, $Q = \left(Q^\dagger\right)^\dagger$, satisfy the sqm(2) superalgebra, defined by the relations

$$Q^2 = 0, \qquad [\mathcal{H}, Q] = 0, \qquad \left\{Q, Q^\dagger\right\} = \mathcal{H}, \tag{3.1}$$

together with their Hermitian conjugates [9]. In such a context, shape invariance [12] provides an integrability condition, yielding all the bound state energy eigenvalues and eigenfunctions, as well as the scattering matrix.

Recently, Sukhatme, Rasinariu, and Khare [13] introduced cyclic shape invariant potentials of period p in SSQM. They are characterized by the fact that the supersymmetric partner Hamiltonians correspond to a series of shape invariant potentials, which repeats after a cycle of p iterations. In other words, one may define p sets of operators $\left\{\mathcal{H}_\mu, Q_\mu^\dagger, Q_\mu\right\}$, $\mu = 0, 1, \ldots, p-1$, each satisfying the sqm(2) defining relations (3.1). The operators may be written as

$$\mathcal{H}_\mu = \begin{pmatrix} \mathcal{H}^{(\mu)} - \mathcal{E}_0^{(\mu)} I & 0 \\ 0 & \mathcal{H}^{(\mu+1)} - \mathcal{E}_0^{(\mu)} I \end{pmatrix}, \quad Q_\mu^\dagger = \begin{pmatrix} 0 & A_\mu^\dagger \\ 0 & 0 \end{pmatrix}, \quad Q_\mu = \begin{pmatrix} 0 & 0 \\ A_\mu & 0 \end{pmatrix}, \tag{3.2}$$

where

$$\mathcal{H}^{(0)} = A_0^\dagger A_0,$$
$$\mathcal{H}^{(\mu)} = A_{\mu-1}A_{\mu-1}^\dagger + \mathcal{E}_0^{(\mu-1)}I = A_\mu^\dagger A_\mu + \mathcal{E}_0^{(\mu)}I, \qquad \mu = 1,2,\ldots,p, \tag{3.3}$$
$$A_\mu = \frac{d}{dx} + W(x,b_\mu), \qquad A_\mu^\dagger = -\frac{d}{dx} + W(x,b_\mu), \qquad \mu = 0,1,\ldots,p,$$

and $\mathcal{E}_0^{(\mu)}$ denotes the ground state energy of $\mathcal{H}^{(\mu)}$ (with $\mathcal{E}_0^{(0)} = 0$). Here the superpotentials $W(x,b_\mu)$ depend upon some parameters b_μ, such that $b_{\mu+p} = b_\mu$, and they satisfy p shape invariance conditions

$$W^2(x,b_\mu) + W'(x,b_\mu) = W^2(x,b_{\mu+1}) - W'(x,b_{\mu+1}) + \omega_\mu, \qquad \mu = 0,1,\ldots,p-1, \tag{3.4}$$

where ω_μ, $\mu = 0,1,\ldots,p-1$, are some real constants.

From the solution of Eq.(3.4), one may then construct the potentials corresponding to the supersymmetric partners $\mathcal{H}^{(\mu)}$, $\mathcal{H}^{(\mu+1)}$ in the usual way, i.e., $V^{(\mu)} = W^2(x,b_\mu) - W'(x,b_\mu) + \mathcal{E}_0^{(\mu)}$, $V^{(\mu+1)} = W^2(x,b_\mu) + W'(x,b_\mu) + \mathcal{E}_0^{(\mu)}$. For $p = 2$, Gangopadhyaya and Sukhatme [14] obtained such potentials as superpositions of a Calogero potential and a δ-function singularity. For $p \geq 3$, however, only numerical solutions of the shape invariance conditions (3.4) have been obtained [13], so that no analytical form of $V^{(\mu)}$ is known. In spite of this, the spectrum is easily derived and consists of p infinite sets of equally spaced energy levels, shifted with respect to each other by the energies $\omega_0, \omega_1, \ldots, \omega_{p-1}$.

Since for some special choices of parameters, spectra of a similar type may be obtained with the bosonic oscillator Hamiltonian (2.10) acting in the bosonic Fock space representation of $\mathcal{A}^{(p)}(G(N))$, one may try to establish a relation between the class of algebras $\mathcal{A}^{(p)}(G(N))$ and SSQM with cyclic shape invariant potentials of period p.

In Ref. [11], we proved that the operators $\mathcal{H}^{(\mu)}$, $A_\mu^\dagger$, and A_μ of Eqs.(3.2) and (3.3) can be realized in terms of the generators of p algebras $\mathcal{A}^{(p)}(G^{(\mu)}(N))$, $\mu = 0,1,\ldots,p-1$, belonging to the class $\left\{\mathcal{A}^{(p)}(G(N))\right\}$. The parameters of such algebras are obtained by cyclic permutations from a starting set $\{\alpha_0, \alpha_1, \ldots, \alpha_{p-1}\}$ corresponding to $\mathcal{A}^{(p)}(G^{(0)}(N)) = \mathcal{A}^{(p)}(G(N))$. Denoting by N, $a_\mu^\dagger$, a_μ the number, creation, and annihilation operators corresponding to the μth algebra $\mathcal{A}^{(p)}(G^{(\mu)}(N))$, where $a_0^\dagger = a^\dagger$, and $a_0 = a$, we may write the fourth relation in the algebra defining relations (2.4) as

$$\left[a_\mu, a_\mu^\dagger\right] = I + \sum_{\nu=0}^{p-1} \alpha_\nu^{(\mu)} P_\nu, \qquad \alpha_\nu^{(\mu)} \equiv \alpha_{\nu+\mu}, \qquad \mu = 0,1,\ldots,p-1, \tag{3.5}$$

while the remaining relations keep the same form.

The realization of $\mathcal{H}^{(\mu)}$, $A_\mu^\dagger$, A_μ, $\mu = 0,1,\ldots,p-1$, is then given by

$$\mathcal{H}^{(\mu)} = F(N+\mu) = N + \mu I + \sum_{\nu=0}^{p-1} \beta_{\nu+\mu} P_\nu = H_0^{(\mu)} - \frac{1}{2}\sum_{\nu=0}^{p-1}\left(1 + \alpha_\nu^{(\mu)}\right) P_\nu + \mathcal{E}_0^{(\mu)} I, \tag{3.6}$$
$$A_\mu^\dagger = a_\mu^\dagger, \qquad A_\mu = a_\mu,$$

where $H_0^{(\mu)} \equiv \frac{1}{2}\left\{a_\mu, a_\mu^\dagger\right\}$ is the bosonic oscillator Hamiltonian associated with $\mathcal{A}^{(p)}(G^{(\mu)}(N))$, $\mathcal{E}_0^{(\mu)} = \sum_{\nu=0}^{\mu-1} \omega_\nu$, and the level spacings are $\omega_\mu = 1 + \alpha_\mu$. For this result to be meaningful, the conditions $\omega_\mu > 0$, $\mu = 0, 1, \ldots, p-1$, have to be fulfilled. When combined with the restrictions (2.9), the latter imply that the parameters of the starting algebra $\mathcal{A}^{(p)}(G(N))$ must be such that $-1 < \alpha_0 < \lambda - 1$, $-1 < \alpha_\mu < \lambda - \mu - 1 - \sum_{\nu=0}^{\mu-1} \alpha_\nu$ if $\mu = 1, 2, \ldots, \lambda - 2$, and $\alpha_{\lambda-1} = -\sum_{\nu=0}^{\lambda-2} \alpha_\nu$.

4 Application to parasupersymmetric quantum mechanics of order p

The sqm(2) superalgebra (3.1) is most often realized in terms of mutually commuting boson and fermion operators. Plyushchay [15], however, showed that it can alternatively be realized in terms of only boson-like operators, namely the generators of the Calogero–Vasiliev algebra $\mathcal{A}^{(2)}(G(N))$ (see also Ref. [16]). Such an SSQM bosonization can be performed in two different ways, by choosing either $Q = a^\dagger P_1$ (so that $\mathcal{H} = H_0 - \frac{1}{2}(K + \kappa)$) or $Q = a^\dagger P_0$ (so that $\mathcal{H} = H_0 + \frac{1}{2}(K + \kappa)$). The first choice corresponds to unbroken SSQM (all the excited states are twofold degenerate while the ground state is nondegenerate and at vanishing energy), and the second choice describes broken SSQM (all the states are twofold degenerate and at positive energy).

SSQM was generalized to parasupersymmetric quantum mechanics (PSSQM) of order two by Rubakov and Spiridonov [17], and later on to PSSQM of arbitrary order p by Khare [18]. In the latter case, Eq. (3.1) is replaced by

$$
\begin{aligned}
& Q^{p+1} = 0 \qquad (\text{with } Q^p \neq 0), \\
& [\mathcal{H}, Q] = 0, \\
& Q^p Q^\dagger + Q^{p-1} Q^\dagger Q + \cdots + Q Q^\dagger Q^{p-1} + Q^\dagger Q^p = 2p Q^{p-1} \mathcal{H},
\end{aligned}
\tag{4.1}
$$

and is retrieved in the case where $p = 1$. The parasupercharges Q, $Q^\dagger$, and the parasupersymmetric Hamiltonian $\mathcal{H}$ are usually realized in terms of mutually commuting boson and parafermion operators.

A property of PSSQM of order p is that the spectrum of $\mathcal{H}$ is $(p+1)$-fold degenerate above the $(p-1)$th energy level. This fact and Plyushchay's results for $p = 1$ hint at a possibility of representing $\mathcal{H}$ as a linear combination of the bosonic oscillator Hamiltonian H_0 associated with $\mathcal{A}^{(p+1)}(G(N))$ and some projection operators, as in Eq.(3.6).

In Ref. [10] (see also Refs. [8, 19]), we proved that PSSQM of order p can indeed be bosonized in terms of the generators of $\mathcal{A}^{(p+1)}(G(N))$ for any allowed (i.e., satisfying Eq.(2.9)) values of the algebra parameters $\alpha_0, \alpha_1, \ldots, \alpha_{p-1}$. For such a purpose, we started from ansätze of the type

$$
Q = \sum_{\nu=0}^{p} \sigma_\nu a^\dagger P_\nu, \qquad \mathcal{H} = H_0 + \frac{1}{2} \sum_{\nu=0}^{p} r_\nu P_\nu,
\tag{4.2}
$$

where σ_ν and r_ν are some complex and real constants, respectively, to be determined in such a way that Eq.(4.1) is fulfilled. We found that there are $p+1$ families of solutions, which may be distinguished by an index $\mu \in \{0, 1, \ldots, p\}$ and from which we may choose the following representative solutions

$$
\begin{aligned}
Q_\mu &= \sqrt{2} \sum_{\nu=1}^{p} a^\dagger P_{\mu+\nu}, \\
\mathcal{H}_\mu &= N + \frac{1}{2}(2\gamma_{\mu+2} + r_{\mu+2} - 2p + 3)I + \sum_{\nu=1}^{p} (p+1-\nu) P_{\mu+\nu},
\end{aligned}
\tag{4.3}
$$

where

$$
r_{\mu+2} = \frac{1}{p}\left[(p-2)\alpha_{\mu+2} + 2\sum_{\nu=3}^{p}(p-\nu+1)\alpha_{\mu+\nu} + p(p-2)\right]. \tag{4.4}
$$

The eigenvectors of $\mathcal{H}_\mu$ are the states (2.7) and the corresponding eigenvalues are easily found. All the energy levels are equally spaced. For $\mu = 0$, PSSQM is unbroken, otherwise it is broken with a $(\mu+1)$-fold degenerate ground state. All the excited states are $(p+1)$-fold degenerate. For $\mu = 0, 1, \ldots, p-2$, the ground state energy may be positive, null, or negative depending on the parameters, whereas for $\mu = p-1$ or p, it is always positive.

Khare [18] showed that in PSSQM of order p, $\mathcal{H}$ has in fact $2p$ (and not only two) conserved parasupercharges, as well as p bosonic constants. In other words, there exist p independent operators Q_r, $r = 1, 2, \ldots, p$, satisfying with $\mathcal{H}$ the set of equations (4.1), and p other independent operators I_t, $t = 2, 3, \ldots, p+1$, commuting with $\mathcal{H}$, as well as among themselves. In Ref. [10], we obtained a realization of all such operators in terms of the $\mathcal{A}^{(p+1)}(G(N))$ generators.

As a final point, let us note that there exists an alternative approach to PSSQM of order p, which was proposed by Beckers and Debergh [20], and wherein the multilinear relation in Eq.(4.1) is replaced by the cubic equation

$$
\left[Q, \left[Q^\dagger, Q\right]\right] = 2Q\mathcal{H}. \tag{4.5}
$$

In Ref. [8], we proved that for $p = 2$, this PSSQM algebra can only be realized by those $\mathcal{A}^{(3)}(G(N))$ algebras that simultaneously bosonize Rubakov–Spiridonov–Khare PSSQM algebra.

5 Application to pseudosupersymmetric quantum mechanics

Pseudosupersymmetric quantum mechanics (pseudoSSQM) was introduced by Beckers, Debergh and Nikitin [21] in a study of relativistic vector mesons interacting with an external constant magnetic field. In the nonrelativistic limit, their theory leads to a pseudosupersymmetric oscillator Hamiltonian, which can be realized in terms of mutually commuting boson and pseudofermion operators, where the latter are intermediate between standard fermion and $p = 2$ parafermion operators.

It is then possible to formulate a pseudoSSQM [21], characterized by a pseudosupersymmetric Hamiltonian $\mathcal{H}$ and pseudosupercharge operators Q, $Q^\dagger$, satisfying the relations

$$
Q^2 = 0, \qquad [\mathcal{H}, Q] = 0, \qquad QQ^\dagger Q = 4c^2 Q\mathcal{H}, \tag{5.1}
$$

and their Hermitian conjugates, where c is some real constant. The first two relations in Eq.(5.1) are the same as those occurring in SSQM, whereas the third one is similar to the multilinear relation valid in PSSQM of order two. Actually, for $c = 1$ or $1/2$, it is compatible with Eq.(4.1) or (4.5), respectively.

In Ref. [10], we proved that pseudoSSQM can be bosonized in two different ways in terms of the generators of $\mathcal{A}^{(3)}(G(N))$ for any allowed values of the parameters α_0, α_1. This time, we started from the ansätze

$$Q = \sum_{\nu=0}^{2} \left(\xi_\nu a + \eta_\nu a^\dagger\right) P_\nu, \qquad \mathcal{H} = H_0 + \frac{1}{2}\sum_{\nu=0}^{2} r_\nu P_\nu, \tag{5.2}$$

and determined the complex constants ξ_ν, η_ν, and the real ones r_ν in such a way that Eq.(5.1) is fulfilled.

The first type of bosonization corresponds to three families of two-parameter solutions, labelled by an index $\mu \in \{0, 1, 2\}$,

$$\begin{aligned} Q_\mu(\eta_{\mu+2}, \varphi) &= \left(\eta_{\mu+2} a^\dagger + e^{\mathrm{i}\varphi}\sqrt{4c^2 - \eta_{\mu+2}^2}\, a\right) P_{\mu+2}, \\ \mathcal{H}_\mu(\eta_{\mu+2}) &= N + \frac{1}{2}(2\gamma_{\mu+2} + r_{\mu+2} - 1)I + 2P_{\mu+1} + P_{\mu+2}, \end{aligned} \tag{5.3}$$

where $0 < \eta_{\mu+2} < 2|c|$, $0 \le \varphi < 2\pi$, and

$$r_{\mu+2} = \frac{1}{2c^2}(1 + \alpha_{\mu+2})\left(|\eta_{\mu+2}|^2 - 2c^2\right). \tag{5.4}$$

Choosing for instance $\eta_{\mu+2} = \sqrt{2}|c|$, and $\varphi = 0$, hence $r_{\mu+2} = 0$ (producing an overall shift of the spectrum), we obtain

$$\begin{aligned} Q_\mu &= c\sqrt{2}\left(a^\dagger + a\right) P_{\mu+2}, \\ \mathcal{H}_\mu &= N + \frac{1}{2}(2\gamma_{\mu+2} - 1)I + 2P_{\mu+1} + P_{\mu+2}. \end{aligned} \tag{5.5}$$

A comparison between Eq.(5.3) or (5.5) and Eq.(4.3) shows that the pseudosupersymmetric and $p = 2$ parasupersymmetric Hamiltonians coincide, but that the corresponding charges are of course different. The conclusions relative to the spectrum and the ground state energy are therefore the same as in Sec. 4.

The second type of bosonization corresponds to three families of one-parameter solutions, again labelled by an index $\mu \in \{0, 1, 2\}$,

$$\begin{aligned} Q_\mu &= 2|c| a P_{\mu+2}, \\ \mathcal{H}_\mu(r_\mu) &= N + \frac{1}{2}(2\gamma_{\mu+2} - \alpha_{\mu+2})I + \frac{1}{2}(1 - \alpha_{\mu+1} + \alpha_{\mu+2} + r_\mu)P_\mu + P_{\mu+1}, \end{aligned} \tag{5.6}$$

where $r_\mu \in \mathrm{R}$ changes the Hamiltonian spectrum in a significant way. We indeed find that the levels are equally spaced if and only if $r_\mu = (\alpha_{\mu+1} - \alpha_{\mu+2} + 3) \bmod 6$. If r_μ is small enough, the ground state is nondegenerate, and its energy is negative for $\mu = 1$, or may have any sign for $\mu = 0$ or 2. On the contrary, if r_μ is large enough, the ground state remains nondegenerate with a vanishing energy in the former case, while it becomes twofold degenerate with a positive energy in the latter. For some intermediate r_μ value, one gets a two or threefold degenerate ground state with a vanishing or positive energy, respectively.

6 Application to orthosupersymmetric quantum mechanics of order two

Mishra and Rajasekaran [22] introduced order-p orthofermion operators by replacing the Pauli exclusion principle by a more stringent one: an orbital state shall not contain more than one particle, whatever be the spin direction. The wave function is thus antisymmetric in spatial indices alone with the order of the spin indices frozen.

Khare, Mishra, and Rajasekaran [23] then developed orthosupersymmetric quantum mechanics (OSSQM) of arbitrary order p by combining boson operators with orthofermion ones, for which the spatial indices are ignored. OSSQM is formulated in terms of an orthosupersymmetric Hamiltonian $\mathcal{H}$, and $2p$ orthosupercharge operators Q_r, $Q_r^\dagger$, $r = 1, 2, \ldots, p$, satisfying the relations

$$Q_r Q_s = 0, \qquad [\mathcal{H}, Q_r] = 0, \qquad Q_r Q_s^\dagger + \delta_{r,s} \sum_{t=1}^{p} Q_t^\dagger Q_t = 2\delta_{r,s}\mathcal{H}, \tag{6.1}$$

and their Hermitian conjugates, where r and s run over $1, 2, \ldots, p$.

In Ref. [10], we proved that OSSQM of order two can be bosonized in terms of the generators of some well-chosen $\mathcal{A}^{(3)}(G(N))$ algebras. As ansätze, we used the expressions

$$Q_1 = \sum_{\nu=0}^{2} \left(\xi_\nu a + \eta_\nu a^\dagger\right) P_\nu, \quad Q_2 = \sum_{\nu=0}^{2} \left(\zeta_\nu a + \rho_\nu a^\dagger\right) P_\nu, \quad \mathcal{H} = H_0 + \frac{1}{2} \sum_{\nu=0}^{2} r_\nu P_\nu, \tag{6.2}$$

and determined the complex constants ξ_ν, η_ν, ζ_ν, ρ_ν, and the real ones r_ν in such a way that Eq.(6.1) is fulfilled. We found two families of two-parameter solutions, labelled by $\mu \in \{0, 1\}$,

$$\begin{aligned}
&Q_{1,\mu}(\xi_{\mu+2}, \varphi) = \xi_{\mu+2} a P_{\mu+2} + e^{\mathrm{i}\varphi} \sqrt{2 - \xi_{\mu+2}^2}\, a^\dagger P_\mu, \\
&Q_{2,\mu}(\xi_{\mu+2}, \varphi) = -e^{-\mathrm{i}\varphi} \sqrt{2 - \xi_{\mu+2}^2}\, a P_{\mu+2} + \xi_{\mu+2} a^\dagger P_\mu, \\
&\mathcal{H}_\mu = N + \frac{1}{2}(2\gamma_{\mu+1} - 1) I + 2P_\mu + P_{\mu+1},
\end{aligned} \tag{6.3}$$

where $0 < \xi_{\mu+2} \leq \sqrt{2}$ and $0 \leq \varphi < 2\pi$, provided the algebra parameter $\alpha_{\mu+1}$ is taken as $\alpha_{\mu+1} = -1$. As a matter of fact, the absence of a third family of solutions corresponding to $\mu = 2$ comes from the incompatibility of this condition (i.e., $\alpha_0 = -1$) with conditions (2.9).

The orthosupersymmetric Hamiltonian $\mathcal{H}$ in Eq.(6.3) is independent of the parameters $\xi_{\mu+2}$, φ. All the levels of its spectrum are equally spaced. For $\mu = 0$, OSSQM is broken: the levels are threefold degenerate, and the ground state energy is positive. On the contrary, for $\mu = 1$, OSSQM is unbroken: only the excited states are threefold degenerate, while the nondegenerate ground state has a vanishing energy. Such results agree with the general conclusions of Ref. [23].

For p values greater than two, the OSSQM algebra (6.1) becomes rather complicated because the number of equations to be fulfilled increases considerably. A glance at the 18 independent conditions for $p = 3$ led us to the conclusion that the $\mathcal{A}^{(4)}(G(N))$ algebra is not rich enough to contain operators satisfying Eq.(6.1). Contrary to what happens for PSSQM, for OSSQM the $p = 2$ case is therefore not representative of the general one.

7 Conclusion

In this communication, we showed that the S_2-extended oscillator algebra, which was introduced in connection with the two-particle Calogero model, can be extended to the whole class of C_λ-extended oscillator algebras $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$, where $\lambda \in \{2, 3, \ldots\}$, and $\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-2}$ are some real parameters. In the same way, the GDOA realization of the former, known as the Calogero–Vasiliev algebra, is generalized to a class of GDOAs $\mathcal{A}^{(\lambda)}(G(N))$, where $\lambda \in \{2, 3, \ldots\}$, for which one can define a bosonic oscillator Hamiltonian H_0, acting in the bosonic Fock space representation.

For $\lambda \geq 3$, the spectrum of H_0 has a very rich structure in terms of the algebra parameters $\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-2}$. This can be exploited to provide an algebraic realization of SSQM with cyclic shape invariant potentials of period λ, a bosonization of PSSQM of order $p = \lambda - 1$, and, for $\lambda = 3$, a bosonization of pseudoSSQM and OSSQM of order two.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 298–303.

Some Problems of Quantum Group Gauge Field Theory

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Some difficulties of the construction of quantum group gauge field theory on the classical and quantum spacetime are clarified. The classical geometric interpretation of the ghost field is generalized to case of the quantum group gauge field theory.

1 Introduction

The notation of a generalized Lie group as a noncommutative and noncocommutative Hopf algebra was done by Drinfeld [1], Jimbo [2], Woronowicz [3]. The first step in the construction of noncommutative dynamics was undertaken by I.G. Biedenharn [4] and McFarlane [5] in their study of the quantum noncommutative harmonic oscillator. From this period attempts were undertaken to construct deformed dynamical theories [6, 7, 8, 9], in particular, the deformed gauge theory named the quantum group gauge field theory with the quantum group playing the role of the gauge group. The conceptual problems concerning of the definition of the gauge field theory where the quantum group is considered as object of the gauge symmetry, i.e. the quantum group gauge field theory are not settled. Such theories investigated by Bernard [10], Aref'eva and Volovich [11], Hietarinta [12], Isaev and Popowicz [13], Bernard [10], Watamura [15], Brzezinski, Majid [16, 17], Hajac [18], Sudbery [19]. The deformed gauge field theory is interesting from various points of view. The enlargement of the rigid frameworks of the gauge theory would help to solve the fundamental theoretical problems of the spontaneos symmetry breaking and the quark confinement. In particular, in the quantized deformed gravity theory the spacetime becomes noncommutative and could possible provide the regularization mechanism. In the quantized gauge theory the deformation could be interpreted as a kind of the symmetry breaking, which does not reduce the symmetry but deforms it. This mechanism could give the masses to some vector bosons without the necessity to consider Higgs fields. There are two approaches in the construction of q-deformed dynamical field theory. The spacetime in one of them is assumed to be the usual manifold and it deforms only the structure of the dynamical variables. In the second approach the spacetime becomes the quantum (noncommutative) manifold.

2 The quantum group gauge field theory on the classical spaces

2.1. The classical gauge field theory. Let T^a, $a = 1, 2, \ldots, N$ be generators of the Lie algebra of some compact Lie group G satisfying the relations

$$[T_a, T_b] = f^{abc}T^c, \tag{1}$$

where f^{abc} are structure constants of this algebra. The basic objects of nonabelian gauge theory are the gauge fields – the Yang–Mills potentials. These are the set of vector fields $A^a_\mu(x)$, $a = 1, 2, \ldots, N$, $\mu = 0, 1, 2, 3$. The matrix gauge potentials

$$A_\mu(x) = T^a A^a_\mu(x) \tag{2}$$

define the matrix strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{3}$$

or in component form

$$F_{\mu\nu} = T^a F^a_{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A^a_\nu - \partial^a_\nu A_\mu + f^{abc} A^b_\mu A^c_\nu. \tag{4}$$

The Lagrangian density of the theory

$$\mathcal{L} = -1/4 F^a_{\mu\nu} F^{a\mu\nu} \tag{5}$$

is invariant under the gauge transformation

$$A_\mu \to g(x)^{-1} A_\mu g(x) + g(x)^{-1} \partial_\mu g(x), \tag{6}$$

where $g(x) = \exp\{\varepsilon^a(x) T^a\}$, and $\varepsilon^a(x)$ are real functions. The transformation (6) can be written in the infinitesimal form as

$$\delta A^a_\mu = f^{abc} \varepsilon^b(x) A^c_\mu(x) - \partial_\mu \varepsilon^a(x), \qquad \delta F^a(x) = f^{abc} \varepsilon^b F^c(x). \tag{7}$$

In the following we shall review the several approaches in the construction of the q-deformed gauge field theory.

2.2. The construction based on the differential extension of the quantum group G_q [13, 11]. The main efforts in this approach were directed to keep the classical form of the gauge transformation for the gauge potentials. The problem is in the following. Let A be an element of some extension of the quantum group G_q. What differential calculus on this group should be considered and from what extension of this group should be taken the potential A to guarantee that the gauge transformed element A' also belongs to that extension. In some cases this problem were solved [13, 11].

2.3. The construction based on the bicovariant differential calculus on the quantum group [14]. There are many methods to deform a Lie algebra. The one of them is the method of the bicovariant differential calculus on the quantum groups.

Definition 2.1. *A bicovariant bimodule over Hopf algebra $\mathcal{A}$ is a triplet $(\Gamma, \Delta_L, \Delta_R)$ bimodule Γ over $\mathcal{A}$ and of linear mappings*

$$\Delta_L : \Gamma \to \mathcal{A} \otimes \Gamma, \qquad \Delta_R : \Gamma \to \Gamma \otimes \mathcal{A}$$

such that diagrams

1.
$$\begin{array}{ccc} \Gamma & \xrightarrow{\Delta_L} & \mathcal{A}\otimes\Gamma \\ \Delta_L \downarrow & & \downarrow 1\otimes\Delta_L \\ \mathcal{A}\otimes\Gamma & \xrightarrow{\Delta\otimes 1} & \mathcal{A}\otimes\mathcal{A}\otimes\Gamma \end{array}, \qquad \begin{array}{ccc} \Gamma & \xrightarrow{\Delta_L} & \mathcal{A}\otimes\Gamma \\ & \searrow & \downarrow \varepsilon\otimes id \\ & & k\otimes\Gamma \end{array}$$

2.
$$\begin{array}{ccc} \Gamma & \xrightarrow{\Delta_R} & \Gamma\otimes\mathcal{A} \\ \Delta_R \downarrow & & \downarrow 1\otimes\Delta_R \\ \Gamma\otimes\mathcal{A} & \xrightarrow{id\otimes\Delta} & \Gamma\otimes\mathcal{A}\otimes\mathcal{A} \end{array}, \qquad \begin{array}{ccc} \Gamma & \xrightarrow{\Delta_R} & \Gamma\otimes\mathcal{A} \\ & \searrow & \downarrow id\otimes\varepsilon \\ & & \Gamma\otimes k \end{array}$$

$$
3.\quad
\begin{array}{ccc}
\Gamma & \xrightarrow{\Delta_R} & \Gamma\otimes\mathcal{A} \\
\Delta_L\downarrow & & \downarrow \Delta_L\otimes id \\
\mathcal{A}\otimes\Gamma & \xrightarrow{id\otimes\Delta_R} & \mathcal{A}\otimes\Gamma\otimes\mathcal{A}
\end{array}
$$

commute and

4. $\Delta_L(a\omega b) = \Delta_L(a)\Delta\omega\Delta_L(b), \qquad \Delta_R(a\omega b) = \Delta_R(a)\Delta\omega\Delta_R(b).$

Definition 2.2. *A first order differential calculus over Hopf algebra $\mathcal{A}$ is a pair (Γ, d), where Γ is bimodule over $\mathcal{A}$, and the liner mapping $d : \mathcal{A} \to \Gamma$ such that $d(ab) = dab + adb$ and $\Gamma = \{adb : a, b \in \mathcal{A}\}$.*

Definition 2.3. *A first order differential calculus is called bicovariant differential calculus on the quantum group if $(\Gamma, \Delta_L, \Delta_R)$ is a bicovariant bimodule and $\Delta_L(da) = (id \otimes d)\Delta(a)$, $\Delta_R(da) = (d \otimes id)\Delta(a)$.*

Definition 2.4. *A first order differential calculus is called universal if $\Gamma = \ker m$, $m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ is multiplication map in algebra $\mathcal{A}$, and $d = 1 \otimes a - a \otimes 1$.*

It easy to see that $d : \mathcal{A} \to \Gamma$ is linear map satisfying the Leibnitz rule, Γ has the bimodule structure $c\left(\sum\limits_k a_k \otimes b_k\right) = \sum\limits_k ca_k \otimes b_k$, $\left(\sum\limits_k a_k \otimes b_k\right) c = \sum\limits_k a_k \otimes b_k c$ and every element of Γ has the form $\sum\limits_k a_k db_k$.

Definition 2.5. *A Z_2-graded complex differential algebra (Ω, d) is Z_2-graded complex algebra $\Omega = \Omega^+ + \Omega^-$,equipped with the graded derivation d which is odd and of square zero $d\Omega^{+,-} \subset \Omega^{-,+}$, $d^2 = 0$.*

Every first order differential calculus (Γ, d) generate Z_2-graded complex differential algebra $(\Omega(\mathcal{A}), d)$. A complex differential algebra generated by universal calculus is called *a differential envelop* of $\mathcal{A}$ and is denoted as $(\Omega\mathcal{A}, d)$. The space dual to the left-invariant subspace $\Gamma_{\rm inv}$ can be introduced as a linear subspace of A' whose basis elements $\chi_i \in A'$ are defined by

$$
da = \chi_i * a\omega^i \qquad \text{for all } a \in \mathcal{A}. \tag{8}
$$

The analogue of the ordinary permutation operator is a bimodule automorphism Λ in $\Gamma\otimes\Gamma$ defined by

$$
\Lambda\left(\omega^i \otimes \eta^j\right) = \eta^j \otimes \omega^i, \tag{9}
$$

i.e. $\Lambda(a\tau) = a\Lambda(\tau)$, $\Lambda(\tau b) = \Lambda(\tau)b$, where $a \in \mathcal{A}$, $\tau \in \Gamma\otimes\Gamma$. With the help of braiding operator Λ the exterior product of the elements $\rho, \rho' \in \Gamma$ is given

$$
\rho \bigwedge \rho' = \rho \otimes \rho' - \Lambda(\rho \otimes \rho'), \tag{10}
$$

$$
\omega^i \bigwedge \omega^j = \omega^i \otimes \omega^j - \Lambda^{ij}_{kl}\left(\omega^k \otimes \omega^l\right). \tag{11}
$$

The exterior product of two left invariant forms satisfies the relation

$$
\omega^i \bigwedge \omega^j = \frac{1}{q^2 + q^{-2}}\left[\Lambda^{ij}_{kl} + \left(\Lambda^{-1}\right)^{ij}_{kl}\right] \omega^k \bigwedge \omega^l. \tag{12}
$$

There exists an *adjoint representation* M^i_j of the quantum group defined by the right action on the left invariant ω^i

$$\Delta_R\left(\omega^i\right) = \omega^i \otimes M^i_j, \qquad M^i_j \in \mathcal{A}. \tag{13}$$

The bicovariant calculus on a G_q is characterized by the functionals χ_i and f^i_j on A satisfying

$$\chi_i\chi_j - \Lambda^{kl}_{ij}\chi_k\chi_l = \mathbf{C}^k_{ij}\chi_k, \tag{14}$$

$$\Lambda^{nm}_{ij} f^i_p f^i_q = f^n_i f^m_j \Lambda^{ij}_{pq}, \tag{15}$$

$$\mathbf{C}^i_{mn} f^m_j f^n_k = \Lambda^{pq}_{jk}\chi_p f^i_q + \mathbf{C}^l_{jk} f^i_p, \tag{16}$$

$$\chi^n_k f^n_l = \Lambda^{ij}_{kl} f^n_i \chi_j, \tag{17}$$

where $\Lambda^{ij}_{kl} = f^i_l(M^j_k)$, $\mathbf{C}^i_{jk} = \chi_k(M^i_j)$. In adjoint representation these conditions have the form

$$\mathbf{C}^n_{ri}\mathbf{C}^s_{nj} - R^{kl}_{ij}\mathbf{C}^n_{rk}\mathbf{C}^s_{nl} = \mathbf{C}^k_{ij}\mathbf{C}^s_{rk}, \tag{18}$$

$$\Lambda^{nm}_{ij}\Lambda^{ik}_{rp}\Lambda^{js}_{kq} = \Lambda^{nk}_{ri}\Lambda^{ms}_{kj}\Lambda^{ij}_{pq}, \tag{19}$$

$$\mathbf{C}^i_{mn}\Lambda^m_i\Lambda^i_j\mathbf{C}^s_{lk} = \Lambda^{nm}_{jk}\Lambda^{il}_{rq}\mathbf{C}^s_{lp} + \mathbf{C}^m_{jk}\Lambda^{is}_{rm}, \tag{20}$$

$$\mathbf{C}^m_{rk}\Lambda^{ns}_{ml} = \Lambda^{ij}_{kl}\Lambda^{nm}_{ri}\mathbf{C}^s_{mj}. \tag{21}$$

In this case the Lie algebra (1) of the gauge group of the classical gauge theory, taking into account of (14), is replaced by the quantum Lie algebra

$$T_aT_b - \Lambda^{cd}_{ab}T_cT_d = \mathbf{C}^c_{ab}T_c. \tag{22}$$

As in the classical case (2) the gauge field is defined by the same formula $A_\mu = A^a_\mu T_a$, but now the gauge potentials are noncommutative and satisfy the commutation relations

$$A^a_{[\mu}A^b_{\nu]} = -\frac{1}{q^2+q^{-2}}\left(\Lambda+\Lambda^{-1}\right)^{ab}_{cd}A^c_{[\mu}A^d_{\nu]}. \tag{23}$$

The field strength can be represented in the form

$$F^a_{\mu\nu} = \partial_{[\mu}A^a_{\nu]} + P^{kl}_{A\ mn}C^a_{kl}A^m_{[\mu}A^n_{\nu]}, \tag{24}$$

where $\mathbf{C}^n_{kl} = C^n_{kl} - \Lambda^{ij}_{kl}C^n_{ij}$. The deformed gauge transformations are assumed to have the form

$$\delta A = -d\varepsilon - A\varepsilon + \varepsilon A, \qquad \varepsilon = \varepsilon^a T_a. \tag{25}$$

The gauge parameters ε are q-numbers and are assumed to have the following commutation relations

$$\varepsilon^a A_a = \Lambda^{ab}_{mn}\varepsilon^n. \tag{26}$$

Then F is transformed as $\delta F = \varepsilon F - F\varepsilon$ and the deformed Lagrangian density $\mathcal{L} = F^a_{\mu\nu}F^b_{\mu\nu}g_{ab}$ is invariant under transformations (25) if

$$\Lambda^{nb}_{rs}C^a_{mn}g_{ab} + C^b_{rs}g_{mb} = 0. \tag{27}$$

2.4. The construction based on the quantum deformation of the BRST algebra [6, 15, 9]. One of the alternative formulations of the gauge field theory is the BRST method. In this approach the BRST transformation s is defined and parameter $\varepsilon(x)$ is replaced by the ghost field $C(x)$. If we restrict ourselves by the pure Yang–Mills field theory, then BRST transformations are reduced to the form

$$sA^a_\mu = \partial C^a + f^a_{bc} A^b_\mu C^C, \qquad sC^a = -\frac{1}{2} f^a_{bc} C^b C^c. \tag{28}$$

The gauge field strength F^a is transformed covariantly $sF^a = f^a_{bc} F^b C^c$. As was noted in [17] if we use the gauge symmetry of the Hopf algebra then it is necessary to formulate all theory in the algebraic frameworks. The gauge transformations should be represented in the abstract language. As we saw at (26) it is not known to what algebra belongs the set of the parameters. The idea of [6] is as follows: replace local gauge parameters of the theory by the ghost fields which now placed at same level as the gauge and matter fields. The formulation of all theory is algebraic.

3 The quantum group gauge field theory on the quantum spaces

3.1. The geometrical meaning of the gauge field potentials [21]. Let P and M be smooth manifolds, a Lie group G smooth acting on P and the differentiable principal fiber bundle $P(M, G)$ over M with the group G. A global (local) cross-section of a principal fiber bundle is a map σ from the base space (neighbourhood U_α) to the bundle space P such that $\pi(\sigma(x)) = x$, $\forall\, x \in M$ ($\pi\sigma_\alpha(x) = x$, $\forall\, x \in U_\alpha$) Let ω_α be a 1-form in U_α. It can be written in terms of its components (Lie-algebra valued functions) $A^\mu_\alpha(x)$

$$\omega = \sum_\mu A^\mu_\alpha(x) dx_\mu. \tag{29}$$

Suppose we transform σ_α into σ'_α by the action of some $g \in G$. If $\sigma'_\alpha(x) = \sigma_\alpha(x) g(x)$, then $\omega'_\alpha = \sigma'^*_\alpha \omega = A'^\mu dx_\mu$, where

$$A'_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g. \tag{30}$$

This reproduces the gauge transformation formula for gauge potentials (6). The connection form ω describes at the same time both the Yang–Mills potential and ghost fields. It is split into two components of the gauge field ϕ which is horisontal and the ghost field χ which is normal to the section σ. From the Cartan–Maurer theorem the equations follow:

$$s\chi + 1/2[\chi, \chi] = 0, \qquad s\phi + B\chi = 0, \tag{31}$$

which are the same as BRST transformation (28).

3.2. The construction based on the quantum group generalization of the fiber bundle [17, 18]. The quantum group gauge field theory is constructed also in the framework of the fiber bundle with the quantum structure group [17]. The Cartan–Maurer equation obtained by the universal bicovariant differential calculus on quantum group is the same as BRST transformation ghost fields of the quantum group gauge field theory. But for general quantum fiber bundle this problem is open.

Acknowledgements

This research was supported by the Ukrainian State Foundation of Fundamental Researches Grant No. 1.4/206.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 304–309.

Use of Quantum Algebras in Quantum Gravity

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After brief survey of appearance of quantum algebras in diverse contexts of quantum gravity, we demonstrate that the particular deformed algebras, which arise within the approach of J. Nelson and T. Regge to $(2+1)$ anti-de Sitter quantum gravity (for space surface of genus g) and which should generate algebras of independent quantum observables, are in fact isomorphic to nonstandard q-deformed analogues $U'_q(so_n)$ (introduced in 1991) of Lie algebras of the orthogonal groups $SO(n)$, n being related to g as $n = 2g + 2$.

1 Introduction

Quantum or q-deformed algebras may appear in quantum (or q-versions of) gravity in various situations. Let us mention some of them.

• Case of n spacetime dimensions ($n \geq 2$), straightforward approach to construct q-gravity (this is accomplished, e.g., in [1]). Basic steps are:

– Start with some version of quantum/q-deformed algebra $iso_q(n)$ (in [1] it is projected out from the standard quantum algebras $U_q(B_r)$, $U_q(D_r)$ of Drinfeld and Jimbo [2]). In the particular Poincare algebra $iso_q(3,1)$ exploited by Castellani, only those commutation relations which involve momenta do depend on the parameter q, while the Lorentz subalgebra remains non-deformed;

– Develop necessary bicovariant differential calculus;

– A q-gravity is constructed by "gauging" the q-analogue of Poincaré algebra. The resulting Lagrangian turns out to be a generalization [1] (see also [3]) of the usual Einstein or Einstein–Cartan one.

It is worth to emphasize that in this approach the obtained results, including physical implications, unambiguously depend on the specific features of chosen the q-algebra.

• Two-dimensional quantum Liouville gravity [4], within particular framework of quantization, leads to the appearance [5] of quantum algebras such as $U_q(sl(2,\mathbf{C}))$.

• Case of 3-dimensional (Euclidean) gravity. The simpl approach developed by Ponzano and Regge [6] employs irreducible representations of the algebra $su(2)$ labelled by spins j and assigned to edges of tetrahedra in triangulation, the main ingredient being $6j$-symbols of $su(2)$. Within natural generalization of this approach by Turaev and Viro [7], see also [8], the underlying symmetry of the action (which can be related to Chern–Simons theory) is that of the quantum algebra $su_q(2)$, and basic objects are $q - 6j$ symbols. Due to this, physical quantities become expressible through topological (knot or link) invariants. The parameter q takes into account the cosmological constant and, on the other hand, is connected with the (quantized) Chern–Simons coupling constant k as $q = \exp \frac{2i\pi}{k+2}$.

• $(2+1)$-dimensional gravity with or without cosmological constant Λ is known to possess important peculiar features [9, 10]. Within the approach to quantization developed by J. Nelson and T. Regge, specific deformed algebras arise [11, 12] for the situation with $\Lambda < 0$, and just this fact will be of our main concern here.

2 Nonstandard q-deformed algebras $U'_q(\mathrm{so}_n)$, their advantages

As defined in [13], the nonstandard q-deformation $U'_q(\mathrm{so}_n)$ of the Lie algebra so_n is given as a complex associative algebra with $n-1$ generating elements I_{21}, $I_{32}, \ldots$, $I_{n,n-1}$ obeying the defining relations (denote $q+q^{-1} \equiv [2]_q$)

$$\begin{aligned} &I^2_{j,j-1} I_{j-1,j-2} + I_{j-1,j-2} I^2_{j,j-1} - [2]_q \; I_{j,j-1} I_{j-1,j-2} I_{j,j-1} = -I_{j-1,j-2}, \\ &I^2_{j-1,j-2} I_{j,j-1} + I_{j,j-1} I^2_{j-1,j-2} - [2]_q \; I_{j-1,j-2} I_{j,j-1} I_{j-1,j-2} = -I_{j,j-1}, \\ &[I_{i,i-1}, I_{j,j-1}] = 0 \qquad \text{if} \quad | \, i-j \, | > 1. \end{aligned} \tag{1}$$

At $q \to 1$, $[2]_q \to 2$ (non-deformed or classical limit), these go over into the defining relations of the $so(n)$ Lie algebras.

Among the *advantages of these nonstandard q-deformed algebras* with regards to the Drinfeld–Jimbo quantum deformations, the following should be pointed out.

(**i**) Existence of the canonical chain of embedded subalgebras (from now on, we omit the prime in the symbol)

$$U_q(so_n) \supset U_q(so_{n-1}) \supset \cdots \supset U_q(so_4) \supset U_q(so_3)$$

in the case of $U_q(so_n)$ and, due to this, implementability of the q-analogue of Gelfand–Tsetlin formalism enabling one to construct finite dimensional representations [13, 14].

(**ii**) Existence, for all the real forms known in the nondeformed case $q=1$, of their respective q-analogues – the "compact" $U_q(so_n)$ and the "noncompact" $U_q(so_{p,s})$ (with $p+s=n$) real forms. Moreover, each such form exists along with the corresponding chain of embeddings. For instance, in the n-dimensional q-Lorentz case we have

$$U_q(so_{n-1,1}) \supset U_q(so_{n-1}) \supset U_q(so_{n-2}) \supset \cdots \supset U_q(so_3).$$

This fact enables one to develop the construction and analysis of infinite-dimensional representations of $U_q(so_{n-1,1})$, see [13, 15].

(**iii**) Existence of embedding $U_q(so_3) \subset U_q(sl_3)$ generalizable [16] to the embedding of higher q-algebras such that $U_q(so_n) \subset U_q(sl_n)$, – the fact which enables construction of the proper quantum analogue [16] of symmetric coset space $SL(n)/SO(n)$.

(**iv**) If one attempts to get a q-analogue of the Capelli identity known to hold for the dual pair $sl_2 \leftrightarrow so_n$, nothing but this nonstandard q-algebra $U_q(so_n)$ inevitably arises [17]. As a result, the relation Casimir$\{U_q(sl_2)\}$ = Casimir$\{U_q(so_n)\}$ is valid [17, 18] within particular representation.

(**v**) Natural appearance, as will be discussed in Sec.4, of these q-algebras within the Nelson–Regge approach to $2+1$ quantum gravity.

As a drawback let us mention the fact that Hopf algebra structure is not known for $U_q(so_n)$, although for the situation (**iii**) the nonstandard q-algebra $U_q(so_n)$ was shown to be a coideal [16] in the Hopf algebra $U_q(sl_n)$.

Recall that it was (**i**), (**ii**) which motivated introducing in [13] this class of q-algebras.

3 Bilinear formulation of $U_q(so_n)$

Along with the definition in terms of trilinear relations (1) above, a *'bilinear' formulation* of $U_q(so_n)$ can as well be provided. To this end, one introduces the generators (set $k > l+1$, $1 \le k, l \le n$)

$$I^{\pm}_{k,l} \equiv [I_{l+1,l}, I^{\pm}_{k,l+1}]_{q^{\pm 1}} \equiv q^{\pm 1/2} I_{l+1,l} I^{\pm}_{k,l+1} - q^{\mp 1/2} I^{\pm}_{k,l+1} I_{l+1,l}$$

together with $I_{k+1,k} \equiv I^+_{k+1,k} \equiv I^-_{k+1,k}$. Then (1) imply

$$
\begin{aligned}
&[I^+_{lm}, I^+_{kl}]_q = I^+_{km}, \qquad [I^+_{kl}, I^+_{km}]_q = I^+_{lm}, \qquad [I^+_{km}, I^+_{lm}]_q = I^+_{kl} \qquad \text{if} \quad k > l > m,\\
&[I^+_{kl}, I^+_{mp}] = 0 \qquad \text{if} \quad k > l > m > p \quad \text{or if} \quad k > m > p > l; \\
&[I^+_{kl}, I^+_{mp}] = (q - q^{-1})(I^+_{lp} I^+_{km} - I^+_{kp} I^+_{ml}) \qquad \text{if} \quad k > m > l > p.
\end{aligned}
\tag{2}
$$

Analogous set of relations exists which involves I^-_{kl} along with $q \to q^{-1}$ (denote this "dual" set by (2′)). In the 'classical' limit $q \to 1$, both (2) and (2′) reduce to those of so_n.

To illustrate, we give the examples of $n = 3$, isomorphic to Fairlie–Odesskii algebra [19], and $n = 4$ (recall that the q-commutator is defined as $[X, Y]_q \equiv q^{1/2} XY - q^{-1/2} YX$):

$$
U_q(\mathrm{so}_3): \quad [I_{21}, I_{32}]_q = I^+_{31}, \qquad [I_{32}, I^+_{31}]_q = I_{21}, \qquad [I^+_{31}, I_{21}]_q = I_{32}. \tag{3}
$$

$$
U_q(\mathrm{so}_4): \quad
\begin{cases}
[I_{32}, I_{43}]_q = I^+_{42}, \quad [I^+_{31}, I_{43}]_q = I^+_{41}, \quad [I_{21}, I^+_{42}]_q = I^+_{41},\\
[I_{43}, I^+_{42}]_q = I_{32}, \quad [I_{43}, I^+_{41}]_q = I^+_{31}, \quad [I^+_{42}, I^+_{41}]_q = I_{21},\\
[I^+_{42}, I_{32}]_q = I_{43}, \quad [I^+_{41}, I^+_{31}]_q = I_{43}, \quad [I^+_{41}, I_{21}]_q = I^+_{42},
\end{cases}
\tag{4}
$$

$$
[I_{43}, I_{21}] = 0, \quad [I_{32}, I^+_{41}] = 0, \quad [I^+_{42}, I^+_{31}] = (q - q^{-1})(I_{21} I_{43} - I_{32} I^+_{41}). \tag{5}
$$

The first relation in (3) is viewed as definition for the third generator I^+_{31}; with this, the algebra is given in terms of q-commutators. Dual copy of $U_q(\mathrm{so}_3)$ involves the generator $I^-_{31} = [I_{21}, I_{32}]_{q^{-1}}$ which enters the relations same as (3), but with $q \to q^{-1}$. Similar remarks concern the generators I^+_{42}, I^+_{41}, as well as (dual copy of) the whole algebra $U_q(\mathrm{so}_4)$.

4 The deformed algebras $A(n)$ of Nelson and Regge

For $(2 + 1)$-dimensional gravity with cosmological constant $\Lambda < 0$, the Lagrangian involves spin connection ω_{ab} and dreibein e^a, $a, b = 0, 1, 2$, combined in the $SO(2, 2)$-valued (anti-de Sitter) spin connection ω_{AB} of the form

$$
\omega_{AB} = \begin{pmatrix} \omega_{ab} & \frac{1}{\alpha} e^a \\ -\frac{1}{\alpha} e^b & 0 \end{pmatrix},
$$

and is given in the Chern–Simons (CS) form [10]

$$
\frac{\alpha}{8} \left(\mathrm{d}\omega^{AB} - \frac{2}{3} \omega^A_{\ F} \wedge \omega^{FB} \right) \wedge \omega^{CD} \epsilon_{ABCD}.
$$

Here $A, B = 0, 1, 2, 3$, the metric is $\eta_{AB} = (-1, 1, 1, -1)$, and the CS coupling constant is connected with Λ, so that $\Lambda = -\frac{1}{3\alpha^2}$. The action is invariant under $SO(2, 2)$, leads to Poisson brackets and field equations. Their solutions (infinitesimal connections) describe space-time which is locally anti-de Sitter.

To describe global features of space-time, of principal importance are the *integrated connections* which provide a mapping $S : \pi_1(\Sigma) \to G$ of the homotopy group for a surface Σ into the group $G = SL_+(2, R) \otimes SL_-(2, R)$ (spinorial covering of $SO(2, 2)$) and thoroughly studied in [11]. To generate the algebra of observables, one takes the traces

$$
c^\pm(a) = c^\pm(a^{-1}) = \frac{1}{2} \mathrm{tr}[S^\pm(a)], \qquad a \in \pi_1, \qquad S^\pm \in SL_\pm(2, R).
$$

For $g = 1$ (torus) surface Σ, the algebra of (independent) quantum observables was derived [11], which turned out to be isomorphic to the cyclically symmetric Fairlie–Odesskii algebra [19]. This latter algebra, however, is known to coincide [15] with the special $n = 3$ case of $U_q(so_n)$. So, natural question arises whether for surfaces of higher genera $g \geq 2$, the nonstandard q-algebras $U_q(so_n)$ also play a role.

Below, the positive answer to this question is given.

For the topology of spacetime $\Sigma \times \mathbf{R}$ (fixed-time formulation; Σ is genus-g surface), the homotopy group $\pi_1(\Sigma)$ is most efficiently described in terms of $2g + 2 = n$ generators $t_1, t_2, \ldots, t_{2g+2}$ introduced in [12] and such that

$$t_1 t_3 \cdots t_{2g+1} = 1, \qquad t_2 t_4, ..., t_{2g+2} = 1, \qquad \text{and} \qquad \prod_{i=1}^{2g+2} t_i = 1.$$

Classical gauge invariant trace elements ($n(n-1)/2$ in total) defined as

$$\alpha_{ij} = \frac{1}{2}\mathrm{Tr}(S(t_i t_{i+1} \cdots t_{j-1})), \qquad S \in SL(2,R), \tag{6}$$

generate concrete algebra with Poisson brackets, explicitly found in [12]. At the quantum level, to the algebra with generators (6) there corresponds quantum commutator algebra $A(n)$ specific for $2+1$ quantum gravity with negative Λ. For each quadruple of indices $\{j, l, k, m\}$, $j, l, k, m = 1, \ldots, n$, obeying (see [12]) 'anticlockwise ordering'

$$\begin{array}{ccccc} & & j & & \\ & \swarrow & & \nwarrow & \\ l & & & & m \\ & \searrow & & \nearrow & \\ & & k & & \end{array} \tag{7}$$

the quantum algebra $A(n)$ reads [12]:

$$\begin{aligned} &[a_{mk}, a_{jl}] = [a_{mj}, a_{kl}] = 0, \\ &[a_{jk}, a_{kl}] = \left(1 - \frac{1}{K}\right)(a_{jl} - a_{kl}a_{jk}), \\ &[a_{jk}, a_{km}] = \left(\frac{1}{K} - 1\right)(a_{jm} - a_{jk}a_{km}), \\ &[a_{jk}, a_{lm}] = \left(K - \frac{1}{K}\right)(a_{jl}a_{km} - a_{kl}a_{jm}). \end{aligned} \tag{8}$$

Here the parameter K of deformation involves both α and Planck's constant, namely

$$K = \frac{4\alpha - ih}{4\alpha + ih}, \qquad \alpha^2 = -\frac{1}{3\Lambda}, \qquad \Lambda < 0. \tag{9}$$

Note that in (6) only one copy of the two $SL_\pm(2,R)$ is indicated. In conjunction with this, besides the deformed algebra $A(n)$ derived with, say, $SL_+(2,R)$ taken in (6) and given by (8), another identical copy of $A(n)$ (with the only replacement $K \to K^{-1}$) can also be obtained starting from $SL_-(2,R)$ taken in place of $SL(2,R)$ in (6). This another copy is independent from the original one: their generators mutually commute.

5 Isomorphism of the algebras $A(n)$ and $U_q(so_n)$

To establish isomorphism between the algebra $A(n)$ from (8) and the nonstandard q-deformed algebra $U_q(so_n)$ one has to make the following two steps.

$$\text{Redefine:} \qquad \left\{K^{1/2}(K-1)^{-1}\right\} a_{ik} \longrightarrow A_{ik},$$
$$\text{Identify:} \qquad A_{ik} \longrightarrow I_{ik}, \qquad K \longrightarrow q.$$

Then, the Nelson–Regge algebra $A(n)$ is seen to translate exactly into the nonstandard q-deformed algebra $U'_q(\mathrm{so}_n)$ described above, see (2). We conclude that these two deformed algebras are isomorphic to each other (of course, for $K \neq 1$). Recall that n is linked to the genus g as $n = 2g + 2$, while $K = (4\alpha - ih)/(4\alpha + ih)$ with $\alpha^2 = -\frac{1}{\Lambda}$.

Let us remark that it is the bilinear presentation (2) of the q-algebra $U_q(so_n)$ which makes possible establishing of this isomorphism. It should be stressed also that the algebra $A(n)$ plays the role of "intermediate" one: starting with it and reducing it appropriately, the algebra of quantum observables (gauge invariant global characteristics) is to be finally constructed. The role of Casimir operators in this process, as seen in [12], is of great importance. In this respect let us mention that the quadratic and higher Casimir elements of the q-algebra $U_q(so_n)$, for q being not a root of 1, are known in explicit form [18, 20] along with eigenvalues of their corresponding (representation) operators [20].

As shown in detail in [11], the deformed algebra for the case of genus $g = 1$ surfaces (tori) reduces to the desired algebra of three independent quantum observables which coincides with $A(3)$, the latter being isomorphic to the Fairlie–Odesskii algebra $U_q(so_3)$. The case of $g = 2$ is significantly more involved: here one has to derive, starting with the 15-generator algebra $A(6)$, the necessary algebra of 6 (independent) quantum observables. J. Nelson and T. Regge have succeeded [21] in constructing such an algebra. Their construction however is highly nonunique and, what is more essential, is not seen to be extendable to general situation of $g \geq 3$.

6 Outlook

Our goal in this note was to attract attention to the isomorphism of the deformed algebras $A(n)$ from [12] and the nonstandard q-deformed algebras $U'_q(\mathrm{so}_n)$ introduced in [13]). The hope is that, taking into account a significant amount of the already existing results concerning diverse aspects of $U'_q(\mathrm{so}_n)$ (the obtained various classes of irreducible representations, knowledge of Casimir operators and their eigenvalues depending on representations, etc.) we may expect for a further progress concerning construction of the desired algebras of quantum observables for space surfaces of genera $g > 2$.

Acknowledgements

This work was supported in part by the CRDF award No. UP1-309.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 310–314.

On Casimir Elements of q-Algebras $U_q'(\mathrm{so}_n)$ and Their Eigenvalues in Representations

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The nonstandard q-deformed algebras $U_q'(\mathrm{so}_n)$ are known to possess q-analogues of Gel'fand–Tsetlin type representations. For these q-algebras, all the Casimir elements (corresponding to basis set of Casimir elements of so_n) are found, and their eigenvalues within irreducible representations are given explicitly.

1 Introduction

The nonstandard deformation $U_q'(\mathrm{so}_n)$, see [1], of the Lie algebra so_n admits, in contrast to standard deformation [2] of Drinfeld and Jimbo, an explicit construction of irreducible representations [1, 3] corresponding to those of Lie algebra so_n in Gel'fand–Tsetlin formalism. Besides, as it was shown in [4], $U_q'(\mathrm{so}_n)$ is the proper dual for the standard q-algebra $U_q(\mathrm{sl}_2)$ in the q-analogue of dual pair $(\mathrm{so}_n, \mathrm{sl}_2)$.

Let us mention that the algebra $U_q'(\mathrm{so}_3)$ appeared earlier in the papers [5]. As a matter of interest, this algebra arose naturally as the algebra of observables [6] in $2+1$ quantum gravity with 2D space fixed as torus. At $n > 3$, the algebras $U_q'(\mathrm{so}_n)$ are no less important, serving as intermediate algebras in deriving the algebra of observables in 2+1 quantum gravity with 2D space of genus $g > 1$, so that n depends on g, $n = 2g + 2$ [7, 8]. In order to obtain the algebra of observables, the q-deformed algebra $U_q'(\mathrm{so}_{2g+2})$ should be quotiented by some ideal generated by (combinations of) Casimir elements of this algebra. This fact, along with others, motivates the study of Casimir elements of $U_q'(\mathrm{so}_n)$.

2 The q-deformed agebras $U_q'(\mathrm{so}_n)$

According to [1], the nonstandard q-deformation $U_q'(\mathrm{so}_n)$ of the Lie algebra so_n is given as a complex associative algebra with $n-1$ generating elements $I_{21}, I_{32}, \ldots, I_{n,n-1}$ obeying the defining relations (denote $q + q^{-1} \equiv [2]_q$)

$$
\begin{aligned}
&I_{j,j-1}^2 I_{j-1,j-2} + I_{j-1,j-2} I_{j,j-1}^2 - [2]_q\ I_{j,j-1} I_{j-1,j-2} I_{j,j-1} = -I_{j-1,j-2},\\
&I_{j-1,j-2}^2 I_{j,j-1} + I_{j,j-1} I_{j-1,j-2}^2 - [2]_q\ I_{j-1,j-2} I_{j,j-1} I_{j-1,j-2} = -I_{j,j-1},\\
&[I_{i,i-1}, I_{j,j-1}] = 0 \qquad \text{if} \quad | i-j | > 1.
\end{aligned}
\tag{1}
$$

Along with definition in terms of trilinear relations, we also give a *'bilinear' presentation.* To this end, one introduces the generators (here $k > l+1$, $1 \le k, l \le n$)

$$
I_{k,l}^{\pm} \equiv [I_{l+1,l}, I_{k,l+1}^{\pm}]_{q^{\pm 1}} \equiv q^{\pm 1/2} I_{l+1,l} I_{k,l+1}^{\pm} - q^{\mp 1/2} I_{k,l+1}^{\pm} I_{l+1,l}
$$

together with $I_{k+1,k} \equiv I^+_{k+1,k} \equiv I^-_{k+1,k}$. Then (1) imply

$$\begin{aligned} &[I^+_{lm}, I^+_{kl}]_q = I^+_{km}, \qquad [I^+_{kl}, I^+_{km}]_q = I^+_{lm}, \qquad [I^+_{km}, I^+_{lm}]_q = I^+_{kl} \qquad \text{if} \quad k>l>m,\\ &[I^+_{kl}, I^+_{mp}] = 0 \qquad \text{if} \quad k>l>m>p \quad \text{or} \quad k>m>p>l;\\ &[I^+_{kl}, I^+_{mp}] = (q-q^{-1})(I^+_{lp}I^+_{km} - I^+_{kp}I^+_{ml}) \qquad \text{if} \quad k>m>l>p. \end{aligned} \tag{2}$$

Analogous set of relations exists involving I^-_{kl} along with $q \to q^{-1}$ (let us denote this "dual" set by ($2'$)). If $q \to 1$ ('classical' limit), both (2) and ($2'$) reduce to those of so_n.

Let us give explicitly the two examples, namely $n = 3$ (the Odesskii–Fairlie algebra [5]) and $n = 4$, using the definition $[X, Y]_q \equiv q^{1/2}XY - q^{-1/2}YX$:

$$U_q'(\mathrm{so}_3): \quad [I_{21}, I_{32}]_q = I^+_{31}, \qquad [I_{32}, I^+_{31}]_q = I_{21}, \qquad [I^+_{31}, I_{21}]_q = I_{32}. \tag{3}$$

$$U_q'(\mathrm{so}_4): \begin{cases} [I_{32}, I_{43}]_q = I^+_{42}, \qquad [I^+_{31}, I_{43}]_q = I^+_{41}, \qquad [I_{21}, I^+_{42}]_q = I^+_{41},\\ [I_{43}, I^+_{42}]_q = I_{32}, \qquad [I_{43}, I^+_{41}]_q = I^+_{31}, \qquad [I^+_{42}, I^+_{41}]_q = I_{21},\\ [I^+_{42}, I_{32}]_q = I_{43}, \qquad [I^+_{41}, I^+_{31}]_q = I_{43}, \qquad [I^+_{41}, I_{21}]_q = I^+_{42}, \end{cases} \tag{4}$$

$$[I_{43}, I_{21}] = 0, \quad [I_{32}, I^+_{41}] = 0, \quad [I^+_{42}, I^+_{31}] = (q-q^{-1})(I_{21}I_{43} - I_{32}I^+_{41}). \tag{5}$$

The first relation in (3) can be viewed as the definition for third generator needed to give the algebra in terms of q-commutators. Dual copy of the algebra $U_q'(\mathrm{so}_3)$ involves the generator $I^-_{31} = [I_{21}, I_{32}]_{q^{-1}}$ and the other two relations similar to (3), but with $q \to q^{-1}$.

In order to describe the basis of $U_q'(\mathrm{so}_n)$ we introduce a lexicographical ordering for the elements $I^+_{k,l}$ of $U_q'(\mathrm{so}_n)$ with respect to their indices, i.e., we suppose that $I^+_{k,l} \prec I^+_{m,n}$ if either $k < m$, or both $k = m$ and $l < n$. We define *an ordered monomial* as the product of non-decreasing sequence of elements $I^+_{k,l}$ with different k, l such that $1 \le l < k \le n$. The following proposition describes the Poincaré–Birkhoff–Witt basis for the algebra $U_q'(\mathrm{so}_n)$.

Proposition. *The set of all ordered monomials is a basis of* $U_q'(\mathrm{so}_n)$.

3 Casimir elements of $U_q'(\mathrm{so}_n)$

As it is well-known, tensor operators of Lie algebras so_n are very useful in construction of invariants of these algebras. With this in mind, let us introduce q-analogues of tensor operators for the algebras $U_q'(\mathrm{so}_n)$ as follows:

$$J^{\pm}_{k_1,k_2,\ldots,k_{2r}} = q^{\mp\frac{r(r-1)}{2}} \sum_{s\in S_{2r}}{}' \varepsilon_{q^{\pm1}}(s) I^{\pm}_{k_{s(2)},k_{s(1)}} I^{\pm}_{k_{s(4)},k_{s(3)}} \cdots I^{\pm}_{k_{s(2r)},k_{s(2r-1)}}. \tag{6}$$

Here $1 \le k_1 < k_2 < \cdots < k_{2r} \le n$, and the summation runs over all the permutations s of indices $k_1, k_2, \ldots, k_{2r}$ such that

$$k_{s(2)} > k_{s(1)}, \qquad k_{s(4)} > k_{s(3)}, \quad \ldots, \quad k_{s(2r)} > k_{s(2r-1)}, \qquad k_{s(2)} < k_{s(4)} < \cdots < k_{s(2r)}$$

(the last chain of inequalities means that the sum includes only ordered monomials). Symbol $\varepsilon_{q^{\pm1}}(s) \equiv (-q^{\pm1})^{\ell(s)}$ stands for a q-analogue of the Levi–Chivita antisymmetric tensor, $\ell(s)$ means the length of permutation s. (If $q \to 1$, both sets in (6) reduce to the set of components of rank $2r$ antisymmetric tensor operator of the Lie algebra so_n.)

Using q-tensor operators given by (6) we obtain the Casimir elements of $U_q'(\mathrm{so}_n)$.

Theorem 1. *The elements*

$$C_n^{(2r)} = \sum_{1\le k_1<k_2<\ldots<k_{2r}\le n} q^{k_1+k_2+\cdots+k_{2r}-r(n+1)} J^+_{k_1,k_2,\ldots,k_{2r}} J^-_{k_1,k_2,\ldots,k_{2r}}, \tag{7}$$

where $r = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ *(* $\lfloor x \rfloor$ *means the integer part of* x*), are Casimir elements of* $U'_q(\mathrm{so}_n)$*, i.e., they belong to the center of this algebra.*

In fact, for even n*, not only the product (which constitutes* $C_n^{(n)}$*) of elements* $C_n^{(n)+} \equiv J^+_{1,2,\ldots,n}$ *and* $C_n^{(n)-} \equiv J^-_{1,2,\ldots,n}$ *belongs to the center, but also each of them.*

We conjecture that, in the case of q being not a root of 1, the set of Casimir elements $C_n^{(2r)}$, $r = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$, and the Casimir element $C_n^{(n)+}$ (for even n) generates the center of $U'_q(\mathrm{so}_n)$, i.e., any element of the algebra $U'_q(\mathrm{so}_n)$ which commutes with all other elements can be presented as a polynomial of elements from this set of Casimir elements.

Let us give explicitly some of Casimir elements. For $U'_q(\mathrm{so}_3)$ and $U'_q(\mathrm{so}_4)$ we have

$$C_3^{(2)} = q^{-1}I_{21}^2 + I_{31}^+I_{31}^- + qI_{32}^2 = qI_{21}^2 + I_{31}^-I_{31}^+ + q^{-1}I_{32}^2,$$
$$C_4^{(2)} = q^{-2}I_{21}^2 + I_{32}^2 + q^2I_{43}^2 + q^{-1}I_{31}^+I_{31}^- + qI_{42}^+I_{42}^- + I_{41}^+I_{41}^-,$$
$$C_4^{(4)+} = q^{-1}I_{21}I_{43} - I_{31}^+I_{42}^+ + qI_{32}I_{41}^+ = qI_{21}I_{43} - I_{31}^-I_{42}^- + q^{-1}I_{32}I_{41}^- = C_4^{(4)-}.$$

For $U'_q(\mathrm{so}_5)$ the fourth order Casimir element is

$$C_5^{(4)} = q^{-2}J^+_{1,2,3,4}J^-_{1,2,3,4} + q^{-1}J^+_{1,2,3,5}J^-_{1,2,3,5} + J^+_{1,2,4,5}J^-_{1,2,4,5} + qJ^+_{1,3,4,5}J^-_{1,3,4,5} + q^2J^+_{2,3,4,5}J^-_{2,3,4,5},$$

where $J^+_{i,j,k,l} = q^{-1}I^+_{ji}I^+_{lk} - I^+_{ki}I^+_{lj} + qI^+_{kj}I^+_{li}$ and $J^-_{i,j,k,l} = qI^-_{ji}I^-_{lk} - I^-_{ki}I^-_{lj} + q^{-1}I^-_{kj}I^-_{li}$. For $U'_q(\mathrm{so}_6)$, we present only the highest order Casimir element:

$$\begin{aligned} C_6^{(6)+} &= q^{-3}I_{21}I_{43}I_{65} - q^{-2}I_{31}^+I_{42}^+I_{65} + q^{-1}I_{32}I_{41}^+I_{65} - q^{-2}I_{21}I_{53}^+I_{64}^+ + q^{-1}I_{31}^+I_{52}^+I_{64}^+ \\ &- I_{32}I_{51}^+I_{64}^+ + q^{-1}I_{21}I_{54}I_{63}^+ - I_{41}^+I_{52}^+I_{63}^+ + qI_{42}^+I_{51}^+I_{63}^+ - I_{31}^+I_{54}I_{62}^+ + I_{41}^+I_{53}^+I_{62}^+ \\ &- q^2I_{43}I_{51}^+I_{62}^+ + qI_{32}I_{54}I_{61}^+ - q^2I_{42}^+I_{53}^+I_{61}^+ + q^3I_{43}I_{52}^+I_{61}^+. \end{aligned}$$

Finally, let us give explicitly the quadratic Casimir element of $U'_q(\mathrm{so}_n)$,

$$C_n^{(2)} = \sum_{1\le i<j\le n} q^{i+j-n-1} I_{ji}^+I_{ji}^-.$$

This formula coincides with that given in [4], and is a particular case of (7).

4 Irreducible representations of $U'_q(\mathrm{so}_n)$

Let us give a brief description of irreducible representations (irreps) of $U'_q(\mathrm{so}_n)$. More detailed description of these irreps can be found in [1, 3].

As in the case of Lie algebra so_n, finite-dimensional irreps T of the algebra $U'_q(\mathrm{so}_n)$ are characterized by the set $\mathbf{m}_n \equiv (m_{1,n}, m_{2,n}, \ldots, m_{\lfloor \frac{n}{2} \rfloor,n})$ (here $\lfloor x \rfloor$ means the integer part of x) of numbers, which are either all integers or all half-integers, and satisfy the well-known dominance conditions

$$m_{1,n} \geq m_{2,n} \geq \cdots \geq m_{\frac{n}{2}-1,n} \geq |m_{\frac{n}{2},n}| \qquad \text{if } n \text{ is even,}$$

$$m_{1,n} \geq m_{2,n} \geq \cdots \geq m_{\frac{n-1}{2},n} \geq 0 \qquad \text{if } n \text{ is odd.}$$

To give the representations in Gel'fand–Tsetlin basis we denote, as in the case of Lie algebra so_n, the basis vectors $|\alpha\rangle$ of representation spaces by Gel'fand–Tsetlin patterns α. The representation operators $T_{\mathbf{m}_n}(I_{2p+1,2p})$ and $T_{\mathbf{m}_n}(I_{2p,2p-1})$ act on $|\alpha\rangle$ by the formulae

$$T_{\mathbf{m}_n}(I_{2p+1,2p})|\alpha\rangle = \sum_{r=1}^{p} \Big(A^r_{2p}(\alpha)|m^{+r}_{2p}\rangle - A^r_{2p}(m^{-r}_{2p})|m^{-r}_{2p}\rangle \Big),$$

$$T_{\mathbf{m}_n}(I_{2p,2p-1})|\alpha\rangle = \sum_{r=1}^{p-1} \Big(B^r_{2p-1}(\alpha)|m^{+r}_{2p-1}\rangle - B^r_{2p-1}(m^{-r}_{2p-1})|m^{-r}_{2p-1}\rangle \Big) + \mathrm{i}C_{2p-1}|\alpha\rangle.$$

Here the matrix elements A^r_{2p},B^r_{2p-1}, C_{2p-1} are obtained from the classical (non-deformed) ones by replacing each factor (x) with its respective q-number $[x] \equiv (q^x - q^{-x})/(q - q^{-1})$; besides, the coefficient $\frac{1}{2}$ in the 'classical' A^r_{2p} is replaced with the $l_{r,2p}$-dependent expression $(([l_{r,2p}][l_{r,2p}+1])/([2l_{r,2p}][2l_{r,2p}+2]))^{1/2}$, where $l_{r,2p} = m_{r,2p} + p - r$.

5 Casimir operators and their eigenvalues

The Casimir operators (the operators which correspond to the Casimir elements), within irreducible finite-dimensional representations of $U'_q(\mathrm{so}_n)$ take diagonal form. To give them explicitly, we employ the so-called *generalized factorial elementary symmetric polynomials* (see [9]). Fix an arbitrary sequence of complex numbers $\mathbf{a} = (a_1, a_2, \ldots)$. Then, for each $r = 0, 1, 2, \ldots, N$, introduce the polynomials of N variables z_1, $z_2, \ldots$, z_N as follows:

$$e_r(z_1, z_2, \ldots, z_N|\mathbf{a}) = \sum_{1\leq p_1<p_2<\cdots<p_r\leq N} (z_{p_1} - a_{p_1})(z_{p_2} - a_{p_2-1}) \ldots (z_{p_r} - a_{p_r-r+1}). \qquad (8)$$

The Casimir operators in the irreducible finite-dimensional representations characterized by the set $(m_{1,n},\ m_{2,n}, \ldots,\ m_{N,n})$, $N = \lfloor \frac{n}{2} \rfloor$, by the Schur Lemma, are presentable as (here $\mathbf{1}$ denotes the unit operator):

$$T_{\mathbf{m}_n}(C^{(2r)}_n) = \chi^{(2r)}_{\mathbf{m}_n}\mathbf{1}.$$

Theorem 2. *The eigenvalue of the operator* $T_{\mathbf{m}_n}\left(C^{(2r)}_n\right)$ *is*

$$\chi^{(2r)}_{\mathbf{m}_n} = (-1)^r e_r([l_{1,n}]^2, [l_{2,n}]^2, \ldots, [l_{N,n}]^2|\mathbf{a})$$

where $\mathbf{a} = ([\epsilon]^2, [\epsilon+1]^2, [\epsilon+2]^2, \ldots)$, $l_{k,n} = m_{k,n} + N - k + \epsilon$. *Here* $\epsilon = 0$ *for* $n = 2N$ *and* $\epsilon = \frac{1}{2}$ *for* $n = 2N + 1$.

In the case of even n, *i.e.,* $n = 2N$,

$$T_{\mathbf{m}_n}\left(C^{(n)+}_n\right) = T_{\mathbf{m}_n}\left(C^{(n)-}_n\right) = \left(\sqrt{-1}\right)^N [l_{1,n}][l_{2,n}] \ldots [l_{N,n}]\mathbf{1}.$$

The eigenvalues of Casimir operators are important for physical applications. Let us quote some of Casimir operators together with their eigenvalues. For $U'_q(\mathrm{so}_3)$,

$$T_{(m_{13})}\left(C^{(2)}_3\right) = -[m_{13}][m_{13}+1]\mathbf{1}.$$

For $U'_q(\mathrm{so}_4)$ we have

$$T_{(m_{14},m_{24})}\left(C_4^{(2)}\right) = -\left([m_{14}+1]^2+[m_{24}]^2-1\right)\mathbf{1},$$
$$T_{(m_{14},m_{24})}\left(C_4^{(4)+}\right) = T_{(m_{14},m_{24})}\left(C_4^{(4)-}\right) = -[m_{14}+1][m_{24}]\mathbf{1}.$$

Finally, for $U'_q(\mathrm{so}_5)$ the Casimir operators are

$$T_{(m_{15},m_{25})}\left(C_5^{(2)}\right) = -\left([m_{15}+3/2]^2+[m_{25}+1/2]^2-[1/2]^2-[3/2]^2\right)\mathbf{1},$$
$$T_{(m_{15},m_{25})}\left(C_5^{(4)}\right) = \left([m_{15}+3/2]^2-[1/2]^2\right)\left([m_{25}+1/2]^2-[1/2]^2\right)\mathbf{1}.$$

6 Concluding remarks

In this note, for the nonstandard q-algebras $U'_q(\mathrm{so}_n)$ we have presented explicit formulae for all the Casimir operators corresponding to basis set of Casimirs of so_n. Their eigenvalues in irreducible finite-dimensional representations are also given. We believe that the described Casimir elements generate the whole center of the algebra $U'_q(\mathrm{so}_n)$ (of course, for q being not a root of unity).

As mentioned, the algebras $U'_q(\mathrm{so}_n)$ for $n > 4$ are of importance in the *construction of algebra of observables for* 2+1 *quantum gravity* (with 2D space of genus $g > 1$) serving as certain intermediate algebras. For that reason, the results concerning Casimir operators and their eigenvalues will be useful in the process of construction of the desired algebra of independent quantum observables for the case of higher genus surfaces, in the important and interesting case of anti-De Sitter gravity (corresponding to negative cosmological constant).

Acknowledgements

The authors express their gratitude to J. Nelson and A. Klimyk for interesting discussions. This research was supported in part by the CRDF Award No. UP1-309, and by the DFFD Grant 1.4/206.

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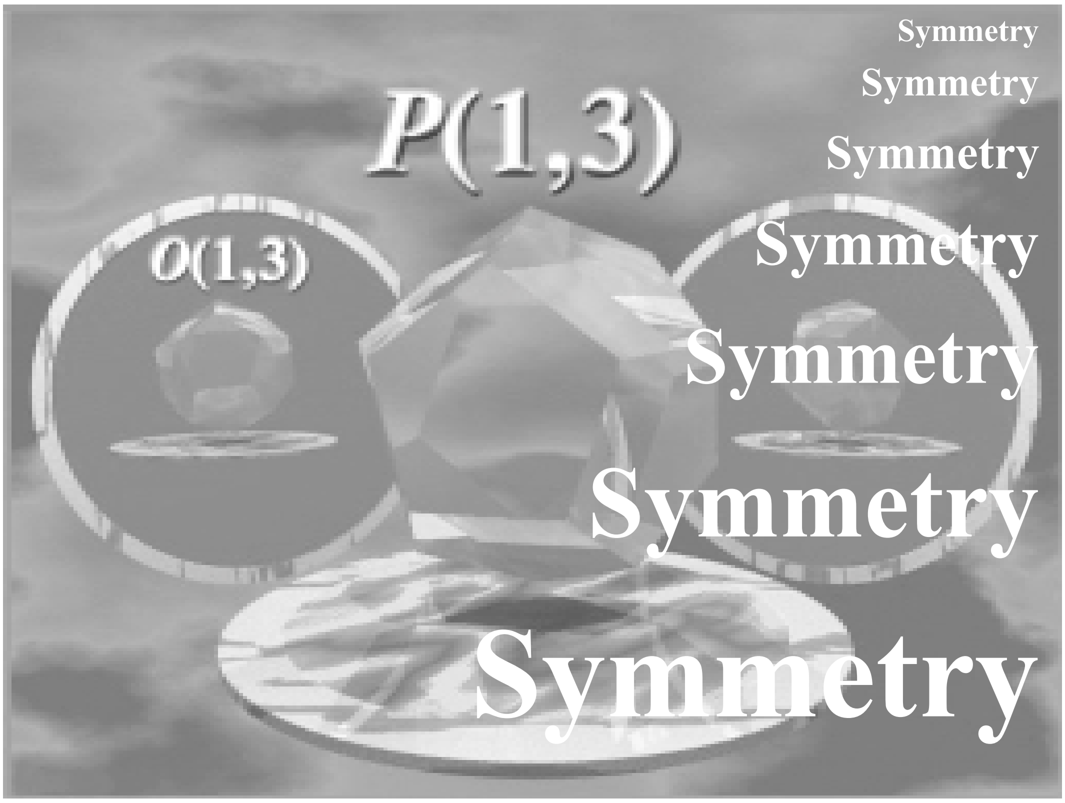
Algebras and Groups: Developments and Applications



Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 317–323.

Discrete Subgroups of the Poincaré Group

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The framework of point form relativistic quantum mechanics is used to construct interacting four-momentum operators in terms of creation and annihilation operators of underlying constituents. It is shown how to write the creation and annihilation operators in terms of discrete momenta, arising from discrete subgroups of the Lorentz group, in such a way that the Poincaré commutation relations are preserved. For discrete momenta the bosonic creation and annihilation operators can be written as multiplication and differentiation operators acting on a holomorphic Fock space. It is shown that with such operators matrix elements of the relativistic Schrödinger equation become an infinite coupled set of first order partial differential equations.

1 Introduction

In nonrelativistic quantum mechanics one often puts a system of interest in a box, in order to deal with discrete momenta and avoid delta functions. This is equivalent to looking at representations of discrete subgroups of the Euclidean group, the group consisting of rotations and Galilei boosts, and itself a subgroup of the full Galilei group. It is of interest to see if the same thing can be done for relativistic systems, and in particular to see what happens to relativistic spin. The point of this paper is to show how to construct discrete subgroups of the Lorentz group and then embed these discrete subgroups in a quasidiscrete Poincaré subgroup, in order to construct interacting four-momentum operators.

The context for this work is point form relativistic quantum mechanics [1], wherein all interactions are put in the four-momentum operators, and the Lorentz generators are free of interactions. The Lorentz generators are readily exponentiated to give global Lorentz transformations; this is important since discrete subgroups do not have an associated Lie algebra. The point form is to be contrasted with the more usual instant form of dynamics, where interactions are present in the Hamiltonian and boost generators, and the momentum and angular momentum generators are free of interactions (for a discussion of the various forms of dynamics, see for example [2]).

One of the main reasons for introducing discrete subgroups is that the creation and annihilation operators that are used to build the interacting four-momentum operators then have discrete momenta in their arguments, and for bosonic creation and annihilation operators, can be realized as multiplication and differentiation operators acting on a holomorphic Hilbert space. In this representation matrix elements of the relativistic Schrödinger equation then take on the form of an infinite coupled set of first order partial differential equations.

2 Point form quantum mechanics

In order to have a relativistic theory it is necessary to satisfy the commutation relations of the Poincaré algebra. In quantum field theory this is done by integrating the stress-energy tensor – made up of polynomials of field operators – over a time constant surface (see for example [3]). It is however also possible to integrate over the forward hyperboloid, in which case the interactions will all be in the four-momentum operators, which must commute with one another, $[P^\mu, P^\nu] = 0$, where μ and ν run between zero and three. Here $P^\mu = P^\mu_{fr} + P^\mu_I$, the sum of free and interacting four-momentum operators. Since Lorentz generators do not contain interactions, it is more convenient to deal with representations of global Lorentz transformations, written as U_Λ, where Λ is a Lorentz transformation and U_Λ the unitary operator representing the Lorentz transformation. The Poincaré relations are then

$$[P^\mu, P^\nu] = 0, \tag{1}$$

$$U_\Lambda P^\mu U_\Lambda^{-1} = (\Lambda^\mu_\nu)^{-1} P^\nu. \tag{2}$$

Since P^μ are the generators for space-time translations, the relativistic Schrödinger equation can be written as

$$i\hbar\partial/\partial x^\mu |\Psi_x\rangle = P_\mu |\Psi_x\rangle, \tag{3}$$

where $|\Psi_x\rangle$ is an element of the Fock space and x $(= x^\mu)$ is a space-time point. The space-time independent Schrödinger equation is then

$$P^\mu |\Psi\rangle = p^\mu |\Psi\rangle, \tag{4}$$

where p^μ is an eigenvalue of P^μ. The mass operator is $M := \sqrt{P^\mu P_\mu}$ and must have a spectrum bounded from below.

For particles of mass m $(m > 0)$ and spin j, it is well known that the irreducible representations of the Poincaré group can be written as

$$U_\Lambda |p, \sigma\rangle = \sum_{\sigma'=-j}^{j} |\Lambda p, \sigma'\rangle D^j_{\sigma',\sigma}(R_W), \tag{5}$$

$$U_a |p, \sigma\rangle = e^{-ip^\mu a_\mu} |p, \sigma\rangle, \tag{6}$$

$$P^\mu |p, \sigma\rangle = p^\mu |p, \sigma\rangle. \tag{7}$$

Here p is a four-momentum vector satisfying $p^\mu p_\mu = m^2$, σ is a spin projection variable, and $|p, \sigma\rangle$ is a (nonnormalizable) state vector. R_W is a Wigner rotation (see, for example [4] and references cited therein) and $D^j_{\sigma',\sigma}(\ )$ an $SU(2)$ D function. For infinitesimal space-time elements a, equation (7) follows from equation (6).

To get a many-body theory, creation and annihilation operators are introduced which take on the transformation properties of the single particle states, equations (5), (6):

$$[a(p, \sigma), a^\dagger(p', \sigma')]_\pm = 2E\delta^3(p - p')\delta_{\sigma,\sigma'}, \tag{8}$$

$$U_\Lambda a^\dagger(p, \sigma) U_\Lambda^{-1} = \sum_{\sigma'=-j}^{j} a^\dagger(\Lambda p, \sigma') D^j_{\sigma',\sigma}(R_W), \tag{9}$$

$$U_a a^\dagger(p, \sigma) U_a^{-1} = e^{-ip^\mu a_\mu} a^\dagger(p, \sigma), \tag{10}$$

where $\pm$ means commutator or anticommutator. Then a free four-momentum operator can be written as

$$P^\mu_{fr} = \sum_\sigma \int dp(p^\mu) a^\dagger(p, \sigma) a(p, \sigma) \tag{11}$$

and satisfies the Poincaré commutation relations, equations (1), (2). $dp := d^3p/2\sqrt{m^2+p^2}$ is the Lorentz invariant measure.

The full interacting four-momentum operator, $P^\mu = P^\mu_{fr} + P^\mu_I$, must also satisfy equation(1):

$$\begin{aligned}[P^\mu, P^\nu] &= 0 \\ &= [P^\mu_{fr} + P^\mu_I, P^\nu_{fr} + P^\nu_I] \\ &= [P^\mu_{fr}, P^\nu_I] + [P^\mu_I, P^\nu_{fr}] + [P^\mu_I, P^\nu_I].\end{aligned} \tag{12}$$

Equation(12) can be satisfied if $[P^\mu_I, P^\nu_I] = 0$ and $[P^\mu_{fr}, P^\nu_I] = [P^\nu_{fr}, P^\mu_I]$.

A natural way to construct an interacting four-momentum operator that satisfies these equations is with an interaction Lagrangian built out of local fields. While the long range goal is to use discrete subgroups for pion-nucleon interactions, in this paper we will consider the simpler case of a charged scalar meson interacting with a neutral meson. If $a(p)$ and $b(p)$ denote the annihilation operators for the positively and negatively charged mesons, while $c(k)$ denotes the annihilation operator for the neutral meson, then local fields $\phi(x)$ and $\phi^\dagger(x)$ for the charged mesons and $\chi(x)$ for the neutral meson are defined by

$$\begin{aligned}&\phi(x) = \int dp \left(e^{-ip^\mu x_\mu} a(p) + e^{ip^\mu x_\mu} b^\dagger(p)\right), \quad \phi^\dagger(x) = \int dp \left(e^{-ip^\mu x_\mu} b(p) + e^{ip^\mu x_\mu} a^\dagger(p)\right), \\ &\chi(x) = \int dk \left(e^{-ik^\mu x_\mu} c(k) + e^{ik^\mu x_\mu} c^\dagger(k)\right),\end{aligned} \tag{13}$$

where $p^\mu p_\mu = m^2$ and $k^\mu k_\mu = m_\pi^2$.

An interacting four-momentum operator can be built out of these local fields by integrating (say a trilinear coupling) over the forward hyperboloid:

$$P^\mu_I = \lambda_0 \int d^4x \delta\left(x^\nu x_\nu - \tau^2\right) \theta(x_0) x^\mu \phi^\dagger(x)\phi(x)\chi(x); \tag{14}$$

if derivative couplings, say of the form $\partial\phi^\dagger(x)/\partial x^\alpha \partial\phi(x)/\partial x_\alpha \chi(x)$, and of all higher order derivatives are also added on to equation (14), then an interacting four-momentum operator with an arbitrary potential will result. That is, each differentiation of a field brings down a power of momentum, and if these are all added together, and the integration over space-time carried out, the interacting four-momentum operator will have the form,

$$P^\mu_I(1) = \int dp\, dp'\, dk \Delta^\mu(p - p' + k) v(p^\alpha p'_\alpha, -p^\alpha k_\alpha, p'_\alpha k^\alpha) a^\dagger(p) a(p') c^\dagger(k), \tag{15}$$

plus seven other terms of the same form involving different creation and annihilation operators. λ_0 is a coupling constant and the coupling constants for all the higher derivatives times the powers of momenta combine to give the potential function $v(\ )$. For example, there is a term of the form $a(p)b(p')c^\dagger(k)$ in which the potential function has the argument $v(-p^\alpha p'_\alpha, p'_\alpha k^\alpha, p^\alpha k_\alpha)$. Thus, $P^\mu_I = \sum_{i=1}^{8} P^\mu(i)$, and by construction satisfies equation (12). $\Delta^\mu(p) := \int d^4x \delta\left(x^\alpha x_\alpha - \tau^2\right) \theta(x_0) x^\mu e^{ip^\alpha x_\alpha}$ and comes from the exponentials in the field operators. The full four-momentum operator, a sum of free four-momentum operators for the three types of particles called a, b, and c and the interacting four-momentum operator, of the form given in equation (15), satisfies the point form equations, equations (1) and (2) (this is shown in reference [5]), and provides the starting point for the discrete subgroups of the Lorentz group.

3 Discrete subgroups of the Lorentz group

In order for the arguments in the creation and annihilation operators introduced in section (2) to become discrete, the Lorentz transformations which boost a particle from its rest frame must be discrete:

$$p_n = \Lambda(n)p(\text{rest}), \tag{16}$$

where $p(\text{rest})$ is the rest frame four-momentum, $p(\text{rest}) = (m, 0, 0, 0)$. To construct discrete subgroups of the Lorentz group, it is convenient to start with Lorentz transformations along the z direction, in which case the discrete elements have the form

$$\Lambda_z(n) = \begin{pmatrix} \operatorname{ch}\beta n & 0 & 0 & \operatorname{sh}\beta n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \operatorname{sh}\beta n & 0 & 0 & \operatorname{ch}\beta n \end{pmatrix}. \tag{17}$$

For a fixed constant β, the set of elements with n ranging over all positive and negative integers forms a discrete subgroup of the group of all z axis Lorentz transformations. The minimum momentum is $p_1 = m \operatorname{sh}\beta$ and in the limit where β goes to zero, n to infinity, such that $\beta n = \alpha$, one recovers the continuum limit.

Next consider the finite rotation subgroups of the full rotation group, $SO(3)$, and in particular the crystallographic subgroups denoted by $SO(3)_D$ (see for example [6], chapters two and four). Now arbitrary Lorentz transformations can be written as

$$\Lambda = R\Lambda_z R', \tag{18}$$

where R, R' are elements of $SO(3)$. Then discrete subgroups $SO(1,3)_D$ of the continuous Lorentz group $SO(1,3)$ have elements of the form

$$\Lambda_D = R_D \Lambda_z(n) R'_D, \tag{19}$$

where R_D, R'_D are elements of $SO(3)_D$. Thus the relevant discrete subgroups of the Lorentz group are indexed by the finite subgroups of the rotation group.

The discrete Lorentz transformations can now be adjoined to space-time translations to give groups with a Poincaré-like structure. Because of the semidirect product nature of the Poincaré group, the Lorentz transformations act on space-time translations, but not the other way around. Hence it is possible to adjoin discrete Lorentz transformations to continuous space-time translations and still get a group structure. These groups will be called quasi-discrete Poincaré groups and denoted by $\mathcal{P}_D$, with a group law given by

$$(\Lambda_D, a)(\Lambda'_D, a') = (\Lambda_D \Lambda'_D, \Lambda_D a + a'), \tag{20}$$

with Λ_D in $SO(1,3)_D$ and a in $\mathcal{R}^4$.

The representations of $\mathcal{P}_D$ are obtained in exactly the same way as the ordinary Poincaré group representations, namely as induced representations [4]. For positive mass representations the little group is now $SO(3)_D$ and the action of a discrete Lorentz transformation on a state with discrete four-momentum p_D and spin projection σ is given by

$$U_{\Lambda_D}|p_D, \sigma\rangle = \sum_{\sigma'=-j}^{j} |\Lambda_D p_D, \sigma'\rangle D^j_{\sigma',\sigma}(R_W), \tag{21}$$

where now R_W is a discrete Wigner rotation given by

$$R_W = B^{-1}(\Lambda_D p_D)\Lambda_D B(p_D), \tag{22}$$

with the discrete momenta given by $p_D = B(p_D)p(\text{rest})$, and $B(p_D)$ a boost (coset) representative of $SO(1,3)_D$ with respect to $SO(3)_D$. It should be noted that since the irreducible representations of all the finite subgroups of $SO(3)$ are bounded in their spin values [6], the irreducible representation label j in equation (21) can only take on low lying spin values. Thus relativistic spin is well-defined for the discrete subgroups of the Lorentz group, but bounded in its possible values.

Given the representations of $\mathcal{P}_D$, creation and annihilation operators with discrete arguments can be defined in the usual way:

$$\begin{aligned} &|p_D,\sigma\rangle = a^\dagger(p_D,\sigma)|0\rangle, \\ &[a(p_D,\sigma), a^\dagger(p'_D,\sigma')]_\pm = \delta_{p_D,p'_D}\delta_{\sigma,\sigma'}, \qquad P^\mu_{fr} = \sum_{\sigma,p_D} p^\mu_D a^\dagger(p_D,\sigma)a(p_D,\sigma). \end{aligned} \tag{23}$$

A problem however arises when attempting to define local fields. As an example take the charged scalar field, equation (13) with

$$\begin{aligned} &\phi_D(x) := \sum_{p_D} e^{-ip^\mu_D x_\mu} a(p_D) + e^{ip^\mu_D x_\mu} b^\dagger(p_D), \\ &U_{\Lambda_D}\phi_D(x)U^{-1}_{\Lambda_D} = \phi_D(\Lambda_D x), \qquad U_a\phi_D(x)U_a^{-1} = \phi_D(x+a). \end{aligned} \tag{24}$$

But

$$[\phi(x),\phi^\dagger(y)] = \sum_{p_D} e^{-ip^\mu_D(x_\mu-y_\mu)} - e^{ip^\mu_D(x_\mu-y_\mu)} := \Delta_D(x-y). \tag{25}$$

Then $\Delta_D(\Lambda_D x) = \Delta_D(x)$ and $\Delta_D(-x) = -\Delta_D(x)$. However, for x spacelike, there is in general no Λ_D such that $\Lambda_D x = -x$; rather there is only a restricted set of space-time points x satisfying this equation. Therefore in general $\Delta_D(x) \neq 0$ for x spacelike and $\phi_D(x)$ is not local.

It is nevertheless still possible to define the discrete analogue of the interacting four-momentum operator, $P^\mu_{I_D}$ so that the sum $P^\mu_{I_D} = \sum_{i=1}^{8} P^\mu_{I_D}(i)$, as in equation (15), satisfies the point form equations, equation (1), with now the Lorentz transformations replaced by discrete Lorentz transformations (proofs of these statements are given in reference [7]).

4 Matrix elements of the point form relativistic Schrödinger equation

To conclude we show how to convert the point form equations, equations (1), (2), and (4) into a set of coupled first order partial differential equations. To keep the formalism as simple as possible we consider one space and one time dimension only. Then the bosonic creation and annihilation operators, $c^\dagger(k_n)$ and $c(k_n)$, can be replaced, respectively, by z_n and $\partial/\partial z_n$, with the bosonic Fock space now a holomorphic Hilbert space:

$$\mathcal{F}(\pi) = \{f(Z)|\|f\|^2 < \infty,\ f \text{ holomorphic}\}, \tag{26}$$

with a differentiation inner product given by $(f,f') = f^\star(\partial/\partial Z)f'(Z)|_{Z=0}$, where Z denotes the set of complex variables $\{z_n\}$ with n running over all the integers (see for example [8] and references therein).

Typical terms in the full four-momentum operator have the following form:

$$P^{\mu}_{fr}(\pi) = \sum_n k^{\mu}_n z_n \partial/\partial z_n, \qquad P^{\mu}_I(1) = \sum_{n,n',m} v^{\mu}(n,n',m) a^{\dagger}(n) a(n') z_m, \tag{27}$$

where now $v^{\mu}(n,n',m)$ includes both the Δ^{μ} and the potential $v()$ terms of the four-momentum operator, equation (15).

The goal is now to convert the relativistic Schrödinger equation, equation (4), into a coupled set of first order partial differential equations; this is possible because in all the terms in the full four-momentum operator, the meson annihilation operator, $\partial/\partial z_n$, never appears as a product with itself, but only as a product with other creation and annihilation operators. Therefore matrix elements of the relativistic Schrödinger equation will give a set of coupled first order differential equations.

The a, b creation and annihilation operators always occur in pairs that conserve baryon number, and hence it is easily shown that P^{μ} commutes with the baryon number operator, defined by

$$\hat{B} = \sum_n a^{\dagger}(n) a(n) + b^{\dagger}(n) b(n), \tag{28}$$

with positive or negative integers as eigenvalues. The Fock space thus decomposes into baryon number sectors, and there will be a coupled set of differential equations for each baryon number sector.

As an example consider the $B = 0$ sector, the physical vacuum, one pion, ..., sector, for which the wave function can be written as

$$|\Psi_{B=0}\rangle = f^{(0)}(Z)|0\rangle + \sum_{n,n'} f^{(2)}(Z,n,n')|n,n'\rangle + \cdots, \tag{29}$$

that is, a superposition of $0, 1, 2, \ldots$ particle-antiparticle pairs, where in each term there is a function of Z representing the meson cloud. $|0\rangle$ in equation (29) designates the a, b type particle bare vacuum only, since the bare meson vacuum is given by $f(Z) = 1$.

Taking $B = 0$ sector matrix elements of the relativistic Schrödinger equation with the wave function given in equation (29),

$$\begin{aligned} \langle 0|P^{\mu}|\Psi_{B=0}\rangle &= (m_{\pi}, 0)^{\mu}\langle 0|\Psi_{B=0}\rangle \\ \langle m, m'|P^{\mu}|\Psi_{B=0}\rangle &= (m_{\pi}, 0)^{\mu}\langle m, m'|\Psi_{B=0}\rangle \\ \vdots &= \vdots \end{aligned} \tag{30}$$

then gives a set of coupled partial differential equations in the 'meson cloud' amplitudes, $f^{(0)}(Z)$ for mesons with no particle-antiparticle pairs, $f^{(2)}(Z,n,n')$ for mesons with one particle-antiparticle pair having discrete momenta n and n' respectively (actually momentum $p = m \operatorname{sh}\beta n$ and $p' = m \operatorname{sh}\beta n'$ respectively).

The first and second of the equations given in equation (30) can be written more explicitly as

$$\begin{aligned} &m_{\pi} \sum_n k^{\mu}_n z_n \partial/\partial z_n f^{(0)}(Z) + \sum_{m,m',n} v^{\mu}(m,m',n)(z_n + \partial/\partial z_n) f^{(2)}(Z,m,m') \\ &= (m_{\pi}(v), 0)^{\mu} f^{(0)}(Z), \end{aligned}$$

$$
\begin{aligned}
&\sum_n v^\mu(m,m',n)(z_n+\partial/\partial z_n)f^{(0)}(Z)+m(p_m^\mu+p_{m'}^\mu)f^{(2)}(Z,m,m') \\
&\quad +m_\pi\sum_n k_n^\mu z_n\partial/\partial z_n f^{(2)}(Z,m,m')+\sum_{n,n'}v^\mu(m,n',n)(z_n+\partial/\partial z_n)f^{(2)}(Z,n',m') \\
&\quad +4\sum_{r,r',n}v^\mu(r,r',n)(z_n+\partial/\partial z_n)f^{(4)}(Z,m,m',r,r')=(m_\pi(v),0)^\mu f^{(2)}(Z,p,p),
\end{aligned}
$$

where $m_\pi(v)$ is the eigenvalue for the physical (renormalized) pion, relative to the potential v^μ. If $m_\pi(v)$ is set equal to zero, the resulting eigenfuction is the eigenfunction for the physical vacuum. In all cases the eigenfunctions are those for a system at rest. Since the Lorentz boost generators are kinematic, these eigenfunctions can always be boosted to an arbitrary momentum. In terms of a column of unknown functions $f^{(0)}, f^{(2)}, f^{(4)}, \ldots$, one gets a tridiagonal matrix of first order partial differential equations. Possible methods of solution include truncating the f's at some order, but that is the subject of another paper.

In conclusion we have shown how quasidiscrete subgroups of the Poincaré group can be used to write the total four-momentum operator in terms of discrete momenta. This allows one to replace the mesonic creation and annihilation operators by multiplication and differentiation operators acting on a holomorphic Hilbert space. By taking matrix elements of the relativistic Schrödinger equation in this representation, one gets a set of coupled first order partial differential equations, in which the unknown functions are related to meson cloud amplitudes. Aside from the specific application discussed in this paper, one can think of the quasidiscrete subgroups of the Poincaré group as supplying a method for putting relativistic particles in a box.

It remains to find (at least approximate) solutions to the partial differential equations. Moreover this last section dealt only with systems in one spatial dimension, and to include spin, it is necessary to go to three dimensions. Finally it would be very interesting to see if clothing transformations [9] (see also [3], chapter 12), unitary transformations on the interacting four momentum operator, could be found that give an interacting four-momentum operator in terms of physical particles.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 324–329.

Matrix Realizations of Four-Dimensional Lie Algebras and Corresponding Invariant Spaces

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We have performed classification of nonequivalent realizations of solvable four-dimensional Lie algebras. Furthermore, the finite-dimensional invariant spaces are obtained which can be utilized for construction of exactly solvable matrix models of one-dimensional Schrödinger equation.

This paper is devoted to the application of realizations of four-dimensional Lie algebras for the construction of exactly solvable matrix models of one-dimensional Schrödinger equation.

The paper is organized as follows. In first section we perform the construction of realizations of solvable four-dimensional Lie algebras. Then from the realizations we pick out those for which we can construct a model. In second section we describe the procedure of obtaining the invariant spaces admitted by realizations of Lie algebras, which was found in first part of this paper, and present the invariant spaces.

1 Realizations of four-dimensional Lie algebras

We will construct nonequivalent realizations of four-dimensional real Lie algebras in class of matrix differential operators

$$Q = \xi(x)\partial_x + \eta(x), \tag{1}$$

where $\xi(x)$ is smooth real function, $\eta(x)$ is a complex matrix. Here and below $\partial_x = \frac{d}{dx}$.

Note that the classification of realizations of three-dimensional Lie algebras was done by R. Zhdanov in [1].

Abstract Lie algebras of dimension $n \leq 5$ have been classified by G.M. Mubarakzyanov in [2]. There are twelve algebras $L_{4,j}$ which are not direct sums of algebras of lower dimensions. Let us consider the algebra $L_{4,6}$ with non-zero commutation relations

$$L_{4,6}: \quad [Q_1, Q_4] = aQ_1, \quad [Q_2, Q_4] = bQ_2 - Q_3, \quad [Q_3, Q_4] = Q_2 + bQ_3, \quad (a \neq 0,\ b \geq 0).$$

From [3] we know that any one of the operators Q_i $(i = 1, \ldots 4)$ may be equal to ∂_x or $\eta(x)$. Let $Q_1 = \partial_x$ and other of operators have the form (1):

$$Q_i = \xi_i(x)\partial_x + \eta_i(x), \qquad i = 2, 3, 4.$$

As $[Q_1, Q_2] = [Q_1, Q_3] = 0$, $[Q_1, Q_4] = aQ_1$ then $\xi_i = \alpha_i$, $\eta_i = A_i$, $i = 2, 3$, $\alpha_4 = ax$, $\eta_4 = A_4$, where $\alpha_i \in R, A_i$ are arbitrary constant matrices $r \times r$. Substituting $Q_1 = \partial_x$, $Q_2 = \alpha_2\partial_x + A_2$, $Q_3 = \alpha_3\partial_x + A_3, Q_4 = ax\partial_x + A_4$ into the commutation relations we obtain $(a - b)^2 = -1$. That is why if $Q_1 = \partial_x$ then there exist no realizations of algebra $L_{4,6}$ in class of operators (1).

Let $Q_1 = \eta(x)$ and other operators have the form (1). Then all $\xi_i(x)$ can not be equal to zero simultaneously.

If $\xi_2(x) \neq 0$, then the operator Q_2 may be reduced to the operator $Q_2 = \partial_x$. In this case from commutation relations it follows that $Q_3 = \alpha_3\partial_x + A_3$, $Q_4 = (b-\alpha_3)x\partial_x - A_3x + A_4$, where $\alpha_3 \in R$, A_i are arbitrary constant matrices. The check of the relation $[Q_3, Q_4] = Q_2 + bQ_3$ gives $\alpha_3^2 = -1$. Hence, in this case algebra $L_{4,6}$ has no realizations in the class of operators (1) too.

If $\xi_2(x) = 0$, then or $\xi_3(x) \neq 0$ or $\xi_3(x) = 0$ and $\xi_4(x) \neq 0$. The checking of commutation relations shows that in this case algebra the $L_{4,6}$ has no realizations in the class of operators (1).

If $\xi_2(x) = \xi_3(x) = 0$ and $\xi_4(x) \neq 0$, then operator Q_4 may be reduced to the operator $Q_4 = \partial_x$ and the checking of commutation relations for the algebra $L_{4,6}$ shows that $Q_1 = A\exp(-ax)$, $Q_2 = \exp(-bx)(B\cos x + C\sin x)$, $Q_3 = \exp(-bx)(C\cos x - B\sin x)$, where A, B, C are arbitrary non-zero $r \times r$ matrices which satisfy the commutation relations

$$[A, B] = [A, C] = [B, C] = 0.$$

Below we give the list of nonequivalent realizations of the four-dimensional Lie algebras $L_{4,j}$.

$L^1_{4,1}:\quad Q_1 = A,\quad Q_2 = B,\quad Q_3 = \partial_x,\quad Q_4 = Bx + C,$
$\qquad [A, B] = [A, C] = 0,\quad [B, C] = A.$

$L^2_{4,1}:\quad Q_1 = A,\quad Q_2 = -Ax + B,\quad Q_3 = \frac{1}{2}Ax^2 - Bx + C,\quad Q_4 = \partial_x,$
$\qquad [A, B] = [B, C] = [A, C] = 0.$

$L^1_{4,2}:\quad Q_1 = \partial_x,\quad Q_2 = \partial_x + A,\quad Q_3 = \beta\partial_x + B,\quad Q_4 = x\partial_x + C,$
$\qquad [A, B] = 0,\quad [A, C] = A,\quad [B, C] = A + B.$

$L^2_{4,2}:\quad Q_1 = A,\quad Q_2 = B,\quad Q_3 = \partial_x,\quad Q_4 = x\partial_x + Bx + C,$
$\qquad [A, B] = 0,\quad [B, C] = B,\quad [A, C] = aA.$

$L^1_{4,3}:\quad Q_1 = \partial_x,\quad Q_2 = A,\quad Q_3 = B,\quad Q_4 = x\partial_x + C,$
$\qquad [A, B] = [A, C] = 0,\quad [B, C] = A.$

$L^2_{4,3}:\quad Q_1 = A,\quad Q_2 = B,\quad Q_3 = \partial_x,\quad Q_4 = Bx + C,$
$\qquad [A, B] = [B, C] = 0,\quad [A, C] = A.$

$L^3_{4,3}:\quad Q_1 = Ae^{-x},\quad Q_2 = B,\quad Q_3 = -Bx + C,\quad Q_4 = \partial_x,$
$\qquad [A, B] = [B, C] = [A, C] = 0.$

$L^1_{4,4}:\quad Q_1 = A,\quad Q_2 = B,\quad Q_3 = \partial_x,\quad Q_4 = x\partial_x + Bx + C,$
$\qquad [A, B] = 0,\quad [A, C] = A,\quad [B, C] = A + B.$

$L^2_{4,4}:\quad Q_1 = Ae^{-x},\quad Q_2 = e^{-x}(Ax + B),\quad Q_3 = e^{-x}\left(\frac{1}{2}Ax^2 - Bx + C\right),\quad Q_4 = \partial_x,$
$\qquad [A, B] = [A, C] = [B, C] = 0.$

$L^1_{4,5}:\quad Q_1 = \partial_x,\quad Q_2 = \alpha\partial_x + A,\quad Q_3 = \beta\partial_x + B,\quad Q_4 = x\partial_x + C,$
$\qquad [A, B] = 0,\quad [A, C] = A,\quad [B, C] = B.$

$L^2_{4,5}:\quad Q_1 = A,\quad Q_2 = \partial_x + \epsilon B,\quad Q_3 = \partial_x + (1-\epsilon)B,\quad Q_4 = (\epsilon b + (1-\epsilon)a)x\partial_x + C,$
$\qquad [A, B] = 0,\quad [A, C] = A,\quad [B, C] = (\epsilon a + (1-\epsilon)b)B.$

$L^3_{4,5}:\quad Q_1 = Ae^{-x},\quad Q_2 = e^{-ax}(\alpha\partial_x + B),\quad Q_3 = e^{-bx}(\beta\partial_x + C),\quad Q_4 = \partial_x,$
$\qquad [A, B] = -\alpha A,\quad [A, C] = -\beta B,\quad [B, C] = \alpha bC - \beta aB,\quad a = b.$

$L_{4,6}:\quad Q_1 = Ae^{-ax},\quad Q_2 = e^{-bx}(B\cos x + C\sin x),\quad Q_3 = e^{-bx}(C\cos x - B\sin x),$
$\qquad Q_4 = \partial_x,\quad [A, B] = [A, C] = [B, C] = 0.$

$L^1_{4,7}:$ $Q_1 = A, \quad Q_2 = -Ax + B, \quad Q_3 = \partial_x, \quad Q_4 = x\partial_x - \frac{1}{2}Ax^2 + Bx + C,$
$[A,B] = 0, \quad [A,C] = 2A, \quad [B,C] = B.$

$L^2_{4,7}:$ $Q_1 = Ae^{-2x}, \quad Q_2 = Be^{-x}, \quad Q_3 = e^{-x}(Bx - C), \quad Q_4 = \partial_x,$
$[A,B] = [A,C] = 0, \quad [B,C] = -A.$

$L^1_{4,8}:$ $Q_1 = A, \quad Q_2 = \epsilon\partial_x + (\epsilon - 1)(Ax - B), \quad Q_3 = (1-\epsilon)\partial_x + \epsilon(Ax + B),$
$Q_4 = (2\epsilon - 1)x\partial_x + C, \quad [A,B] = [A,C] = 0, \quad [B,C] = (1-2\epsilon)B.$

$L^2_{4,8}:$ $Q_1 = A, \quad Q_2 = e^{-x}(\alpha\partial_x + B), \quad Q_3 = e^{x}(\beta\partial_x + C), \quad Q_4 = \partial_x,$
$[A,B] = [A,C] = 0, \quad [B,C] = -\beta B - \alpha C + A, \quad \alpha\beta = 0.$

$L^1_{4,9}:$ $Q_1 = A, \quad Q_2 = \partial_x - \epsilon(Ax + B), \quad Q_3 = \partial_x + (1-\epsilon)(Ax + B), \quad Q_4 = x\partial_x + C,$
$[A,B] = 0, \quad [A,C] = 2A, \quad [B,C] = (1-2\epsilon)B.$

$L^2_{4,9}:$ $Q_1 = Ae^{-(1+b)x}, \quad Q_2 = e^{-x}(\epsilon\alpha\partial_x + B), \quad Q_3 = e^{-bx}((1-\epsilon)\beta\partial_x + C),$
$Q_4 = \partial_x, \quad [A,B] = -\epsilon(1+b)A\alpha, \quad [A,C] = (\epsilon - 1)(1+b)\beta A,$
$[B,C] = A + \epsilon\alpha bC + (\epsilon - 1)\beta B.$

$L_{4,10}:$ $Q_1 = A, \quad Q_2 = B\cos x + C\sin x, \quad Q_3 = C\cos x - B\sin x, \quad Q_4 = \partial_x,$
$[A,B] = [A,C] = 0, \quad [B,C] = A.$

$L_{4,11}:$ $Q_1 = Ae^{-2ax}, \quad Q_2 = e^{-ax}(B\cos x + C\sin x), \quad Q_3 = e^{-ax}(C\cos x - B\sin x),$
$Q_4 = \partial_x, \quad [A,B] = [A,C] = 0, \quad [B,C] = A.$

$L^1_{4,12}:$ $Q_1 = Ae^{-x}, \quad Q_2 = Be^{-x}, \quad Q_3 = \partial_x, \quad Q_4 = C,$
$[A,B] = 0, \quad [A,C] = -B, \quad [B,C] = A.$

$L^2_{4,12}:$ $Q_1 = A\cos x + B\sin x, \quad Q_2 = B\cos x - A\sin x, \quad Q_3 = \alpha\partial_x + C, \quad Q_4 = \partial_x,$
$[A,B] = 0, \quad [A,C] = A + \alpha B, \quad [B,C] = B - \alpha A.$

Here A, B, C are arbitrary constant $r \times r$ matrices, α, β are arbitrary constants, $\epsilon = 0, 1$.

In what follows we shall consider only 2×2 matrices. It is known [4], that any matrix may be reduced to one of the forms $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. After corresponding procedure we conclude that realizations of algebras $L^1_{4,1}$, $L^1_{4,2}$, $L^1_{4,3}$, $L^1_{4,4}$, $L^1_{4,7}$, $L^2_{4,7}$, $L^1_{4,9}$, $L_{4,10}$, $L_{4,11}$, $L^1_{4,12}$ has no models. Thus we will seek models for realizations of Lie algebras $L^2_{4,1}$, $L^2_{4,2}$, $L^2_{4,3}$, $L^3_{4,3}$, $L^2_{4.4}$, $L^1_{4,5}$, $L^2_{4,5}$, $L^3_{4,5}$, $L_{4,6}$, $L^1_{4,8}$, $L^2_{4,8}$, $L^2_{4,9}$, $L^2_{4,12}$.

2 Invariant spaces admitted by four-dimensional Lie algebras

The second step in construction of matrix models is description of invariant spaces for each of obtained realizations of four-dimensional Lie algebras. This step we will show for an example of realization of the Lie algebra $L_{4,6}$.

It is known [3], that invariant space corresponding to the operator $Q_4 = \partial_x$ have such form:

$$\Pi = \Pi^1 \oplus \Pi^2 = \sum_j \exp(\lambda_j x) P^{[m_j]} \vec{e}_1 + \sum_j \exp(\lambda_j x) R^{[n_j]} \vec{e}_2,$$

where $P^{[m_j]}$, $R^{[n_j]}$ are m_j, n_j-th degree polynomials in x.

Acting on Π by the operator $Q_1 = A\exp(-ax)$ we get

$$
\begin{aligned}
Q_1\Pi &= A\exp(-ax)\sum_j \exp(\lambda_j x)P^{[m_j]}\vec{e}_1 + A\exp(-ax)\sum_j \exp(\lambda_j x)R^{[n_j]}\vec{e}_2 \\
&= \sum_j \exp((\lambda_j - a)x)P^{[m_j]}\lambda\vec{e}_1 + \sum_j \exp((\lambda_j - a)x)R^{[n_j]}(\lambda\vec{e}_2 + \vec{e}_1) \\
&= \sum_j \exp((\lambda_j - a)x)\left(\lambda P^{[m_j]} + R^{[n_j]}\right)\vec{e}_1 + \sum_j \exp((\lambda_j - a)x)R^{[n_j]}\lambda\vec{e}_2.
\end{aligned}
$$

Let $\lambda \neq 0$. Fix the minimum λ_1. Then $\lambda_1 - a < \lambda_1$. But this inequality is impossible. Hence, a polynomial near $\exp(\lambda_1 - a)$ must be $R^1 = 0$, and respectively $P^1 = 0$. Thus all polynomials are zero, and invariant space is empty. This case is not interesting for us, that is why we do not consider the case $\lambda = 0$. So, the result of the action Q_1 on Π has such form:

$$
Q_1\Pi = \sum_j \exp((\lambda_j - a)x)R^{[n_j]}\vec{e}_1.
$$

The invariant space will have the form:

$$
\Pi_1 = \sum_k \left(\exp((\lambda_k - a)x)P^{[m_k]}\vec{e}_1 + \exp(\lambda_k x)R^{[n_k]}\vec{e}_2\right), \qquad n_k \leq m_k.
$$

We act on Π_1 by the operator $Q_2 = \exp(-bx)(B\cos x + C\sin x)$:

$$
\begin{aligned}
Q_2\Pi_1 = &\sum_k \exp((\lambda_k - b - i)x)R^{[n_k]}((b_2 + ic_2)\vec{e}_1 + (b_1 + ic_1)\vec{e}_2) \\
&+ \sum_k \exp((\lambda_k - (a+b) - i)x)P^{[m_k]}(b_1 + ic_1)\vec{e}_1.
\end{aligned}
$$

Again we fix minimum λ_1. Then degrees $\lambda_k - b - i < \lambda_1$, $\lambda_k - (a+b) - i < \lambda_k$. That is why $R^1 = P^1 = 0$ or $b_1 + ic_1 = 0$. In the first case the invariant space is empty. Thus we take the case for which $b_1 + ic_1 = 0$. Hence, the invariant space admitted by the operators Q_1, Q_2, Q_4 should have such form:

$$
\begin{aligned}
\Pi_2 = &\sum_k \left(\exp((\lambda_k - a)x)P^{[m_k]}\vec{e}_1 + \exp((\lambda_k - b - i)x)S^{[r_k]}\vec{e}_1 + \exp(\lambda_k x)R^{[n_k]}\vec{e}_2\right), \\
&m_k, r_k \geq n_k.
\end{aligned}
$$

Finally, we act on the space Π_2 by operator $Q_3 = \exp(-bx)(C\cos x - B\sin x)$:

$$
\begin{aligned}
Q_3\Pi_2 = &\sum_k \Big(\exp((\lambda_k - b + i)x)R^{[n_k]}((b_2 - ic_2)\vec{e}_1 + (b_1 - ic_1)\vec{e}_2) \\
&+ \exp((\lambda_k - a - b + i)x)P^{[m_k]}(b_1 - ic_1)\vec{e}_1 + \exp((\lambda_k - 2b)x)S^{[r_k]}(b_1 - ic_1)\vec{e}_1\Big).
\end{aligned}
$$

After similar actions we obtain that $b_1 - ic_1 = 0$. This equality is possible when $b_1 = c_1 = 0$. Hence invariant space admitted by the Lie algebra $L_{4,6}$ has the following form:

$$
\begin{aligned}
\Pi = &\sum_k \Big(\exp((\lambda_k - a)x)P^{[m_k]}\vec{e}_1 + \exp((\lambda_k - b - i)x)S^{[r_k]}\vec{e}_1 \\
&+ \exp((\lambda_k - b + i)x)T^{[q_k]}\vec{e}_1 + \exp(\lambda_k x)R^{[n_k]}\vec{e}_2\Big), \qquad q_k, m_k, r_k \geq n_k.
\end{aligned}
$$

Moreover, matrices A, B, C are of the following form:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}.$$

Below we adduce invariant spaces for rest of four-dimensional Lie algebras.

$$L^2_{4,1}: \quad \Pi = \sum_k \exp(\lambda_k x) P^{[m_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_2, \quad m_k \geq n_k + 2,$$
$$A = \sigma_0, \quad B = b_2 \sigma_0, \quad C = c_1 E + c_2 \sigma_0.$$

$$L^3_{4,3}: \quad \Pi = \sum_k \exp((\lambda_k - 1)x) S^{[r_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_2,$$
$$r_k \geq n_k + 1, \quad A = \sigma_0, \quad B = b_2 \sigma_0, \quad C = c_1 E + c_2 \sigma_0.$$

$$L^2_{4,4}: \quad \Pi = \sum_k \exp((\lambda_k - 1)x) S^{[r_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_2,$$
$$r_k \geq n_k + 2, \quad A = \sigma_0, \quad B = b_2 \sigma_0, \quad C = c_2 \sigma_0.$$

$$L^3_{4,5}: \quad \Pi = \sum_{k=0}^{K_1} \exp\left(\left(-\frac{c_1}{\beta} + k\right) x\right) d_k \vec{e}_1 + \sum_{k=0}^{K_2} \exp\left(\left(-\frac{c_1}{\beta} + 1 + k\right) x\right) d^*_k \vec{e}_2,$$
$$K_1 \geq K_2 + 2, \quad d_k, d^*_k = \text{const}, \quad A = \lambda E + \sigma_0, \quad B = a\sigma_0,$$
$$C = \frac{c_1}{\beta} E + \frac{c_2}{\beta} \sigma_0 - a\sigma_+, \quad a = \pm 1.$$

$$L_{4,6}: \quad \Pi = \sum_k \exp((\lambda_k - a)x) P^{[m_k]} \vec{e}_1 + \sum_k \exp((\lambda_k - b - i)x) S^{[r_k]} \vec{e}_1$$
$$+ \sum_k \exp((\lambda_k - b + i)x) T^{[q_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_2,$$
$$q_k, m_k, r_k \geq n_k, \quad A = \sigma_0, \quad B = b_2 \sigma_0, \quad C = c_2 \sigma_0.$$

$$L^2_{4,8}: \quad 1.\ \Pi = \sum_{k=0}^{K_1} \exp\left(\left(-\frac{c_1}{\beta} - k\right) x\right) d_k \vec{e}_1 + \sum_{k=0}^{K_2} \exp\left(\left(-\frac{c_1}{\beta} - k\right) x\right) d^*_k \vec{e}_2,$$
$$K_1 > K_2, \quad d_k, d^*_k = \text{const}, \quad A = \sigma_0, \quad B = \frac{1}{\beta} \sigma_0, \quad C = \frac{c_1}{\beta} E.$$
$$2.\ \Pi = \sum_{k=0}^{K_1} \exp\left(\left(-\frac{b_1}{\alpha} + k\right) x\right) d_k \vec{e}_1 + \sum_{k=0}^{K_2} \exp\left(\left(-\frac{b_1}{\alpha} + k\right) x\right) d^*_k \vec{e}_2,$$
$$K_1 > K_2, \quad d_k, d^*_k = \text{const}, \quad A = \sigma_0, \quad B = \frac{b_1}{\alpha} E, \quad C = \frac{1}{\alpha} \sigma_0.$$

$$L^2_{4,9}: \quad 1.\ \epsilon = 0,\ \Pi = \sum_{k=0}^{K_1} \exp\left(\left(-\frac{c_1}{\beta} + k\right) x\right) d_k \vec{e}_1 + \sum_{k=0}^{K_2} \exp\left(\left(-\frac{c_1}{\beta} + 2 + k\right) x\right) d^*_k \vec{e}_2,$$
$$K_1 \geq K_2, \quad d_k, d^*_k = \text{const}, \quad A = \sigma_0, \quad B = \frac{1}{b} \sigma_0, \quad C = c_1 E + c_2 \sigma_0 - 2\sigma_+.$$
$$2.\ \epsilon = 1,\ \Pi = \sum_{k=0}^{K_1} \exp\left(\left(-\frac{b_1}{\alpha} + k\right) x\right) d_k \vec{e}_1 + \sum_{k=0}^{K_2} \exp\left(\left(-\frac{b_1}{\alpha} + 1 + b + k\right) x\right) d^*_k \vec{e}_2,$$
$$K_1 \geq K_2, \quad d_k, d^*_k = \text{const}, \quad A = \sigma_0, \quad B = \frac{b_1}{\alpha} E + \frac{b_2}{\alpha} \sigma_0 - (1+b)\sigma_+, \quad C = \frac{1}{\alpha} \sigma_0.$$

$$L^2_{4,12}: \quad 1.\ \Pi = \sum_k \exp(\lambda_k x) P^{[m_k]}\vec{e}_1 + \sum_k \exp((\lambda_k - i)x) S^{[r_k]}\vec{e}_1$$
$$+ \sum_k \exp((\lambda_k + i)x) W^{[s_k]}\vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]}\vec{e}_2$$
$$+ \sum_k \exp((\lambda_k - i)x) T^{[q_k]}\vec{e}_2 + \sum_k \exp((\lambda_k + i)x) V^{[t_k]}\vec{e}_2,$$
$$n_k, t_k, q_k \le m_k, r_k, s_k, \quad A = \lambda E, \quad B = \alpha\lambda E, \quad C = c_1 E + (c_2 - c_1)\sigma_+.$$
$$2.\ \Pi = \sum_k \exp(\lambda_k x) P^{[m_k]}\vec{e}_1 + \sum_k \exp((\lambda_k - i)x) S^{[r_k]}\vec{e}_1$$
$$+ \sum_k \exp((\lambda_k + i)x) W^{[s_k]}\vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]}\vec{e}_2$$
$$+ \sum_k \exp((\lambda_k - i)x) T^{[q_k]}\vec{e}_2 + \sum_k \exp((\lambda_k + i)x) V^{[t_k]}\vec{e}_2,$$
$$n_k, t_k, q_k \le m_k, r_k, s_k, \quad A = \lambda E, \quad B = \alpha\lambda E, \quad C = \nu E + \sigma_0.$$
$$3.\ \Pi = \sum_k \exp(\lambda_k x) P^{[m_k]}\vec{e}_1 + \sum_k \exp((\lambda_k - i)x) S^{[r_k]}\vec{e}_1$$
$$+ \sum_k \exp((\lambda_k + i)x) W^{[s_k]}\vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]}\vec{e}_2,$$
$$q_k, m_k, r_k \ge n_k, \quad A = \sigma_0, \quad B = b_2\sigma_0, \quad C = c_1 E + c_2\sigma_0 + \sigma_+.$$

Here $P^{[m_k]}$, $R^{[n_k]}$, $S^{[r_k]}$, $W^{[s_k]}$, $T^{[q_k]}$, $V^{[t_k]}$ are m_k, n_k, r_k, s_k, q_k, t_k-th degree polynomials in x correspondingly, $\sigma_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, λ, α, β, ν, b_1, b_2, c_1, c_2 are arbitrary constants, $\alpha, \beta \ne 0$.

3 Conclusions

The above realizations of Lie algebras and the corresponding invariant spaces will be used for construction of exactly solvable 2×2 matrix Schrödinger models in future works. What is more, Hermitian models present special interest since they describe physical models with real eigenvalues of Hamiltonians.

The author would like to thank R.Z. Zhdanov for formulation of the problem and S.V. Spichak and V.I. Lahno for discussion of some issues of this work.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 330–332.

Operators on $(A, *)$-Spaces and Linear Classification Problems

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In the paper we consider the problem about the canonical form of linear operators on vector spaces that are gradable by a partially ordered set with involution. In a natural and important (from the practical point of view) case we establish a connection between this problem and the problem of description representations of partially ordered sets with involution.

The problem of about the canonical form of linear operators on a finite-dimensional S-space over a field k, where $S = (A, *)$ is a partially ordered set with involution, was introduced by the author in [1] (see also [2–4]). In present paper we reduce a natural and important (from the practical point of view) case of this problem to the problem of description of representations of partially ordered sets with involution.

1. Main concepts. Through out the paper all vector spaces are finite-dimensional and all partially ordered sets (posets) are finite. A poset A with trivial involution $*$ is identified with A. Under consideration of linear maps, morphisms, functors, etc. we use the right-side notations (in particular, vector spaces are right).

For a poset with involution $S = (A, *)$ and a field k we denote by $\mathrm{mod}_S k$ the category with objects the vector k-spaces $U = \oplus_{x\in A} U_x$, where $U_{x^*} = U_x$ for all $x \in A$ (such k-spaces are called S-spaces over k), and with morphisms $\delta : U \to U'$ those linear maps $\delta \in \mathrm{Hom}_k(U, U')$ for which $\delta_{x^*x^*} = \delta_{xx}$ for all $x \in A$ and $\delta_{xy} = 0$ if $x \not\leq y$ (such maps are called S-maps) [1]; here δ_{xy} denotes (as usual in analogous situations) the linear map of U_x into U'_y, induced by the map δ). If $|A| = 1$, the category $\mathrm{mod}_A k$ coincide with the category $\mathrm{mod}\, k$ of all (finite-dimensional) vector k-spaces.

The set of all S-maps of U into U' (U and U' are S-spaces) is denoted by $\mathrm{Hom}_{S.k}(U, U')$. If U is a S-space and $C \subset A$, U_C denotes the subspace $\oplus_{x\in C} U_x \subset U$; if, moreover, V is a k-space and $\gamma \in \mathrm{Hom}_k(V, U)$, γ_C denotes the map of V into U_C induced by γ; if γ is a map of a S-space U into a S-space U', $\gamma_{C,D}$ denotes the map of U_C into U'_D induced by γ.

Let $S = (A, *)$ be a poset with involution and $f = f(t)$ be a polynomial over k. We denote by $\Lambda_{S,k}$ the category whose objects are the linear operators on S-spaces, i.e. the pairs (U, φ) formed a S-space U and a map $\varphi \in \mathrm{Hom}_k(U, U)$. A morphism from (U, φ) to (U', φ') is determined by a S-map $\delta : U \to U'$ such that $\varphi\delta = \delta\varphi'$. By $\Lambda_{S,k,f}$ we denote the full subcategory of $\Lambda_{S,k}$ consisting of all objects (U, φ) such that $f(\varphi) = 0$.

2. Formulation of the main result. Let $f = f(t)$ be a polynomial over k. If x is a root of $f(t)$, $r(x)$ denotes its multiplicity. We assume that each root of $f(t)$ belongs to k and has multiplicity < 3. For $f = f(t)$, define a poset with involution $\widehat{P}_f = (P_f, *_f)$ in the following way:

1) P_f consists of the triples (x, p, i) formed by a root x of $f(t)$, and integer numbers p and i such that $1 \leq i \leq p \leq r(x)$;

2) $(x, p, i) \leq (y, q, j)$ if and only if $x = y$, and $p \geq q$, $i \leq j$ or $p \leq q$, $j - i \geq q - p$;

3) $\overline{x}^{*_f} = \overline{y}$ for unequal $\overline{x} = (x, p, i)$ and $\overline{y} = (y, q, j)$ if and only if $x = y$ and $p = q$.

Let $S=(A,*)$ be a poset with involution. A representation of S is (in our terms) a triple (V,U,γ) formed by vector k-space $V\in \operatorname{mod} k, U\in \operatorname{mod}_S k$ and a linear map $\gamma\in \operatorname{Hom}_k(V,U)$; a morphism of representations $(V,U,\gamma)\to(V',U',\gamma')$ is given by a pair (μ,ν) of linear maps $\mu\in\operatorname{Hom}_k(V,V')$ and $\nu\in\operatorname{Hom}_{S,k}(U,U')$ such that $\gamma\nu=\mu\gamma'$ (see [5]). Thus defined category is denoted by $\mathcal{R}_k(S)$.

Denote by $\overline{\mathcal{R}}_k(S\coprod\widehat{P}_f)$, where $S\coprod\widehat{P}_f$ is the direct sum of S and $\widehat{P}_f$ (i.e. $S\coprod\widehat{P}_f=(A\cup P_f,*)$, $A\cap P_f=\varnothing$ and $*$ on $A\cup P_f$ is induced by $*$ on A and $*_f$ on P_f), the full subcategory of $\mathcal{R}_k(S\coprod\widehat{P}_f)$ consisting of all objects (V,U,γ) with $\gamma_A:V\to U_A$ and $\gamma_{P_f}:V\to U_{P_f}$ being isomorphisms $\operatorname{mod} k$.

In this paper we shall prove the following statement.

Theorem A. *Let $S=(A,*)$ be a poset with involution, and $f=f(t)$ be a polynomial (over k) such that each its root belong to k and has multiplicity <3. Then the categories $\Lambda_{S,k,f}$ and $\overline{\mathcal{R}}_k(S\coprod\widehat{P}_f)$ are equivalent.*

3. Proof of Theorem A. Recall that a functor $F:\Phi\to\Psi$ is called faithful (respectively, full) if, for an arbitrary $X,Y\in\operatorname{Ob}\Phi$, the map $F:\operatorname{Hom}_\Phi(X,Y)\to\operatorname{Hom}_\Psi(X,Y)$ is injective (respectively, surjective); a functor F is called dense if each $Y\in\operatorname{Ob}\Psi$ is isomorphic to some XF (a special case of a dense functor is a surjective on objects functor, i.e. such one that the map $F:\operatorname{Ob}\Phi\to\operatorname{Ob}\Psi$ is surjective). According to the well-known theorem a functor F is equivalence of categories if and only if it is full, faithful and dense.

For a $\widehat{P}_f$-space U, we denote by $[U]$ a linear operator on U with the following (Jordan) matrix $([U]_{\overline{x}\overline{y}})$, where $\overline{x}=(x,p,i)$ and $\overline{y}=(y,q,j)$ run through the set P_f:

$$[U]_{\overline{x}\,\overline{x}}=x\mathbf{1}\quad\text{for every }\overline{x};$$
$$[U]_{\overline{x}\,\overline{y}}=\mathbf{1}\quad\text{if } x=y,\ p=q,\ i=j-1;$$
$$[U]_{\overline{x}\,\overline{y}}=O\quad\text{in the remaining cases},$$

where $\mathbf{1}=\mathbf{1}_{U(\overline{x})}$.

Define a functor $F:\overline{\mathcal{R}}_k(S\coprod\widehat{P}_f)\to\Lambda_{S,k,f}$ as follows:

$$(V,U,\gamma)F=(U_A,\gamma_A^{-1}\gamma_{P_f}[U_{P_f}]\gamma_{P_f}^{-1}\gamma_A)\quad\text{for an object }(V,U,\gamma);$$
$$(\mu,\nu)F=\nu_{A,A}\quad\text{for a morphism }(\mu,\nu):(V,U,\gamma)\to(V',U',\gamma').$$

Here S-map $\nu_{A,A}$ is a morphism in $\Lambda_{S,k,f}$ because the equality $\varphi\nu_{A,A}=\nu_{A,A}\varphi'$, where $\varphi=\gamma_A^{-1}\gamma_{P_f}[U_{P_f}]\gamma_{P_f}^{-1}\gamma_A$ and $\varphi'=(\gamma'_A)^{-1}\gamma'_{P_f}[U'_{P_f}](\gamma'_{P_f})^{-1}\gamma'_A$, easily reduces to the obviously equality $[U_{P_f}]\nu_{P_f,P_f}=\nu_{P_f,P_f}[U'_{P_f}]$ (using the equalities $\gamma_A\nu_{A,A}=\mu\gamma'_A$ and $\gamma_{P_f}\nu_{P_f,P_f}=\mu\gamma'_{P_f}$ which are equivalent to the equality $\gamma\nu=\mu\gamma'$)

It follows from the equalities $\gamma_A\nu_{A,A}=\mu\gamma'_A$ and $\gamma_{P_f}\nu_{P_f,P_f}=\mu\gamma'_{P_f}$ (taking into account the invertibility of the maps γ_A and γ_{P_f}) that $\mu=0$ and $\nu=0$ whenever $(\mu,\nu)F=0$. Hence the functor F is faithful.

The functor F is full — for a morphism $\alpha:(V,U,\gamma)F\to(V',U',\gamma')F$, we can take as a morphism $(\mu,\nu):(V,U,\gamma)\to(V',U',\gamma')$ such that $(\mu,\nu)F=\alpha$ the following morphism:

$$\mu=\gamma_A\alpha(\gamma'_A)^{-1},\quad \nu_{A,A}=\alpha,\quad \nu_{P_f,P_f}=\gamma_{P_f}^{-1}\gamma_A\alpha(\gamma'_A)^{-1}\gamma'_{P_f}\quad(\nu_{A,P_f}=0,\nu_{P_f,A=0}).$$

Finally, let us verify that the functor F is surjective on objects. If $(W,\varphi)\in\Lambda_{S,k,f}$, denote by W^0 the $\widehat{P}_f$-space $\oplus_{\overline{x}=(x,p,i)\in P_f}W_{\overline{x}}$, where $W_{\overline{x}}=\operatorname{Ker}(\varphi-x\mathbf{1}_W)$, and fix a map $\lambda\in\operatorname{Hom}_k(W,W^0)$ such that $\varphi=\lambda[W^0]\lambda^{-1}$. Then we can take as the objects $(V,U,\gamma)\in\overline{\mathcal{R}}_k(S\coprod\widehat{P}_f)$ such that $(V,U,\gamma)F=(W,\varphi)$ the following object: $V=W$, $U_A=W$, $U_{P_f}=W^0$ and $\gamma_A=\mathbf{1}_{U_A}$, $\gamma_{P_f}=\lambda$.

Thus the functor F is full, faithful and dense. Theorem A is proved.

4. Generalization of Theorem A. Our theorem can be generalized to the case of an arbitrary polynomial $f(t)$. Here we consider the case when each root of $f(t)$ belong to k and have an arbitrary multiplicity r ($1 \le r \le \deg f$).

In this situation we shall also need representations of so-called completed posets [6]. A completed poset consists of a poset B and an equivalence relation $\sim$ on $B^{\le} = \{(x,y) \in B \times B |\, x \le y\}$. These data are subjected to the condition that $x \le z \le y$ and $(x,y) \sim (x',y')$ imply the existence of a unique z' satisfying $x' \le z' \le y'$, $(x,z) \sim (x',z')$ and $(z,y) \sim (z',y')$. In case $(x,x) \sim (x',x')$ we shall write $x \sim x'$; therefore it is possible to describe restriction of the relation $\sim$ on B. A completed poset is called weakly completed if $(x,y) \sim (x',y')$ implies $x = y$ and $x' = y'$.

Let $T = (B, \sim)$ be a completed poset. A T-space (over k) is a B-space $U = \oplus_{b\in B} U_b$ such that $U_x = U_y$ if $x \sim y$. A T-map of U into U' (U and U' are T-spaces) is a B-map $\varphi : U \to U'$ such that $\varphi_{xy} = \varphi_{zt}$ if $(x,y) \sim (z,t)$; $\mathrm{Hom}_{T.k}(U,U')$ denotes the set of all T-maps of U into U'. The category of T-spaces over k (whose objects and morphisms are, respectively, the T-spaces and T-maps) is denoted by $\mathrm{mod}_T k$.

Representations of a completed poset $T = (B,\sim)$ are defined in a way analogous to that for a poset with involution $S = (A, *)$. A representation of $T = (B,\sim)$ is (in our terms) a triple (V,U,γ) formed by vector k-spaces $V \in \mathrm{mod}\, k$, $U \in \mathrm{mod}_T k$ and a linear map $\gamma \in \mathrm{Hom}_k(V,U)$. A morphism of representations $(V,U,\gamma) \to (V',U',\gamma')$ is given by a pair (μ,ν) of linear maps $\mu \in \mathrm{Hom}_k(V,V')$ and $\nu \in \mathrm{Hom}_{T,k}(U,U')$ such that $\gamma\nu = \mu\gamma'$. The category of all representations of $T = (B,\sim)$ is denoted by $\mathcal{R}_k(T)$.

For $f = f(t)$, we shall consider (instead of the poset with involution $\widehat{P}_f = (P_f, *_f)$ which was considered above) the completed posets $(\widetilde{P}_f, \sim_f)$, where P_f is defined by the conditions 1) and 2), and the relations $\sim_f$ on $P_f^{\le}$, defined in the following way: $((x,p,i,),(x,q,j)) \sim_f ((x',p',i'),(x',q',j'))$ if, and only if, $x = x'$, $p = p'$, $q = q'$ and $i - i' = j - j'$. This obviously implies that $(x,p,i) \sim_f (x',p',i'))$ if and only if $x = x'$ and $p = p'$.

Let $S = (A, *)$ be a poset with involution. We identify S with the weakly completed poset $(A, \sim_*)$, where $x \sim_* x'$ for $x \ne x'$ if, and only if, $x^* = x'\,(x,x' \in A)$. Consider the direct sum of S and $\widetilde{P}_f$: $S \coprod \widetilde{P}_f = (A,\sim_*) \coprod (P_f, \sim_f)$ .As in the case above denote by $\overline{\mathcal{R}}_k(S \coprod \widetilde{P}_f)$ the full subcategory of $\mathcal{R}_k(S \coprod \widetilde{P}_f)$ consisting of all objects (V,U,γ) with $\gamma_A : V \to U_A$ and $\gamma_{P_f} : V \to U_{P_f}$ being isomorphisms in $\mathrm{mod}\, k$.

We have the following generalization of Theorem A.

Theorem B. *Let $S = (A, *)$ be a poset with involution, and $f = f(t)$ be a polynomial (over k) with roots belonging to k. Then the categories $\Lambda_{S,k,f}$ and $\overline{\mathcal{R}}_k(S \coprod \widetilde{P}_f)$ are equivalent.*

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 333–340.

Poincaré Parasuperalgebra with Central Charges and Parasupersymmetric Wess–Zumino Model

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In the present paper we consider irreducible representations of Poincaré parasuperalgebra with central charges for even N and find the internal symmetry group for case $P_\mu P^\mu > 0$. The generalization of Wess–Zumino model for $N = 1$ and arbitrary p is also obtained.

1 Introduction

Supersymmetry (SUSY), introduced in theoretical physics and mathematics, has a lot of interesting applications [1]. One of them consists in mixing fermionic and bosonic states. It has important consequences in the quantum field theory, namely, this property provides a mechanism for cancellation of the ultraviolet divergences. Moreover, supersymmetric quantum field theory (SSQT) allows to unify the space symmetries of the Poincaré group with internal symmetries [2]. It allows to overcome the"no-go" theorem of Coleman and Mandula.

Supersymmetric quantum field theory (SSQFT) induced appearance of supersymmetric quantum mechanics (SSQM) [3]. SSQM stimulated deeper understanding of ordinary quantum mechanics and provided new ways for solving some problems [4].

SSQM has been generalized to the parasupersymmetric quantum mechanics (PSSQM) [5]. The latter deals with bosons and $p = 2$ parafermions having parastatistical properties. Here p is the so-called paraquantization order [6]. Soon an independent version of PSSQM yielding to positive defined Hamiltonians was proposed [7].

The crucial step in developing PSSQM was made by Beckers and Debergh [8] who required Poincaré invariance of the theory and formulated the group-theoretical foundations of the so-called parasupersymmetric quantum field theory (PSSQFT). This theory is a natural generalization of SSQFT, dealing with the Poincaré parasupergroup (or Poincaré parasuperalgebra (PPSA)) instead of the Poincaré supergroup (or Poincaré superalgebra (PSA)).

Recently IRs of the PPSA for $N = 1$ have been described [9] and then IRs for arbitrary N and internal symmetry group have been found [10, 11].

The present paper consists of two main parts. In the first part we consider the Poincaré parasuperalgebra with central charges. The second part includes the physical model, which is invariant under the Poincaré parasuperalgebra.

2 Extended Poincaré parasuperalgebra

Definition of the PPSA and the main Casimir operators. The Poincaré parasuperalgebra [8–11] is generated by ten generators P_μ, $J_{\mu\nu}$ of the Poincaré group, satisfying the commutation relations

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0, \qquad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu),\\
&[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}),\\
&J_{\mu\nu} = -J_{\nu\mu}, \qquad \mu,\nu = 0,1,2,3
\end{aligned}
\tag{2.1}
$$

and N parasupercharges Q^J_α, $(Q^J_\alpha)^\dagger$ ($\alpha = 1,2,\ J = 1,2,\ldots,N$), which satisfy the following double commutation relations

$$
\begin{aligned}
&[Q^I_\alpha, [Q^J_\beta, Q^K_\gamma]] = 4\varepsilon_{\alpha\beta}Z^{IJ}Q^K_\gamma - 4\varepsilon_{\alpha\gamma}Z^{IK}Q^J_\beta,\\
&[(Q^I_\alpha)^\dagger, [(Q^J_\beta)^\dagger, (Q^K_\gamma)^\dagger]] = 4\varepsilon_{\alpha\beta}Z^{*IJ}(Q^K_\beta)^\dagger - 4\varepsilon_{\alpha\gamma}Z^{*IK}(Q^J_\beta)^\dagger,\\
&[Q^I_\alpha, [Q^J_\beta, (Q^K_\gamma)^\dagger]] = 4\varepsilon_{\alpha\beta}Z^{IJ}(Q^K_\gamma)^\dagger - 4Q^J_\beta(\sigma_\mu)_{\alpha\gamma}P^\mu,\\
&[(Q^I_\alpha)^\dagger, [Q^J_\beta, t(Q^K_\gamma)^\dagger]] = 4(Q^K_\gamma)^\dagger(\sigma_\mu)_{\alpha\beta}P^\mu - 4\varepsilon_{\alpha\beta}Z^{*IK}Q^J_\beta,
\end{aligned}
\tag{2.2}
$$

where σ_ν are the Pauli matrices, $(\cdot)_{\alpha\gamma}$ stand for the matrix elements. Relations (2.1), (2.2) include operators Z^{IJ}, which we call the central charges. This definition is a direct generalization of Poincaré superalgebra with central charges.

In analogy with the Poincaré superalgebra the central charges must satisfy the relations $Z^*_{IJ} = Z^{IJ}$, $Z^{IJ} = -Z^{JI}$ and commute with generators of the PPSA. The spinor indices are risen and dropped using the universal spinor $\varepsilon^{\alpha\beta}$ ($\varepsilon^{11} = \varepsilon_{11} = \varepsilon^{22} = \varepsilon_{22} = 0$, $\varepsilon^{12} = \varepsilon_{21} = 1$, $\varepsilon^{21} = \varepsilon_{12} = -1$).

In addition, we have the following commutation relations between the generators of the Poincaré group and the parasupercharges:

$$
\begin{aligned}
&[J_{\mu\nu}, Q^J_\alpha] = -\frac{1}{2i}\,(\sigma_{\mu\nu})_{\alpha\beta}\,Q^J_\beta, \qquad [P_\mu, Q^J_\alpha] = 0,\\
&[J_{\mu\nu}, \left(Q^J_\alpha\right)^\dagger] = -\frac{1}{2i}\left(\sigma^*_{\mu\nu}\right)_{\alpha\beta}\left(Q^J_\beta\right)^\dagger, \qquad [P_\mu, \left(Q^J_\alpha\right)^\dagger] = 0.
\end{aligned}
\tag{2.3}
$$

The PPSA, as well as PSA, can be extended by adding the generators Σ_l of the internal symmetry group, which satisfy the following relations:

$$
[Q^I_\alpha, \Sigma_l] = S^I_{lJ}Q^J_\alpha, \qquad [\Sigma_l, \left(Q^I_\alpha\right)^\dagger] = S^{*I}_{lJ}\left(Q^I_\alpha\right)^\dagger, \qquad [\Sigma_l, \Sigma_m] = f^k_{lm}\Sigma_k. \tag{2.4}
$$

By analogy with the PSA, P_σ and $J_{\mu\nu}$ are called even and Q^J_α, $\left(Q^J_\alpha\right)^\dagger$ are called odd elements of the PPSA.

In the papers [8–11] two main Casimir operators were found. They have the form

$$
C_1 = P_\mu P^\mu, \qquad C_2 = P_\mu P^\mu B_\nu B^\nu - (B_\mu P^\mu)^2, \tag{2.5}
$$

where

$$
B_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma + (\sigma_\mu)_{AB}\,\bar{Q}^I_A Q^I_B.
$$

The eigenvalues of C_1, C_2 are used for the classification of irreducible representations (IRs) of PPSA.

IRs with the central charges. Like the case of the ordinary Poincaré group, IRs of the PPSA can be divided into three main classes I. $P_\mu P^\mu > 0$, II. $P_\mu P^\mu = 0$, III. $P_\mu P^\mu < 0$. It is known [8–11] that for classes I and II there exists the additional Casimir operator $C_3 = P_0/|P_0|$, whose eigenvalues are $\varepsilon = \pm 1$. In other words, the classes I and II can be splitted into two subclasses corresponding to the fixed values of C_3, and we will mark them by I^+ and II^+ (for class $C_3 = 1$) and I^- , II^- (for class $C_3 = -1$).

The IRs for these main classes are described in [8–11]. Here we note that for the Poincaré superalgebra only classes I^+ and II^+ exist (since the relevant generators P_0 should be positive defined).

Now let us consider the IRs of PPSA with central charges (2.1)–(2.3). In the present paper we shall obtain the IRs for class I^+. The rest of IRs will be considered in our forthcoming papers.

Thus, for class I^+ we have $C_3 = 1$ and $P_\mu = (M, 0, 0, 0)$ in the rest frame. The central charges Z^{IJ} have to be equal to the unit matrix multiplied by the numeric coefficients which are elements of the $N \times N$ antisymmetric matrix Z.

By means of the unitary transformation $Z \longrightarrow \bar{Z} = UZU^\dagger$, any such matrix can be reduced to the quasidiagonal form

$$\tilde{Z}^{IJ} = U^I_L U^J_M Z^{LM}, \tag{2.6}$$

where

$$\tilde{Z}^{IJ} = \varepsilon \otimes D \quad (\text{even } N), \qquad \tilde{Z}^{IJ} = \begin{pmatrix} \varepsilon \otimes D & 0 \\ 0 & 0 \end{pmatrix} \quad (\text{odd } N), \tag{2.7}$$

D is a diagonal matrix with the positive real entries Z_m, and ε is the universal spinor. Relations (2.2) are invariant under the simultaneous transformation

$$Z^{IJ} \longrightarrow \tilde{Z}^{IJ} = U^{IK} U^{LJ} Z^{KJ}, \qquad Q^I_A \longrightarrow \tilde{Q}^I_A = U^{JK} Q^K_A, \tag{2.8}$$

where all nonzero Z^{IJ} are exhausted by the following ones $Z^{2m-1,2m} = -Z^{2m,2m-1} = Z^m$.

Choosing a new basis

$$\begin{aligned} Q^{2m-1}_1 &= \frac{1}{\sqrt{2}}(\hat{Q}^{2m-1}_1 + \hat{Q}^{2m}_1), \qquad & Q^{2m-1}_2 &= \frac{1}{\sqrt{2}}(\hat{\bar{Q}}^{2m}_2 - \hat{\bar{Q}}^{2m-1}_2), \\ Q^{2m}_1 &= \frac{1}{\sqrt{2}}(\hat{Q}^{2m-1}_2 + \hat{Q}^{2m}_2), \qquad & Q^{2m}_2 &= \frac{1}{\sqrt{2}}(\hat{\bar{Q}}^{2m}_1 - \hat{\bar{Q}}^{2m-1}_1), \end{aligned} \tag{2.9}$$

we reduce relations (2.2) in the rest frame $P = (M, 0, 0, 0)$ to the form

$$\begin{aligned} &[\hat{Q}^{2k-1}_A, [\hat{\bar{Q}}^{2m-1}_B, \hat{Q}^J_C]] = \delta_{AB}\delta_{km}(2M - Z_k)\hat{Q}^J_C, \\ &[\hat{Q}^{2k}_A, [\hat{\bar{Q}}^{2m}_B, \hat{Q}^J_C]] = \delta_{AB}\delta_{km}(2M + Z_m)\hat{Q}^J_C, \\ &[\hat{\bar{Q}}^{2k-1}_A, [\hat{Q}^{2m-1}_B, \hat{\bar{Q}}^J_C]] = \delta_{AB}\delta_{km}(2M - Z_m)\hat{\bar{Q}}^J_C, \\ &[\hat{\bar{Q}}^{2k}_A, [\hat{Q}^{2m}_B, \hat{\bar{Q}}^J_C]] = \delta_{AB}\delta_{km}(2M + Z_m)\hat{Q}^J_C, \end{aligned} \tag{2.10}$$

the remaining double commutators of the parasupercharges are equal to zero.

Below we shall restrict ourselves to the case, where all $Z_m < 2M$. Then, using the analogy of (2.10) with the formulas from [11], we easily find the general solution of (2.10):

$$\begin{aligned} \hat{Q}^{2m-1}_A &= (-1)^{A-1}\sqrt{2M - Z_m}(S_{4N+1,\,8m-11+4A} - iS_{4N+1,\,8m-10+4A}), \\ \hat{Q}^{2m}_A &= (-1)^{A-1}\sqrt{2M + Z_m}(S_{4N+1,\,8m-9+4A} - iS_{4N+1,\,8m-8+4A}), \end{aligned} \tag{2.11}$$

where S_{IJ} are generators of $SO(1, 4N + 1)$ satisfying the commutation relations

$$[S_{kl}, S_{mn}] = -i(g_{km}S_{ln} + g_{ln}S_{km} - g_{kn}S_{lm} - g_{lm}S_{km}).$$

Substituting (2.11) into (2.9), we obtain parasupercharges in the rest frame

$$\begin{aligned} Q_A^{2m-1} = & \sqrt{M - Z_m/2}((-1)^{A-1} S_{4N+1,\,8m-11+4A} - S_{4N+1,\,8m-10+4A}) \\ & + \sqrt{M + Z_m/2}(S_{4N+1,\,8m-9+4A} + i(-1)^A S_{4N+1,\,8m-8+4A}), \\ Q_A^{2m} = & \sqrt{M - Z_m/2}(-S_{4N+1,\,8m-7+4A} + i(-1)^{A-1} S_{4N+1,\,8m-6+4A}) \\ & + \sqrt{M + Z_m/2}((-1)^A S_{4N+1,\,8m-5+4A} + i S_{4N+1,\,8m-4+4A}). \end{aligned} \tag{2.12}$$

In accordance with [11] the IRs of the class I^+ of the PPSA with central charges $Z_m < 2M$ are labeled by the following sets of numbers $(M, j, n_1, n_2, \ldots, n_{2N}, Z_1, Z_2, \ldots, Z_{\{\frac{n}{2}\}})$, here j label the IRs of $SO(3)$; $n_1 \geq n_2 \geq \ldots \geq n_{2N}$ label the IRs of $SO(1, 4N+1)$; n_1, n_2, $\ldots$, n_{2N} are either integer or half integer. Using the Lorentz transformation, we can find the basis elements P_μ, $J_{\mu\nu}$, Q_α^j, $\bar{Q}_\alpha^J$ in arbitrary frame.

Internal symmetry group. The internal symmetry group for PPSA without central charges was described in [11]. In paper [11] the authors showed that the internal symmetry group for IRs of classes I^+ and II^+ and arbitrary N is $SU(N)$.

Therefore, it is interesting to generalize these results to the case of nontrivial central charges. In this paper we shall obtain the IRs of class I^+, $Z_m < 2M$, $Z_m \neq 0$ and even N.

In analogy with SUSY it is easy to see that the internal symmetry group is smaller than in the case of central charges, equal to zero. Indeed, considering the first of relations (2.2) for $\alpha = \beta = 1$, $\gamma = 2$:

$$[Q_1^J, [Q_2^J, Q_1^K]] = 4Z^{IJ} Q_1^K \tag{2.13}$$

and evaluating the commutators of l.h.s. and r.h.s. of (2.13) with Σ_l and using (2.4) we come to the following condition

$$S_{lJ}^I Z^{JK} = S_{lJ}^K Z^{JI}. \tag{2.14}$$

In other words, the products of generators of the internal group with the matrix of central charges should be symmetric matrices.

Now let us present the explicit description of internal symmetry algebra for IRs of class I^+ and even N. In our case the internal symmetry algebra is isomorphic to $Sp(N)$. The basis elements have the following form

$$\begin{aligned} & A^{kk} = Z_k^{-1}(-S_{8k-7,\,8k-6} - S_{8k-5,\,8k-4} + S_{8k-3,\,8k-2} + S_{8k-1,\,8k}), \\ & B^{kk} = Z_k^{-1}(S_{8k-5,\,8k} - S_{8k-4,\,8k-1} + S_{8k-7,\,8k-2} - S_{8k-6,\,8k-3}) \\ & \qquad + i(S_{8k-5,\,8k-1} + S_{8k-4,\,8k} + S_{8k-7,\,8k-3} + S_{8k-6,\,8k-2}), \\ & C^{kk} = (B^{kk})^\dagger, \\ & A^{kn} = (f_{kn}^- + f_{nk}^-)\Sigma_{kn} + (f_{kn}^+ + f_{nk}^+)\Sigma_{k+2,\,n+2}, \\ & B^{kn} = f_{nk}^- \tilde{\Sigma}_{kn} + f_{kn}^- \Sigma_{kn}^\dagger + f_{nk}^+ \tilde{\Sigma}_{k+2,\,n+2} - f_{nk}^+ \tilde{\Sigma}_{k+2,\,n+2}^\dagger, \quad n > k, \\ & C^{kn} = (B^{kn})^\dagger, \quad n < k, \end{aligned} \tag{2.15}$$

where

$$f_{kn}^\pm = \frac{1}{Z_n}\sqrt{\frac{2M \pm Z_k}{2M \pm Z_n}}, \qquad f_{nk}^\pm = \frac{1}{Z_k}\sqrt{\frac{2M \pm Z_n}{2M \pm Z_k}},$$

$$\Sigma_{kn} = S_{8k-7,\,8n-6} - S_{8k-6,\,8n-7} - S_{8k-3,\,8n-2} + S_{8k-2,\,8n-3}$$
$$- i(S_{8k-7,\,8n-7} + S_{8k-6,\,8n-6} + S_{8k-3,\,8n-3} + S_{8k-2,\,8n-2}),$$
$$\tilde{\Sigma}_{kn} = -S_{8k-7,\,8n-2} + S_{8k-6,\,8n-3} + S_{8k-3,\,8n-6} - S_{8k-2,\,8n-7}$$
$$- i(S_{8k-7,\,8n-3} + S_{8k-6,\,8n-2} + S_{8k-3,\,8n-7} + S_{8k-2,\,8n+6}),$$
$$n \neq k, \quad k, n = 1, 2, ..., N/2.$$

Matrices (2.18) commute with the generators of Poincaré group and satisfy the following relations

$$[A^{kk}, Q_A^J] = Z_k^{-1}(\delta_{J,\,2k-1} - \delta_{J,\,2k})Q_A^J,$$
$$[B^{kk}, Q_A^J] = 2Z_k^{-1}\delta_{J,\,2k-1}Q_{2k}^J,$$
$$[C^{kk}, Q_A^J] = 2Z_k^{-1}\delta_{J,\,2k}Q_A^{2k-1},$$
$$[A^{kn}, Q_A^J] = \delta_{J,\,2k-1}Z_k^{-1}Q_A^{2k-1} - \delta_{J,\,2n-1}Z_n^{-1}Q_A^{2k-1} + \delta_{J,\,2k}Z_K^{-1}Q_A^{2n} - \delta_{J,\,2k}Z_n^{-1}Q_A^{2k},$$
$$[B^{kn}, Q_A^J] = \delta_{j,\,2k-1}Z_k^{-1}Q_A^{2n} + \delta_{J,\,2n-1}Z_n^{-1}Q_A^{2k},$$
$$[C^{kn}, Q_A^J] = \delta_{J,\,2k}Z_k^{-1}Q_A^{2n-1} + \delta_{J,\,2n}Z_n^{-1}Q_A^{2k-1},$$
$$[A^{mn}, A^{kl}] = Z_k^{-1}\delta^{kn}A^{ml} - Z_m^{-1}\delta^{ml}A^{nk},$$
$$[A^{mn}, B^{kl}] = Z_n^{-1}(\delta^{nk}B^{ml} + \delta^{nl}B^{mk}),$$
$$[A^{mn}, C^{kl}] = [C^{mn}, C^{kl}] = 0,$$
$$[B^{mn}, C^{kl}] = Z_k^{-1}(\delta^{nk}A^{ml} + \delta^{mk}A^{nl}) + Z_k^{-1}(\delta^{nl}A^{mk} + \delta^{ml}A^{nk}).$$

Thus the internal symmery group for IRs of class I^+, $Z_m < 2M$, $Z_m \neq 0$ and even N is $Sp(N)$.

3 Parasupersymmetric Wess–Zumino model

Now let us consider the PPSA without the central charges in the terms of the paragrassmannian variables for arbitrary p. Then the generators of PPSA will be of the form [8]:

$$P_\mu = p_\mu = i\frac{\partial}{\partial x^\mu},$$
$$J_{12} = x_1p_2 - x_2p_1 + \frac{1}{4}(\theta^1Q_2 - Q_2\theta^1 - \theta^2Q_1 + Q_1\theta^2),$$
$$J_{13} = x_1p_3 - x_3p_1 + \frac{i}{4}(\theta^1Q_1 - Q_1\theta^1 - \theta^2Q_2 + Q_2\theta^2),$$
$$J_{23} = x_2p_3 - x_3p_2 + \frac{1}{4}(\theta^1Q_1 - Q_1\theta^1 - \theta^2Q_2 + Q_2\theta^2),$$
$$J_{01} = x_0p_1 - x_1p_0 + \frac{i}{4}(Q_1\theta^1 - \theta^1Q_1 - \theta^2Q_2 + Q_2\theta^2),$$
$$J_{02} = x_0p_2 - x_2p_0 + \frac{1}{4}(Q_2\theta^1 - \theta^1Q_2 + \theta^2Q_1 - Q_1\theta^2),$$
$$J_{03} = x_0p_3 - x_3p_0 + \frac{i}{4}(\theta^2Q_1 - Q_1\theta^2 - \theta^1Q_2 + Q_2\theta^1),$$
$$Q_1 = \frac{\partial}{\partial\theta^1}, \qquad (Q_1)^\dagger = -2((p_3 - p_0)\theta^1 + (p_1 + ip_2)\theta^2),$$
$$Q_2 = \frac{\partial}{\partial\theta^2}, \qquad (Q_2)^\dagger = 2((p_3 + p_0)\theta^2 - (p_1 - ip_2)\theta^1),$$

where θ_α are paragrassmanian variables defined by the Green anzats:

$$\theta_\alpha = \sum_{i=1}^{p} \theta_\alpha^{(i)}, \qquad \frac{\partial}{\partial\theta_\alpha} = \sum_{i=1}^{p} \frac{\partial}{\partial\theta_\alpha^{(i)}}, \qquad \left[\theta_\alpha^{(i)}, \theta_\beta^{(i)}\right]_+ = 0, \qquad \left[\theta_\alpha^{(i)}, \theta_\beta^{(j)}\right] = 0,$$
$$\left[\frac{\partial}{\partial\theta_\alpha^{(i)}}, \frac{\partial}{\partial\theta_\beta^{(i)}}\right]_+ = 0, \quad \left[\theta_\alpha^{(i)}, \frac{\partial}{\partial\theta_\beta^{(i)}}\right]_+ = \delta_{\alpha\beta}, \quad \left[\theta_\alpha^{(i)}, \frac{\partial}{\partial\theta_\beta^{(j)}}\right]_+ = 0, \quad \left[\theta_\alpha^{(i)}, \frac{\partial}{\partial\theta_\beta^{(j)}}\right] = 0, \tag{3.1}$$

p is paraquantization order, $i \neq j$. It should be noted that θ_α are Majorana spinors.

There exists the realization of $N = 1$ PPSA in terms of four paragrassmanian variables θ^1, θ^2, $(\theta^1)^\dagger$, $(\theta^2)^\dagger$. In this case we have

$$P_\mu = p_\mu = i\frac{\partial}{\partial x_\mu},$$
$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu - \frac{1}{2}\left((\sigma_{\mu\nu})_{\alpha\beta}\left[\theta^\alpha, \frac{\partial}{\partial\theta_\beta}\right] + (\sigma^\dagger{}_{\mu\nu})_{\alpha\beta}\left[(\theta^\alpha)^\dagger, \frac{\partial}{\partial(\theta^\beta)^\dagger}\right]\right), \tag{3.2}$$
$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - i(\sigma_\mu)_{\alpha\beta}(\theta^\beta)^\dagger P^\mu, \qquad (Q_\alpha)^\dagger = i\frac{\partial}{\partial(\theta^\alpha)^\dagger} + i(\theta^\beta)^\dagger(\sigma_\mu)_{\beta\alpha}P^\mu.$$

Then we can define the covariant derivatives

$$(D_\alpha)^\dagger = \frac{\partial}{\partial\theta^\alpha} - (\sigma_\mu)_{\alpha\beta}(\theta^\beta)^\dagger P^\mu, \qquad (D_\alpha)^\dagger = -\frac{\partial}{\partial(\theta^\alpha)^\dagger} + \theta^\beta(\sigma_\mu)_{\beta\alpha}P^\mu. \tag{3.3}$$

Derivatives (3.3) have important property which will be used below, namely, the operators $L = D_\alpha D^\alpha = [D_1, D_2]$ and $(L)^\dagger = (D_\alpha)^\dagger(D^\alpha)^\dagger = [(D_1)^\dagger, (D_2)^\dagger]$ commute with the PPSA generators in representation (3.2).

We notice that the representation of PPSA in the terms of paragrassmannian variables can be found, using the covariant representation for the PPSA [11]. Then we put

$$\sqrt{2M}(S_{51} - iS_{52}) \longrightarrow -i\frac{\partial}{\partial\theta^1} - i(\theta^1)^\dagger M, \qquad \sqrt{2M}(S_{53} - iS_{54}) \longrightarrow -i\frac{\partial}{\partial\theta^2} - i(\theta^2)^\dagger M,$$
$$\sqrt{\frac{2}{M}}(S_{51} + iS_{52}) \longrightarrow -i\frac{\partial}{\partial(\theta^2)^\dagger} - i\theta^1 M, \qquad \sqrt{\frac{2}{M}}(S_{53} + iS_{54}) \longrightarrow -i\frac{\partial}{\partial(\theta^1)^\dagger} - i\theta^2 M.$$

Operators (3.2)–(3.3) act on the space of functions $\Phi(x, \theta, (\theta)^\dagger)$ which depend on space coordinates x_μ and paragrassmanian variables θ, $(\theta)^\dagger$. We shall call such functions parasuperfields. They form the linear space. In general case this space is reducible. In other words, the expansion $\Phi(x, \theta, (\theta)^\dagger)$ by powers of θ and $\theta^\dagger$ has superficial components. We can eliminate these components, if we impose the covariant constraints on Φ. But these constraints should not lead to the differential consequences in the x-space which restrict the dependence of the parasuperfields on x_μ. For the case $N = 1$ and arbitrary p these constraints have the following form

$$D_\alpha\Phi(x, \theta, (\theta)^\dagger) = 0 \qquad \text{or} \qquad (D_\alpha)^\dagger\Phi(x, \theta, (\theta)^\dagger) = 0. \tag{3.4}$$

Constraints (3.4) pick out the invariant subspaces, containing smaller number of component fields, from the space of the parasuperfields. The fields satisfying conditions (3.4) are called chiral.

In order to investigate equations (3.4) it is convenient to choose a new representation for the parasuperfields

$$\Phi^\pm = \exp(\mp G)\Phi, \qquad \text{where} \qquad G = \frac{1}{2}(\sigma_\mu)_{\alpha\beta}[(\theta^\alpha)^\dagger, \theta^\beta]P^\mu \tag{3.5}$$

(these representations will be called "+" and "−" representations). The operators A^+ and A^- in "+" and "−" representations are connected with the operator A in the initial representation by the formula

$$A^{\pm} = \exp(\mp G) A \exp(\pm G) \tag{3.6}$$

Using (3.5), we can show that Φ^+ doesn't depend on $(\theta)^\dagger$ and Φ^+ doesn't depend on θ.

In the case $N = 1$ and $p = 2$ the invariant spaces, picked out by equations (3.4), will contain 6 independent fields: 3 fields with spin 0, 2 filds with spin $\frac{1}{2}$ and 1 field with spin 1 (see [8]).

Using the fact that the operator L defined above commutes with the generators Poincaré parasuperalgebra, we can write down the equation for $\Phi^+(x,\theta)$ (without interaction), which will be invariant under the Poincaré parasuperalgebra:

$$(L^+)^\dagger \exp(-2G)\Phi_+^*(x,\theta) = 0, \tag{3.7}$$

where $(L^+)^\dagger = [(D_1^+)^\dagger, (D_2^+)^\dagger]$, $(D^+)^\dagger$ is the covariant derivative in "+" representation. For $p = 2$ we find, taking into account (3.3), (3.5), that the equations for the component fields A, ϕ_α, $\psi_{\alpha\beta}$, λ_α, B (below we omit the indices"+") are

$$(p_0 + \vec{S}\vec{p})\vec{\Lambda} = 0, \qquad \operatorname{div}\vec{\Lambda} = 0, \tag{3.8}$$

$$(p_0 + \vec{\sigma}\vec{p})\varphi = 0, \tag{3.9}$$

$$\Box A = 0, \tag{3.10}$$

$$\chi(x) = B(x) = \lambda_1 = \lambda_2 = 0, \tag{3.11}$$

where $\vec{\Lambda} = (\psi_{22} - \psi_{11},\ -i(\psi_{22} + \psi_{11}),\ \psi_{12} + \psi_{21})$, $\chi(x) = \psi_{12} - \psi_{21}$, the matrices S_a have the form

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is obvious that (3.8) is the system of Maxwell's equations for the massless field with spin 1. System (3.9) is the system of the equations for the massless field with spin $\frac{1}{2}$. Equation (3.10) is the equation for the massless field with spin 0. In addition, it is obvious from (3.11) that two scalar fields and spinor field are equal zero.

Now let us consider the equation for $\Phi^+(x,\theta)$ including the interaction (parasupersymmetric Wess–Zumino model). In this case parasupersymmetric equation for $\Phi^+(x,\theta)$ has the form

$$((L^+)^\dagger)^p \exp(-2G)\Phi_+^*(x,\theta) = g\Phi_+^2(x,\theta), \tag{3.12}$$

where g is interaction constant. For $p = 1$ we recover supersymmetric Wess–Zumino model. For $p = 2$ it yields the model described in [8].

The Lagrangian which corresponds to equation (3.12) has the form

$$\mathcal{L} = (\Phi_+^* \exp(2G)\Phi_+)_{(\theta^1)^p(\theta^2)^p((\theta^1)^\dagger)^p((\theta^2)^\dagger)^p} + \frac{1}{3}(g\Phi_+^3)_{(\theta^1)^p(\theta^2)^p} + (\text{c.c.}),$$

(c.c. is complex conjugation). In conclusion of this paper let us note that the Wess–Zumino model for $p > 1$ is incompatible with the description of massive particles (see [8]).

Acknowledgements

The author thanks Professor A.G. Nikitin for fruitful discussions.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 341–342.

On $*$-Wild Algebras

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In this article, we consider discrete groups and study the complexity of their representations from the point of view of the theory of $*$-representations.

Let F_2 be the free group with two generators, $C^*(F_2)$ be the group C^*-algebra.

Definition (see [1]). *We call a discrete group F $*$-wild if there exist $n \in \mathbb{N}$ and an epimorphism $\varphi : C^*(F) \to M_n(C^*(F_2))$.*

In this note, we give two constructions that allow to construct examples of $*$-wild groups (other than the semi-direct products $F \rtimes G$, where F is a wild group). Note that the group $W = F \rtimes G$ is $*$-wild if F is a $*$-wild group.

1. Let G be a discrete group that has a faithful irreducible unitary representation in the algebra $M_n(\mathbb{C})$. In what follows, we assume that the group G is already an irreducible group of unitary operators in the algebra $M_n(\mathbb{C})$, i.e., such that if $[x, g] = 0$ for all $g \in G$ and some $x \in M_n(\mathbb{C})$, then $x = \lambda I_n$, $\lambda \in \mathbb{C}$, I_n is the identity operator in $M_n(\mathbb{C})$. Let F be a free group with generators $u_1, \ldots, u_m$ (the number of generators can be finite or infinite). Let also B_1, $\ldots$, B_m be unitary operators in the algebra $M_n(\mathbb{C})$. Consider the group $G_1 = \langle e \otimes g \mid g \in G$, e is the identity of the group $F\rangle \subset M_n(C^*(F))$. Note that $G \cong G_1$.

Theorem. *The group $G_2 = \langle G_1, u_1 \otimes B_1, \ldots, u_m \otimes B_m\rangle$ (the smallest subgroup of $M_n(C^*(F))$ containing the set of generating elements: elements of G_1, $u_1 \otimes B_1, \ldots, u_m \otimes B_m \in M_n(C^*(F))$) is $*$-wild.*

Proof. Let us construct a homomorphism $\phi\colon C^*(G_2) \to M_n(C^*(F))$ by extending the embedding $G_2 \subset M_n(C^*(F))$ to a homomorphism of the group C^*-algebra $C^*(G_2)$. Consider a unitary representation $\pi\colon F \to B(H)$ on a Hilbert space H. Denote by $\hat{\pi}$ the lift of the representation π, i.e., $\hat{\pi} = \pi \otimes I_n$. Define a functor $\Phi\colon \operatorname{Rep} F \to \operatorname{Rep} G_2$ as follows: $\Phi(\pi) = \hat{\pi} \circ \phi$. Let K be an intertwining operator for the representation π of the group F. Set $\Phi(K) = K \otimes I_n$.

Let us show that ϕ is an epimorphism. To prove that, it is sufficient to show (see [2]) that the functor $\Phi\colon \operatorname{Rep} F \to \operatorname{Rep} G_2$ is full (here $\operatorname{Rep} G$ is a category where the points are unitary representations of the group G, and morphisms are intertwining operators).

Recall that a functor Φ is called full, if it defines a one-to-one correspondence between intertwining operators for the corresponding representations of the groups F and G.

Let us show that the defined functor is full, i.e., there is a one-to-one correspondence between intertwining operators for the representation π of the group F and intertwining operators for the representation $\hat{\pi} \circ \phi$ of the group G_2. Let us consider an intertwining operator L for the representation $\hat{\pi} \circ \phi$ of the group G_2, i.e., the operator L commutes with all operators of the representation $\hat{\pi} \circ \phi$. Since the operator L commutes with elements of the group $\hat{\pi} \circ \phi(G_1)$, it follows that L must have the form $K \otimes I_n$, where $K \in B(H)$, I_n is the identity in the algebra $M_n(\mathbb{C})$. Since the operator L commutes with the elements $\pi(u_1) \otimes B_1, \ldots, \pi(u_m) \otimes B_m$, it follows that K commutes with all of the elements $\pi(u_1), \ldots, \pi(u_m)$, i.e., the functor Φ defines a one-to-one correspondence between intertwining operators. In virtue of paragraph 1.10 [1], the constructed functor $\Phi\colon \operatorname{Rep} F \to \operatorname{Rep} G_2$ is full and faithful, and since F is a $*$-wild group, there exists an epimorphism $\psi\colon C^*(F) \to M_n(C^*(F_2))$. Then the composition $\psi \circ \phi$ is an epimorphism and $\psi \circ \phi\colon C^*(G_2) \to M_n(C^*(F_2))$, hence G_2 is $*$-wild.

2. We give one more construction of $*$-wild groups that have the following form: they are extensions of a group F by a group G, where F is a $*$-wild group.

Consider the group $G = \langle g_1, \ldots, g_p\rangle$, where the number of generators could be both finite or infinite. The conditions that are imposed on the group G are the same as in **1**. Consider the set $T = \{D$ is a discrete group, $D \subset U_n(\mathbb{C}) \mid$ so that there exists an extension of the group G by the group $D\}$. Consider the discrete groups $V = \langle w \otimes g \mid w \in F_m, g \in G\rangle$, $Z_1 = \langle e \otimes z \mid e$ is the identity in $F_m, z \in Z\rangle$. Note that $V \cong F_m \times G$. By construction, V is $*$-wild. Consider the group generated by all products of elements of the groups V and Z_1, and denote it by $Y = \langle V, Z_1\rangle$. Note that Y is an extension of the group V by Z_1.

Proposition. *The group Y is $*$-wild.*

Proof. The proof is similar to the proof of Theorem. Construct a homomorphism $\phi\colon C^*(Y) \to M_n(C^*(V))$ by extending the embedding $Y \subset M_n(C^*(V))$ to a homomorphism of the group C^*-algebra $C^*(Y)$. We only prove that the correspondence for the intertwining operators for a representation π of the group V and the intertwining operators for the representation $\hat{\pi} \circ \phi$ of the group Y is one-to-one.

Consider an operator K that intertwines the representation $\hat{\pi} \circ \phi$ of the group Y. The operator K commutes with all operators of the group $\hat{\pi} \circ \phi(Y)$. Hence, K has the form $p \otimes I_n$, where $p \in B(H)$, H is the corresponding Hilbert space, and I_n is the identity of the algebra $M_n(\mathbb{C})$. Since K commutes with elements of the group F_m, it follows that p commutes with all elements $w \in F_m$, i.e., there exists an epimorphism $\phi\colon C^*(Y) \to M_n(C^*(V))$, i.e., Y is $*$-wild.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 343–349.

Canonical Realization of the Poincaré Algebra for a Relativistic System of Charged Particles Plus Electromagnetic Field

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The procedure of reducing of canonical field degrees of freedom for a system of charged particles plus electromagnetic field in the constraint Hamiltonian formalism is developed up to the first order in the coupling constant expansion. The canonical realization of the Poincaré algebra in the terms of physical variables is found. The relation between covariant and physical particle variables in the Hamiltonian description is studied.

1 Introduction

Usually, an interaction within a system of N charged particles is described by means of the electromagnetic field with its own degrees of freedom represented by the 4-potential $A_\mu(x)$, $x \in \mathbb{M}_4$, over the Minkowski space-time[1] [1, 2, 3].

Such a system of particles plus electromagnetic field is completely determined by the following action

$$
\begin{aligned}
S = &-\int \sum_{a=1}^{N} m_a \sqrt{u_a^\mu(\tau) u_{a\mu}(\tau)}\, d\tau \\
&-\int \sum_{a=1}^{N} e_a u_a^\mu(\tau) A_\mu[x_a(\tau)]\, d\tau - \frac{1}{16\pi} \int F_{\mu\nu}(x) F^{\mu\nu}(x)\, d^4x.
\end{aligned}
\tag{1.1}
$$

Here $F_{\mu\nu}(\mathbf{x},\tau) = \partial_\mu A_\nu(\mathbf{x},\tau) - \partial_\nu A_\mu(\mathbf{x},\tau)$ is the field strength; m_a, e_a are the mass and the charge of particle a, respectively, $u_a^\mu(\tau) = dx_a^\mu(\tau)/d\tau$, and $\tau \mapsto x_a^\mu(\tau)$ gives parametric equation of the particle world line in the Minkowski space-time.

But often it is desirable to exclude the field degrees of freedom and formulate the description of the system only in the terms of particle variables. The elimination of the field variables can be performed exactly in the action (1.1). This leads to the time-symmetric Wheeler–Feynman electrodynamics [4, 5] with the Fokker action. Nonlocality of the Fokker-type actions result in serious difficulties in transition to the Hamiltonian description [6]. The same problems occur when Fokker-type action is replaced by the single-time Lagrangian depending on the infinity order derivatives of the particle coordinates [7, 8]. Although this problem can be solved within the corresponding approximation schemes [6, 9]. Here we shall consider an alternative way to

[1] The Minkowski space-time $\mathbb{M}_4$ is endowed with a metric $\|\eta_{\mu\nu}\| = \mathrm{diag}(1,-1,-1,-1)$. The Greek indices $\mu,\nu,\ldots$ run from 0 to 3; the Latin indices from the middle of alphabet, $i,j,k,\ldots$ run from 1 to 3 and both types of indices are subject of the summation convention. The Latin indices from the beginning of alphabet, a,b, label the particles and run from 1 to N. The sum over such indices is indicated explicitly.

overcome these difficulties. The main idea consists in the elimination of field degrees of freedom *after* transition to the Hamiltonian description of the particles plus field theory.

Then, we must solve the field Hamiltonian equations of motion and make the canonical transformation to the free field variables. After that the canonical free field variables will be eliminated by means of canonical constraint method. This procedure gives us the canonical realization of the Poincaré algebra in the terms of particle variables.

However, the field equations of motion are nonlinear, so we will find the solutions of these equations and other relations in the first order in the coupling constant expansion. Therefore, the Lienard–Wiechert potentials will be the expected solutions of the field equations.

The present paper is organized as follows. In Section 2, there is a canonical realization of the Poincaré algebra for the system of N point charged particles plus electromagnetic field (field theory).

In Section 3, we find solutions of the field equations of motion of first order in the coupling constant expansion, make canonical transformation to the free field variables and eliminate them with help of constraints. We obtain a canonical realization of the Poincaré generators depending on the particle coordinates and momenta. It is shown that the new generators form an algebra. There is a study of relations between new canonical coordinates and positions of particles of the reduced system.

The conclusions in Section 4 contain some final remarks and the outline of future research.

2 Field theory Poincaré generators

Action (1.1) for the system of field and particles is manifestly Poincaré-invariant. Its invariance leads to the conservation of the symmetric energy-momentum tensor [1, 3]

$$\begin{aligned}\theta^{\mu\nu}(z) = {} & \sum_{a=1}^{N} m_a \int \frac{u_a^{\mu}(\tau) u_a^{\nu}(\tau)}{\sqrt{u_a^2(\tau)}} \delta^4(x_a(\tau) - z) d\tau \\ & + \frac{1}{4\pi}\left(-F^{\mu\lambda}(z) F^{\nu}{}_{\lambda}(z) + \frac{1}{4} F_{\lambda\sigma}(z) F^{\lambda\sigma}(z)\eta^{\mu\nu}\right).\end{aligned} \tag{2.1}$$

For transition to the Hamiltonian description we use $3+1$ splitting of the Minkowski space-time corresponding to the instant form of dynamics [10, 11]. In geometric approach the instant form of dynamics is determined by foliation of the Minkowski space-time by the hyperplanes $x^0 = \tau$, $\tau \in \mathbb{R}$.

In this case the Lagrangian of the system is

$$\begin{aligned}L = {} & -\sum_{a=1}^{N} m_a \sqrt{1 - \mathbf{u}_a^2(\tau)} d\tau - \sum_{a=1}^{N} e_a \left(u_a^i(\tau) A_i[\mathbf{x}_a(\tau), \tau] + A_0[\mathbf{x}_a(\tau), \tau]\right) \\ & - \frac{1}{16\pi} \int F_{\mu\nu}(\mathbf{x}, \tau) F^{\mu\nu}(\mathbf{x}, \tau) d^3x d\tau,\end{aligned}$$

where $\mathbf{x}_a = (x_a^i)$, $\mathbf{u}_a = (u_a^i)$, $\mathbf{A}(\mathbf{x}, \tau) = (A^i(\mathbf{x}, \tau))$.

The canonical momenta are given by

$$p_{ai}(\tau) = -\frac{\partial L}{\partial u_a^i} = \frac{m_a u_{ai}(\tau)}{\sqrt{1 - \mathbf{u}_a^2(\tau)}} + e_a A_i[\mathbf{x}_a(\tau), \tau],$$

$$E^i(\mathbf{x}, \tau) = \frac{\delta L}{\delta \dot{A}_i(\mathbf{x}, \tau)} = \frac{1}{4\pi} F^{i0}(\mathbf{x}, \tau), \qquad E^0(\mathbf{x}, \tau) = \frac{\delta L}{\delta \dot{A}_0(\mathbf{x}, \tau)} = 0.$$

The canonical and Dirac Hamiltonians are

$$H = \sum_{a=1}^{N} \left[\sqrt{m_a^2 + [\mathbf{p}_a - e_a \mathbf{A}(\mathbf{x}_a)]^2} + e_a A_0(\mathbf{x}_a) \right] + \int \left(\frac{1}{16\pi} F_{ij} F_{ij} + 2\pi E^i E^i - A_0 \partial_i E^i \right) d^3x,$$

$$H_D = H + \int \lambda E^0 d^3x,$$

where λ is the Dirac multiplier.

The basic Poisson brackets are

$$\{x_a^i(\tau), p_{bj}(\tau)\} = -\delta_{ab}\delta_j^i, \qquad \{A^\mu(\mathbf{x}, \tau), E^\nu(\mathbf{y}, \tau)\} = \eta^{\mu\nu}\delta^3(\mathbf{x} - \mathbf{y}). \tag{2.2}$$

The constraint $E^0(\mathbf{x}, \tau) \approx 0$ ($\approx$ means "weak equality" in the sense of Dirac) reflects the gauge invariance of S; its time constancy produces the only secondary constraint, $\partial_i E^i(\mathbf{x}, \tau) - \rho(\mathbf{x}, \tau) \approx 0$, where $\rho(\mathbf{x}, \tau) = \sum_{a=1}^{N} e_a \delta^3(\mathbf{x} - \mathbf{x}_a(\tau))$. The two constraints $E^0(\mathbf{x}, \tau) \approx 0, \partial_i E^i(\mathbf{x}, \tau) - \rho(\mathbf{x}, \tau) \approx 0$ are first class with vanishing Poisson brackets. Therefore, the corresponding conjugate variables $A_0(\mathbf{x}, \tau)$, $\int \Delta^{-1}(\mathbf{x} - \mathbf{y})\partial_i A_i(\mathbf{y}, \tau) d^3y$, $(\Delta^{-1}(\mathbf{x}) = -1/(4\pi|\mathbf{x}|))$ are arbitrary functions.

Conservation of the energy-momentum tensor (2.1) leads to ten conserved Poincaré generators:

$$P^\mu = \int \theta^{\mu 0}(\mathbf{x}, \tau) d^3x, \qquad M^{\mu\nu} = \int \left(x^\mu \theta^{\nu 0}(\mathbf{x}, \tau) - x^\nu \theta^{\mu 0}(\mathbf{x}, \tau) \right) d^3x.$$

They can be rewritten in terms of canonical variables as

$$P^0 = \sum_{a=1}^{N} \sqrt{m_a^2 + [\mathbf{p}_a - e_a \mathbf{A}(\mathbf{x}_a)]^2} + \int \left(\frac{1}{16\pi} F_{ij} F_{ij} + 2\pi E^i E^i \right) d^3x,$$

$$P^k = \sum_{a=1}^{N} \left[p_a^k - e_a A^k(\mathbf{x}_a) \right] + \int E^l F^{lk} d^3x,$$

$$M^{k0} = \sum_{a=1}^{N} x_a^k \sqrt{m_a^2 + [\mathbf{p}_a - e_a \mathbf{A}(\mathbf{x}_a)]^2} + \int \left(\frac{1}{16\pi} F_{ij} F_{ij} + 2\pi E^i E^i \right) x^k d^3x - \tau P^k,$$

$$M^{ik} = \sum_{a=1}^{N} (x_a^i p_a^k - x_a^k p_a^i) + \int \left(x^k E^l \partial^i A^l - x^i E^l \partial^k A^l \right) d^3x + \int \left(A^i E^k - A^k E^i \right) d^3x,$$

where $\mathbf{p}_a = (p_a^i)$. They satisfy the commutation relations of the Poincaré algebra,

$$\{P^\mu, P^\nu\} = 0, \qquad \{P^\mu, M^{\nu\lambda}\} = \eta^{\mu\nu} P^\lambda - \eta^{\mu\lambda} P^\nu,$$

$$\{M^{\mu\nu}, M^{\lambda\sigma}\} = -\eta^{\mu\lambda} M^{\nu\sigma} + \eta^{\nu\lambda} M^{\mu\sigma} - \eta^{\nu\sigma} M^{\mu\lambda} + \eta^{\mu\sigma} M^{\nu\lambda},$$

in terms of the Poisson brackets (2.2).

3 Reduction of field degrees of freedom

The equations of motion in first order in the coupling constant expansion are

$$\dot{x}_a^i = \frac{p_a^i}{\sqrt{m_a^2+\mathbf{p}_a^2}} + \frac{e_a \Pi_a^{ij}}{\sqrt{m_a^2+\mathbf{p}_a^2}} A_j(\mathbf{x}_a), \qquad \dot{p}_{ai} = \partial_{ai} A_j(\mathbf{x}_a) \frac{e_a p_a^j}{\sqrt{m_a^2+\mathbf{p}_a^2}} + e_a \partial_{ai} A_0(\mathbf{x}_a),$$

$$\dot{A}_i = -4\pi E_i + \partial_i A_0, \qquad \dot{E}^i = -j^i - \frac{\Delta}{4\pi} A^i - \frac{1}{4\pi} \partial^i \left(\partial_j A^j\right), \tag{3.1}$$

$$\dot{E}^0 = \partial_i E^i - \rho \approx 0, \qquad \dot{A}_0 = \lambda,$$

where $\Pi_a^{ij} \equiv \delta^{ij} - p_a^i p_a^j/(m_a^2+\mathbf{p}_a^2)$, $j^i = \sum\limits_{a=1}^{N} \left(e_a p_a^i/\sqrt{m_a^2+\mathbf{p}_a^2}\right) \delta^3(\mathbf{x}-\mathbf{x}_a(\tau))$ is current density, and λ is an arbitrary function of the evolution parameter τ. They are generated by the Dirac Hamiltonian

$$H_D = \sum_{a=1}^{N} \left[\sqrt{m_a^2+\mathbf{p}_a^2} + \frac{e_a p_{ai}}{\sqrt{m_a^2+\mathbf{p}_a^2}} A_i(\mathbf{x}_a) + e_a A_0(\mathbf{x}_a) \right]$$
$$+ \int \left(\frac{1}{16\pi} F_{ij} F_{ij} + 2\pi E^i E^i - A_0 \partial_i E^i \right) d^3x + \int \lambda E^0 d^3x.$$

From Eqs.(3.1) one gets

$$\ddot{A}_k - \Delta A_k - \partial_k \left(\dot{A}_0 - \partial_l A_l \right) = 4\pi j_k.$$

If we require that $\dot{A}_0 - \partial_l A_l = 0$ (the Lorentz gauge), then by using the constraint $\partial_i E^i - \rho \approx 0$ we obtain wave equations for the potentials

$$\ddot{A}_k - \Delta A_k = 4\pi j_k, \qquad \ddot{A}_0 - \Delta A_0 = 4\pi\rho. \tag{3.2}$$

The general solutions of the inhomogeneous Eqs.(3.2) can be presented in the form

$$A_\mu = A_\mu^{\mathrm{rad}} + A_\mu^1,$$

where A_μ^{rad} is the general solution of the corresponding homogeneous equation and

$$\begin{aligned} A_k^1(\mathbf{x},\tau) &= 4\pi \sum_{a=1}^{N} \int D(\tau-\tau'|\mathbf{x}-\mathbf{x}_a(\tau')) \frac{e_a p_{ak}(\tau')}{\sqrt{m_a^2+\mathbf{p}_a^2(\tau')}} d\tau', \\ A_0^1(\mathbf{x},\tau) &= 4\pi \sum_{a=1}^{N} e_a \int D(\tau-\tau'|\mathbf{x}-\mathbf{x}_a(\tau')) d\tau', \end{aligned} \tag{3.3}$$

with the real Green function D which satisfies the equation $(\partial_\tau^2 - \Delta) D(\tau|\mathbf{x}) = \delta(\tau)\delta^3(\mathbf{x})$.

In a given approximation, the expressions (3.3) do not depend on the concrete choice of the Green function (retarded or advanced) and after integration and using free-particle equations we obtain

$$A_k^1(\mathbf{x},\tau) = \sum_{a=1}^{N} \frac{e_a u_{ak}}{\sqrt{[\mathbf{u}_a(\mathbf{x}-\mathbf{x}_a(\tau))]^2 + (1-\mathbf{u}_a^2)(\mathbf{x}-\mathbf{x}_a(\tau))^2}},$$

$$A_0^1(\mathbf{x},\tau) = \sum_{a=1}^{N} \frac{e_a}{\sqrt{[\mathbf{u}_a(\mathbf{x}-\mathbf{x}_a(\tau))]^2 + (1-\mathbf{u}_a^2)(\mathbf{x}-\mathbf{x}_a(\tau))^2}}, \tag{3.4}$$

where $u_a^k = p_a^k/\sqrt{m_a^2+\mathbf{p}_a^2}$ is the free-particle velocity.

Let us perform the canonical transformation to the new field variables:

$$\phi_\mu(\mathbf{x},\tau) = A_\mu(\mathbf{x},\tau) - A_\mu^1(\mathbf{x},\tau),\ \ \chi^k(\mathbf{x},\tau) = E^k(\mathbf{x},\tau) - E_1^k(\mathbf{x},\tau), \tag{3.5}$$

where $E_1^k(\mathbf{x},\tau)$ is

$$\begin{aligned}
E_1^k(\mathbf{x},\tau) &= -\frac{1}{4\pi}\left(\dot{A}_1^k(\mathbf{x},\tau) - \partial^k A_0^1(\mathbf{x},\tau)\right) \\
&= -\frac{1}{4\pi}\sum_{a=1}^{N} \frac{e_a(1-\mathbf{u}_a^2)(x^k - x_a^k(\tau))}{\sqrt{[\mathbf{u}_a(\mathbf{x}-\mathbf{x}_a(\tau))]^2 + (1-\mathbf{u}_a^2)(\mathbf{x}-\mathbf{x}_a(\tau))^2}}.
\end{aligned}$$

This transformation changes the particle variables: $(x_a^i, p_{ai}) \mapsto (q_a^i, k_{ai})$, where

$$\begin{aligned}
x_a^i &= q_a^i + \int \left[\left(\phi_k + \frac{1}{2}A_k^1\right)\frac{\partial E_1^k}{\partial k_{ai}} - \left(\chi^k + \frac{1}{2}E_1^k\right)\frac{\partial A_k^1}{\partial k_{ai}} - E^0\frac{\partial A_0^1}{\partial k_{ai}}\right] d^3x, \\
p_{ai} &= k_{ai} - \int \left[\left(\phi_k + \frac{1}{2}A_k^1\right)\frac{\partial E_1^k}{\partial q_a^i} - \left(\chi^k + \frac{1}{2}E_1^k\right)\frac{\partial A_k^1}{\partial q_a^i} - E^0\frac{\partial A_0^1}{\partial q_a^i}\right] d^3x.
\end{aligned} \tag{3.6}$$

In the considered approximation the equalities (3.5) may be put into the form

$$\begin{aligned}
A_\mu &= \phi_\mu + A_\mu^1(\mathbf{q}_a,\mathbf{k}_a) = \phi_\mu + A_\mu^1(\mathbf{x}_a,\mathbf{p}_a), \\
E^k &= \chi^k + E_1^k(\mathbf{q}_a,\mathbf{k}_a) = \chi^k + E_1^k(\mathbf{x}_a,\mathbf{p}_a).
\end{aligned}$$

Let us note some useful transformation properties of A_k^1, E_1^k, A_0^1

$$\begin{aligned}
&\{A_1^l(\mathbf{x},\tau), x^ip^k - x^kp^i - m^{ik}\} = \delta^{il}A_1^k(\mathbf{x},\tau) - \delta^{kl}A_1^i(\mathbf{x},\tau), \\
&\{A_1^l(\mathbf{x},\tau), x^kp^0 - m^{k0}\} = \delta^{kl}A_0^1(\mathbf{x},\tau), \\
&\{E_1^l(\mathbf{x},\tau), x^ip^k - x^kp^i - m^{ik}\} = \delta^{il}E_1^k(\mathbf{x},\tau) - \delta^{kl}E_1^i(\mathbf{x},\tau), \\
&\{E_1^l(\mathbf{x},\tau), x^kp^0 - m^{k0}\} = \frac{1}{4\pi}\left(\partial^l A_1^k(\mathbf{x},\tau) - \partial^k A_1^l(\mathbf{x},\tau)\right), \\
&\{A_0^1(\mathbf{x},\tau), x^ip^k - x^kp^i - m^{ik}\} = 0, \\
&\{A_0^1(\mathbf{x},\tau), x^kp^0 - m^{k0}\} = A_1^k(\mathbf{x},\tau),
\end{aligned}$$

here $p^0 = \sum\limits_{a=1}^{N}\sqrt{m_a^2+\mathbf{k}_a^2}$, $p^i = \sum\limits_{a=1}^{N} k_a^i$, $m^{k0} = \sum\limits_{a=1}^{N} q_a^k\sqrt{m_a^2+\mathbf{k}_a^2}$, $m^{ik} = \sum\limits_{a=1}^{N}(q_a^ik_a^k - q_a^kk_a^i)$.

The conserved quantities after the canonical transformation may be rewritten as

$$\begin{aligned}
P^0 &= \sum_{a=1}^{N}\sqrt{m_a^2+\mathbf{k}_a^2} + \frac{1}{2}\sum_{a=1}^{N}\left[\frac{e_ak_a^i}{\sqrt{m_a^2+\mathbf{k}_a}}A_i^1(\mathbf{q}_a) + A_0(\mathbf{q}_a)\right] \\
&\quad + \int\left[\frac{1}{16\pi}\Phi_{ij}\Phi_{ij} + 2\pi\chi^i\chi^i\right] d^3x, \\
P^i &= \sum_{a=1}^{N} k_a^i + \int \chi^k\Phi^{ki} d^3x,
\end{aligned}$$

$$M^{k0} = \sum_{a=1}^{N} q_a^k \sqrt{m_a^2 + \mathbf{k}_a^2} + \frac{1}{2}\sum_{a=1}^{N} q_a^k \left[\frac{e_a k_a^i}{\sqrt{m_a^2 + \mathbf{k}_a}} A_i^1(\mathbf{q}_a) + A_0(\mathbf{q}_a)\right]$$
$$+ \int \left[\frac{1}{16\pi}\Phi_{ij}\Phi_{ij} + 2\pi\chi^i\chi^i\right] x^k d^3x - \tau P^k,$$
$$M^{ik} = \sum_{a=1}^{N}(q_a^i k_a^k - q_a^k k_a^i) + \int \left(x^k\chi^l\partial^i\phi^l - x^i\chi^l\partial^k\phi^l\right) d^3x + \int \left(\phi^i\chi^k - \phi^k\chi^i\right) d^3x,$$

here $\Phi_{ij} = \partial_i\phi_j - \partial_j\phi_i$.

We reduce field degrees of freedom using the following set of constraints

$$(\Psi_\alpha) = (\phi_k, \chi^k, \phi_0, E^0) \approx 0. \tag{3.7}$$

The constraints depending on gauge A_k, A_0 potentials already contain gauge-fixing constraints. Indeed, the equations of motion lead to the conclusion that A_0 is an arbitrary function. However, the additional constraint $\dot{A}_0 - \partial_l A_l \approx 0$ together with the pure secondary constraint $\partial_i E^i - \rho \approx 0$ defines A_0 as a function of particle variables (see Eq.(3.4)). In this case, $\partial_l A_l$ can be found from the additional constraint in the terms of the coordinates and the momenta of particles too. Using Hodge decomposition for A_k

$$A_k(\mathbf{x},\tau) = A_k^{\perp}(\mathbf{x},\tau) + \partial_k \int \Delta^{-1}(\mathbf{x}-\mathbf{y})\partial_l A_l(\mathbf{y},\tau) d^3y$$
$$\approx A_k^{\perp}(\mathbf{x},\tau) + \partial_k \int \Delta^{-1}(\mathbf{x}-\mathbf{y})\partial_l A_l^1(\mathbf{y},\tau) d^3y,$$

we see that the constraint $A_k - A_k^1 \approx 0$ or $\phi_k \approx 0$ analogously determines $\partial_l A_l$ as $\dot{A}_0 - \partial_l A_l \approx 0$. This means that the gauge-fixing constraints and the constraints Eqs.(3.7) does not need to be separated.

The constraints Eqs.(3.7) are second class, so we can eliminate them by means of use of the Dirac brackets:

$$\{F,G\}_D = \{F,G\} - \int \{F, \Psi_\alpha(\mathbf{x},\tau)\} C_{\alpha\beta}^{-1}(\mathbf{x}-\mathbf{y}) \{\Psi_\beta(\mathbf{y},\tau), G\}\, d^3x d^3y$$
$$= \sum_{a=1}^{N}\left(\frac{\partial F}{\partial q_a^i}\frac{\partial G}{\partial k_a^i} - \frac{\partial G}{\partial q_a^i}\frac{\partial F}{\partial k_a^i}\right),$$

where $\|C_{\alpha\beta}^{-1}(\mathbf{x}-\mathbf{y})\|$ is the inverse matrix to $\|\{\Psi_\alpha(\mathbf{x},\tau), \Psi_\beta(\mathbf{y},\tau)\}\|$.

Thus we obtain the Poincaré generators of the reduced system

$$P^0 = \sum_{a=1}^{N} \sqrt{m_a^2 + \mathbf{k}_a^2} + \frac{1}{2}\sum_{a=1}^{N} \left[\frac{e_a k_a^i}{\sqrt{m_a^2 + \mathbf{k}_a}} A_i^1(\mathbf{q}_a) + A_0(\mathbf{q}_a)\right], \qquad P^i = \sum_{a=1}^{N} k_a^i,$$
$$M^{k0} = \sum_{a=1}^{N} q_a^k \sqrt{m_a^2 + \mathbf{k}_a^2} + \frac{1}{2}\sum_{a=1}^{N} q_a^k \left[\frac{e_a k_a^i}{\sqrt{m_a^2 + \mathbf{k}_a}} A_i^1(\mathbf{q}_a) + A_0(\mathbf{q}_a)\right] - \tau P^k,$$
$$M^{ik} = \sum_{a=1}^{N}(q_a^i k_a^k - q_a^k k_a^i),$$

which act on the particle phase space $T^*\mathbb{R}^{3N}$. They satisfy the commutation relations of the Poincaré algebra in a given approximation.

According to the Eq.(3.6) the covariant particle positions x_a^i are connected with the canonical variables as

$$x_a^i = q_a^i + \frac{1}{2}\int \left[A_k^1 \frac{\partial E_1^k}{\partial k_{ai}} - E_1^k \frac{\partial A_k^1}{\partial k_{ai}}\right] d^3x. \qquad (3.8)$$

These relations cannot be complemented to the canonical transformation to the reduced phase space $T^*\mathbb{R}^{3N}$ in full accordance with the famous no-interaction theorem [12]. It can be verified directly that in a given approximation the expression (3.8) satisfies the world line condition

$$\{x_a^i, M^{k0}\}_D = \{x_a^i, P^0\} x_a^k - \tau\delta^{ik}.$$

The Poisson brackets between particle positions are

$$\{x_a^i, x_b^j\}_D = \int \left(\frac{\partial A_k^1}{\partial k_{bj}} \frac{\partial E_1^k}{\partial k_{ai}} - \frac{\partial E_1^k}{\partial k_{bj}} \frac{\partial A_k^1}{\partial k_{ai}}\right) d^3x \not\equiv 0.$$

4 Conclusions

In this paper a method of reduction the field degrees of freedom by means of canonical constraints has been developed for a system of N charged particles plus electromagnetic field. In the first order in the coupling constant expansion it is shown that the properties of the Poincaré algebra are preserved after field reduction.

We found the solutions of the inhomogeneous field equations of motion as the sum of the Lienard–Wiechert potentials and the free fields. By means of the canonical transformation to the free field variables we got new form for the Poincaré generators. It is turned out that the Poincaré generators may be presented as the sum of free field and particle terms. In our approximation we eliminate the radiation phenomenon connected with the free electromagnetic fields. The commutation and transformation properties of particle positions are studied.

The obtained description may be used for the statistical description of the system of charged particles interacting without field. The elaborated procedure of reduction can be realized for the gravity in near future.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 350–356.

Irreducible Representations of the Dirac Algebra for a System Constrained on a Manifold Diffeomorphic to S^D

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The irreducible representations of the Dirac algebra for a particle constrained to move on S^D are generalized to a system on a manifold diffeomorphic to S^D. It is shown that there exists a one-one correspondence between irreducible representations of two Dirac algebras given respectively on S^D and on the manifold diffeomorphic to it. Among diffeomorphic mappings connecting S^D to the manifold the area-preserving one plays a crucial role to derive out our main result. It is observed that the representation space of the Dirac algebra is kept unchanged through area-preserving mappings.

1 Dirac algebra

Let us consider a system constrained to move on a D-dimensional manifold embedded in the $(D+1)$-Euclidean space R^{D+1} whose coordinates will be denoted as $x_1, x_2, \ldots, x_{D+1}$. The Hamiltonian in R^{D+1} is assumed to be

$$H = \frac{1}{2}\sum_{\alpha=1}^{D+1} p_\alpha^2 + V(x) \tag{1.1}$$

and the D-dimensional smooth manifold on which the system is constrained will be written as

$$f(x) = 0 \tag{1.2}$$

with $f(x) \in C^\infty$. We further assume the manifold to be diffeomorphic to S^D.

Equation (1.2) is the so called primary constraint. According to the prescription by Dirac [1] the consistency of (1.2) under the time development leads us to the secondary constraint that can be written as

$$\{f_{,\alpha}(x), p_\alpha\} = 0, \tag{1.3}$$

where and in what follows $f_{,\alpha}(x) \equiv \partial_\alpha f(x)$, $f_{,\alpha\beta}(x) \equiv \partial_\alpha\partial_\beta f(x)$, $\{A, B\} \equiv AB + BA$ and repeated two Greek indices in a single term indicate a summation of such terms in which the pair of those indices run over 1 to $D+1$. The fundamental Dirac brackets [1] for canonical variables in classical mechanics are seen to be converted to

$$[x_\alpha, x_\beta] = 0, \tag{1.4}$$

$$[x_\alpha, p_\beta] = i\Lambda_{\alpha\beta}(x), \tag{1.5}$$

$$[p_\alpha, p_\beta] = -\frac{i}{2}\left\{\frac{1}{R^2(x)}\left(f_{,\alpha}(x)f_{,\beta\gamma}(x) - f_{,\beta}(x)f_{,\alpha\gamma}(x)\right), p_\gamma\right\}, \tag{1.6}$$

where

$$R^2(x) \equiv f_{,\alpha}(x) f_{,\alpha}(x) \tag{1.7}$$

and

$$\Lambda_{\alpha\beta}(x) \equiv \delta_{\alpha\beta} - \frac{f_{,\alpha}(x) f_{,\beta}(x)}{R^2(x)}. \tag{1.8}$$

With direct calculations one easily finds that Eqs. (1.4)∼(1.6) are compatible with the constraints (1.2) and (1.3). The inner product of two wave functions $\chi(x)$ and $\varphi(x)$ is given by

$$\langle \chi | \varphi \rangle = \int d^{D+1} x \delta(f(x)) \chi^*(x) \varphi(x). \tag{1.9}$$

We call the algebra described by (1.2)∼(1.6) the Dirac algebra on $f(x) = 0$.

2 Relation between two Dirac algebras

In order to examine the Dirac algebra on $f(x) = 0$ we introduce another manifold in R^{D+1} which is also diffeomorphic to S^D. We denote it as

$$g(x) = 0. \tag{2.1}$$

Then we have the following Dirac algebra just corresponding to (1.3)∼(1.8):

$$\{g_{,\alpha}(x), p_\alpha\} = 0, \tag{2.2}$$

$$[x_\alpha, x_\beta] = 0, \tag{2.3}$$

$$[x_\alpha, p_\beta] = i \Lambda'_{\alpha\beta}(x), \tag{2.4}$$

$$[p_\alpha, p_\beta] = -\frac{i}{2} \left\{ \frac{1}{R'^2(x)} \left(g_{,\alpha}(x) g_{,\beta\gamma}(x) - g_{,\beta}(x) g_{,\alpha\gamma}(x) \right), p_\gamma \right\}, \tag{2.5}$$

where

$$R'^2(x) = g_{,\alpha}(x) g_{,\alpha}(x) \quad \text{and} \quad \Lambda'_{\alpha\beta}(x) = \delta_{\alpha\beta} - \frac{g_{,\alpha}(x) g_{,\beta}(x)}{R'^2(x)}. \tag{2.6}$$

Since the manifold (1.2) is connected with (2.1) through a diffeomorphic mapping

$$x'_\alpha = x'_\alpha(x) \qquad (\text{or equivalently} \quad x_\alpha = x_\alpha(x')), \tag{2.7}$$

we may write the relation between them as

$$g(x') = f(x). \tag{2.8}$$

For the sake of simplicity by applying a scale transformation we will set up the following normalization condition for the volume of the manifold

$$\int d^{D+1} x \delta(f(x)) = \int d^{D+1} x \delta(g(x)) \tag{2.9}$$

without loss of generality[1].

[1] Applying the scale transformation $x_\alpha \to \rho x_\alpha$, $p_\alpha \to (1/\rho) p_\alpha$ with $\rho > 0$ and introducing $\bar{g}(x) \equiv g(\rho x)$ we find that (2.1)∼(2.5) remain unchanged under the replacement $g(x) \to \bar{g}(x)$, and we have

$$\int d^{D+1} x \delta(g(x)) = \rho^{D+1} \int d^{D+1} \delta(\bar{g}(x)).$$

Now assuming that there exist operators x_α and p_α ($\alpha = 1, 2, \ldots, D+1$) that satisfy the Dirac algebra on $f(x) = 0$ we introduce a transformation such that

$$\begin{cases} x'_\alpha = x'_\alpha(x), \\ p'_\alpha = \dfrac{1}{2}\left\{(\Lambda'(x')[\partial x/\partial x'])_{\alpha\beta}, p_\beta\right\}, \end{cases} \tag{2.10}$$

where $\Lambda'(x')$ and $[\partial x/\partial x']$ stand for $(D+1)\times(D+1)$-matrices whose (α,β)-elements are given by ${\Lambda'}_{\alpha\beta}(x')$ and $\partial x_\beta/\partial x'_\alpha$, respectively. Similarly the matrices $\Lambda(x)$ and $[\partial x'/\partial x]$ are defined by $(\Lambda(x))_{\alpha\beta} = \Lambda_{\alpha\beta}(x)$ and $[\partial x'/\partial x]_{\alpha\beta} = \partial x'_\beta/\partial x_\alpha$. The equations (2.10) provide us with a variable transformation $(x,p) \to (x',p')$ in the operator form. It must be written in the representation space of x_α and p_α. Then there holds the following:

Theorem. *Given x_α and p_α that satisfy the Dirac algebra on $f(x) = 0$ then the operators x'_α and p'_α defined by (2.10) satisfy the Dirac algebra on $g(x) = 0$ for an arbitrary diffeomorphic mapping described with (2.7) and (2.8).*

Before entering a proof of the Theorem we remark the followings:

1. *Let A, B and C be operators. If $[[C,A],B] = 0$, then*

$$\{A,\{B,C\}\} = \{\{A,B\},C,\}. \tag{2.11}$$

Thus if further $[A,B] = 0$, we have

$$\frac{1}{2}\{A,\{B,C\}\} = \{AB,C\}. \tag{2.12}$$

Proof: omitted.

2. *There holds true the identity*

$$\Lambda(x)[\partial x'/\partial x]\Lambda'(x') = \Lambda(x)[\partial x'/\partial x]. \tag{2.13}$$

Proof: Inserting ${\Lambda'}_{\gamma\beta}(x')$, which is defined by (2.6), into the left hand side (*l.h.s.*) of the above we find

$$\begin{aligned} (\alpha,\beta)\text{-element of } l.h.s. &= (\Lambda(x)[\partial x'/\partial x])_{\alpha\beta} - \frac{1}{R'^2(x')}\Lambda_{\alpha\rho}(x)\frac{\partial x'_\gamma}{\partial x_\rho}g_{,\gamma}(x')g_{,\beta}(x') \\ &= (\Lambda(x)[\partial x'/\partial x])_{\alpha\beta} - \frac{1}{R'^2(x')}\Lambda_{\alpha\rho}(x)f_{,\rho}(x)g_{,\beta}(x') = (\Lambda(x)[\partial x'/\partial x])_{\alpha\beta}, \end{aligned}$$

where use has been made of (2.8) together with $\Lambda_{\alpha\rho}(x)f_{,\rho}(x) = 0$. (*q.e.d.*)

3. *We can uniquely solve the second equation of (2.10) with respect p_α to obtain*

$$p_\alpha = \frac{1}{2}\{(\Lambda(x)[\partial x'/\partial x])_{\alpha\beta}, p'_\beta\}. \tag{2.14}$$

Proof: Taking symmetrized products of $(\Lambda(x)[\partial x'/\partial x])_{\gamma\alpha}/2$ with the both sides of the second equation of (2.10) and making a sum over α we obtain with help of (2.11) and (2.13)

$$\begin{aligned} \frac{1}{2}\{(\Lambda(x)[\partial x'/\partial x])_{\gamma\alpha}, p'_\alpha\} &= \frac{1}{4}\left\{(\Lambda(x)[\partial x'/\partial x])_{\gamma\alpha}, \{(\Lambda'(x')[\partial x/\partial x'])_{\alpha\beta}, p_\beta\}\right\} \\ &= \frac{1}{2}\{(\Lambda(x)[\partial x'/\partial x]\Lambda'(x')[\partial x'/\partial x])_{\gamma\beta}, p_\beta\} \\ &= \frac{1}{2}\{(\Lambda(x)[\partial x'/\partial x][\partial x/\partial x'])_{\gamma\beta}, p_\beta\} = \frac{1}{2}\{\Lambda_{\gamma\beta}(x), p_\beta\} \end{aligned}$$

which reduces to

$$p_\gamma - \frac{1}{2}\left\{\frac{f_{,\gamma}(x)f_{,\beta}(x)}{R^2(x)}, p_\beta\right\} = p_\gamma - \frac{1}{4}\left\{\frac{f_{,\gamma}(x)}{R^2(x)}, \{f_{,\beta}(x), p_\beta\}\right\} = p_\gamma,$$

where we have used (1.8) together with (2.12) and (1.3). Thus we have proved (2.14).

With these preparations we will give a proof of the Theorem. To this end we first examine the constraint (2.2) starting with the Dirac algebra on $f(x) = 0$. Taking the anti-symmetrized products of $g_{,\alpha}(x')$ with the both sides of the second equation of (2.10) we find

$$\begin{aligned}\{g_{,\alpha}(x'), p'_\alpha\} &= \frac{1}{2}\left\{g_{,\alpha}(x'), \{(\Lambda'(x')[\partial x/\partial x'])_{\alpha\beta}, p_\beta\}\right\} \\ &= \left\{g_{,\alpha}(x'){\Lambda'}_{\alpha\gamma}(x')[\partial x/\partial x']_{\gamma\beta}, p_\beta\right\} = 0,\end{aligned}$$

where use has been made of (2.12) and the identity $g_{,\alpha}(x'){\Lambda'}_{\alpha\gamma}(x') = 0$. Thus the constraint (2.4) has been derived

Next to derive (2.6) we make a commutator of x'_α with p'_β. Then from (2.10) we obtain

$$\begin{aligned}[x'_\alpha, p'_\beta] &= \frac{1}{2}\left[x'_\alpha, \{(\Lambda'(x')[\partial x/\partial x'])_{\beta\gamma}, p_\gamma\}\right] \\ &= (\Lambda'(x')[\partial x/\partial x'])_{\beta\gamma}[x'_\alpha, p_\gamma] = i(\Lambda'(x')[\partial x/\partial x'])_{\beta\gamma}[\partial x'/\partial x]_{\rho\alpha}\Lambda_{\rho\gamma}(x) \\ &= i{\Lambda'}_{\alpha\beta}(x') - \frac{i}{R^2(x)}(\Lambda'(x')[\partial x/\partial x'])_{\beta\gamma}f_{,\gamma}(x)f_{,\rho}(x)[\partial x'/\partial x]_{\rho\alpha} \\ &= i{\Lambda'}_{\alpha\beta}(x') - \frac{i}{R^2(x)}{\Lambda'}_{\beta\gamma}(x')g_{,\gamma}(x')f_{,\rho}(x)[\partial x'/\partial x]_{\rho\alpha} = i{\Lambda'}_{\alpha\beta}(x'),\end{aligned}$$

which proves (2.4).

Finally we will derive (2.5). To avoid complications we will proceed in the following way: As seen from (1.5) and (1.6) the commutator $[p'_\alpha, p'_\beta]$ is linear in p_γ's, thereby applying (2.14) we can write it as

$$[p'_\alpha, p'_\beta] = \frac{i}{2}\left\{c^{[\alpha\beta]}_\gamma(x'), p'_\gamma\right\} \tag{2.15}$$

with undetermined functions of x', which have been denoted as $c^{[\alpha\beta]}_\gamma(x')$. Taking the commutators of x'_γ with the both sides of (2.15) we obtain from the left hand side

$$\begin{aligned}[x'_\gamma, [p'_\alpha, p'_\beta]] &= [[x'_\gamma, p'_\alpha], p'_\beta] + [p'_\alpha, [x'_\gamma, p'_\beta]] \\ &= -i\left[\frac{g_{,\gamma}(x')g_{,\alpha}(x')}{R'^2(x')}, p'_\beta\right] + i\left[\frac{g_{,\gamma}(x')g_{,\beta}(x')}{R'^2(x')}, p'_\alpha\right]\end{aligned}$$

by virtue of (2.6), while from the right hand side

$$\left\{c^{[\alpha\beta]}_\rho(x'), [x'_\gamma, p'_\rho]\right\} = -{\Lambda'}_{\gamma\rho}(x')c^{[\alpha\beta]}_\rho(x') = -c^{[\alpha\beta]}_\gamma(x') + \frac{1}{R'^2(x')}g_{,\gamma}(x')g_{,\rho}(x')c^{[\alpha\beta]}_\rho(x').$$

Since the right hand sides of the above two equations are the same we find

$$c^{[\alpha\beta]}_\gamma(x') = \frac{1}{R'^2(x')}g_{,\gamma}(x')g_{,\rho}(x')c^{[\alpha\beta]}_\rho(x') + i\left[\frac{g_{,\gamma}(x')g_{,\alpha}(x')}{R'^2(x')}, p'_\beta\right] - i\left[\frac{g_{,\gamma}(x')g_{,\beta}(x')}{R'^2(x')}, p'_\alpha\right].$$

Then inserting this relation into the right hand side of (2.15) we find

$$[p'_\alpha, p'_\beta] = \frac{i}{2}\left\{\frac{1}{R'^2(x')}c_\rho^{[\alpha\beta]}(x')g_{,\rho}(x')g_{,\gamma}(x'), p'_\gamma\right\}$$
$$-\frac{1}{2}\left(\left\{\left[\frac{1}{R'^2(x')}g_{,\alpha}(x')g_{,\gamma}(x'), p'_\beta\right], p'_\gamma\right\} - (\alpha \leftrightarrow \beta)\right),$$

where the first term of the right hand side is found to vanish owing to (2.12) and (2.2). On the other hand, with the aid of (2.4) we have by direct calculation

$$\left[\frac{1}{R'^2(x')}g_{,\alpha}(x')g_{,\gamma}(x'), p'_\beta\right] = i\Lambda'_{\rho\beta}(x')\frac{\partial}{\partial x'_\rho}\left(\frac{1}{R'^2(x')}g_{,\alpha}(x')g_{,\gamma}(x')\right)$$
$$= \frac{i}{R'^2(x')}\left(g_{,\alpha\beta}(x')g_{,\gamma}(x') - g_{,\alpha}(x')g_{,\beta}(x')g_{,\rho}(x')g_{,\gamma}(x')\frac{\partial}{\partial x'_\rho}\left(\frac{1}{R'^2(x')}\right)\right.$$
$$\left.-\frac{g_{,\alpha}(x')g_{,\beta}(x')g_{,\rho}(x')g_{,\gamma\rho}(x')}{R'^2(x')}\right) + \frac{i}{R'^2(x')}g_{,\alpha}(x')g_{,\beta\gamma}(x')$$
$$-\frac{i}{R'^4(x')}\left(2g_{,\beta\rho}(x')g_{,\alpha}(x')g_{,\rho}(x') + g_{,\alpha\rho}(x')g_{,\rho}(x')g_{,\beta}(x')\right)g_{,\gamma}(x'),$$

which immediately leads to

$$\left[\frac{1}{R'^2(x')}g_{,\alpha}(x')g_{,\gamma}(x'), p'_\beta\right] - (\alpha \leftrightarrow \beta) = \frac{i}{R'^2(x')}\left(g_{,\alpha}(x')g_{,\beta\gamma}(x') - g_{,\beta}(x')g_{,\alpha\gamma}(x')\right)$$
$$-\frac{i}{R'^4(x')}\left(g_{,\alpha}(x')g_{,\beta\rho}(x')g_{,\rho}(x') - g_{,\beta}(x')g_{,\alpha\rho}(x')g_{,\rho}(x')\right)g_{,\gamma}(x').$$

Then taking the anti-commutators of p'_γ with the both sides of the above equation we find the contribution from the second term of the right hand side turns zero due to (2.12) and (2.2), and finally obtain

$$[p'_\alpha, p'_\beta] = -\frac{i}{2}\left\{\frac{1}{R'^2(x')}\left(g_{,\alpha}(x')g_{,\beta\gamma}(x') - g_{,\beta}(x')g_{,\alpha\gamma}(x')\right), p'_\gamma\right\},$$

thereby proving (2.5). Thus we have completed the proof of the Theorem.

It is noted that among diffeomorphic mappings satisfying (2.9) there always exist [2] those which obey the condition

$$d^{D+1}x\delta(f(x)) = d^{D+1}x'\delta(g(x')). \tag{2.16}$$

We call them area-preserving mappings. Eq.(2.6) is of course equivalent to $\delta(f(x)) = \det[\partial x'/\partial x] \times\delta(f(x))$. After normalizing the constraints in a form of (2.9) we will apply this type of mapping[2] under which the transformation of the wave function $\varphi(x)$ is given by

$$\varphi'(x') = \varphi(x). \tag{2.17}$$

Then we are led to the invariance of the inner product of wave functions under the area-preserving mapping, i.e.,

$$\int d^{D+1}x\delta(f(x))\chi'^*(x)\varphi'(x) = \int d^{D+1}x\delta(g(x))\chi^*(x)\varphi(x). \tag{2.18}$$

[2]Physically the existence of the are-preserving mapping under the condition (2.9) could be understood by considering an incompressible fluid which uniformly covers the manifold.

Since, as was mentioned already, the transformation (2.10) has the inverse, the two descriptions based on the respective Dirac algebras on $f(x) = 0$ and $g(x) = 0$ are seen to be equivalent. Thus, if conversely starting with the canonical variables x_α and p_α that satisfy the Dirac algebra on $g(x) = 0$ we will then obtain those on $f(x) = 0$ by applying the inverse transformation of (2.10). It can be written as

$$\begin{cases} x'_\alpha = x'_\alpha(x), \\ p'_\alpha = \dfrac{1}{2}\left\{(\Lambda(x')[\partial x/\partial x'])_{\alpha\beta}, p_\beta\right\}, \end{cases} \tag{2.19}$$

where the first line stands for an area-preserving mapping from the manifold of $g(x) = 0$ to that of $f(x) = 0$ so that it satisfies $f(x') = g(x)$ together with (2.9). It is noted that as seen from the process of deriving (2.14) the transformation (2.19) is uniquely given by (2.10). Furthermore it is also remarkable that owing to (2.18) the irreducible representation space of (x_α, p_α) is found to be the same as that of (x'_α, p'_α), that is, in this case the irreducible representation space of the Dirac algebra is kept unchanged under a smooth deformation of the manifold.

Based on this fact we will determine, in the next section, all possible irreducible representations of the Dirac algebra on $f(x) = 0$. To this end we will use S^D for the manifold $g(x) = 0$ in (2.19), since the irreducible representations of the Dirac algebra on S^D have been known completely [3].

3 Irreducible representations and remarks

The operators p_β in the irreducible representation space of the Dirac algebra on S^D are given by [3]

$$\begin{cases} p_1 = -\dfrac{1}{2}\{x_2, L_{12}\} - \alpha x_2, \\ p_2 = \dfrac{1}{2}\{x_1, L_{12}\} + \alpha x_1 \end{cases} \quad \text{for} \quad D = 1 \tag{3.1}$$

with $0 \le \alpha < 1$, and

$$p_\beta = \frac{1}{2}\{x_\rho, L_{\rho\beta}\} \qquad \text{for} \quad D \ge 2, \tag{3.2}$$

where, in x-diagonal representation, $L_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \dots, D+1)$ are defined by

$$L_{\alpha\beta} \equiv \frac{1}{i}\left(x_\alpha \frac{\partial}{\partial x_\beta} - x_\beta \frac{\partial}{\partial x_\alpha}\right). \tag{3.3}$$

In (3.1) and (3.2) we have assumed the radius of S^D to be 1 for simplicity.

For $D = 1$ the irreducible representations are uniquely specified by α, while for each of $D \ge 2$ we have one and only one irreducible representation. Furthermore it is known [3] that the irreducible representations of the Dirac algebra on S^D are exhausted by the above. Thus inserting p_β in (3.1) and (3.2) into the right hand side of (2.19) we can completely determine all possible irreducible representations of the Dirac algebra on $f(x) = 0$. They are expressed as

$$p'_\beta = \frac{1}{2}\{(\Lambda(x')[\partial x'/\partial x])_{\beta\gamma} x_\rho, L_{\rho\gamma}\} - \alpha(\Lambda(x')[\partial x'/\partial x])_{\beta\gamma} x_\rho \epsilon_{\rho\gamma} \qquad \text{for} \quad D = 1 \tag{3.4}$$

and

$$p'_\beta = \frac{1}{2}\{(\Lambda(x')[\partial x'/\partial x])_{\beta\gamma} x_\rho, L_{\rho\gamma}\} \qquad \text{for} \quad D \ge 2. \tag{3.5}$$

It is to be noted that for $D = 1$ there exist an infinite number of inequivalent irreducible representations corresponding to values of the parameter α, while in the case of $D \geq 2$ the irreducible representation is uniquely given except for unitary equivalent representations.

Finally in concluding the present note we make a few remarks. It has been shown [3] that each of p_β's in (3.1) and (3.2) is a self-adjoint operator. Hence it is obvious that the operators p'_β given by (3.4) and (3.5) are all symmetric (hermitian) as easily seen from (2.18) and (2.19). Perhaps, however, they will be self-adjoint as well, although the proof has not yet been known. Moreover from the arguments made in this note it could be expected that if there exists a representation space of the Dirac algebra on a given manifold it is uniquely determined only by the topology of the manifold.

A detailed study on these problems would highly be desired.

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[3] Ohnuki Y. and Kitakado S., *J. Math. Phys.*, 1993, V.34, 2827; especially see Section VI and Appendix.

Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 357–363.

On Generalization of the Cuntz Algebras

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We study the representations of generalisation of the Cuntz algebra O_n. The algebra $O_{n,\{\alpha_k\}_{k=1}^n}$ is a C^*-algebra generated by isometries $s_1, \ldots, s_n$ such that $\sum\limits_{k=1}^n \alpha_k s_k s_k^* = e$, where $0 < \alpha_k < 1, k = 1, \ldots, n$. The fact that some algebra is $*$-wild implies that the problem of unitary description of all representations of the algebra is very complicated. We show that the algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ is $*$-wild and establish the criterion of $*$-wildness of the algebra $O_{3,\{\alpha_k\}_{k=1}^3}$.

This paper is concerned with the complexity problem of unitary description of representations for C^*-algebras generated by isometries connected with some relation on an infinite-dimensional Hilbert space. These algebras were suggested by Yuriĭ Samoilenko.

The fact that some algebra is $*$-wild implies that the problem of unitary description of all representations is very complicated.

On C^*-algebra $O_n, \{\alpha_k\}_{k=1}^n$

As in [1, 2], we consider a C^*-algebra $\mathfrak{P}_n = \mathfrak{P}(\alpha_1, \ldots, \alpha_n)$ generated by orthoprojectors $p_1, \ldots,$ p_n such that

$$\sum_{k=1}^n \alpha_k p_k = e, \tag{1}$$

where e is the identity of the algebra and $0 < \alpha_k < 1,\ k = 1, \ldots,\ n$. Let us note that the condition $0 < \alpha_k < 1,\ k = 1, \ldots, n$ is not a restriction. It was shown in [6] that it is always possible to reduce the values of the coefficients $\alpha_1, \ldots, \alpha_n$ by a linear change of the variables $p_1, \ldots, p_n$ to this form.

For the same set α_k we deal with the C^*-algebra $O_{n,\{\alpha_k\}_{k=1}^n}$ generated by isometries $s_1, s_2, \ldots,$ s_n such that

$$\sum_{k=1}^n \alpha_k s_k s_k^* = e. \tag{2}$$

For $\alpha_k = 1, k = 1, \ldots, n$ it is the Cuntz algebra O_n. The Cuntz algebra O_n is nuclear, simple and non type I (see [3]).

In this paper we prove that the algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ is $*$-wild and establish the criterion of $*$-wildness of the algebra $O_{3,\{\alpha_k\}_{k=1}^3}$. In [4] we considered $O_{3,\{\alpha_k\}_{k=1}^3}$ when $\alpha_1 + \alpha_2 + \alpha_3 = 2$.

In [5, 6] all irreducible representations of the algebras $\mathfrak{P}_3 = \mathfrak{P}(\alpha_1, \alpha_2, \alpha_3)$ and $\mathfrak{P}_4 = \mathfrak{P}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ were described. The algebra $\mathfrak{P}_3$ has one-dimensional and two-dimensional irreducible representations. All irreducible representations of the algebras $\mathfrak{P}_4$ are finite-dimensional Jacobian matrices. In case $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2$ all irreducible representations of the algebra $\mathfrak{P}_4$ are one-dimensional and two-dimensional. It means that the algebras $\mathfrak{P}_3$ and $\mathfrak{P}_4$ are tame.

For $n \geq 5$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$; $\alpha_4 = \alpha_5 = \beta$, $\alpha + \beta = 1$, the problem of description of the collection of orthoprojectors $\{P_k\}_{k=1}^{5}$ such that

$$\alpha \sum_{k=1}^{3} P_k + \beta \sum_{i=4}^{5} P_i = I, \tag{3}$$

is $*$-wild (see [7]).

On majorization of C^*-algebras

Let us give definitions of majorization of C^*-algebras and $*$-wildness following to [1, 8].

The problem of description of pairs of self-adjoint (or unitary) operators up to unitary equivalence (representations of the $*$-algebra $\mathfrak{S}_2$ (or $\mathfrak{U}_2$) generated by a pair of free self-adjoint (or unitary) generators) was choosen as the standard $*$-wild problem in the theory of $*$-representations in [7].

The problem of unitary classification of representation of pairs of self-adjoint operators contains as a subproblem the problem of unitary classification of representation of any $*$-algebra with a countable number of generators (see [1]).

A problem containing the standard $*$-wild problem is called $*$-wild.

A number of $*$-wild algebras have been studied in recent years ([1, 7, 9]).

Let $\mathfrak{A}$ be a C^*-algebra. We will denote by $\operatorname{Rep}\mathfrak{A}$ the category of representations of $\mathfrak{A}$. The objects of this category are the representations $\mathfrak{A}$ to $L(H)$ (the algebra of linear bounded operators in a Hilbert space H), the morphisms are intertwining operators. Let $\mathfrak{N}$ be a nuclear C^*-subalgebra of $L(H_0)$. Let $\pi : \mathfrak{A} \to L(H)$ be a representation of $\mathfrak{A}$. It induces the representation

$$\tilde{\pi} = \pi \otimes id\colon \mathfrak{A} \otimes \mathfrak{N} \mapsto L(H \otimes H_0)$$

of the algebra $\mathfrak{A} \otimes \mathfrak{N}$.

Definition 1. *We say that a C^*-algebra $\mathfrak{B}$ majorizes a C^*-algebra $\mathfrak{A}$ (and denote it by $\mathfrak{B} \succ \mathfrak{A}$), if there exist a nuclear C^*-algebra $\mathfrak{N}$ and a unital $*$-homomorphism $\psi\colon \mathfrak{B} \mapsto \mathfrak{A} \otimes \mathfrak{N}$ such that the functor $F\colon \operatorname{Rep}\mathfrak{A} \mapsto \operatorname{Rep}\mathfrak{B}$ defined by the following rule:*

$$F(\pi) = \tilde{\pi} \circ \psi \quad \text{for any } \pi \in \operatorname{Rep}\mathfrak{A}, \tag{4}$$

$$F(A) = A \otimes I \quad \text{for any operator } A \text{ intertwining } \pi_1 \text{ and } \pi_2, \tag{5}$$

is full.

Denote by $\pi(\mathfrak{A})'$ a commutant of $\pi(\mathfrak{A})$.

Remark 1. In order to verify whether F is full it is enough to check for any representation $\pi \in \operatorname{Rep}(\mathfrak{A})$ in $L(H)$ that the condition $\mathcal{A} \in F(\pi)(\mathfrak{B})'$ implies $\mathcal{A} = A \otimes I \in \pi(\mathfrak{A})$ and $A \in \pi(\mathfrak{A})'$.

Remark 2. To prove that functor F is full it is enough to show that the $*$-homomorphism ψ is a surjection (see [1]).

The proofs of these remarks see in [1].

Let F_2 denote the free group on two generators u, v. Denote by $C^*(F_2)$ an enveloping C^*-algebra of F_2.

Definition 2. *A C^*-algebra is called $*$-wild if $\mathfrak{A} \succ C^*(F_2)$.*

Let us repeat that the fact that some algebra is $*$-wild implies that the problem of unitary description of all representations is very complicated.

On representations of the algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$

The C^*-algebra $\mathfrak{P}_4 = \mathfrak{P}(1/2, 1/2, 1/2, 1/2)$ has such irreducible representations (see [5]):

1) one-dimensional representation is

$$P_1 = P_2 = I, \qquad P_3 = P_4 = 0;$$

2) two-dimensional representation is

$$P_1 = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \sin^2\phi & -\cos\phi\sin\phi \\ -\cos\phi\sin\phi & \cos^2\phi \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

here $0 < \phi < \pi/2$.

Let us consider the corresponding C^*-algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$.

Theorem 1. *The C^*-algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ is $*$-wild.*

We will prove three lemmas for the proof of this theorem. In accordance by the definition of $*$-wildness to prove $*$-wildness of the algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$, we give a $*$-homomorphism

$$\psi : O_{4,\{\alpha_k=1/2\}_{k=1}^4} \to M_2(C^*(F_2)) \otimes \mathfrak{N}.$$

Here $\mathfrak{N}$ is a nuclear C^*-algebra, $\mathfrak{N} \subset L(H_0)$. As $\mathfrak{N}$ we take the Cuntz algebra

$$O_2 = \mathbb{C}\langle T_1, T_2, T_1^*, T_2^* \mid T_1^*T_1 = T_2^*T_2 = I_0,\ T_1T_1^* + T_2T_2^* = I_0\rangle.$$

We take the operators T_1, T_2 acting in a separable Hilbert space H_0 such that

$$T_1 : e_j \to e_{2j-1}, \qquad T_2 : e_j \to e_{2j}, \tag{6}$$

where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of H_0.

We set

$$\psi(s_1) = S_1 = \begin{pmatrix} (\cos\phi)u & 0 & 0 & 0 & 0 & 0 & \dots \\ (\sin\phi)e & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & (\cos\phi)v & 0 & 0 & 0 & 0 & \dots \\ 0 & (\sin\phi)e & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & (\cos\phi)u & 0 & 0 & 0 & \dots \\ 0 & 0 & (\sin\phi)e & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & (\cos\phi)v & 0 & 0 & \dots \\ 0 & 0 & 0 & (\sin\phi)e & 0 & 0 & \dots \\ \vdots & & \ddots & & \ddots & & \ddots \end{pmatrix},$$

$$\psi(s_2) = S_2 = \begin{pmatrix} (\sin\phi)u & 0 & 0 & 0 & 0 & 0 & \dots \\ -(\cos\phi)e & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & (\sin\phi)v & 0 & 0 & 0 & 0 & \dots \\ 0 & -(\cos\phi)e & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & (\sin\phi)u & 0 & 0 & 0 & \dots \\ 0 & 0 & -(\cos\phi)e & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & (\sin\phi)v & 0 & 0 & \dots \\ 0 & 0 & 0 & -(\cos\phi)e & 0 & 0 & \dots \\ \vdots & & \ddots & & \ddots & & \ddots \end{pmatrix}, \tag{7}$$

$$\psi(s_3) = S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ e & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & e & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & e & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & e & 0 & 0 & \dots \\ \vdots & & \ddots & & \ddots & & \ddots \end{pmatrix},$$

$$\psi(s_4) = S_4 = \begin{pmatrix} e & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & e & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & e & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & e & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & & \ddots & & \ddots & & \ddots \end{pmatrix},$$

here $0 < \phi < \pi/2$.

Lemma 1. *The map ψ defined by (7) is a $*$-homomorphism from $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ to $M_2(C^*(F_2)) \otimes O_2$.*

Proof. It is easy to check that S_1, S_2, S_3, S_4 satisfy the relations of the C^*-algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$.

One can see that the map ψ has the form:

$$S_1 = \begin{pmatrix} (\cos\phi)u & 0 \\ (\sin\phi)e & 0 \end{pmatrix} \otimes T_1 + \begin{pmatrix} 0 & (\cos\phi)v \\ 0 & (\sin\phi)e \end{pmatrix} \otimes T_2,$$

$$S_2 = \begin{pmatrix} (\sin\phi)u & 0 \\ -(\cos\phi)e & 0 \end{pmatrix} \otimes T_1 + \begin{pmatrix} 0 & (\sin\phi)v \\ 0 & -(\cos\phi)e \end{pmatrix} \otimes T_2, \tag{8}$$

$$S_3 = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \otimes T_1 + \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \otimes T_2, \qquad S_4 = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \otimes T_1 + \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \otimes T_2,$$

here T_1, T_2 are the same as in (6). ■

Let us note that $M_2(C^*(F_2)) \otimes O_2 \simeq (C^*(F_2)) \otimes O_2$ because $O_2 \simeq M_2(O_2)$ [10]. Therefore the $*$-homomorphism ψ is the needed homomorphism for the proof of $*$-wildness of the algebra.

Let π be a representation of $C^*(F_2)$ in a Hilbert space $\hat{H}$. Then the map ψ induces the representation $F(\pi)$ of $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ in a Hilbert space H.

Lemma 2. *If $\pi \in \operatorname{Rep} C^*(F_2)$ in $L(\hat{H})$ and $\mathcal{A} \in (F_\psi(\pi)(O_{4\{\alpha_k=1/2\}_{k=1}^4}))'$ then $\mathcal{A} = A \otimes I$ and $A \in \pi(C^*(F_2))'$ (here I is the identity in $L(H)$).*

The proof follows by direct computation.

Lemma 3. *The $*$-homomorphism $\psi : O_{4,\{\alpha_k=1/2\}_{k=1}^4} \to M_2(C^*(F_2)) \otimes O_2$ is a surjection.*

Proof. In the algebra $M_2(C^*(F_2)) \otimes O_2$ we choose the following generators:

$$a_{11} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \otimes I_0, \qquad a_{12} = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \otimes I_0,$$

$$a_{21} = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \otimes I_0, \qquad a_{22} = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \otimes I_0, \tag{9}$$

$$b = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \otimes I_0, \qquad c_1 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \otimes T_1, \qquad c_2 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \otimes T_2.$$

It is easy to see that the linear combinations of the generators a_{11}, a_{12}, a_{21}, a_{22}, b, c_1, c_2 give everywhere dense set of $M_2(C^*(F_2)) \otimes O_2$. The closure by norm gives our C^*-algebra. To prove that ψ is a surjection we point out the elements of the algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ which give the generators of $M_2(C^*(F_2)) \otimes O_2$:

$$S_4S_4^* = a_{11}, \qquad S_4S_3^* = a_{12}, \qquad a_{21} = a_{12}^*, \qquad S_3S_3^* = a_{22},$$

$$S_4^*(S_1 + S_2) = b, \qquad S_4^2S_4^* + S_3S_4S_3^* = c_1, \qquad S_3^2S_3^* + S_4S_3S_4^* = c_2.$$ ■

The proof of Theorem 1 follows from Remark 1 and Lemmas 1, 2. Another proof follows from Remark 2 and Lemmas 1, 3.

The criterion of $*$-wildness of the algebra $O_{3,\{\alpha_k\}_{k=1}^3}$

For the algebras $\mathfrak{P}_3 = \mathfrak{P}(\alpha_1, \alpha_2, \alpha_3)$ all irreducible representations were described in [5]. The irreducible representations of these algebras exist only in the cases:

1) $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $0 < \alpha_k < 1$, $k = 1, 2, 3$, $P_1 = P_3 = P_3 = I$;
2) $\alpha_i \in R\backslash\{1\}$, $\alpha_j + \alpha_k = 1$,$0 < \alpha_j < 1$, $0 < \alpha_k < 1$, here i, j, k are pairwise different integers from the set $\{1, 2, 3\}$, $P_j = P_k = I$; $P_i = I$ if $\alpha_i = 0$ and $P_i = 0$ otherwise;

3) $\alpha_1+\alpha_2+\alpha_3=2$, $0<\alpha_k<1$, $k=1,2,3$, P_1, P_2, P_3 are two-dimensional matrices:

$$\begin{gathered} P_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ P_2=\begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}, \qquad P_3=\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \end{gathered} \tag{10}$$

here

$$\begin{gathered} \cos\phi=\sqrt{\frac{(1-\alpha_2)(\alpha_2+\alpha_3-1)}{\alpha_2(2-\alpha_2-\alpha_3)}}, \qquad \sin\phi=\sqrt{\frac{1-\alpha_3}{\alpha_2(2-\alpha_2-\alpha_3)}}, \\ \cos\theta=\sqrt{\frac{(1-\alpha_3)(\alpha_2+\alpha_3-1)}{\alpha_3(2-\alpha_2-\alpha_3)}}, \qquad \sin\theta=-\sqrt{\frac{1-\alpha_2}{\alpha_3(2-\alpha_2-\alpha_3)}}. \end{gathered} \tag{11}$$

Theorem 2. *The C^*-algebra $O_{3,\{\alpha_1,\alpha_2,\alpha_3\}}$ is $*$-wild if one of the following conditions holds:*

1) $\alpha_1+\alpha_2+\alpha_3=1$, $0<\alpha_k<1$, $k=1,2,3$;

2) $\alpha_i=0$, $\alpha_j+\alpha_k=1$, $0<\alpha_j<1$, $0<\alpha_k<1$, here i, j, k are pairwise different integers from the set $\{1,2,3\}$;

3) $\alpha_j+\alpha_k=1$, $\alpha_i=\alpha_j$ or $\alpha_i=\alpha_k$, $0<\alpha_l<1$, $l=1,2,3$; here i, j, k are pairwise different integers from the set $\{1,2,3\}$.

Proof. One-dimensional representations of the algebra $\mathfrak{P}_3$ exist only when conditions 1, 2, 3 hold. In the first case we set the $*$-homomorphism $\psi: O_{3,\{\alpha_k\}_{k=1}^3}\to C^*(F_2)$ by the following way: $\psi(s_1)=e$, $\psi(s_2)=u$, $\psi(s_3)=v$ and $\psi(s_j)=u$, $\psi(s_k)=v$, $\psi(s_i)=e$ in the second case (here u, v are the generators of $C^*(F_2)$. It is easy to see that the map ψ is a surjection.

In the third case we restrict ourselves the case $\alpha_i=\alpha_j$. We give a $*$-homomorphism $\psi: O_{3,\{\alpha_k\}_{k=1}^3}\to M_2(C^*(F_2))\otimes O_2$ in such a way:

$$\begin{gathered} \psi(s_i)=S_i=\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}\otimes T_1+\begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}\otimes T_2, \\ \psi(s_j)=S_j=\begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}\otimes T_1+\begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}\otimes T_2, \qquad \psi(s_k)=S_k=\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\otimes I_0, \end{gathered} \tag{12}$$

here T_1, T_2 are the same as in (6). It is easy to check that the functor F induced by the $*$-homomorphism ψ is full. Therefore the algebra $O_{3,\{\alpha_1,\alpha_2,\alpha_3\}}$ majorizes $C^*(F_2)$ and is $*$-wild. ■

Theorem 3. *If $\alpha_1+\alpha_2+\alpha_3=2$, $0<\alpha_k<1$, $k=1,2,3$, then the C^*-algebra $O_{3,\{\alpha_1,\alpha_2,\alpha_3\}}$ is $*$-wild.*

Proof. We set the $*$-homomorphism $\psi: O_{3,\{\alpha_k\}_{k=1}^3}\to M_2(C^*(F_2))\otimes O_2$ in such a way:

$$\begin{gathered} \psi(s_1)=S_1=\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}\otimes T_1+\begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}\otimes T_2, \\ \psi(s_2)=S_2=\begin{pmatrix} (\cos\phi)u & 0 \\ (\sin\phi)e & 0 \end{pmatrix}\otimes T_1+\begin{pmatrix} 0 & (\cos\phi)v \\ 0 & (\sin\phi)e \end{pmatrix}\otimes T_2, \\ \psi(s_3)=S_3=\begin{pmatrix} (\cos\theta)u & 0 \\ (\sin\theta)e & 0 \end{pmatrix}\otimes T_1+\begin{pmatrix} 0 & (\cos\theta)v \\ 0 & (\sin\theta)e \end{pmatrix}\otimes T_2, \end{gathered} \tag{13}$$

here T_1, T_2 are the same as in (6), $\cos\phi$, $\sin\phi$, $\cos\theta$, $\sin\theta$ are such as in (11).

It is easy to verify that the functor F generated by the $*$-homomorphism ψ is full. ■

Remark 3. One can see that these theorems together give also the needed conditions of $*$-wildness of $O_{3,\{\alpha_k\}_{k=1}^3}$ since either there are no representations of the corresponding algebra $\mathfrak{P}_3$ for other α_k, $k = 1, 2, 3$ (see [5]) or there are no representations of the algebra $O_{3,\{\alpha_k\}_{k=1}^3}$ (in the case $\alpha_i \neq 0$, if $\alpha_i \neq \alpha_j$ and $\alpha_i \neq \alpha_k$, here i, j, k are pairwise different integers from the set $\{1, 2, 3\}$).

The criterion of $*$-wildness of the C^*-algebra $O_{3,\{\alpha_k\}_{k=1}^3}$ follows from Theorems 2, 3 and Remark 3.

Theorem 4. *The algebra* $O_{3,\{\alpha_k\}_{k=1}^3}$ *is* $*$*-wild if and only if* α_1, α_2, α_3 *satisfy one of the following conditions:*

1) $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $0 < \alpha_k < 1$, $k = 1, 2, 3$;

2) $\alpha_i = 0$, $\alpha_j + \alpha_k = 1$, $0 < \alpha_j < 1$, $0 < \alpha_k < 1$, *here* i, j, k *are pairwise different integers from the set* $\{1, 2, 3\}$;

3) $\alpha_j + \alpha_k = 1$, $\alpha_i = \alpha_j$ *or* $\alpha_i = \alpha_k$, $0 < \alpha_l < 1$, $l = 1, 2, 3$, *here* i, j, k *are pairwise different integers from the set* $\{1, 2, 3\}$;

4) $\alpha_1 + \alpha_2 + \alpha_3 = 2$, $0 < \alpha_k < 1$, $k = 1, 2, 3$.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 364–370.

C^*-Algebras Associated with Quadratic Dynamical System

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In this paper we consider enveloping C^*-algebras of $*$-algebras given by generators and defining relations of the following form $A = \mathbb{C}\langle X, X^* |\ XX^* = f(X^*X)\rangle$, where f is a Hermitian mapping. Some properties of these algebras associated with simple dynamical systems $(f, \mathbb{R})$ are studied. As an example quadratic dynamical systems are considered.

1 Introduction

It is well known that there is close connection between the representation theory of C^*-algebras and structure of dynamical systems $(f(), X)$. In the case when f is one-to-one mapping, the C^*-algebra associated with the transformation group had been studied by many authors, for example by Glimm, Effros and Hahn. The general theory of cross-products of C^*-algebras was elaborated by Doplicher, Kastler and Robinson.

In recent papers (see [8] and references given there in) a special class of $*$-algebras given by generators and relations was considered and some of the results from the theory of cross-product C^*-algebras were transferred into non-bijective settings, which may be important in studying of multi-dimensional non-linear deformation (see [11, 10, 3]), such as Witten's first deformation of $su(2)$, Quesne and Beckers non-linear deformation of $su(2)$ etc. Examples were studied in connection with different quantum deformations of algebras, such as Quantum Unit Disc (Klimek and Lesnievski), one-dimensional q-CCR and their non linear transformation, ets see [7, 8].

Thus, for example, for one-parameter Quantum Unit Disc there corresponds the dynamical system: $f(\lambda) = \frac{(q+\mu)\lambda-\mu}{\mu\lambda+1-\mu}$, where μ is a parameter of deformation, for two-parameter Quantum Unit Disc there corresponds $f(\lambda) = \frac{(q+\mu)\lambda+1-q-\mu}{\mu\lambda+1-\mu}$, for Witten's first deformation of $su(2)$ there corresponds two-dimensional quadratic map $f(x, y) = (p^{-1}(1 + p^{-1}x), g(gy - x + (p - p^{-1})x^2))$, where $g = \pm 1$ depending on the chosen real form and p is a parameter of deformation.

In the present paper we will deal with a one-dimensional polynomial map $f : \mathbb{R} \to \mathbb{R}$ and consider $*$-algebra $\mathcal{A}_f = \mathbb{C}\langle X, X^* |\ XX^* = f(X^*X)\rangle$. Under condition of simplicity of the dynamical system $(f, \mathbb{R})$ we prove that the enveloping C^*-algebra is GCR (type I) C^*-algebra and investigate some other properties. We also discuss the question what relation between dynamical systems $(f_1, \mathbb{R})$ and $(f_2, \mathbb{R})$ corresponds to the isomorphism of enveloping C^*-algebras of $*$-algebras $\mathcal{A}_{f_1}$ and $\mathcal{A}_{f_2}$.

In the last section we consider an example: Unharmonical Quantum Oscillator, i.e. the two-parametric family of $*$-algebras $\mathcal{A}_{a,b} = \mathbb{C}\langle X, X^* |\ XX^* = 1 + aX^*X - b(X^*X)^2\rangle$, where a and b are real parameters with $b > 0$. Some partitioning of parametric domain into parts depending on isomorphism class of C^*-enveloping algebra are given.

2 Simple dynamical systems

For the convenience of the reader we repeat the relevant material from [12, 8] without proofs, thus making our exposition self-contained. By the dynamical system we mean a continuous map $f : \mathbb{R} \to \mathbb{R}$ or $f : I \to I$, where $I \subset \mathbb{R}$ is a closed bounded interval. By the orbit of dynamical system $(f, \mathbb{R})$ we mean a sequence $\delta = (x_k)_{k \in P}$, where P is one of the sets $\mathbb{Z}$, $\mathbb{N}$, such that $f(x_k) = x_{k+1}$. But sometimes we will consider orbit as the set $\{x_k | \ k \in P\}$. The set of all orbits will be denoted by $\mathrm{Orb}(f)$. For $x \in \mathbb{R}$ denote by $\mathcal{O}_+(x)$ the forward orbit, i.e. $(f^k(x))_{k \geq 0}$. For every orbit $\delta \in \mathrm{Orb}(f)$ define $\omega(\delta)$ be the set of accumulation points of forward half-orbit and $\alpha(\delta)$ be the set of accumulation points of backward half-orbit.

By the positive orbit of $(f(), \mathbb{R})$ we mean a sequence $\omega = (x_k)_{k \in \mathbb{Z}}$ such that $f(x_k) = x_{k+1}$ and $x_k > 0$ for all integer k. Unilateral positive orbit is a sequence $\omega = (x_k)_{k \in \mathbb{N}}$ (Fock-orbit) such that $x_1 = 0$ and $f(x_k) = x_{k+1}$, $x_k > 0$ for $k > 1$ or $\omega = (x_{-k})_{k \in \mathbb{N}}$ (anti-Fock-orbit) such that $x_{-1} = 0$ and $f(x_k) = x_{k+1}$, $x_k > 0$ for $k < -1$. Define $\mathrm{Orb}_+(f)$ be the set of all positive orbits which are either periodic (cycles) or contain no cycles. Note that $\omega(\delta) = \emptyset$ for any anti-Fock orbit δ and $\alpha(\delta_1) = \emptyset$ for the Fock orbit δ_1.

Cycle $\beta = \{\beta_1, \ldots, \beta_m\}$ is called attractive if there is a neighborhood U of β such that $f(U) \subseteq U$ and $\cap_{i>0} f^i(U) = \beta$.

Point $x \in \mathbb{R}$ is called non-wandering if for every its neighborhood U there exists a positive integer m such that $f^m(U) \cap U \neq \emptyset$.

Since we will consider only bounded from above functions f and positive orbits we can always consider our dynamical system on closed interval $[0, \sup f]$.

In this paper we will deal with a simple dynamical system which possesses one of the equivalent properties listed in the following theorem:

Theorem 1 ([12], 3.14]). *Let $(f(), I)$ be dynamical system with $f \in C(I)$, ($I \subset \mathbb{R}$ is closed bounded interval). The following conditions are equivalent:*

1) for every $x \in I$ $\omega(x) = \omega(\mathcal{O}_+(x))$ is cycle;

2) $\mathrm{Per}(f)$ is closed;

3) every non-wandering point is periodic.

f is called partially monotone, if I decomposes into a finite union of sub intervals, on which f is monotone.

Let us mention the following statement from [8].

Theorem 2. *Let (f, I) be a dynamical system with partially monotone and continuous f. Then the following conditions are equivalent:*

1) $\mathrm{Per}(f)$ is closed;

2) for some positive integer m the relation $\mathrm{Fix}(f^{2^{m+1}}) = \mathrm{Fix}(f^{2^m})$ holds;

3) any quasi-invariant ergodic measure is concentrated on a single element of the trajectory decomposition.

The class of dynamical systems which satisfies equivalent conditions 1–3 of Theorem 2 is denoted by $\mathcal{F}_{2^m}$. Let us note that when $\mathrm{Per}(f)$ is closed Theorem ([12], 3.12) implies that the length of every cycle is a power of 2 and there no homoclinical orbits (i.e. orbit δ such that $\alpha(\delta) = \omega(\delta)$ is a cycle).

We will need the following lemma:

Lemma 1. *Let $(f, \mathbb{R})$ be simple dynamical system with bounded f and the set of periodic points which are not the points of attractive cycles, i.e. the set $[0, \sup f] \cap \mathrm{Per}(f) \setminus \cup_{\beta\text{-attractive cycle}} \beta$ be finite then for every orbit $\delta \in \mathrm{Orb}_+(f)$ the α-boundary $\alpha(\delta)$ is cycle which is not attractive.*

Proof. 1. Let us show that every α-boundary point is non-wandering: if $x \in \alpha(\delta)$ then for arbitrary $\epsilon > 0$ and positive integer n there is $y \in B_\epsilon(x)$ and integer $l \geq n$ such that $f^l(y) \in B_\epsilon(x)$. Indeed, if $\delta = (x_k)_{k\in\mathbb{Z}}$ then there is subsequence $x_{-n_k} \to x$. For a given $\epsilon > 0$ we can find an integer k_0 such that $x_{-n_k} \in B_\epsilon(x)$ for all $k \geq k_0$. Take $k_1 > k_0$ such that $n_{k_1} \geq n_{k_0} + n$ and put $y = x_{-n_{k_1}} \in B_\epsilon(x)$ and $l = n_{k_1} - n_{k_0} \geq n$. Then $f^l(y) = x_{-n_{k_0}} \in B_\epsilon(x)$.

2. Since for a simple dynamical system every non-wandering point is periodic we obtain that $\alpha(f) = \cup_{\delta\in \mathrm{Orb}_+(f)}\alpha(\delta)$ is contained in $\mathrm{Per}(f) \cap [0, \sup f]$.

3. Let β be an attractive cycle and assume that $\beta \subseteq \alpha(\delta)$, then $\alpha(\delta) = \beta$. Indeed, let $\beta_1 \in \alpha(\delta)$ be another cycle, then there is $\epsilon > 0$ such that $\beta_1 \cap B_\epsilon(\beta) = \emptyset$. Since β is an attractive cycle there is $\eta > 0$ and $\eta < \epsilon$ such that for arbitrary $y \in B_\eta(\beta)$ we have $\mathcal{O}_+(y) \subseteq B_\epsilon(\beta)$. Let $\delta = (x_k)_{k\in\mathbb{Z}}$. Since $\beta_1 \in \alpha(\delta)$ there is a positive integer k_0 such that $x_{-k_0} \in B_{\epsilon_1}(\beta_1)$, where $\epsilon_1 > 0$ chosen such that $B_{\epsilon_1}(\beta_1) \cap B_\epsilon(\beta) = \emptyset$. But $\beta \in \alpha(\delta)$ so there is a positive integer $k_1 > k_0$ with the property $x_{-k_1} \in B_\eta(\beta)$ then $\mathcal{O}_+(x_{-k_1}) \subseteq B_\epsilon(\beta)$ and, obviously, $x_{-k_0} \in \mathcal{O}_+(x_{-k_1})$. This is a contradiction. Thus we have proved that $\alpha(\delta) = \beta$. But this implies that $\alpha(\delta) = \omega(\delta) = \beta$. So δ is a homoclinical orbit and so $(f, \mathbb{R})$ is not simple. Which is a contradiction. Thus $\alpha(\delta)$ has no attractive cycles.

4. Let us prove that $\alpha(\delta)$ is a single cycle for every orbit δ. We already know that $\alpha(\delta) \subset [0, \sup f] \cap \mathrm{Per}(f) \setminus \cup_{\beta\text{-attractive cycle}}\beta$. Since the last one is a finite set there is $\epsilon > 0$ such that for arbitrary distinct cycles $\beta_1, \beta_2 \in \alpha(\delta)$ we have $B_\epsilon(\beta_1) \cap B_\epsilon(\beta_2) = \emptyset$ and $f(B_\epsilon(\beta_1)) \cap B_\epsilon(\beta_2) = \emptyset$ (the last is a possible since $f(\beta_1) = \beta_1$ and so $f(\beta_1) \cap \beta_2 = \emptyset$). There is a positive integer n such that $x_{-k} \in B_\epsilon(\alpha(\delta))$ for every $k > n$. Thus for every cycle $\beta \in \alpha(\delta)$ there is positive $\epsilon_1 < \epsilon$ such that $f(y) \in B_{\epsilon_1}(\beta_1)$ for every $y \in B_{\epsilon_1}(\beta)$, where $\beta_1 \neq \beta$ (in the opposite case would be $\alpha(\delta) = \beta$). This contradicts $f(B_\epsilon(\beta)) \cap B_\epsilon(\beta_1) = \emptyset$. ■

3 Representation theory of $*$-algebras associated with $\mathcal{F}_{2^m}$ dynamical systems

The following theorem is due to Samoilenko and Ostrovskii [8].

Theorem 3. *Let f be partially monotone continuous map and $(f, \mathbb{R})$ be $\mathcal{F}_{2^m}$ dynamical system. Let $A = \mathbb{C}\langle X, X^* |\ XX^* = f(X^*X)\rangle$ be corresponding $*$-algebra.*

1. To every positive non-cyclic orbit $\omega(x_k)_{k\in Z}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(Z)$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k \in Z$ and $X = UC$ is a polar decomposition.

2. To positive Fock-orbit $\omega = (x_k)_{k\in N}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(N)$ given by the formulae: $Ue_0 = 0$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$.

3. To positive anti-Fock-orbit $\omega = (x_{-k})_{k\in N}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(N)$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$.

4. To cyclic positive orbit $\omega = (x_k)_{k\in N}$ of length m there corresponds a family of m-dimensional irreducible representation $\pi_{\omega,\phi}$ in Hilbert space $l_2(\{1, \dots, m\})$ given by the formulae: $Ue_0 = e^{i\phi}e_{m-1}$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k = 1, \dots, m$; $0 \leq \phi \leq 2\pi$ and $X = UC$.

This is a complete list of unequivalent irreducible representation of a given $*$-algebra.

4 Enveloping C^*-algebra

Let f be a bounded from above Hermitian polynomial (hence f is always partially monotone and continuous). Let $A_f = \mathbb{C}\langle X, X^* | \; XX^* = f(X^*X)\rangle$ be $*$-algebra given by generators and relations which has at least one representation. Let $C = \sup f$. Then for any representation π of $*$-algebra A_f we have $\|X\| \le \sqrt{C}$. Thus there is an enveloping C^*-algebra, which we denote by $\mathcal{E}_f$. Let us note that by Theorem 3.3 [12] for $f \in C^1(I, I)$ simplicity of a dynamical system is equivalent to $(f, I) \in \mathcal{F}_{2^m}$ for some integer m.

Theorem 4. *Let a dynamical system $(f, \mathbb{R})$ be simple and $\delta \in \mathrm{Orb}_+(f)$.*

1. If δ is non-cyclic bilateral orbit than $C^*(\pi_\delta) = Z \times_\delta C(\overline{\delta})$ is a cross-product of C^*-algebra, where $\overline{\delta} = \delta \cup \omega(\delta) \cup \alpha(\delta)$.

The set of irreducible representation $\mathrm{Irr}(C^*(\pi_\delta))$ is $\pi_\delta, \pi_{\omega(\delta),\phi}, \pi_{\alpha(\delta),\phi}$, where $0 \le \phi \le 2\pi$.

2. Assume that 0 is not a periodic point. If δ is a Fock-orbit then $C^*(\pi_\delta) \cong M_m(\mathcal{T}(C(\mathbb{T})))$ is a matrix algebra of dimension $m =| \omega(\delta) |$ over C^*-algebra $\mathcal{T}(C(\mathbb{T}))$ of the Toeplitz operators.

The same is true for anti-Fock orbit with $m =| \alpha(\delta) |$.

Proof. Let $\pi \in \mathrm{Irr}(C^*(\pi_\delta))$ then $\sigma(\pi(C^2)) \subseteq \sigma(\pi_\delta(C^2)) = \delta \cup \omega(\delta) \cup \alpha(\delta)$. Since every irreducible representation of $C^*(\pi_\delta)$ is also an irreducible representation of $*$-algebra $\mathcal{A}$ there is orbit δ' such that $\pi = \pi_{\delta'}$. Then we will have $\delta' \subseteq \delta \cup \omega(\delta) \cup \alpha(\delta)$. Since no cycle can be properly contained in an orbit δ' we conclude that $\delta' = \delta$ or $\delta' = \omega(\delta)$ or $\delta' = \alpha(\delta)$. So π must be one of the representations listed in the theorem. Let us prove that all of them are actually representations of the algebra $C^*(\pi_\delta)$. Let $A = \mathrm{diag}(\ldots, x_{-1}, x_0, x_1, \ldots)$ be diagonal operator in Hilbert space $l_2(\mathbb{Z})$ with orthonormal basis $\{e_k\}$, where $\delta = (\ldots, x_1, x_0, x_1, \ldots)$. Let U be a bilateral shift operator $Ue_k = e_{k-1}$. The equality $UAU^* = f(A)$ implies $UAU^* \in C^*(A)$. Since $B = U^*AU$ is a diagonal operator $Be_k = x_{k+1}e_k$ and the mapping $x_k \to x_{k+1}$ is mutually continuous on the closure of δ(which is $\sigma(A)$) $B \in C^*(A)$. Thus the mapping $\rho(D) = UDU^*$ is a automorphism of $C^*(A)$. Let us prove that $C^*(\pi(\delta))$ is a cross-product C^*-algebra. Consider the linear subspace $L_t = \{\sum_{-m}^{n} A_i u^i e_t | A_i \in C^*(A);\; m, n \ge 0\}$. Then L_t is dense in H and L_t is isomorphic to a dense subspace in $L_2(\mathbb{Z}, \mathbb{C})$ via isomorphism $\sum A_i u^i e_t \to f()$, where $f(i) = (A_i e_{t+i}, e_{t+i})$. Direct computations show that $A(\sum \alpha_i e_i) = \sum \alpha_i \phi_t(U^{-i}AU^{*-i})$, where $\phi_t(D) = (De_t, e_t)$ for all $D \in C^*(A)$ is a one-dimensional representation of $C^*(A)$. Thus π_δ is a regular representation, λ_ϕ, associated with the representation ϕ_t. Since $\phi = \oplus_{t\in\mathbb{Z}}\phi_t$ is a faithful representation of $C^*(A)$ we conclude that λ_ϕ is faithful on the cross-product $\mathbb{Z} \times_\rho C^*(A)$ (see [9], Theorems 7.7.5 and 7.7.7). Since all representations λ_{ϕ_t} are isomorphic to π_δ we conclude that $C^*(\pi_\delta)$ is $\mathbb{Z} \times_\rho C^*(A)$.

Consider the case of unilateral orbits. Since point 0 is not periodic and the dynamical system is simple we conclude that for every orbit δ there is $\eta > 0$ such that $\delta \subseteq [\eta, \sup f]$, i.e. δ is separeted from zero. Let $\delta = (x_k)_{k\in\mathbb{N}}$ be the Fock orbit. Then $\pi_\delta(X)$ is a weighted shift operator with all weights separated from zero. If $X = UC$ is polar decomposition of X then $U, C \in C^*(X)$ and the algebra of compact operators $\mathcal{K} \subseteq C^*(X)$ (see [1], Lemma 2.1). We know that $\omega(\delta) = (y_k)_{k\in\mathbb{N}}$ is periodic orbit and $x_k - y_k \to 0$. By Theorem 3 $C = \mathrm{diag}(\sqrt{x_k})$. Let us put $C_1 = \mathrm{diag}(\sqrt{y_k})$ then since $C - C_1 \in \mathcal{K}$ and $\mathcal{K} \subseteq C^*(U, C)$ and $\mathcal{K} \subseteq C^*(U, C_1)$ we conclude that $C^*(U, C) = C^*(U, C_1)$ as operator algebras. If $m =| \omega(\delta) |$ then $C^*(U, C_1) = C^*(UC_1)$ is an algebra generated by m-periodic weighted shift. It is known that C^*-algebra generated by all m-periodic weighted shifts in a given separable Hilbert space is isomorphic to $\mathcal{T}(C(\mathbb{T}))$ (the Toeplitz operators) and there is m-periodic shift which generate this algebra. But if $D = \mathrm{diag}(d_k)$ and $D_1 = \mathrm{diag}(d^1_k)$ are diagonal operators with m-periodic coefficients and m is least possible, then

$C^*(U,D) = C^*(U,D_1)$ (since the map $g: \ d_k \to d_k^1$ is continuous on $\sigma(D)$ and by functional calculus $f(D) = D_1$ and obviously, $f(UDU^*) = Uf(D)U^*$). From these facts follows that $C^*(X) = \mathcal{T}(C(\mathbb{T}))$. For anti-Fock orbits arguments are the same. ■

Define support of the dynamical system $(f,\mathbb{R})$ to be the union $X = \bigcup_{\delta\in \mathrm{Orb}_+(f)} \delta$ and the finite support X_{fin} to be union of positive cycles.

Theorem 5. *If a dynamical system $(f(),\mathbb{R})$ is simple then the C^*-algebra $\mathcal{E}_f$ is GCR (type I C^*-algebra), and the finite spectrum is homeomorphic to $X_{\mathrm{fin}}/\sim$, where $\sim$ is an orbit equivalence relation.*

Proof. First let us show that the finite-dimensional spectrum $\mathrm{Irr}(\mathcal{E}_f) \simeq (X_{\mathrm{fin}}/\ ) \times S^1$ (X_{fin} is finite and so compact set). Indeed, it is obvious that $f: X_{\mathrm{fin}} \to X_{\mathrm{fin}}$ is one-to-one map. Thus we can apply the results from [5]. If $\delta \in \mathrm{Orb}_+(f)$ is a bilateral non-periodic orbit then by previous theorem, the set of irreducible representation $\mathrm{Irr}(C^*(\pi_\delta))$ is $\pi_\delta, \pi_{\omega(\delta),\phi}, \pi_{\alpha(\delta),\phi}$, where $0 \le \phi \le 2\pi$. Since we know the topology on finite dimensional representations we conclude that $\mathrm{Irr}(C^*(\pi_\delta))$ is T_0 space. Hence $C^*(\pi_\delta)$ is GCR-C^*-algebra. It is known that $\mathcal{T}(C(\mathbb{T}))$ is also a GCR algebra. Cyclic orbits generate finite-dimensional and so GCR algberas. Hence $\mathcal{E}_f$ is GCR C^*-algebra. ■

The question of isomorphism of enveloping C^*-algebras may turn to be very difficult. Even in one-to-one case there are only fragmentary results in this direction, for example it is known that for minimal dynamical systems on Cantor sets the isomorphism of cross-product C^*-algebras equivalent to orbit-equivalence of corresponding dynamical systems (with condition on K_0-groups) see [4], Theorem 4. However, in particular “discrete” case we have the following:

Theorem 6. *If dynamical systems $(f_1,\mathbb{R})$ and $(f_2,\mathbb{R})$ are simple and for every non-cyclic orbit $\delta \in \mathrm{Orb}_+(f_k)$ there exists a point $x \in \delta$ isolated in the support space of a dynamical system $(f_k,\mathbb{R})$, then $\mathcal{E}_{f_1} \cong \mathcal{E}_{f_2}$ if and only if there is a one-to-one map $\phi : \mathrm{Orb}_+(f_1)/\sim \to \mathrm{Orb}_+(f_2)/\sim$, such that $|\phi(\delta)| = |\delta|$ and $\phi(\omega(\delta)) = \omega(\phi(\delta))$, $\phi(\alpha(\delta)) = \alpha(\phi(\delta))$. Moreover in this case the topology on $\mathrm{Irr}(\mathcal{E}_{f_k})$ is given by its base consisting of closed sets $\{\pi_\delta, \pi_{\omega(\delta),\phi}, \pi_{\alpha(\delta),\phi} | \phi \in S^1\}$, where $\delta \in \mathrm{Orb}_+(f_k)\backslash \mathrm{Cyc}(f)$ and $\{\pi_{\beta,\phi} | \phi \in M\}$, where β is a positive cycle and M is a closed subset in S^1. Thus $\mathcal{E}_{f_1} \cong \mathcal{E}_{f_2}$ if and only if their dual spaces are homeomorphic.*

Proof. Let $\psi : \mathcal{E}(f_1) \to \mathcal{E}(f_2)$ be an isomorphism. Then ψ induces a homeomorphism of spectra spaces $\psi^* : \mathcal{E}(\hat{f_2}) \to \mathcal{E}(\hat{f_1})$, i.e the spaces of irreducible representation with Jacobson's topology. We know that $\mathcal{E}(\hat{f_1})$ can be identified with $\mathrm{Orb}_+(f_1)/$. With this identification we have one-to-one map $\psi^* : \mathrm{Orb}_+(f_2) \to \mathrm{Orb}_(f_1)$. Let $\delta \in \mathrm{Orb}_+(f_2)$. Since ψ^* is homeomorphism $\psi^*(\overline{\pi_\delta}) = \overline{\psi^*(\pi)}$. As we know that $\overline{\pi_\delta} = \{\pi_{\omega(\delta),\phi}, \pi_{\alpha(\delta),\phi}\}$ we have proved necessity of conditions of the theorem.

Let ω_1 be a non-attractive cycle or an empty set and ω_2 be non-repellent cycles or empty sets. Denote $\Omega_{\omega_1}^{\omega_2}(f_1) = \{\delta \in \mathrm{Orb}_+(f_1) | \alpha(\delta) = \omega_1,\ \omega(\delta) = \omega_2\}$. Then $\mathrm{Orb}_+(f_1)$ is the disjoint union of these sets. For all $\delta \in \Omega_{\omega_1}^{\omega_2}(f_1)$ we will realize the corresponding representation π_δ in the same Hilbert space $H_{\omega_1}^{\omega_2}$. Consider an atomic representation of $\mathcal{E}(f_1)$ which is realized in Hilbert space $H = \otimes_{\omega_1,\omega_2} (H_{\omega_1}^{\omega_2})^{\otimes n(\omega_1,\omega_2)}$, where $n(\omega_1,\omega_2) = |\Omega_{\omega_1}^{\omega_2}(f_1)|$. Then $\mathcal{E}(f_1)$ is isomorphic to the algebra generated by the diagonal operator $C = \mathrm{diag}(C_{\omega_1,\omega_2})$, where $C_{\omega_1,\omega_2} = \mathrm{diag}(C_\delta | \delta \in \Omega_{\omega_1,\omega_2})$ and $C_\delta = \pi_\delta((XX^*)^{1/2})$ and block-diagonal with respect to direct sum decomposition of H operator $U = \mathrm{diag}(I \otimes U_{\omega_1,\omega_2})$. Discreteness of the dynamical system implies that all block-diagonal with respect to an expanded direct sum decomposition $H = \oplus_{\omega_1,\omega_2} \oplus_{n(\omega_1,\omega_2)} H_{\omega_1}^{\omega_2}$ compact operators belong to $\mathcal{E}(f_1)$. We will denote this subalgebra of compact operators by $\mathcal{K}_1$. Modulo this compact operators C is $C_{\mathrm{normal}} = \mathrm{diag}(I \otimes A_{\omega_1,\omega_2})$, where $A_{\omega_1,\omega_2} e_k = x_k^1$ if $k < 0$ and $A_{\omega_1,\omega_2} e_k = x_k^2$ if $k > 0$ and $\omega_1 = (x_{-k}^1)_{k\in\mathbb{N}}$ and $\omega_2 = (x_k^2)_{k\in\mathbb{N}}$ regarded as periodic orbit.

Moreover, it is obvious that $C^*(C_{\text{normal}}, U, \mathcal{K}_1) = C^*(C, U)$. If there is ϕ which satisfies all conditions of the theorem then we can consider $\mathcal{E}(f_1)$ and $\mathcal{E}(f_2)$ in the same Hilbert space and using functional calculus obtain $\phi^*(C_{\text{normal}}(f_1)) = C_{\text{normal}}(f_2)$ (where ϕ^* is continuous map $\text{Per}(f_1)_+ \to \text{Per}(f_2)_+$ which is lifting of ϕ). and so $\mathcal{E}(f_1) = \mathcal{E}(f_2)$ as operator algebras. This completes the proof. ■

5 Quadratic dynamical system

In this section we consider an example of one-dimensional quadratic dynamical system. Let $f_{a,b}(x) = 1 + ax - bx^2$ with $\{a, b\} \in \mathbb{R}$ and $b > 0$ to provide boundedness. Since when $a < 0$ dynamical system is one-to-one on $\mathbb{R}_+$ (and so all irreducible representations are one-dimensional) we assume that $a > 0$. This dynamical system is conjugated to $f_\mu(x) = \mu x(1 - x)$, where $\mu = 1 + \sqrt{a^2 - 2a + 1 + 4b}$. The values of parameter μ when bifurcations of cycles of one parametric family $\{f_\mu\}$ occur are given in [12]. However a conjugacy relation does not preserve positiveness, i.e. $\text{Orb}_+(f_{a,b})$ may not map into $\text{Orb}_+(f_\mu)$.

If (a, b) belong to domain $D = \{(a, b)|\ b < \frac{1}{2} - \frac{a^2}{4} + \frac{a}{2} + \frac{\sqrt{1+2a}}{2}\}$ bounded by curve G (see Fig. 1) then for every $x \in [0; \sup f_{a,b}]$ $\mathcal{O}_+(x) \subset [0; \sup f_{a,b}]$. Thus for such (a, b) algebra $A_{a,b}$ has Fock representation and as it easily can be shown has no anti-Fock representations. In the complement of D algebra $A_{a,b}$ has anti-Fock representations.

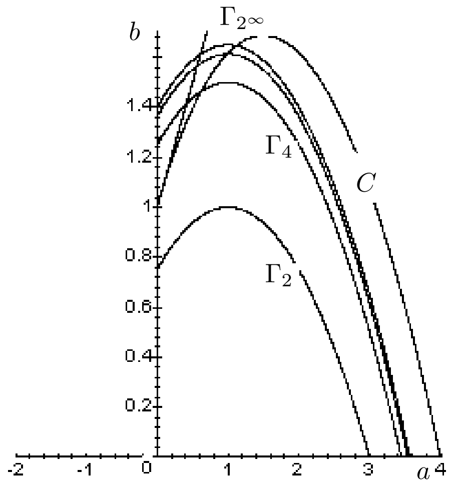

Figure 1.

Proposition 1. *If (a, b) belong to domain $P_1 = \{(a, b)|\ b < 1 - \frac{(a-1)^2}{4}\}$ bounded by curve Γ_2 (see picture) then $\mathcal{E}_{a,b}$ has one dimensional and Fock irreducible representations only. Moreover $\mathcal{E}_{a,b} \simeq \mathcal{T}(C(\mathbb{T}))$.*

Proof. For $(a, b) \in P_1$ dynamical system has two fix point $\beta_+ > 0$, $\beta_- < 0$ but has no other cycles.

Let us show that $\text{Orb}_+(f) = \{\beta_+,\ \delta_1\}$, where δ_1 is the Fock orbit. If $\delta \in \text{Orb}_+(f)$ and $\delta \neq \beta_+$, $\delta \neq \delta_1$ then $\alpha(\delta)$ is a cycle which cannot be the attractive point β_+ (see Lemma 1). Hence $\alpha(\delta) = \beta_- < 0$ which is contradiction.

Since $P_1 \subset D$ then δ_1 is positive orbit. And Theorem 4 implies that $\mathcal{E}_{a,b} \simeq \mathcal{T}(C(\mathbb{T}))$. ■

Proposition 2. *Let $P_2 = \{(a, b)|\ 1 - \frac{(a-1)^2}{4} < b < -\frac{a^2}{4} + \frac{a}{2} + \frac{5}{4}\}$ bounded by curves Γ_2 and Γ_4. Domain P_2 is divided into three domains P_2^1, P_2^2, P_2^3: $P_2^1 = P_2 \cap D$; $P_2^2 = \{(a, b) \in P_2 \setminus D|\ b < a + 1\}$; $P_2^3 = \{(a, b) \in P_2 \setminus D|\ b > a + 1\}$.*

Then for $(a,b) \in P_2^1$ C^*-algebra $\mathcal{E}_{a,b}$ has the family of one-, two-dimensional and the Fock irreducible representations, but has no anti-Fock representation. For $(a,b) \in P_2^2$ C^*-algebra $\mathcal{E}_{a,b}$ has one-, two-dimensional and anti-Fock irreducible representations, but has no Fock representation. For $(a,b) \in P_2^3$ C^*-algebra $\mathcal{E}_{a,b}$ has the family of one-dimensional representations and anti-Fock irreducible representations, but has no two-dimensional and Fock representations.

For (a,b) from domain bounded by curves Γ_4 and Γ_8 the dynamical system has 4-cycle, 2-cycle, two fix points and no other cycles.

The curve $\Gamma_{2^\infty} \approx 1.651225 - \frac{(a-1)^2}{4}$ separates the domain, where the dynamical system is simple from the one (including Γ_{2^∞}), where the dynamical system is not simple.

Acknowledgments

The authors are greatly indebted to Prof. Yu.S. Samoilenko for suggesting the problem and many stimulating conversations, and wish to thank Profs. Yu.L. Maistrenko and V.V. Fedorenko for helpful comments concerning the theory of dynamical systems.

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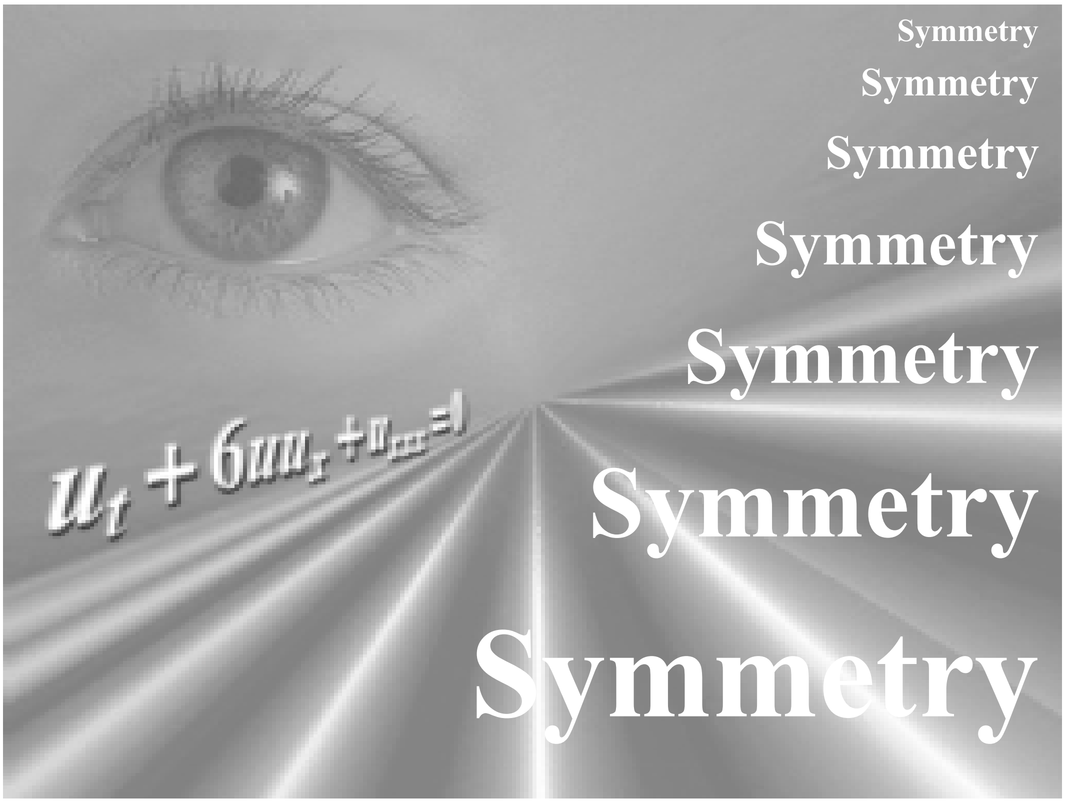
Related Problems of Mathematical Physics



Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 373–381.

On Asymptotic Decompositions for Solutions of Systems of Differential Equations in the Case of Multiple Roots of the Characteristic Equation

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This paper is a surwey of results on asymptotic expansions of solutions of linear systems $\varepsilon\frac{dx}{dt} = A(t)x$, when the roots of matrix $A(t)$ are multiple. The results, obtained by the author, as well as by other mathematicians, are briefly reviewed, and some open problems are listed.

1 A short historical review

The paper presents the results on investigation of linear differential systems with coefficients depending on "slow" time $\tau = \varepsilon t$ ($\varepsilon > 0$ is a small parameter). Fundamental results on investigation of such systems were obtained by S.F. Feshchenko, S.G. Krein, Yu.L. Daletskii, I.Z. Shtokalo, I.M. Rapoport. The works of these authors appeared under direct influence of asymptotic methods developed by N.M. Krylov, N.N. Bogoliubov, Yu.A. Mitropolskii.

The sources of construction of asymptotic decompositions for solutions of systems of differential equations containing a parameter, can be found in the papers by Liouville, Birkhoff, Schlesinger, Tamarkin.

In particular, Liouville considered the issue of decomposition of arbitrary functions on fundamental functions of the equation

$$\frac{d^2y}{dx^2} + (\lambda q(x) - r(x))y = 0. \tag{1.1}$$

The fundamental functions obtained by Liouville for the equation (1.1) in the case of large values of the parameter λ, possess the property of orthogonality. For this reason the form of decomposition of a given function with respect to fundamental functions of the equation (1.1) can be determined directly. It is necessary only to show that 1) the constructed series converges; 2) it represents the given function. Liouville showed the convergence of the series by means of asymptotic formulae for fundamental functions that he obtained. The proof of the statement 2 was obtained by means of certain Sturm's results.

After the papers of Sturm and Liouville the theory of asymptotic representation of functions begun to develop quickly.

However, all these studies were concerned with self-conjugate differential equations. These limitations were removed in the investigations by Schlesinger, Birkhoff, Tamarkin.

Birkhoff considered construction of an asymptotic solution for the differential equation

$$\frac{d^ny}{dx^n} + \rho a_{n-1}(x,\rho)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + \rho^n a_0(x,\rho)y = 0, \tag{1.2}$$

where $a_i(x,\rho)$, $(i = 0, 1, \ldots, n-1)$ are analytical functions with respect to the complex parameter ρ on infifnity and have derivatives of all orders by real variable $x \in [a; b]$. Unlike Schlesinger who proved the asymptotic property of solutions only on some fixed ray $\arg\rho = \alpha$ for large $|\rho|$, Birkhoff proves the same properties for the area $\theta < \arg\rho < \psi$.

Tamarkin generalized Birkhoff's results for systems of linear differential equations

$$\frac{dy_i}{dx} = \sum_{k=1}^{n} a_{ik}(x,\rho) y_k, \qquad i = 1,\ldots,n, \tag{1.3}$$

where $a_{ik}(x,\rho)$ are single-value functions of complex parameter ρ, analytical near the point $\rho = \infty$ but having singularities with $\rho = \infty$ (a pole of the order $r \geq 1$). The asymptotic expressions for solutions of the system (1.3), derived by Birkhoff, contain as particular cases similar formulae, established by other methods by Schlesinger for systems of the form (1.3) and by Birkhoff for a differential equation of the order n (the latter considered the case $r = 1$).

In 1936 the paper by Trzitzinsky appeared where he gave a complete exposure of the issue of asymptotic representation for solutions of systems of ordinary differential equations with generalization of the Schlesinger–Birkhoff–Tamarkin theory for the case of linear integral-differential equations.

During the period of 1940–1945 a series of V.S. Pugachiov's papers appeared in which, unlike the previous researchers, the author presented the asymptotic representation for solutions in more general form.

We can also speak about papers by G.L. Turritin and M. Hukuchara as papers on asymptotic issues, where the asymptotic decomposition of a system of linear differential equations, with coefficients depending on a parameter, into lower-order systems.

At the end of a short historical review of classical papers on asymptotic representation for solutions of linear differential equations, we shall note that these methods were comprehensively and fruitfully developed in the following. The extensive lists of references related to these investigations are given in the books [1, 2].

As we have mentioned above, under the influence of asymptotic methods of Krylov–Bogoliubov–Mitropolskii the investigations on linear differential equations containing a small parameter in a singular way, started to develop extensively.

S.F. Feshchenko obtained the first results in this direction in 1948–1949. For the equation

$$\frac{d^2y}{dt^2} + \varepsilon\rho(\tau,\varepsilon)\frac{dy}{dt} + q(\tau,\varepsilon)y = \varepsilon f(\tau,\varepsilon)\cdot e^{i\theta(t,\varepsilon)}, \tag{1.4}$$

where $\rho(\tau,\varepsilon)$, $q(\tau,\varepsilon)$, $f(\tau,\varepsilon)$ are slowly changing functions, allowing decomposition by degress of the small parameter ε. The case when the function $\nu(\tau)$ $\left(\nu(\tau) = \frac{d\theta(t,\varepsilon)}{dt}\right)$ with certain τ from the area of its variation coincides with one of the simple roots of the characteristic equation, constructed for the equation (1.4) was considered, that is very important from mathematical physics applications perspective, and also from the theoretical side. This case was named "resonance" by the author.

The theorems proved by S.F. Feshchenko allow to construct an asymptotic solution for the equation (1.4) in the "resonance" and "non-resonance" (when $\nu(\tau)$ for any τ does not coincide with any root of the characteristic equation) cases.

The similar theorems were obtained by S.F. Feshchenko for the system of linear differential equations of the form (1.4).

Then S.F. Feshchenko obtained very important results on asymptotic decomposition of systems of linear differential equations of the form

$$\frac{dx}{dt} = A(\tau,\varepsilon)x, \tag{1.5}$$

where x is an n-dimensional vector, $A(\tau,\varepsilon)$ is real square matrix of the order n allowing the representation

$$A(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_s(\tau). \tag{1.6}$$

In particular he proved the following theorems.

Theorem 1.1. *Let us assume that the roots of the characteristic equation*

$$\det \|A_0(\tau) - \lambda \cdot E\| = 0 \tag{1.7}$$

(E is a unit matrix) can be splitted into two groups $\lambda_1(\tau),\ldots,\lambda_r(\tau)$ and $\lambda_{r+1}(\tau),\ldots,\lambda_n(\tau)$ so that no root from the first group for all $\tau \in [0;L]$ is equal to roots from the second group. Then, if $A(\tau,\varepsilon)$ on the interval $[0,L]$ has derivatives on τ of all orders, the system of differential equations (1.5) has a formal solution of the form

$$x = U_1(\tau,\varepsilon)\xi_1 + U_2(\tau,\varepsilon)\xi_2, \tag{1.8}$$

where $U_1(\tau,\varepsilon)$, $U_2(\tau,\varepsilon)$ are rectangular matrices of the size correspondingly $(n \times r)$, $(n \times n-r)$ and ξ_1 is an r-dimensional vector, ξ_2 is a $n-r$-dimensional vector, determined by systems of differential equations

$$\frac{d\xi_1}{dt} = W_1(\tau,\varepsilon)\xi_1, \qquad \frac{d\xi_2}{dt} = W_2(\tau,\varepsilon)\xi_2 \tag{1.9}$$

of the order correspondingly r and $n-r$.

Theorem 1.2. *If $A(\tau,\varepsilon)$ satisfies the conditions of the theorem 1.1 and eigenvalues of the matrices*

$$\Delta_i(\tau) = \frac{1}{2}\left(W_i(\tau) + W_i^*(\tau)\right), \qquad i = 1,2,$$

where $W_1(\tau)$, $W_2(\tau)$ are diagonal cells of the matrix $T^{-1}(\tau)A_0(\tau)T(\tau)$ ($T(\tau)$ is a matrix of transformation, $T^{-1}(\tau)$ is the inverse of $T(\tau)$), $W_1^*(\tau)$, $W_2^*(\tau)$ are matrices conjugate respectively to the matrices $W_1(\tau)$, $W_2(\tau)$ and are non-positive, then for any $L > 0$ and $0 < \varepsilon \le \varepsilon_0$ it is possible to find such constant $c > 0$ not depending on ε, that if only $x\,|_{t=0} = x_m\,|_{t=0}$ (x_m is an m-approximation), then

$$\|x - x_m\| \le \varepsilon^m c. \tag{1.10}$$

Using of Theorems 1.1, 1.2 it is possible to asymptotically lower the order of the system (1.5). In particular, if all roots of the equation (1.7) are distinct at the interval $[0,L]$, then these theorems allow to obtain an asymptotic solution for the system (1.5).

However, by means of theorems on asymptotic decomposition it is possible mainly only to lower the order of the initial system. In the case of multiple roots of the characteristic equation it is impossible to get a solution of the initial system differential equations by means of these theorems. Though this case is frequently encountered both in investigation of theoretical issues and in solution of practical problems. Even in investigation of one of the simplest equations – the Sturm–Liouville equation – we encounter a multiple root. These roots are also encountered in investigation of systems of differential equations with a small parameter at certain derivatives in the problems of optimal control. Let us note that the case of multiple roots, especially when multiple elementary divisors correspond to multiple roots, is rather complicated. It is the consequence of the fact that the initial system of differential equations in general does not have solutions allowing decomposition by integer degrees of the parameter ε. Such solutions, unlike the case of simple roots, are represented by formal series by different fractional orders of this parameter, and these orders depend not only on multiplicity of a root of the characteristic equation, but also on corresponding elementary divisors and on some relations among coefficients of the system under consideration.

The case of multiple roots of the characteristic equation was comprehensively studied by M.I. Shkil. These results are partially presented in the following paragraphs.

2 Asymptotic decomposition in the case of multiple roots of the characteristic equation

Let us consider the system of the form (1.5). We assume that the characteristic equation (1.7) has at least one root $\lambda = \lambda_0(\tau)$ of the constant multiplicity $k, (2 \leq k < n)$, with the corresponding elementary divisor of the same multiplicity.

Theorem 2.1. *If $A(\tau, \varepsilon)$ has at the interval $[0; L]$ derivatives by τ of all orders and the matrix*

$$C(\tau) = T^{-1}(\tau) \left(\frac{dT(\tau)}{d\tau} - A_1(\tau) \cdot T(\tau) \right), \tag{2.1}$$

where $T(\tau)$ is the matrix transforming $A_0(\tau)$ to the Jordan form, and $T^{-1}(\tau)$ is inverse of $T(\tau)$, such that for every $\tau \in [0; L]$ its element

$$c_{k1} \neq 0, \tag{2.2}$$

then the system of differential equations (1.5) has a formal solution of the form

$$x = u(\tau, \mu) \exp \left(\int_0^t \lambda(\tau, \mu) dt \right), \tag{2.3}$$

where an n-dimensional vector $u(\tau, \mu)$ and a scalar function $\lambda(\tau, \mu)$ allow decompositions

$$u(\tau, \mu) = \sum_{S=0}^{\infty} \mu^s u_s(\tau), \qquad \lambda(\tau, \mu) = \lambda_0(\tau) + \sum_{S=1}^{\infty} \mu^s \lambda_s(\tau), \tag{2.4}$$

where

$$\mu = \varepsilon^{1/k}. \tag{2.5}$$

Let us note that if $c_{k1}(\tau) \equiv 0$ but at the same time $c_{k-1,1}(\tau) + c_{k2}(\tau) \neq 0$, then the initial system has a formal solution of the form (2.3), where $u(\tau, \mu)$, $\lambda(\tau, \mu)$ can be represented by formal series by degrees of the parameter $\mu = \varepsilon^{\frac{1}{k-1}}$.

Let us adduce a more general result.

Let the following conditions be fulfilled:

1) the matrix $A(\tau, \varepsilon)$ has derivatives by τ of all orders at the interval $[0, L]$;

2) the characteristic equation (1.7) has one root of constant multiplicity k;

3) there are $r \geq 1$ elementary divisors, corresponding to the root $\lambda_0(\tau)$, of the form

$$(\lambda - \lambda_0(\tau))^{k_1}, \ldots, (\lambda - \lambda_0(\tau))^{k_r};$$

4) one of the following conditions is satisfied:

a) $k_1 = k_2 = \cdots = k_r = k$, *b)* $k_1 > k_2 > \cdots > k_r$.

Then for the case a) the following theorem is true:

Theorem 2.2. *If the conditions 1)–4) are fulfilled, then for the vector*

$$x = u(\tau,\mu)\exp\left(\int_0^t \lambda(\tau,\mu)dt\right), \tag{2.6}$$

where an n-dimensional vector $u(\tau,\mu)$ and a scalar function $\lambda(\tau,\mu)$ can be represented by formal series of the form

$$u(\tau,\mu) = \sum_{S=0}^{\infty} \mu^s u_s(\tau), \qquad \lambda(\tau,\mu) = \sum_{S=0}^{\infty} \mu^s \lambda_s(\tau), \tag{2.7}$$

where $\mu = \varepsilon^{\frac{1}{k}}$, to be a formal vector solution of the system (1.5), it is necessary and sufficient that the function $(\lambda_1(\tau))^k$ for every $\tau \in [0;L]$ be a root of the equation

$$\det \begin{Vmatrix} \rho + c_{k1}(\tau) & c_{k\,k+1}(\tau) & \cdots & c_{k\,l_{r-1}+1}(\tau) \\ c_{2\,k1}(\tau) & \rho + c_{2k\,k+1}(\tau) & \cdots & c_{2k\,l_{r-1}+1}(\tau) \\ \vdots & \vdots & \cdots & \vdots \\ c_{n1}(\tau) & c_{n\,k+1}(\tau) & \cdots & \rho + c_{n\,l_{r-1}+1}(\tau) \end{Vmatrix} = 0, \tag{2.8}$$

where $c_{k1}(\tau), \ldots, c_{n\,l_{r-1}+1}(\tau)$, $l_{r-1} = (r-1)k$ are elements of the matrix (2.1).

Let us note that the proof of the sufficient condition of this theorem simultaneously gives a method for construction of coefficients of the formal series (2.7).

The similar theorem is true for the case b). It was proved also that for the both cases formal solutions are asymptotic decompositions by the parameter ε of the true solutions of the system (1.5).

3 Turning points

The theorems, adduced in the Section 1.2, hold true under the condition that the roots of the characteristic equation and the corresponding elementary divisors preserve the constant multiplicity for all $\tau \in [0;L]$. If these conditions are violated (turning points appear, see [4]), then the construction of asymptotic solutions for solutions of the systems under study is rather difficult. Some results for the cases with turning points were obtained only for one second-order differential equation [3], and for systems of two second-order differential equations [4]. Investigation of this case by different authors was carried out with the use of Airy functions or by reduction of the differential equations under study to certain model equations. One of these equations is e.g. the Airy equation. More details can be found in the book [4].

The author of the present paper was the first to attempt constructing of formal decompositions in elementary functions for solutions of the systems of differential equations (1.5) [5].

Theorem 3.1. *Let the following conditions be fulfilled for the system of differential equations (1.5):*

1. *The matrix $A(\tau,\varepsilon)$ admits a decomposition*

$$A(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_s(\tau).$$

2. *The matrices $A_s(\tau)$ $(s = 0, 1, \ldots)$ are infinitely differentiable at the interval* $[0, L]$.

3. There exists such integer number $k \geq 1$ that the roots of the equation

$$\det \|A_0(\tau) + \varepsilon A_1(\tau) + \cdots + \varepsilon^k A_k(\tau) - \lambda E\| = 0 \tag{3.1}$$

are simple for all $\tau \in [0; L]$.

Then there exists a formal vector that is a solution of the system (1.5) such as

$$x(\tau, \varepsilon) = U(\tau, \varepsilon) \exp\left(\frac{1}{\varepsilon}\int\limits_0^\tau \Lambda(\sigma, \varepsilon) d\sigma\right) \cdot a, \tag{3.2}$$

where $U(\tau, \varepsilon)$ is $(n \times n)$-matrix which allows a formal decomposition

$$U(\tau, \varepsilon) = \sum_{S=0}^{\infty} \varepsilon^s U_s(\tau, \varepsilon), \tag{3.3}$$

$\Lambda(\tau, \varepsilon)$ is a diagonal matrix, constructed of the roots of the equation (1.7)*, a is a constant n-dimensional vector.*

Let us note that unlike formal decompositions, adduced in the paragraph 1.2, coefficients in the decomposition (3.3) depend on ε, what presents considerable difficulties for investigation of asymptotic properties of these decompositions. Some results in this direction were obtained in the papers of the author and his students [6, 7].

4 Simplification of formal decompositions. Problems

The proof of existence of formal solution for the system (1.5) can be simplified considerably by means of consideration of another algebraic equation, related to the system (1.5). However in this case new and rather difficult problems appear, related to substantiation of asymptotic properties of formal solutions, obtained by means of this method. We will illustrate the above statements by consideration of the simplified system of the form

$$\frac{dx}{dt} = A(\tau)x, \tag{4.1}$$

where $n \times n$-matrix $A(\tau)$ is differentiable sufficient number of times at the interval $[0, L]$ ($\tau = \varepsilon t$, $\varepsilon > 0$ is a small parameter).

We shall assume that the characteristic equation

$$\det \|A(\tau) - \lambda E\| = 0 \tag{4.2}$$

at the interval $[0, L]$ has only one identically multiple root $\lambda = \lambda_0(\tau)$ of the multiplicity n, with corresponding elementary divisor of the same multiplicity.

Then by means of the substitution

$$x = V(\tau)y, \tag{4.3}$$

where $V(\tau)$ is a matrix, reducing the matrix $A(\tau)$ to the Jordan form, the system (4.1) can be reduced to the form

$$\frac{dy}{dt} = B(\tau, \varepsilon)y, \tag{4.4}$$

where

$$B(\tau, \varepsilon) = W(\tau) - \varepsilon V^{-1}(\tau)V'(\tau), \tag{4.5}$$

$W(\tau)$ is a Jordan cell, corresponding to the root $\lambda_0(\tau)$, $V^{-1}(\tau)$ is the inverse of $V(\tau)$, $V'(\tau)$ is a derivative of $V(\tau)$.

Let us construct an equation

$$\det \|B(\tau,\varepsilon) - \rho E\| = 0. \tag{4.6}$$

We will assume that the roots $\rho_1(\tau,\varepsilon),\dots,\rho_n(\tau,\varepsilon)$ of equation (4.6) are simple for $\forall\, x \in [0;L]$ and $\forall\, \varepsilon \in (0;\varepsilon_0]$, or that

$$\rho_i(\tau,\varepsilon) \neq \rho_j(\tau,\varepsilon), \qquad i \neq j, \quad \forall\, i,j = \overline{1,n}. \tag{4.7}$$

Then making the following substitution in the system (4.4)

$$y = U_m(\tau,\varepsilon,\varepsilon)z, \qquad U_m(\tau,\varepsilon,\varepsilon) = \sum_{S=0}^{m} \varepsilon^s U_s(\tau,\varepsilon), \tag{4.8}$$

($m \geq 1$ is a natural number) and defining the matrices $U_s(\tau,\varepsilon)$ ($s = \overline{0,m}$) by means of the method [2], we arrive at the system of differential equations of the form

$$U_m(\tau,\varepsilon,\varepsilon)\frac{dz}{dt} = U_m(\tau,\varepsilon,\varepsilon)\left(\Lambda_m(\tau,\varepsilon,\varepsilon) + \varepsilon^{m+1}C_m(\tau,\varepsilon)\right)z, \tag{4.9}$$

where a diagonal matrix

$$\Lambda_m(\tau,\varepsilon,\varepsilon) = \sum_{S=0}^{m} \varepsilon^s \Lambda_s(\tau,\varepsilon), \tag{4.10}$$

and an $n \times n$-matrix $C_m(\tau,\varepsilon)$ are determined by means of the formulae from [2].

Let for all $\tau \in [0;L]$ and for a sufficiently small $\varepsilon \in (0;\varepsilon_0]$ the following conditions are fulfilled:

1. The matrix $U_m(\tau,\varepsilon,\varepsilon)$ is non-singular. Then the system (4.9) can be written in the form

$$\frac{dz}{dt} = \left(\Lambda_m(\tau,\varepsilon,\varepsilon) + \varepsilon^{m+1}C_m(\tau,\varepsilon)\right)z, \tag{4.11}$$

2. $\operatorname{Re}(\rho_j(\tau,\varepsilon,\varepsilon))_{j=\overline{1,n}} \leq 0$,

$$C_m(\tau,\varepsilon) = O\left(\varepsilon^{-\alpha}\right) \qquad (\varepsilon \to 0), \tag{4.12}$$

where $0 \leq \alpha < m$, then the system (4.11) can be integrated by means of the method of sequential approximations (conditions 2 ensure the applicability of this method). Whence for the vector z we obtain an asymptotic formula by the parameter ε ($\varepsilon \to 0$):

$$z = \exp\left(\frac{1}{\varepsilon}\int_0^\tau \sum_{s=0}^{m} \varepsilon^s \Lambda_s(\sigma,\varepsilon)d\sigma\right) a + O\left(\varepsilon^{m-\alpha}\right), \tag{4.13}$$

where a is a constant n-dimensional vector.

3. The matrix $U_m(\tau,\varepsilon,\varepsilon)$ is limited by the norm. Then using (4.3), (4.8), (4.13) we obtain an asymptotic formula for the vector x.

Finally we note that the formula (4.13) was obtained in assumption of conditions 1–2, where coefficients of the system (4.1) do not appear explicitly. The following question a rises:

What should be the requirements for the matrix $A(\tau)$ for the conditions 1–3 to be fulfilled? The answer for this question presents the problems mentioned at the beginning of the Section 4.

The solution of these problems requires further research.

Example. Let us consider a scalar equation

$$\frac{d^2x}{dt^2} + \varepsilon p(\tau)x = 0, \tag{4.14}$$

where $p(\tau) \neq 0$ at the interval $[0; L]$ and has continuous derivatives up to the second order.

The equation (4.14) can be represented in the form of the system (4.4) where $y = (y_1, y_2)$ $\left(y_1 = x,\ y_2 = \frac{dx}{dt}\right)$ is a two-dimensional vector, $B(\tau, \varepsilon)$ is a square matrix of the form

$$B(\tau, \varepsilon) = \begin{Vmatrix} 0 & 1 \\ -\varepsilon p(\tau) & 0 \end{Vmatrix}. \tag{4.15}$$

Then according to the assumption the equation has simple roots

$$\rho_1(\tau, \varepsilon) = \sqrt{-\varepsilon p(\tau)}, \qquad \rho_2(\tau, \varepsilon) = -\sqrt{-\varepsilon p(\tau)}. \tag{4.16}$$

We apply the transformation (4.8) to the system (4.4) with the matrix (4.15), putting $m = 1$. Then we obtain a system of the form (4.9) where the matrices $U_1(\tau, \varepsilon, \varepsilon)$, $\Lambda_1(\tau, \varepsilon, \varepsilon)$ are:

$$\begin{gathered} U_1(\tau, \varepsilon, \varepsilon) = \begin{Vmatrix} 1 + \dfrac{\varepsilon p'(\tau)}{8p(\tau)\sqrt{-\varepsilon p(\tau)}} & 1 - \dfrac{\varepsilon p'(\tau)}{8p(\tau)\sqrt{-\varepsilon p(\tau)}} \\ \sqrt{-\varepsilon p(\tau)} - \dfrac{\varepsilon p'(\tau)}{8p(\tau)} & -\dfrac{\varepsilon p'(\tau)}{8p(\tau)} \end{Vmatrix}, \\ \Lambda_1(\tau, \varepsilon, \varepsilon) = \operatorname{diag}\left(\sqrt{-\varepsilon p(\tau)} + \frac{\varepsilon p'(\tau)}{4p(\tau)},\ -\sqrt{-\varepsilon p(\tau)} + \frac{\varepsilon p'(\tau)}{4p(\tau)}\right), \\ U_1^{-1}(\tau, \varepsilon, \varepsilon) = \frac{1}{a(\tau, \varepsilon)} \begin{Vmatrix} -\dfrac{\varepsilon p'(\tau)}{8p(\tau)} & \dfrac{\varepsilon p'(\tau)}{8p(\tau)\sqrt{-\varepsilon p(\tau)}} - 1 \\ \dfrac{\varepsilon p'(\tau)}{8p(\tau)} - \sqrt{-\varepsilon p(\tau)} & 1 + \dfrac{\varepsilon p'(\tau)}{8p(\tau)\sqrt{-\varepsilon p(\tau)}} \end{Vmatrix}, \\ a(\tau, \varepsilon) = \frac{\varepsilon p'(\tau)}{8p(\tau)} - \sqrt{-\varepsilon p(\tau)} - \frac{\varepsilon^2 (p'(\tau))^2}{32p^2(\tau)\sqrt{-\varepsilon p(\tau)}}. \end{gathered} \tag{4.17}$$

The matrix $C_1(\tau, \varepsilon)$ is determined by means of the formula

$$C_1(\tau, \varepsilon) = -U_1^{-1}(\tau, \varepsilon, \varepsilon)(U_1(\tau, \varepsilon)\Lambda_1(\tau, \varepsilon) + U_1'(\tau, \varepsilon)), \tag{4.18}$$

where

$$\begin{gathered} U_1(\tau, \varepsilon) = \frac{p'(\tau)}{8p(\tau)\sqrt{-\varepsilon p(\tau)}} \begin{Vmatrix} 1 & -1 \\ -\sqrt{-\varepsilon p(\tau)} & -\sqrt{-\varepsilon p(\tau)} \end{Vmatrix}, \\ \Lambda_1(\tau, \varepsilon) = \operatorname{diag}\left(\frac{p'(\tau)}{4p(\tau)}, \frac{p'(\tau)}{4p(\tau)}\right). \end{gathered} \tag{4.19}$$

The direct computation of the elements of the matrix $C_1(\tau, \varepsilon)$ (we will omit it, as it is very cumbersome) shows that they have the order $O\left(\varepsilon^{-\frac{1}{2}}\right)$ in the neighbourhood of the point $\varepsilon = 0$ for all $\tau \in [0; L]$. So, having required the fulfilment of the condition

$$\operatorname{Re} \Lambda_1(\tau, \varepsilon, \varepsilon) \leq 0 \tag{4.20}$$

(this condition will be satisfied when the function $p(\tau) > 0$ for $\forall\ \tau \in [0; L]$), we get the following asymptotic formula for the vector z:

$$z = \exp\left(\frac{1}{\varepsilon}\int\limits_0^\tau \Lambda_1(\sigma, \varepsilon, \varepsilon)d\sigma\right) a + O\left(\varepsilon^{\frac{1}{2}}\right), \tag{4.21}$$

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 382–386.

On Asymptotic Formulae for Solutions of Differential Equations with Summable Coefficients

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Using the technique of asymptotic expansions, we prove the theorem about the form of system $\frac{dx}{dt} = A(t)x$, when the eigenvalues of $A(t)$ are multiple and the corresponding elementary divisors have constant multiplicity.

In this paper we consider a system of differential equations

$$\frac{dx}{dt} = A(t)x, \qquad t \ge t_0 \tag{1}$$

in the case when eigenvalues of the matrix $A(t)$ are multiple and the corresponding elementary divisors are of constant multiplicity.

The problem of asymptotic behaviour of solutions of the system (1) was partially solved by I.M. Rappoport [1] and M.I. Shkil [2] by reducing it to the generalized L-diagonal form

$$\frac{dx}{dt} = (\Lambda(t) + C(t))x, \tag{2}$$

where $\Lambda(t) = \operatorname{diag}\{W_1(t), \ldots, W_m(t)\}$, $W_k(t) = ||w_{ij}(t)||_1^{n_k}$, $w_{ii}(t) = w_i(t)$, $w_{ii+1}(t) = 1$, $w_{ij}(t) = 0$ $(i \ne j,\ j \ne i+1)$.

In particular I.M. Rappoport, having imposed some restrictions on elements of the matrices $\Lambda(t)$ and $C(t)$ [1], obtained asymptotic formulae for solutions of the system (2). He also adduced the simplest substitutions by means of which the system (1) can be reduced to the system (2). M.I. Shkil and his students in investigation of asymptotic properties of the system (1) used the previously developed methods of asymptotic integration for systems of differential equations with slowly changing coefficients

$$\frac{dx}{dt} = A(\tau, \varepsilon)x,$$

where

$$A(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_s(\tau), \qquad \tau \in [0; L], \quad \tau = \varepsilon t$$

and ϵ is a small parameter, in the case of multiple eigenvalues of the matrix $A_0(\tau)$. Here constant multiplicity elementary divisors of $A_0(\tau)$ caused substantial difficulties.

In this paper we suggest a method for construction of asymptotic solutions of the system (1) by reducing it to the L-diagonal form

$$\frac{dx}{dt} = (\Lambda(t) + C(t))x, \tag{3}$$

where $\Lambda(t) = \operatorname{diag}\{\lambda_1(t), \ldots, \lambda_n(t)\}$.

It follows from conditions imposed on the matrix $A(t)$ that there exist non-degenerate for all $t \geq t_0$ matrix $T(t)$ such that

$$T^{-1}(t)A(t)T(t) = \Lambda(t),$$

where $\Lambda(t)$ is a Jordan matrix corresponding to the matrix $A(t)$. In the system (1) we set

$$x = T(t)y$$

and multiply the obtained system by $T^{-1}(t)$. We get

$$\frac{dy}{dt} = \left(\Lambda(t) - T^{-1}T^{'}(t)\right) y. \tag{4}$$

In the following we will assume that the matrix $D(t) = \Lambda(t) - T^{-1}T^{'}(t)$ has simple eigenvalues at the interval $[t_0; +\infty)$. Instead of the system (4) we will consider the system

$$\varepsilon \frac{dy}{dt} = D(t)y, \tag{5}$$

where $\varepsilon > 0$ is a real parameter (the system (5) with $\varepsilon = 1$ coincides with system (4)). When we assume

$$y = U_m(t,\varepsilon)z, \qquad U_m(t,\varepsilon) = \sum_{s=0}^{m} \varepsilon^s U_s(t),$$

where z is an n-dementional vector and $U_s(t)$ are square matrices of the order n, we obtain

$$\varepsilon U_m(t,\varepsilon)\frac{dz}{dt} = (D(t)U_m(t,\varepsilon) - \varepsilon U_m'(t,\varepsilon))z. \tag{6}$$

We will construct the matrices $U_s(t)$ $(s = 0, 1, \ldots, m)$ for the following matrix equality to be satisfied:

$$D(t)U_m(t,\varepsilon) - \varepsilon U_m'(t,\varepsilon) = U_m(t,\varepsilon)\left(\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}C_m(t,\varepsilon)\right), \tag{7}$$

where $\Lambda_m(t,\varepsilon)$ is diagonal matrix of the form $\Lambda_m(t,\varepsilon) = \sum\limits_{s=0}^{m} \varepsilon^s \Lambda_s(t)$ and $C_m(t,\varepsilon)$ is a square matrix of the order n to be determined.

Matrices $U_m(t,\varepsilon), \Lambda_m(t,\varepsilon)$ we will determine from the equality (7) where we require coefficients at $\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^m$ to be equal. Then we obtain the following system of matrix equations

$$D(t)U_0(t) - U_0(t)\Lambda_0(t) = 0, \tag{8}$$

$$D(t)U_s(t) - U_s(t)\Lambda_0(t) = U_{s-1}'(t) + \sum_{j=1}^{s} U_{s-j}(t)\Lambda_j(t). \tag{9}$$

Let us write the matrix equation (8) in the vector form. For this purpose we designate the columns of the matrix $U_0(t)$ as $u_{0i}(t)$ $(i = 1, 2, \ldots, n)$ and write the matrix $\Lambda_0(t)$ in the form

$$\Lambda_0(t) = \operatorname{diag}\{\lambda_1(t), \lambda_2(t) \ldots, \lambda_n(t)\},$$

where $\lambda_i(t)$ $(i = 1, 2, \ldots, n)$ are eigenvalues of the matrix $D(t)$. Then we obtain the following system of equations from (8):

$$(D(t) - \lambda_i(t)E)u_{0i}(t) = 0. \tag{10}$$

Thus, if $\mu_i(t)$ $(i = 1, 2, \ldots, n)$ are eigenvectors of the matrix $D(t)$, we can set

$$u_{0i}(t) = \mu_i(t).$$

Let us note that we showed in [3] that it is possible to construct $\mu_i(t)$ $(i = 1, 2, \ldots, n)$ so that

$$(\mu_i(t), \psi_j(t)) = \begin{cases} 1, & i = j; \\ 0, & i \neq j, \ i, j = 1, \ldots, n. \end{cases}$$

where $\psi_j(t)$ $(j = 1, 2, \ldots, n)$ are elements of the zero space of the matrices $(D(t) - \lambda_j(t)E)^*$.

Let us consider the system of matrix equations (9) with $s = 1$, having written it in the vector form

$$(D(t) - \lambda_i(t)E)u_{1i}(t) = g_{1i}(t), \qquad i = 1, 2, \ldots, n, \tag{11}$$

where $u_{1i}(t)$ are columns of the matrix $U_1(t)$ and vector $g_{1i}(t)$ is determined as follows:

$$g_{1i}(t) = u'_{0i}(t) + u_{0i}(t)\lambda_{1i}(t). \tag{12}$$

The equation (11) is solvable with respect to $u_{1i}(t)$ if and only if the vector $g_{1i}(t)$ $(i = 1, 2, \ldots, n)$ is orthogonal to all vectors that are solutions of the corresponding homogeneous associated system. Thus, for the system (11) to have a solution, it is necessary and sufficient that for all $t \geq t_0$ the following equality is satisfied:

$$(g_{1i}(t), \psi_i(t)) = 0, \qquad i = 1, 2, \ldots, n.$$

Substituting to the latter equation the value of the vector $g_{1i}(t)$, we obtain a scalar equation with respect to $\lambda_{1i}(t)$

$$(\mu'_i(t), \psi_i(t)) + (\mu_i(t)\lambda_{1i}(t), \psi_i(t)) = 0.$$

Whence we get that

$$\lambda_{1i}(t) = -(\mu'_i(t), \psi_i(t)), \qquad i = 1, 2, \ldots, n. \tag{13}$$

Therefore we can set

$$\Lambda_1(t) = \operatorname{diag}\{\lambda_{11}(t), \lambda_{12}(t), \ldots, \lambda_{1n}(t)\}.$$

Then, substituting the values of $\lambda_{1i}(t)$ into (11), we get a system that has a solution with respect to the vector $u_{1i}(t)$. We will look for this solution in the form

$$u_{1i}(t) = \sum_{r=1}^{n} c_{ri}^{(1)}(t)\mu_r(t), \qquad i = 1, 2, \ldots, n, \tag{14}$$

where $c_{ri}^{(1)}(t)$ are functions that have to be determined for the vector (14) to satisfy the system (11). For this purpose we substitute (14) into the system (11) and scalarly multiply the obtained equality by the vector $\psi_j(t)$ $(j = 1, 2, \ldots, n)$. We obtain

$$c_{ji}^{(1)}(t)(\lambda_j(t) - \lambda_i(t)) = (g_{1i}(t), \psi_j(t)), \qquad j = 1, 2, \ldots, n.$$

When $i = j$ we obtain the equality

$$c_{jj}^{(1)}(t) \cdot 0 \equiv 0.$$

Thus we can take an arbitrary function $c_{jj}^{(1)}(t)$, e.g.

$$c_{jj}^{(1)}(t) \equiv 0, \qquad t \geq t_0.$$

In case when $i \neq j$ we get that

$$c_{ji}^{(1)}(t) = \frac{(g_{1i}(t), \psi_j(t))}{\lambda_j(t) - \lambda_i(t)}.$$

Then the vector $u_{1i}(t)$ is as follows:

$$u_{1i}(t) = \sum_{r=1, r\neq i}^{n} \frac{(g_{1i}(t), \psi_r(t))}{\lambda_r(t) - \lambda_i(t)} \mu_r(t).$$

So we determined the matrices $U_1(t)$ and $\Lambda_1(t)$.

Using the method of mathematical induction we can show that similarly it is possible to find all further matrices $U_s(t)$ and $\Lambda_s(t)$ $(s = 2, 3, \ldots, n)$. from the equations (9) [2, 3].

Let us proceed to the finding of the matrix $C_m(t, \varepsilon)$.

Taking into account that arbitrary elements of the matrix $U_m(t, \varepsilon)$ can be chosen so that [3]

$$\det U_m(t, \varepsilon) \neq 0, \qquad t \geq t_0,$$

then we get from the system (7) with $\varepsilon = 1$

$$C_m(t, 1) = -U_m^{-1}(t, 1) \left(U_m'(t) + \sum_{k=1}^{m} \sum_{j=k}^{m} U_j(t) \Lambda_{m+k-j}(t) \right).$$

Thus, the system (4) is reduced to the system of the form

$$\frac{dz}{dt} = (\Lambda_m(t, 1) + C_m(t, 1))z. \tag{15}$$

If also

a) neither of the differences

$$\operatorname{Re} \lambda_i(t, 1) - \operatorname{Re} \lambda_j(t, 1) \tag{16}$$

changes the sign for all $t \geq t_1 \geq t_0$, where $\lambda_i(t, 1)$ $(i = 1, 2, \ldots, n)$ are diagonal elements of the matrix $\Lambda_m(t, 1)$;

b)

$$\int_{t_0}^{\infty} ||C_m(t, 1)|| dt < \infty, \tag{17}$$

then for the vector $x_s(t)$, which is solution of the system (1), we have the formula

$$x_s(t) = \mu_{sj}(t) \exp \int_{t_0}^{t} \lambda_j(t, 1) dt, \qquad s, j = 1, 2, \ldots, n, \tag{18}$$

where $\mu_{sj}(t)$ are continuous functions in the interval $[t_0; +\infty)$.

Thus, the following theorem hold true.

Theorem. *Let the matrix $D(t)$ of the system (4) on the segment $[t_0; +\infty)$ have simple eigenvalues and the condidtions (16), (17) be fulfilled. Then n solutions of the system (1) have the form (18).*

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 387–391.

PP-Test for Integrability of Some Evolution Differential Equations

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The connection between transcendents of Painlevé and evolution equations is discussed. The calculus PP-procedure is proposed.

The Painlevé singularity analysis is one of the systematic and powerful method to identify the integrability conditions of nonlinear partial differential equations (NPDEs). In recent years, this method has been applied to a very large number of NPDEs and systematically established the complete integrability properties like Lax pair, Bäcklund, Darboux and Miura transformations, bilinear transformation, soliton solutions and so on.

In the last decade of the nineteenth century some mathematicians focused their attention on the classification of ordinary differential equations (ODEs) on the basis of the type of singularity their of solutions.

It is essential to distinguish between two types of singularities. Fixed singularities determined by the coefficients of the equation and its location do not therefore depend on initial conditions. Movable singularities are such whose location on the complex plane does indeed depend on the initial conditions.

The beginning of the study of singularities in the complex plane for differential equations was always attributed to Cauchy, whose idea was to consider local solutions on the complex plane and to use methods of analytical continuation to obtain general solutions. For this procedure to work a complete knowledge of singularities of the equation and its location in the complex plane is required.

Some French mathematicians (Painlevé, Gambier, Garnier and Chazy), following the ideas of Fuchs, Kovalevskaya, Picard and other, completely classified first order equations and studied second order differential equations. In this case, Paul Painlevé [1] found 50 types of second order equations whose only movable singularities were ordinary poles. This special analytical property now carries his name and in what follow will be referred to as the Painlevé Property (PP). Of these 50 types of equations 44 can be integrated in terms of known functions (Riccati equations, elliptic functions, linear equations) and the other six in spite of having meromorphic solutions do not have algebraic integrals that would allow to reduce the equation to quadratures. Today these are known as Painlevé Transcendents:

$$P_1: \quad w''(z) = 6w^2(z) + az;$$

$$P_2: \quad w''(z) = 2w^3(z) + zw(z) + b;$$

$$P_3: \quad w''(z) = \frac{w'^2}{w} + e^z(aw^2 + b) + e^{2z}\left(cw^3 + \frac{d}{w}\right);$$

$$P_4: \quad w''(z) = \frac{w'^2}{2w} + \frac{3w^3}{2} - 4zw^2 + 2\left(z^2 - a\right)w + \frac{b}{w};$$

$$P_5: \quad w''(z) = w'^2\left(\frac{1}{2w} + \frac{1}{w-1}\right) - \frac{w'}{z} + \frac{(w-1)^2(aw + b/w)}{z^2} + \frac{cw}{z} + \frac{dw(w+1)}{w-1};$$

$$P_6: \quad w''(z) = \frac{w'^2}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right) - w'\left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)$$
$$+\frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(a + \frac{bz}{w^2} + c(z-1)(w-1)^2 + dz(z-1)(w-z)^2\right).$$

P_1, P_2, P_3, P_4 being simple meromorphic functions; P_5 has fixed transcendent critical points $z = 0$, $z = \infty$; P_6 has fixed transcendent critical points $z = 0$, $z = 1$, $z = \infty$.

The main contribution of Paul Painlevé lies in that he established the basis for a theory that would allow one a priory, by singularity analysis, to decide on integrability of the partial differential equations (PDEs) without previously solving them. Singularity analysis turns out to be a test of integrability for an equation.

An ordinary differential equation (ODE) is said to possess the Painlevé property if all of its movable singularities are poles. The relation between integrability and the absence of movable critical points was made more explicit through the work [2] in which it was established such ARS (Ablowitz, Ramani, Segur) conjecture: every ODE obtained by similarity reduction of a partial differential equation (PDE) solvable with the inverse scattering method posses the Painlevé property. For the equations that do not have symmetries the ARS conjucture is quite useless as it is not possible to obtain similarity reductions from usual group-theory procedures.

The definition of the PP for PDEs was proposed in [3]. According to these authors, we say that a PDE has the PP if its solutions are singlevalued in a neighbourhood of the manifold of movable singularities. When this manifold depends on the initial conditions it is called a movable singularity manifold.

It is known that the singularities of a function $f(z_1, z_2, \ldots, z_n)$ of $n > 1$ complex variables cannot be isolated; rather they occur along analytic manifolds of (complex)-dimension $n-1$ determined by equation of the form

$$\chi(z_1, z_2, \ldots, z_n) = 0, \tag{1}$$

being an analytical function of its variables in a neighbourhood of the singularity manifold defined by (1).

To test for the presence of PP one assumes that a solution $u(z_1, z_2, \ldots, z_n)$ of a PDE can be expanded around the singularity manifold (1) as following Laurent series of the form

$$u = \chi^{-\alpha} \sum_{k=0}^{\infty} u_k \chi^k, \tag{2}$$

where the coefficients $u_k(z_1, z_2, \ldots, z_n)$ are analytical in a neighbourhood of $\chi = 0$.

It is possible in any to truncate the expansion series at a certain term in order to obtain particular solutions of the equation. If the expansion is truncated at the constant term, expression (2) reduces to:

$$u = u_0\varphi^{-\alpha} + u_1\varphi^{1-\alpha} + \cdots + u_\alpha. \tag{3}$$

Substitution of (3) in the corresponding PDE leads to an overdetermined system of equations for φ, u_j and their derivatives. The truncation of the Painlevé series is the basis of a method called as Singular Manifold Method (SMM). One then substitutes the above expansion (2) in the PDE to determine the value of α and the recurrence relations among the u_k's. If all the allowed values of α turn out to be integers and the set of recurrence relations consistently allows for the arbitrariness of initial conditions, then the given PDE is said to posses the PP and is conjectured to integrable.

An algorithmic procedure has recently been put forward for determining similarity reductions for PDEs. The essence of the procedure is to study the Lie symmetries.

It has been found that Painlevé Transcendents often appear in similarity reductions of the evolution equations with solitons.

The soliton is an object describing solitary wave solutions interacting among themselves without any change in shape except for a small change in its phase. The solitary waves were studied and were described in hydrodynamics problems by Scott Rassel (1844), Boussinesq (1872), Korteveg-de-Vries (1895), M.A. Lavrentjev (1945), Friedrichs (1954). But the concept of “soliton” emerged for the first time in 1965 with Zabusky and Kruskal [4] and the Korteveg-de-Vries (KDV) equation reappeared between 1955 and 1960 in the context of plasma physics.

The Hirota’s bilinear method [5] is known as a powerful procedure for generating multisoliton solutions for PDEs. It is essentially consists in bilinearizing the differential equation by an transformation reminiscent of the Painlevé truncated expansion. The WTC (Weiss, Tabor, Carnevale) method also provides an iterative procedure for generating solitons from the Lax pair and from the corresponding auto-Bäcklund transformation,where the corresponding singularity manifold φ is determined in each step and after n-steps the solution can be expressed in terms of the product $\varphi_1, \varphi_2, \ldots, \varphi_n$ from which it is then possible to construct the Hirota τ-function associated with the solution with n solitons.

The Inverse Scattering Method (ISM) was developed initially allowing one to solve many integrable evolution equations with soliton solutions,in particular, the KdV equation:

$$6u_t = 3uu_x - \frac{1}{2}u_{xxx}, \tag{4}$$

and its different modifications:

$$6v_t = 3v^2 v_x - \frac{1}{2}v_{xxx}, \tag{5}$$

$$u_t + u^p u_x + u_{xxx} = 0; \tag{6}$$

the sine-Gordon equation:

$$u_{xt} = \sin u; \tag{7}$$

the Kline–Gordon equation:

$$u_{xt} = f'(u), \qquad f(u) = -\cos u; \tag{8}$$

the Schrödinger equation:

$$iu_t = u_x^2 - 4iu^2 u_x + 8|u|^4 u; \tag{9}$$

the Boussinesq equation:

$$u_{tt} = u_{xx} + 6\left(u^2\right)_{xx} - u_{xxx}; \tag{10}$$

and the Born–Infeld equation:

$$\left(1 - u_t^2\right) u_{xx} + 2u_x u_{xt} - \left(1 + u_x^2\right) u_{tt} = 0. \tag{11}$$

In recent years Miura transformation

$$f(z) = w'(z) + w^2(z), \qquad w(z) \equiv P_2 \tag{12}$$

widely was applied to find the automodel solutions of evolution equations of the type (4)–(11).

There is a known connection between first Painlevé transcendent P_1 and the automodel solution of KdV equation of such type

$$u_t + u_{xxx} - 6u_x u = 0, \qquad u(x,t) \equiv u, \tag{13}$$

which is received by following relation:

$$u(x,t) = 2[w(z) - t], \qquad z = x - 6t^2, \quad w(z) \equiv P_1. \tag{14}$$

The substitution

$$u(x,t) \equiv w(z), \qquad z = x - t \tag{15}$$

relates P_1 with automodel solution of Boussinesq equation (10). Second Painlevé transcendent P_2 relates with modifications of KdV equation (6) and

$$v_t + v_{xxx} - 6v_x v^2 = 0; \qquad v \equiv v(x,t). \tag{16}$$

In fact, if we make such transformation

$$u(x,t) = [3(t-t_0)]^{-2/3} f(z), \qquad z = [3(t-t_0)]^{-1/3}(x - x_0) \tag{17}$$

and then carry out some mathematical procedure of differential calculus using Miura transformation (12) we obtain automodel solution (17) for equation (13) where $w(z)$ being P_2. KdV evolution equation of type (16) has the automodel solution

$$v(x,t) = [3(t-t_0)]^{-1/3} w(z), \qquad z = [3(t-t_0)]^{-1/3}(x - x_0), \qquad w(z) \equiv P_2. \tag{18}$$

Other evolution equations also have relation with Painlevé transcendents, particularly, the sine–Gordon equation (7) relates with P_3, the Schrödinger equation (9) relates with P_4, the KdV equation of type (5) relates with P_5 and the Born–Infeld equation (11) relates with P_6.

We apply two types of expansions, described in [6], to construct the Painlevé transcendents at explicit form.

In brief, the calculus of Painlevé Property Procedure comes to the following:

1. The initial Cauchy problem for Painlevé equations $P_1 - P_6$ solves, by exact initial conditions, in a holomorphic neighbourhood with help of truncated expansions to consider local solutions on the complex plane and use the method of analytical continuation for obtaining general solution. Unknown coefficients a_n of the power series can be obtained from recurrence relations [7].
2. In order to find a location of unknown poles of m-th order (to the point, Painlevé transcendents have the poles of the different orders for every number $(1, 2, \ldots, 6)$ we apply the algorithm of isolation of pole, proposed by the author [8].
3. On the one hand, the meromorphic integrals of $[P_1 - P_6]$ can be represented by superposition of finite polynomial $\mathcal{P}_\nu(z)$ and some general summarized geometrical progression $\mathcal{R}_\nu(z)$ in a neighbourhood of singularities for Painlevé transcendents. Corresponding correlations for the coefficients of the regular power series, m-orders of poles and value $q = 1/R$, defining location of poles, were found.
4. On the other hand, meromorphic integrals of $[P_1 - P_6]$ can be expanded around poles in form of Laurent series in a neighbourhood of the found poles and then both type of expansions (regular, described in 1, and irregular, described in 3) stick together.
5. Transition across the pole realizes with help of procedure of the analytical continuations, which also is used in the case of realization of procedure for isolation of poles.

6. All algebraic operations with the power series and Laurent series and obtaining of recurrence relations were made according to the Method of generalized power series, proposed by Prof. P.F. Fil'chakov for solving of wide classes of linear and nonlinear problems and described in his book [9]. This method is based on Euler's method with using of Cauchy's formula for multiplying of power series.

We hope that there will be further study in this direction.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 392–400.

Asymptotic Integration of System of the Differential Equations of Third Order

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We present the theorems about the asymptotic expansion in ε for the solution of the three-component system $\varepsilon \frac{dx}{dt} = A(t)x$ for the case, when 3×3 matrix $A(t)$ has multiple roots.

1. Development of asymptotic methods of solution of differential equations with variable coefficients started in 19 century in Fourier's, Liouville's and Sturm's papers. The systematic research of the linear differential equations with slowly variable coefficients bigin from fifty yeares after Feschenko papers were published. However for new, more accurate and rational the serch methods of solutions such equations and their systems continus. For the systems which roots of characteristic equations are simple the solutions are easily found by classical Birkhoff method. The first result in the case of multiple roots of characteristics equations in general case was obtained in 60–70 years in Shkil's works.

In my work I suggest a method for reduction of identical multiple roots of characteristic equations to simple roots of some algebraic equations. It enables one to make use of well-known formula for the construction of the asymptotic solution of the system under study what considerably simplifies calculations.

2. Consider the system of first order linear differential equations

$$\varepsilon \frac{dx}{dt} = A(t)x, \tag{1}$$

where x is three dimensional vector, $A(t) = \|a_{ij}\|$, $i, j = 1, 2, 3$ is matrix, whose elements are infinitely differentiable on the segment $[0, L]$, $\varepsilon > 0$ is the small parameter.

Let us build the characteristic equation

$$\det \|A(t) - \lambda E\| = 0, \tag{2}$$

(where E is unit matrix), that is

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix} = -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda(a_{11}a_{33} + a_{22}a_{33}$$

$$+a_{11}a_{22} - a_{31}a_{13} - a_{32}a_{23} - a_{12}a_{21}) + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31}$$

$$+a_{11}a_{22}a_{33} - a_{31}a_{13}a_{22} - a_{12}a_{21}a_{33} - a_{32}a_{23}a_{11} = 0.$$

The substitution

$$\lambda = \nu + \frac{a_{11} + a_{22} + a_{33}}{3}$$

transforms it into the cubic equation of the canonical form

$$\nu^3 + p(t)\nu + g(t) = 0,$$

where $p(t)$, $q(t)$ is linear combination of the matrix $A(t)$ coefficients.

Let the matrix $A(t)$ be such that

$$D(t) = -27q^2(t) - 4p^3(t) = 0, \qquad \forall\, t \in [0, L].$$

Then equation (1) has multiple roots. There are two possible cases

a) $p(t) \neq 0$, $q(t) \neq 0$ – root of multiplicity two and one root of multiplicity one;

b) $p(t) = q(t) = 0$ – one root of multiplicity three.

We don't consider the ase a), since then the methods of [1] allow to split the system under study into subsystem, for which the characteristic equations has one simple root and one root of multiplicity two. This case will be considered separately. In case b) we have one elementary divisor

$$\lambda_1(t) \equiv \lambda_2(t) \equiv \lambda_3(t) \equiv \lambda_0(t) \equiv \frac{a_{11}(t) + a_{22}(t) + a_{33}(t)}{3} = \frac{\operatorname{tr} A(t)}{3}. \tag{3}$$

of multiplicity three.

Then, as it is shown in [2], there exists the nondegenerate matrix of the transformation of similarity $V(t)$ that transforms the matrix $A(t)$ to the quasidiagonal canonical form $W(t) = V^{-1}(t)A(t)V(t)$, where

$$W(t) = \begin{pmatrix} \lambda_0(t) & 1 & 0 \\ 0 & \lambda_0(t) & 1 \\ 0 & 0 & \lambda_0(t) \end{pmatrix},$$

The substitution $x = V(t)y$ transforms system (1) into the system

$$\varepsilon \frac{dy}{dt} = D(t, \varepsilon)y, \tag{4}$$

where

$$D(t, \varepsilon) = \begin{pmatrix} \lambda_0(t) & 1 & 0 \\ 0 & \lambda_0(t) & 1 \\ 0 & 0 & \lambda_0(t) \end{pmatrix} + \varepsilon \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = V^{-1}(t)V'(t).$$

Using Cardano formula, we obtain the eidenvalues of system (4).

$$\rho_1 = \lambda_0 + \frac{\varepsilon b_1}{3} + \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c + \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}}$$
$$+ \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c - \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}} = \lambda_0 + O\left(\varepsilon^{1/3}\right),$$

$$\rho_2 = \lambda_0 + \frac{\varepsilon b_1}{3} + \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c + \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}} \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$+ \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c - \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}} \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \lambda_0 + O\left(\varepsilon^{1/3}\right),$$

$$\rho_3 = \lambda_0 + \frac{\varepsilon b_1}{3} + \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c + \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}} \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$+ \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c - \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}} \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \lambda_0 + O\left(\varepsilon^{1/3}\right),$$

where $b_1, a, b, c, d_6, d_5, \ldots, d_2$ are linear combinations of the coefficient of matrix $A(t)$. Assume that $\varepsilon^6 d_6 + \cdots + \varepsilon^2 d_2 \neq 0$. When ρ_i are different and the eigenvectors $\mu_i(t,\varepsilon)$, $\mu_i^*(t,\varepsilon)$ of the matrix $D(t,\varepsilon)$ and conjugate matrix $D^*(t,\varepsilon)$ may be chosen in such a way that their scalar products have the form

$$(\mu_i \cdot \mu_j^*) = \begin{cases} 1, & i = j, \\ 0, & i \neq j,\ i,j = 1,2,3. \end{cases} \tag{5}$$

For example take

$$\mu_i(t,\varepsilon) = \begin{pmatrix} 1 \\ \dfrac{\varepsilon b_{31}(1+\varepsilon b_{23}) - \varepsilon b_{21}(\lambda_0 + \varepsilon b_{33} - \rho_i)}{(\lambda_0 + \varepsilon b_{22} - \rho_i)(\lambda_0 + \varepsilon b_{33} - \rho_i) - \varepsilon b_{32}(1+\varepsilon b_{23})} \\ \dfrac{\varepsilon^2 b_{32} b_{21} - \varepsilon b_{31}(\lambda_0 + \varepsilon b_{22} - \rho_i)}{(\lambda_0 + \varepsilon b_{22} - \rho_i)(\lambda_0 + \varepsilon b_{33} - \rho_i) - \varepsilon b_{32}(1+\varepsilon b_{23})} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \dfrac{\varepsilon b_{31}(1+\varepsilon b_{23}) - \varepsilon b_{21}\left(\varepsilon b_{33} - O\left(\varepsilon^{1/3}\right)\right)}{\left(\varepsilon b_{22} - O\left(\varepsilon^{1/3}\right)\right)\left(\varepsilon b_{33} - O\left(\varepsilon^{1/3}\right)\right) - \varepsilon b_{32}(1+\varepsilon b_{23})} \\ \dfrac{\varepsilon^2 b_{32} b_{21} - \varepsilon b_{31}\left(\varepsilon b_{22} - O\left(\varepsilon^{1/3}\right)\right)}{\left(\varepsilon b_{22} - O\left(\varepsilon^{1/3}\right)\right)\left(\varepsilon b_{33} - O\left(\varepsilon^{1/3}\right)\right) - \varepsilon b_{32}(1+\varepsilon b_{23})} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \varepsilon^{1/3} \dfrac{b_{31}(1+\varepsilon b_{23}) - \varepsilon^{1/3} b_{21}\left(\varepsilon^{2/3} b_{33} - O\left(\varepsilon^{0}\right)\right)}{\left(\varepsilon^{2/3} b_{22} - O\left(\varepsilon^{0}\right)\right)\left(\varepsilon^{2/3} b_{33} - O\left(\varepsilon^{0}\right)\right) - \varepsilon^{1/3} b_{32}(1+\varepsilon b_{23})} \\ \varepsilon^{2/3} \dfrac{\varepsilon^{2/3} b_{32} b_{21} - b_{31}\left(\varepsilon^{2/3} b_{22} - O\left(\varepsilon^{0}\right)\right)}{\left(\varepsilon^{2/3} b_{22} - O\left(\varepsilon^{0}\right)\right)\left(\varepsilon^{2/3} b_{33} - O\left(\varepsilon^{0}\right)\right) - \varepsilon^{\frac{1}{3}} b_{32}(1+\varepsilon b_{23})} \end{pmatrix} = \begin{pmatrix} O\left(\varepsilon^{0}\right) \\ O\left(\varepsilon^{1/3}\right) \\ O\left(\varepsilon^{2/3}\right) \end{pmatrix}.$$

As we can see, vector μ_i can be write in form

$$\mu_i = \left(1, \varepsilon^{1/3}\mu_{i2}^{a}, \varepsilon^{2/3}\mu_{i3}^{a}\right). \tag{6}$$

Here and below index a denotes analitical in point $\varepsilon = 0$ function. So, it is easy to notice that for the condition (5) to hold, the coordinates of the vector μ_i^* must have the form

$$\mu_i^* = \left(\mu_{i1}^{a*}, \varepsilon^{-1/3}\mu_{i2}^{a*}, \varepsilon^{-2/3}\mu_{i3}^{a*}\right). \tag{6'}$$

The following theorems hold true.

Theorem 1. *If the functions $a_{ij}(t)$, $i,j = 1,2,3$ are infinitely differentiable on the segment $[0,L]$, then system (4) has the formal particular solution*

$$y(t,\varepsilon) = U(t,\varepsilon,\varepsilon) \exp\left(\frac{1}{\varepsilon}\int_0^t \lambda(\tau,\varepsilon,\varepsilon)\, d\tau\right), \tag{7}$$

where $U(t,\varepsilon,\varepsilon)$ is an 3-component vector, and $\lambda(t,\varepsilon,\varepsilon)$ is a scalar function which are represented by the following formal series

$$U(t,\varepsilon,\varepsilon)=\sum_{s=0}^{\infty}\varepsilon^s U_s(t,\varepsilon), \qquad \lambda(t,\varepsilon,\varepsilon)=\sum_{s=1}^{\infty}\varepsilon^s\lambda_s(t,\varepsilon)+\rho_1(t,\varepsilon). \tag{8}$$

Proof. In order to prove this theorem let us substitute the vector y, given by the relation (7), into system (4). We have

$$\varepsilon U'(t,\varepsilon,\varepsilon)\equiv(D(t,\varepsilon)-\lambda(t,\varepsilon,\varepsilon)E)U(t,\varepsilon,\varepsilon). \tag{9}$$

The coefficients of series (8) are to be determined from the following system of algebraic equations

$$(D(t,\varepsilon)-\rho_1(t,\varepsilon)E)U_0(t,\varepsilon)=0, \tag{10}$$

$$(D(t,\varepsilon)-\rho_1(t,\varepsilon)E)U_m(t,\varepsilon)=U'_{m-1}(t,\varepsilon)+\sum_{j=1}^{m}\lambda_j(t,\varepsilon)U_{m-j}(t,\varepsilon), \quad m=1,2,\ldots. \tag{11}$$

Here $(\ )'$ denotes the derivative with respect to t.

Consider vector equation (10). It is obvious that

$$U_0(t,\varepsilon)=\mu_1(t,\varepsilon). \tag{12}$$

Let us turn to the system (11). Since $\det\|D(t,\varepsilon)-\rho_1(t,\varepsilon)E\|=0$, for the existence of a solution of the inhomogeneous system of algebraic equations of such form it is necessary and sufficient that the scalar product of the vector on the right hand side with any solution of the associated system, that is the system

$$(D(t,\varepsilon)-\rho_1(t,\varepsilon)E)^*y=0,$$

vanishes [3]. That's why

$$\left(\left(U'_{m-1}(t,\varepsilon)+\sum_{j=1}^{m}\lambda_j(t,\varepsilon)U_{m-j}(t,\varepsilon)\right)\cdot\mu_1^*(t,\varepsilon)\right)=0, \quad m=1,2,\ldots.$$

From this, making use of properties of the scalar product and formulas (5), (12), we obtain

$$\lambda_m(t,\varepsilon)=-(U'_{m-1}(t,\varepsilon)\cdot\mu_1^*(t,\varepsilon))-\sum_{j=1}^{m-1}\lambda_j(t,\varepsilon)(U_{m-j}(t,\varepsilon)\cdot\mu_1^*(t,\varepsilon)), \quad m=1,2,\ldots. \tag{13}$$

We shall look for the vector $U_m(t,\varepsilon)$ in the form

$$U_m(t,\varepsilon)=c_1^{(m)}(t,\varepsilon)\cdot\mu_1(t,\varepsilon)+c_2^{(m)}(t,\varepsilon)\cdot\mu_2(t,\varepsilon)+c_3^{(m)}(t,\varepsilon)\cdot\mu_3(t,\varepsilon), \tag{14}$$

where $c_k^{(m)}(t,\varepsilon)$, $k=1,2,3$ are scalar functions. Substituting (14) into (11), we obtain that the functions $c_1^{(m)}(t,\varepsilon)$ are arbitrary. Let $c_1^{(m)}(t,\varepsilon)\equiv 0$. Then

$$\begin{aligned}U_m(t,\varepsilon)&=\sum_{k=2}^{3}c_k^{(m)}(t,\varepsilon)\cdot\mu_k(t,\varepsilon)\\&=\sum_{k=2}^{3}\frac{(U'_{m-1}(t,\varepsilon)\cdot\mu_k^*(t,\varepsilon))+\sum\limits_{j=1}^{m}\lambda_j(t,\varepsilon)U_{m-j}(t,\varepsilon)\cdot\mu_k^*(t,\varepsilon))}{\rho_k(t,\varepsilon)-\rho_1(t,\varepsilon)}\cdot\mu_k(t,\varepsilon).\end{aligned} \tag{15}$$

Taking into account (5) and (15), formula (13) can be rewritten in the form

$$\lambda_m(t,\varepsilon) = -(U'_{m-1}(t,\varepsilon)\cdot\mu_1^*(t,\varepsilon)), \quad m = 1,2,\dots. \tag{16}$$

So, the solution of the system (10), (11) can be written in the form of recurrent formulas (15), (16). Construction of the formulas for the coefficientes of series (8) completes the proof.

3. Let's evaluate the coefficients of series (8). Estimate of the $U_0(t,\varepsilon)$ give in (6). That's why, taking into account (12), write

$$U_0(t,\varepsilon) = \left(1, \varepsilon^{1/3}U_{02}^a(t,\varepsilon), \varepsilon^{2/3}U_{03}^a(t,\varepsilon)\right). \tag{17}$$

The diferentiation with respect to t have not an influence on the function's analitical in ε. If ε is function's zero, then we carry out it from the differentiation sign as a const, that is ε will not become a function's pole. The differentiation differential can increase only the order of the functions zero in the point $\varepsilon = 0$, but it only can improve the rezult. Look for

$$\lambda_1(t,\varepsilon) = \begin{pmatrix} 0 \\ \varepsilon^{1/3}U_{02}^{a'}(t,\varepsilon) \\ \varepsilon^{2/3}U_{03}^{a'}(t,\varepsilon) \end{pmatrix} \cdot \begin{pmatrix} \mu_{11}^{a*}(t,\varepsilon) \\ \varepsilon^{-1/3}\mu_{12}^{a*}(t,\varepsilon) \\ \varepsilon^{-2/3}\mu_{13}^{a*}(t,\varepsilon) \end{pmatrix} = \lambda_1^a(t,\varepsilon). \tag{18}$$

Then

$$U_1(t,\varepsilon) = \sum_{k=2}^{3} \frac{(U'_0(t,\varepsilon)\cdot\mu_k^*(t,\varepsilon)) + \lambda_1(t,\varepsilon)(U_0(t,\varepsilon)\cdot\mu_k^*(t,\varepsilon))}{\rho_k(t,\varepsilon)-\rho_1(t,\varepsilon)}\mu_k(t,\varepsilon).$$

Proceed from (6′), (17), (18) we can conclude that the numerator of the fraction of this sum is analitical function. Evaluate the denominator.

$$\rho_2-\rho_1 = \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c + \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \dots + \varepsilon^2 d_2}}\left(-\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$+ \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c - \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \dots + \varepsilon^2 d_2}}\left(-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right) = O\left(\varepsilon^{1/3}\right),$$

$$\rho_3-\rho_1 = \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c + \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \dots + \varepsilon^2 d_2}}\left(-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$+ \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c - \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \dots + \varepsilon^2 d_2}}\left(-\frac{3}{2} + i\frac{\sqrt{3}}{2}\right) = O\left(\varepsilon^{\frac{1}{3}}\right).$$

So

$$U_1(t,\varepsilon) = \sum_{k=2}^{3} \frac{f^a(t,\varepsilon)\cdot\left(1, \varepsilon^{1/3}\mu_{k2}^a(t,\varepsilon), \varepsilon^{2/3}\mu_{k3}^a(t,\varepsilon)\right)}{\varepsilon^{1/3}(\rho_k(t,\varepsilon)-\rho_1(t,\varepsilon))^a} = \left(\varepsilon^{-1/3}U_{11}^a, \varepsilon^0 U_{12}^a, \varepsilon^{1/3}U_{13}^a\right).$$

Then

$$\lambda_2(t,\varepsilon) = -\begin{pmatrix} \varepsilon^{-1/3}U_{11}^{a'} \\ \varepsilon^0 U_{12}^{a'} \\ \varepsilon^{1/3}U_{13}^{a'} \end{pmatrix} \cdot \begin{pmatrix} \mu_{11}^{a*} \\ \varepsilon^{-1/3}\mu_{12}^{a*} \\ \varepsilon^{-2/3}\mu_{13}^{a*} \end{pmatrix} = \varepsilon^{-1/3}\lambda_2^a(t,\varepsilon).$$

Assume that for all $U_j(t,\varepsilon)$, $\lambda_j(t,\varepsilon)$, $j<m$ next formulas are true

$$U_j(t,\varepsilon)=\left(\frac{U^a_{1j}(t,\varepsilon)}{\varepsilon^{j/3}},\frac{U^a_{2j}(t,\varepsilon)}{\varepsilon^{(j-1)/3}},\frac{U^a_{3j}(t,\varepsilon)}{\varepsilon^{(j-2)/3}}\right)=\frac{U^a_j(t,\varepsilon)}{\varepsilon^{j/3}},$$
$$\lambda_j(t,\varepsilon)=\frac{\lambda^a_j(t,\varepsilon)}{\varepsilon^{(j-1)/3}},\qquad j=1,2,\dots,m-1, \tag{19}$$

m is some fixed natural number.

Let's show the correctness this assumption for $j=m$, $m\in N$. Whis usage of (19), (6′), formula (16) can be rewritten as

$$\lambda_m(t,\varepsilon)=-\left(\frac{U^{a'}_{1(m-1)}(t,\varepsilon)}{\varepsilon^{(m-1)/3}},\frac{U^{a'}_{2(m-1)}(t,\varepsilon)}{\varepsilon^{(m-2)/3}},\frac{U^{a'}_{3(m-1)}(t,\varepsilon)}{\varepsilon^{(m-3)/3}}\right)$$
$$\times\left(\mu^{a*}_{11}(t,\varepsilon),\varepsilon^{-1/3}\mu^{a*}_{12}(t,\varepsilon),\varepsilon^{-2/3}\mu^{a*}_{13}(t,\varepsilon)\right)=\frac{\lambda^a_m(t,\varepsilon)}{\varepsilon^{(m-1)/3}}.$$

So, $\lambda_m(t,\varepsilon)$ have pole in ε and its order is $(m-1)/3$, what confirms the correctness of our assumption for $j=m$.

Likewise, rewrite (15):

$$U_m(t,\varepsilon)=\sum_{k=2}^{3}\left(\left(\frac{U^{a'}_{1(m-1)}}{\varepsilon^{(m-1)/3}},\frac{U^{a'}_{2(m-1)}}{\varepsilon^{(m-2)/3}},\frac{U^{a'}_{3(m-1)}}{\varepsilon^{(m-3)/3}}\right)\cdot\left(\mu^{a*}_{k1},\varepsilon^{-1/3}\mu^{a*}_{k2},\varepsilon^{-2/3}\mu^{a*}_{k3}\right)\right.$$
$$\left.+\sum_{j=1}^{m}\frac{\lambda^a_j(t,\varepsilon)}{\varepsilon^{(j-1)/3}}\left(\frac{U^a_{1(m-j)}}{\varepsilon^{(m-j)/3}},\frac{U^a_{2(m-j)}}{\varepsilon^{(m-j-1)/3}},\frac{U^a_{3(m-j)}}{\varepsilon^{(m-j-2)/3}}\right)\cdot\left(\mu^{a*}_{k1},\varepsilon^{-1/3}\mu^{a*}_{k2},\varepsilon^{-2/3}\mu^{a*}_{k3}\right)\right)$$
$$\times\frac{\left(\mu^a_{k1},\varepsilon^{1/3}\mu^a_{k2},\varepsilon^{2/3}\mu^a_{k3}\right)}{\varepsilon^{1/3}(\rho_k-\rho_1)^a}=\sum_{k=2}^{3}\frac{f^a_1}{\varepsilon^{(m-1)/3}}\cdot\frac{\left(\mu^a_{k1},\varepsilon^{1/3}\mu^a_{k2},\varepsilon^{2/3}\mu^a_{k3}\right)}{\varepsilon^{1/3}(\rho_k-\rho_1)^a}$$
$$=\left(\frac{\mu^a_{k1}}{\varepsilon^{m/3}},\frac{\mu^a_{k2}}{\varepsilon^{(m-1)/3}},\frac{\mu^a_{k3}}{\varepsilon^{(m-2)/3}}\right)\cdot\frac{f^a_1}{(\rho_k-\rho_1)^a}=\frac{U^a_m(t,\varepsilon)}{\varepsilon^{j/3}}.$$

So, using the mathematical induction, we conclude that the formulas (19) hold true for all $j\in N$.

Then the following expansions for series (8) take place

$$U(t,\varepsilon,\varepsilon)=\sum_{s=0}^{\infty}\varepsilon^{2s/3}U^a_s(t,\varepsilon),\qquad \lambda(t,\varepsilon,\varepsilon)=\sum_{s=1}^{\infty}\varepsilon^{(2s+1)/3}\lambda^a_s(t,\varepsilon)+\rho_1(t,\varepsilon). \tag{20}$$

Here $U^a_s(t,\varepsilon)$, $\lambda^a_s(t,\varepsilon)$ are analytical in ε.

4. Next Theorem 2 proves the asymptotic property of formal solution in the sense of [4].

Theorem 2. *If the conditions of Theorem 1 are fulfilled and*

$$\operatorname{Re}(\rho_1(t,\varepsilon))\le 0\qquad \textit{for}\quad \forall\, t\in[0,L],\ 0<\varepsilon\le\varepsilon_0, \tag{21}$$

on the segment $[0,L]$ m-th approximations satisfies the diferential system (4) up to $O\left(\varepsilon^{(2m+3)/3}\right)$.

Proof. For the proof, similarly to [1], let us introduce the vector

$$y_m(t,\varepsilon,\varepsilon)=U_m(t,\varepsilon,\varepsilon)\exp\left(\frac{1}{\varepsilon}\int_0^t\lambda_m(\tau,\varepsilon,\varepsilon)\,d\tau\right), \tag{22}$$

where

$$\begin{aligned} U_m(t,\varepsilon,\varepsilon) &= \sum_{s=0}^{m} \varepsilon^s U_s(t,\varepsilon) = \sum_{s=0}^{m} \varepsilon^{2s/3} U_s^a(t,\varepsilon), \\ \lambda_m(t,\varepsilon,\varepsilon) &= \sum_{s=1}^{m} \varepsilon^s \lambda_s(t,\varepsilon) + \rho_1(t,\varepsilon) = \sum_{s=1}^{m} \varepsilon^{(2s+1)/3} \lambda_s^a(t,\varepsilon) + \rho_1(t,\varepsilon), \qquad m \geq 1. \end{aligned} \tag{23}$$

Substitute (22) to the differential expression

$$L(y_m) = \varepsilon \frac{dy_m}{dt} - D(t,\varepsilon) y_m. \tag{24}$$

We have

$$\begin{aligned} L(y_m(t,\varepsilon)) = {} & [\varepsilon U_m'(t,\varepsilon,\varepsilon) - D(t,\varepsilon) U_m(t,\varepsilon,\varepsilon) \\ & + \lambda_m(t,\varepsilon,\varepsilon) U_m(t,\varepsilon,\varepsilon)] \exp\left(\frac{1}{\varepsilon} \int_0^t \lambda_m(\tau,\varepsilon,\varepsilon)\, d\tau \right). \end{aligned} \tag{25}$$

The magnitude of the function $\exp\left(\frac{1}{\varepsilon}\int_0^t \lambda_m(\tau,\varepsilon,\varepsilon)\,d\tau\right)$ is limited on the set $\{K : 0 < \varepsilon \leq \varepsilon_0; t \in [0,L]\}$. In fact

$$\begin{aligned} \left| \exp\left(\frac{1}{\varepsilon} \int_0^t \lambda_m(\tau,\varepsilon,\varepsilon)\, d\tau \right) \right| &= \exp\left(\frac{1}{\varepsilon} \int_0^t \alpha_m(\tau,\varepsilon,\varepsilon)\, d\tau \right) \\ &= \exp\left(\frac{1}{\varepsilon} \int_0^t \left[\alpha_0(\tau,\varepsilon) + \varepsilon \alpha_1(\tau,\varepsilon) + \ldots + \varepsilon^{(2m+1)/3} \alpha_m(\tau,\varepsilon) \right] d\tau \right) \end{aligned} \tag{26}$$

(here $\alpha_m(\tau,\varepsilon,\varepsilon) = \sum\limits_{s=1}^{m} \varepsilon^{\frac{2}{3}s+\frac{1}{3}} \alpha_s(\tau,\varepsilon)$ is the real part of the function $\lambda_m(\tau,\varepsilon,\varepsilon)$, defined by (23), $\alpha_0(\tau,\varepsilon)$ is the real part of the function $\rho_1(\tau,\varepsilon)$, $\alpha_s(\tau,\varepsilon)$ are analytical, $s = 1, 2, \ldots$).

Since the functions $\alpha_1(\tau,\varepsilon), \ldots, \alpha_m(\tau,\varepsilon)$ according to theorem 1 are infinitely differentiable on the segment $[0,L]$, (26) can be rewritten as

$$\begin{aligned} & \exp\left(\frac{1}{\varepsilon} \int_0^t \alpha_0(\tau,\varepsilon)\, d\tau \right) \exp\left(\frac{1}{\varepsilon} \int_0^t \left[\varepsilon \alpha_1(\tau,\varepsilon) + \cdots + \varepsilon^{(2m+1)/3} \alpha_m(\tau,\varepsilon) \right] d\tau \right) \\ & \quad \leq \exp\left(\frac{1}{\varepsilon} \int_0^t \alpha_0(\tau,\varepsilon)\, d\tau \right) \exp\left(\int_0^t \sum_{s=1}^{m} \varepsilon^{(2s-2)/3} \mid \alpha_s(\tau,\varepsilon) \mid \, d\tau \right) \\ & \quad \leq \exp\left(\frac{1}{\varepsilon} \int_0^t \alpha_0(\tau,\varepsilon)\, d\tau \right) \exp\left(M \int_0^t \left(1 + \varepsilon^{2/3} + \ldots + \varepsilon^{(2m-2)/3} \right) d\tau \right) \\ & \quad \leq \exp\left(\frac{1}{\varepsilon} \int_0^t \alpha_0(\tau,\varepsilon)\, d\tau \right) \exp\left(ML \frac{1 - \varepsilon_0^{m/3}}{1 - \varepsilon_0^{2/3}} \right), \end{aligned}$$

here $(M = \max \mid \alpha_s(t,\varepsilon) \mid)$, $s = 1,\dots,m$. In virtue of (21) we can state that this value is limited.

Let's evaluate the vector which is the multiplier at $\exp\left(\frac{1}{\varepsilon}\int\limits_0^t \lambda_m(\tau,\varepsilon,\varepsilon)\,d\tau\right)$ in right hand side of the equality (25). Since when we determined coeficients of series $U_m(t,\varepsilon,\varepsilon), \lambda_m(t,\varepsilon,\varepsilon)$ we compared the coefficientes at the external powers of ε parameter up to order including m itself, it is clear that this vector will have nonzero coefficients only at the powers $\varepsilon^{m+1}, \varepsilon^{m+2}, \dots, \varepsilon^{2m}$. That is why, taking into account said above on derivative with respect to t of function $U_m(t,\varepsilon,\varepsilon)$ and (19), we obtain

$$\varepsilon(U_0'(t,\varepsilon) + \varepsilon U_1'(t,\varepsilon) + \cdots + \varepsilon^m U_m'(t,\varepsilon)) - D(t,\varepsilon)(U_0(t,\varepsilon) + \varepsilon U_1(t,\varepsilon) + \cdots + \varepsilon^m U_m(t,\varepsilon))$$
$$+(\rho_1(t,\varepsilon) + \varepsilon\lambda_1(t,\varepsilon) + \cdots + \varepsilon^m \lambda_m(t,\varepsilon))(U_0(t,\varepsilon) + \varepsilon U_1(t,\varepsilon) + \cdots + \varepsilon^m U_m(t,\varepsilon))$$
$$= \varepsilon^{m+1} U_m'(t,\varepsilon) + \sum_{j=1}^{m} \varepsilon^{m+j} \sum_{s=j}^{m} \lambda_s(t,\varepsilon) U_{m+j-s}(t,\varepsilon) = \varepsilon^{m+1} \frac{U_m^{a'}(t,\varepsilon)}{\varepsilon^{m/3}}$$
$$+ \sum_{j=1}^{m} \varepsilon^{m+j} \sum_{s=j}^{m} \frac{\lambda_s^a(t,\varepsilon)}{\varepsilon^{(s-1)/3}} \frac{U_{m+j-s}^a(t,\varepsilon)}{\varepsilon^{(m+j-s)/3}} = \varepsilon^{(2m+3)/3} U_m^{a'}(t,\varepsilon) + \sum_{j=1}^{m} \varepsilon^{(2m+2j+1)/3} \sum_{s=j}^{m} \lambda_s^a(t,\varepsilon)$$
$$\times U_{m+j-s}^a(t,\varepsilon) = \varepsilon^{(2m+3)/3} U_m^{a'}(t,\varepsilon) + \varepsilon^{(2m+3)/3} \sum_{j=1}^{m} \varepsilon^{(2j-2)/3} \sum_{s=j}^{m} \lambda_s^a(t,\varepsilon) U_{m+j-s}^a(t,\varepsilon)$$
$$= \varepsilon^{(2m+3)/3} \left(U_m^{a'}(t,\varepsilon) + \sum_{j=1}^{m} \varepsilon^{(2j-2)/3} \sum_{s=j}^{m} \lambda_s^a(t,\varepsilon) U_{m+j-s}^a(t,\varepsilon) \right) = O\left(\varepsilon^{(2m+3)/3}\right), \ m \in N.$$

The theorem is proved.

5. If system (1) is system of the linear differential equations of the second order, then theorem likewise Theorem 1, 2 take place. Formulas (20) for $n = 2$ have form

$$U(t,\varepsilon,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^{s/2} U_s^a(t,\varepsilon), \qquad \lambda(t,\varepsilon,\varepsilon) = \sum_{s=1}^{\infty} \varepsilon^{(s+1)/2} \lambda_s^a(t,\varepsilon) + \rho_1(t,\varepsilon).$$

The solution, found by the method of Theorem 1, satisfies the system (4) up to $O\left(\varepsilon^{(m+2)/2}\right)$.

The advantage of this method in contradistinction to well know M.I. Shkil's method [1], is the possibility bringing the solution of the equation with multiple roots to the classical theory of the simple roots.

Let us illustrate it on the example

$$\varepsilon \frac{d^2 y}{dt^2} + p(t) y = 0,$$

here $p(t) \neq 0, t \in [0, L]$.

Let us write this equation in the form of system (4). We use here the following notations $y = \varepsilon y_1$, $\frac{dy}{dt} = y_2$.

Then we obtain the system

$$\varepsilon \frac{dy_1}{dt} = y_2,$$
$$\varepsilon \frac{dy_2}{dt} = \varepsilon \frac{d^2 y}{dt} = -p(t) \varepsilon y_1$$

or

$$\varepsilon\frac{dy}{dt} = \begin{pmatrix} 0 & 1 \\ -\varepsilon p(t) & 0 \end{pmatrix} y,$$

where y is a two dimensional vector.

In this case the roots of the characteristic equation are different.

$$\rho_1(t,\varepsilon) = \sqrt{-\varepsilon p(t)}, \qquad \rho_2(t,\varepsilon) = -\sqrt{-\varepsilon p(t)}.$$

Then the conditions of Theorem 1 are satisfied. That's why using the recurrent formulas for $U_m(t,\varepsilon)$, $\lambda_m(t,\varepsilon)$, we obtain

$$U_0(t,\varepsilon) = \left(1, \sqrt{-\varepsilon p(t)}\right), \quad \lambda_1(t,\varepsilon) = -\frac{p'(t)}{4p(t)}, \quad U_1(t,\varepsilon) = \left(\frac{p'(t)}{8\sqrt{\varepsilon}(-p(t))^{3/2}}, -\frac{p'(t)}{8p(t)}\right),$$

$$\lambda_2(t,\varepsilon) = \frac{p'^2(t)}{32\sqrt{\varepsilon}(-p(t))^{5/2}}, \quad U_2(t,\varepsilon) = \left(\frac{p''}{16\varepsilon p^2} - \frac{3p'^2}{32\varepsilon p^3}, \frac{p''}{16\sqrt{\varepsilon}(-p)^{3/2}} - \frac{3p'^2}{32\sqrt{\varepsilon}(-p)^{5/2}}\right).$$

Substitute the vector

$$y_1(t,\varepsilon,\varepsilon) = \left((1,\sqrt{-\varepsilon p}) + \varepsilon\left(\frac{p'}{8\sqrt{\varepsilon}(-p)^{3/2}}, -\frac{p'}{8p}\right)\right)\exp\left(\frac{1}{\varepsilon}\int_0^t\left(\sqrt{-\varepsilon p} - \varepsilon\frac{p'}{4p}\right)d\tau\right)$$

to differential expression (24). We have

$$L(y_1(t,\varepsilon,\varepsilon)) = \varepsilon\sqrt{\varepsilon}\left(-\frac{p''}{8p^{3/2}} + \frac{7p'^2}{32(-p)^{5/2}}, \frac{\sqrt{\varepsilon}p''}{8p} - \frac{3\sqrt{\varepsilon}p'^2}{64p^2}\right)\exp\left(\frac{1}{\varepsilon}\int_0^t\left(\sqrt{-\varepsilon p} - \varepsilon\frac{p'}{4p}\right)d\tau\right).$$

As you can see, the vector $y_1(t,\varepsilon,\varepsilon)$ satisfies the system of the differential equations up to $O\left(\varepsilon^{3/2}\right)$.

Likewise

$$y_2 = \left(\left(1,\sqrt{-\varepsilon p}\right) + \varepsilon\left(\frac{p'}{8\sqrt{\varepsilon}(-p)^{3/2}}, -\frac{p'}{8p}\right) + \varepsilon^2\left(\frac{p''}{16\varepsilon p^2} - \frac{3p'^2}{32\varepsilon p^3},\right.\right.$$

$$\left.\left.\frac{p''}{16\sqrt{\varepsilon}(-p)^{3/2}} - \frac{3p'^2}{32\sqrt{\varepsilon}(-p)^{5/2}}\right)\right)\exp\left(\frac{1}{\varepsilon}\int_0^t\left(\sqrt{-\varepsilon p} - \varepsilon\frac{p'}{4p} + \varepsilon^2\frac{p'^2}{32\sqrt{\varepsilon}(-p)^{5/2}}\right)d\tau\right),$$

and asymptotical evaluation

$$L(y_2) = \varepsilon^2\left(\frac{77p'^3}{2^8p^4} - \frac{21p'p''}{2^6p^3} + \frac{\sqrt{\varepsilon}p'^2p''}{2^9(-p)^{9/2}} - \frac{3\sqrt{\varepsilon}p'^4}{2^{10}(-p)^{11/2}},\right.$$

$$\left.-\frac{7p'^2}{2^6p^2} - \frac{19\sqrt{\varepsilon}p'p''}{2^6(-p)^{5/2}} - \frac{33\sqrt{\varepsilon}p'^3}{2^7(-p)^{7/2}} + \frac{\sqrt{\varepsilon}p''}{2^4(-p)^{3/2}} - \frac{\varepsilon p'^2p''}{2^9p^4} + \frac{3\varepsilon p'^4}{2^{10}p^5}\right) = O\left(\varepsilon^2\right)$$

holds true.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 401–405.

On Hyperelliptic Solutions of Spectral Problem for the Two-Dimensional Schrödinger Equation

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On the basis of the special Abelian 2-differentials of the second kind corresponding to hypereliptic curves of the g genus addition formulae for hyperelliptic functions determined on the Jacobi manifold is considered. In the case of hyperelliptic curves of the second genus these formulae yield both relations for hypereliptic $\wp$-functions which can be rewritten as integrable nonlinear differential equations and 2-dimensional differential relations which are the generalization of the one-dimensional two-gap Schrödinger equation with potentials which have the form of the linear combinations of hyperelliptic $\wp$-functions with shift arguments.

Introduction

The hyperelliptic Abel functions expressed as derivatives from the hyperelliptic sigma function σ (which is proportional to the n-dimensional Riemann theta function) are an n-dimensional generalization of the elliptic one-dimensional Weierstrass functions [1, 2, 3]. First and second derivatives of these hypereliptic functions are hyperelliptic ζ- and $\wp$-functions dependent on vector arguments $\boldsymbol{u}$ which is the Abel map of corresponding hypereliptic curve V to the Jacobi manifold $\mathrm{Jac}(V)$, where $V = \left\{(y,x) \in \boldsymbol{C}^2 : y^2 - \sum\limits_{i=0}^{2g} \lambda_i x^i = 0\right\}$ means the hyperelliptic curve.

An algebraic curve V is characterized by canonical differential 1-forms including holomorphic, meromorhic differentials of second and third kinds and the special differential 2-form of second kind. The fundamental relation between the differential 2- and 1-forms which is established with a help of the Riemann vanishing theorem for the theta functions leads to the fundamental Baker relations between hyperelliptic σ- and $\wp$-functions. This permits to construct special multi-dimensional linear differential equations with known (see [4, 5]) solutions. Also, the Baker relation leads to the relation connecting derivatives of $\wp$-functions part of which can be rewritten in the form of known integrable equations (see [5]).

1 Relations between differential 1- and 2-forms

The hyperelliptic curve V with cuts connecting branching points realizes the hyperelliptic Riemann curve which is characterized by the canonical system of differential 1-forms holomorphic and meromorphic on the Reimann surface. Holomorphic differentials $du_i = x^{i-1}dx/y$, $i = \overline{1,g}$ on the Riemann curve with the canonical basis of cycles $\overline{a_1, a_g}$ and $\overline{b_1, b_g}$ determine $g \times g$-matrices of a- and b-periods

$$2\omega = \left(\oint_{a_k} u_l\right), \qquad 2\omega' = \left(\oint_{b_k} du_l\right).$$

The relations

$$\mathrm{d}\mathbf{v} = (2\omega)^{-1}\mathrm{d}\mathbf{u},$$

and

$$\boldsymbol{\tau}=\oint_{b_k}\mathrm{d}v_l=\omega^{-1}\omega'$$

determine the normalized g-dimensional vector $\mathbf{v}$ and the $\boldsymbol{\tau}$-matrix of the Riemann surface Γ, respectively. These vector and matrix determine the the Riemann theta function

$$\theta[^{\epsilon'}_{\epsilon''}](\tilde{\boldsymbol{z}}|\boldsymbol{\tau})=\sum_{\boldsymbol{n}\in Z^g}\exp\left\{\imath\pi\left(\boldsymbol{n}+\frac{1}{2}\epsilon',\tau\boldsymbol{n}+\frac{1}{2}\epsilon\right)+2\imath\pi\left(\tilde{\boldsymbol{z}}+\frac{1}{2}\epsilon'',\boldsymbol{n}+\frac{1}{2}\epsilon'\right)\right\},$$

$$\tilde{\boldsymbol{z}}=\int_{x_0}^{x_k}d\boldsymbol{v}-\sum_{k=1}^{g}\int_{x_0}^{x_k}d\boldsymbol{v}+\boldsymbol{K},\qquad K_j=\frac{1+\tau_{jj}}{2}-\sum_{i\neq j}\int_{a_i}\left(dv_i(x)\int_{x_0}^{x}dv_j\right),$$

at $x\in\Gamma$ where Γ is the Riemann surface corresponding to the hyperelliptic curve V. Here $\epsilon'^{('')}=\left({\epsilon'_1}^{('')},\ldots,{\epsilon'_g}^{('')}\right)$, ${\epsilon'_i}^{('')}\in(0,1)$. The vanishing property of this theta function in g points of the Riemann surface which constitutes the essence of the Riemann vanishing theorem (see [6]) is used for calculating principal relations between proposed by Klein [1] above mentioned hyperelliptic Abel functions.

The meromorphic differentials of second kind of the form

$$\mathrm{d}r_j=\sum_{k=j}^{2g+1-j}(k+1-j)\lambda_{k+1+j}\frac{x^k\mathrm{d}x}{4y},\qquad j=\overline{1,g} \tag{1}$$

determine η-matrices of a and b-periods

$$2\eta=\left(-\oint_{a_k}\mathrm{d}r_l\right),\qquad 2\eta'=\left(-\oint_{b_k}\mathrm{d}r_l\right).$$

The latter together with ω-matrices enter in definition of the above mentioned basis hyperelliptic Abel function

$$\sigma(\boldsymbol{u})=C(\boldsymbol{\tau})\exp\left\{\boldsymbol{u}^T\boldsymbol{\kappa}\boldsymbol{u}\right\}\theta[\varepsilon]\left((2\omega)^{-1}\boldsymbol{u}-\boldsymbol{K}_a|\boldsymbol{\tau}\right). \tag{2}$$

Here $\boldsymbol{\kappa}=(2\omega)^{-1}\boldsymbol{\eta}$, $\boldsymbol{K}_a$ is the vector of Riemann constants with the base point a and $C(\boldsymbol{\tau})$ is the constant which is determined by the parameters of the hyperelliptic curve V (see [5]).

Principal relations between the hyperelliptic σ-functions (refsithet) and ζ and $\wp$-functions are provided with help of the fundamental 2-differential

$$\mathrm{d}\omega(z_1,z_2)=\frac{2y_1y_2+F(x_1,x_2)}{4(x_1-x_2)^2}\frac{\mathrm{d}x_1}{y_1}\frac{\mathrm{d}x_2}{y_2}, \tag{3}$$

where

$$F(x_1,x_2)=2\lambda_{2g+2}x_1^{g+1}x_2^{g+1}+\sum_{i=0}^{g}x_1^ix_2^i(2\lambda_{2i}+\lambda_{2i+1}(x_1+x_2)). \tag{4}$$

Taking into account (4) we can rewrite (3) in the form

$$\mathrm{d}\omega(x_1,x_2)=\frac{\partial}{\partial x_2}\left(\frac{y_1+y_2}{2y_1(x_1-x_2)}\right)\mathrm{d}x_1\mathrm{d}x_2+\mathrm{d}\boldsymbol{u}^T(x_1)\mathrm{d}\boldsymbol{r}(x_2) \tag{5}$$

of the Abelian 2-differential with the pole of the second order.

Applying the Abel map (defined by the equality $\hat{\boldsymbol{A}}(\cdots) = \sum_{k=1}^{g} \int_{x_{0k}}^{x_k} dx(\cdots)$) to the fundamental 2-differential (3) with respect to the variable x_2, integrating over the variable x_1 taking into account (5) and the Riemann vanishing theorem [6] we can obtain an expression in the form of rations of logarithm of the Riemann theta functions [2] (also see [5]). Then, a substitution the theta-representation of σ functions (2) leads to the fundamental relation

$$\begin{aligned}
&\int_{\mu}^{x} \sum_{i=1}^{g} \int_{\mu_i}^{x_i} \frac{2yy_i + F(x, x_i)}{4(x - x_i)^2} \frac{\mathrm{d}x}{y} \frac{\mathrm{d}x_i}{y_i} \\
&\quad = \ln \left\{ \frac{\sigma\left(\int_{a_0}^{x} \mathrm{d}\mathbf{u} - \sum_{i=1}^{g} \int_{a_i}^{x_i} \mathrm{d}\mathbf{u}\right)}{\sigma\left(\int_{a_0}^{x} \mathrm{d}\mathbf{u} - \sum_{i=1}^{g} \int_{a_i}^{\mu_i} \mathrm{d}\mathbf{u}\right)} \right\} - \ln \left\{ \frac{\sigma\left(\int_{a_0}^{\mu} \mathrm{d}\mathbf{u} - \sum_{i=1}^{g} \int_{a_i}^{x_i} \mathrm{d}\mathbf{u}\right)}{\sigma\left(\int_{a_0}^{\mu} \mathrm{d}\mathbf{u} - \sum_{i=1}^{g} \int_{a_i}^{\mu_i} \mathrm{d}\mathbf{u}\right)} \right\},
\end{aligned} \tag{6}$$

where the F-function is defined above.

By definition, ζ and $\wp$ hyperelliptic functions are determined via σ functions by differential relations

$$\zeta_i(\boldsymbol{u}) = \frac{\partial}{\partial_{u_i}} \ln \sigma(\boldsymbol{u}), \qquad \wp_{ij}(\boldsymbol{u}) = -\frac{\partial^2}{\partial_{u_j}\partial_{u_j}} \ln \sigma(\boldsymbol{u}), \qquad i, j = \overline{1, g},$$

where the vector $\boldsymbol{u}$ belongs to the Jacobian $\mathrm{Jac}(V)$ of the hyperelliptic curve. A differentiation of (6) with respect to variables u_j leads to relation for ζ and $\wp$ hyperelliptic functions corresponding to hyperelliptic curves with the arbitrary genus g.

Differentiating $\partial^2/\partial_{x_i}\partial_{x_j}$ the relation (6) yields the equality

$$P(x; \boldsymbol{u}) = 0, \qquad P(x; \boldsymbol{u}) = \sum_{j=0}^{g-1} \wp_{g,j+1} x^j \tag{7}$$

which gives the solution of the inverse Jacobi problem consisting in calculating points x_i, $i = \overline{1, g}$ of the Riemann surface via the values of the vector $\boldsymbol{u}$.

Differentiating both sides (6) with respect to ∂/∂_{x_i} from both sides of (6) we can come to the relations

$$-\zeta_j\left(\int_a^{x_0} \mathrm{d}\mathbf{u} + \boldsymbol{u}\right) = \int_a^{x_0} \mathrm{d}r_j + \sum_{k=1}^{g} \int_{a_k}^{x_k} \mathrm{d}r_j - \frac{1}{2} \sum_{k=0}^{g} y_k \left(\left. \frac{D_j(R'(z))}{R'(z)} \right|_{z=x_k} \right), \tag{8}$$

where $R(z) = \prod_0^g (z - z_j)$ and $R'(z) = (\partial/\partial_z) R(z)$ and

$$-\zeta_j(\boldsymbol{u}) = \sum_{k=1}^{g} \int_{a_k}^{x_k} \mathrm{d}r_j - \frac{1}{2} \wp_{gg,j+1}(\boldsymbol{u}). \tag{9}$$

The relations (8) and (9) are the basis for obtaining principal relations between hyperelliptic functions.

2 Basis relations for hyperelliptic functions

A differentiation of the relations (8) and (9) with the respect to u_j leads to the fundamental Baker addition formula ([3], see [5])

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma^2(\boldsymbol{u})\sigma^2(\boldsymbol{v})} = M(\boldsymbol{u},\boldsymbol{v}), \tag{10}$$

where $M(\boldsymbol{u},\boldsymbol{v})$ is a polynomial in $\wp$-functions. Here M-function is determined by differential equation [5]

$$\left\{\left(\frac{\partial^2}{\partial_{u_g^2}} - \frac{\partial^2}{\partial_{v_g^2}}\right)\ln M_{k-1}(\boldsymbol{u},\boldsymbol{v}) + 2M_1(\boldsymbol{u},\boldsymbol{v})\right\}M_{k-1}^2(\boldsymbol{u},\boldsymbol{v}) - 4M_k(\boldsymbol{u},\boldsymbol{v})M_{k-1}(\boldsymbol{u},\boldsymbol{v}) = 0. \tag{11}$$

This equation is a recursive relation for M_k-functions where the subscript k means the genus of the corresponding hyperelliptic curve V.

In the case $g=2$ under consideration it can be shown that

$$M(\boldsymbol{u},\boldsymbol{v}) = \wp_{22}(\boldsymbol{u})\wp_{12}(\boldsymbol{v}) - \wp_{12}(\boldsymbol{u})\wp_{22}(\boldsymbol{v}) + \wp_{11}(\boldsymbol{v}) - \wp_{11}(\boldsymbol{u}). \tag{12}$$

On the basis of the Baker addition formula and the recursive relation (11) we can obtain the system of possible relations between derivatives of $\wp$-functions with respect to variables u_i, $i=\overline{1,g}$.

Taking logarithm of both sides of the equality (10) and with help of differentiating ith respect to u_j and v_j we can obtain well known addition formulae for the hyperelliptic ζ-functions of the form

$$\zeta_j(\boldsymbol{u}+\boldsymbol{v}) - \zeta(\boldsymbol{u}) - \zeta_j(\boldsymbol{v}) = \frac{1}{2}\frac{1}{M(\boldsymbol{u},\boldsymbol{v})}\left(\frac{\partial}{\partial_{u_j}} + \frac{\partial}{\partial_{u_j}}\right)M(\boldsymbol{u},\boldsymbol{v}) \tag{13}$$

and

$$\zeta_j(\boldsymbol{u}+\boldsymbol{v}) + \zeta_j(\boldsymbol{u}-\boldsymbol{v}) = \zeta(\boldsymbol{u}) + \zeta_j(\boldsymbol{v}) + \frac{1}{2}\frac{1}{M(\boldsymbol{u},\boldsymbol{v})}\left(\frac{\partial}{\partial_{u_j}} + \frac{\partial}{\partial_{u_j}}\right)M(\boldsymbol{u},\boldsymbol{v}), \tag{14}$$

$$\zeta_j(\boldsymbol{u}+\boldsymbol{v}) - \zeta_j(\boldsymbol{u}-\boldsymbol{v}) = 2\zeta(\boldsymbol{v}) + \frac{1}{M(\boldsymbol{u},\boldsymbol{v})}\frac{\partial}{\partial_{u_j}}M(\boldsymbol{u},\boldsymbol{v}). \tag{15}$$

Then, expanding the functions

$$\Omega_+ = \ln\sigma(\boldsymbol{u}+\boldsymbol{v}) + \ln\sigma(\boldsymbol{u}-\boldsymbol{v}), \qquad \Omega_- = \ln\sigma(\boldsymbol{u}+\boldsymbol{v}) - \ln\sigma(\boldsymbol{u}-\boldsymbol{v})$$

into power series in the small value $\boldsymbol{v}$ and taking into account the relations (14) and (15) we can obtain all possible differential relations between $\wp$-functions. In the case of the genus $g=2$ such power expansion leads to relations

$$\wp_{2222} = 6\wp_{22}^2 + \frac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{12},$$

$$\wp_{1111} = 6\wp_{11}^2 - 3\lambda_0\wp_{22} + \lambda_1\wp_{12} + \lambda_2\wp_{11} - \frac{1}{2}\lambda_0\lambda_4 + \frac{1}{8}\lambda_1\lambda_3,$$

$$\wp_{2221} = 6\wp_{22}\wp_{12} + \lambda_4\wp_{12} - 2\wp_{11},$$

$$\wp_{2111} = 6\wp_{12}\wp_{11} + \lambda_2\wp_{12} - \frac{1}{2}\lambda_1\wp_{22} - \lambda_0,$$

$$\wp_{2211} = 2\wp_{22}\wp_{11} + 4\wp_{12}^2 + \frac{1}{2}\lambda_3\wp_{12}.$$

Here the first equation can be rewritten as the two-gap KdV equation with respect to $\wp_{22}$ and the last equation can be rewritten as sine-Gordon equation with respect to $\ln \wp_{12}$. Analogously we can obtain another differential relations between $\wp$ functions (see [5]).

The relation (13) can be rewritten as a differential equation

$$\frac{\partial}{\partial_{u_j}}\Phi(\boldsymbol{u}) = \Lambda\Phi(\boldsymbol{u}), \qquad \Lambda = \frac{1}{2}\frac{1}{M(\boldsymbol{u},\boldsymbol{v})}\left(\frac{\partial}{\partial_{u_j}} + \frac{\partial}{\partial_{u_j}}\right) M(\boldsymbol{u},\boldsymbol{v}) \tag{16}$$

with respect to the Bloch function

$$\Phi(u_0, \boldsymbol{u}; (y,x)) = \frac{\sigma(\boldsymbol{\alpha} - \boldsymbol{u})}{\sigma(\boldsymbol{\alpha})\sigma(\boldsymbol{u})} \exp\left(-\frac{1}{2}yu_0 + \boldsymbol{\zeta}^T(\boldsymbol{\alpha})\boldsymbol{u}\right),$$

for which the hyperelliptic curve $V(y,x)$ is the spectral variety. Here $\boldsymbol{\zeta}^T(\boldsymbol{\alpha}) = (\zeta_1(\boldsymbol{\alpha}), \ldots, \zeta_g(\boldsymbol{\alpha}))$ and $(y,x) \in V$, $\boldsymbol{u}$ and $\boldsymbol{\alpha} = \int_a^x \mathrm{d}\boldsymbol{u} \in \mathrm{Jac}(V)$. Using the equation (refDE) we can construct the system of linear differential equations of the second order both for the Φ-function and for linear combinations of their derivatives. This system yields the solution of the spectral problem for linear differential equations of the second order which can considerate as generalization of Schrödinger equation.

In the case of the genus $g = 2$ this system of differential equations with respect to Φ-function has the form [5]

$$\begin{aligned}
&\left(\partial_2^2 - 2\wp_{22}\right)\Phi = \frac{1}{4}(4x + \lambda_4)\Phi,\\
&\left(\partial_2\partial_1 - \wp_{22}\partial_2^2 + \frac{1}{2}\wp_{222}\partial_2 + \wp_{22}^2 - 2\wp_{12}\right)\Phi = \frac{1}{4}\left(4x^2 + \lambda_4 x + \lambda_3\right)\Phi,\\
&\left(\partial_1^2 - 2\wp_{12}\partial_2^2 + \wp_{122}\partial_2 + 2\wp_{22}\wp_{12}\right)\Phi = \frac{1}{4}\left(4x^3 + \lambda_4 x^2 + \lambda_3 x + \lambda_2\right)\Phi,
\end{aligned}$$

where λ_i denotes coefficients of the hyperelliptic curve V. Analogously spectral problem is solved for the function in the form of linear combinations of different derivatives of Φ-function with different shifts of argument $\boldsymbol{u}$. In doing so, corresponding linear differential equations have a similar form and after reduction from the hyperelliptic to elliptic curve V turn to be the one-dimension Schrödinger equations with the Treibich–Verdier potentials [7].

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 406–410.

Explicit Solutions of Nonlinear Evolution Equations via Nonlocal Reductions Approach

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We consider a system of nonlinear partial differential equations admitting the operator Zakharov–Shabat representation. By means of nonlocal reductions approach explicit solutions of the equations under consideration are found.

In the last 30 years a great progress in the investigation of nonlinear partial differential equations of mathematical and theoretical physics was achieved due to application of different approaches mainly based on modern functional and algebraic-geometric methods [1–3]. It gave possibility to study different properties of solutions important for applications in physics, mechanics and other fields of knowledge and to construct exact formulae for solutions of many systems of nonlinear differential equations having numerous applications.

In the present communication we consider an approach connected with studying of a system of nonlinear partial differential equations possessing so-called operator Zakharov–Shabat representation which is based on reduction principle. The formula for exact solutions of a system of nonlinear partial differential equations describing resonance interaction of M waves is given.

1. Let Φ be a function of variables $x, y, t \in \mathbf{R}^1$ satisfying to a system of linear partial differential equations

$$\alpha\frac{\partial\Phi}{\partial y} = L_1\Phi, \qquad \beta\frac{\partial\Phi}{\partial t} = L_2\Phi, \qquad \alpha,\beta \in \mathbf{R}^1, \tag{1}$$

where differential operators

$$L_1 = \sum_{i=0}^{p} u_i \frac{\partial^i}{\partial x^i}, \qquad u_i = u_i(x,y,t), \quad i = \overline{0,p}, \tag{2}$$

$$L_2 = \sum_{0}^{q} v_j \frac{\partial^j}{\partial x^j}, \qquad v_j = v_j(x,y,t), \quad j = \overline{0,q}, \tag{3}$$

are defined in the corresponding functional space.

A system (1)–(3) is compatible if the following Frobenius (operator) condition

$$\beta\frac{\partial L_1}{\partial t} - \alpha\frac{\partial L_2}{\partial y} + [L_1, L_2] = 0 \tag{4}$$

takes place, where in the operator equation $[L_1, L_2] = L_1L_2 - L_2L_1$ is a commutator of differential operators L_1 and L_2.

The operator relation (4) is called the Zakharov–Shabat representation or the generalized Lax representation [1, 2].

In general the equality (4) is fulfilled if the potentials (coefficients of operators L_1 and L_2)

$$u_i = u_i(x, y, t), \quad i = \overline{0, p}, \qquad v_j = v_j(x, y, t), \quad j = \overline{0, q}, \tag{5}$$

satisfy a system of partial differential equations usually written in the form

$$K_l[\mathbf{u}, \mathbf{v}] = 0, \qquad l = \overline{0, p+q}, \tag{6}$$

where $\mathbf{u} = \{u_1, u_2, \ldots, u_p\}$, $\mathbf{v} = \{v_1, v_2, \ldots, v_q\}$.

The equations (4) and (6) are equivalent in the certain sense [1, 2].

The given compatible linear system (1)–(3) has an important significance and practical applications when the coefficients of the operators (2), (3) satisfy to some additional conditions, which are called *reductions.*

One of the important problem of the modern theoretical and mathematical physics and, in particular, the theory of nonlinear dynamical system [2, 3], is the problem of classification and description of the reductions under which the system (1)–(3) is compatible.

There is an important type of restrictions under which the system of differential equations (1)–(3) is compatible. This restrictions are called nonlocal reductions [4, 5].

Definition. *A system of equations of general form*

$$F[\mathbf{u}, \mathbf{v}; \Phi; x, y, t] = 0 \tag{7}$$

is called a nonlocal reduction for the system (1)–(3) if the equations (7) and (1)–(3) are compatible.

If nontrivial functions

$$u_i = F_i[\Phi; x, y, t], \quad i = \overline{0, p}, \qquad v_j = G_j[\Phi; x, y, t], \quad j = \overline{0, q}, \tag{8}$$

satisfy the equations (7), then after substitution of functions (8) into system (1)–(3) we obtain a system of nonlinear partial differential equations of the following form

$$\alpha \frac{\partial \Phi}{\partial y} = \sum_{i=0}^{p} F_i[\Phi; x, y, t] \frac{\partial^i \Phi}{\partial x^i}, \tag{9}$$

$$\beta \frac{\partial \Phi}{\partial t} = \sum_{j=0}^{q} G_j[\Phi; x, y, t] \frac{\partial^j \Phi}{\partial x^j}. \tag{10}$$

The system (9), (10) is compatible due to the conditions (4). In other words, the functions (8) satisfy the system of partial differential equations (6).

Thus, the problem of solving the system of nonlinear differential equations (7) equivalent to the operator equation (4) in $(2+1)$-dimensions is reduced to the corresponding problem for the system (9), (10) in $(1+1)$-dimensions.

2. Nonlocal reductions in linear hyperbolic systems. Explicit solutions to a system of nonlinear differential equations describing resonance interaction of M waves. Let us consider a hyperbolic system of linear partial differential equations of first order

$$\frac{\partial \Phi}{\partial y} = A \frac{\partial \Phi}{\partial x} + P\Phi, \tag{11}$$

$$\frac{\partial \Phi}{\partial t} = B\frac{\partial \Phi}{\partial x} + Q\Phi, \tag{12}$$

$$\frac{\partial \Phi^*}{\partial y} = A\frac{\partial \Phi^*}{\partial x} - \bar{P}^T\Phi^*, \tag{13}$$

$$\frac{\partial \Phi^*}{\partial y} = B\frac{\partial \Phi^*}{\partial x} - \bar{Q}^T\Phi^*, \tag{14}$$

where $A = \operatorname{diag}(a_1, a_2, \dots, a_n)$ and $B = \operatorname{diag}(b_1, b_2, \dots, b_n)$ are diagonal $(n \times n)$-matrices, elements of which are real numbers satisfying to the conditions $a_i \neq a_j$, $b_i \neq b_j$ when $i \neq j$, $i, j = \overline{1, n}$.

Here $\Phi = \Phi(x, y, t)$ and $\Phi^* = \Phi^*(x, y, t)$ are $(n \times m)$-matrix functions, elements of which are second-degree integrable with respect to variable x, i.e.,

$$\int\limits_s^{+\infty} |\Phi_{km}(x, y, t)|^2\, dx < +\infty, \qquad \int\limits_s^{+\infty} |\Phi^*_{km}(x, y, t)|^2\, dx < +\infty, \tag{15}$$

where $k = \overline{1, n}$, $j = \overline{1, m}$ and s is an arbitrary (fixed) real number.

The property (15) of the matrices $\Phi = \Phi(x, y, t)$ and $\Phi^* = \Phi^*(x, y, t)$ described above we will denote in the following way

$$\Phi = \Phi(x, y, t), \qquad \Phi^* = \Phi^*(x, y, t) \in \mathrm{Mat}_{N\times nN}(\mathbf{R}^3; L_2^x(s, +\infty)). \tag{16}$$

The matrix potentials $P = P(x, y, t)$ and $Q = Q(x, y, t)$ belong to the space $\mathrm{Mat}_{n\times m}(\mathbf{R}^3;$ $L_2^x(s, +\infty))$, i.e., they are $(n \times m)$-matrix functions, elements of which are second-degree integrable with respect to variable x. In addition, we suppose that the diagonal elements of matrices P and Q are equal to zero, i.e.,

$$P_{ii}(x, y, t) \equiv 0, \qquad Q_{ii}(x, y, t) \equiv 0, \qquad i = \overline{1, n}. \tag{17}$$

The compatibility condition (4) for the system (11), (12) as well as for the conjugate system (13), (14), implies the following relation

$$[A, Q] = [B, P], \tag{18}$$

$$P_t - Q_y + AQ_x - BP_x + [P, Q] = 0. \tag{19}$$

It is easy to verify that the matrix-functions

$$P = [V, A], \qquad Q = [V, B] \tag{20}$$

satisfy the condition (18) for some arbitrary $(n \times n)$-matrix-function $V = V(x, y, t)$.

Thus the corresponding system of partial differential equations of the form (6) can be written as follows

$$[V_t, A] - [V_y, B] + AV_xB - BV_xA + [[V, A], [V, B]] = 0. \tag{21}$$

In the case when V is an Hermitian matrix, the equation (21) is one of fundamental nonlinear models of theoretical physics since it describes a resonance interaction of M waves, where $M = n(n-1)/2$ waves [1–3]. The equation (21) is a basic system of differential equations of nonlinear optics [6] when $n = 3$.

The equality (21) is considered as a system of nonlinear partial differential equations, solutions of which should be found.

To find the exact formula for the solutions of equation (21) let us consider now the unperturbed system of the form (11), (12) (or with zero potential P and Q) of the following form

$$\frac{\partial \varphi}{\partial y} = A\frac{\partial \varphi}{\partial x}, \qquad \frac{\partial \varphi}{\partial t} = B\frac{\partial \varphi}{\partial x}. \tag{22}$$

where the matrix function $\varphi = \varphi(x, y, t) \in \mathrm{Mat}_{n\times m}(\mathbf{R}^3, L_2^x(s, +\infty))$.

The following theorem is valid.

Theorem 1. *Let C is $(n \times n)$-matrix with constant and real elements such that $\bar{C}^T = C$ and* $\det C \neq 0$.

Then under the mapping

$$\varphi \to \Phi = \varphi\Omega^{-1}, \tag{23}$$

where $(m \times m)$-matrix

$$\Omega = C + \int_x^{+\infty} \bar{\varphi}^T(x, y, t)\varphi(x, y, t)\, dx \tag{24}$$

has in some domain $\sigma = \{(y, t) \in \mathbf{R}^2\}$ nonzero determinant, the system of differential equations (23) is transformed into the following system

$$\frac{\partial \Phi}{\partial y} = A\frac{\partial \Phi}{\partial x} + \left[\Phi\tilde{\Omega}^{-1}\bar{\Phi}^T, A\right]\Phi, \tag{25}$$

$$\frac{\partial \Phi}{\partial t} = B\frac{\partial \Phi}{\partial x} + \left[\Phi\tilde{\Omega}^{-1}\bar{\Phi}^T, B\right]\Phi, \tag{26}$$

where $(m \times m)$-matrix $\tilde{\Omega} = \tilde{\Omega}(x, y, t)$ is represented by the formula

$$\tilde{\Omega} = C^{-1} - \int_x^{+\infty} \bar{\Phi}^T(x, y, t)\Phi(x, y, t)\, dx. \tag{27}$$

Theorem 1 is proved by the direct calculation and by using the following lemma.

Lemma 1. *The matrix $\tilde{\Omega}$ is inverse to the matrix Ω, i.e.,*

$$\Omega\tilde{\Omega} = \tilde{\Omega}\Omega = E, \tag{28}$$

where E is identity $(m \times m)$-matrix.

To prove the lemma 1 it is sufficient to note that $\det \tilde{\Omega} \neq 0$ if $(y, t) \in \sigma$ and x is enough large and to consider the derivative of matrix $\tilde{\Omega}$ with respect to variable $x \in \mathbf{R}^1$, i.e.,

$$\frac{\partial}{\partial x}\tilde{\Omega} = \bar{\Phi}^T(x, y, t)\Phi(x, y, t). \tag{29}$$

Taking into consideration formula (23) and compairing the value (29) with relations

$$\frac{\partial}{\partial x}\tilde{\Omega}^{-1} = -\Omega^{-1}\Omega_x\Omega^{-1} = \Omega^{-1}\bar{\varphi}^T\varphi\Omega^{-1} = \bar{\Phi}^T\Phi, \tag{30}$$

it is easy to conclude the Lemma 1.

From the argument mentioned above we deduce the following theorem.

Theorem 2. *The identity*

$$\Phi\tilde{\Omega}^{-1}\bar{\Phi}^T \equiv \varphi\Omega^{-1}\bar{\varphi}^T \tag{31}$$

is true.

The system (25), (26) is compatible since the system (22) has the same property. It is a simple implication from compatibility conditions (18) and (19).

By comparison of equations (11), (12) and (22) it is easy to conclude that the constraints

$$P = [V, A], \qquad Q = [V, B], \tag{32}$$

where

$$V = \Phi\tilde{\Omega}^{-1}\bar{\Phi}^T = \Phi\left(C^{-1} - \int_x^{+\infty} \bar{\Phi}^T\Phi\, dx\right)^{-1}\bar{\Phi}^T \tag{33}$$

are admissible nonlocal reductions for the system of linear partial differential equations (11), (12).

Thus, it allows us to state the following result.

Theorem 3. *The solution of partial differential equations (21) are represented with the following formula*

$$V = \Phi\left(C^{-1} - \int_x^{+\infty} \bar{\Phi}^T\Phi\, dx\right)^{-1}\bar{\Phi}^T \equiv \varphi\left(C + \int_x^{+\infty} \bar{\varphi}^T\varphi\, dx\right)^{-1}\bar{\varphi}^T$$

or in component form

$$V_{ij} = \varphi_i\left(C + \int_x^{+\infty} \bar{\varphi}_k^T \otimes \varphi_k\, dx\right)^{-1}\bar{\varphi}_j^T,$$

where $\varphi_i = \varphi_i(x + a_i y + b_i t)$ is i-tuple of matrix function $\varphi(x, y, t)$.

Matrix function $\varphi(x, y, t)$ is a solution of unperturbed systems (22) and has elements of the following form $\varphi_{km} = f_{km}(x + a_k y + b_k t)$, where $f_{km}(\tau)$, $k = \overline{1, n}$, $m = \overline{1, m}$, are arbitrary continuos differentiable functions of variable $\tau \in \mathbf{R}^1$.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 411–416.

Exact Solutions of the Inhomogeneous Problems for Polyparabolic Operator

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The theorem establishing the correctness of the inhomogeneous problem for polyparabolic equation with righ-hand side belonging to set of bounded functions in $\mathbf{R}^n$ is proved. Exact formulas for constants of evaluations for potentials with the these densities are represented; exact solutions for particular cases are obtained.

We consider inhomogeneous problem for a linear partial differential equation

$$T^{m+1}u \equiv \sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\frac{\partial^{m-j+1}}{\partial t^{m-j+1}}\nabla^{2j}u(t,\mathbf{x}) = f(t,\mathbf{x}), \tag{1}$$

where $t \in \mathbf{R}^1_+$, $\mathbf{x} \in \mathbf{R}^n$ $(n \in \mathbf{N})$, ∇^2 is the Laplace operator, $f(t,\mathbf{x}) \in L^1_{\mathrm{loc}}(\mathbf{R}^{1+n})$, $\binom{m+1}{j}$ are binomial coefficients.

In the case $m = 0$ this equation is transformed to the classical heat transfer equation. Let us introduce an arbitrary exact solution for equation $T^{m+1}u = 0$, which is defined at a domain of space $\mathbf{R}^{n+1}$, the polycaloric function [1, 2], which takes the form

$$u(t,\mathbf{x}) = u_0(t,\mathbf{x}) + tu_1(t,\mathbf{x}) + \cdots + t^m u_m(t,\mathbf{x}), \tag{2}$$

where $u_k(t,\mathbf{x})$ are solutions of the equation $Tu = 0$. We find by induction

$$T^{m+1}(t^k u) = \sum_{j=0}^{m+1} j!\binom{m+1}{j}\binom{k}{j}t^{k-j}T^{m-j+1}u. \tag{3}$$

The fundamental solution for operator T^{m+1} from space $\mathcal{D}'(\mathbf{R}^{1+n})$ is [3]

$$\mathcal{E}_{m,n}(t,\mathbf{x}) = \frac{\theta(t)t^{m-n/2}}{(2\sqrt{\pi})^n m!}e^{-\frac{|\mathbf{x}|^2}{4t}}. \tag{4}$$

It is positive, vanishing for $t < 0$, infinitely differentiable for $(t,\mathbf{x}) \neq 0$ and has additional properties

$$\int\limits_{\mathbf{R}^n} \mathcal{E}_{m,n}(t,\mathbf{x})\, d^n\mathbf{x} = \frac{t^m}{m!}, \tag{5}$$

$$\frac{\partial^k}{\partial t^k}\mathcal{E}_{m,n}(+0,\mathbf{x}) = 0 \quad (0 \le k \le m-1), \qquad \frac{\partial^m}{\partial t^m}\mathcal{E}_{m,n}(+0,\mathbf{x}) = 1. \tag{6}$$

From $\mathcal{E}_{m,n} \in L^1_{\mathrm{loc}}(\mathbf{R}^{1+n})$ the solution of the problem (1) can be written as convolution [4]

$$u(t,\mathbf{x}) = \mathcal{E}_{m,n}(t,\mathbf{x}) * f(t,\mathbf{x}), \tag{7}$$

which define a polycaloric potential with density $f(t,\mathbf{x})$. Then $u \in L^1_{\mathrm{loc}}(\mathbf{R}^{1+n})$, if

$$h(t,\mathbf{x}) = [\mathcal{E}_{m,n}(t,\mathbf{x}) * |f(t,\mathbf{x})|] \in L^1_{\mathrm{loc}}(\mathbf{R}^{1+n}). \tag{8}$$

The following theorem gives one of density classes with convolution (7). For simplification we do not write further indices m and n.

Let us denote a class of functions vanishing for $t < 0$ and bounded in the sphere $0 \le t \le t_0$: $|f| \le A_f = \sup |f(\tau,\xi)|$ $(0 \le \tau \le t\ ,\ \xi \in \mathbf{R}^n)$ as K_0.

Theorem. *If $f(t,\mathbf{x}) \in K_0$, a polycaloric potential $U(t,\mathbf{x})$ of m-th order is in K_0, can be written in the form (7) and satisfies the following estimates:*

$$|U| \le A_f \frac{t^{m+1}}{(m+1)!}, \tag{9}$$

$$\left|\frac{\partial^k U}{\partial t^k}\right| \le A_f a^{(k)}_{m,n} t^{m-k+1}, \tag{10}$$

$$\left|\nabla^{2p} U\right| \le A_f b^{(p)}_{m,n} t^{m-p+1}, \tag{11}$$

Here $a^{(k)}_{m,n}$ and $b^{(p)}_{m,n}$ are positive constants, and initial conditions are

$$U(+0,\mathbf{x}) = 0, \tag{12}$$

$$\left.\frac{\partial^k U}{\partial t^k}\right|_{t=+0} = 0 \quad (1 \le k \le m), \qquad \nabla^{2p} U|_{t=+0} = 0 \quad (1 \le p \le m), \tag{13}$$

$$T^s U(+0,\mathbf{x}) = 0 \quad (1 \le s \le m). \tag{14}$$

Proof. Under Fubini theorem from (5) we get

$$h(t,\mathbf{x}) \le A_f \frac{t^{m+1}}{(m+1)!}.$$

As $|U| \le h$, since $U = 0$ for $t < 0$, estimate (9) is satisfied and thus $U \in K_0$.

Using the formula of convolution differentation with respect to t and property (6), for $t > 0$ we come to

$$\frac{\partial^k U}{\partial t^k} = \int_0^t \int_{\mathbf{R}^n} f(\tau,\xi) \frac{\partial^k}{\partial t^k} \mathcal{E}(t-\tau, \mathbf{x}-\xi)\, d^n\xi\, d\tau.$$

Then

$$\left|\frac{\partial^k U}{\partial t^k}\right| \le A_f \int_{\mathbf{R}^n} \frac{\partial^{k-1}}{\partial t^{k-1}} \mathcal{E}(t,\xi)\, d^n\xi. \tag{15}$$

Further

$$\frac{\partial^{k-1}\mathcal{E}}{\partial t^{k-1}} = \frac{1}{(2\sqrt{\pi})^n m!} \sum_{l=0}^{k-1} \binom{k-1}{l} \frac{d^{k-l-1} t^{m-n/2}}{dt^{k-l-1}} \frac{\partial^l}{\partial t^l} e^{-\frac{|\xi|^2}{4t}},$$

and

$$\frac{d^{k-l-1}t^{m-n/2}}{dt^{k-l-1}} = \frac{\Gamma\left(m-\frac{n}{2}+1\right)}{\Gamma\left(m-\frac{n}{2}-k+l+2\right)} t^{m-\frac{n}{2}-k+l-1},$$

$$\frac{\partial^l}{\partial t^l} e^{-\frac{|\xi|^2}{4t}} = t^{-l} e^{-\frac{|\xi|^2}{4t}} \sum_{j=0}^{l-1} (-1)^j (l-1)\dots(l-j) \binom{l}{j} \left(\frac{|\xi|^2}{4t}\right)^{l-j},$$

follows term in (15)

$$\int\limits_{\mathbf{R}^n} \frac{\partial^{k-1}}{\partial t^{k-1}} \mathcal{E}(t,\xi)\, d^n\xi = \frac{t^{m-\frac{n}{2}+k+1}}{(2\sqrt{\pi})^n m!} \sum_{l=0}^{k-1} \binom{k-1}{l} \frac{\Gamma\left(m-\frac{n}{2}+1\right)}{\Gamma\left(m-\frac{n}{2}-k+l+2\right)}$$
$$\times \sum_{j=0}^{l-1} (-1)^j (l-1)(l-2)\dots(l-j) \binom{l}{j} \int\limits_{\mathbf{R}^n} \left(\frac{|\xi|^2}{4t}\right)^{l-j} e^{-\frac{|\xi|^2}{4t}}\, d^n\xi.$$

Here $\Gamma(z)$ is the Gamma function [5].

But inserting

$$\begin{aligned} &\int\limits_{\mathbf{R}^n} |\xi|^{2l-2j} e^{-\frac{|\xi|^2}{4t}}\, d^n\xi \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int\limits_0^\infty \rho^{2(l-j)+n-1} e^{-\frac{\rho^2}{4t}}\, d\rho = \frac{(2\sqrt{\pi})^n}{\Gamma\left(\frac{n}{2}\right)} 2^{2l-2j} \Gamma\left(l-j+\frac{n}{2}\right) t^{l-j+\frac{n}{2}}, \end{aligned} \tag{16}$$

into (15) yields estimate (10), where a constant $a_{m,n}^{(k)}$ depends on the order of polycaloric potential, order of its derivative with respect to time, dimension of space and is calculated exactly:

$$a_{m,n}^{(k)} = \frac{\Gamma\left(m-\frac{n}{2}+1\right)}{m!\Gamma\left(\frac{n}{2}\right)} \sum_{l=1}^{k-1} \sum_{j=0}^{l-1} \frac{\binom{k-1}{l} \left(\binom{l}{j}\right)^2 \Gamma\left(l-j+\frac{n}{2}\right)(l-j)j!}{l\,\Gamma\left(m-\frac{n}{2}-k+l+2\right)}.$$

Following the usual procedure for finding estimates let us consider operator

$$\nabla^{2p} U = \nabla^{2p}[\mathcal{E}(t,\mathbf{x}) * f(t,\mathbf{x})] = f(t,\mathbf{x}) * \nabla^{2p}\mathcal{E}(t,\mathbf{x}).$$

Since $f \in K_0$, then

$$\left|\nabla^{2p} U\right| \le A_f \int\limits_0^t \int\limits_{\mathbf{R}^n} \nabla^{2p} \mathcal{E}(\tau,\mathbf{x})\, d^n\mathbf{x}\, d\tau = \frac{A_f}{(2\sqrt{\pi})^n m!} \int\limits_0^t \tau^{m-n/2} \int\limits_{\mathbf{R}^n} \nabla^{2p} e^{-\frac{|\mathbf{x}|^2}{4\tau}}\, d^n\mathbf{x}\, d\tau. \tag{17}$$

We can easily prove that

$$\nabla^{2p} e^{-\frac{|\mathbf{x}|^2}{4\tau}} = 2^{-2p} e^{-\frac{|\mathbf{x}|^2}{4\tau}} \sum_{j=0}^{p} \frac{(-1)^j (2j)! \binom{2p}{2j}}{j!\, \tau^{2p-j}} |\mathbf{x}|^{2p-2j},$$

hence in (17)

$$\int\limits_{\mathbf{R}^n} \nabla^{2p} e^{-\frac{|\mathbf{x}|^2}{4\tau}}\, d^n\mathbf{x} = 2^{-2p} \sum_{j=0}^{p} \frac{(-1)^j (2j)! \binom{2p}{2j}}{j!\, \tau^{2p-j}} \int\limits_{\mathbf{R}^n} |\mathbf{x}|^{2p-2j} e^{-\frac{|\mathbf{x}|^2}{4\tau}}\, d^n\mathbf{x}. \tag{18}$$

Here the integral in the right hand side exists and is calculated according to the formula (16). Inserting the result of calculation into inequality (17), we obtain final result (11). Here a constant $b_{m,n}^{(p)}$ can be written in the form:

$$b_{m,n}^{(p)} = \frac{(2p)!}{(m-p+1)m!\,\Gamma\left(\frac{n}{2}\right)} \sum_{j=0}^{p} \frac{\Gamma\left(p-j+\frac{n}{2}\right)}{2^{2j}(2p-2j)!}.$$

So we proved the estimates (9)–(11). Hence the polycaloric potential satisfies conditions (12) and (13), and the initial condition (14) follows from previous formula (3).

From Theorem we can apply the results for equation (1) with inhomogeneous initial conditions corresponding conditions (14):

$$T^k u(+0,\mathbf{x}) = \varphi_k(\mathbf{x}) \quad (1 \le k \le m).$$

We look for exact solutions of problem (1) in the form corresponding (7):

$$u(t,\mathbf{x}) = \frac{1}{(2\sqrt{\pi})^n m!} \int_0^t \tau^{m-n/2} e^{-\frac{|\mathbf{x}|^2}{4\tau}}\, d\tau \int_{\mathbf{R}^n} f(t-\tau,\xi) e^{-\frac{|\xi|^2 - 2(\mathbf{x}\cdot\xi)}{4\tau}}\, d^n\xi. \tag{19}$$

Namely, we consider cases of the general formula (19). If $f = f(t,|\mathbf{x}|)$, then for $n \ge 2$ from (19) we obtain

$$u(t,\mathbf{r}) = \frac{1}{2^{n-1}\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_0^t \tau^{m-n/2} e^{-\frac{|\mathbf{r}|^2}{4\tau}}\, d\tau$$
$$\times \int_0^\infty \rho^{n-1} e^{-\frac{\rho^2}{4\tau}} f(t-\tau,\rho) d\rho \int_0^\pi e^{\frac{\mathbf{r}\rho}{2\tau}\cos\varphi} \sin^{n-2}\varphi\, d\varphi,$$

where $\mathbf{r}^2 = x_1^2 + x_2^2 + \cdots + x_n^2$. Since for $\nu > 0$

$$\int_0^\pi e^{\pm z\cos\varphi} \sin^{2\nu}\varphi\, d\varphi = \sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)\left(\frac{2}{z}\right)^\nu I_\nu(z),$$

then for $n \ge 2$

$$u(t,\mathbf{r}) = \frac{1}{2\mathbf{r}^{\frac{n}{2}-1} m!} \int_0^t \tau^{m-1} e^{-\frac{|\mathbf{r}|^2}{4\tau}}\, d\tau \int_0^\infty \rho^{\frac{n}{2}} e^{-\frac{\rho^2}{4\tau}} I_{\frac{n}{2}-1}\left(\frac{\mathbf{r}\rho}{2\tau}\right) f(t-\tau,\rho)\, d\rho. \tag{20}$$

Here $I_\nu(z)$ is the Bessel function [5].

The degenerate case $n = 1$ has the following solution

$$u(t,\mathbf{x}) = \frac{1}{2\sqrt{\pi} m!} \int_0^t \tau^{m-\frac{1}{2}} e^{-\frac{\mathbf{x}^2}{4\tau}}\, d\tau \int_{-\infty}^\infty e^{-\frac{\xi^2 - 2\mathbf{x}\xi}{4\tau}} f(t-\tau,\xi)\, d\xi. \tag{21}$$

If the dimension of space is odd, then the integral with respect to ρ in (16) yields the integral from elementary functions. Namely, since for $n = 3$

$$I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\mathrm{sh} z,$$

solution (20) is

$$u(t,\mathbf{r}) = \frac{1}{\sqrt{\pi}m!\,\mathbf{r}} \int_0^t \tau^{m-\frac{1}{2}} e^{-\frac{\mathbf{r}^2}{4\tau}}\, d\tau \int_0^\infty \rho^2 e^{-\frac{\rho^2}{4\tau}} \mathrm{sh}\left(\frac{\mathbf{r}\rho}{2\tau}\right) f(t-\tau,\rho)\, d\rho. \tag{22}$$

Separately, we consider the case when $f(t,\mathbf{x})$ is a finite function in $\mathbf{R}^n$. If

$$f(t,\mathbf{x}) = A\omega(t)F(|\mathbf{x}|)\,\theta(R^2-|\mathbf{x}|^2) \qquad (A,R=\text{const}>0,\ n\geq 2).$$

then for same density from (20) we obtain

$$u(t,\mathbf{r}) = \frac{1}{2\mathbf{r}^{\frac{n}{2}-1}m!} \int_0^t \tau^{m-1} e^{-\frac{\mathbf{r}^2}{4\tau}} \omega(t-\tau)\, d\tau \int_0^R \rho^{\frac{n}{2}} e^{-\frac{\rho^2}{4\tau}} I_{\frac{n}{2}-1}\left(\frac{\mathbf{r}\rho}{2\tau}\right) F(\rho)\, d\rho.$$

Let us consider $n=1$ and $f(t,\mathbf{x}) = A\theta(t)\theta(R-|\mathbf{x}|)$. From (21) we get solution in the form

$$u(t,\mathbf{x}) = \frac{A}{2m!} \int_0^t \tau^m \left[\mathrm{erf}\left(\frac{R+\mathbf{x}}{2\sqrt{\tau}}\right) + \mathrm{erf}\left(\frac{R-\mathbf{x}}{2\sqrt{\tau}}\right)\right] d\tau, \tag{23}$$

where $\mathrm{erf}(z)$ is the probabilistic integral [5]. Then using

$$\int_0^t \tau^m \mathrm{erf}\left(\frac{z}{2\sqrt{\tau}}\right) d\tau = 2\left(\frac{z}{2}\right)^{2m+2} \int_{\frac{z}{2\sqrt{t}}}^\infty \xi^{-2m-3}\mathrm{erf}(\xi)\, d\xi,$$

and the formula [3]

$$\int_\xi^\infty \xi^{-2m-3}\mathrm{erf}(\xi)\, d\xi = \frac{1}{2(m+1)} \left\{ \frac{(-1)^{m+1}\sqrt{\pi}}{\Gamma\left(m+\frac{3}{2}\right)} \mathrm{erfc}(\xi) \right.$$

$$\left. +\xi^{-2m-2}\left[\mathrm{erf}(\xi) - \frac{1}{\pi} e^{-\xi^2} \sum_{k=1}^{m+1} (-1)^k \frac{\Gamma\left(m-k+\frac{3}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)} \xi^{2k+1}\right]\right\},$$

we obtain exact solution from (23)

$$u(t,\mathbf{x}) = \frac{At^m}{2(m+1)!} \left[\Phi_m\left(\frac{R+\mathbf{x}}{2\sqrt{t}}\right) + \Phi_m\left(\frac{R-\mathbf{x}}{2\sqrt{t}}\right)\right], \tag{24}$$

where

$$\Phi_m(z) = \mathrm{erf}(z) + \frac{(-1)^{m+1}\sqrt{\pi}}{\Gamma\left(m+\frac{3}{2}\right)} z^{2m+2}\mathrm{erfc}(z) - \frac{1}{\pi} e^{-z^2} \sum_{k=1}^{m+1} (-1)^k \frac{\Gamma\left(m-k+\frac{3}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)} z^{2k+1}.$$

We can verify easily that reduced exact solutions of problem (1) for density from K_0 satisfied proved above theorem.

Thus we can apply the results for a problem

$$\sin Tu = F(t,\mathbf{x}). \tag{25}$$

Using the expansion of left side in series and generalizing (2) and (3), we can write the fundamental solution of operator $\sin T$ from space $\mathcal{D}'(\mathbf{R}^{1+n})$ in the form of expansion into series

$$\begin{aligned}\mathcal{S}_n(t,\mathbf{x}) &= \sum_{m=0}^{\infty} \frac{(-1)^m \mathcal{E}_{m,n}(t,\mathbf{x})}{(2m+1)!} \\ &= \frac{\theta(t)e^{-\frac{|\mathbf{x}|^2}{4t}}}{(2\sqrt{\pi})^n} \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m-n/2}}{(2m)!(2m+1)!} = \frac{\theta(t)e^{-\frac{|\mathbf{x}|^2}{4t}} t^{-n/2-1}}{(2\sqrt{\pi})^n} \int_0^t \mathrm{ber} 2\sqrt{\tau}\, d\tau.\end{aligned} \tag{26}$$

This expansion is absolutely convergent for $t > 0$ and has the following properties:

$$\int_{\mathbf{R}^n} \mathcal{S}_n(t,\mathbf{x})\, d^n\mathbf{x} = \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{(2m)!(2m+1)!},$$

$$\frac{\partial^k}{\partial t^k}\mathcal{S}_n(+0,\mathbf{x}) = 0 \quad (k = 2p-1), \qquad \frac{\partial^k}{\partial t^k}\mathcal{S}_n(+0,\mathbf{x}) = \frac{(-1)^p}{(2p+1)!} \quad (k = 2p).$$

Then we obtain a solution of problems (25) for $f(t,\mathbf{x}) \in \mathbf{K_0}$ in the form of convolution

$$u(t,\mathbf{x}) = \mathcal{S}_n(t,\mathbf{x}) * f(t,\mathbf{x}), \tag{27}$$

that has form similarly to (20) and (21) for $n \geq 2$

$$\begin{aligned}u(t,\mathbf{r}) = {}& \frac{1}{2\mathbf{r}^{\frac{n}{2}-1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!(2m+1)!} \int_0^t \tau^{2m-1} e^{-\frac{|\mathbf{r}|^2}{4\tau}}\, d\tau \\ & \times \int_0^{\infty} \rho^{\frac{n}{2}} e^{-\frac{\rho^2}{4\tau}} I_{\frac{n}{2}-1}\left(\frac{\mathbf{r}\rho}{2\tau}\right) f(t-\tau,\rho)\, d\rho,\end{aligned} \tag{28}$$

and for $n = 1$

$$u(t,\mathbf{x}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2\sqrt{\pi}(2m)!(2m+1)!} \int_0^t \tau^{2m-\frac{1}{2}} e^{-\frac{\mathbf{x}^2}{4\tau}}\, d\tau \int_{-\infty}^{\infty} e^{-\frac{\xi^2-2\mathbf{x}\xi}{4\tau}} f(t-\tau,\xi)\, d\xi. \tag{29}$$

If $n = 1$ and $f(t,\mathbf{x}) = A\theta(t)\theta(R - |\mathbf{x}|)$, we find the exact solution of problem (25)

$$u(t,\mathbf{x}) = \sum_{m=0}^{\infty} \frac{(-1)^m A t^{2m}}{2((2m+1)!)^2} \left[\Phi_{2m}\left(\frac{R+\mathbf{x}}{2\sqrt{t}}\right) + \Phi_{2m}\left(\frac{R-\mathbf{x}}{2\sqrt{t}}\right)\right]. \tag{30}$$

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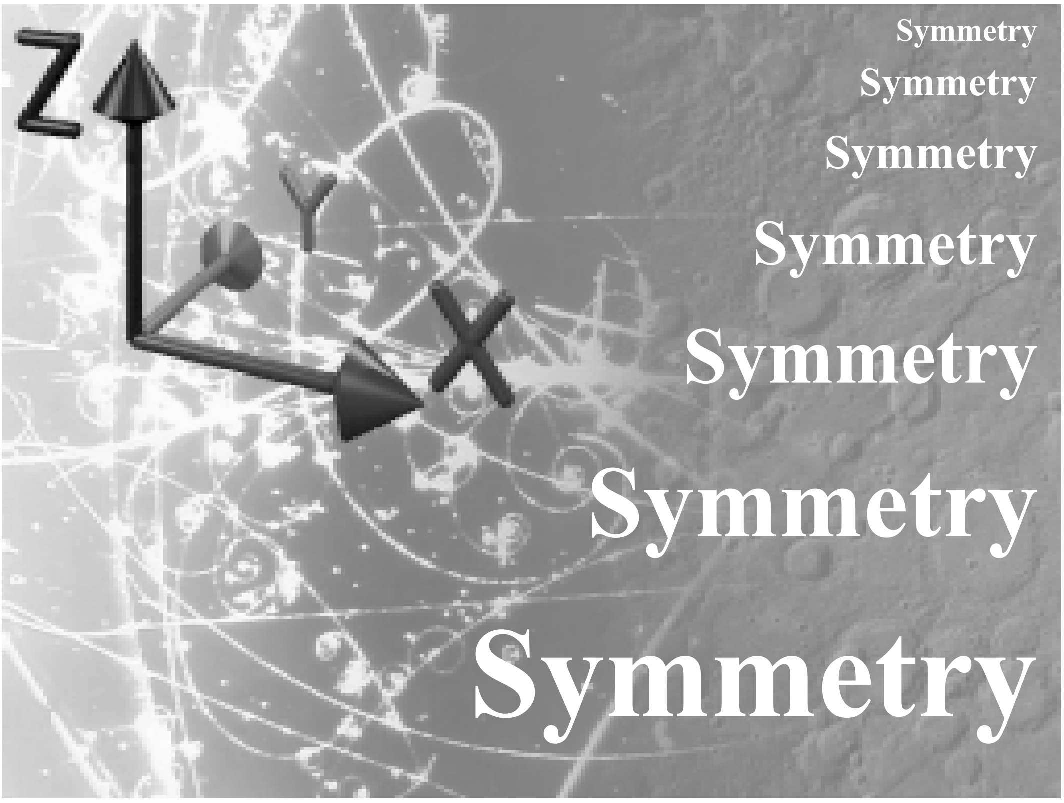

Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 419–428.

A Unitary Parallel Filter Bank Approach to Magnetic Resonance Tomography

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The purpose of the present paper is to establish a unitary parallel multichannel filter bank approach to clinical magnetic resonance tomography and magnetic resonance microscopy. The approach which is based on the Stern–Gerlach filter explains the high resolution capabilities of these non-invasive cross-sectional imaging modalities which revolutionized the field of clinical diagnostics.

To a physicist, the 20*th century begins in* 1895, *with* Wilhelm Conrad Röntgen*'s unexpected discovery of* X*-rays.*

Steven Weinberg

One morning about the 10 *July* 1925 *I suddenly saw light:* Heisenberg*'s symbolic manipulation was nothing but the matrix calculus well-known to me since my student days.*

Max Born

The amount of theoretical work one has to cover before being able to solve problems of real practical value is rather large, but this circumstance is an inevitable consequence of the fundamental part played by transformation theory and is likely to become more pronounced in the theoretical physics of the future.

Paul Adrien Maurice Dirac

Magnetic resonance imaging has the potential of totally replacing computed tomography. If history was rewritten, and CT *invented after* MRI, *nobody would bother to pursue* CT.

Philip Drew

From the beginning, it was evident that all of the powerful and sophisticated methods of nuclear MR *spectroscopy could be used in imaging, and those branches in the field of* MR *have continued to enrich one another.*

Paul Christian Lauterbur

1 The Stern–Gerlach filter

In 1921 and 1922, the pioneers of nuclear magnetic resonance (NMR), Otto Stern (Nobel Prize 1943) and Walther Gerlach, came up with a technique to test Arnold Sommerfeld's theory of spatial quantization using molecular beams. The basic idea of the molecular beam technique – first demonstrated by L. Dunoyer in 1911 – was to vaporize an element by heating it in a furnace and then allowing atoms or molecules to escape through a narrow slit located at one end of the furnace adjacent to a vacuum chamber. As the atoms or molecules entered the chamber, they were collimated into a narrow beam of isolated, non-colliding particles via a series of apertures and directed toward the opposite end of the vacuum chamber where they could be detected by a flat piece of cold glass that caused the heated molecules or atoms to condense and be deposited for post-flight analysis.

About 25 years earlier, Pieter Zeeman (Nobel Prize 1902) had found by a spectroscopic experiment that sodium's single band of yellow spectral light split into multiple, narrow bands when placed in a strong magnetic field. To experimentally verify the spatial quantization phenomenon, Stern and Gerlach added a strong magnetic field to a molecular beam – actually, in this case, a beam of silver atoms – and found that they were deposited on the beam detector in two distinct and well defined separate bands. However, two separate beam channels were not what they had expected.

Midway through the normal flight path of the beam, Stern and Gerlach had placed a C-shaped electromagnet, positioned in such a way that a beam of silver atoms traveling along and between its north and south poles would encounter, perpendicular to the beams's direction of motion, a strong magnetic field acting upon the magnetic moments of the atoms in the beam. How would the external magnetic field affect the individual magnetic moments? The answer they expected, based on classical physics, was that their degree of reorientation would be determined by their initial orientation. In other words, they expected that as the randomly oriented atoms encountered the magnetic field, those atoms whose magnetic moments tended toward parallel orientation with the external field would be attracted toward the south pole of the external magnet, those that tended toward anti-parallel orientation with the external field would be repelled or pushed toward the north pole of the external field, and those that were basically at right angles to the field direction would proceed to the destination unaffected. Under that scenario, the magnetic field would tend to widen the narrow beam, because of the reorientations, resulting in a fairly solid smear of silver being deposited in the center of the detector and decreasing in density toward the edge of the beam.

That is not what happened. The silver atoms did initially enter the chamber randomly oriented, as expected, but when they encountered the magnetic field, they were given only two choices, either parallel or antiparallel, either beam channel 1 or beam channel 2. No other options were allowed. And apparently all the magnetic moments complied, as two distinct arcing bands of silver were deposited on the detector glass with nothing in between. Where the heaviest deposit should have accumulated, according to conventional understanding, there was nothing! According to George E. Uhlenbeck and Samuel A. Goudsmit, the spatial quantization phenomenon is due to the spin of the farthest electron of the silver atom.

It is noteworthy, incidentally, that the two ways of looking at quantum effects – Bohr's quantum transitions between energy levels and Sommerfeld's spatial quantization theory – would foreshadow the two approaches to magnetic resonance taken by Isidor I. Rabi (Nobel Prize 1944) nearly two decades later as well as the nearly-simultaneous, but independent discoveries of NMR in condensed matter in late 1945 and early 1946. Rabi and his collaborators would use both approaches. Edward M. Purcell (Nobel Prize 1952, jointly with Felix Bloch) would focus on quantum transitions between energy levels and Felix Bloch, former student of Werner Heisenberg, on physical reorienting of magnetic moments. Of course, neither Purcell nor Bloch had digital computers available which were powerful and fast enough to apply the *dynamics* inherent to NMR to the purposes of non-invasive clinical imaging. Finally, Nicolaas Bloembergen (Nobel Prize 1981) helped set stage for human MRI scanning and Richard R. Ernst (Nobel Prize 1990) provided the Fourier transform method to the realm of NMR.

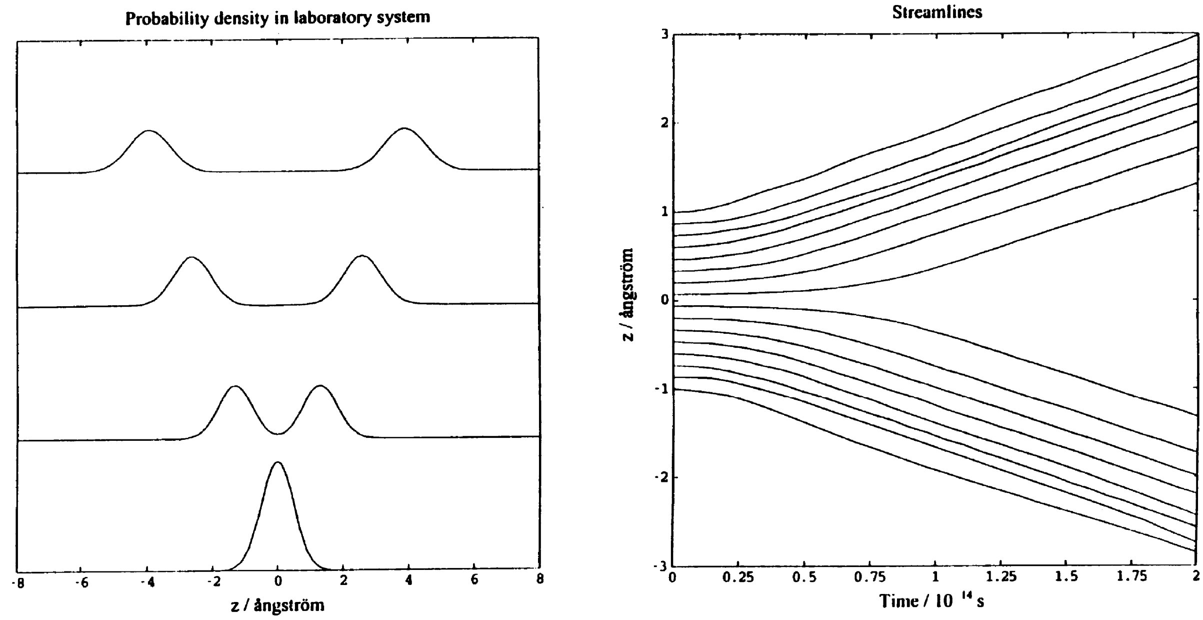

Figure 1. Splitting of a wave packet caused by an impulsive magnetic field. The figure on the left displays the probability density, the figure on the right displays streamlines.

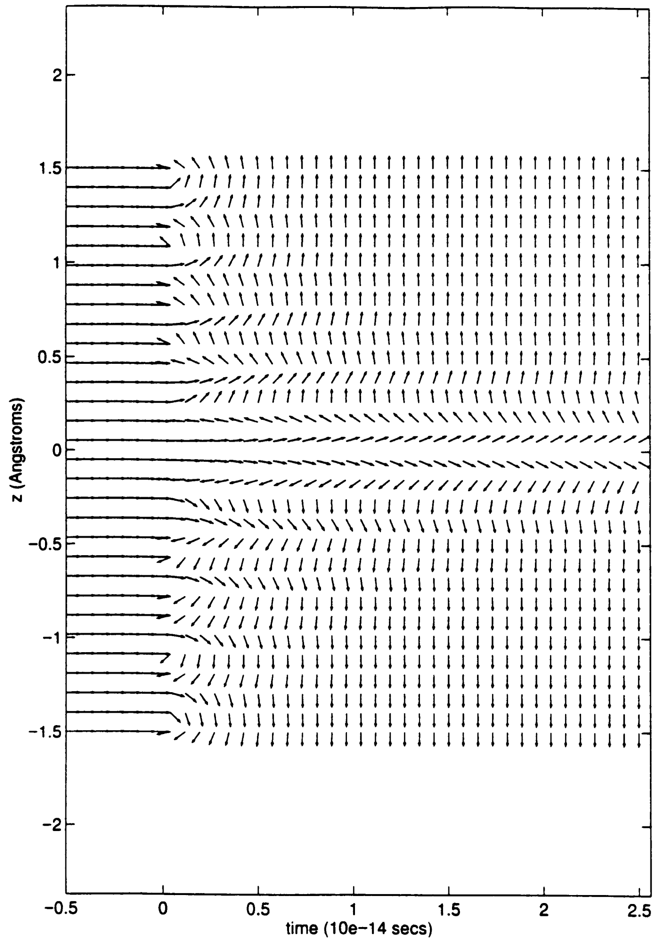

Figure 2. Evolution of the spin vector. Immediately after the shock the spin vectors point in all directions, but after about $2 \times 10^{-14}s$, they sort themselves into the wave packets of the two channels.

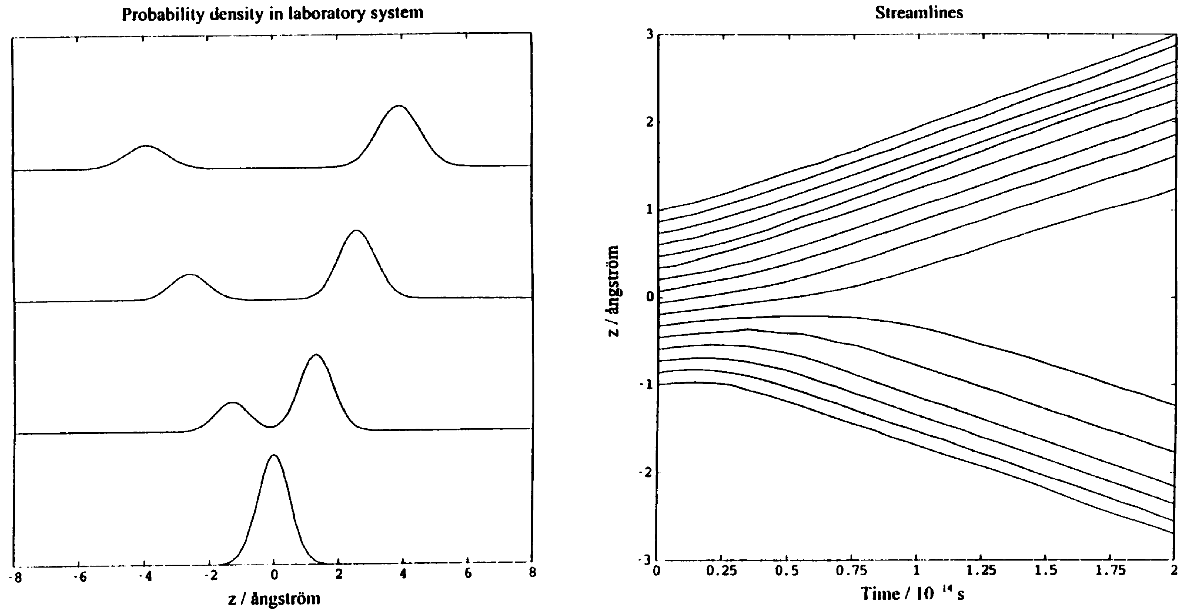

Figure 3. Splitting of a wave packet with different mixtures of spin-up and spin-down components.

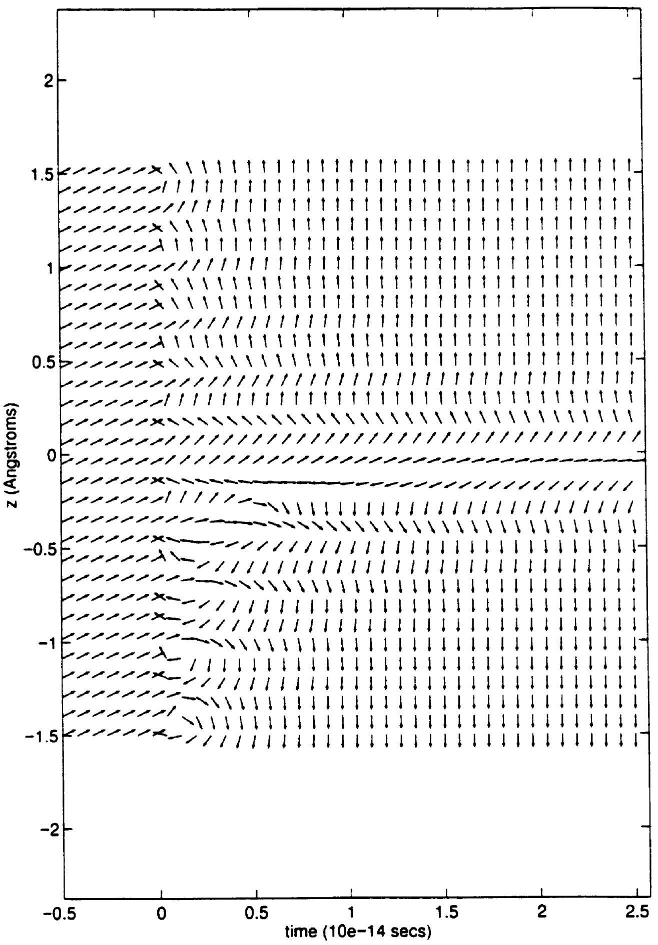

Figure 4. The evolution of the spin vector when different mixtures of spin-up and spin-down components are initially present.

2 Magnetic resonance tomography

The Stern–Gerlach experiment admits a high tech system application to clinical magnetic resonance tomography [22, 18] and magnetic resonance spectroscopy [10] which is suggested by the holographic viewpoint of high resolution radar imaging [16]. Actually clinical MRI is based on a generalization of the Stern–Gerlach filter.

Magnetic resonance tomography has changed the practice of medicine perhaps more than any other clinical technology developed over the past quarter of a century. This cross-sectional imaging modality allows to generate high precision images inside a tomographic slice when the patient is placed in a strong magnetic flux density of high uniformity [12, 17, 21, 25]. The spins of the protons in the tissue start to precess in a coherent way. Because spin ensembles are objects of quantum physics [23], the orbital plane W must be interpreted in terms of quantum physics. This can be done by means of projective spinor quantization which does *not* consider the spin as an intrinsic angular momentum but an orientation of the quantum field information channel. The ensuing method is based on the concept of three-dimensional real Heisenberg Lie group [19]

$$G \hookrightarrow \mathrm{Loop}(\mathbf{T})$$

and its natural fibration

$$G \longrightarrow \mathbf{C}.$$

The planar affine symplectic coadjoint orbits of G are realizations of the affine symplectic plane W not containing the origin, within the dual vector space $\mathrm{Lie}(G)^\star$ of the Lie algebra $\mathrm{Lie}(G)$ [22]. The embedding

$$W \hookrightarrow \mathrm{Lie}(G)$$

makes it apparent that the affine symplectic plane W is also self-dual with respect the Jacobi bracket of the Lie algebra $\mathrm{Lie}(G)$ and that G forms a central *extension* of the horizontal plane $\exp_G W$. The one-dimensional center of G pointed by the origin carries a logarithmic scale along the longitudinal channel direction. The scale comes from the homeomorphism

$$\exp_G : \mathrm{Lie}(G) \longrightarrow G.$$

By duality, the moment map

$$\mu : W \longrightarrow \mathrm{Lie}(G)^\star$$

is injective and defines a geometric quantization procedure. This quantization procedure allows to derive Keppler's Third Law from the Second Law via hyperbolic geometry. Moreover, the moment map μ associates with the coordinate swapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{pmatrix} -y \\ x \end{pmatrix}$$

the Fourier cotransform

$$\bar{\mathcal{F}}_{\mathbf{R}} : \mathrm{L}^2(\mathbf{R}) \longrightarrow \mathrm{L}^2(\mathbf{R}).$$

Thus

$$\bar{\mathcal{F}}_{\mathbf{R}} = \mu(J).$$

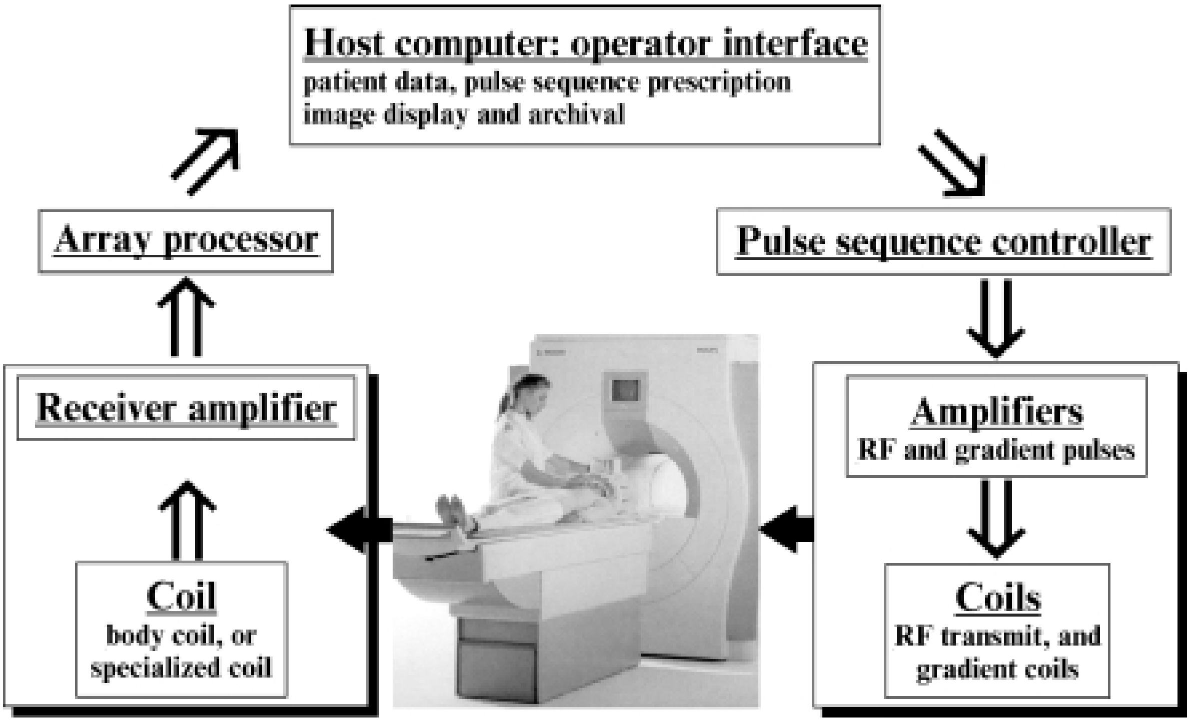

Figure 5. Architecture of a clinical MRI scanner. The system block diagram shows the main hardware components of magnetic resonance tomography organized according to function. The system includes the main magnet, a set of gradient coils, a radiofrequency (RF) coil, a transmitter based on a pulse shape generator of hard and soft RF pulse trains, a receiver endowed with a superheterodyne circuitry, and a work station for image processing. Also shown is the pathway followed by operator commands, pulse sequences, and image data. The imaging subsystem displayed on the right consists of amplifiers (RF and gradients), the excitation RF coil, and the gradient coils used to encode the signals. The detection subsystem displayed on the left consists of the receiver coil, and the hardware involved in the signal amplification pathway.

Due to the moment map μ, the affine symplectic structures of W and the planar coadjoint orbits of G in the dual vector space $\mathrm{Lie}(G)^\star$ are compatible. Due to the action of the one-dimensional compact group

$$\mathbf{U}(1, \mathbf{C}) = \mathbf{T}$$

of gauge transformations, which leaves the one-dimensional center of G invariant, they are suitable structures for describing the *resonance* phenomena of NMR. By making the flux density non-uniform via the controlled application of $\mathbf{R}$-linear distortions, called magnetic field *gradients*, the synchronized phase-frequency coordinates of the points in a spin ensemble inside the selected tomographic slice can be spatially separated. The switching of orthogonal gradients converts the Fourier cotransform $\bar{\mathcal{F}}_{\mathbf{R}}$ via affine $\mathbf{R}$-linear Möbius transformations of the Keppler flow within the affine symplectic plane $W \hookrightarrow \mathbf{P}(\mathbf{R} \times W)$ into a unitary parallel multichannel filter bank. The bank of polyphase filters implemented in the *ruled* plane W transforms the spin density into a clinically valuable planar image (Fig. 7). One of the various advantages of the MRI procedure over X-ray imaging modalities such as CT scanning is that it works in a non-invasive manner [12, 17, 21, 25]. Unlike CT, which is uniplanar, MRI can produce cross-sectional images from any plane.

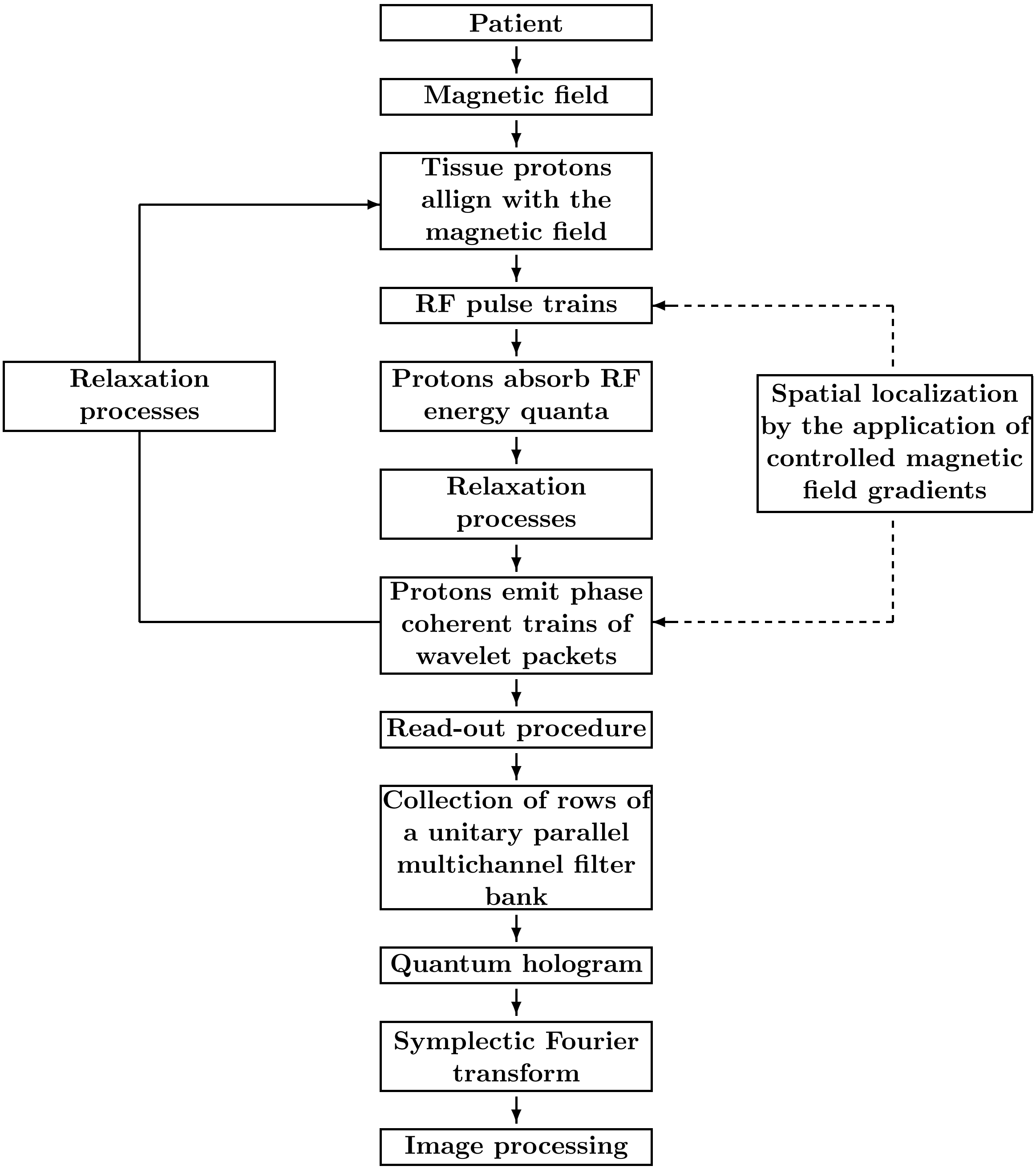

Figure 6. Simplified flowchart displaying the main principles on which a clinical MRI scanner operates. The reconstruction process from a collection of multichannel phase histories is through probing the magnetic moments of nuclei employing strong magnetic flux densities and radiofrequency (RF) electromagnetic radiation. The whole process of clinical MRI is based on a controlled perturbation of the equilibrium magnetization of the object with a train of RF pulses and observing the resulting time-evolving phase coherent response wavelet packet trains in a coil. In terms of line geometry of dimension 3, the collection of a series of 256–512 views establishes a unitary parallel multichannel filter bank which reduces the quantum entropy [8]. In the read-out procedure of the quantum hologram [20], the algorithm of the symplectic Fourier transform reflects the basic structural feature of the clinical MRI modality. The HASTE (half-Fourier acquisition single-shot turbo spin-echo) technique takes advantage of the symmetry properties of the symplectic Fourier transform.

Figure 7. Clinical magnetic resonance tomography: A high resolution sagittal image of the head. In terms of line geometry of dimension 3, the unitary parallel multichannel filter bank generalizes the Stern–Gerlach filter to clinical MRI. It generates the high resolution image which is simulated by a phased-array coil on the receiver side.

Keppler's geometric method of planet tracking by synchronized clockworks, however, admits a completely unexpected high tech system application to the field of non-invasive clinical imaging by NMR [7, 11, 12, 13, 17, 18, 21, 22, 25]. Spin ensembles are excited by the Keppler flow and combine to a planar image within the affine symplectic plane $W \hookrightarrow \mathbf{P}(\mathbf{R} \times W)$ that can be decoded by a unitary parallel multichannel filter bank. In terms of line geometry of dimension 3, the filter bank generalizes the Stern–Gerlach filter (Fig. 7 supra).

Within ten years, clinical MRI had become a billion-dollar business and continues to expand rapidly. Although already well established in the clinical diagnostic centers all over the world, the future of the MRI modality continues to be bright [11, 12, 21, 25]. MRI will become the most important clinical imaging modality by the year 2010. It will be the major morphological staging method for the abdomen and pelvis. MR-guided surgery will be commonplace, and thermal ablation of disc and soft-tissue tumors will be performed under MR control. Morphology, early stroke detection, and even emergency trauma cases will be done by clinical MRI. In most cases, the clinical MRI modality will replace conventional CT whole-body scan interpretation, as well as helical CT scanning [15]. Specifically, brain imaging will no longer make use of CT by 2010 [9].

Quantum teleportation shows that projective spinor quantization is not in conflict with projective relativity [6]. The method is not restricted to RF pulse trains. It extends to the light-in-flight recording by ultrafast laser pulse trains in the pico- and femtosecond regime [1, 2, 3, 4, 5]. Moreover, the technique of projective spinor quantization provides a new approach to the Becchi–Rouet–Stora–Tyutin (BRST) quantization procedure [14, 7]). This underlines the power of the projective viewpoint adopted for the matching of quantum physics and special relativity.

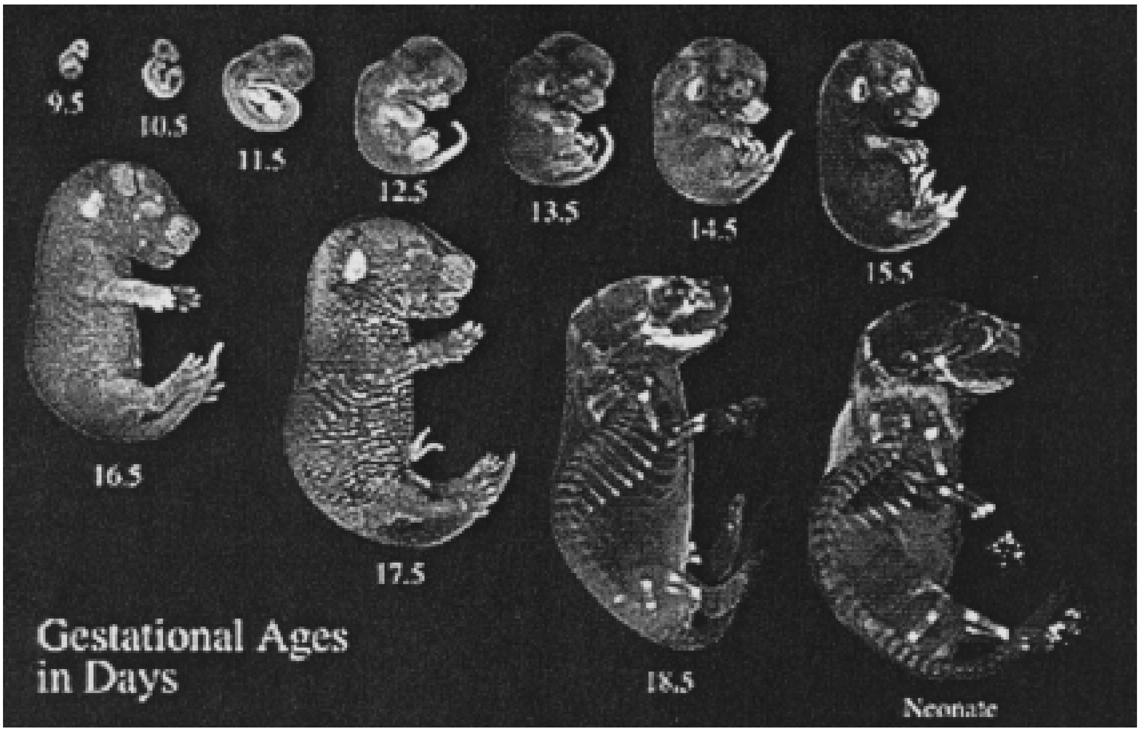

Figure 8. Magnetic resonance microscopy: An embryologic study.

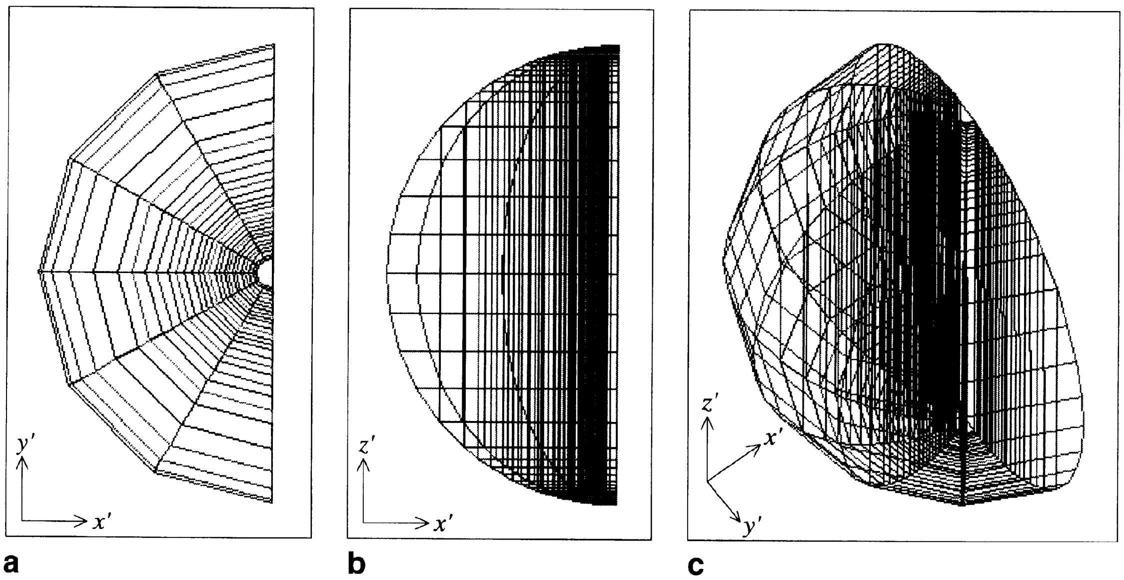

Figure 9. Finite element analysis provides the two types of projectively dual totally orthogonal planes occurring in line geometry of dimension 3. The two planes are cohomologically of different type. The bundle of radial lines as well as the ruled planes FEM visualize the one-dimensional compact group $\mathbf{U}(1, \mathbf{C})$ of gauge transformations acting on $\mathbf{C}^2$. The bank polyphase filters in the ruled planes transform the spin density into a clinically valuable image.

3 Conclusion

As a conclusion, the paper establishes via the Stern–Gerlach filter and gauge group analysis that the higt resolution capability of clinical magnetic resonance tomography is due to a unitary parallel multichannel filter bank generated by quantum holography.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 429–439.

Models of Covariant Quantum Theories of Extended Objects from Generalized Random Fields

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The problem of supplying nontrivial models of local covariant and obeying the spectral condition quantum theories of extended objects is discussed. In particular, it was demonstrated that starting from sufficiently regular generalized random fields the construction of the corresponding quantum dynamics describing extended objects is possible. Several particular examples of such generalized random fields are presented.

1 Introduction

The importance of nonlocal, gauge invariant functionals was firstly recognized in local quantum field theories of gauge type [28]. In such theories the compatibility of standard positivity, locality and covariance is hard to achieve if at all, see e.g. [26, 25]. The restrictions of the allowed set of observables to the set of gauge invariant observables and the arising space of states seem to be correct choice of subspace of physical states. Also the role played by certain nonlocal order parameters in studying the phase structure (the complicated vacuum structure) of gauge quantum field theories must to be pointed out [4, 6, 23, 27, 29]. The still continued attempts [14] to formulate local, covariant and positive quantum theories of extended objects like strings, membranes, etc. also justify the importance of searching new mathematical techniques for constructing nontrivial models of this type. Let us recall also that the recent attempts to formulate quantum gravity in terms of loop variables seem to be very attractive idea [2]. Finally let us mention the application of the loop variables in the topological quantum field theories to the classical problems of geometry [3].

An interesting approach to the construction physically reasonable models of extended objects was proposed in [23] in the context of quantum field theories of gauge type. The approach presented in [23] can be called the Euclidean approach and is of axiomatic type. However there are not too many nontrivial models obeying the system of axioms proposed in [23]. To our best knowledge the Wilson loop Schwinger functions in the continuum limit of QCD_2, and in the free QED_d are the only examples discussed explicitely in the literature [23], see also [18, 22]. It is the main aim of the present contribution to provide some new examples of theories obeying the proposed axiomatic scheme of [23] and to outline a general constructive approach for constructing models of this sort from the generalized random fields.

2 The Fröhlich–Osterwalder–Seiler axiomatic approach

Let $\mathcal{C}_k(d)$ be a variety of k-dimensional piecewise C^1 cycles in the space $\mathbf{R}^d$, i.e. elements Γ of $\mathcal{C}_k(d)$, a k-dimensional boundaryless piecewise C^1 compact submanifolds of the d-dimensional

Euclidean space $\mathbf{R}^d$. The allowed topologies τ on $\mathcal{C}_k(d)$ are such that only small C^∞ local deformations are allowed and they define a basis of neighborhoods of a given $\Gamma \in \mathcal{C}_k(d)$, in particular local continuous but not differentiable deformations $\delta\Gamma$ of Γ send $\delta\Gamma$ far from Γ. The allowed topologies (as above) on the variety $\mathcal{C}_k(d)$ can be prescribed explicitly in the metric form (an example in the case of loops is provided in [23]).

From now on we shall assume that τ is an allowed topology on $\mathcal{C}_k(d)$.

A system $\mathbf{S} = \{S_n\}_{n\geq 0}$ of functionals, where each S_n is jointly τ-continuous functional on the space $(\mathcal{C}_k(d),\tau)^{\times n}_{\#}$, where $(\Gamma_1,\ldots,\Gamma_n) \in \mathcal{C}_k(d)^{\times n}_{\#}$ iff $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, is called k-cycles Schwinger functional iff it fulfills the following conditions:

FOS0-1 Let $\Gamma_i^{a_i}$, $i = 1,\ldots,n$ be a translation of $\Gamma_i \in \mathcal{C}_k(d)$ by the vector $a_i \in \mathbf{R}^d$ and let $\delta(\Gamma_1^{a_1},\ldots,\Gamma_n^{a_n}) = \inf_{i,j}\{\mathrm{dist}(\Gamma_1^{a_i},\Gamma_j^{a_j})\}$. If $\delta(\Gamma_1^{a_1},\ldots,\Gamma_n^{a_n}) > 0$, then there exist constants K_n, c_n, p such that:

$$|S_n(\Gamma_1^{a_1},\ldots,\Gamma_n^{a_n})| \leq K_n \exp c_n\delta^{-p}.$$

FOS0-2 Let

$$\delta_t(\Gamma_1,\Gamma_2) = \inf\left\{|t_1 - t_2|; t_1 : \ \bigvee(t_1,\mathbf{x}_1) \in \Gamma_1, \quad t_2 : \ \bigvee(t_2,\mathbf{x}_2) \in \Gamma_2\right\}$$

be a temporal distance between Γ_1 and Γ_2. Then there exist constants $K_{\Gamma_i,\epsilon}$ (depending on Γ_i and $\epsilon > 0$) such that:

$$|S_n(\Gamma_1,\ldots,\Gamma_n)| \leq K_{\Gamma_1,\epsilon}\cdot\cdots\cdot K_{\Gamma_n,\epsilon}$$

providing $\delta_t(\Gamma_i,\Gamma_j) \geq \epsilon$, $i,j = 1,\ldots,n$.

FOS1 For any $n \geq 1$, any ensemble $\{\Gamma_1,\ldots,\Gamma_n\} \subset \mathcal{C}_k(d)^{\times n}_{\#}$ and any permutation $\pi \in s_n$ ($\equiv$ the symmetric group):

$$S_n(\Gamma_1,\ldots,\Gamma_n) = S_n(\Gamma_{\pi(1)},\ldots,\Gamma_{\pi(n)}).$$

FOS2 For any Euclidean motion $(a,\Lambda) \in T \lhd O(d)$ (where $O(d)$ stands for the orthogonal group, T are translations and $\lhd$ means the standard semidirect product) and any ensemble $\Gamma_1,\ldots,\Gamma_n \in \mathcal{C}_k(d)$ we have:

$$S_n(\Gamma_1,\ldots,\Gamma_n) = S_n\left(\Gamma_1^{(a,\Lambda)},\ldots,\Gamma_n^{(a,\Lambda)}\right),$$

where $\Gamma^{(a,\Lambda)} = \{\Lambda^{-1}(x-a) \mid x \in \Gamma\}$.

FOS3 Reflection Positivity. Let $\mathbf{V}_{+(-)}$ be a subset of $\mathbf{C}_k(d) \equiv \bigcup_{n\geq 0}\mathcal{C}_k(d)^{\times n}_{\#}$ consisting of the ensembles of families of nonintersecting cycles

$$\left(\emptyset,\Gamma^1,\left(\Gamma_1^2,\Gamma_2^2\right),\ldots,\left(\Gamma_1^n,\ldots,\Gamma_n^n\right),\ldots\right)$$

that are supported in $\mathbf{R}^d_{+(-)} = \{(t,\mathbf{x}) \in \mathbf{R}^d \mid t > 0(< 0)\}$. Let Θ be a natural involution from $\mathbf{V}_+$ onto $\mathbf{V}_-$. Then for any

$$\underline{\Gamma} \equiv \left(\emptyset,\Gamma^1,\left(\Gamma_1^2,\Gamma_2^2\right),\ldots,\left(\Gamma_1^n,\ldots,\Gamma_n^n\right),\ldots\right) \in \mathbf{V}_+$$

we have

$$\mathbf{S}(\underline{\Gamma}\Theta\underline{\Gamma}) = \sum_{l,m} c_l \overline{c}_m S_{l+m}(\Gamma_1^l, \ldots, \Gamma_l^l, \Theta\Gamma_1^m, \ldots, \Theta\Gamma_m^m) \geq 0$$

and for any $\underline{c} = (c_0, c_1, \ldots)$ (finite sequence of complex numbers).

FOS4 For any $n = k + l$, $k, l > 0$ and $|a| \to \infty$

$$\lim_{|a|\to\infty} S_n(\Gamma_1, \ldots, \Gamma_k, \Gamma'^a_1, \ldots, \Gamma'^a_l) = S_k(\Gamma_1, \ldots, \Gamma_k) S_l(\Gamma'_1, \ldots, \Gamma'_l).$$

It was demonstrated (originally for the case of 1-cycles but the arguments are easily extendable to the case of k-cycles with $1 \leq k \leq d-1$) in [23] that certain real time quantum dynamical system can be reconstructed from any system of Schwinger functions obeying **FOS**0-**FOS**4.

Theorem 2.1 *Let* $\mathbf{S}$ *be a system of k-cycles Schwinger functions on* $(\mathbf{C}_k(d), \tau)$*. Then there exists: a separable Hilbert space* $\mathcal{H}$*, a continuous unitary representation of the universal covering group of the proper orthochronous Poincaré group* $\mathcal{P}_+^\uparrow(d)$ *obeying a spectral condition (i.e. the joint spectrum of the generators of translations is included in the closed forward light cone). Moreover there exists a unique vector* $\Omega \in \mathcal{H}(\mathbf{S})$ *which is invariant under the action of* $\mathcal{P}_+^\uparrow(d)$*.*

In particular, with any time-ordered ensemble of k-cycles $\{\Gamma_1, \ldots, \Gamma_n\}$ and such that $\inf_{i,j}\{d_t(\Gamma_i, \Gamma_j)\} > 0$ one can associate (in a unique manner) a system of holomorphic functionals $\mathcal{W}_{(\Gamma_1,\ldots,\Gamma_n)}(z_1, \ldots, z_n)$ in the tubular region

$$\mathcal{T}_n = \left\{ (z_1, \ldots, z_n) \in \mathbf{C}^{dn} \mid \Im m(z_i - z_{i-1}) \in V_+^d \right\},$$

where $V_+^d = \{x \in \mathbf{R}^d \mid x \cdot x > 0, x^0 > 0\}$ (where $x \cdot x = (x^0)^2 - \mathbf{x}^2$ means Minkowski space scalar product) and such that

(i) restriction of $\mathcal{W}^n_{(\Gamma_1,\ldots,\Gamma_n)}(z_1, \ldots, z_n)$ to the "Euclidean" piece of the boundary $\partial_E \mathcal{T}_n$ of $\mathcal{T}_n$ defined as:

$$\partial_E \mathcal{T}_n = \left\{ \underline{z} \in \mathbf{C}^{nd} \mid \Re z_i^0 = 0, \Im m \mathbf{z}_i = 0, \Im m z_i^0 < \Im m z_{i+1}^0 \right\}$$

is equal to $\mathbf{S}_n(\Gamma_1, \ldots, \Gamma_n)$, i.e.

$$\mathcal{W}^n_{(\Gamma_1,\ldots,\Gamma_n)}\left(\left(ia_1^0, \mathbf{a}_1\right), \ldots, \left(ia_n^0, \mathbf{a}_n\right) \right) = \mathbf{S}_n \left(\Gamma_1^{(ia_1^0,\mathbf{a}_1)}, \ldots, \Gamma_n^{(ia_n^0,\mathbf{a}_n)} \right);$$

(ii) for any collection $(\Gamma_1, \ldots, \Gamma_n)$ of k-cycles located in space-like hyperplanes there exists

$$\lim_{\substack{z_l = x_l + i\eta_l \to 0 \\ \eta_l - \eta_{l-1} \in V_+^d}} \mathcal{W}^n_{(\Gamma_1,\ldots,\Gamma_n)}(z_1, \ldots, z_n) = \mathcal{W}^n_{(\Gamma_1,\ldots,\Gamma_n)}(x_1, \ldots, x_n)$$

in the space of ultradistributions of Jaffe type and with the corresponding indicator function compatible with the singularity behaviour of **FOS**0–**FOS**1.

The boundary ultradistributions $\mathcal{W}^n_{(\Gamma_1,\ldots,\Gamma_n)}(x_1, \ldots, x_n)$ are called k-cycles Wightman ultradistributions corresponding to Schwinger functional $\mathbf{S}$. The problem of formulating conditions on the system of $\mathcal{W}$ of Wightman ultradistributions that lead to k-cycles Schwinger functions $\mathbf{S}$ obeying **FOS**0-**FOS**4 still seems to be open.

3 The scalar models

Let μ_λ be an infinite-volume limit of the so called $P(\Phi)_2$ interaction [13, 24], where $\lambda > 0$ refers to the major coupling constant. The case $\lambda = 0$ corresponds to the Nelson free field measure, i.e. μ_0 stands for the centered Gaussian measure on the space of (real valued) tempered distributions $\mathcal{S}'(\mathbf{R}^2)$ defined by

$$\mu_0(\exp(i(\varphi, f))) \equiv \exp\left\{-\frac{1}{2}\|f\|_{-1}^2\right\}, \tag{1}$$

where $\|f\|_{-1}^2 = (-\triangle + 1)^{-1}(f \otimes f)$, $S_0 \equiv (-\triangle + 1)^{-1}$ being a principial Green function of the operator $(-\triangle + 1)$. Let Γ be a Jordan type curve which is assumed to be sufficiently smooth (see below). We would like first to give a rigorous mathematical meaning to the (formal) expression $\oint_\Gamma \varphi$. For this goal we use a theory of Lions–Magenes traces of distributions [20] together with some arguments from [1].

Lemma 3.1 *Let Γ be a Jordan type 1-cycle in $\mathbf{R}^2$. If the generalized random field μ_λ on $\mathcal{S}'(\mathbf{R}^2)$ obeys the estimate*

$$\mu_\lambda(\varphi^2(f)) \le c_{-1}\|f\|_{-1}^2 + c_p\|S_0 * f\|_{L^p} + c_1\|S_0 * f\|_{L^1}, \tag{2}$$

where $p \in [2, \infty)$ and c_{-1}, c_p, c_1 are some nonnegative constants then for μ_λ a.e. $\varphi \in \mathcal{S}'(\mathbf{R}^2)$ there exists a trace of φ on Γ in the Lions–Magenes sense, denoted as $\varphi|_\Gamma$ and moreover $\varphi|_\Gamma \in \bigcap_{\alpha>0} \mathcal{H}^{-\alpha}(\Gamma)$, where $\mathcal{H}^{-\alpha}(\Gamma)$ are negative-order Sobolev spaces on Γ (defined as in [20]).

Using the fact that χ_Γ($\equiv$ the characteristic function of Γ) belongs to $\bigcap_{\alpha>0} \mathcal{H}^{+\alpha}(\Gamma)$ (as being a constant function) it follows easily by dualization that for any μ obeying the estimate (2) we can define $\langle \chi_\Gamma, \varphi\rangle$ and this number is defined to be $\oint_\Gamma \varphi$. Proceeding in this way we can define for any collection $\{\Gamma_1, \dots, \Gamma_n\}$ a measurable and defined μ a.e. function

$$\mathcal{L}^*_{\Gamma_1,\dots,\Gamma_n}(\varphi) \equiv \prod_{j=1}^n \mathrm{e}^{i\oint_{\Gamma_j}\varphi}.$$

This is an almost sure version of the result on the existence of random loop function for models of Euclidean Quantum Field Theory obeying (2).

However, due to the problem of exceptional sets the above a.e. result is not sufficient and certain computable L^p version of the random loop functions has to be given.

Proposition 3.2 *Let μ be generalized random field on $\mathcal{S}'(\mathbf{R}^2)$ obeying the following estimate:*

$$|\mu(\varphi^2(f))| \le c\|f\|_{-1}^2 \tag{3}$$

for any f with compact support. Let $(\chi_\epsilon)_{\epsilon>0}$ be any smooth mollifier i.e. $0 \le \chi_\epsilon \in C_0^\infty(\mathbf{R}^2)$ for any $\epsilon > 0$, $\int \chi_\epsilon(x)d^2x = 1$ and $\lim_{\epsilon\downarrow 0}\chi_\epsilon = \delta$ ($\equiv$ Dirac delta) in the sense of weak convergence. Let $\{\Gamma_1, \dots, \Gamma_n\}$ be any ensemble of nonintersecting loops of Jordan type.

Then for any $p \ge 1$ the unique limit

$$\lim_{\epsilon\downarrow 0}\mathcal{L}^\mu_\epsilon(\Gamma_1, \dots, \Gamma_n)(\varphi) \equiv \prod_{j=1}^n e^{i\oint_{\Gamma_j}\varphi_\epsilon} \equiv \mathcal{L}^\mu(\Gamma_1, \dots, \Gamma_n)(\varphi)$$

exists in $L^p(d\mu)$ sense.

Thus, defining the loop Schwinger functions

$$S^{\mu}(\Gamma_1,\ldots,\Gamma_n)=\int_{\mathcal{S}'(\mathbf{R}^2)}\mathcal{L}^{\mu}(\Gamma_1,\ldots,\Gamma_n)(\varphi)d\mu(\varphi)$$

for any generalized random field μ obeying (3), we can expect that they are good candidates for nontrivial models obeying the systems of axioms proposed in Section 2.

Theorem 3.3 *Let μ be a Euclidean homogeneous generalized random field obeying the estimate (3). Then the corresponding system of loop Schwinger functions $\mathbf{S}^{\mu}$ obeys the system of* **FOS**0–**FOS**2 *axioms with the possible exception of reflection positivity. If moreover μ is a reflection positive random field then the corresponding loop Schwinger functions obey the reflection positivity axiom too.*

It is well known that many of the constructed two-dimensional scalar models of Euclidean Quantum Field Theory [13, 24] obey the estimates like (2) with the values of p as indicated in (2) and it is known that the following estimates are valid (see e.g. Lemma 2.1 in [1]):

$$c_{-1}\|f\|_{-1}^2+c_p\|S_0*f\|_{L^p}+c_1\|S_0*f\|_{L^1}\le c\|f\|_{-1}^2$$

for any f with compact support and some $c>0$.

A similar theorem is valid for the case of renormalized ϕ_3^4 theory [24] and 2-cycles of Jordan type in $\mathbf{R}^3$. The proof being similar to that above.

However, the weak point of these examples is that the corresponding quantum systems reproduce the basic quantum field theoretical structures.

Theorem 3.4 *Let $({}^c\mathcal{H}^{\mu_\lambda};{}^c\Omega^{\mu_\lambda};{}^cU_t^{\mu_\lambda})$ be a quantum dynamical system obtained from the $P(\varphi)_2$ loop Schwinger functions and let $(\mathcal{H}^{\mu_\lambda};\Omega^{\lambda};U_t^{\mu_\lambda})$ be the corresponding quantum dynamical system obtained from the point (field theoretical) Schwinger functions [13, 24]. Then there exists a unitary map J:*

$$J:{}^c\mathcal{H}^{\mu}_{\lambda}\to\mathcal{H}^{\mu}_{\lambda}$$

such that $J:{}^c\Omega^{\mu_\lambda}\to\Omega^{\lambda}$ and $J^{-1}U_t^{\lambda}J={}^cU_t^{\lambda}$.

For a complete proof see [12].

4 Regular, covariant, generalized random fields

Let $(A_0,\mathbf{A})$ be a generalized random field indexed by $\mathcal{S}(\mathbf{R}^d)\otimes\mathbf{R}^d$, where $d\ge 2$ and $\mathbf{A}$ stands for the space components of A according to the decomposition

$$\mathbf{R}^d=\left\{(x^0,\mathbf{x})\mid x^0\in\mathbf{R},\quad \mathbf{x}\in\mathbf{R}^{d-1}\right\}.$$

Let us denote by μ the corresponding law of A, i.e. the probability, Borel, cylindric measure on $\mathcal{S}'(\mathbf{R}^d)\otimes\mathbf{R}^d$. Here $\mathcal{S}'(\mathbf{R}^d)$ stands for the space of tempered distributions. A field A is called vector field iff for any pair (a,Λ), where $a\in\mathbf{R}^d$, $\Lambda\in SO(d)$ the following equality $(A,f_{(a,\Lambda)})\cong(A,f)$ in law holds, where $f_{(a,\Lambda)}(x)=\sum_{j=0}^{d-1}\Lambda_i^j f_j(\Lambda^{-1}(x-a))$. A vector field A is called reflection invariant iff $(A,rf)\cong(A,f)$ (in law), where $(rf)^0(x^0,\mathbf{x})=-f^0(-x^0,\mathbf{x})$ and $(rf)^i(x^0,\mathbf{x})=f^i(-x^0,\mathbf{x})$ for $i=1,\ldots,d-1$. Let us recall that a vector field A which is Markoff and reflection invariant is reflection positive. The main question addressed in this section is now to find sufficient conditions on the field A that enable us to define a family of loop Schwinger functions obeying the system of axioms **FOS**0–**FOS**4. Let $\omega \in C_0^\infty(\mathbf{R}^d)$ be a non-negative function with support in the unit ball $\{x : \|x\| \leq 1\}$ and such that $\int \omega(x)dx = 1$. Then the we define $\omega^N(x) = N^d\omega(Nx)$ and we note that $\lim_{N\to\infty} \omega^N(x) = \delta(x)$. For any loop Γ, parametrized by $\gamma(t)$, $t \in [0,1]$, we define the following family of test functions from $C_0^\infty(\mathbf{R}^d) \otimes \mathbf{R}^d$:

$$\Delta_{\Gamma,k}^N(x) = \oint_\Gamma \omega^N(x-y)dy^k = \int_0^1 \omega^N(\gamma(t)-x)\dot{\gamma}^k(t)dt.$$

For a given ensemble $\{\Gamma_1, \ldots, \Gamma_n\}$ of loops we define the sequence of functionals

$${}^N\mathcal{L}(\Gamma_1, \ldots, \Gamma_n)(A) = \prod_{l=1}^n \exp\{i\langle \Delta_{\Gamma_l}^N, A\rangle)\}$$

and the corresponding Schwinger functions

$${}^N S(\Gamma_1, \ldots, \Gamma_n) = \mathbf{E}^N\mathcal{L}(\Gamma_1, \ldots, \Gamma_n)(A).$$

Theorem 4.1 *Let A be a vector, reflection positive generalized random field on the space $\mathcal{S}'(\mathbf{R}^d) \otimes \mathbf{R}^d$, $d \geq 2$ and let $\{\mathcal{G}_{ij}(x-y)\}$ be a two-point Schwinger function of A. Assume that for any loop $\Gamma \in \mathcal{C}_1^g(\mathbf{R}^d)$ the following integrals*

$$\oint_\Gamma \oint_\Gamma |\mathcal{G}_{ii}(x-y)| dx^i dy^i \tag{4}$$

and for all $i = 0, \ldots, d-1$ do exist. Then, there exists a system of loop Schwinger functions $\{S_n\}$ on $\bigcup_{n\geq 0} \mathcal{C}_1^g(\mathbf{R}^d)^{\times n}$ obeying the system of axioms **FOS**0–**FOS**3*, where $\mathcal{C}_1^g$ means globally $\mathcal{C}_1$-curves.*

In particular, the assumptions of Theorem 4.1 are valid for the 2-dimensional versions of Abelian, free QED. In higher dimensions we should expect some infinite renormalizations connected to the divergence of the integrals (4) see e.g. [18, 22]. A suitable version of Theorem 4.1 to handle this case can also be formulated [12].

Proof of Theorem 4.1. Using

$$\mathbf{E}\left|{}^N\mathcal{L}(\Gamma)(A) - {}^{N'}\mathcal{L}(\Gamma)(A)\right| \leq \mathbf{E}\left|\langle \Delta_\Gamma^N - \Delta_\Gamma^{N'}, A\rangle\right| \leq \left\{\mathbf{E}\left|\langle \Delta_\Gamma^N - \Delta_\Gamma^{N'}, A\rangle\right|^2\right\}^{\frac{1}{2}},$$

but

$$\mathbf{E}\left|\langle \Delta_\Gamma^N - \Delta_\Gamma^{N'}, A\rangle\right|^2 = \sum_{i,j} G_{ij}\left(\Delta_{\Gamma,i}^N - \Delta_{\Gamma,i}^{N'}, \Delta_{\Gamma,j}^N - \Delta_{\Gamma,j}^{N'}\right),$$

where $G_{ij}(x,y) = \mathbf{E}A_i(x)A_j(y)$. We see that the problem of $L^1(d\mu)$-convergence of functionals ${}^N\mathcal{L}(\Gamma)$ is reduced to the question of existence of $\lim_{N\to\infty} G_{ii}\left(\Delta_{\Gamma,i}^N, \Delta_{\Gamma,i}^N\right)$. For this

$$\begin{aligned}\left|G_{ij}\left(\left(\Delta_\Gamma^N - \Delta_\Gamma^{N'}\right)_i, \left(\Delta_\Gamma^N - \Delta_\Gamma^{N'}\right)_j\right)\right| &= \left|\mathbf{E}A_i\left(\Delta_\Gamma^N - \Delta_\Gamma^{N'}\right)_i A_j\left(\Delta_\Gamma^N - \Delta_\Gamma^{N'}\right)_j\right| \\ &\leq \left\{\mathbf{E}\langle A_i, \left(\Delta_{\Gamma,i}^N - \Delta_{\Gamma,i}^{N'}\right)\rangle^2\right\}^{\frac{1}{2}} \left\{\mathbf{E}\langle A_j, \left(\Delta_{\Gamma,j}^N - \Delta_{\Gamma,j}^{N'}\right)\rangle^2\right\}^{\frac{1}{2}}.\end{aligned}$$

Thus, we need to prove that $\lim_{N\to\infty} G_{ii}\left(\Delta^N_{\Gamma,i}, \Delta^N_{\Gamma i}\right)$ exists for all i and then

$$\lim_{N\to\infty} G_{ii}\left(\Delta^N_{\Gamma,i}, \Delta^N_{\Gamma i}\right) = \lim_{N\to\infty} \int_{\mathbf{R}^4} dx \int_{\mathbf{R}^4} dy G_{ii}(x-y)$$

$$\int\int_{[0,1]^{\times 2}} \omega^N(\gamma(t_1)-x)\dot{\gamma}^i(t_1)\omega^N(\gamma(t_2)-y)\dot{\gamma}^i(t_2)dt_1 dt_2$$

formally is equal to:

$$\oint_\Gamma \oint_\Gamma G_{ii}(x-y)dx^i dy^i = \int_0^1 dt_1 \int_0^1 dt_2 G_{ii}(\gamma(t_1)-\gamma(t_2))\dot{\gamma}^i(t_1)\dot{\gamma}^i(t_2)$$

so we need to justify only the change of limit operation $\lim_{N\to\infty}$ with integral but this is allowed by the Lebesgue dominated convergence theorem. ■

5 Some solvable interacting models

A large class of covariant, Markovian generalized random fields can be obtained as a solution of systems of covariant partial differential stochastic equations [7, 9, 10, 11].

For this let (τ, τ') be a pair of real representations of the special orthogonal transformation group $SO(d)$, where d is the dimension of the Euclidean space-time. We assume that dimension of τ (resp. τ') is equal to n_τ (resp. $n_{\tau'}$) and we denote the natural lifting of τ to the space $\mathcal{S}(\mathbf{R}^d)\otimes\mathbf{R}^{n_\tau}$ (resp. $\mathcal{S}(\mathbf{R}^d)\otimes\mathbf{R}^{n_{\tau'}}$) as T_τ (resp. $T_{\tau'}$). A first order differential operator $\mathcal{D} = \sum_{\mu=0}^{3} B_\mu\partial_\mu + \mathbf{M}$, where $B_\mu, \mathbf{M} \in \mathrm{Hom}(\mathbf{R}^{n_\tau}, \mathbf{R}^{n_{\tau'}})$ is called (τ,τ')-covariant operator iff the following diagram

$$\begin{array}{ccc} \mathcal{S}'(\mathbf{R}^d)\otimes\mathbf{R}^{\dim\tau} & \xrightarrow{\mathcal{D}} & \mathcal{S}'(\mathbf{R}^d)\otimes\mathbf{R}^{\dim\tau'} \\ T_\tau \Big\downarrow & & \Big\downarrow T_{\tau'} \\ \mathcal{S}'(\mathbf{R}^d)\otimes\mathbf{R}^{\dim\tau} & \xrightarrow{\mathcal{D}} & \mathcal{S}'(\mathbf{R}^d)\otimes\mathbf{R}^{\dim\tau'} \end{array} \quad (5)$$

commutes. The complete list of such operators for the case $d = 2, 3, 4$ is well known for any pair (τ, τ'). See, e.g. [8, 19, 11, 21].

Let $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{d-1})$ be any multiindex of length d, i.e. $\alpha_\mu \in \mathbf{N}\cap\{0\}$, $\mu = 0, \ldots, d-1$ and let $|\underline{\alpha}| = \alpha_0 + \cdots + \alpha_{d-1}$. We denote by $\mathbf{I}_K(d)$ (for a given integer $K > 0$) the set of all multiidices $\underline{\alpha}$ as above and such that $|\underline{\alpha}| \le K$ and let $\mathbf{C}_K(d)$ be a cardinality of the set $\mathbf{I}_K(d)$. For a given $\underline{\alpha}$, let $D^{\underline{\alpha}} = \frac{\partial^{\alpha_0+\cdots+\alpha_{d-1}}}{\partial x_0^{\alpha_0}\cdots\partial x_{d-1}^{\alpha_{d-1}}}$.

Let us consider the operator $\mathbf{D}$ defined as

$$(\mathbf{D}^{jl}_{\underline{\alpha}}) \equiv \sum_{\underline{\beta}\in\mathbf{I}_K(d)} E^{jl}_{\underline{\alpha}\underline{\beta}} D^{\underline{\beta}} \quad (6)$$

for $\underline{\alpha}, \underline{\beta} \in \mathbf{I}_K(d)$, $j, l = 1, \ldots, N$, where $E^{jl}_{\underline{\alpha}\underline{\beta}}$ are some real numbers. The endomorphism E of the space $\mathbf{R}^{N\mathbf{C}_K(d)}$ corresponding to $E^{jl}_{\underline{\alpha}\underline{\beta}}$ in the canonical basis of $\mathbf{R}^{N\mathbf{C}_K(d)}$ will be useful in the following. For $f \in \mathcal{S}(\mathbf{R}^d)\otimes\mathbf{R}^N$ the operator $\mathbf{D}$ corresponding to (6) is given by $(\mathbf{D})^j_{\underline{\alpha}}(x) = \sum_l (\mathbf{D})^{jl}_\alpha f^l(x)$, so $\mathbf{D}$ maps $\mathcal{S}(\mathbf{R}^d)\otimes\mathbf{R}^N$ into $\mathcal{S}(\mathbf{R}^d)\otimes\mathbf{R}^{N\mathbf{C}_K(d)}$. We fix a pair $(\mathbf{D}_G, \mathbf{D}_P)$ of operators defined as above.

A noise corresponding to the pair $(\mathbf{D}_G, \mathbf{D}_P)$ (a general noise of order K) is defined as a generalized random field ν on the space $\mathcal{S}(\mathbf{R}^d) \otimes \mathbf{R}^N$ the characteristic functional Γ_ν of which is given by the product:

$$\Gamma_\nu(f) = \Pi_\nu^G(f)\Pi_\nu^P(f), \tag{7}$$

where the characteristic functional (of Gaussian part of ν) Π_ν^G is defined

$$\Pi_\nu^G(f) = \exp\left\{-\frac{1}{2}\int_{\mathbf{R}^d} \langle \mathbf{D}_G f, \mathbf{A}\mathbf{D}_G f\rangle(x)dx\right\}, \tag{8}$$

where $\mathbf{A}$ $\in$End$(\mathbf{R}^{N\mathbf{C}_K(d)})$, $\mathbf{A} \geq 0$, and the characteristic functional (of the Poisson part of ν) Π_ν^P is explicitly displayed as:

$$\Pi_\nu^P(f) = \exp\left\{-\int_{\mathbf{R}^d} \Psi^P(\mathbf{D}_P f(x))dx\right\}, \tag{9}$$

where

$$\Psi^P(y) = -\int_{\mathbf{R}^{N\mathbf{C}_K}\setminus\{0\}} \left[\mathrm{e}^{i\langle\Lambda,y\rangle} - 1 - i\langle\Lambda, y\rangle\right] dL(\Lambda) \tag{10}$$

or

$$\Psi^P(y) = -\int_{\mathbf{R}^{N\mathbf{C}_K}\setminus\{0\}} \left[\mathrm{e}^{i\langle\Lambda,y\rangle} - 1\right] dL(\Lambda) \tag{11}$$

for some Borel measure dL on the space $\mathbf{R}^{N\mathbf{C}_K}\setminus\{0\}$ with all finite moments.

It is easy to observe that a given noise ν on the space $\mathcal{S}'(\mathbf{R}^d) \otimes \mathbf{R}^{n_\tau}$ is T_τ-covariant iff the following covariance conditions are fulfilled

$$\left(\tau^T \otimes \gamma\right)(g)\mathbf{B}\left(\tau \otimes \gamma^T\right)(g) = \mathbf{B}, \tag{12}$$

$$dL_{(E_P^T)^{-1}}(\tau \otimes \gamma)(g)(\Lambda) = dL_{(E_P^T)^{-1}}(\Lambda), \tag{13}$$

where $\mathbf{B} \equiv E_G^T \mathbf{A} E_G$, $dL_{(E_P^T)^{-1}}$ is the transport of the Levy measure dL by the endomorphism E_P^T and finally γ is an orthogonal representation of the group $SO(d)$ in the space $\mathbf{R}^{\mathbf{C}_K(d)}$ defined explicitly as:

$$\gamma_{\underline{\alpha}\underline{\beta}}(g) = \sum_{\Pi_{\mu\nu}} \prod_{\mu,\nu=1}^{d-1} g_{\mu\nu}^{\Pi_{\mu\nu}}, \tag{14}$$

where the sum $\sum\limits_{\Pi_{\mu\nu}}$ runs over all matrices $(\Pi_{\mu\nu})_{\mu,\nu=0}^{d-1}$ built from the elements of $\{1, \ldots, K\}$ and chosen in such a way that $\alpha_\mu = \sum\limits_{\nu=0}^{d-1} \Pi_{\nu\mu}$, $\beta_\mu = \sum\limits_{\nu=0}^{d-1} \Pi_{\mu\nu}$ for $\underline{\alpha}, \underline{\beta} \in \mathbf{I}_K(d)$.

The interesting class of non-Gaussian covariant generalized (Markovian) in a suitable sense, see e.g. [17, 15], random fields is obtained as a solution of covariant SPDE's of the type

$$\mathcal{D}\varphi = \eta, \tag{15}$$

where $\mathcal{D}$ is some (τ, τ')-covariant operator which obeys certain additional conditions for the existence of not too singular Green function (from the infrared divergencies point of view, see [11, 21] for details), η is a noise of order K which is assumed to be $T_{\tau'}$-covariant noise.

It was proven in [9, 10, 11, 21] that under these conditions the solutions of the equation (15) do exist in certain sense and give rise to a new T_τ-covariant, generalized Markovian random fields, the moments of which can be analytically continued to Minkowski space-time yielding a system of covariant Wightman distributions obeying the spectral conditions (in the weak form) and the quantum field theoretical locality principle as well (see [7, 11] for details).

We would like to address here the question whether with solutions of (15) obtained in [9, 10, 11] one can associate systems of k-loop Schwinger functions on $\mathbf{R}^d$ that might be good candidates for explicit models obeying **FOS**0-**FOS**2. The important question on the existence of the reflection positive solutions of equations of the type (15) being still unsolved in general, presses the necessity to develop a weaker scheme for obtaining results on the real-time dynamics of extended objects from the corresponding Euclidean data of the spirit as in the general indefinite metric quantum field theory [16].

The following localization property of the noise ν is crucial for the existence of the almost sure version of the corresponding k-cycles Schwinger functionals.

Proposition 5.1 *Let $\Gamma(\mathbf{R}^d)$ be the space of locally finite configurations of the space $\mathbf{R}^d$ and let $\ni$ be a Poisson noise with the characteristics $(\mathbf{D}_P, E_p)$. Then, the set*

$$\left\{ \eta \in \mathcal{D}'(\mathbf{R}^d) \otimes \mathbf{R}^N \mid \eta = \sum_{k=1}^{N} \sum_{\underline{\alpha} \in \mathbf{I}_K(d)} \sum_{\delta_{k,\underline{\alpha}}=1}^{\infty} (-1)^{|\underline{\alpha}|} D^{\underline{\alpha}} \delta_{x_{\delta_{k,\underline{\alpha}}}} \otimes \left(E_P^T \Lambda_{\delta_{k,\underline{\alpha}}} \right)_{\underline{\alpha}}^{k} \right\}. \tag{16}$$

As a corollary we obtain

Theorem 5.2 *Let φ be a solution of (15) in the sense explained in [7, 9, 11] and let us assume that the underlying Green function $\mathcal{G}_{\mathcal{D}}$ of the operator $\mathcal{D}^T$ has a decay at least as $\frac{1}{|x|^{d+\epsilon}}$ if $|x| \to \infty$ and such that τ contains the appropriate subrepresentation corresponding to k-skew symmetric tensor. Then, for any fixed configuration $(\Gamma_1, \ldots, \Gamma_n)$ of k-cycles on $\mathbf{R}^d$, there exists a measureable functional defined:*

$$\mathcal{S}'(\mathbf{R}^d) \otimes \mathbf{R}^{n_\tau} \ni \varphi \longrightarrow \prod_{l=1}^{n} e^{i \oint_{\Gamma_l} \varphi|_{\tau(k)}}, \tag{17}$$

where $\phi_{\tau(k)}$ is the corresponding stochastic differential k-form which is perfectly well defined for μ_φ-a.e. $\varphi \in \mathcal{S}'(\mathbf{R}^d) \otimes \mathbf{R}^{n_\tau}$.

By simple argumentation, the existence of the unique measurable, defined μ_φ-a.e. maps

$$\mathcal{S}'(\mathbf{R}^d) \otimes \mathbf{R}^{n_\tau} \times \mathcal{C}_k(d)^{\times n} \ni (\varphi, (\Gamma_1, \ldots, \Gamma_n)) \longrightarrow \prod_{l=1}^{n} \mathrm{e}^{i \oint_{\Gamma_l} \varphi|_{\tau(k)}}$$

can be proven.

The computable, i.e. L^1-version of the above result is provided by the following theorem.

Theorem 5.3 *Let $\mathcal{D}$, $\tau^{(k)}$ be as in the previous theorem. We impose the following estimates on the behaviour of the Green function $G|_{\tau^{(k)}}$ and its first deriviatives:*

$$|G|_{\tau^{(k)}}|(x) \leq \frac{\underline{c}}{|x|^{\underline{p}}} \qquad \text{for} \quad 0 < |x| < 1 \quad \text{and} \quad 0 < \underline{p} < 4,$$

$$|G|_{\tau^{(k)}}|(x) \leq \frac{\overline{c}}{|x|^{\overline{p}}} \qquad \text{for} \quad 1 < |x| \quad \text{and} \quad 0 < \overline{p},$$

$$\left| \frac{\partial}{\partial x^\mu} G \right|_{\tau^{(k)}} |(x) \leq \frac{\overline{d}}{|x|^{\overline{q}}} \qquad \text{for} \quad 1 < |x|, \quad \mu = 0, 1, 2, 3 \quad \text{and} \quad 0 < \overline{q}$$

and the estimates on the behaviour of the characteristic function ψ (negative defined function, see e.g. [5]) of the noise η.

$$|\psi(y)| \leq \underline{M}|y|^{1+\underline{\eta}} \qquad \textit{for} \quad |y| < 1 \quad \textit{and} \quad (-1 + \frac{4}{\underline{q}}, 1].$$

In the case of k-cycles, if we demand the estimate

$$|\psi(y)| \leq \overline{M}|y|^{1+\overline{\eta}} \qquad \textit{for} \quad 1 < |y| \quad \textit{with} \quad \overline{\eta} \in \left(-1, -1 + \frac{4-k}{\underline{p}}\right) \cap (-1, 1]$$

then for any collection $\left\{\Gamma_1^{(k)}, \ldots, \Gamma_n^{(k)}\right\}$ of k-cycles there exists a Cauchy sequence of functionals $\left\{{}^N\hat{\mathbf{S}}_n\left(\Gamma_1^{(k)}, \ldots, \Gamma_n^{(k)}\right)\right\}_{N=1}^{+\infty} \subset L^p(\mathcal{S}'(\mathbf{R}^4) \otimes \mathbf{R}^{n_\tau}, \mu_\varphi)$ for all $p \in [1, +\infty)$.

Let $\hat{\mathbf{S}}_n\left(\Gamma_1^{(k)}, \ldots, \Gamma_n^{(k)}\right)$ denote the limit of that sequence treated as an element in the space $L^1(\mathcal{S}'(\mathbf{R}^4) \otimes \mathbf{R}^{n_\tau}, \mu_\varphi)$ $(p = 1)$ and let us define

$$\mathbf{S}_n\left(\Gamma_1^{(k)}, \ldots, \Gamma_n^{(k)}\right) = \int_{\mathcal{S}'(\mathbf{R}^4)\otimes\mathbf{R}^{n_\tau}} \hat{\mathbf{S}}_n\left(\Gamma_1^{(k)}, \ldots, \Gamma_n^{(k)}\right)(T)\mu_\varphi(T).$$

If, in addition, the condition $\underline{\eta} \in \left(-1 + \frac{4}{\underline{p}}, 1\right]$ is fulfilled then

$$\mathbf{S}_n\left(\Gamma_1^{(k)}, \ldots, \Gamma_n^{(k)}\right) = \exp\left\{-\int_{\mathbf{R}^4} \psi\left(\sum_{l=1}^n G_{\Gamma_l^{(k)}}|_{\tau^{(k)}}(x)\right) d^4x\right\},$$

where we introduced the auxiliary function

$$G_{\Gamma^{(k)}}|_{\tau^{(k)}}(x) = \begin{cases} \int_{\Gamma^{(k)}} G|_{\tau^{(k)}}(\Omega - x)d\Omega & \textit{for} \quad x \notin \Gamma^{(k)} \\ 0 & \textit{for} \quad x \in \Gamma^{(k)} \end{cases}$$

with integration in the sense of k-forms.

The proof of the above results follows the chain of arguments as presented in our earlier paper [11], where the case of the Wilson loops is discussed. All the details can be found in [21].

Theorem 5.4 *Let $\mathcal{D}$, η, $\tau^{(k)}$ be as in Theorem 5.3. Then the correspondinig k-loop Schwinger functionals:*

$$\mathbf{S}(\Gamma_1, \ldots, \Gamma_n) = \int_{\mathcal{S}'(\mathbf{R}^d)\otimes\mathbf{R}^{n_\tau}} \prod_{l=1}^n e^{i\oint_{\Gamma_l} \varphi|_{\tau^{(k)}}} d\mu(\varphi) \tag{18}$$

obey the axioms **FOS**0–**FOS**2 *and also* **FOS**4.

The important problem to reconstruct the corresponding quantum, real-time dynamics from the data contained in the k-loop Schwinger functionals and the existence of the corresponding Wightman functions is left to another publication [12].

Acknowledgements

One of the authors (R.G.) would like to thank the Organizing Committee of this interesting Conference (especially to Prof. A. Nikitin) for their remarkable hospitality in Kyiv. We would like to thank to Prof. B. Jancewicz for his careful reading of the manuscript.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 440–446.

Generalized Gauge Invariants for Certain Nonlinear Schrödinger Equations

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In previous work, Doebner and I introduced a group of nonlinear gauge transformations for quantum mechanics, acting in a certain family of nonlinear Schrödinger equations. Here the idea for a further generalization is presented briefly. It makes possible the treatment of the logarithmic amplitude and the phase of the wave function on an equal footing, suggesting a more radical reinterpretation of these variables in linear and nonlinear quantum theory.

1 Background

Motivated by our desire to interpret a certain class of nonrelativistic current algebra representations as descriptive of quantum mechanical systems, H.-D. Doebner and I proposed a parameterized family of nonlinear Schrödinger equations (NLSEs) whose solutions would satisfy the appropriate equation of continuity [1, 2, 3]. It was then logically necessary to extend the usual gauge group for quantum mechanics to include transformations that could act nonlinearly [4, 5]. Writing the complex-valued wave function $\psi(\mathbf{x},t)$, describing a single spinless particle in a pure state, as $\psi = R(\mathbf{x},t)\exp[iS(\mathbf{x},t)]$, where the amplitude R and the phase S are real, these nonlinear gauge trasnformations act by

$$R' = R, \qquad S' = \Lambda S + \gamma \ln R + \theta, \tag{1.1}$$

where Λ is a smooth, real-valued, nonzero function of t, γ is a smooth, real-valued function of t, and θ is a smooth, real-valued function of $\mathbf{x}$ and t. The transformations (1.1) map members of our family of NLSEs into each other, and have other desirable properties. In particular, they extend naturally to act on a hierarchy of N-particle wave functions $\psi_N(\mathbf{x}_1,\ldots,\mathbf{x}_N,t)$, defined on the (positional) configuration space, in a way that is strictly local and satisfies a separation condition for product states [6].

The justification for considering them to be gauge transformations is as follows. For all of the nonlinear quantum theories under discussion, we interpret $\rho = |\psi|^2 = R^2$ as the probability density in configuration space. We adopt as a working hypothesis the view (taken by many theorists) that all measurements in ordinary quantum mechanics can be regarded as a sequence of positional measurements, made at different times, where external fields exerting forces may be imposed on the system between measurements [7, 8]. Then for any wave function ψ obeying a Schrödinger equation (linear or nonlinear) in our family, the wave function ψ' transformed by (1.1) obeys a transformed Schrödinger equation, still in the family, with gauge-transformed external fields; while the outcomes of all physical measurements remain invariant.

To be explicit, with

$$\rho = \overline{\psi}\psi, \qquad \hat{\mathbf{j}} = \frac{1}{2i}\,[\,\overline{\psi}\nabla\psi - (\nabla\overline{\psi})\psi], \tag{1.2}$$

define the real, homogeneous nonlinear functionals

$$R_1 = \frac{\nabla \cdot \hat{\mathbf{j}}}{\rho}, \qquad R_2 = \frac{\nabla^2 \rho}{\rho}, \qquad R_3 = \frac{\hat{\mathbf{j}}^2}{\rho^2}, \qquad R_4 = \frac{\hat{\mathbf{j}} \cdot \nabla \rho}{\rho^2}, \qquad R_5 = \frac{(\nabla \rho)^2}{\rho^2}, \tag{1.3}$$

and consider the following family of one-particle NLSEs (where for mathematical convenience both sides have been divided by ψ):

$$\begin{aligned} i\frac{\dot{\psi}}{\psi} = i &\left[\sum_{j=1}^{2} \nu_j R_j[\psi] + \frac{\nabla \cdot (\mathcal{A}(\mathbf{x},t)\rho)}{\rho} \right] \\ &+ \left[\sum_{j=1}^{5} \mu_j R_j[\psi] + U(\mathbf{x},t) + \frac{\nabla \cdot (\mathcal{A}_1(\mathbf{x},t)\rho)}{\rho} + \frac{\mathcal{A}_2(\mathbf{x},t) \cdot \hat{\mathbf{j}}}{\rho} + \alpha_1 \ln \rho + \alpha_2 S \right]. \end{aligned} \tag{1.4}$$

Here the coefficients ν_j $(j = 1,2)$, μ_j $(j = 1,\ldots,5)$, and α_j $(j = 1,2)$ are smooth, real-valued functions of t; U is an external real-valued, time-dependent scalar field; and $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ are distinct, external real-valued, time-dependent vector fields. Using the fact that $\nabla^2\psi/\psi = iR_1[\psi] + (1/2)R_2[\psi] - R_3[\psi] - (1/4)R_5[\psi]$, it is straightforward that Eq.(1.4) reduces to the usual, time-dependent linear Schrödinger equation

$$i\hbar\dot{\psi} = \frac{[-i\hbar\nabla - (e/c)\mathbf{A}(\mathbf{x},t)]^2}{2m}\psi + e\Phi(\mathbf{x},t)\psi \tag{1.5}$$

with external electromagnetic potentials $\mathbf{A}$, Φ, when

$$\begin{aligned} &\nu_1 = -\frac{\hbar}{2m}, \qquad \nu_2 = 0, \qquad \mathcal{A} = \frac{e}{2mc}\mathbf{A}, \\ &\mu_1 = 0, \qquad \mu_2 = -\frac{\hbar}{4m}, \qquad \mu_3 = \frac{\hbar}{2m}, \qquad \mu_4 = 0, \qquad \mu_5 = \frac{\hbar}{8m}, \\ &U = \frac{e}{\hbar}\Phi + \frac{e^2}{2m\hbar c^2}\mathbf{A}^2, \qquad \mathcal{A}_1 = 0, \qquad \mathcal{A}_2 = -\frac{e}{mc}\mathbf{A}, \qquad \alpha_1 = \alpha_2 = 0. \end{aligned} \tag{1.6}$$

Eq.(1.4) generalizes the class of nonlinear equations that Doebner and I first derived, to include external electromagnetic potentials, two additional external vector fields that can act nonlinearly (one of which was studied some time ago by Haag and Bannier [9]), and terms of the type proposed by Kostin [10] and by Bialynicki–Birula and Micielski [11]. An exploration of the relation of some of these terms with the separation property was begun in joint work with Svetlichny [12]. Though obtained on fundamental grounds, Eq.(1.4) contains as special cases a remarkable variety of independently-proposed nonlinear terms [13–19].

Since $\mathrm{Re}\,[\dot{\psi}/\psi] = (1/2)[\dot{\rho}/\rho]$, we see from inspection of the imaginary part of the right-hand side of (1.4) that $\dot{\rho}$ is the divergence of a vector field. As long as this falls off sufficiently rapidly at infinity, we have that $(d/dt)\int \rho(\mathbf{x},t)\,d\mathbf{x}$ is zero – thus the interpretation of ρ as a conserved probability density makes sense.

When the gauge transformations (1.1) is applied, we have

$$\begin{aligned} &\rho' = \overline{\psi}'\psi' = \rho, \\ &\hat{\mathbf{j}}' = \frac{1}{2i}\left[\overline{\psi}'\nabla\psi' - (\nabla\overline{\psi}')\psi'\right] = \Lambda\hat{\mathbf{j}} + \frac{\gamma}{2}\nabla\rho + \rho\nabla\theta. \end{aligned} \tag{1.7}$$

Thus ρ is gauge-invariant, while $\hat{\mathbf{j}}$ is not. Furthermore, if ψ satisfies an equation in the family defined by (1.4), then ψ' satisfies a transformed equation, with gauge-transformed coefficients

ν'_j, μ'_j, α'_j, and external fields $\mathcal{A}'$, U', $\mathcal{A}'_j$ that can be expressed in terms of the unprimed quantities. We have a gauge-invariant current (see below), and gauge-invariant expressions for the usual, observable electric and magnetic fields. We also have formulas for independent gauge-invariant combinations of the coefficients ν_j, μ_j, and α_j, and the external vector fields. Here "gauge invariant" refers to the group of nonlinear transformations specified by (1.1). Naturally it is the gauge-invariant quantitities that must encode the physical content of a quantum theory described by one of equations in our family. Details of these transformations, and discussions of the gauge-invariant combinations, are published elsewhere.

2 Generalization of the Gauge Group

Next I shall describe and justify the idea for further generalization of this framework. It begins with the observation that the combination

$$\mathbf{j}^{gi} = \nu_1 \hat{\mathbf{j}} + \nu_2 \nabla \rho + \rho \mathcal{A} \tag{2.1}$$

is invariant under the transformation (1.1), so that $\mathbf{J} = -2\mathbf{j}^{gi}$ is a gauge-invariant current obeying $\dot{\rho} = -\nabla \cdot \mathbf{J}$. This means that our original working hypothesis, that all observations could be expressed as a succession of positional measurements at different times – i.e., in terms of $\rho(\mathbf{x}, t)$ – together with the imposition of external physical fields, may be unnecessarily stringent. Measurement procedures that detect $\mathbf{J}(\mathbf{x}, t)$, whether or not they can be expressed exclusively in terms of ρ and external fields, are equally compatible with (i.e., invariant under) the nonlinear gauge transformations (1.1).

Note also that unlike the formula for ρ, the expression for $\mathbf{J}$ involving (2.1) depends explicitly on two of the coefficients and one of the fields in Eq.(1.4). Indeed, there is no *a priori* reason why the expression for ρ could not also depend on these quantities. The important properties of the functions ρ and $\mathbf{J}$ are that they are invariant under the action of the group of nonlinear gauge transformations, that ρ is positive definite, and that they are related by an equation of continuity. Thus we might entertain the possibility of replacing the equation $\rho = |\psi|^2$ by a more general expression, that would have to be gauge invariant and reduce to $\rho = |\psi|^2$ in the case of the linear Schrödinger equation.

Now in standard, linear nonrelativistic quantum mechanics, the amplitude R and phase S of the wave function describing a pure state have very different status. The former is gauge invariant, and considered as physically observable; the latter is gauge dependent, and not observable. Likewise in the nonlinear quantum mechanics discussed in the previous section, R is manifestly gauge invariant, while S is not. When one reflects on this asymmetry, it seems increasingly extraordinary that we write a Schrödinger equation (linear or nonlinear) for the time-evolution by relating the *gauge* fields S, U and $\mathcal{A}$ to the *physical* field R, via the complex combination $R \exp[iS]$. Why should we not be able to couple gauge fields to gauge fields, and correspondingly, physical fields to physical fields? The purpose of this paper is to suggest a way to do just that, using a natural generalization of the nonlinearity Doebner and I proposed. The analysis applies even when the underlying physics is that of linear quantum mechanics!

If we return to Eqs.(1.1)–(1.4), we see that everything can be written very naturally in terms of the variables $\ln R$ and S. In particular, setting $T = \ln R$, Eq.(1.1) becomes

$$\begin{pmatrix} S' \\ T' \end{pmatrix} = \begin{pmatrix} \Lambda & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S \\ T \end{pmatrix} + \begin{pmatrix} \theta \\ 0 \end{pmatrix}, \tag{2.2}$$

where Λ and γ depend on t and θ depends on $\mathbf{x}$ and t. The condition $\Lambda \neq 0$ is just the requirement that the determinant of the matrix be nonvanishing. If we like, we can also write $\ln \rho = 2T$ so that $\nabla\rho/\rho = 2\nabla T$, and $\hat{\mathbf{j}}/\rho = \nabla S$. Then we can re-express the nonlinear functionals in (1.3) in terms of $\nabla^2 S$, $\nabla^2 T$, $(\nabla S)^2$, $\nabla S \cdot \nabla T$, and $(\nabla T)^2$; for example, $R_1 = \nabla^2 S + 2\nabla S \cdot \nabla T$, while $R_3 = (\nabla S)^2$. Since $\dot{\psi}/\psi$ is just $\dot{T} + i\dot{S}$, Eq.(1.4) becomes a pair of coupled partial differential equations for S and T. These logarithmic variables are familiar from earlier hydrodynamical and stochastic versions of quantum theory [20, 21].

It is time to take the leap. Eq.(2.2) practically cries out to be generalized to affine transformations modeled on the general linear group $GL(2, \mathbf{R})$:

$$\begin{pmatrix} S' \\ T' \end{pmatrix} = \begin{pmatrix} \Lambda & \gamma \\ \lambda & \kappa \end{pmatrix} \begin{pmatrix} S \\ T \end{pmatrix} + \begin{pmatrix} \theta \\ \phi \end{pmatrix}, \tag{2.3}$$

where Λ, γ, λ and κ are smooth, real-valued functions of t, and θ, ϕ are smooth, real-valued functions of $\mathbf{x}$ and t. This is essentially equivalent to complexifying the coefficients in (1.1). We can permit $\Lambda = 0$, but require that $\Delta = \kappa\Lambda - \lambda\gamma \neq 0$.

Immediately it is evident that the family of NLSEs must also be generalized for it to remain invariant under (2.3). The necessary (and natural) generalization is to introduce into the imaginary part of the right-hand side the terms $\nu_3 R_3$, $\nu_4 R_4$ and $\nu_5 R_5$, as well as external scalar and vector fields, so that there is symmetry between the real and imaginary parts. Thus

$$\begin{aligned} i\frac{\dot{\psi}}{\psi} = i\dot{T} - \dot{S} = i&\left[\sum_{j=1}^{5} \nu_j R_j[\psi] + \mathcal{T}(\mathbf{x},t) + \frac{\nabla\cdot(\mathcal{A}(\mathbf{x},t)\rho)}{\rho} + \frac{\mathcal{D}(\mathbf{x},t)\cdot\hat{\mathbf{j}}}{\rho} + \delta_1 \ln\rho + \delta_2 S\right] \\ &+ \left[\sum_{j=1}^{5} \mu_j R_j[\psi] + U(\mathbf{x},t) + \frac{\nabla\cdot(\mathcal{A}_1(\mathbf{x},t)\rho)}{\rho} + \frac{\mathcal{A}_2(\mathbf{x},t)\cdot\hat{\mathbf{j}}}{\rho} + \alpha_1 \ln\rho + \alpha_2 S\right], \end{aligned} \tag{2.4}$$

where $\mathcal{T}$ is a new external scalar field, and $\mathcal{D}$ a new external vector field. Note that the heat equation and other interesting equations of mathematical physics fall within this family; as well as the linear Schrödinger equation, with $\nu_3 = \nu_4 = \nu_5 = \delta_1 = \delta_2 = 0$, $\mathcal{T} = 0$, $\mathcal{D} = 0$, and the other values as in Eq.(1.6). Some equations with soliton-like solutions are also included [22].

As with the smaller family of nonlinear equations (1.4) and the smaller group of nonlinear gauge transformations (1.1), if ψ solves an equation within the class (2.4), then the wave function transformed under (2.3), $\psi' = R' \exp iS'$ with $R' = \ln T'$, solves another equation in the same class, but with transformed coefficients and transformed external fields. The question now is whether we can identify appropriate invariants under the group of transformations (2.3), in terms of which all the quantum observables can be expressed. If so, we are justified in considering R (or, alternatively, T) and S both as gauge fields, obeying one or another NLSE from the class (2.4), and deriving the physical fields from them as invariants under the enlarged nonlinear gauge group. We will have succeeded in treating S and $\ln R$ on an equal footing. It will even be possible to entertain quantum mechanics in a (nonlinear) gauge where $\ln R$ and S have been exchanged.

3 Generalized Gauge Invariants

From this point on, it is more convenient to work using the variables S and T. Consider then the coupled pair of general second-order quadratic partial differential equations,

$$\begin{aligned} \dot{S} = a_1\nabla^2 S + a_2\nabla^2 T + a_3(\nabla S)^2 + a_4\nabla S\cdot\nabla T + a_5(\nabla T)^2 \\ + a_6 S + a_7 T + u_0 + \mathbf{u}_1\cdot\nabla S + \mathbf{u}_2\cdot\nabla T, \end{aligned}$$

$$\dot{T} = b_1\nabla^2 S + b_2\nabla^2 T + b_3(\nabla S)^2 + b_4\nabla S\cdot\nabla T + b_5(\nabla T)^2 \\ + b_6 S + b_7 T + v_0 + \mathbf{v}_1\cdot\nabla S + \mathbf{v}_2\cdot\nabla T, \tag{3.1}$$

where the relation between (3.1) and (2.4) is given by

$$\begin{aligned} &a_1 = -\mu_1, \quad a_2 = -2\mu_2, \quad a_3 = -\mu_3, \quad a_4 = -2\mu_1 - 2\mu_4, \quad a_5 = -4\mu_2 - 4\mu_5, \\ &a_6 = -\alpha_2, \quad a_7 = -2\alpha_1, \quad u_0 = -U - \nabla\cdot\mathcal{A}_1, \quad \mathbf{u}_1 = -\mathcal{A}_2, \quad \mathbf{u}_2 = -2\mathcal{A}_1, \\ &b_1 = \nu_1, \quad b_2 = 2\nu_2, \quad b_3 = \nu_3, \quad b_4 = 2\nu_1 + 2\nu_4, \quad b_5 = 4\nu_2 + 4\nu_5, \\ &b_6 = \delta_2, \quad b_7 = 2\delta_1, \quad v_0 = \mathcal{T} + \nabla\cdot\mathcal{A}, \quad \mathbf{v}_1 = \mathcal{D}, \quad \mathbf{v}_2 = 2\mathcal{A}. \end{aligned} \tag{3.2}$$

Now the coefficients a_j, b_j obey the following transformation laws under (2.3), with the determinant $\Delta = \kappa\Lambda - \lambda\gamma$:

$$\begin{bmatrix} a_1' \\ a_2' \\ b_1' \\ b_2' \end{bmatrix} = \Delta^{-1} \begin{bmatrix} \kappa\Lambda & -\lambda\Lambda & \kappa\gamma & -\lambda\gamma \\ -\gamma\Lambda & \Lambda^2 & -\gamma^2 & \gamma\Lambda \\ \kappa\lambda & \lambda^2 & \kappa^2 & -\kappa\lambda \\ -\lambda\gamma & \lambda\Lambda & -\kappa\gamma & \kappa\Lambda \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}, \tag{3.3}$$

$$\begin{bmatrix} a_3' \\ a_4' \\ a_5' \\ b_3' \\ b_4' \\ b_5' \end{bmatrix} = \Delta^{-2}\mathcal{M} \begin{bmatrix} a_3 \\ a_4 \\ a_5 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}, \tag{3.4}$$

where

$$\mathcal{M} = \begin{bmatrix} \kappa^2\Lambda & -\kappa\lambda\Lambda & \lambda^2\Lambda & \kappa^2\gamma & -\kappa\lambda\gamma & \lambda^2\gamma \\ -2\kappa\gamma\Lambda & \Lambda(\kappa\Lambda+\lambda\gamma) & -2\lambda\Lambda^2 & -2\kappa\gamma^2 & \gamma(\kappa\Lambda+\lambda\gamma) & -2\lambda\gamma\Lambda \\ \gamma^2\Lambda & -\gamma\Lambda^2 & \Lambda^3 & \gamma^3 & -\gamma^2\Lambda & \gamma\Lambda^2 \\ \kappa^2\lambda & -\kappa\lambda^2 & \lambda^3 & \kappa^3 & -\kappa^2\lambda & \kappa\lambda^2 \\ -2\kappa\lambda\gamma & \lambda(\kappa\Lambda+\lambda\gamma) & -2\lambda^2\Lambda & -2\kappa^2\gamma & \kappa(\kappa\Lambda+\lambda\gamma) & -2\kappa\lambda\Lambda \\ \lambda\gamma^2 & -\lambda\gamma\Lambda & -\lambda\Lambda^2 & \kappa\gamma^2 & -\kappa\gamma\Lambda & \kappa\Lambda^2 \end{bmatrix},$$

and

$$\begin{bmatrix} a_6' \\ a_7' \\ b_6' \\ b_7' \end{bmatrix} = \Delta^{-1} \begin{bmatrix} \kappa\Lambda & -\lambda\Lambda & \kappa\gamma & -\lambda\gamma \\ -\gamma\Lambda & \Lambda^2 & -\gamma^2 & \gamma\Lambda \\ \kappa\lambda & \lambda^2 & \kappa^2 & -\kappa\lambda \\ -\lambda\gamma & \lambda\Lambda & -\kappa\gamma & \kappa\Lambda \end{bmatrix} \begin{bmatrix} a_6 \\ a_7 \\ b_6 \\ b_7 \end{bmatrix} + \Delta^{-1} \begin{bmatrix} \kappa\dot{\Lambda} - \lambda\dot{\gamma} \\ \Lambda\dot{\gamma} - \gamma\dot{\Lambda} \\ \kappa\dot{\lambda} - \lambda\dot{\kappa} \\ \Lambda\dot{\kappa} - \gamma\dot{\lambda} \end{bmatrix}. \tag{3.5}$$

For brevity we omit the transformation laws for the external fields.

The final task for this paper is to suggest invariant combinations of S and T. For simplicity, we consider only the matrix part of the transformation (2.3), i.e., we take $\theta = \phi = 0$. First suppose that X and Y obey

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \Lambda & \gamma \\ \lambda & \kappa \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{3.6}$$

and c_1, c_2 are coefficients. Then $c_1X + c_2Y$ is invariant under A if and only if $[c_1\, c_2]A^{-1} = [c_1'\, c_2']$. But one can verify from (3.4) that with $d_1 = 2a_3 + b_4$ and $d_2 = a_4 + 2b_5$, we have

$[d_1\, d_2]A^{-1} = [d_1'\, d_2']$. Hence $d_1 S + d_2 T$ can serve as one of the desired invariant combinations. Next let $L_1 = a_1 S + a_2 T$ and $L_2 = b_1 S + b_2 T$. We have

$$\begin{pmatrix} L_1' \\ L_2' \end{pmatrix} = \begin{pmatrix} \Lambda & \gamma \\ \lambda & \kappa \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = A \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}. \tag{3.7}$$

Therefore $d_1 L_1 + d_2 L_2$ is also an invariant. In fact, we can consider $d_1(\sigma L_1 + \tau S) + d_2(\sigma L_2 + \tau T)$ as a general linear combination of the invariants we have found, where σ and τ are fully invariant combination of the coefficients. For example, it is straightforward to verify that $a_1 + b_2$ and $a_1 b_2 - a_2 b_1$, which were earlier identified as gauge invariants for (2.2), are also invariants under (2.3). In short the desired invariant combinations of S and T exist, and we even have some flexibility in our choice: we can choose combinations that reduce to the usual formulas in the case of the linear Schrödinger equation!

This permits us to obtain a positive definite, gauge-invariant probability density and gauge-invariant current. Finally, a large subfamily of the equations (2.4) have solutions for which the gauge-invariant density and current obey a continuity equation. Details of these results are to be presented elsewhere.

4 Conclusion

Consideration of nonlinear gauge transformations modeled on the general linear group $GL(2, \mathbf{R})$ leads to a beautiful, apparently unremarked symmetry or duality between the phase and the logarithm of the amplitude in quantum mechanics. Both can be treated as gauge fields, suggesting the possibility of a fundamental reappraisal of the meaning of the wave function (and of gauge transformation). In particular, the linear Schrödinger equation is embedded in a natural class of nonlinear time-evolution equations, invariant as a class under nonlinear gauge transformations, extending (necessarily) the family that I proposed earlier in joint work with H.-D. Doebner. Formulas for gauge-invariant probability density and flux exist that apply across the whole class of nonlinear equations. The usual expressions for these quantities, along with the Schrdinger equation, are recovered for linearizable theories in a particular gauge.

Acknowledgments

I am grateful to Anatoly Nikitin and the other organizers of the Third International Conference on Symmetry in Nonlinear Mathematical Physics in Kyiv, July 1999, for the invitation to talk about this work; and to H.-D. Doebner for many interesting related discussions. I would also like to acknowledge the Alexander von Humboldt Foundation for its generous support of this work during my 1998-99 sabbatical year in Germany, and the Arnold Sommerfeld Institute for Mathematical Physics at the Technical University of Clausthal for hospitality during my visits there.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 447–455.

Supersymmetry and Supergroups in Stochastic Quantum Physics

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Supersymmetry was first applied to high energy physics. In the early eighties, it found a second and very fruitful field of applications in stochastic quantum physics. This relates to Random Matrix Theory, the topic I focus on in this contribution. I review several aspects of more mathematical interest, in particular, supersymmetric extensions of the Itzykson–Zuber and the Berezin–Karpelevich group integral, supersymmetric harmonic analysis and generalized Gelfand–Tzetlin constructions for the supergroup $\mathrm{U}(k_1/k_2)$. The consequences for the representation theory for supergroups are also addressed.

1 Introduction

The spectral fluctuations of very different quantum systems show a remarkably high degree of similarity: if one measures the eigenenergies of the system in units of the local mean level spacing, the energy correlation functions of nuclei, atoms, molecules and disordered systems become almost indistinguishable. In many cases, the distribution $P(x)$ of the spacings x between adjacent levels on the local scale is well described by the Wigner surmise

$$P_{\mathrm{Wigner}}(x) = \frac{\pi}{2}\, x \exp\left(-\frac{\pi}{4}x^2\right) . \tag{1}$$

We notice that the probability for finding small spacings is suppressed and vanishes linearly. This means that the levels are *correlated* and repel each other. A spectrum of uncorrelated levels behaves very differently. Such a spectrum can easily be modeled by producing a sequence of *uncorrelated* numbers with a random number generator. One then finds the Poisson law

$$P_{\mathrm{Poisson}}(x) = \exp\left(-x\right) \tag{2}$$

as the distribution of the spacings x. It is remarkable, that the wealth of conceivable correlations due to different types of interactions leads in so many cases to the distribution (1). This surprising universality is reflected in the simplicity of the phenomenological and statistical model, Random Matrix Theory (RMT), that quantitatively describes these correlations, or, their absence. It is easy to model the latter: the Hamiltonian in the energy bases is written as a diagonal matrix,

$$H = \mathrm{diag}\left(E_1, \ldots, E_N\right) \tag{3}$$

whose entries, the eigenvalues, are chosen as uncorrelated random numbers. To model the presence of correlations, one has to add off-diagonal matrix elements,

$$H = \begin{bmatrix} H_{11} & \cdots & H_{1N} \\ \vdots & & \vdots \\ H_{N1} & \cdots & H_{NN} \end{bmatrix} , \tag{4}$$

such that the eigenvalues of H are correlated. It is convenient to chose the matrix elements H_{nm} as Gaussian distributed random numbers.

RMT was founded by Wigner [1] about forty years ago and worked out mathematically by Mehta [2] and Dyson [3] in the following decade. Due to the general symmetry constraints, a time reversal invariant system with conserved or broken rotation invariance is modeled by the Gaussian Orthogonal (GOE) of real-symmetric matrices or the Gaussian Symplectic Ensemble (GSE) of self-dual matrices, respectively, while the Gaussian Unitary Ensemble (GUE) of Hermitean matrices models the fluctuation properties of a system under broken time reversal invariance. These generic fluctuation properties are referred to as Wigner–Dyson fluctuations. In 1984, a conceptually new element was brought into the discussion when Bohigas, Gianonni and Schmit [4] stated the following famous conjecture: *The quantization of a classical, conservative and fully chaotic system is expected to show Wigner–Dyson fluctuations, i.e. $P(x)$ is of the form (1). On the other hand, the quantization of a classical, conservative, integrable and regular system is expected to have significantly different spectral fluctuation properties, i.e. $P(x)$ is often of the form (2).* Although not yet rigorously proven, the Bohigas conjecture is supported by an overwhelming number of classical, semiclassical and quantal studies [5, 6].

It is now believed that RMT can be viewed as *thermodynamics for spectral fluctuations and related properties.* Based on little more than symmetry constraints and the assumption of sheer randomness, RMT grasps the crucial statistical features of a rich variety of systems. A detailed review was recently given in Ref. [7].

Unfortunately, the mathematical difficulties encountered in random matrix models were so serious that many interesting problems could only partially be solved. In 1983, Efetov [8] discovered, in condensed matter physics, a connection of paramount importance between RMT and supersymmetry in condensed matter physics. By supersymmetry we mean theories involving commuting and anticommuting degrees of freedom. We emphasize that these bosons and fermions have no direct physical interpretation as particles or so. They serve to map the stochastic model, without approximation, onto a model in superspace. The great merit of supersymmetry in stochastic quantum physics is a dramatic reduction of the number of integration variables due to this exact mapping. This can be viewed as an *irreducible representation* of the statistical system in question.

Verbaarschot, Weidenmüller and Zirnbauer [9] derived the same supersymmetric non-linear σ model for the statistical discussion of compound nuclear scattering starting from a random matrix model. Since then, the supersymmetric technique has experienced a burst of activities. The treatment of numerous, previously inaccessible, problems became possible.

2 Graded Eigenvalue Method

The mathematical solution of Efetov's supersymmetric non-linear σ models is still non-trivial. For purely spectral fluctuations, an alternative technique, the Graded Eigenvalue Method, was presented in Ref. [12], as a variant of Efetov's original approach.

The main motivation for this method will be given in the following section. The essence of the Graded Eigenvalue Method is the exact calculation of integrals over supergroups. The supersymmetric version of the Harish-Chandra–Itzykson–Zuber integral [13, 14] was evaluated in Ref. [12]. The Hermitean $2k \times 2k$ supermatrix σ has k eigenvalues s_{p1}, $p = 1, \ldots, k$ in the boson boson and k eigenvalues is_{p2}, $p = 1, \ldots, k$ in the fermion fermion sector ordered in the diagonal matrix s, it is diagonalized by a unitary supermatrix u such that $\sigma = u^{-1}su$. Moreover, we introduce a second supermatrix $\rho = v^{-1}rv$ with the same symmetries. The supersymmetric version of the Harish-Chandra–Itzykson–Zuber integral can then be written in the form [12]

$$\begin{aligned}\int \exp\left(i\,\mathrm{trg}\,\sigma\rho\right)\,d\mu(u) &= \int \exp\left(i\,\mathrm{trg}\,u^{-1}sur\right)\,d\mu(u)\\ &= \frac{\det\left[\exp(is_{p1}r_{q1})\right]_{p,q=1,\ldots,k}\det\left[\exp(is_{p2}r_{q2})\right]_{p,q=1,\ldots,k}}{B_k(s)B_k(r)},\end{aligned} \tag{5}$$

where trg stands for a properly defined trace in superspace. The function $B_k(s) = \det\left[1/(s_{p1} - is_{q2})\right]_{p,q=1,\ldots,k}$ is the square root of the Jacobian for the transformation of the volume element of the Hermitean supermatrix σ to eigenvalue-angle coordinates s and u.

With the Graded Eigenvalue Method, a quick rederivation of all GUE k-level correlation functions could be given [12].

3 Transitions towards Quantum Chaos

What is the merit of the Graded Eigenvalue Method for physics? – It is, for example, capable of exactly solving a fundamental problem of chaos theory, the regularity-chaos transition [15]. Within Efetov's original approach this problem can only asymptotically be studied. Consider the Hydrogen atom in a magnetic field and its classical analogue. The classical system is fully integrable for zero magnetic field, but becomes chaotic as the magnetic field grows because the spherical symmetry is broken. Following the Bohigas–Giannoni–Schmitt conjecture, it is easily conceivable that $P(x)$ undergoes a transition from the Poisson distribution to the Wigner surmise. Indeed, as Fig. 1 shows, this was confirmed in the experiment and in numerical calculations [10].

Similar transitions are encountered in many physical situations. In heavy ion reactions, a spreading of the electrical quadrupole transition strength has been observed which can be understood in terms of a regularity chaos transition [11]. In condensed matter physics, the phenomenon of localization can also be related to this crossover. Billiard systems [5] show similar transitions as well.

Naturally, the statistical model for this transition is the weighted sum of the two limiting Hamiltonians (3) and (4),

$$H(\alpha) = H^{(0)} + \alpha H^{(1)}, \tag{6}$$

where α is the dimensionless transition parameter. The matrices $H^{(1)}$ are drawn from a Gaussian Ensemble. Although the regularity chaos transition is our main interest, we make no assumptions yet for the probability distribution of the matrices $H^{(0)}$. Detailed numerical simulations of this transition can be found in Ref. [11] for two different ensembles of matrices $H^{(0)}$.

The key to the exact solution is the observation that the generating functions of the spectral correlators for the transition ensemble (6) obey an diffusive process. The fictitious time of the diffusion is related to the transition parameter through $\tau = \alpha^2/2$. The diffusion can be formulated in the curved space of the eigenvalues r of a supermatrix which provide the fictitious spatial coordinates. Moreover, this can be done not only for the GUE but for all three Gaussian Ensembles. For the generating functions $z_{\beta k}(r,\tau)$ the diffusive process reads

$$\frac{\beta}{4}\,\Delta_{\beta r}\,z_{\beta k}(r,\tau) = \frac{\partial}{\partial\tau} z_{\beta k}(r,\tau), \tag{7}$$

where $\beta = 1,2,4$ for the GOE, GUE and GSE, respectively. The operator $\Delta_{\beta r}$ is the radial part of the Laplacean in the space of supermatrices. It has $2k$ degrees of freedom for the

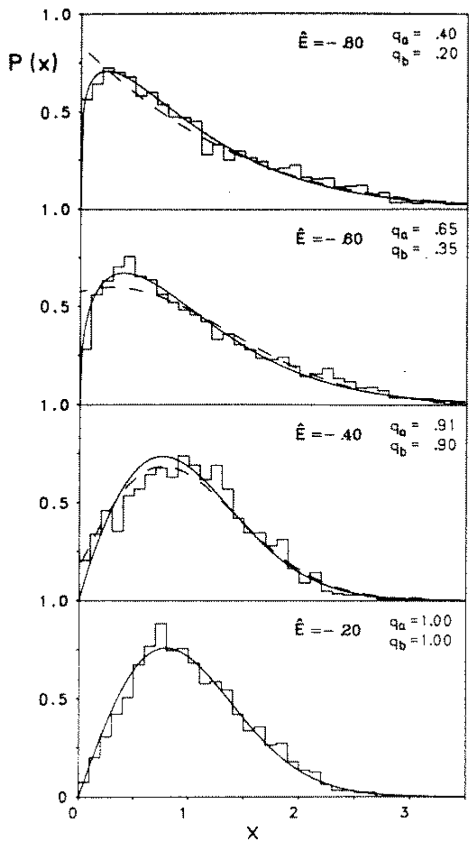

Figure 1. The nearest neighbor spacing distribution for the Hydrogen atom in a magnetic field. Since this system exhibits a certain scaling, the transition from regularity to chaos is governed by one single parameter $\widehat{E}$ which is a combination of energy and magnetic field. Taken from Ref. [10].

GUE and $4k$ for the GOE and the GSE. The initial condition is the generating function of the arbitrary correlations, $\lim_{\tau\to 0} z_{\beta k}(r,\tau) = z_{\beta k}^{(0)}(r)$. The diffusion kernel is, apart from trivial factors, the supersymmetric Harish-Chandra–Itzykson–Zuber integral for the three symmetry classes. It should be emphasized that the explicit knowledge of this kernel is not necessary to derive the diffusion process.

For the GUE, the k-level correlation functions can be expressed as a $2k$-fold integral over the eigenvalues s of a $2k \times 2k$ Hermitean supermatrix by using the integral (5). We arrive at the result

$$X_k(\xi_1,\ldots,\xi_k,\tau) = \frac{(-1)^k}{\pi^k} \int G_k(s-\xi,\tau)\, \Im z_k^{(0)}(s)\, B_k(s) d[s] \tag{8}$$

on the scale of the unfolded energies ξ_p, $p = 1,\ldots,k$. Here, the transition parameter τ is rescaled by the mean level spacing D such that $\tau \to \tau/D^2$. The pure GUE result is recovered in the limit $\tau \to \infty$. The function $G_k(s-\xi,\tau)$ is a normalized Gaussian with variance τ. The integral representation (8) is valid for an arbitrary ensemble of the matrices $H^{(0)}$. The function $\Im z_k^{(0)}(s)$ is the generating function of the corresponding correlations. In the case of two-level correlations, i.e. $k = 2$, two of the four integrals in Eq.(8) can be performed due to translation invariance for an arbitrary ensemble of the matrices $H^{(0)}$.

The regularity-chaos transition is obtained by choosing the matrices $H^{(0)}$ from the ensemble (3). This case is also discussed in detail in Refs. [15, 16].

4 Chiral Random Matrix Theory

While RMT was originally developed for non-relativistic quantum mechanics described by the Schrödinger equation, Shuryak and Verbaarschot [17] showed that this concept also works very well for relativistic systems where the Dirac equation applies. However, due to chiral symmetry, the RMT ansatz has to be modified. This leads to chiral RMT (chRMT).

For massless fermions, the Euclidean Dirac operator has the form

$$i\not{D} = i\not{\partial} + g\sum_a \frac{\eta^a}{2}\not{A}^a, \tag{9}$$

where g is the coupling constant, η^a are the generators of the gauge group and $\not{A}^a$ are the gauge fields. Physically, the gauge fields represent the gluons, i.e. the exchange particles of the strong interaction. The eigenfunctions of the Dirac operator are the *constituent quarks* which eventually form pion, proton, neutron, etc. In the chiral basis, the Dirac operator has an off-diagonal matrix structure, indicated in the relation (10). The main idea of chRMT is the replacement of the operator $i\not{D}$ in relation (10) by a random matrix W, such that

$$i\not{D} \quad\longrightarrow\quad \begin{bmatrix} 0 & i\not{D} \\ (i\not{D})^\dagger & 0 \end{bmatrix} \quad\longrightarrow\quad \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix}, \tag{10}$$

where W is a complex $N\times N$ matrix which has no further symmetries. The chirality of the Dirac operator implies that all eigenvalues come in pairs $(-\lambda, +\lambda)$. Thus, the center of the spectrum, where the eigenvalues are zero is distinguished. The existence of this region of zero virtuality states a fundamental difference to ordinary RMT.

What is the relevance of chRMT for QCD? – First, chRMT correctly reproduces low energy sum rules of QCD. Second, detailed studies have shown that the spectra from lattice gauge calculations indeed exhibit the correlations predicted by chRMT, see the review in Ref. [7]. To calculate the spectral correlators in non-trivial cases, it is highly convenient to extend the Graded Eigenvalue Method to chiral symmetry. The generating function is mapped onto superspace and expressed as an integral over $2k \times 2k$ complex supermatrices σ. Due to chirality, and in contrast to the cases discussed in the previous sections, σ has no further symmetries, in particular, it is not Hermitean. We proceed by introducing polar coordinates $\sigma = us\bar{v}$, where $u \in \mathrm{U}(k/k)$, $\bar{v} \in \mathrm{U}(k/k)/\mathrm{U}^{2k}(1)$, and $s = \mathrm{diag}\,(s_1, is_2)$ with $s_j = \mathrm{diag}\,(s_{1j}, \ldots, s_{kj})$ for $j = 1, 2$. The s_{pj} are real and non-negative. The transformation of the Cartesian volume element to radial and angular coordinates involves the Jacobian $B_k^2(s^2)$. The integral over the supergroups is non-trivial. It is the supersymmetric extension [18] of the Berezin–Karpelevich integral and reads

$$\int d\mu(u) \int d\mu(\bar{v}) \exp\left(i\mathrm{Re}\,\mathrm{trg}\, us\bar{v}r\right) = \frac{1}{2^{2k^2}(k!)^2} \frac{\det[J_0(s_{p1}r_{p'1})]\det[J_0(s_{q2}r_{q'2})]}{B_k(s^2)B_k(r^2)}, \tag{11}$$

where r is diagonal and where J_0 is a Bessel function. We stress that this integral is not contained in the supersymmetric analogue of the celebrated Harish-Chandra formula.

By using the supergroup integral (11), we succeeded in presenting exact calculations of the spectral correlators in the presence of arbitrarily many Matsubara frequencies [19]. We found a remarkable scaling property of all correlators near zero virtuality. Moreover, we also calculated the correlators for the massive Dirac operator, i.e. if non-zero *sea quark* masses are taken into account [20]. Recently, we managed to combine these two scenarios which is the physically realistic case. Again, we could give an exact and complete solution [21].

5 Representation Theory for Supergroups

The use of supersymmetric techniques in physical applications raises numerous non-trivial mathematical questions. Most of them lead directly to the representation theory of supergroups. Interestingly, a previously unknown class of representations emerges as a natural consequence of supersymmetry in stochastic quantum physics.

To begin with, we work out the simplest case of an harmonic analysis in a matrix space by studying 2×2 Hermitean matrices [22]. First, we consider two ordinary matrices H and K with eigenvalue matrices x and k, respectively. The expansion of the plane wave in this space reads trivially

$$\exp(i \operatorname{tr} HK) = \sum_{LM} T_L(x,k) Y_{LM}^*(\Omega_H) Y_{LM}(\Omega_K), \tag{12}$$

where L and M are the usual angular momentum and its magnetic projection. The function $T_L(x,k)$ is related to the spherical Bessel function $j_L(z)$ and the $Y_{LM}(\Omega)$ are the usual spherical harmonics depending on the solid angle Ω.

Surprisingly, a completely analogous expansion can be found in superspace. For two supermatrices σ and ρ with eigenvalue matrices s and r, respectively, we find

$$\exp(i \operatorname{trg} \sigma\rho) = \int t_{|\mu|}(s,r) y_{\mu\mu^*}^*(\omega_\sigma) y_{\mu\mu^*}(\omega_\rho) d[\mu] \tag{13}$$

such that the summation over L and M is replaced by an integral over an anticommuting variable μ and its complex conjugate μ^*. The graded Bessel function $t_{|\mu|}(s,r)$ depends only on the *length* $|\mu|^2 = \mu\mu^*$ of the anticommuting variable similar to the fact that $T_L(x,k)$ depends only on L. The graded spherical harmonics $y_{\mu\mu^*}(\omega)$ depend on a solid angle ω consisting of anticommuting variables which can be viewed as the analogue of Euler angles. Remarkably and crucially, these functions span something like a Hilbert space whose states are labeled by something like anticommuting angular momentum quantum numbers μ and μ^*. There are orthogonality and completeness relations. This is of fundamental importance for an application of the expansion (13), particularly for a Fourier–Bessel analysis in superspace.

The occurrence of this Hilbert-space like object spanned by the graded spherical harmonics raises the question whether one can construct representation functions of the supergroup U(1/1) which involve these anticommuting quantum numbers. The answer is affirmative [23]. Completely analogous to the Wigner representation functions of the ordinary group SU(2), graded Wigner representation functions can be constructed for U(1/1). The anticommuting variables μ and μ^* label these representations.

For the general case of the supergroup $\mathrm{U}(k_1/k_2)$, the construction of something like Euler angles is completely out of question. Thus, a method has to be devised which incorporates a recursion in the dimension of the supergroup. For the ordinary unitary group $\mathrm{U}(k)$, Gelfand [25] constructed such representations based on a recursive embedding in the group chain

$$\mathrm{U}(k) \supset \mathrm{U}(k-1) \supset \cdots \supset \mathrm{U}(2) \supset \mathrm{U}(1).$$

Moreover, Gelfand and Tzetlin [26, 27] constructed explicit coordinates on the manifold of $\mathrm{U}(k)$ which are closely related to these representations. The recursive structure of the representations is also reflected in the coordinates implying several most useful features. A detailed discussion can be found in Ref. [28].

The original, more geometric construction [26, 27] for ordinary unitary matrices needs to be modified to a more algebraic procedure in the case of supermatrices for reasons which will become clear in the following. We consider a Hermitean supermatrix σ with k_1 bosonic and k_2 fermionic dimensions. Let u be a unitary supermatrix in the supergroup $\mathrm{U}(k_1/k_2)$ such that $\sigma = u^{-1}su$ with the eigenvalues ordered in a diagonal matrix s. Define $u_p, p = 1, \ldots, k_1 + k_2$ as the columns of u. Since u_1 is an unit vector, the number of independent variables is $2(k_1 + k_2) - 1$, and the elements of u_1 cannot be used directly as independent variables. The idea is to project onto the $k_1 + k_2 - 1$ dimensional subspace spanned by the vectors $u_2, \ldots, u_{k_1+k_2}$. The corresponding projection $(1 - u_1u_1^\dagger)s(1 - u_1u_1^\dagger)$ of the eigenvalue matrix s has has $k_1 + k_2 - 1$ eigenvalues $s_p^{(1)}, p = 2, \ldots, k_1 + k_2$. We refer to them as the generalized Gelfand–Tzetlin eigenvalues. The ensuing system of equations involves a new type of singularity. It can be solved and yields explicit, comparatively simple expressions for the elements of u_1 in terms of the eigenvalues s, the generalized Gelfand–Tzetlin eigenvalues $s^{(1)}$ and phases. The eigenvalues in the fermion fermion block have the important property $|\xi_p^{(1)}|^2 = is_{p2}^{(1)} - is_{p2}$, $p = 1, \ldots, k_2$ where $\xi_p^{(1)}$ is a complex anticommuting variable and $\xi_p^{(1)*}$ its complex conjugate. By making appropriate basis rotations, this coordinate system is recursively continued to $k_1 + k_2$ levels with Gelfand–Tzetlin eigenvalues $s^{(m)}$, $m = 1, \ldots, k_1 + k_2$ and anticommuting variables $\xi^{(m)}$, $m = 1, \ldots, k_1$. On each level, the number of Gelfand–Tzetlin eigenvalues is lowered by one. Thus, we arrive at a complete, explicit coordinate system for $\mathrm{U}(k_1/k_2)$.

In view of the discussion in the previous section, we are led to conclude that the construction of the representations of the unitary supergroup $\mathrm{U}(1/1)$ can now be generalized to the unitary supergroup $\mathrm{U}(k_1/k_2)$. Analogously to the Gelfand–Tzetlin representations of the ordinary unitary group $\mathrm{U}(k)$, we interpret the commuting Gelfand–Tzetlin eigenvalues $s_{p1}^{(m)}$ and $is_{p2}^{(m)}$ as positive integers subject to a *betweenness condition.* Naturally, there is no interpretation of this sort for the anticommuting variables $\xi_p^{(m)}$ and $\xi_p^{(m)*}$. We obtain the generalized Gelfand pattern

$$
\begin{array}{ccccccccc}
s_{11} & & s_{21} & & s_{31} & \cdots & s_{k_1 1} & & \\
& s_{21}^{(1)} & & s_{31}^{(1)} & \cdots & s_{k_1 1}^{(1)} & & & \\
& & & \vdots & & & & & \\
& & s_{(k_1-1)1}^{(k_1-2)} & & s_{k_1 1}^{(k_1-2)} & & & & \\
& & & s_{k_1 1}^{(k_1-1)} & & & & &
\end{array}
$$

$$
\begin{array}{cccc}
|\xi_1^{(1)}|^2 & |\xi_2^{(1)}|^2 & \cdots & |\xi_{k_2}^{(1)}|^2 \\
|\xi_1^{(2)}|^2 & |\xi_2^{(2)}|^2 & \cdots & |\xi_{k_2}^{(2)}|^2 \\
\vdots & & & \vdots \\
|\xi_1^{(k_1)}|^2 & |\xi_2^{(k_1)}|^2 & \cdots & |\xi_{k_2}^{(k_1)}|^2
\end{array}
$$

$$
\begin{array}{ccccccccc}
is_{12}^{(k_1)} & & is_{22}^{(k_1)} & & is_{32}^{(k_1)} & \cdots & is_{k_2 2}^{(k_1)} & & \\
& is_{22}^{(k_1+1)} & & is_{32}^{(k_1+1)} & \cdots & is_{k_2 2}^{(k_1+1)} & & & \\
& & & \vdots & & & & & \\
& & is_{(k_2-1)2}^{(k_1+k_2-2)} & & is_{k_2 2}^{(k_1+k_2-2)} & & & & \\
& & & is_{k_2 2}^{(k_1+k_2-1)} & & & & &
\end{array}
$$

which labels representations of the unitary supergroup $\mathrm{U}(k_1/k_2)$. The generalized Gelfand pattern consists of two triangular sub-patterns for the commuting and one rectangular sub-pattern for the anticommuting variables. The two triangular sub-patterns label irreducible bases of the ordinary unitary groups $\mathrm{U}(k_1)$ and $\mathrm{U}(k_2)$ and hence both together label irreducible bases for the direct product $\mathrm{U}(k_1) \otimes \mathrm{U}(k_2)$ which is a subgroup of the supergroup $\mathrm{U}(k_1/k_2)$. The remaining coset $\mathrm{U}(k_1/k_2)/(\mathrm{U}(k_1) \otimes \mathrm{U}(k_2))$ is represented by the rectangular pattern of anticommuting variables.

From the construction, we may conclude that the generalized Gelfand pattern labels the basis which corresponds to the supergroup chain

$$\mathrm{U}(k_1/k_2) \supset \mathrm{U}(k_1-1/k_2) \supset \cdots \supset \mathrm{U}(1/k_2) \supset \mathrm{U}(k_2) \supset \cdots \supset \mathrm{U}(2) \supset \mathrm{U}(1)$$

and that the basis functions are eigenfunctions of the complete set of commuting operators in this chain.

There is already a theory [29] of finite-dimensional representations of the superalgebras $\mathrm{gl}(k_1/k_2)$ and $\mathrm{u}(k_1/k_2)$ involving a Gelfand pattern. Although anticommuting variables do not appear in those, there ought to be a connection to the generalized Gelfand pattern for the supergroup $\mathrm{U}(k_1/k_2)$ which contains anticommuting variables explicitly. Moreover, Balantekin and Bars [30] constructed representations of the unitary supergroup in terms of extended Young supertableaux. Again, anticommuting variables do not appear explicitly in those tableaux and it remains an open problem to find the relation to the generalized Gelfand pattern.

6 Summary and Outlook

The Graded Eigenvalue Method has proven to be a powerful technique for the exact calculation of various problems in stochastic quantum physics such as the regularity-chaos transition. Importantly, this method could be extended to chiral RMT where it also made several exact calculations feasible. The method is based on the computation of certain supergroup integrals, the supersymmetric versions of the Harish-Chandra–Itzykson–Zuber and the Berezin–Karpelevich integral.

Very naturally, these studies lead to a new representation theory for supergroups. A complete harmonic analysis on the supergroup $\mathrm{U}(1/1)$ and a representation in terms of graded Wigner functions are constructed involving anticommuting labels of the representations. The Gelfand–Tzetlin method was generalized for the supergroup $\mathrm{U}(k_1/k_2)$ in arbitrary dimensions k_1 and k_2.

Work on various physical applications of these results is in progress. Moreover, some interesting mathematical questions are still unanswered yet, such as the precise mathematical interpretation of the anticommuting group labels. This will also be studied in the near future.

Acknowledgments

I thank the Heisenberg foundation for financial support.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 456–464.

Nonlinear Symmetry and Unity of Spacetime and Matter*

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Based upon the invariances under the nonlinear global supersymmetry (NL SUSY) and the general coordinate transformation ($GL(4,R)$), a unified description of spacetime and matter is proposed. Except the graviton all elementary particles accomodated in a single irreducible representation of $N=10$ extended super-Poincaré (SP) algebra are the composites of more fundamental objects *superons* with spin 1/2, which are Nambu–Goldstone (N–G) fermions accompnying the spontaneous breakedown of the global supertranslation of spacetime. The electroweak standard model (SM) and $SU(5)$ ($SO(10)$) grand unified model (GUTs) are investigated systematically by using the superon diagrams. The stability of the proton, the suppression of the flavour changing neutral currents (FCNC), $K^0-\overline{K}^0$ and $B^0-\overline{B}^0$ mixings, CP-violation, the atmospheric ν_μ deficit, the charmless nonleptonic B decay and the absence of the electroweak lepton-flavor-mixing are understood naturally in the superon pictures of GUTs and some predictions are presented. The fundamental action of superon-graviton model (SGM) for supersymmetric spacetime and matter is obtained.

1 Introduction

As a unified gauge field theory of all particles and all forces, the strong-electroweak standard model (SM), the grand unified theories (GUT) like $SU(5)$ ($SO(10)$) and their variants with supersymmetry (SUSY) still leave many fundamental problems, for example, the lack of the explanations of the generation structure of quarks and leptons and the absence of the electroweak mixings only among the lepton generations, the stability of the proton and the missing of the gravitational interaction, ... etc. The (local) supersymmetry (SUSY) [1], although unclear yet in the low energy particle physics, is the most promising notion for explaining the rationale of beings of all elementary particles including the graviton. As shown by Gell-Mann [2], $SO(8)$ maximally extended supergravity theory (SUGRA) is too small to accommodate all observed particles as elementary fields within the framework of the local gauge field theory. However it may be interesting, even at the risk of the local gauge field theory at the moment, from the viewpoints of simplicity and beauty of nature to attempt the accomodation of all observed elementary particles in a single irreducible representation of a certain group (algebra).

In ref. [3], by extending the group theoretical arguments beyond $N=8$ we have shown that among all $SO(N)$ extended super-Poincaré (SP) symmetry, the massless irreducible representations of $SO(10)$ SP algebra gives minimally and uniqely the framework for the unification of all observed particles and forces. However the fundamental theory has left unknown.

In ref. [4], we have pointed out that the anticommutators of the supercharges of $SO(10)$ SP algebra in the light-cone frame (the massless irreducible representations) can be interpreted as canonical anticommutators of creation and annihilation operators of spin 1/2 fermions and

*Plenary talk given at the 3rd International Conference on Symmetry in Nonlinear Mathematical Physics, Kyiv, Ukraine, July 12–18.

that it may indicate the existence of certain fundamental objects which are the constituents of all elementary particles except the graviton. We have identified the fundamental objects with Nambu–Goldstone (N–G) fermions *superons* of the spontaneous breakdown of supertranslation of spacetime. We have proposed *superon-graviton model* (SGM) as a fundamental theory for supersymmetric structure of spacetime and matter by using Volkov–Akulov nonlinear (NL) SUSY action for N–G fermion [5] in the curved spacetime. In this article, with a concise review of ref. [3] and [4] for the selfcontained arguments we study SGM further from the viewpoints of the internal structure of the quarks, leptons and gauge bosons except the graviton. The symmetry breaking of SGM and its cosmological implications are discussed briefly.

2 $SO(10)$ super-Poincaré algebra

In ref. [3] and [4], by noting that 10 generators Q^N $(N = 1, 2, \ldots, 10)$ of $SO(10)$ SP algebra are the fundamental represemtations of $SO(10)$ internal symmetry and that $SO(10) \supset SU(5) \supset SU(3) \times SU(2) \times U(1)$ we have decomposed 10 generators Q^N of $SO(10)$ SP algebra as follows with respect to $SU(5)$

$$\begin{aligned} \underline{10} &= \underline{5} + \underline{5}^* \\ &= \left\{ \left(\underline{3}, \underline{1}; -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \right) + (\underline{1}, \underline{2}; 1, 0) \right\} + \left\{ \left(\underline{3}^*, \underline{1}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) + (\underline{1}, \underline{2}^*; -1, 0) \right\}, \end{aligned} \tag{1}$$

where we have specified ($\underline{SU(3)}$, $\underline{SU(2)}$; electric charges). This assignment is the extention of that of $SO(8)$ SUGRA of Gell-Mann and interestingly coincides with $\underline{5}$ of $SU(5)$ GUT of Georgi and Glashow [6]. To obtain a smaller single irreducible representation we have studied the massless representation. For massless case the little algebra for the supercharges in the light-like frame $P_\mu = \epsilon(1, 0, 0, 1)$ becomes after a suitable rescaling

$$\left\{ Q^M_\alpha, Q^N_\beta \right\} = \left\{ \bar{Q}^M_{\dot\alpha}, \bar{Q}^N_{\dot\beta} \right\} = 0, \qquad \left\{ Q^M_\alpha, \bar{Q}^N_{\dot\beta} \right\} = \delta_{\alpha 1} \delta_{\dot\beta \dot 1} \delta^{MN}, \tag{2}$$

where $\alpha, \beta = 1, 2$ and $M, N = 1, 2, \ldots, 5$. Note that the spinor charges Q^M_1, $\bar{Q}^M_{\dot 1}$ satisfy the algebra of annihilation and creation operators respectively and can be used to construct a finite-dimensional supersymmetric Fock space with positive metric. It is natural to identify the graviton with the Clifford vacuum $\mid \Omega(\lambda)\rangle$ ($SO(10)$ singlet but not necessarily the lowest energy state) satisfying $Q^M_\alpha \mid \Omega(\lambda)\rangle = 0$, for the adjoint representation with helicity ± 1 appears automatically. We obtain $2 \cdot 2^{10}$ dimensional irreducible representation of the little algebra (2) of SO(10) SP algebra as follows: $\left[\underline{1}(+2), \underline{10}\left(+\frac{3}{2}\right), \underline{45}(+1), \underline{120}\left(+\frac{1}{2}\right), \underline{210}(0), \underline{252}\left(-\frac{1}{2}\right), \underline{210}(-1), \underline{120}\left(-\frac{3}{2}\right), \underline{45}(-2), \underline{10}\left(-\frac{5}{2}\right), \underline{1}(-3)\right]$ + [CPT-conjugate], where $\underline{d}(\lambda)$ represent $SO(10)$ dimension $\underline{d}$ and the helicity λ.

3 Superon quintet model (SQM) for matter

3.1 Particles in SQM

By noting that the helicities of these states are automatically determined by $SO(10)$ SP algebra and that Q^M_1 and $\bar{Q}^M_{\dot 1}$ satisfy the algebra of the annihilation and the creation operators for the spin $\frac{1}{2}$ particle, we speculate boldly that these states spanned upon the mathematical (not the physical true vacuum with the lowest energy) Clifford vacuum $\mid \Omega(\pm 2)\rangle$ are the *relativistic* (gravitationally induced) massless composite eigenstates made of the fundamental massless object Q^N *superon* with spin $\frac{1}{2}$. Therefore we regard (1) as *a superon-quintet* and *an antisuperon-quintet*. The unfamiliar identification of the generators of $SO(10)$ SP algebra with the fundamental objects is discussed later. Now we envisage the Planck scale physics as follows.

Nature (spacetime and matter) have the symmetric structure described by $SO(10)$ SP algebra at (above) the Planck energy scale, where the gravity dominates and induces the spontaneous breakdown of the supertranslation of spacetime acompanying the pair production of N–G formions (the superon-quintet and the antisuperon-quintet) from the vacuum in such a way as all the possible nontrivial multiplicative combinations of superons span the massless irreducible representations (i.e. eigenstates) of $SO(10)$ SP algebra. As shown later, the interaction of superons is highly nonlinear.

Now from the viewpoints of the superon-quintet model (SQM) for matter we can study more concretely the physical meaning of the results obtained in ref. [3] and [4].

Hereafter we use the following symbols for superons Q^N $(N = 1, 2, \ldots .10)$.

For the superon-quintet $\underline{5}$: $\left[\left(\underline{3}, \underline{1}; -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right), (\underline{1}, \underline{2}; 1, 0)\right]$, we use

$$[Q_a, Q_m;\ a = 1, 2, 3; m = 4, 5] \tag{3}$$

and for the antisuperon-quintet $\underline{5}^*$: $\left[\left(\underline{3}^*, \underline{1}; +\frac{1}{3}, +\frac{1}{3}, +\frac{1}{3}\right), (\underline{1}, \underline{2}^*; -1, 0)\right]$, we use

$$[Q_a^*, Q_m^*;\ a = 1, 2, 3; m = 4, 5]\,, \tag{4}$$

i.e. Q_a and Q_m represent color- and electroweak-components of superon quintets respectively. Accordingly all the states are specified explicitly with respect to ($SU(3)$, $SU(2)$; electric charges). In order to see the low energy particle contents, suppose that through the symmetry breaking:

$$[SO(10) \text{ SP symmetry}] \longrightarrow [\cdots] \longrightarrow [SU(3) \times SU(2) \times U(1)] \longrightarrow [SU(3) \times U(1)]$$

which is discussed later, the states with higher helicities $\left(\pm 3, \pm\frac{5}{2}, \pm 2, \pm\frac{3}{2}, \pm 1\right)$ of $SO(10)$ SP algebra absorb the lower helicity states$\left(\pm 2, \pm\frac{3}{2}, \pm 1, \pm\frac{1}{2}, 0\right)$ in $SU(3) \times SU(2) \times U(1)$ invariant way and become massive as many as possible in so far as the SM with three generations of quarks and leptons survive in the residual massless states. We have carried out the recombinations among $2 \cdot 2^{10}$ helicity states and found surprisingly just three generations of quarks and leptons of the SM appear in the surviving massless states of the fermions. At least one new spin $\frac{3}{2}$ lepton-like (gravitino) electroweak doublet (ν_Γ, Γ^-) with the mass of the electroweak scale is predicted [3]. ((ν_Γ, Γ^-) may be included in $\underline{5}$ of $\underline{10} = \underline{5} + \underline{5}^*$ of helicity $\pm\frac{3}{2}$ state.)

Towards the construction of the fundamental theory of SQM and for surveying the physical (phenomenological) implications of the superons for the unified gauge models (SM and GUTs) it is very important to understand all the gauge and the Yukawa couplings of the unified gauge models in terms of the superon pictures. For simplicity we neglect the mixing between superons and take the following left-right symmetric assignment for quarks and leptons by using the conjugate representations naively , i.e. $(\nu_l, l^-)_R = (\bar{\nu}_l, l^+)_L$, etc. [3].

For three generations of leptons [(ν_e, e), (ν_μ, μ), (ν_τ, τ)], we take

$$[(Q_m \varepsilon_{ln} Q_l^* Q_n^*), (Q_a Q_a^* Q_m^*), (Q_a Q_a^* Q_b Q_b^* Q_m^*)] \tag{5}$$

and the conjugate states respectively.

For three generations of quarks [(u, d), (c, s), (t, b)], we have *uniquely*

$$[(\varepsilon_{abc} Q_b^* Q_c^* Q_m^*), (\varepsilon_{abc} Q_b^* Q_c^* Q_l \varepsilon_{mn} Q_m^* Q_n^*), (\varepsilon_{abc} Q_a^* Q_b^* Q_c^* Q_d Q_m^*)] \tag{6}$$

and the conjugate states respectively.

For $[SU(2) \times U(1)$ gauge bosons], we have from $\underline{2} \times \underline{2}^*$

$$\left[-Q_4Q_5^*, \frac{1}{\sqrt{2}}(Q_4Q_4^* - Q_5Q_5^*), Q_5Q_4^*; \frac{1}{\sqrt{2}}(Q_4Q_4^* + Q_5Q_5^*)\right].$$

For $[SU(3)$ gluons], we have

$$\left[Q_1Q_3^*, Q_2Q_3^*, Q_1Q_2^*, \frac{1}{\sqrt{2}}(Q_1Q_1^* - Q_2Q_2^*), Q_2Q_1^*, \frac{-1}{\sqrt{6}}(2Q_3Q_3^* - Q_2Q_2^* - Q_1Q_1^*), Q_3Q_2^*, Q_3Q_1^*\right].$$

For $[SU(2)$ Higgs Boson], we have $[\varepsilon_{abc}Q_aQ_bQ_cQ_m]$ and the conjugate state.

For $[(X,Y)$ leptoquark bosons] in GUTs, we have $[Q_a^*Q_m]$ and the conjugate state.

For [a color- and $SU(2)$-singlet neutral gauge boson] from $\underline{3} \times \underline{3}^*$ (which we call simply S boson to represent the singlet) we have $Q_aQ_a^*$.

The specification of (X,Y) gauge boson is important for the proton decay in $SU(5)$ GUT. The specification of S boson may be interesting as an additional $U(1)$ of the gauge structure $SU(3) \times SU(2) \times U(1) \times U(1)$ of SUSY SM. As shown later S boson plays crucial roles in the process concerning the third generation of quarks and leptons. We have considered only two-superons states $\underline{45}$ of the adjoint representation of $SO(10)$ SP algebra for the vector gauge bosons.

3.2 Superon diagrams

Now in order to see the physical implications of SQM for SM (GUTs) for matter we try to interpret the Feynman diagrams of SM (GUTs) in terms of the Feynman diagrams of SQM. The superon-line Feynman diagrm of SQM is obtained by replacing the single line in the Feynman diagram of the unified gauge models by the corresponding multiple superon lines. To translate the vertex of the Feynman diagram of the unified gauge models into that of SQM, we assume that the superon-antisuperon pair creation and pair annihilation within a single state for a quark, a lepton and a (gauge) boson (i.e. within a single $SO(10)$ SP eigenstate) are rigorously forbidden. This rule seems natural because every state is an irreducible representation of $SO(10)$ SP algebra and is prohibitted from the decay without any remnants, i.e. without the interaction between the superons contained in another state. As discussed later this means the absence of the excited states of quarks, leptons, gauge bosons despite their compositeness. Here we just mention that all the states necessary to the SM and GUTs with three genarations of quarks and leptons appear up to five-superons states (i.e. one half of the full occupation ten-superons). As mentioned later this observation may be crucial for the spontaneous symmetry breaking with a large mass hierarchy. At the moment we naievely assume that all exotic states besides higher spin states composed of more than five superons have large masses in the low energy.

Now the translation is unique and straightforward. We see that in the Yukawa coupling of SQM the observed quark (lepton) interacts with the Higgs boson and a new quark(lepton) which is exotic with respect to $SU(2)$ and/or spin. Then the Yukawa coupling of SM (GUTs) can be reproduced effectively only in the higher orders of the Yukawa couplings of SQM, which gives potentially the Yukawa coupling of SM (GUTs) a small factor of the order of the inverse of the large mass of the exotic quark and lepton. This mechanism may be the origin of CKM mixing matrix for the quark sector but may be dangerous so far for the lepton sector because of the disastrous violations of the lepton family quantum numbers by the lepton mixings. However we find that at every gauge coupling vertex there is a stringent selection rule for generations which is characteristic to SQM, for each generation is identical only with respect to $SU(3) \times SU(2) \times U(1)$ quantum numbers but has another superon content corresponding to the flavor quantum number. This selection rule is the matching of the superons, i.e. the superon number conservation, at the gauge coupling vertex. For the quark sector, surprisingly, the selection rule respects the CKM mixing of the Yukawa coupling sector and maintains the successful gauge current structure of SM. While for the lepton sector, remarkably the selection rule prohibits basically the lepton flavor changing electroweak currents between lepton generations at the tree level and reproduces the success of SM. We regard that SQM can explain the absence of the lepton-flavor-mixing in the *electroweak* gauge interaction.

As a few examples of the gauge interactions and the selection rule at the gauge coupling vertex we have discussed the following typical processes [4], i.e. (i) β decay: $n \longrightarrow p + e^- + \bar{\nu}_e$, (ii) $\pi^0 \longrightarrow 2\gamma$, (iii) the proton decay: $p \longrightarrow e^+ + \pi^0$, (iv) a flavor changing neutral current process (FCNC): $K^+ \longrightarrow \pi^+ + \nu_e + \bar{\nu}_e$ and (v) an advocated typical process of the (non-gauged) compositeness: $\mu \longrightarrow e + \gamma$.

Now the translation is unique and straightforward. For the processes (i) and (ii) we can draw the corresponding similar tree-like superon line diagrams easily, where the triangle-like superon diagram does not appear. For the process (iii) we examine the Feynman diagrams for the proton decay of GUTs and find that the corresponding superon line diagrams do not exist due to the selection rule, i.e. the mismatch of the superons contained in the quarks(u and d) and the gauge bosons(X and Y) at the gauge coupling vertices. This means that irrespective of the massses of the gauge bosons the proton is stable at the tree level against $p \longrightarrow e^+ + \pi^0$. For FCNC process (iv) the penguin-type and the box-type superon line diagrams are to be studied corresponding to the penguin- and box-Feynmann diagrams for $K^+ \longrightarrow \pi^+ + \nu_e + \bar{\nu}_e$ of GUTs. Remarkablely the superon line diagrams which have only the u and c quarks for the internal quark line exist due to the selection rule and GIM mechanism of the SM is reproduced. The third generation t quark for the internal line is decoupled due to the selection rule. This is the indication of the strong suppression of the FCNC process, $K^+ \longrightarrow \pi^+ + \nu_e + \bar{\nu}_e$. This simple mechanism may hold in general for FCNC processes. For the process (v) the corresponding tree-like superon line diagram does not exist due to the selection rule at the gauge coupling vertex, i.e. $\mu \longrightarrow e + \gamma$ decay mode is absent at the tree-level in the superon (composite) model. The process $\tau \longrightarrow e(\mu) + \gamma$ is suppressed similarly.

As for the CP-violation, the mixing K^0-$\overline{K^0}$ is natural in SQM, for remarkably K^0 and $\overline{K^0}$ have *the same superon contents* (i.e. indistinguishable and superposing at the superon level) but have the different superon combinations distinguished by the interactions which lead to mass differences. GIM mechanism works for the superon picture of K^0-$\overline{K^0}$ mixing box diagram of SM but remarkably t quark (the third generation) decouples due to the selection rule at the gauge coupling vertices. However in SQM there is another higher order box (ladder-like) diagram contributing to K^0-$\overline{K^0}$ mixing amplitude, where S gauge boson emitted by the transition $(u,d) \leftrightarrow (t,b)$ and t quark play crucial (dominant) roles besides W boson. The relative phase of these two amplitudes may be an origin of CP-violation in the neutral K-meson decay. This mechanism of CP-violation without requiring complex gauge coupling constants seem natural from the viewpoint of the unification of all forces including gravity (which is a singlet, neutral and universal force) in a (semi)simple gauge group with one universal gauge coupling constant. It is interesting that t quark (the third generation of quarks) which appears automatically in SQM is needed for CP-violation in SQM context. The mixings B^0-$\overline{B^0}$ and D^0-$\overline{D^0}$ are natural in the same reason but the preliminary analyses suggest the similar new mechanisms for mixing and CP-violation characteristic of the SQM. SQM explains qualitatively the Weinberg angle (i.e. the mixing of the neutral electroweak gauge bosons) and predicts the mixing of a gluon and S boson by the same reason. The low energy $SU(3)$ color symmetry may be a residual gauge symmetry like $U(1)$ electromagnetic gauge symmetry in SM. As for the charmless nonleotonic B decay [7] in SQM the transition $(t,b) \leftrightarrow (c,s)$ occurrs not at the tree level of the weak charged current but at the higher orders of the gauge couplings due to the selection rule for the quark sector, where the transition $(t, b) \leftrightarrow (c, s)$ is achieved by the emissions of S boson and W boson and may give an explanation of the excess of the charmless (or the suppression of the charm mode) nonleotonic B decay. Furthermore for the lepton sector amazingly S gauge boson induces the transition *only* $\nu_\mu \leftrightarrow \nu_\tau$ (i.e. between the second and the third generation) at the tree level due to the selection rule, which may solve simply and naturally the ν_μ deficit problem of the atomospheric neutrino [8].

Next we just mention the excited states of quarks, leptons and gauge bosons. As stated before these particles (i.e. the massless eigenstates of $SO(10)$ SP symmetry) do not have the low energy excited states in SQM, because each particle is a single (massless) eigenstates of $SO(10)$ SP symmetry composed of superons and transits to *another eigenstate* through the interaction, i.e. through the absorption or the emission of superons (i.e. eigenstates).

4 Superon-graviton model (SGM) for spacetime and matter

4.1 Fundamental action for SGM

Finally we consider the fundamental theory of superon-graviton model (SGM) for supersymmetric spacetime and matter. In carrying through the canonical quantization of the elementary N–G spinor field $\psi(x)$ of two dimensional Volkov–Akulov model [4] of the NL SUSY, we have shown that the supercharges Q given by the supercurrents

$$J^\mu(x) = \frac{1}{i}\sigma^\mu\psi(x) - \kappa\ \{\text{the higher orders of } \kappa,\ \psi(x) \text{ and } \ \partial\psi(x)\} \tag{7}$$

obtained by the ordinary Noether procedures can satisfy the super-Poincaré algebra at the cnonically quantized level [9], where κ is a fundamental volume of the superspace of the NL SUSY with the mass dimension -2 (for the two dimensional case). Remarkably (7) means the field-current identity between the fundamental Nambu–Goldstone spinor $\psi(x)$ field and the supercurrent, which justify our basic assumption that the generator(supercharge) Q^N $(N = 1, 2, \ldots, 10)$ of $SO(10)$ SP algebra for the massless case represents the fundamental object *superon with spin* $\frac{1}{2}$. And our qualitative arguments are valid in the leading order for the small κ and/or in the low energy (momentum) as seen from (7). Therefore we speculate that the fundamental theory of SQM for matter is $SO(10)$ NL SUSY and that the fundamental theory of SGM for spacetime and matter at (above) the Planck scale is $SO(10)$ NL SUSY in the curved spacetime which corresponds to the Clifford vacuum $\mid \Omega(\pm 2)\rangle$. We regard that all the helicity-states of $SO(10)$ SP algebra including the observed quarks, leptons and gauge bosons except the graviton are the relativistic (gravitational) composite massless states of N–G fermion superons. SGM may show that the relativistic version of the composite (quark) model [11] of matter is realized as eigenstates of $SO(10)$ SP algebra at the superon level.

We propose the following Lagrangian as the fudamental theory of SGM of spacetime and matter.

$$L_{SGM} = -\frac{c^3}{16\pi G} e(R + \Lambda)|W|, \tag{8}$$

$$|W| = \det W^\nu_\mu = \det\left(\delta^\nu_\mu + \kappa T^\nu_\mu\right), \qquad T^\nu_\mu = \frac{1}{2i}\sum_{i,j=1}^{10}\left(\bar{s}^i O_{ij}\gamma_\mu D^\nu s^j - D^\nu \bar{s}^i \gamma_\mu O_{ij} s^j\right), \tag{9}$$

where κ is a fundamental volume of the superspace of the NL SUSY with the mass dimension -4, $e = \det e^a_\mu$, $D_\mu = \partial_\mu + \frac{1}{2}\omega^{ab}_\mu \sigma_{ab}$ and R and Λ are the scalar curvature and the cosmological constant, respectively. O_{ij} is a 10×10 unitary matrix representing the mixing among the superons, which may be probable but unpleasant from the elementary nature of the superon. The multiplication of the Einstein-Hilbert action by SQM action $|W|$ in (8) is essential and unique for the fundamental theory if we require that (i) it should be reduced to $SO(10)$ NL SUSY *a la* Volkov–Akulov in the flat spacetime by taking only $R \to 0$, (ii) also to the Einstein–Hilbert action (i.e. Clifford vacuum action) by taking the superonless limit $s^i \to 0$, (iii) except the graviton all fields participating in the superHiggs(recombination) mechanism should be the composites of superons and (iv) the action (8) should be invariant under the global $SO(10)$ NL SUSY,

$$\delta s^M = \frac{\epsilon^M}{\sqrt{\kappa}} - 2i\sqrt{\kappa}\left(\bar{\epsilon}^L \gamma^\mu s^L\right) D_\mu s^M, \tag{10}$$

$$\delta e^a{}_\mu = i\sqrt{\kappa}\left(\bar{\epsilon}^L \gamma^\rho s^L\right) \mathcal{D}_\rho e^a{}_\mu, \tag{11}$$

where ϵ^M $(M = 1, 2, \ldots, 10)$ is a constant spinor parameter with spin 1/2. (8) is manifestly invariant at least under the general coordinate transformation and global $SO(10)$. Furthermore the all order invariance of (8) under the global $SO(10)$ NL SUSY (10) and (11) in the similar sense of ref. [12] and [13] can be anticipated, which may be included in the scope of ref. [12] and [13]. The states with helicity ± 3, $\pm\frac{5}{2}$ and ± 2 (except the graviton) made of 10-, 9- and 8-superons appear afetr specifying the contorsion in the spin conection $\omega^{\mu}_{ab}(e^\mu_a, s^i)$ [10]. The fundamental Lagrangian (8) can be rewritten in the following simple form $L_{SGM} = -\frac{c^3}{16\pi G} n(R + \Lambda)$, where $n = \det n^a_\mu = \det(e^a_\nu W^\nu_\mu)$.

4.2 Symmetry breaking of SGM

As for the abovementioned spontaneous symmetry breaking it is urgent to study the structure of the true vacuum of (8). To see clearly the (low energy) mass spectrum of the particles spanned upon the true vacuum, we should convert the highly nonlinear SGM Lagrangian (8) into the equivalent linearized broken SUSY $SO(10)$ (or SM) Lagrangian. The orders of the mass scales of spontaneous SUSY and $SO(10)$ breaking are given by κ and Λ. The low-energy structure of the linearized broken SUSY Lagrangian should involve GUTs, at least the SM with three generations. For carrying through the complicated scenario it is encouraging that the linearlization of such a nonlinear fermionic system was already carried out explicitly [12, 13]. They investigated in detail the conversions between $N = 1$ NL SUSY (Volkov–Akulov) model and the equivalent linear (broken) $N = 1$ SUSY Lagrangian in the flat spacetime. The extension of the generic and the systematic arguments by using the superspace [13] may be useful for the linerization of SGM. From the mathematical viewpoint an equivalent linear theory would exist. It is a challenge to pursue the scenario. We expect that by taking *non-perturbatively* the true vaccum of (8) the conversions into the linear representation is achieved, where SUSY is broken spontaneously at the tree level and the bosonic and the fermionic high-spin massless states turn out to be massive states. This may be only the possible way to circumvent the no-go theorem [14] and to accomodate successfully high spin (massless) states in the local field theoretical GUTs. The massless tensor fields (states) in the adjoint representation $\underline{45}$ of $SO(10)$ may play important roles in the early spontaneous symmetry breakings: $[SO(10)\ \mathrm{SP}] \longrightarrow [\cdots] \longrightarrow [SU(3) \times SU(2) \times U(1)] \longrightarrow [SU(3) \times U(1)]$.

By generalizing the idea of the strong gravity [15] all tensor fields of the adjoint representation can have $U(M) \times U(N) \times \cdots$ invariant masses by the spontaneous symmetry breaking induced by the Higgs potential analogue gauge invariant self-interactions, provided these tensor fields are the gauge fields of the nonlinear realization of $SL(2M, C) \times SL(2N, C) \times \cdots$ with $45 = M^2 + N^2 + \cdots$. $GL(4, R)$ does not break spontaneously. $SL(12, C) \times SL(6, C)$ and $[SL(6, C) \times SL(4, C) \times SL(2, C)]^3$ which allow $U(6) \times U(3)$ and $[U(3) \times U(2) \times U(1)]^3$ invariant masses respectively are interesting from simplicity and may be relevant to SGM scenario. Especially this mechanism of the spontaneous symmetry breaking is worthwhile to be studied in detail to see whether it generates large masses spontaneously to all the states composed of more than five superons that are irrelevant to the (low energy) GUTs as mentioned before. It is very interesting if we can regard the yet hypothetical SGM (8) may be for the unified gauge models (SM and GUTs) what the BCS (electron-phonon) theory is for the Landau–Ginzburg theory of the superconductivity. The boundary condition (the global structure) of spacetime(universe) may be crucial.

Alternatively, disregarding the linearlization it is interesting from the purely phenomenological viewpoint to fit all the decay data of leptons and low lying hadrons in terms of the quark model [16] analogue $SO(10)$ superon current algebra including the higher order terms of (7), which potentially gives all the transition matrix elements in terms of superon pictures and may describe the nonlinear superon dynamics at the short ditance of the spacetime and may give a qualitative test of SQM [4]. Also it is worth studying other assignment for quarks and leptons than $R = L^*$ symmetric SQM ((5) and (6)). The left-right assymmetric assignment for quarks and leptons is also possible from only group theoretical investigations.

The cosmological implications of SGM (8) is also worth studying. Because SGM (8) describes a pre-history of quark-lepton era, i.e. N–G superons are created (i.e. pre-big bang is ignited) by the spontaneous breakdown of the supertranslation of spacetime and $SO(10)$ SP invariant massless superon composite states (quark-lepton era) are spanned, which lead to the big bang of the universe inducing the spontaneous breakedown of $SO(10)$ SUSY by the interactions among the massless composite states.

Finally we just mention that in SGM the singularities of the gravitational collapse may be prohibitted by the phase transition to the N-G pahase achieved *gravitationally.* It is a challenge to test these conjecture quantitatively by starting from the SGM action (8), where the higher order terms of κ and momentum (derivatives) become dominant.

5 Conclusion

We have shown by the qualitative arguments that the unified gauge models (SM and GUTs) are strengthened or revived by taking account of the topology of the superon diagram of SGM, while drawing the superon diagram (i.e. extracting the low energy physical implications) of SGM is guided by the Feynman diagram of SM (GUTs). We regard that these beautiful complimentality between the gauge unified models (SM and GUTs) and SGM may be an evidence of $SO(10)$ SP symmetric structure of spacetime and matter behind the gauge models, i.e. an evidence of the superon-quintet hypothesis for matter (SQM) and superon-graviton model (SGM) (8) for spacetime and matter. The experimental searches for a predicted new spin $\frac{3}{2}$ lepton-type (gravitino) doublet (ν_Γ, Γ^-) with the mass of the electroweak scale [3] and a new gauge boson S are important. Also SQM predicts two doubly-charged, electroweak- and color-singlet (unconfined) particles E^{2+} and M^{2+} with spin $\frac{1}{2}$ [3]. Their masses are left unknown within this study. From the present experimental data for τ^- decay S boson mass seems much larger than the W boson mass. The clear signals of (ν_Γ, Γ^-) may be similar to the top-quark pair production event without jets production, i.e. $e + \bar{e} \longrightarrow l + \bar{l}$ + *missing large* P_T *(energy)* [3]. The evidence of S boson may be seen already and will become clear in the high energy B meson experiment.

Besides those interesting aspects of SGM (8), much more open questions are left.

Acknowledgments

The author would like to express gratitude to J. Wess for his encouragement, interest in this work and the hospitality at University of München and Max–Planck–Institut für Physik und Astrophysik (Werner–Heisenberg–Institut). Also he would like to thank K. Mori, T. Shirafuji and Y. Tanii for useful discussions and the hospitality at Physics Department of Saitama University and K. Akama, T. Matsuda and T. Uematsu for useful discussions.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 465–472.

On the Global Conjugacy of Smooth Flows

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We aim to give useful integral criteria for smooth flows to be globally conjugate when their infinitesimal generators are close to each other. We study and use the Möller wave operator which is familiar in quantum mechanics. Theorems on smooth global straightening out of nowhere zero vector fields and on smooth global linearization are given. The stability of trajectories at perturbations and properties of the adjoint operators are studied.

1 Preliminaries

The problem of conjugacy of differentiable flows is equally the problem of equivalence of complete vector fields or, as well, the problem of equivalence of differentiable dynamical systems. A part of the latter is the question of normal forms for a system of autonomous differential equations at a singular point.

The so far results, started by Poincaré and Siegel in the real analytic case, to those by Sternberg in the C^∞ setting, have been concentrated on the linearization of dynamical systems. All they have local character.

We wish to present a global treatment of the problem. If one also insist to work in the C^∞ setting, the Fréchet space calculus and relevant techniques seem to be adequate. In the noncompact case we consider manifolds which are countable at infinity and are endowed with a Riemannian metric. In R^n, having fixed a globally Lipschitz vector field X, perturbations $X + Z$ are performed only by vector fields Z which are globally bounded together with all derivatives. Such vector fields are globally Lipschitz and consequently they are complete.

Let D be the induced Riemannian covariant derivative operator. $\|D^k X\|$ will mean the operator norm of the k-th covariant derivative of X as a multilinear map on $(TM)^k$ valued in the tangent bundle TM. When $M = R^n$ it denotes the usual operator norm of the k-th derivative of vector function X.

For a differentiable map f on M, Tf will mean the induced linear tangent map defined on TM. A diffeomorphism f of M induces the adjoint linear operator

$$f_* X = (Tf.X) \circ f^{-1}$$

on the Lie algebra $\mathcal{X}$ of all C^∞ vector fields on M.

Since the smoothness is a local property, and since the boundedness with respect to the time t and the convergence of the considered improper time-integrals will be required to be uniform in x only on compact subsets of M, we may perform calculations in R^n. Eventually, we may glue the limits over different local charts. Therefore we assume $M = R^n$ throughout the paper, except the following basic definition

Definition 1. *Let X be a smooth, globally Lipschitz vector field on a manifold M. Let E be a subspace of $\mathcal{X}(M)$ equiped with a nondecreasing countable system of supremum seminorms $\|\cdot\|_k$, $k \geq 0$, related to the Riemannian norm $\|\cdot\|$.*

We shall say that the adjoint flow $(\phi_t)_*$ decays *on* E *(to infinite order) in a set* $\Omega \subset M$*, if for every integer* $k \geq 0$ *there is* $l_k \geq k$ *and a continuous function* $\nu_k(t,x) > 0$ *defined on* $(0,\infty)\times\Omega$ *such that for any vector field* $Z \in E$ *it holds*

$$\|D^k \phi Z(x)\| \leq \nu_k(t,x)\|Z\|_{l_k} \tag{1}$$

and

$$\int_0^\infty \nu_k(t,x)dt \tag{2}$$

converges uniformly with respect to x *on compact subsets of* Ω.

Alternatively, $(\phi_t)_*$ *in (1) can be replaced by* $\phi_t^* := (\phi_{-t})_*$*, depending on the asymptotic behavior of the flow* ϕ_t.

In particular ϕ *decays exponentially if* $\nu_k(t,x) \leq e^{-c_k t} M_k(\|x\|)$ *for some* $c_k > 0$ *and a positive continuous function* M_k.

Example 1. Let $\Omega = R^n$ and $E = \{Z \in \mathcal{X}^\infty(R^n);\ Z \in \Im^n\}$, where $\Im$ is the Schwartz space of all functions on R^n which are fast falling together with all derivatives. The norms in $\Im^n$ are

$$\|Z\|_r = \max_{i+k\leq r} \sup_{x\in R^n} (1+\|x\|^2)^{i/2}\|D^k Z(x)\|.$$

where $\|\cdot\|$ is the Euclidean norm. Take $X = v$(const), $\|v\| = 1$. Then $\phi_t = \exp tv = id + tv$ and $\phi Z(x) = Z(x - tv)$. We have

$$\|D^k(\phi Z)(x)\| \leq \frac{1}{1+\|x-tv\|^2}\|Z\|_{k+2}. \tag{3}$$

Hence $l_k = k+2$ and

$$\nu_k(t,x) = \frac{1}{1+a(x)+(t-\langle x,v\rangle)^2} \qquad \text{for} \qquad a(x) = \|x\|^2 - \langle x,v\rangle^2 \geq 0, \tag{4}$$

where $\langle x,v\rangle$ means the scalar product. The convergence of (2) on compact sets is evident. Thus ϕ decays on E but not exponentially.

Example 2. Let $\Omega = R^n$ and let $C_b^\infty = C_b^\infty(R^n,R^n)$ be the space of vector fields in R^n with globally bounded derivatives

$$E = \{Z \in C_b^\infty(R^n,R^n), \qquad \|Z(x)\| = o(\|x\|) \text{ as } x \to 0\}.$$

We equip E with the standard operator norms in the spaces of bounded symmetric multilinear mappings.

Take $X(x) = -cx$ with $c > 0$. Then $\phi_t(x) = e^{-ct}x$, $\phi_t^* Z(x) = e^{ct}Z(e^{-ct}x)$ and

$$\|D^k\phi_t^* Z(x)\| = e^{(1-k)ct}\|(D^k Z)(e^{-ct}x)\| \leq e^{(1-k)ct}\|D^k Z\| \leq e^{(1-k)}\|Z\|_k$$

for $t > 0$, $x \in M$ and $k \geq 2$.

For $k = 0,1$ we have $\|\phi_t^* Z(x)\| \leq e^{-ct}\|x\|^2\|Z\|_2$ and $D\phi Z(x)\| \leq e^{-ct}\|x\|\|Z\|_2$. The integrals (2) are convergent for all $k \geq 0$ uniformly for x in any ball $\{\|x\| \leq r\}$.

2 Estimates of perturbations

The purpose of this section is to study the question of whether the property of decaying is preserved under small perturbations of X. We assume $M = R^n$. Let $Z \in E$ and let ϕ decay on Z in R^n. We consider the perturbed vector field $X + Z$. As both the X, Z are globally Lipschitz, the flow $\psi_t = \exp t(X + Z)$ is defined for all $t \in R$.

Lemma 1 (On orbital stability at perturbation). *Suppose that $\|Z\| < \infty$ and for some $L \geq 0$, $r > 0$ and all $x, y \in R^n$ such that $\|x - y\| \geq r$ we have*

$$\langle x - y, X(x) - X(y)\rangle \leq -L\|x - y\|^2 \tag{5}$$

If $L > 0$ then

$$\|\phi_t(x) - \psi_t(x)\| \leq r_1 = \max\{L^{-1}\|Z\|,\ r\} \tag{6}$$

for all $t \geq 0$ and $x \in R^n$.

If $X = v \neq 0$ is a constant vector field, and Z is fast falling to order 2 at infinity with small $\|Z\|$, then

$$\|\phi_t(x) - \psi_t(x)\| \leq C \qquad (t \geq 0) \tag{7}$$

on compact sets (where C is constant).

For $X = v$ and $Z \in C^1$ with $\|Z\| < \infty$ one gets the (sharp) estimate

$$\|\phi_t(x) - \psi_t(x)\| \leq \|Z\|t. \tag{8}$$

Proof. For the solutions of the Cauchy problems

$$x' = X(x), \qquad y' = X(y) + Z(y), \qquad x(0) = y(0)$$

the standard computation yields

$$\frac{1}{2}\left(\|x - y\|^2\right)' = \langle x - y, x' - y'\rangle = \langle x - y, X(x) - X(y)\rangle - \langle x - y, Z(y)\rangle,$$

hence

$$\left(\|x(t) - y(t)\|^2\right)' \leq 2(\|Z\| - L\|x(t) - y(t)\|)\|x(t) - y(t)\|.$$

For $L > 0$ this is possible only if $\|x(t) - y(t)\| \leq r_1$, which translates into (6).

Now assume $X(x) = v \neq 0$. Then it is easy to see that $\|x + tv - \psi_t(x)\| \leq \|Z\|t$. Thus for small $\|Z\|$ we have $\|\psi_t(x)\| \geq |\|x\| - bt|$ for some $b > 0$. But on the other hand we have for $X = v$

$$\psi_t(x) - (x + tv) = \int_o^t Z(\psi_s(x))ds \tag{9}$$

and (cf. Example 1)

$$\int_o^\infty \|Z(\psi_s(x))\|ds \leq \|Z\|_2 \int_o^\infty \frac{1}{1 + (\|x\| - bt)^2}dt \leq C < \infty$$

on compact sets. Therefore each trajectory $t \to \psi_t(x)$ has globally a finite distance from that of $\phi_t(x)$ uniformly in x on compact sets. In the last case the estimate (8) follows directly from (9). In particular, for $Z = w$ constant, (8) is equality.

Thus we see that the falling at infinity of Z is essential for the global proximity of the perturbation flow.

Remark. The assumption that both X and Z in Lemma 1 are globally Lipschitz can be relaxed. As we see from the proof, in all the cases considered in the Lemma, if X is complete then $\psi_t(x)$ can not be unbounded in finite time. Thus it can be defined over R^+ (i.e., the vector field $X+Z$ is positively semicomplete).

As a byproduct we obtain the following

Proposition 1. *If a vector field X in R^n satisfies condition (5) with $L > 0$, $X(x_o) \neq 0$, then it is positively semicomplete and for any fixed x_o and x in R^n and all $t \geq 0$*

$$\|\phi_t(x) - x_o\| \leq \max\{L^{-1}\|X(x_o)\|, r\}.$$

Moreover, if $X(x_o) = 0$ and $r = 0$ then

$$\|\phi_t(x) - x_o\| \leq \|x - x_o\| e^{-Lt}.$$

Proof. This time, for solutions of equation $x' = X(x)$, we have

$$\frac{1}{2}\left(\|x - x_o\|^2\right)' = \langle x - x_o, X(x) - X(x_o)\rangle + \langle x - x_o, X(x_o)\rangle.$$

Further arguments are analogous as in the proof of Lemma 1 or routine.

Definition 2. *We shall say that the adjoint flow ϕ C_m-decays on a subspace E if it decays on E and the functions $\nu_k(t,x)$ in (1) do not depend on x for $0 \leq k \leq m$, so that $\|D^k\phi Z(x)\| \leq \nu_k(t)\|Z\|_{l_k}$ for all x, and $\nu_k(t)$ is integrable over R^+.*

We say that ϕ decays C_∞ on E if it decays C_m for all integers $m \geq 0$.

This definition will apply also for the flow ϕ_t^* (generated by $-X$).

Lemma 2. *Suppose that ϕ generated by X decays on E and one of the following conditions is satisfied.*

(A.1) The vector field X fullfils the hypothesis (5) with $L > 0$, or

(A.2) $X = v$ (const), or

(A.3) ϕ C_o-decays on E.

Then also the adjoint flow ∂, where $\psi_t = \exp t(X+Z)$, decays on E provided Z is sufficiently small in the seminorm $\|\cdot\|_{l_1}$. Moreover, if X fulfills (A.1) then so does $X+Z$, and if ϕ C_1-decays on E then ∂ decays C_o on E.

A corresponding result is true for ϕ_t^*.

Proof. Put $f_t = \phi_t \circ \psi_{-t}$. Since $T\phi_t.X = X \circ \phi_t$, we get by differentiating in t

$$f_t' = -(T\phi_t.Z) \circ \psi_{-t} = -(\phi Z) \circ f_t, \qquad f_o = id, \tag{10}$$

we can integrate (10) to obtain

$$f_t = id - \int_o^t (\phi_s) * Z \circ f_s ds. \tag{11}$$

Now we aim to show that on compact subsets $f_t - id$ and all its derivatives $T^n(f_t - id)$ are bounded uniformly in $t \in (0,\infty)$.

First, suppose that X satisfies (A.1) or (A.2). By substituting $\psi_t^{-1}(x)$ in place of (x) in (6) or (7) we get $\|f_t(x) - x\| \leq C$ for all $t \geq 0$ uniformly for x in compact sets.

Similar result can be obtained when the condition (A.3) is satisfied. In fact, consider equation $x' = F(t,x)$, where $F(t,x) = -\phi Z(x)$. Putting $u = \|x\|$ we have

$$u\frac{du}{dt} = x^T F(t,x) \leq \|x\|\|\phi Z(x)\| \leq \nu_0(t)u\|Z\|_{l_0}.$$

Hence

$$u(t) - u(0) \le \left(\int_0^t \nu(s) ds \right) \|Z\|_{l_0},$$

where the integral is bounded as $t \to \infty$. Therefore $u(t) - u(0)$ remains bounded over R^+. For $x(t) = f_t$ it gives that $\|f_t(x)\| - \|x\|$ is bounded for $t \in R^+$. By the principle of the integral continuity of solutions it also remains bounded when x runs over a compact set.

Next we are going to prove the boundedness of the first derivative Tf_t. For this we differentiate (11) and take estimates

$$\|Tf_t(x)\| \le 1 + \int_o^t \|D(\phi_s)_* Z(f_s(x))\| \|Tf_s(x)\| ds \le \int_o^t \nu_1(s, f_s(x)) \|Z\|_{l_1} \|Tf_s(x)\| ds.$$

Using the Bellman's lemma, we get

$$\|Tf_t(x)\| \le \exp\left(\int_o^t \nu_1(s, f_s(x)) \|Z\|_{l_1} ds \right) = e^{B\|Z\|_{l_1}} < \infty,$$

where we put $B = \int_o^\infty \nu_1(s, f_s(x)) ds$. The integral is convergent since $\|f_s(x)\|$ differs from $\|x\|$ by a constant, so they may be placed in the same compact set for all $s \ge 0$.

In particular, if ϕ decays C_1 on E then ν_1 does not depend on x and then $Tf_t(x)$ is bounded globally for all $t \ge 0$ and $x \in M$.

Now, assuming that $T^{n-1}f_t$ is bounded for $n-1 \ge 1$, we wish to show this for $T^n f_t$. We make use of the standard formulae for higher order derivatives of composition maps. We have

$$D^n(\phi Z \circ f_t)(x)) = (D\phi Z)(f_t(x)).T^n f_t(x) + R_n(t, x), \tag{12}$$

where

$$R_n(t, x) = \sum_{k=2}^n \sum_{j_1 + \cdots + j_k = n} C_{k, j_1, \ldots, j_k} D^k(\phi Z)(f_t(x)) \{T^{j_1} f_t(x), \ldots, T^{j_k} f_t(x)\}$$

with $j_1, \ldots, j_k \ge 1$. Passing to the estimates we have

$$\|D^n(\phi Z(f_t(x))\| \le \|D\phi Z\| \|T^n f_t(x)\| + \|R_n(t, x)\|,$$

where

$$\|R_n(t, x)\| \le C \sum_{k=2}^n \sum_{j_1 + \cdots + j_k = n} \nu_k(t, f_t(x)) \|Z\|_{l_k} \|T^{j_1} f_t(x)\| \cdots \|T^{j_k} f_t(x)\|$$

with $1 \le j_1, \ldots, j_k \le n-1$. All this and (11) yields for the derivatives of order $n \ge 2$

$$\|T^n f_t\| \le \int_o^t \|R_n(s)\| ds + \|Z\|_{l_1} \int_o^t \nu_1(s, f_s(x)) \|T^n f_t\| ds.$$

Again by Bellman's inequality

$$\|T^n f_t\| \le \left(\int_o^t \|R_n(s, x)\| ds \right) \cdot \exp\left(\|Z\|_{l_1} \int_o^t \nu_1(s, f_s(x)) ds \right) < \infty$$

uniformly for $t \ge 0$. Thus $\|T^n f_t(x)\|_\infty$ is finite for $n \ge 1$.

Now, by the definition of f_t we have $\psi_t = f_t^{-1} \circ \phi_t$. Put $g_t = f_t^{-1}$. Then clearly $\|g_t(x)\| - \|x\|$ is also uniformly bounded in $t \ge 0$. We wish to prove that $g_t - id$ has all x-derivatives unformly bounded in t.

From (11) it follows

$$\|Tf_t - I\| \le \int_o^t \|D(\phi_s)_* Z\| \|Tf_s\| ds. \tag{13}$$

Put $\delta = \|Z\|_{l_1}$. Then, by (12), $\|Tf_t\| \le e^{B\delta}$ for $t > 0$. Now the estimate (13) can be written

$$\|Tf_t - I\| \le B\delta e^{B\delta}.$$

For δ sufficiently small, and B being independent of Z, we shall have $\|Tf_t - I\| < \epsilon < 1$. But if $\|Tf_t(x) - I\| < \epsilon$ then

$$\|Tg_t(f_t(x))\| = \|Tf_t(x)^{-1}\| \le \frac{1}{1-\epsilon} < \infty$$

for all $t > 0$, uniformly in x in any compact set.

Now, since $f_t \circ g_t = id$, the uniform boundedness of higher order derivatives of g_t follows recurently from the relations

$$T^n g_t(x) = -Tg_t(x) \sum_{k=2}^{n} \sum_{j_1+\cdots+j_k=n} C_{k,j_1,\ldots,j_k} T^k f_t(g_t(x)) \left\{T^{j_1} g_t(x), \ldots, T^{j_k} g_t(x)\right\},$$

where $j_1, \ldots, j_k \le n-1$, and hence the estimate

$$\|T^n g_t\| \le C \sum_{k=2}^{n} \sum_{j_1+\cdots+j_k=n} \|T^k f_t\| \|T^{j_1} g_t\| \cdots \|T^{j_k} g_t\| < \infty.$$

Having $g_t - id$ uniformly bounded in t to infinite order, we deduce easily that for any smooth vector field Y with bounded derivatives it holds

$$\|D^n (g_t)_* Y(x)\| \le C \sum_{k \le n} \|D^k Y\| \qquad (n \ge 0)$$

for some constant $C > 0$ depending on n and independent of x in compact sets.

From $\psi_t = g_t \circ \phi_t$ it follows $\partial Z = (g_t)_*(\phi Z)$. In the above inequality we replace Y by ϕZ, which decays on Z. Consequently, $(\psi_t)_*$ decays on Z.

Finally, suppose that X satifies condition (5) with $L > 0$. Then for $X + Z$ we have whenever $\|x - y\| > r$

$$\begin{aligned}&\langle x - y, (X+Z)(x) - (X+Z)(y)\rangle \\ &\qquad \le -L\|x-y\|^2 + \|x-y\| \|Z(x) - Z(y)\| \le (-L+K)\|x-y\|^2,\end{aligned}$$

where K is the global Lipschitz constant of Z. If $\|Z\|_{l_1}$ is small then so is K and $-L + K < 0$, as required.

In the case where ϕ decays C_1 on E, then $Tf_t(x)$ and hence also $Tg_t(x)$ are bounded globally in t and x. Therefore we have

$$\|\partial Z(x)\| = \|(g_t)_*(\phi_t)_* Z(x)\| \le C\|(\phi_t)_* Z\| \le C\nu_o(t)\|Z\|_{l_1}$$

for all x. This completes the proof of the Lemma.

3 Conjugacy of flows

Lemma 3. *Let X, Z be C^k complete vector fields on a smooth manifold M. Suppose that the integrals*

$$f = id - \int_0^\infty (T\exp tX).Z \circ \exp(-t(X+Z))dt \tag{14}$$

converges to class C^k $(k \ge 1)$ and

$$g = id - \int_0^\infty (T\exp t(X+Z)).Z \circ \exp(-tX)dt \tag{15}$$

converges to class C^1, both uniformly on compact subsets of M.

Then f and g are C^k diffeomorphisms of M, $g = f^{-1}$, and

$$f_*(X + Z) = X.$$

Note that the assumptions are satisfied if ϕ_t fulfills the hypothesis of Lemma 2 and Z is sufficiently small.

Proof. In the proof we use the Möller wave operator [1] which is known in quantum mechanics. Put as before $\phi_t = \exp tX$ and $\psi_t = \exp t(X + Z)$. The idea is that if the diffeomorphisms $\phi_t \circ \psi_{-t}$ have the limit

$$\lim_{t\to\infty} \phi_t \circ \psi_{-t} = f \qquad \text{(wave operator)} \tag{16}$$

and f is invertible then $f^{-1} \circ \phi_t \circ f = \psi_t$. This is so because from (16) it follows $\phi_t \circ f \circ \psi_{-t} = f$.

We introduce the integral formulae for the wave operator in order to simplify the proof of its existence. For this again define

$$f_t = \phi_t \circ \psi_{-t} \qquad \text{and} \qquad g_t = \psi_t \circ \phi_{-t} = f_t^{-1}. \tag{17}$$

Hence

$$f_t' = -(T\phi_t.Z) \circ \psi_{-t}, \qquad g_t' = -(T\psi_t.Z) \circ \phi_{-t}. \tag{18}$$

The existence of both the limits $f = \lim_{t\to\infty} f_t$ and $g = \lim_{t\to\infty} g_t$ ensures the invertibility of f. Thus, we can integrate (18) in the interval $[0, t]$ and pass to limit as $t \to \infty$. It results in $\psi_t = f^{-1} \circ \phi_t \circ f$, so the flows are conjugate by f. From this, by differentiating in t, we get $(f^{-1})_*X = X + Z$ or equivalently $f_*(X + Z) = X$. It is well known that if f is C^k and has a C^1 inverse, then its inverse is C^k. So, g is also C^k.

Remark. Alternatevaly, by reversing time, we may look for the wave operator of the form $f = \lim_{t\to\infty} \phi_{-t}\psi_t$ which satisfies $\phi_{-t} \circ f \circ \psi_t = f$. It can be calculated from the integral formulae

$$f = id + \int_o^\infty (T\exp(-tX)).Z \circ \exp t(X + Z)dt,$$

$$g = id + \int_o^\infty (T\exp -t(X + Z)).Z \circ \exp tX dt.$$

If the integrals converge then $g = f^{-1}$ and $f_*(X + Z) = X$. This version may be used if the asymptotic behavior of $\exp(-tX)$ is more suitable than that of $\exp(tX)$.

Definition 3. *Let the subspace $E \subset C_b^\infty(R^n, R^n)$ be closed in the standard supremum norms $\|\cdot\|_k$ ($k \geq 0$). Let X be a globally Lipschitz vector field such that the X-flow ϕ_t leaves E invariant, i.e., $(\phi_t)_*E \subset E$ for any $t \in R$. We say that E has a hyperbolic structure for ϕ_t if there is a continuous splitting $E = E_1 + E_2$, such that E_2 is $\exp(X + E_1)$ invariant, $(\phi_t)_*$ fulfills the hypothesis (A.1) or (A.3) of Lemma 2 on E_1 and so does ϕ_t^* on E_2.*

Note that we do not assume any invariance of subspaces E_i individually.

Lemma 4. *Suppose that E has a hyperbolic structure for the X-flow. Let $Z = Z_1 + Z_2$, where Z_i are sufficiently small. Then $X + Z$ is C^∞ conjugate to X.*

Proof. By Lemma 3 for X and Z_1, there is a diffeomorphism f such that $f_*X = X + Z_1$. Since ϕ_t^* fulfills (A.1) or (A.3) on E_2, for small Z_1 also $(\exp t(X + Z_1))^*$ fulfills (A.1) or (A.3) on E_2. Therefore, for small Z_2 there exists a diffeomorphism h such that $h_*(X + Z_1) = (X + Z_1) + Z_2 = X + Z$. This completes the proof.

4 Main results

With the notations of preceding sections the integrals (14) and (15) can be written in the form

$$f = id - \int_0^\infty \phi Z \circ f_t dt \qquad \text{and} \qquad g = id - \int_0^\infty (\psi_t)_* Z \circ g_t dt$$

with $f_t = \phi_t \circ \psi_{-t}$ and $g_t = f_t^{-1}$.

Accordingly, we express the alternative formulae (14) and (15) by putting $-t$ in place of t.

Now we can apply the results of previous sections and Examples 1, 2 to prove the convergence of the integrals and of all their x-derivatives. This will result in the following theorems.

Theorem 1 (On conjugacy of perturbations). *Let $X \in \mathcal{X}(R^n)$ be a globally Lipschitz vector field and let E be a linear space of vector fields on R^n. Suppose that the adjoint flow generated by X decays on E with respect to a collection of seminorms $\{\|\cdot\|_k, k \in N\}$. Assume also that one of the conditions* (A.1) *to* (A.3) *is satisfied.*

Then there is a neighborhood $U = \{Z \in E;\ \|Z\|_{l_1} < \delta\}$ such that for every $Z \in U$ there exists a C^∞ diffeomorphism f of M which conjugate X to $X + Z$, that is $f_*X = X + Z$.

Theorem 2 (Global straightening out theorem). *Let X be a non-zero constant vector field on R^n. There is a $\delta > 0$ such that for every fast falling vector field Z on R^n with $\|Z\|_3 < \delta$ the vector fields X and $X + Z$ are C^∞ conjugate on R^n.*

Thus, any sufficiently small perturbation (as above) of $X = \frac{\partial}{\partial x_1}$ can be transformed globally to $\frac{\partial}{\partial x_1}$ by a C^∞ change of coordinates.

Theorem 3. *Let $X(x) = -cx$, $c > 0$, $x \in R^n$, and let Z be a vector field with globally bounded derivatives and satisfying $\|Z(x)\| = o(\|x\|)$ as $x \to 0$. Then the perturbed vector field $X + Z$ is C^∞ conjugate to X in R^n.*

This theorem can be easily generalized to the case where $X = Ax$ with negative real parts of the eigenvalues of the matrix A and without the familiar resonance relations. Thus the Sternberg [2] local linearization theorem for contractions can be given a global version.

Moreover, applying Lemma 4, we may also obtain the globalization of the Sternberg's linearization theorem for arbitrary hyperbolic point with no resonance. This will be subject to another article.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 473–480.

Classical Mechanics of Relativistic Particle with Colour

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Classical description of relativistic pointlike particle with intrinsic degrees of freedom such as isospin or colour is proposed. It is based on the Lagrangian of general form defined on the tangent bundle over a principal fibre bundle. It is shown that the dynamics splits into the external dynamics which describes the interaction of particle with gauge field in terms of Wong equations, and the internal dynamics which results in a spatial motion of particle via integrals of motion only. A relevant Hamiltonian description is built too.

1 Introduction

Wong equations of motion of classical pointlike relativistic particle with isospin or colour [1] permit various Lagrangian and Hamiltonian formulations. Two of them [2, 3] known to the author are based on geometric notions brought from gauge field theories. Namely, the configuration space of particle is a principal fibre bundle with the structure group being the gauge group; the interaction of particle with an external gauge field is introduced via the connection on this bundle. The only difference between two approaches consists in the choice of Lagrangian function. That proposed in [2] is linear in gauge potentials (as in classical electrodynamics) while the nonlinear Lagrangian [3, 2] arises naturally within the Kaluza–Klein theory. Nevertheless, both of them lead to the same Wong equations.

In the present paper, starting from the mentioned above geometrical treatment of the kinematics of relativistic particle with isospin or colour (Section 2), we construct in Section 3 the Lagrangian of a fairy general form. In particular cases this Lagrangian reduces to those of Refs. [2, 3]. The generalization is not trivial since it leads to two types of dynamics, according to what variables are used for the description of intrinsic degrees of freedom. The external dynamics is described by the Wong equations which include gauge potentials, but the form of which is indifferent to the choice of Lagrangian. On the contrary, the internal dynamics is governed by the particular choice of Lagrangian function while gauge potentials completely fall out of this dynamics. This fact becomes more transparent within the framework of the appropriate Hamiltonian formalism developed in Section 4. In Section 5 we sum up our results and discuss the perspectives of quantization.

2 Kinematics on a principal fibre bundle

Following Refs. [2, 3] we take for a configuration space of relativistic particle with isospin or colour the principal fibre bundle P over Minkowski space $\mathbb{M}$ with structure group G and projection $\pi : P \to \mathbb{M}$ (see Ref. [4] for these notions). A particle trajectory $\gamma : \mathbb{R} \to P$; $\tau \mapsto p(\tau) \in P$ is parameterized by evolution parameter τ. A state of particle is determined by $(p, \dot{p}) \in \mathrm{T}P$.

In this paper we are not interested in the global structure of P, and use its local coordinatization: $P \ni p = (x, g) = (x^\mu, g^i)$, $\mu = \overline{0,3}$, $i = \overline{1, \dim G}$, where $x = \pi(p) \in U \subset \mathbb{M}$ (U is an open subset of $\mathbb{M}$), $g = \varphi(p) \in G$, and $\varphi : P \to G$ defines the choice of gauge. Respectively, $\dot p = (\dot x, \dot g)$, where $\dot x \in \mathrm{T}_x\mathbb{M}$ and $\dot g \in \mathrm{T}_g G$. We call $(x, \dot x)$ and $(g, \dot g)$ the space and the intrinsic (local) variables of particle respectively.

Let the principal fibre bundle P be endowed with a connection defined by 1-form $\boldsymbol{\omega}$ on P which takes values in Lie algebra $\mathcal{G}$ of G. Locally it can be represented as follows [4]:

$$\boldsymbol{\omega} = \mathrm{Ad}_{g^{-1}}\pi^*(\mathbf{A}_\mu(x)dx^\mu) + g^{-1}dg \equiv g^{-1}\left(\pi^*(\mathbf{A}_\mu(x)dx^\mu)\right) g + g^{-1}dg, \tag{1}$$

where π^* is the pull back mapping onto P, Ad denotes the adjoint representation of G in $\mathcal{G}$, and $\mathcal{G}$-valued functions $\mathbf{A}_\mu(x)$ are gauge potentials. Under a right action of G defined in P by

$$R_h : p \mapsto p' \equiv R_h(x, g) = (x, gh), \qquad h \in G, \tag{2}$$

the connection form transforms via a pull back mapping R_h^* and is equivariant, i.e., $R_h^*\boldsymbol{\omega} = \mathrm{Ad}_{h^{-1}}\boldsymbol{\omega}$, $h \in G$.

A gauge transformation arises in a geometrical treatment as a bundle automorphism defined by

$$\Phi_{h(x)} : p \mapsto p' \equiv \Phi_{h(x)}(x, g) = (x, h(x)g), \qquad h(x) \in G. \tag{3}$$

It induces the transformation of the connection form defined by the inverse of the pull back mapping (actually, a push forward mapping), $\boldsymbol{\omega} \to \boldsymbol{\omega}' = [\Phi_{h^{-1}(x)}]^*\boldsymbol{\omega}$, so that the value of the connection form on each vector field is gauge-invariant by definition. The transformed form $\boldsymbol{\omega}'$ is also expressed by eqs.(1), but with new potentials

$$\mathbf{A}'_\mu(x) = h(x)\mathbf{A}_\mu(x)h^{-1}(x) + h(x)\partial_\mu h^{-1}(x). \tag{4}$$

The Minkowski metrics $\eta \equiv \eta_{\mu\nu}dx^\mu \otimes dx^\nu$, $\|\eta_{\mu\nu}\| = \mathrm{diag}\,(+,-,-,-)$ is defined on base space $\mathbb{M}$. It is invariant under the Poincaré group acting in $\mathbb{M}$. Being pulled back by π^* onto P it becomes also right- and gauge-invariant, but appears degenerate.

Here we suppose that the Lie algebra $\mathcal{G}$ of structure group G is endowed with non-degenerate Ad-invariant metrics $\langle\cdot,\cdot\rangle$. The example is the Killing–Cartan metrics in the case of semi-simple group. In terms of this metrics, the connection form, and the Minkowski metrics one can construct a nondegenerate metrics on the bundle P [3],

$$\Xi = \pi^*\eta - a^2\langle\boldsymbol{\omega}, \boldsymbol{\omega}\rangle \tag{5}$$

(a is a constant), which is right- and gauge-invariant but not Poincaré-invariant (the latter is broken by $\boldsymbol{\omega}$). In the case of bundle over a curved base space the Minkowski metrics on the right-hand side (r.h.s.) of eq.(5) is replaced by the Riemanian one. In this form the metrics Ξ arises in the Kaluza–Klein theory [5] which allows to unify the description of gravitational and Yang–Mills fields.

3 Lagrangian dynamics of particle with isospin or colour

The dynamical description of the relativistic particle with isospin or colour should, at least, satisfy the following conditions:

i) gauge invariance;

ii) invariance under an arbitrary change of evolution parameter;

iii) Poincaré invariance provided gauge potentials vanish.

These requrements can be embodied in the action $I = \int d\tau L(p, \dot{p})$ with the following Lagrangian

$$L = |\dot{x}| F(\mathbf{w}), \tag{6}$$

where $\mathbf{w} \equiv \boldsymbol{\omega}(\dot{p})/|\dot{x}|$, $|\dot{x}| \equiv \sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = \sqrt{\pi^*\eta(\dot{p}, \dot{p})}$, and $F : \mathcal{G} \to \mathbb{R}$ is an arbitrary function. We note that the quantities $|\dot{x}|$, $\boldsymbol{\omega}(\dot{p})$, and thus the variable $\mathbf{w}$ and the Lagrangian (6) are gauge-invariant.

In order to calculate the variation δI of the action I it is convenient to use, instead of intrinsic velocity $\dot{g}$ and variation δg, the following variables: $\mathbf{v} \equiv \dot{g}\, g^{-1}$ and $\boldsymbol{\delta} g \equiv \delta g\, g^{-1}$. They take values in Lie algebra $\mathcal{G}$ of group G. Then the argument $\mathbf{w}$ of F can be presented in the form

$$\mathbf{w} = \mathrm{Ad}_{g^{-1}}(\mathbf{v} + \mathbf{A}_\mu \dot{x}^\mu)/|\dot{x}|. \tag{7}$$

Using the formal technique (see Ref. [6] for rigorous substantiation)

$$\begin{aligned}
&\delta g^{-1} = -g^{-1}\delta g\, g^{-1}, \\
&\delta \mathbf{v} = \delta(\dot{g}\, g^{-1}) = \delta\dot{g}\, g^{-1} + \dot{g}\delta g^{-1} = \frac{d}{d\tau}\boldsymbol{\delta} g - [\mathbf{v}, \boldsymbol{\delta} g], \\
&\delta(\mathrm{Ad}_{g^{-1}} \mathbf{V}) = \delta(g^{-1}\mathbf{V}\, g) = \mathrm{Ad}_{g^{-1}}(\delta\mathbf{V} + [\mathbf{V}, \boldsymbol{\delta} g])
\end{aligned} \tag{8}$$

etc., where $[\cdot,\cdot]$ are Lie brackets in $\mathcal{G}$ and $\mathbf{V}$ is an arbitrary $\mathcal{G}$-valued quantity, we obtain the following Euler–Lagrange equations:

$$\dot{p}_\mu = \mathsf{q} \cdot (\partial_\mu \mathbf{A}_\nu)\dot{x}^\nu, \tag{9}$$

$$\dot{\mathsf{q}} \;=\; \mathrm{ad}^*_{\mathbf{A}\cdot\dot{x}}\, \mathsf{q} \tag{10}$$

with

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} = \left(F - \frac{\partial F}{\partial \mathbf{w}} \cdot \mathbf{w}\right)\frac{\dot{x}_\mu}{|\dot{x}|} + \mathsf{q} \cdot \mathbf{A}_\mu, \tag{11}$$

$$\mathsf{q} \equiv \frac{\partial L}{\partial \mathbf{v}} = \mathrm{Ad}^*_{g^{-1}} \frac{\partial F}{\partial \mathbf{w}}, \tag{12}$$

where p_μ are spatial momentum variables, and q is an intrinsic momentum-type variable which takes values in co-algebra $\mathcal{G}^*$. The linear operators ad^* and Ad^* define co-adjoint representations of $\mathcal{G}$ and G respectively, dot " $\cdot$ " denotes a contraction.

First of all we show that the function

$$M(\mathbf{w}) \equiv F - \frac{\partial F}{\partial \mathbf{w}} \cdot \mathbf{w} \tag{13}$$

is an integral of motion. For this purpose let us introduce the following $\mathcal{G}^*$-valued variable:

$$\mathsf{s} \equiv \frac{\partial F}{\partial \mathbf{w}} = \mathrm{Ad}^*_g\, \mathsf{q}. \tag{14}$$

In contrast to q, it is gauge-invariant. Taking into account the equations (14) and (10) we obtain after a bit calculation the equation:

$$\dot{\mathsf{s}} \;=\; \mathrm{ad}^*_{|\dot{x}|\mathbf{w}}\, \mathsf{s}. \tag{15}$$

Then $\dot{M} = -\dot{\mathsf{s}} \cdot \mathbf{w} = -\mathsf{s} \cdot [\mathbf{w}, \mathbf{w}]|\dot{x}| \equiv 0$ q.e.d.

Using this fact and (10) in (9) yields the equations of spatial motion

$$M\frac{d}{d\tau}\frac{\dot{x}_\mu}{|\dot{x}|} = \mathsf{q}\cdot\mathbf{F}_{\mu\nu}\dot{x}^\nu, \tag{16}$$

where $\mathbf{F}_{\mu\nu} \equiv \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu]$. Equations (16) together with the equation (10) or (15) of intrinsic motion determine the particle dynamics on a principal fibre bundle.

Now we suppose the existence in Lie algebra $\mathcal{G}$ of non-degenerate metrics $\langle\cdot,\cdot\rangle$. It allows to identify $\mathcal{G}^*$ and $\mathcal{G}$. In particular, for co-vector $\mathsf{q} \in \mathcal{G}^*$ we introduce the corresponding vector $\mathbf{q} \in \mathcal{G}$, such that $\mathsf{q} = \langle\mathbf{q},\cdot\rangle$. Then the equations of motion (16) and (10) take the form

$$M\frac{d}{d\tau}\frac{\dot{x}_\mu}{|\dot{x}|} = \langle\mathbf{q}, \mathbf{F}_{\mu\nu}\rangle\dot{x}^\nu, \tag{17}$$

$$\dot{\mathbf{q}} \ = \ [\mathbf{q}, \mathbf{A}_\mu]\dot{x}^\mu. \tag{18}$$

Besides, if this metrics is Ad-invariant, the quantity $\langle\mathbf{q},\mathbf{q}\rangle$ is an integral of motion.

At this stage we have obtained the well-known Wong equations (17)–(18) which describe the dynamics of a relativistic particle with mass M and isospin or colour $\mathbf{q}$. Despite that we started with the Lagrangian (6) of a fairy general form, this arbitrariness is obscured in the Wong equations. The reason resides in definitions of M and $\mathbf{q}$ which are, in general, complicated functions on TP. This feature is better understood by analyzing equations (16) and (10) which are very similar to the Wong equations but do not involve the metrics in $\mathcal{G}$.

In general, the set of eqs.(16) and (10) is of the second order with respect to configuration variables x and g. In this regard it is quite equivalent to the set of eqs.(16) and (15). On the other hand, the equations (16) and (10) form a self-contained set in terms of variables x and q. They involve explicitly potentials of external gauge field, but their form is indifferent to a choice of Lagrangian.

The equation (15) is the closed first-order equation with respect to $\mathbf{w}$ or, if eq.(14) is invertible, with respect to s (the quantity $|\dot{x}|$ is not essential because of a parametric invariance of dynamics; we can put, for instance, $|\dot{x}| = 1$). In contrast to the set (16), (10), the equation (15) is determined by a choice of the Lagrangian, but, in terms of $\mathbf{w}$ or s, it does not include gauge potentials.

Hence, the dynamics of isospin particle splits into the *external dynamics* described by the equations (16), (10) in terms of variables x and q, and the *internal dynamics* determined by the equation (15) in terms of $\mathbf{w}$ or s. The only coupling of these realizations of dynamics is provided via integrals of motion, namely, the particle mass M and (if Ad-invariant metrics is involved) the isospin module $|\mathbf{q}| \equiv \sqrt{\langle\mathbf{q},\mathbf{q}\rangle} = |\mathbf{s}|$.

In the following examples we show that the general description of isospin particle includes, as particular cases, results known in the literature. Besides, we demonstrate some new features concerning the internal dynamics.

1. Linear Lagrangian. Electrodynamics. The simplest choice of Lagrangian (6) leading to non-trivial intrinsic dynamics corresponds to the following function $F(\mathbf{w})$:

$$F(\mathbf{w}) = m + \mathsf{k}\cdot\mathbf{w}, \tag{19}$$

where $m \in \mathbb{R}$ and $\mathsf{k} \in \mathcal{G}^*$ are constants. Up to notation it coincides with the Lagrangian proposed by Balachandran et al. in Ref. [2]. In this case the isospin $\mathsf{q} = \mathrm{Ad}^*_{g^{-1}}\mathsf{k}$ is purely configuration variable (it does not depend on velocities) and the equation (10) is truly the first-order Euler–Lagrange equation. Besides, M and s are constants, i.e., $M = m$, $\mathsf{s} = \mathsf{k}$, thus the internal dynamics completely degenerates.

The Lagrangian (6), (19) is linear with respect to gauge potentials $\mathbf{A}_\mu$. In the case of one-parametric gauge group $U(1)$ it reduces to that of electrodynamics. Indeed, in this case we have $g = \exp(\mathrm{i}\theta)$, $\mathbf{v} = \mathrm{i}\dot{\theta}$, $\mathbf{A}_\mu = \mathrm{i}A_\mu$. Choosing $\mathsf{k} = -\mathrm{i}e$, where e is the charge of electron, one can present the Lagrangian in the form:

$$L = m|\dot{x}| + eA_\mu\dot{x}^\mu + e\dot{\theta}. \tag{20}$$

The third term on r.h.s. of (20) is a total derivative and thus it can be omitted. Hence, intrinsic variables disappear in this Lagrangian, and the latter takes the standard form.

The similar situation occurs when considering an arbitrary Abelian gauge group.

2. Right-invariant Lagrangian. Kaluza–Klein theory. The following choice of the function $F(\mathbf{w})$:

$$F(\mathbf{w}) = f(|\mathbf{w}|), \qquad |\mathbf{w}| \equiv \sqrt{\langle \mathbf{w}, \mathbf{w}\rangle} = |\mathbf{v} + \mathbf{A}_\mu\dot{x}^\mu|/|\dot{x}|, \tag{21}$$

where $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function of $|\mathbf{w}|$, corresponds to Lagrangian which is invariant under the right action of G. Following the Noether theorem there exist corresponding integrals of motion. In the present case they form $\mathcal{G}^*$-valued co-vector s defined by eq.(14) or, equivalently, $\mathcal{G}$-valued vector

$$\mathbf{s} = |\mathbf{w}|^{-1}f'(|\mathbf{w}|)\mathbf{w}, \tag{22}$$

where $f'(|\mathbf{w}|) \equiv df/d|\mathbf{w}|$. We note that the integrals of motion $\mathbf{s}$ and $M(\mathbf{w})$ are not independent. Indeed, if $f'(|\mathbf{w}|)$ is not constant, the mass M can be presented as a function of $|\mathbf{s}| = |\mathbf{q}|$. Otherwise both these quantities are constants. Thus the mass M and the isospin module $|\mathbf{q}|$ are completely determined by the external dynamics.

The right-invariant Lagrangian of special kind arises naturally in the framework of Kaluza–Klein theory [5]. It has the following form [3, 2]:

$$L = m\sqrt{\Xi(\dot{p}, \dot{p})}, \tag{23}$$

where the metrics Ξ on a principle bundle P is introduced by eq.(5). In our notations this Lagrangian corresponds to the choice:

$$f(|\mathbf{w}|) = m\sqrt{1 - a^2|\mathbf{w}|^2}. \tag{24}$$

This function determines the following relation between M and $|\mathbf{q}|$:

$$M(|\mathbf{q}|) = \sqrt{m^2 + |\mathbf{q}|^2/a^2}. \tag{25}$$

3. Isospin top. In the above two examples the internal dynamics does not affect the external dynamics. Here we consider a contrary example. Let

$$F(\mathbf{w}) = f(\nu), \qquad \nu \equiv \sqrt{\langle \mathbf{w}, T\mathbf{w}\rangle}, \tag{26}$$

where T is a self-adjoint (in the metrics $\langle\cdot,\cdot\rangle$) linear operator. In this case we have

$$\mathbf{s} = \nu^{-1}f'(\nu)T\mathbf{w}, \qquad M = f(\nu) - \nu f'(\nu). \tag{27}$$

If the function $f(\nu)$ is not linear, the quantity ν turns out to be an integral of motion which is independent of $|\mathbf{q}|$. Then using the parameterization $|\dot{x}| = 1$ one can reduce eq.(15) to the following equation of internal motion:

$$T\dot{\mathbf{w}} = [T\mathbf{w}, \mathbf{w}]. \tag{28}$$

This is nothing but the compact form of Euler equations (i.e., the equations of motion of a free top) generalized to the case of arbitrary group [7]. A solution of this equation is necessary for evaluation of the observable mass of particle.

The relation between the external dynamics and the internal one becomes more transparent within the Hamiltonian formalism which we consider in the next section.

4 Transition to Hamiltonian description

The Lagrangian description on the configuration space P enables a natural transition to the Hamiltonian formalism with constraints [8] on the cotangent bundle T^*P over P. Locally, $\mathrm{T}^*P \simeq \mathrm{T}^*U \times \mathrm{T}^*G$ and, in turn, $\mathrm{T}^*G \simeq G \times \mathcal{G}^*$. The latter isomorphism is established by right or left action of group G on T^*G (see, for instance, [9]). It is implicitly meant in our notation. Namely, we coordinatize T^*G by variables (g, q) or (g, s).

Let us introduce basis vectors $\mathbf{e}_i \in \mathcal{G}$ satisfying the Lie-bracket relations $[\mathbf{e}_i, \mathbf{e}_j] = c_{ij}^k \mathbf{e}_k$, where c_{ij}^k are the structure constant of G, and basis co-vectors $\mathsf{e}^i \in \mathcal{G}^*$ such that $\mathsf{e}^j \cdot \mathbf{e}_i = \delta_i^j$. Then the standard symplectic structure on the cotangent bundle T^*P over the manifold P can be expressed in terms of local coordinates x^μ, p_ν, g^i and $q_j \equiv \mathsf{q} \cdot \mathbf{e}_j$ by the following Poisson-bracket relations:

$$\{x^\mu, p_\nu\} = \delta_\nu^\mu, \qquad \{q_i, q_j\} = c_{ij}^k q_k, \qquad \{g^i, q_j\} = \zeta_j^i(g) \tag{29}$$

(other brackets are equal to zero), where $\zeta_i^j(g)$ are components of right-invariant vector fields on G. Equivalently, we can use variables $s_j \equiv \mathsf{s} \cdot \mathbf{e}_j$ instead of q_j. Then the Poisson-bracket relations take the form:

$$\{x^\mu, p_\nu\} = \delta_\nu^\mu, \qquad \{s_i, s_j\} = -c_{ij}^k s_k, \qquad \{g^i, s_j\} = \xi_j^i(g), \tag{30}$$

where $\xi_i^j(g)$ are the components of left-invariant vector fields on G.

Once the Poisson brackets are defined, we can use in calculations both sets of variables. In particular, taking into account the relation $\mathsf{s} = \mathrm{Ad}^*_g \mathsf{q}$ we obtain:

$$\{q_i, s_j\} = 0. \tag{31}$$

The transition from the Lagrangian description to the Hamiltonian one lies through the Legendre transformation defined by eqs.(11) and (12) or (14). It is degenerate and leads to vanishing canonical Hamiltonian, due to parametrical invariance. Instead, the dynamics is determined by constraints.

In order to obtain constraints explicitly let us consider the relations (13) and (14). They present, in fact, the Legendre mapping $\mathbf{w} \mapsto \mathsf{s}$ and thus allow to consider the mass M as a function of s only. Then eq.(11) reduces to

$$\Pi_\mu \equiv p_\mu - \mathsf{q} \cdot \mathbf{A}_\mu = M(\mathsf{s})\dot{x}_\mu/|\dot{x}| \tag{32}$$

which yields immediately the *mass-shell constraint*:

$$\phi \equiv \Pi^2 - M^2(\mathsf{s}) = 0. \tag{33}$$

There are no more constraints if

$$\det \left\| \frac{\partial^2 F(\mathbf{w})}{\partial w^i \partial w^j} \right\| \neq 0. \tag{34}$$

Otherwise, eq.(14) leads to additional constraints of the following general structure:

$$\chi_r(\mathsf{s}) = 0, \qquad r = \overline{1, \kappa} \leq \dim G \tag{35}$$

which together with the mass-shell constraint (33) form the set of primary constraints. Hence, the Dirac Hamiltonian is $H_D = \lambda_0 \phi + \lambda^r \chi_r$, where λ_0, λ^r are Lagrange multipliers. It is evident from eqs.(33), (35) and (29), (30), (31) that secondary constraints (should they exist) are of the same general structure as in eq.(35). At the final stage of analysis the mass-shell constraint can be considered as the first-class one. This is provided as soon the mass squared $M^2(\mathsf{s})$ is conserved. If it is not, there exists another integral of motion $\tilde{M}^2(\mathsf{s})$ such that $\tilde{M}^2(\mathsf{s})|_{\chi=0} = M^2(\mathsf{s})$. Then the first-class mass-shell constraint has the form (33) but with the function $\tilde{M}^2(\mathsf{s})$ instead of $M^2(\mathsf{s})$.

The total set of constraints is gauge-invariant. This follows from the transformation properties of gauge potentials (4) and variables:

$$x^{\mu\prime} = x^\mu, \qquad {p_\mu}' = p_\mu - \mathsf{q} \cdot \left(h^{-1}(x)\partial_\mu h(x)\right), \qquad \mathsf{q}' = \mathrm{Ad}^*_{h^{-1}(x)}\mathsf{q}, \qquad \mathsf{s}' = \mathsf{s}. \tag{36}$$

In particular, the variables Π_μ defined in eq.(32) are gauge-invariant.

At this stage the splitting of particle dynamics into the external and internal ones becomes obvious. Indeed, the Hamiltonian equations

$$(\dot{x}, \dot{p}, \dot{\mathsf{q}}) = \{(x, p, \mathsf{q}), H_D\} \approx \lambda_0\{(x, p, \mathsf{q}), \Pi^2\}, \tag{37}$$

where $\approx$ is Dirac's symbol of weak equality, are closed with respect to variables (x, p, q). They describe the external dynamics and can be reduced to the equations (16) and (10) by eliminating the variables p_μ and the multiplier λ_0. The equation

$$\dot{\mathsf{s}} = \{\mathsf{s}, H_D\} \approx -\lambda_0\{\mathsf{s}, M^2(\mathsf{s})\} + \lambda^r\{\mathsf{s}, \chi_r(\mathsf{s})\} \tag{38}$$

is closed in terms of s and can be reduced to the equation (15) of the internal dynamics. We note that the group variable g falls out of the equations (37) and (38) which is due to the structure of Poisson-bracket relations (29), (30), (31) and constraints (33), (35). Thus in the present formulation of isospin particle dynamics this variable can be considered as redundant unobservable quantity.

The further treatment of Hamiltonian dynamics, i.e., the classification of constrains as first- and second-class ones etc., demands a consideration of some specific examples.

5 Conclusions

In this paper we consider the formulation of classical dynamics of the relativistic particle in an external Yang–Mills field. We have deduced the Wong equations from the Lagrangian of rather general form defined on the tangent bundle over principle fibre bundle. Besides, we have shown that this Lagrangian leads to some internal particle dynamics. The only quantities coupling this dynamics with the Wong equations are the mass M and isospin (or colour) module $|\mathbf{q}|$, intrinsic characteristics of particle. In the present description they are integrals of internal motion.

The physical treatment of internal dynamics should become better understood within an appropriate quantum-mechanical description. It can be constructed on the base of Hamiltonian particle dynamics proposed in Section 4. Here we only suggest some features of such a description.

Following the procedure of canonical quantization one replaces dynamical variables $x, p, \mathsf{q}, \mathsf{s}$ etc. by operators $\hat{x}, \hat{p}, \hat{\mathsf{q}}, \hat{\mathsf{s}}$, and Poisson brackets by commutators. Let us suppose that the classical dynamics is determined by the only mass-shell constraint (33). Its quantum analogue determines physical states of the system. Eigenvalues q and M of operators $|\hat{\mathbf{q}}|^2 = |\hat{\mathsf{s}}|^2$ and $M^2(\hat{\mathsf{s}})$ which commute with mass-shell constraint and with one another can be treated as the isospin (or colour) and the rest mass of particle. In the case of right-invariant dynamics (as in Kaluza–Klein theory) M is unambiguous function of q. In the general case, the spectrum of $M^2(\hat{\mathsf{s}})$ can consist of few levels M_{qn} which correspond to the same value of q. Thus it is tempting to relate the quantum number n with a flavour or generation. Of course, this supposition is by no means substantiated and needs a following elaboration. It may suggest a phenomenological quantum description of relativistic particles with intrinsic degrees of freedom type of isospin, colour, flavour etc.

Acknowledgments

The author would like to thank Prof. V. Tretyak and Drs. R. Matsyuk, A. Panasyuk, V. Shpytko, and Yu. Yaremko for stimulating discussions of this work.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 481–488.

Crystal Basis Model of the Genetic Code: Structure and Consequences

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The main features of a model of genetic code based on the crystal basis of $U_{q\to 0}(sl(2)\oplus sl(2))$ is presented. The experimentally observed correlation between the values of the codon usage in quartets and sextets fits naturally in the model.

1 Introduction

Let us, briefly, remind how the DNA rules the synthesis of proteins, which constitute the most abundant organic substances in living matter systems. The DNA macromolecule is made of two linear chains of nucleotides wrapped in a double helix structure. Each nucleotide is characterized by one of the four elementary bases: adenine (A) and guanine (G) deriving from purine, and cytosine (C) and thymine (T) coming from pyrimidine. The DNA is localized in the nucleus of the cell and the transmission of the genetic information in the cytoplasm is achieved , schematically speaking, by the ribonucleic acid or RNA. This operation is called the transcription, the A, G, C, T bases in the DNA being respectively associated in RNA to the U, C, G, A bases, U denoting the uracile base. The correspondence law between triples of nucleotides, called *codons*, in the desoxyribonucleic acid (DNA) sequence and the amino-acids is called the genetic code. As a codon is an ordered sequence of three bases (e.g. AAG, AGA, etc.), obviously there are $4^3 = 64$ and different codons. Except the three following triples UAA, UAG and UGA, each of the 61 others is related through a ribosome to an amino-acid (a.a). In the universal eukariotic code, which constitutes the so called universal genetic code, the correspondence is given in Table 1. Thus the chain of nucleotides in the RNA – and also in the DNA – can also be viewed as a sequence of triples, each corresponding to aa a.a., except the three above mentioned codons. These last codons are called *Nonsense* or *Stop* codons, and their role is to stop the biosynthesis.

One can distinguish 20 different amino-acids: Alanine (Ala), Arginine (Arg), Asparagine (Asn), Aspartic acid (Asp), Cysteine (Cys), Glutamine (Gln), Glutamic acid (Glu), Glycine (Gly), Histidine (His), Isoleucine (Ile), Leucine (Leu), Lysine (Lys), Methionine (Met), Phenylalanine (Phe), Proline (Pro), Serine (Ser), Threonine (Thr), Tryptophane (Trp), Tyrosine (Tyr), Valine (Val). It follows that the different codons are associated to the same a.a., i.e. the genetic code is degenerated.

For the eukariotic code (see Table 1), the codons are organized in the following pattern of multiplets, each multiplet orresponding to a specific amino-acid:

1. 3 sextets: Arg, Leu, Ser
2. 5 quadruplets: Ala, Gly, Pro, Thr, Val
3. 2 triplets: Ile, Stop
4. 9 doublets: Asn, Asp, Cys, Gln, Glu, His, Lys, Phe, Tyr
5. 2 singlets: Met, Trp

It is natural, but not at all trivial, to ask if symmetry consideration can explain the existence of such an intriguing degenerate pattern. In our approach [1, 2] we consider the 4 nucleotides as elementary constituents of the codons. Actually, this approach mimicks the group theoretical classification of baryons made out from three quarks in elementary particles physics, the building blocks being here the A, C, G, T/U nucleotides. The main and essential difference stands in the property of a codon to be an *ordered* set of three nucleotides, which is not the case for a baryon. For example, there are three different codons made of the A, A, U nucleotides, namely AAU, AUA and UAA, while the proton appears as a weighted combination of the two u quarks and one d quark, that is $|p\rangle \sim |uud\rangle + |udu\rangle + |duu\rangle$. Constructing such pure states is made possible in the framework of the *crystal bases*, which can be defined in the limit $q \to 0$ of the deformation $\mathcal{U}_q(\mathcal{G})$ of any (semi)-simple classical Lie algebra $\mathcal{G}$.

2 The model

Introducing in $\mathcal{U}_{q\to 0}(\mathcal{G})$ the operators $\tilde{e}_i$ and $\tilde{f}_i$ ($i = 1, \ldots, \text{rank } \mathcal{G}$), whose action on the elements of $\mathcal{U}_q(\mathcal{G})$-module is well-defined in the limit $q \to 0$, a particular kind of basis in a $\mathcal{U}_q(\mathcal{G})$-module can be defined [3]. Such a basis is called a crystal basis and carries the property to undergo in a specially simple way the action of the $\tilde{e}_i$ and $\tilde{f}_i$ operators: as an example, for any couple of vectors u, v in the crystal basis $\mathcal{B}$, one gets $u = \tilde{e}_i v$ if and only if $v = \tilde{f}_i u$. More interesting for our purpose is the property that, in the crystal basis, the basis vectors of the tensor product of two irreducible representations are *pure states*, [3]. Let us emphasize once more the motivation for our choice of the crystal basis. It is an observed fact that in the codons the order of the nucleotides is of fundamental importance (e.g. CCU → `Pro`, CUC → `Leu`, UCC → `Ser`). We want to consider the codons as composite states of the (elementary) nucleotides, but this surely cannot be done in the framework of Lie (super)algebras. Indeed in the Lie theory the composite states, obtained by the tensor product of the fundamental irreducible rpresentations, are linear combinations of the elementary states, with symmetry properties determined by the tensor product (i.e. for $sl(n)$ by the structure of the corresponding Young tableau). The crystal basis on the contrary provides us with the mathematical structure to build composite states as *pure* states, characterized by the order of the constituents. In order to dispose of such a basis, we need to consider the limit $q \to 0$. Note that in this limit we do not deal anymore with a Lie algebra either with an universal deformed enveloping algebra.

We consider the four nucleotides as basic states of the $(\frac{1}{2}, \frac{1}{2})$ representation of the $\mathcal{U}_q(sl(2) \oplus sl(2))$ quantum enveloping algebra in the limit $q \to 0$. A triplet of nucleotides will then be obtained by constructing the tensor product of three such four dimensional representations. The algebra $\mathcal{G} = su(2) \oplus su(2))$ appears the most natural for our purpose. First of all, it is "reasonable" to represent the four nucleotides in the fundamental representation of $\mathcal{G}$. Moreover, the complementary rule in the DNA–RNA transcription may suggest to assign a *quantum number* with opposite values to the couples (A,T/U) and (C,G). The distinction between the purine bases (A,G) and the pyrimidine ones (C,T/U) can be algebraically represented in an analogous way. Thus considering the representation $\left(\frac{1}{2}, \frac{1}{2}\right)$ of the group $SU(2) \times SU(2)$ and denoting $\pm$ the basis vector corresponding to the eigenvalues $\pm\frac{1}{2}$ of the J_3 generator in any of the two $su(2)$ corresponding algebras, we will assume the following "biological" spin structure:

$$\begin{array}{ccc} C \equiv (+,+) & \overset{su(2)_H}{\longleftrightarrow} & U \equiv (-,+) \\ su(2)_V \updownarrow & & \updownarrow su(2)_V \\ G \equiv (+,-) & \underset{su(2)_H}{\longleftrightarrow} & A \equiv (-,-) \end{array} \tag{1}$$

the subscripts H (:= horizontal) and V (:= vertical) being just added to specify the group actions.

To represent a codon, we will have to perform the tensor product of three $\left(\frac{1}{2},\frac{1}{2}\right)$ representations of $\mathcal{U}_{q\to 0}(sl(2)\oplus sl(2))$. We get, using the Kashiwara theorem [3], the following tables

$$\left(\tfrac{3}{2},\tfrac{3}{2}\right) \equiv \begin{pmatrix} \mathrm{CCC} & \mathrm{UCC} & \mathrm{UUC} & \mathrm{UUU} \\ \mathrm{GCC} & \mathrm{ACC} & \mathrm{AUC} & \mathrm{AUU} \\ \mathrm{GGC} & \mathrm{AGC} & \mathrm{AAC} & \mathrm{AAU} \\ \mathrm{GGG} & \mathrm{AGG} & \mathrm{AAG} & \mathrm{AAA} \end{pmatrix}$$

$$\left(\tfrac{3}{2},\tfrac{1}{2}\right)^1 \equiv \begin{pmatrix} \mathrm{CCG} & \mathrm{UCG} & \mathrm{UUG} & \mathrm{UUA} \\ \mathrm{GCG} & \mathrm{ACG} & \mathrm{AUG} & \mathrm{AUA} \end{pmatrix}$$

$$\left(\tfrac{3}{2},\tfrac{1}{2}\right)^2 \equiv \begin{pmatrix} \mathrm{CGC} & \mathrm{UGC} & \mathrm{UAC} & \mathrm{UAU} \\ \mathrm{CGG} & \mathrm{UGG} & \mathrm{UAG} & \mathrm{UAA} \end{pmatrix}$$

$$\left(\tfrac{1}{2},\tfrac{3}{2}\right)^1 \equiv \begin{pmatrix} \mathrm{CCU} & \mathrm{UCU} \\ \mathrm{GCU} & \mathrm{ACU} \\ \mathrm{GGU} & \mathrm{AGU} \\ \mathrm{GGA} & \mathrm{AGA} \end{pmatrix} \qquad \left(\tfrac{1}{2},\tfrac{3}{2}\right)^2 \equiv \begin{pmatrix} \mathrm{CUC} & \mathrm{CUU} \\ \mathrm{GUC} & \mathrm{GUU} \\ \mathrm{GAC} & \mathrm{GAU} \\ \mathrm{GAG} & \mathrm{GAA} \end{pmatrix}$$

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right)^1 \equiv \begin{pmatrix} \mathrm{CCA} & \mathrm{UCA} \\ \mathrm{GCA} & \mathrm{ACA} \end{pmatrix} \qquad \left(\tfrac{1}{2},\tfrac{1}{2}\right)^2 \equiv \begin{pmatrix} \mathrm{CGU} & \mathrm{UGU} \\ \mathrm{CGA} & \mathrm{UGA} \end{pmatrix}$$

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right)^3 \equiv \begin{pmatrix} \mathrm{CUG} & \mathrm{CUA} \\ \mathrm{GUG} & \mathrm{GUA} \end{pmatrix} \qquad \left(\tfrac{1}{2},\tfrac{1}{2}\right)^4 \equiv \begin{pmatrix} \mathrm{CAC} & \mathrm{CAU} \\ \mathrm{CAG} & \mathrm{CAA} \end{pmatrix}$$

3 The Reading (or Ribosome) operator $\mathcal{R}$

Our model does not gather codons associated to one particular a.a. in the same irreducible multiplet. However, it is possible to construct an operator $\mathcal{R}$ out of the algebra $\mathcal{U}_{q\to 0}(sl(2)\oplus sl(2))$, acting on the codons, that will describe the Evarious genetic codes in the following way:

Two codons have the same eigenvalue under $\mathcal{R}$ if and only if they are associated to the same amino-acid. This operator $\mathcal{R}$ will be called the reading operator. It is possible to construct a $\mathcal{R}$ for the various genetic codes. Here we limit orselves to present in detail only the Reading operator for the Eukaryotic code

$$\begin{aligned} \mathcal{R}_{EC} = {} & \tfrac{4}{3}c_1\, C_H + \tfrac{4}{3}c_2\, C_V - 4c_1\, \mathcal{P}_H\, J_{H,3} - 4c_2\, \mathcal{P}_V\, J_{V,3} + (-8c_1\, \mathcal{P}_D + (8c_1 + 12c_2)\, \mathcal{P}_S)\, J_{V,3} \\ & + (-4c_1 + 14c_2)\, \mathcal{P}_{AG} \left(\tfrac{1}{2} - J^{(3)}_{V,3}\right) \\ & + \Big[12c_2\, \mathcal{P}_{AU} + (6c_1 + 6c_2)\, \mathcal{P}_{UG}\Big] \left(\tfrac{1}{2} - J^{(3)}_{V,3}\right)\left(\tfrac{1}{2} - J^{(3)}_{H,3}\right), \end{aligned} \tag{2}$$

where

- the operators $J_{\alpha,3}$ ($\alpha = H, V$) are the third components of the total spin generators of the algebra $\mathcal{U}_{q\to 0}(sl(2) \oplus sl(2))$;
- the operator C_α is a Casimir operator of $\mathcal{U}_{q\to 0}(sl(2))$ in the crystal basis. It commutes with $J_{\alpha\pm}$ and $J_{\alpha,3}$ (where $J_{\alpha\pm}$ are the generators with a well-defined behaviour for $q \to 0$) and its eigenvalues on any vector basis of an irreducible representation of highest weight J is $J(J+1)$, that is the same as the undeformed standard second degree Casimir operator of $sl(2)$. Its explicit expression is

$$C_\alpha = (J_{\alpha,3})^2 + \tfrac{1}{2} \sum_{n\in\mathbb{Z}_+} \sum_{k=0}^{n} (J_{\alpha-})^{n-k} (J_{\alpha+})^n (J_{\alpha-})^k; \tag{3}$$

- $\mathcal{P}_H$, $\mathcal{P}_V$, $\mathcal{P}_D$, $\mathcal{P}_S$, $\mathcal{P}_{AG}$, $\mathcal{P}_{AU}$ and $\mathcal{P}_{UG}$ are projectors operators given by:

$$\mathcal{P}_H = J^d_{H+} J^d_{H-} \qquad \text{and} \qquad \mathcal{P}_V = J^d_{V+} J^d_{V-}, \tag{4}$$

$$\begin{aligned} \mathcal{P}_D = {} & \left(1 - J^d_{V+} J^d_{V-}\right) \left(J^d_{H+} J^d_{H-}\right) \left(J^d_{H-} J^d_{H+}\right) \\ & + \left(1 - J^d_{H+} J^d_{H-}\right) \left(1 - J^d_{V+} J^d_{V-} \left(1 - J^d_{H-} J^d_{H+}\right)\right), \end{aligned} \tag{5}$$

$$\begin{aligned} \mathcal{P}_S = {} & \left(J^d_{H-} J^d_{H+}\right) \left[\left(J^d_{H+} J^d_{H-}\right) \left(1 - J^d_{V+} J^d_{V-}\right) \right. \\ & \left. + \left(J^d_{V+} J^d_{V-}\right) \left(J^d_{V-} J^d_{V+}\right) \left(1 - J^d_{H+} J^d_{H-}\right)\right], \end{aligned} \tag{6}$$

$$\mathcal{P}_{AG} = \Big(J^d_{H+} J^d_{H-})(J^d_{H-} J^d_{H+}\Big) \Big(1 - J^d_{V+} J^d_{V-}\Big) \Big(J^d_{V-} J^d_{V+}\Big), \tag{7}$$

$$\mathcal{P}_{AU} = \Big(1 - J^d_{H+} J^d_{H-}\Big) \Big(J^d_{H-} J^d_{H+}\Big) \Big(J^d_{V+} J^d_{V-}\Big) \Big(J^d_{V-} J^d_{V+}\Big), \tag{8}$$

$$\mathcal{P}_{UG} = \Big(J^d_{H+} J^d_{H-}\Big) \Big(J^d_{H-} J^d_{H+}\Big) \Big(1 - J^d_{V+} J^d_{V-}\Big) \Big(1 - J^d_{V-} J^d_{V+}\Big). \tag{9}$$

We get the following eigenvalues of the reading operators for the amino-acids (after a rescaling, setting $c \equiv c_1/c_2$):

a.a.	value of $\mathcal{R}$	a.a.	value of $\mathcal{R}$	a.a.	value of $\mathcal{R}$
Ala	$-c+3$	Gly	$-c+5$	Pro	$-c-1$
Arg	$-c+1$	His	$-3c+1$	Ser	$3c-1$
Asn	$9c+5$	Ile	$5c+9$	Thr	$3c+3$
Asp	$5c+5$	Leu	$c-1$	Trp	$3c-5$
Cys	$3c+7$	Lys	$17c+5$	Tyr	$c+1$
Gln	$5c+1$	Met	$5c-3$	Val	$c+3$
Glu	$13c+5$	Phe	$-7c-1$	Ter	$9c+1$

(10)

Remark that the reading operators $\mathcal{R}(c)$ can be used for any real value of c, except a finite set of rational values confering the same eigenvalue to codons relative to two different amino-acids. Moreover from our algebra it is possible to construct a hamiltonian which gives a very satisfactory fit of the 16 values of the free energy released in the folding of RNA [1].

codon	a.a.	J_H	J_V	codon	a.a.	J_H	J_V
CCC	Pro	3/2	3/2	UCC	Ser	3/2	3/2
CCU	Pro	(1/2	$3/2)^1$	UCU	Ser	(1/2	$3/2)^1$
CCG	Pro	(3/2	$1/2)^1$	UCG	Ser	(3/2	$1/2)^1$
CCA	Pro	(1/2	$1/2)^1$	UCA	Ser	(1/2	$1/2)^1$
CUC	Leu	(1/2	$3/2)^2$	UUC	Phe	3/2	3/2
CUU	Leu	(1/2	$3/2)^2$	UUU	Phe	3/2	3/2
CUG	Leu	(1/2	$1/2)^3$	UUG	Leu	(3/2	$1/2)^1$
CUA	Leu	(1/2	$1/2)^3$	UUA	Leu	(3/2	$1/2)^1$
CGC	Arg	(3/2	$1/2)^2$	UGC	Cys	(3/2	$1/2)^2$
CGU	Arg	(1/2	$1/2)^2$	UGU	Cys	(1/2	$1/2)^2$
CGG	Arg	(3/2	$1/2)^2$	UGG	Trp	(3/2	$1/2)^2$
CGA	Arg	(1/2	$1/2)^2$	UGA	Ter	(1/2	$1/2)^2$
CAC	His	(1/2	$1/2)^4$	UAC	Tyr	(3/2	$1/2)^2$
CAU	His	(1/2	$1/2)^4$	UAU	Tyr	(3/2	$1/2)^2$
CAG	Gln	(1/2	$1/2)^4$	UAG	Ter	(3/2	$1/2)^2$
CAA	Gln	(1/2	$1/2)^4$	UAA	Ter	(3/2	$1/2)^2$
GCC	Ala	3/2	3/2	ACC	Thr	3/2	3/2
GCU	Ala	(1/2	$3/2)^1$	ACU	Thr	(1/2	$3/2)^1$
GCG	Ala	(3/2	$1/2)^1$	ACG	Thr	(3/2	$1/2)^1$
GCA	Ala	(1/2	$1/2)^1$	ACA	Thr	(1/2	$1/2)^1$
GUC	Val	(1/2	$3/2)^2$	AUC	Ile	3/2	3/2
GUU	Val	(1/2	$3/2)^2$	AUU	Ile	3/2	3/2
GUG	Val	(1/2	$1/2)^3$	AUG	Met	(3/2	$1/2)^1$
GUA	Val	(1/2	$1/2)^3$	AUA	Ile	(3/2	$1/2)^1$
GGC	Gly	3/2	3/2	AGC	Ser	3/2	3/2
GGU	Gly	(1/2	$3/2)^1$	AGU	Ser	(1/2	$3/2)^1$
GGG	Gly	3/2	3/2	AGG	Arg	3/2	3/2
GGA	Gly	(1/2	$3/2)^1$	AGA	Arg	(1/2	$3/2)^1$
GAC	Asp	(1/2	$3/2)^2$	AAC	Asn	3/2	3/2
GAU	Asp	(1/2	$3/2)^2$	AAU	Asn	3/2	3/2
GAG	Glu	(1/2	$3/2)^2$	AAG	Lys	3/2	3/2
GAA	Glu	(1/2	$3/2)^2$	AAA	Lys	3/2	3/2

Table 1: The eukariotic code. The upper label denotes different IR.

	Biological organism	Type	number of sequences	number of codons
1	Homo sapiens	v	14 529	7 168 914
2	Saccharomyces cerevisiae	f	11 771	5 691 597
3	Caenorhabditis elegans	i	12 638	5 514 021
4	Rattus norvegicus	v	4 430	2 135 734
5	Arabidopsis Thaliana	p	3 533	1 497 366
6	Drosophila melanogaster	i	2 625	1 443 176
7	Schizosaccharomyces pombe	f	2 289	1 093 794
8	Gallus gallus	v	1 454	701 782
9	Xenopus laevis	v	1 255	551 494
10	Bos taurus	v	1 217	528 790
11	Oryctolagus cuniculus	v	674	335 049
12	Sus scrofa	v	589	238 579
13	Zea mays	p	603	222 493

Table 2: v) Vertebrates – i) Invertebrata – p) Plants – f) Fungi

Species	Pro	Ala	Thr	Ser	Val	Leu	Arg	Gly
Homo sapiens	2,36	2,05	2,26	2,52	0,23	0,16	0,53	1,00
Saccharomyces c.	3,44	2,64	2,22	2,19	1,10	1,28	1,73	1,83
Caenorhabditis e.	3,17	2,78	2,48	1,88	0,74	0,69	2,82	7,80
Rattus Norveg.	2,45	2,18	2,34	2,34	0,22	0,17	0,62	1,04
Arabidopsis Thal.	1,93	1,97	2,03	1,95	0,53	0,96	1,29	2,45
Drosophila mel.	0,78	0,89	0,75	0,41	0,21	0,18	0,96	4,02
Schizosaccharomyces	2,74	2,94	2,12	2,25	1,49	1,39	2,63	3,66
Gallus gallus	1,82	1,92	1,97	1,90	0,25	0,14	0,51	1,02
Xenopus laevis	4,09	4,32	4,04	3,39	0,48	0,32	1,00	1,70
Bos taurus	1,88	1,62	1,77	2,03	0,19	0,13	0,51	0,95
Oryctolagus cun.	1,51	1,49	1,31	1,50	0,15	0,10	0,45	0,88
Sus scrofa	1,59	1,59	1,50	1,62	0,16	0,12	0,45	0,89
Zea mays	0,87	0,69	0,83	0,98	0,19	0,24	0,39	0,85

Table 3: Branching ratio B_{AG}

Species	Pro	Ala	Thr	Ser	Val	Leu	Arg	Gly
Homo sapiens	2,45	2,44	1,96	3,22	0,36	0,30	0,41	0,66
Saccharomyces c.	2,57	3,44	2,54	2,75	2,06	1,17	3,73	4,00
Caenorhabditis e.	1,07	3,14	2,38	1,58	1,85	1,97	2,78	2,67
Rattus Norveg.	2,67	2,84	1,99	3,28	0,32	0,28	0,48	0,71
Arabidopsis Thal.	2,21	3,34	2,46	2,78	1,53	2,48	2,06	2,30
Drosophila mel.	0,40	1,03	0,63	0,37	0,38	0,22	1,12	3,13
Schizosaccharomyces	4,75	5,70	3,52	3,85	3,58	4,08	5,48	5,18
Gallus gallus	1,72	2,24	1,62	2,32	0,42	0,27	0,57	0,66
Xenopus laevis	3,63	4,68	3,57	4,80	0,73	0,59	1,12	1,06
Bos taurus	1,96	2,10	1,52	2,72	0,32	0,26	0,38	0,65
Oryctolagus cun.	1,63	1,82	1,18	2,03	0,28	0,21	0,34	0,53
Sus scrofa	1,72	2,10	1,43	2,42	0,27	0,23	0,36	0,59
Zea mays	0,78	1,00	0,94	1,08	0,54	0,59	0,67	1,06

Table 4: Branching ratio B_{UG}

Species	Pro	Ala	Thr	Ser	Val	Leu	Arg	Gly
Homo sapiens	2,90	3,82	3,13	3,99	0,51	0,49	0,96	1,41
Saccharomyces c.	1,29	2,06	1,58	1,66	1,09	0,51	1,49	1,62
Caenorhabditis e.	0,49	1,65	1,25	0,93	0,98	1,29	1,22	1,43
Rattus Norveg.	2,93	4,12	3,25	4,20	0,54	0,50	1,00	1,47
Arabidopsis Thal.	0,66	1,24	1,45	1,25	0,74	1,65	0,80	0,90
Drosophila mel.	1,12	2,52	1,59	1,17	0,54	0,36	2,38	6,53
Schizosaccharomyces	1,80	2,23	1,67	1,51	1,35	1,18	2,08	1,99
Gallus gallus	2,29	2,73	2,33	3,03	0,50	0,43	1,25	1,29
Xenopus laevis	2,85	4,02	3,38	4,09	0,58	0,48	1,14	1,17
Bos taurus	2,70	3,76	2,84	3,76	0,53	0,48	1,00	1,46
Oryctolagus cun.	2,58	3,83	2,51	3,70	0,54	0,48	1,18	1,55
Sus scrofa	2,58	3,94	2,95	3,75	0,56	0,50	1,05	1,56
Zea mays	0,90	1,48	1,79	1,70	0,80	0,98	1,75	2,22

Table 5: Branching ratio B_{CG}

4 Correlations of codon usage

In the following the labels X, Y, Z, V represent any of the 4 bases C, U, G, A. Let XYZ be a codon in a given multiplet, say m_i, encoding an a.a., say A_i. We define the probability of usage of the codon XYZ as the ratio between the frequency of usage n_Z of the codon XYZ in the biosynthesis of A_i and the total number N of synthetized A_i, i.e. as the relative codon frequency, in the limit of *very large* N. The frequency rate of usage of a codon in a multiplet is connected to its probability of usage $P(XYZ \to \text{a.a.})$. We define the *branching ratio* B_{ZV} as

$$B_{ZV} = \frac{P(XYZ \to A_i)}{P(XYV \to A_i)}, \tag{11}$$

where XYV is another codon belonging to the same multiplet m_i. It sounds reasonable to argue that in the limit of very large number of codons, for a fixed biological organism and amino-acid, the branching ratio depends essentially on the properties of the codon. In our model this means that in this limit B_{ZV} is a function, depending on the type of the multiplet, on the *quantum numbers* of the codons XYZ and XYV, i.e. on the labels J_α, J_α^3, and on an other set of quantum labels leaving out the degeneracy on J_α; in Table 1 different irreducible representations with the same values of J_α are distinguished by an upper label. Moreover we assume that B_{ZV}, in the limit above specified, depends only on the irreducible representation (IR) of the codons, i.e.:

$$B_{ZV} = F_{ZV}(b.o.; IR(XYZ); IR(XYV)), \tag{12}$$

where we have explicitly denoted by *b.o.* the dependence on the biological species. Let us point out that the branching ratio has a meaning only if the codons XYZ and XYU are in the same multiplet, i.e. if they code the same amino-acid.

In the following, we consider the quartets and the quartet sub-parts of the sextets, i.e. the 4 codons which differ only for the codon in third position. There are five quartets and three sextets in the eukariotic code: that will allow a rather detailed analysis. We recall that the 5 amino-acids coded by the quartets are `Pro`, `Ala`, `Thr`, `Gly` ,`Val` and the 3 amino-acids coded by the sextets are `Leu`, `Arg`, `Ser`. There are, for the quartets, 6 branching ratios, of which only 3 are independent. We choose as fundamental ones the ratios B_{AG}, B_{CG} and B_{UG}. It happens that we can define several functions B_{ZV}, considering ratios of probability of codons differing for the first two nucleotides XY, i.e.

$$\begin{aligned} B_{ZV} &= F_{ZV}(b.o.; IR(XYZ); IR(XYV)), \\ B'_{ZV} &= F_{ZV}(b.o.; IR(X'Y'Z); IR(X'Y'V)). \end{aligned} \tag{13}$$

Then if the codon XYZ (XYV) and $X'Y'Z$ ($X'Y'V$) are respectively in the same irreducible representation, it follows that

$$B_{ZV} = B'_{ZV}. \tag{14}$$

The analysis was performed on a set of data retrieved (May 1999) from the data bank of "Codon usage tabulated from GenBank" [4]. In particular in [5] we analyzed the data set with more than 64.000 codons and we found 34 biological species (neglecting 3 biological species belonging to protozoo, bacteries and mushrooms). This has to be compared with the result of [2] where such a correlation has been remarked for 12 biological species belonging only to the vertebrate series. Here we present th results only for the subset of 13 species with more than 200.000 codons, see Table 2.

In Table 3, 4 and 5 the B_{AG}, B_{UG} and B_{CG} are reported for the 13 amino-acids coded by the quartets and sextets showing:

- a clear correlation between the four amino-acids `Pro`, `Ala`, `Thr` and `Ser`. From Table 1 we see that for these amino-acids the irreducible representation involved in the numerator of the branching ratios (see (11)) is always the same: $(1/2,1/2)^1$ for B_{AG}, $(1/2,3/2)^1$ for B_{UG}, $(3/2,3/2)$ for B_{CG}, while the irreducible representation in the denominator is $(3/2,1/2)^1$ for the whole set.
- a clear correlation between the two amino-acids `Val` and `Leu`. From Table 1 we see that also for these two amino-acids the irreducible representation in the numerator of (11) is the same: $(1/2,1/2)^3$ for B_{AG}, $(1/2,3/2)^2$ for B_{UG}, $(1/2,3/2)^2$ for B_{CG}, and the irreducible representation in the denominator is $(1/2,1/2)^3$.
- no correlation of the `Arg` and also of the `Gly` with the others amino-acids, in agreement with the irreducible representation assignment of Table 1.

5 Conclusion

The model we propose is based on symmetry principles. The symmetry algebra $\mathcal{U}_{q\to 0}(sl(2) \oplus sl(2))$ that we have chosen has two main characteristics. First it encodes the stereochemical property of a base, and also reflects the complementarity rule, by confering quantum numbers to each nucleotide. Secondly, it admits representation spaces or crystal bases in which an ordered sequence of nucleotides or codon can be suitably characterized. Let us add that it is a remarkable property of a quantum algebra in the limit $q \to 0$ to admit representations, obtained from the tensorial product of basic ones, in which each state appears as a unique sequence of ordered basic elements. In this framework, the correspondence codon/amino-acid is realized by the operator $\mathcal{R}_c$, constructed out of the symmetry algebra, and acting on codons: the eigenvalues provided by $\mathcal{R}_c$ on two codons will be the same or different following the two codons are associated to the same or to two different amino-acids. The model does not necessarily assign the codons in a multiplet (in particular the quartets, sextets and triplet) to the same irreducible representation. This feature is relevant as it may explain the different codon usage between codons encoding the same a.a.. Indeed, as we have shown in this paper, it fits very well with our model the observed fact that for *any biological organism*, in the limit of large number of biosynthetized amino-acids, the ratios B_{AG}, B_{UG} and B_{CG} for, Pro, Ala, Thr, Ser, in one side, and Val, Leu, in other side, are very close. Let us remark that obviously these ratios depend on the biological organism and we are unable to make any prevision on their values, but only that their values should be correlated.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 489–492.

On a Class of Equations for Particles with Arbitrary Spin

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The relativistic wave equations proposed by Moshinsky and Smirnov [1] are analysed. It is proved that these equations are causal. A simple algorithm for solving of these equations for particles interacting with a constant magnetic fields proposed.

1 Introduction

Let us consider the wave equation for particle with arbitrary spins proposed by Moshinsky and Smirnov [1]. For $s = 1$ these equations have the form

$$p_0\psi = H\psi, \qquad H = \left(\bar{\alpha}^{(1)} + \bar{\alpha}^{(2)}\right)\bar{p} + \left(\beta^{(1)} + \beta^{(2)}\right)m, \tag{1}$$

where $\bar{\alpha}^{(1)}$, $\bar{\alpha}^{(2)}$, $\beta^{(1)}$, $\beta^{(2)}$ are 16×16 matrices which can be written in the form

$$\alpha_a^{(i)} = \gamma_0^{(i)}\gamma_a^{(i)}, \qquad \beta^{(i)} = \gamma_0^{(i)}, \qquad i = 1, 2, \quad a = 1, 2, 3 \tag{2}$$

and the matrices $\gamma_\mu^{(1)}$, $\gamma_\mu^{(2)}$ are connected to Dirac matrices γ_μ as

$$\gamma_\mu^{(1)} = \gamma_\mu \otimes I_4, \qquad \gamma_\mu^{(2)} = I_4 \otimes \gamma_\mu. \tag{3}$$

The matrices $\beta_\mu = \frac{1}{2}(\gamma_\mu^{(1)} + \gamma_\mu^{(2)})$ satisfy the Kemmer–Duffin relations

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = g_{\mu\nu}\beta_\lambda + g_{\nu\lambda}\beta_\mu. \tag{4}$$

Using (4), equation (1) can be rewritten as

$$p_0\psi = \bar{H}\psi, \qquad \bar{H} = [\beta_0, \beta_a]p_a + \beta_0 m. \tag{5}$$

2 Transformation to the quasidiagonal form

Now let us show that equation (5) can be reduced to the system of three uncoupled equations. To do this, we transform β_μ using the unitary transformation $\beta_\mu \to \hat{\beta}_\mu = U\beta_\mu U^\dagger$, where [3]

$$\begin{aligned} U = \frac{1-i}{2}(&e_{1,1} + e_{1,13} + e_{2,2} + e_{2,14} + e_{3,3} + e_{3,15} - e_{10,8} + e_{10,12} - e_{11,4} - e_{11,16} \\ &+ e_{13,15} - e_{13,9} + e_{14,6} - e_{14,10} + e_{15,7} - e_{15,11}) \\ + \frac{1+i}{2}(&-e_{4,5} - e_{4,9} - e_{5,6} - e_{5,10} - e_{6,7} - e_{6,11} - e_{7,1} + e_{7,13} - e_{8,2} + e_{8,14} \\ &- e_{9,3} + e_{9,15} - e_{12,4} + e_{12,16} + e_{16,8} + e_{16,12}). \end{aligned} \tag{6}$$

Here $e_{i,j}$ stand for the square matrices, whose only nonzery entry, equal to unity, is located at the intersection of i-th row and j-th column.

As a result we obtain

$$\hat{\beta}_\mu = \begin{pmatrix} \beta_\mu^{(10)} & \cdot & \cdot \\ \cdot & \beta_\mu^{(5)} & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}, \qquad \mu = 0, 1, 2, 3, \tag{7}$$

where $\beta_\mu^{(10)}$, $\beta_\mu^{(5)}$ are Kemmer–Duffin 5×5 and 10×10 matrices correspondingly:

$$\begin{aligned} &\beta_0^{(10)} = i(e_{1,7} + e_{2,8} + e_{3,9} - e_{7,1} - e_{8,2} - e_{9,3}), && \beta_0^{(5)} = -i(e_{1,2} - e_{2,1}), \\ &\beta_1^{(10)} = -i(e_{1,10} - e_{5,9} + e_{6,8} + e_{8,6} - e_{9,5} + e_{10,1}), && \beta_1^{(5)} = i(e_{1,3} + e_{3,1}), \\ &\beta_2^{(10)} = -i(e_{2,10} + e_{4,9} - e_{6,7} - e_{7,6} + e_{9,4} + e_{10,2}), && \beta_2^{(5)} = i(e_{1,4} + e_{4,1}), \\ &\beta_3^{(10)} = -i(e_{3,10} - e_{4,8} + e_{5,7} + e_{7,5} - e_{8,4} + e_{10,3}), && \beta_3^{(5)} = i(e_{1,5} + e_{5,1}). \end{aligned} \tag{8}$$

Then equation (5) will be reduced to the system of three equations

$$p_0\psi_{(10)} = \left([\beta_0^{(10)}, \beta_a^{(10)}]p_a + \beta_0^{(10)}m\right)\psi_{(10)}, \tag{9.a}$$

$$p_0\psi_{(5)} = \left([\beta_0^{(5)}, \beta_a^{(5)}]p_a + \beta_0^{(5)}m\right)\psi_{(5)}, \tag{9.b}$$

$$p_0\psi_{(1)} = 0. \tag{9.c}$$

Equation (9.c) is not hyperbolic and hence system (5) is not causal.

Equations (9.a), (9.b) can be represented in the form

$$p_0\psi_{(k)} = \hat{H}\psi_{(k)}, \qquad \hat{H} = \frac{1}{n_1}(S_{a4}p_a + S_{45}m), \qquad k = 5, 10, \tag{13}$$

where $S_{0a} = [\beta_0, \beta_a]$, $S_{ab} = i[\beta_a, \beta_b]$, $S_{45} = \beta_4$, $S_{4a} = -i\beta_a$ are matrices which belong to algebra $AO(5)$. The irreducible representations of $AO(5)$ are labelled by pairs of numbers (n_1, n_2) (simultaneously integer or half-integer). Any of equations proposed in [1] can be reduced to a system of uncoupled equations, corresponding to these irreducible representations.

3 Analysis of hyperbolicity

The Hamiltonian $\hat{H}$ can be reduced to the diagonal form using operator

$$U_1 = \exp\left(i\frac{S_{i5}p_i}{p}\arctan\frac{p}{m}\right) = \exp(iA), \qquad \text{where} \qquad p = \sqrt{p_1^2 + p_2^2 + p_3^2}. \tag{14}$$

Taking into account the Campbell–Hausdorff formula

$$\exp(-iA)B\exp(iA) = B - i[A, B] - \frac{1}{2!}[A, [A, B]] + \cdots$$

and the commutation relations of $AO(4)$

$$[S_{\mu\nu}, S_{\rho\sigma}] = i(\delta_{\mu\rho}S_{\nu\sigma} + \delta_{\nu\sigma}S_{\mu\rho} - \delta_{\mu\sigma}S_{\nu\rho} - \delta_{\nu\rho}S_{\mu\sigma}),$$

we find $\bar{H} = U_1 \hat{H} U_1^{-1} = \frac{1}{n_1} S_{45} \sqrt{p^2 + m^2}$. Thus we come to the equations

$$p_0 \psi' = \bar{H} \psi', \qquad \bar{H} = \frac{1}{n_1} S_{45} \sqrt{p^2 + m^2}, \tag{15}$$

where $\psi'_{(k)} = e^{iA} \psi_{(k)}$ $(k = 5, 10)$.

Form (15) is suitable for studying of hyperbolicity of equation (1). Let us take the matrix S_{45} in the diagonal form and consider two cases: n_1, n_2 are half-integer and n_1, n_2 are integer.

For the first case the reduction of $O(5)$ to $O(4)$ and $O(4)$ to $O(3)$ shows that system (15) is hyperbolical.

Indeed, bearing in mind that the eigenvalues of the matrix S_{45} are

$$s = \frac{n_1 + n_2}{2}, \frac{n_1 + n_2 - 1}{2}, \frac{n_1 + n_2 - 2}{2}, \ldots, 0$$

and their multiplicity is given by the formula [2]

$$M_s = \begin{cases} (n_1 - n_2 + 1)(n_1 + n_2 + 1 - 2s), & s \geq (n_1 - n_2)/2, \\ (2n_1 + 1)(2s + 1), & s < (n_1 - n_2)/2, \end{cases}$$

we find that Ψ' satisfies the equation

$$\prod_s \left(p_0^2 - \frac{1}{s} c^2 p^2 \right)^{M_s} \Psi' = 0. \tag{16}$$

It follows from (16) that system (15) is hyperbolical and the velocity of waves described by this system can have different values $\tilde{c}_s$ and $\tilde{c}_s \leq c$. So the velocity of propagation does not exceed the velocity of light and the causality is not broken.

For n_1, n_2 integer we have equation (9.c), and hence the causality is broken.

4 Equation for particle interacting with constant magnetic field

Consider the generalized equation (13) which describes a particle interacting with constant magnetic field which is directed along x_3.

In accordance with the principle of minimal coupling we change

$$p_\mu \to \pi_\mu = p_\mu - eA_\mu, \tag{17}$$

where $A_0 = A_2 = A_3 = 0, A_1 = eHx_2$.

It is straightforward to check that H satisfies the following relations

$$\hat{H}^3 - \hat{H}\xi \mp MH = 0, \tag{18}$$

where $\xi = \pi_1^2 + \pi_2^2 - 2S_{12}H + M^2$, $M^2 = m^2 + p_3^2$.

Inasmuch as the operators H, $S_{12}H$, $\pi^2 - 2S_{12}H$ commute, relations (17) can be replaced by the relations for eigenvalues of these operators

$$E^3 - E\left((2n + 1)\omega \pm \omega + m^2 + p_3^2\right) \mp \left(m^2 + p_3^2\right) H = 0, \tag{19}$$

where $\omega = (eH)^{1/2}$.

In [1] Moshinsky and Smirnov obtained sixth order algebraic equations for the eigenvalues of H. We show that these equations can be factorized into the product of two third order equations.

For the case of spin 3/2 the authors of [1] found tenth order algebraic equations for the eigenvalues of H. Using our approach, it is possible to show that in fact these equations also can be factorized into the product of sixth and fourth order equations.

5 Problem of complex energies

Analysing equation (19), we see that for

$$H > \frac{M^2}{4} \tag{20}$$

the eigenvalues of Hamiltonian become complex.

The appearance of complex energies in the problem of interaction of a particle with the constant magnetic field is typical for the equations with higher spins [3]. Thus, the equations which were proposed in [1] also have nonphysical solutions corresponding to complex energies.

Let us note that the magnetic field, satisfying (20), is exteremely strong one, and hardly can be encountered in practice. Therefore, the appearance of complex energy eigenvalues should not be considered as an obstacle for application of the equation in question.

6 Conclusions

Analysing the equations for arbitrary spin given above, we found that

1. For the case of half-integer spin these equations are hyperbolical.
2. For the case of integer spin the hyperbolicity is broken in view of appearing of the solutions which correspond to zero energies and these solutions must be rejected.
3. Hyperbolical solutions of equation (13) describe waves having velocity less than c.
4. The equation for energy eigenvalues of (13) with magnetic field can be reduced to algebraic equations, whose order is less than that of the equations given in paper [1].
5. This equation, like other equations with higher spins, has solutions with complex energies.

Acknowledgements

I am sincerely grateful to Prof. A.G. Nikitin and A. Sergyeyev for reading this manuscript and making many useful remarks, which allowed to improve it considerably.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 493–500.

Symmetries, their Breaking and Anomalies in the Self-Consistent Renormalization. Discrete Symmetry Shadow in Chiral Anomalies

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Using the self-consistent renormalization (SCR), a careful study of complicated tangle of problems associated with renormalizations, symmetries conservation, their breaking and anomalies is performed for some set of UV-divergent Feynman amplitudes (FA's) connected with mass-anysotropic AVV- and AAA-triangles in the space-time $n = 4$. Most general quantum corrections (QC's) to the canonical Ward identities (WI's) and some nontrivial "daughter reduction identities" (DRI's) are obtained. The results are new both for a nondegenerate case and for the chiral case. For a nondegenerate case ($m_1 \neq m_2 \neq m_3$, $m_l \neq 0$), the QC's are the zero degree homogeneous functions of masses and are expressed in terms of the Appel hypergeometric functions F_1. For the chiral case ($m = 0$) and the chiral limit ($m \to 0$) the behaviour of the AVV- and AAA-amplitudes depends crucially on the discrete symmetry of these amplitudes in the cases $m = 0$ and $m \to 0$. In the chiral case the QC's to "left-handed" WI's vanish. This may give some insight into why just the left-handed neutrino exists in Nature.

1. Symmetries of quantum field theories (QFT) often manifest themselves via certain formal relations between UV-divergent FA's known as the canonical WI's (CWI's). Anomalies of QFT's occur as breakdown of the CWI's at the level of regular (finite) values of FA's [1–3]. We hope to clarify some obscure points in these violations by employing the SCR [4] to spinor triangle FA's, as the most important subject in such investigations, and to illustrate possibilities of the SCR. Recall that the SCR is an effective realization of the Bogoliubov–Parasiuk R-operation [5] which is complemented with recurrence, compatibility and differential relations fixing a renormalization arbitrariness of the R-operation in some universal way based on mathematical properties of FA's only.

2. The main Feynman amplitude corresponding to the triangle spinor graph of the most general kind (different masses, arbitrary Clifford structure of vertices, the n-dimensional space-time with the (q,p)-signature) looks as follows:

$$I^{\gamma_1\gamma_2\gamma_3}(m,k) := \int_{-\infty}^{\infty} (d^n p)\; \delta(p,k) \frac{\mathrm{tr}[\gamma_1(m_1+\hat p_1)\gamma_2(m_2+\hat p_2)\gamma_3(m_3+\hat p_3)]}{\left(m_1^2-p_1^2-i\epsilon_1\right)\left(m_2^2-p_2^2-i\epsilon_2\right)\left(m_3^2-p_3^2-i\epsilon_3\right)},$$
$$(d^n p) := d^n p_1 d^n p_2 d^n p_3, \qquad \hat p_l := \gamma^\mu p_{l\mu}, \qquad m := (m_1,m_2,m_3), \qquad k := (k_1,k_2,k_3),$$
$$\delta(p,k) := \delta(-k_1+p_3-p_1)\,\delta(-k_2+p_1-p_2)\,\delta(-k_3+p_2-p_3). \tag{1}$$

The matrices γ_i, γ_μ, I_g act in the N_g-dimensional space of the faithful representation π_g of lowest dimension for the Clifford algebra $Cl(g)_{\mathbf{K}}$, $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, with $\gamma_\mu \in \Lambda_1(g)$, $\mu = 1,\ldots,n$, being the generating elements of the $Cl(g)_{\mathbf{K}}$-algebra in its matrix representation π_g; also $\gamma_i \in \Lambda_k(g)$,

$i = 1, 2, 3$, are some k-degree ($k = 0, 1, \ldots, n$) homogeneous elements of the $Cl(g)_{\mathbf{K}}$-algebra in the π_g-representation; I_g is N_g-dimensional unit matrix. The natural analog of the Dirac γ^5-matrix is $\gamma_* := \gamma_1\gamma_2\cdots\gamma_n \in \Lambda_n(g)$, with properties

$$\gamma_\mu\gamma_* = (-1)^{n+1}\gamma_*\gamma_\mu, \quad \mu = 1, \ldots, n, \quad \gamma_*^2 = \varepsilon(g)I_g, \quad \varepsilon(g) := (-1)^q(-1)^{n(n-1)/2}. \tag{2}$$

3. The UV-divergent FA's (1) satisfy formally the *canonical* WI's (CWI's):

$$\begin{aligned}
&k_{1\mu}I^{(\gamma^\mu\gamma)\gamma_2\gamma_3}(m,k) = D_1^{\dot\gamma\gamma_2\gamma_3}(m,k) = \\
&\quad = (-1)^{\pi_1}P_1^{\gamma\gamma_2\gamma_3}(m,k) - P_3^{\gamma\gamma_2\gamma_3}(m,k) + (m_3 - (-1)^{\pi_1}m_1)\, I^{\gamma\gamma_2\gamma_3}(m,k),\\
&k_{2\alpha}I^{\gamma_1(\gamma^\alpha\gamma)\gamma_3}(m,k) = D_2^{\gamma_1\dot\gamma\gamma_3}(m,k) = \\
&\quad = (-1)^{\pi_2}P_2^{\gamma_1\gamma\gamma_3}(m,k) - P_1^{\gamma_1\gamma\gamma_3}(m,k) + (m_1 - (-1)^{\pi_2}m_2)\, I^{\gamma_1\gamma\gamma_3}(m,k),\\
&k_{3\beta}I^{\gamma_1\gamma_2(\gamma^\beta\gamma)}(m,k) = D_3^{\gamma_1\gamma_2\dot\gamma}(m,k) = \\
&\quad = (-1)^{\pi_3}P_3^{\gamma_1\gamma_2\gamma}(m,k) - P_2^{\gamma_1\gamma_2\gamma}(m,k) + (m_2 - (-1)^{\pi_3}m_3)\, I^{\gamma_1\gamma_2\gamma}(m,k).
\end{aligned} \tag{3}$$

Here the quantities $D_1^{\dot\gamma\gamma_2\gamma_3}(m,k)$, $D_2^{\gamma_1\dot\gamma\gamma_3}(m,k)$, $D_3^{\gamma_1\gamma_2\dot\gamma}(m,k)$, $P_l^{\gamma_1\gamma_2\gamma_3}(m,k)$ are similar to the main amplitude $I^{\gamma_1\gamma_2\gamma_3}$ and differ from it only in polynomials of the integrand:

$$\begin{aligned}
D_1^{\dot\gamma\gamma_2\gamma_3}(m,k) &\longleftrightarrow (p_3 - p_1)_\mu \operatorname{tr}[\gamma^\mu\gamma(m_1+\hat p_1)\gamma_2(m_2+\hat p_2)\gamma_3(m_3+\hat p_3)],\\
D_2^{\gamma_1\dot\gamma\gamma_3}(m,k) &\longleftrightarrow (p_1 - p_2)_\alpha \operatorname{tr}[\gamma_1(m_1+\hat p_1)\gamma^\alpha\gamma(m_2+\hat p_2)\gamma_3(m_3+\hat p_3)],\\
D_3^{\gamma_1\gamma_2\dot\gamma}(m,k) &\longleftrightarrow (p_2 - p_3)_\beta \operatorname{tr}[\gamma_1(m_1+\hat p_1)\gamma_2(m_2+\hat p_2)\gamma^\beta\gamma(m_3+\hat p_3)];
\end{aligned} \tag{4}$$

$$\begin{aligned}
P_1^{\gamma_1\gamma_2\gamma_3}(m,k) &\longleftrightarrow \operatorname{tr}[\gamma_1(m_1^2-p_1^2)\gamma_2(m_2+\hat p_2)\gamma_3(m_3+\hat p_3)],\\
P_2^{\gamma_1\gamma_2\gamma_3}(m,k) &\longleftrightarrow \operatorname{tr}[\gamma_1(m_1+\hat p_1)\gamma_2(m_2^2-p_2^2)\gamma_3(m_3+\hat p_3)],\\
P_3^{\gamma_1\gamma_2\gamma_3}(m,k) &\longleftrightarrow \operatorname{tr}[\gamma_1(m_1+\hat p_1)\gamma_2(m_2+\hat p_2)\gamma_3(m_3^2-p_3^2)].
\end{aligned} \tag{5}$$

In Eqs.(3) the vector CWI's ($\gamma = I_g$) and the axial-vector CWI's ($\gamma = \gamma^*$) are represented in the uniform manner. The factors $(-1)^{\pi_i}$ stem from the commutation relations $\gamma^\sigma\gamma = (-1)^\pi\gamma\gamma^\sigma$, $s = 1, \ldots, n$, and are equal: $(-1)^{\pi_i} = 1$ if $\gamma = I_g$, $\forall n$, or $\gamma = \gamma^*$, $n = 2r+1$; $(-1)^{\pi_i} = -1$ if $\gamma = \gamma^*$, $n = 2r$.

4. The *reduction identities* (RI's) is a name given to the obvious identities:

$$\begin{aligned}
&P_{l\epsilon}^{\gamma_1\gamma_2\gamma_3}(m,k) = \overline{P}_{l\epsilon}^{\gamma_1\gamma_2\gamma_3}(\overline{m}_{(l)},k), \qquad l = 1,2,3,\\
&\overline{m}_{(1)} \equiv (m_2,m_3), \qquad \overline{m}_{(2)} \equiv (m_1,m_3), \qquad \overline{m}_{(3)} \equiv (m_1,m_2),
\end{aligned} \tag{6}$$

in which we use the simple idea of cancelling the equal factors in factorized polynomials in numerators and the denominator of integrands. For example, for $l = 1$,

$$P_{1\epsilon}^{\gamma_1\gamma_2\gamma_3}(m,k) := \int\limits_{-\infty}^{\infty} (d^np)\ \delta(p,k)\frac{\operatorname{tr}\left[\gamma_1\left(m_1^2 - p_1^2 - i\epsilon_1\right)\gamma_2\left(m_2+\hat p_2\right)\gamma_3\left(m_3+\hat p_3\right)\right]}{\left(m_1^2-p_1^2-i\epsilon_1\right)\left(m_2^2-p_2^2-i\epsilon_2\right)\left(m_3^2-p_3^2-i\epsilon_3\right)}, \tag{7}$$

$$\overline{P}_{1\epsilon}^{\gamma_1\gamma_2\gamma_3}(m_2,m_3,k) := \int\limits_{-\infty}^{\infty} (d^np)\ \delta(p,k)\frac{\operatorname{tr}\left[\gamma_1\gamma_2\left(m_2+\hat p_2\right)\gamma_3\left(m_3+\hat p_3\right)\right]}{\left(m_2^2-p_2^2-i\epsilon_2\right)\left(m_3^2-p_3^2-i\epsilon_3\right)}. \tag{8}$$

The RI's (6) naturally induce primitive *daughter* RI's (DRI's) via decompositions involving: i) the Clifford tensors $\operatorname{tr}(\gamma_1\gamma_2 m_2\gamma_3 m_3)$, $\operatorname{tr}(\gamma_1\gamma_2 m_2\gamma_3\gamma_\sigma)$, $\operatorname{tr}(\gamma_1\gamma_2\gamma_\sigma\gamma_3 m_3)$, $\operatorname{tr}(\gamma_1\gamma_2\gamma_\sigma\gamma_3\gamma_\tau)$, for

$l = 1$; ii) the symmetric and antisymmetric parts $\frac{1}{2}(p_a^\sigma p_b^\tau \pm p_b^\sigma p_a^\tau)$ of the $p_a^\sigma p_b^\tau$; iii) the tensor structures 1, k_i^σ, $g^{\sigma\tau}$, $(k_i, k_j)^{\sigma\tau} := (k_i^\sigma k_j^\tau + k_j^\sigma k_i^\tau)$, $[\,k_i, k_j\,]^{\sigma\tau} := (k_i^\sigma k_j^\tau - k_j^\sigma k_i^\tau)$, with independent external momenta (e.g., k_2, k_3, or k_1, k_2, or k_1, k_3). There are 10 primitive DRI's, $\forall\, l = 1, 2, 3$. The difference between $P_l^{\gamma_1\gamma_2\gamma_3}(m,k)$ involved in Eqs.(3) and $P_{l\epsilon}^{\gamma_1\gamma_2\gamma_3}(m,k)$ results from $i\epsilon_l$-terms in polynomials of numerators.

5. The amplitude $I^{\gamma_1\gamma_2\gamma_3}(m,k)$ has the divergence index $\nu = n-3$, whereas the amplitudes $D_1^{\dot\gamma\gamma_2\gamma_3}(m,k)$, $D_2^{\gamma_1\dot\gamma\gamma_3}(m,k)$, $D_3^{\gamma_1\gamma_2\dot\gamma}(m,k)$, $P_l^{\gamma_1\gamma_2\gamma_3}(m,k)$, $P_{l\epsilon}^{\gamma_1\gamma_2\gamma_3}(m,k)$, $\overline{P}_{l\epsilon}^{\gamma_1\gamma_2\gamma_3}(\overline{m}_{(l)},k)$, $l = 1, 2, 3$, have the divergence index $\nu + 1 = n - 2$. The regular values for all of them are obtained according to [4] and are given in [6] in the most general form (for arbitrary Clifford structure of vertices and for n-dimensional space-time with the (q,p)-signature). It turns out that so calculated regular values satisfy the identities:

$$
\begin{aligned}
&k_{1\mu}(R^\nu I)^{(\gamma^\mu\gamma)\gamma_2\gamma_3}(m,k) = (R^{\nu+1}D_1)^{\dot\gamma\gamma_2\gamma_3}(m,k)\\
&\qquad = (-1)^{\pi_1}(R^{\nu+1}P_1)^{\gamma\gamma_2\gamma_3} - (R^{\nu+1}P_3)^{\gamma\gamma_2\gamma_3} + (m_3 - (-1)^{\pi_1}m_1)\,(R^{\nu+1}I)^{\gamma\gamma_2\gamma_3},\\
&k_{2\alpha}(R^\nu I)^{\gamma_1(\gamma^\alpha\gamma)\gamma_3}(m,k) = (R^{\nu+1}D_2)^{\gamma_1\dot\gamma\gamma_3}(m,k)\\
&\qquad = (-1)^{\pi_2}(R^{\nu+1}P_2)^{\gamma_1\gamma\gamma_3} - (R^{\nu+1}P_1)^{\gamma_1\gamma\gamma_3} + (m_1 - (-1)^{\pi_2}m_2)\,(R^{\nu+1}I)^{\gamma_1\gamma\gamma_3},\\
&k_{3\beta}(R^\nu I)^{\gamma_1\gamma_2(\gamma^\beta\gamma)}(m,k) = (R^{\nu+1}D_3)^{\gamma_1\gamma_2\dot\gamma}(m,k)\\
&\qquad = (-1)^{\pi_3}(R^{\nu+1}P_3)^{\gamma_1\gamma_2\gamma} - (R^{\nu+1}P_2)^{\gamma_1\gamma_2\gamma} + (m_2 - (-1)^{\pi_3}m_3)\,(R^{\nu+1}I)^{\gamma_1\gamma_2\gamma},
\end{aligned}
\tag{9}
$$

which are referred to as the *regular analog* (RA) of the CWI's [6]. It is important to note that the last terms in the identities (9) are calculated by the renormalization index $\nu + 1$, although their proper divergence index is ν. It is this peculiarity that permits the RA of the CWI's (9) both to imitate the CWI's (3) and to differ from them simultaneously. It is this peculiarity that permits to obtain some effective formulae for calculating of the *quantum corrections* (QC's) to the CWI's in the most general nonchiral case [6].

6. The primitive DRI's stemming from tensors $\mathrm{tr}(\gamma_1\gamma_2\gamma_\sigma\gamma_3\gamma_\tau)$, $\mathrm{tr}(\gamma_1\gamma_\sigma\gamma_2\gamma_3\gamma_\tau)$, $\mathrm{tr}(\gamma_1\gamma_\sigma\gamma_2\gamma_\tau\gamma_3)$, $\frac{1}{2}(p_a^\sigma p_b^\tau - p_b^\sigma p_a^\tau)$, and $\frac{1}{2}[k_2, k_3]^{\sigma\tau}$ are as follows:

$$
\left(R^{\nu+1}P_{l\epsilon[a,b]}\right)^{\sigma\tau}(m,k) = \left(R^{\nu+1}\overline{P}_{l\epsilon[a,b]}\right)^{\sigma\tau}(\overline{m}_{(l)},k), \qquad a, b \neq l, \quad a < b, \quad l = 1, 2, 3, \tag{10}
$$

$$
\left(R^{\nu+1}P_{l\epsilon[a,b]}\right)^{\sigma\tau}(m,k) = (2\pi)^n\delta(k)b(g)\mathrm{tr}(\cdot)\frac{1}{2}[k_2,k_3]^{\sigma\tau}(-1)^{l-1}\left(R^{\nu+1}P_{l\epsilon}^{[2,3]}\right)(m,k), \tag{11}
$$

$$
\begin{aligned}
\left(R^{\nu+1}P_{l\epsilon}^{[2,3]}\right)(m,k) := \int_{\Sigma^2}\frac{d\mu(\alpha)}{\Delta^{n/2}}\Big\{&\frac{\alpha_l}{\Delta}(m_l^2 - i\epsilon_l)\left(R^{\nu+1}\mathcal{F}\right)_{20} - \frac{\alpha_l}{\Delta}Y_l^2\left(R^{\nu+1}\mathcal{F}\right)_{40}\\
&+\left[\frac{\alpha_l}{\Delta}\left(\frac{n}{2}+1\right) - 1\right]\Delta^{-1}\left(R^{\nu+1}\mathcal{F}\right)_{41}\Big\} = 0.
\end{aligned}
\tag{12}
$$

The zero result in Eq.(12) is due to $\left(R^{\nu+1}\overline{P}_{l\epsilon[a,b]}\right)^{\sigma\tau}(\overline{m}_{(l)},k) = 0$, which in turn follows from the antisymmetry of the $\frac{1}{2}\left(p_a^\sigma p_b^\tau - p_b^\sigma p_a^\tau\right)$ and from the special external momentum dependence in this case (independent momenta are: k_3 or $k_1 + k_2$ for $l = 1$, etc.). Hereafter the integration measure is $d\mu(\alpha) := \delta\left(1 - \sum\limits_{l=1}^{3}\alpha_l\right)d\alpha_1\,d\alpha_2\,d\alpha_3$, the integration region is $\Sigma^2 := \left\{\alpha_l |\, \alpha_l \geq 0,\ l = 1,2,3,\ \sum\limits_{l=1}^{3}\alpha_l = 1\right\}$, overall δ-function is $\delta(k) := \delta(-k_1 - k_2 - k_3)$, and the metric dependent constant is $b(g) := \left(\pi^{n/2}i^p\right)/(2\pi)^n$, where p is the number of positive squares in the space-time metric g. The basic functions $(R^{\nu+1}\mathcal{F})_{sj}$ and the determining numbers ν^1_{sj}, λ^1_{sj}, and ω appearing in them are defined as:

$$\begin{aligned}
&\left(R^{\nu+1}\mathcal{F}\right)_{sj} := M_\epsilon^{\omega+j}\, Z_\epsilon^{1+\nu^1_{sj}}\Gamma(\lambda^1_{sj})/\Gamma(2+\nu^1_{sj})\, {}_2F_1\left(1,\lambda^1_{sj};\, 2+\nu^1_{sj};\, Z_\epsilon\right), \\
&\nu^1_{sj} := [(\nu+1-s)/2]+j, \qquad \lambda^1_{sj} := 1+\nu^1_{sj}-\omega-j, \qquad \omega := n/2-3.
\end{aligned} \tag{13}$$

The $[(\nu+1-s)/2]$ in Eqs.(13) is the integral part of the number $(\nu+1-s)/2$. The α-parametric functions Z_ϵ, M_ϵ, A, Δ, Y_l, involved in Eqs.(11)–(13) have the form:

$$\begin{aligned}
&Z_\epsilon := A/M_\epsilon, \qquad M_\epsilon := \sum_{l=1}^{3}\alpha_l(m_l^2-i\epsilon_l), \qquad \Delta := \alpha_1+\alpha_2+\alpha_3, \\
&A := \Delta^{-1}\left[\alpha_1(\alpha_2+\alpha_3)k_2^2+\alpha_3(\alpha_2+\alpha_1)k_3^2+2\alpha_1\alpha_3(k_2\cdot k_3)\right], \\
&Y_1 := \Delta^{-1}[(\alpha_2+\alpha_3)k_2+\alpha_3k_3], \qquad Y_2 := \Delta^{-1}[-\alpha_1k_2+\alpha_3k_3], \\
&Y_3 := \Delta^{-1}[-\alpha_1k_2-(\alpha_1+\alpha_2)k_3].
\end{aligned} \tag{14}$$

7. Now let us consider the AVV ($\gamma_1=\gamma^\mu\gamma^*$, $\gamma_2=\gamma^\alpha$, $\gamma_3=\gamma^\beta$) and the AAA ($\gamma_1=\gamma^\mu\gamma^*$, $\gamma_2=\gamma^\alpha\gamma^*$, $\gamma_3=\gamma^\beta\gamma^*$) spinor amplitudes for $n=4$ [7]. There is the relation

$$I^{\mu\alpha\beta(AAA)}(m_1,m_2,m_3,k) = \varepsilon(g)I^{\mu\alpha\beta(AVV)}(m_1,-m_2,m_3,k) \tag{15}$$

between them. Therefore, in the chiral case ($m_l=0$, $\forall\, l$) they may differ only by the sign.

Using Eqs.(12) and the compatibility relations $(R^\nu\mathcal{F})_{sj} = \left(R^{\nu+1}\mathcal{F}\right)_{s+1,j}$ one finds that the regular values of the main triangle amplitudes (1) after calculating nonzero traces have the followimg representation (here $\nu=1$ and $\omega=-1$):

$$\begin{aligned}
&(R^\nu I)^{\mu\alpha\beta(\cdots)}(m_1,m_2,m_3,k) = (2\pi)^4\delta(k)C^{(\cdots)}(g)\int_{\Sigma^2}\frac{d\mu(\alpha)}{\Delta^2} \\
&\quad\times\Big\{\varepsilon^{\mu\alpha\beta\tau}k_{2\tau}\,(R^\nu\mathcal{I}_1)^{(\cdots)}(m,\alpha,k)+\varepsilon^{\mu\alpha\beta\tau}k_{3\tau}\,(R^\nu\mathcal{I}_2)^{(\cdots)}(m,\alpha,k) \\
&\quad+\varepsilon^{\mu\alpha\sigma\tau}k_{2\sigma}k_{3\tau}(R^\nu\mathcal{I}_3)^\beta(m,\alpha,k)+\varepsilon^{\mu\beta\sigma\tau}k_{2\sigma}k_{3\tau}(R^\nu\mathcal{I}_4)^\alpha(m,\alpha,k)\Big\},
\end{aligned} \tag{16}$$

where integrands $(R^\nu\mathcal{I}_l)^{(\cdots)}(m,\alpha,k)$, etc., and constants $C^{(\cdots)}(g)$ are given as follows:

$$\begin{aligned}
&(R^\nu\mathcal{I}_1)^{(\cdots)} := -\left[\pm m_2m_3\frac{\alpha_2+\alpha_3}{\Delta}+(m_3\mp m_2)m_1\frac{\alpha_1}{\Delta}+\mu_1\frac{\alpha_1}{\Delta}\right](R^\nu\mathcal{F})_{10} \\
&\qquad+\left[k_2^2\frac{\alpha_1(\alpha_2+\alpha_3)}{\Delta^2}-k_3^2\frac{\alpha_3(\alpha_2+\alpha_1)}{\Delta^2}\right](R^\nu\mathcal{F})_{30}, \\
&(R^\nu\mathcal{I}_2)^{(\cdots)} := \left[\pm m_2m_1\frac{\alpha_2+\alpha_1}{\Delta}+(m_1\mp m_2)m_3\frac{\alpha_3}{\Delta}+\mu_3\frac{\alpha_3}{\Delta}\right](R^\nu\mathcal{F})_{10} \\
&\qquad+\left[k_2^2\frac{\alpha_1(\alpha_2+\alpha_3)}{\Delta^2}-k_3^2\frac{\alpha_3(\alpha_2+\alpha_1)}{\Delta^2}\right](R^\nu\mathcal{F})_{30}, \\
&(R^\nu\mathcal{I}_3)^\beta := -2\left[k_2^\beta\frac{\alpha_1\alpha_3}{\Delta^2}+k_3^\beta\frac{\alpha_3(\alpha_2+\alpha_1)}{\Delta^2}\right](R^\nu\mathcal{F})_{30}, \\
&(R^\nu\mathcal{I}_4)^\alpha := 2\left[k_2^\alpha\frac{\alpha_1(\alpha_2+\alpha_3)}{\Delta^2}+k_3^\alpha\frac{\alpha_1\alpha_3}{\Delta^2}\right](R^\nu\mathcal{F})_{30}, \qquad \mu_l := (m_l^2-i\epsilon_l),
\end{aligned} \tag{17}$$

$$C^{(AAA)}(g) := \varepsilon(g) C^{(AVV)}(g), \qquad C^{(AVV)}(g) := \varepsilon(g)\, \mathrm{tr}\,(I_g)\left(\pi^2 i^p\right)/(2\pi)^4. \tag{18}$$

The basic functions $(R^\nu \mathcal{F})_{sj}$, along with the determining numbers ν_{sj}, λ_{sj}, are defined as:

$$\begin{aligned} &(R^\nu \mathcal{F})_{sj} := M_\epsilon^{\omega+j}\, Z_\epsilon^{1+\nu_{sj}} \Gamma(\lambda_{sj})/\Gamma(2+\nu_{sj})\, {}_2F_1(1, \lambda_{sj};\, 2+\nu_{sj};\, Z_\epsilon), \\ &\nu_{sj} := [(\nu - s)/2] + j, \qquad \lambda_{sj} := 1 + \nu_{sj} - \omega - j, \qquad \omega := n/2 - 3. \end{aligned} \tag{19}$$

In Eqs.(16)–(17), the notation $(\cdots)$ means (AVV) or (AAA). Hereafter the upper signs in $(\pm)$ or $(\mp)$ correspond to the AVV-amplitudes while the lower ones to the AAA-amplitudes. The relation (15) and its chiral ($m_l = 0$, $\forall\, l$) form are obeyed at the regular level as well.

8. The first and second amplitudes in the first lines of Eqs.(9) take the form:

$$\begin{bmatrix} k_{1\mu}(R^\nu I)^{\mu\alpha\beta(\cdots)}(m,k) \\ k_{2\alpha}(R^\nu I)^{\mu\alpha\beta(\cdots)}(m,k) \\ k_{3\beta}(R^\nu I)^{\mu\alpha\beta(\cdots)}(m,k) \end{bmatrix} = (2\pi)^4 \delta(k) C^{(\cdots)}(g) \begin{bmatrix} \varepsilon^{\alpha\beta\sigma\tau} k_{2\sigma} k_{3\tau} \left(R^{\nu+1} D_1^{[2,3]}\right)^{(:..)} \\ \varepsilon^{\mu\beta\sigma\tau} k_{2\sigma} k_{3\tau} \left(R^{\nu+1} D_2^{[2,3]}\right)^{(.:.)} \\ \varepsilon^{\mu\alpha\sigma\tau} k_{2\sigma} k_{3\tau} \left(R^{\nu+1} D_3^{[2,3]}\right)^{(..:)} \end{bmatrix}, \tag{20}$$

$$\left(R^{\nu+1} D_l^{[2,3]}\right)^{(\cdots)}(m,k) := \int_{\Sigma^2} \frac{d\mu(\alpha)}{\Delta^2} \left(R^{\nu+1} \mathcal{D}_l^{[2,3]}\right)^{(\cdots)}(m,\alpha,k), \qquad l = 1,2,3, \tag{21}$$

$$\begin{aligned} &\left(R^{\nu+1}\mathcal{D}_1^{[2,3]}\right)^{(:..)}(m,\alpha,k) := \left[(m_3+m_1) m_{20}^{(:..)} + i\left(\epsilon_1 \frac{\alpha_1}{\Delta} + \epsilon_3 \frac{\alpha_3}{\Delta}\right)\right] \left(R^{\nu+1}\mathcal{F}\right)_{20}, \\ &\left(R^{\nu+1}\mathcal{D}_2^{[2,3]}\right)^{(.:.)}(m,\alpha,k) := \left[(m_1 \mp m_2) m_{20}^{(.:.)} - i\left(\epsilon_1 \frac{\alpha_1}{\Delta} + \epsilon_2 \frac{\alpha_2}{\Delta}\right)\right] \left(R^{\nu+1}\mathcal{F}\right)_{20}, \\ &\left(R^{\nu+1}\mathcal{D}_3^{[2,3]}\right)^{(..:)}(m,\alpha,k) := \left[(m_2 \mp m_3) m_{20}^{(..:)} + i\left(\epsilon_2 \frac{\alpha_2}{\Delta} + \epsilon_3 \frac{\alpha_3}{\Delta}\right)\right] \left(R^{\nu+1}\mathcal{F}\right)_{20}, \end{aligned} \tag{22}$$

$$\begin{aligned} &m_{20}^{(:..)}(m,\alpha) := -(\ m_1\alpha_1 \pm m_2\alpha_2 + m_3\alpha_3)\Delta^{-1}, \\ &m_{20}^{(.:.)}(m,\alpha) := -(-m_1\alpha_1 \pm m_2\alpha_2 + m_3\alpha_3)\Delta^{-1}, \\ &m_{20}^{(..:)}(m,\alpha) := -(\pm m_1\alpha_1 + m_2\alpha_2 \mp m_3\alpha_3)\Delta^{-1}. \end{aligned} \tag{23}$$

The notations: $(:..) := (\dot{\mathrm{A}}\mathrm{VV})$ or $(\dot{\mathrm{A}}\mathrm{AA})$, $(.:.) := (\mathrm{A}\dot{\mathrm{V}}\mathrm{V})$ or $(\mathrm{A}\dot{\mathrm{A}}\mathrm{A})$, $(..:) := (\mathrm{AV}\dot{\mathrm{V}})$ or $(\mathrm{AA}\dot{\mathrm{A}})$ are used in Eqs.(20)–(23) and further on.

9. The first and second amplitudes in the second lines of Eqs.(9) are as follows:

$$\begin{aligned} &\begin{bmatrix} \left(R^{\nu+1}P_1\right)^{\alpha\beta(:..)}(m,k) \\ \left(R^{\nu+1}P_3\right)^{\alpha\beta(:..)}(m,k) \end{bmatrix} = (2\pi)^4\delta(k) C^{(\cdots)}(g) \varepsilon^{\alpha\beta\sigma\tau} k_{2\sigma}k_{3\tau} \begin{bmatrix} -\left(R^{\nu+1}P_1^{[2,3]}\right) \\ -\left(R^{\nu+1}P_3^{[2,3]}\right) \end{bmatrix}, \\ &\begin{bmatrix} \left(R^{\nu+1}P_2\right)^{\mu\beta(.:.)}(m,k) \\ \left(R^{\nu+1}P_1\right)^{\mu\beta(.:.)}(m,k) \end{bmatrix} = (2\pi)^4\delta(k) C^{(\cdots)}(g) \varepsilon^{\mu\beta\sigma\tau} k_{2\sigma}k_{3\tau} \begin{bmatrix} \mp\left(R^{\nu+1}P_2^{[2,3]}\right) \\ \left(R^{\nu+1}P_1^{[2,3]}\right) \end{bmatrix}, \\ &\begin{bmatrix} \left(R^{\nu+1}P_3\right)^{\mu\alpha(..:)}(m,k) \\ \left(R^{\nu+1}P_2\right)^{\mu\alpha(..:)}(m,k) \end{bmatrix} = (2\pi)^4\delta(k) C^{(\cdots)}(g) \varepsilon^{\mu\alpha\sigma\tau} k_{2\sigma}k_{3\tau} \begin{bmatrix} \pm\left(R^{\nu+1}P_3^{[2,3]}\right) \\ -\left(R^{\nu+1}P_2^{[2,3]}\right) \end{bmatrix}. \end{aligned} \tag{24}$$

The $\left(R^{\nu+1}P_l^{[2,3]}\right)(m,k)$ in Eqs.(24) are almost the same as the $\left(R^{\nu+1}P_{l\epsilon}^{[2,3]}\right)(m,k)$ in Eq.(12) in which the $(m_l^2 - i\epsilon_l)$ must be replaced by the m_l^2 in the braces. Notice that Eq.(12) is the only nontrivial primitive DRI, $\forall\, l = 1,2,3$, in the AVV- and AAA-cases, $n = 4$. Taking into account the vanishing r.h.s. of Eq.(12), one obtains the important result:

$$\left(R^{\nu+1}P_l^{[2,3]}\right)(m,k) = \int_{\Sigma^2} \frac{d\mu(\alpha)}{\Delta^2} i\epsilon_l \frac{\alpha_l}{\Delta} \left(R^{\nu+1}\mathcal{F}\right)_{20}, \qquad l = 1,2,3. \tag{25}$$

Due to properties of the hypergeometric function ${}_2F_1$ it follows (for $l = 1,2,3$) that

$$\left(R^{\nu+1}P_l^{[2,3]}\right)(m,k) = \begin{cases} 0, & \text{if } (\epsilon_s \to 0,\ m_s \neq 0 \text{ or } m_s = m \to 0,\ \forall\, s); \\ 1/6, & \text{if } (m_s \to 0,\ \epsilon_s = \epsilon \to 0,\ \forall\, s). \end{cases} \tag{26}$$

10. The third amplitudes in the second lines of Eqs.(9) calculated by the renormalization index $\nu + 1 = 2$ are as follows:

$$\begin{bmatrix} \left(R^{\nu+1}I\right)^{\alpha\beta(:..)}(m,k) \\ \left(R^{\nu+1}I\right)^{\mu\beta(.:.)}(m,k) \\ \left(R^{\nu+1}I\right)^{\mu\alpha(..:)}(m,k) \end{bmatrix} = (2\pi)^4\delta(k)C^{(\cdots)}(g) \begin{bmatrix} \varepsilon^{\alpha\beta\sigma\tau}k_{2\sigma}k_{3\tau}\left(R^{\nu+1}I^{[2,3]}\right)^{(:..)} \\ \varepsilon^{\mu\beta\sigma\tau}k_{2\sigma}k_{3\tau}\left(R^{\nu+1}I^{[2,3]}\right)^{(.:.)} \\ \varepsilon^{\mu\alpha\sigma\tau}k_{2\sigma}k_{3\tau}\left(R^{\nu+1}I^{[2,3]}\right)^{(..:)} \end{bmatrix}, \tag{27}$$

where

$$\left(R^{\nu+1}I^{[2,3]}\right)^{(:..)}(m,k) := \int_{\Sigma^2} \frac{d\mu(\alpha)}{\Delta^2} m_{20}^{(:..)}(m,\alpha)\left(R^{\nu+1}\mathcal{F}\right)_{20}, \qquad \text{etc.}, \tag{28}$$

and the quantities $m_{20}^{(:..)}(m,\alpha)$, etc., are defined in Eq.(23).

11. As a result the regular analogs of the CWI's (9) take the form:

$$\begin{aligned} \left(R^{\nu+1}D_1^{[2,3]}\right)^{(:..)} &= \left(R^{\nu+1}P_1^{[2,3]}\right) + \left(R^{\nu+1}P_3^{[2,3]}\right) + (m_3 + m_1)\left(R^{\nu+1}I^{[2,3]}\right)^{(:..)}, \\ \left(R^{\nu+1}D_2^{[2,3]}\right)^{(.:.)} &= -\left(R^{\nu+1}P_2^{[2,3]}\right) - \left(R^{\nu+1}P_1^{[2,3]}\right) + (m_1 \mp m_2)\left(R^{\nu+1}I^{[2,3]}\right)^{(.:.)}, \\ \left(R^{\nu+1}D_3^{[2,3]}\right)^{(..:)} &= \left(R^{\nu+1}P_3^{[2,3]}\right) + \left(R^{\nu+1}P_2^{[2,3]}\right) + (m_2 \mp m_3)\left(R^{\nu+1}I^{[2,3]}\right)^{(..:)}. \end{aligned} \tag{29}$$

Limiting values of quantities in Eqs.(29) depend strongly on the limit employed.

12. Let us first consider a *nonchiral* case. Here, due to Eqs.(25)–(26), the r.h.s. of Eqs.(24) and terms in Eqs.(20)–(22) containing ϵ_l are zero for $\epsilon_l \to 0$, $l = 1,2,3$. The *quantum corrections* (anomalous contributions) to the CWI's appear as

$$\begin{bmatrix} a^{\alpha\beta(:..)}(m,k) \\ a^{\mu\beta(.:.)}(m,k) \\ a^{\mu\alpha(..:)}(m,k) \end{bmatrix} = (2\pi)^4\delta(k)C^{(\cdots)}(g) \begin{bmatrix} \varepsilon^{\alpha\beta\sigma\tau}k_{2\sigma}k_{3\tau}a^{(:..)}(m_1,m_2,m_3) \\ \varepsilon^{\mu\beta\sigma\tau}k_{2\sigma}k_{3\tau}a^{(.:.)}(m_1,m_2,m_3) \\ \varepsilon^{\mu\alpha\sigma\tau}k_{2\sigma}k_{3\tau}a^{(..:)}(m_1,m_2,m_3) \end{bmatrix}, \tag{30}$$

where the mass functions $a^{(\cdots)}(m_1,m_2,m_3)$ have the integral representation:

$$a^{(:..)}(m_1,m_2,m_3) := \int_{\Sigma^2} \frac{d\mu(\alpha)}{\Delta^2} m_{20}^{(:..)}(m,\alpha)\left[(R^{\nu+1}\mathcal{F})_{20} - (R^{\nu}\mathcal{F})_{20}\right], \qquad \text{etc.}, \tag{31}$$

$$\left[(R^{\nu+1}\mathcal{F})_{20} - (R^{\nu}\mathcal{F})_{20}\right] = -M_\epsilon^{-1}, \quad \text{as for } n = 4, \qquad \nu_{20} = -1, \quad \lambda_{20} = 1, \quad \omega = -1;$$

for a *nonchiral nondegenerate* case they are expressed in terms of the Appel hypergeometric functions F_1 of two variables (e.g., $x := m_1/m_2$, $y := m_3/m_2$ if $m_2 \neq 0$) [6, 7]:

$$\begin{aligned}
a^{(\dot{\cdot}\cdot\cdot)}(m_1, m_2, m_3) &= \frac{y+x}{6}\Big[xF_1\left(1,2,1;4;1-x^2,1-y^2\right) \\
&\quad \pm F_1\left(1,1,1;4;1-x^2,1-y^2\right) + yF_1\left(1,1,2;4;1-x^2,1-y^2\right)\Big], \\
a^{(\cdot\dot{\cdot}\cdot)}(m_1, m_2, m_3) &= \frac{x\mp 1}{6}\Big[-xF_1\left(1,2,1;4;1-x^2,1-y^2\right) \\
&\quad \pm F_1\left(1,1,1;4;1-x^2,1-y^2\right) + yF_1\left(1,1,2;4;1-x^2,1-y^2\right)\Big], \\
a^{(\cdot\cdot\dot{\cdot})}(m_1, m_2, m_3) &= \frac{1\mp y}{6}\Big[\pm xF_1\left(1,2,1;4;1-x^2,1-y^2\right) \\
&\quad + F_1\left(1,1,1;4;1-x^2,1-y^2\right) \mp yF_1\left(1,1,2;4;1-x^2,1-y^2\right)\Big].
\end{aligned} \tag{32}$$

This confirms the Frampton's conjecture [8] about a possibility of a mass dependence of the axial-vector anomaly. But the nature of such a dependence revealed here is strongly different from the Frampton's one. Actually it is closely tied with a mass spectrum of fermions and with flavor current structures producing non-conserved vector currents. The Frampton's mechanism appeals to prorerties of the dimensional regularization.

For the *degenerate nonchiral* case ($m_1 = m_2 = m_3 \equiv m \neq 0$), the Eqs.(32) display the famous mass-independent Adler–Bell–Jackiw result [1, 2]:

$$\begin{aligned}
&a^{(\dot{A}VV)}(m,m,m) = 1, \qquad a^{(A\dot{V}V)}(m,m,m) = a^{(AV\dot{V})}(m,m,m) = 0, \\
&a^{(\dot{A}AA)}(m,m,m) = -a^{(A\dot{A}A)}(m,m,m) = a^{(AA\dot{A})}(m,m,m) = 1/3,
\end{aligned} \tag{33}$$

about the axial-vector anomaly (trivial QC's to the CWI's in our terminology).

13. Now we turn to the *chiral* behaviour. Let us consider two ways leading to the chiral state in renormalized amplitudes at hand: (i) the (ϵ, m)-limit, when first $\epsilon_l \to 0$ and then $m_l = m \to 0$, $l = 1,2,3$; (ii) the (m,ϵ)-limit, when first $m_l \to 0$ and then $\epsilon_l = \epsilon \to 0$, $l = 1,2,3$. In the (ϵ, m)-limit, all the amplitudes for AVV- and AAA-cases *inherit* the behaviour of those in the degenerate nonchiral case considered in [6]; the QC's to CWI's are the same as in Eqs.(33). In the (m,ϵ)-limit all amplitudes for the AVV- and AAA-cases *coincide with each other* (apart from the factor $\varepsilon(g) = (-1)^q$ of course). Here the QC's to CWI's are caused by the nonzero contributions of the amplitudes $\left(R^{\nu+1}P_l^{[2,3]}\right)^{(\cdots)}(m,k)$. The results are summarized in Table 1.

Thus, the chiral limit ($m \to 0$) and the chiral case ($m = 0$) are *equivalent* for the AAA-amplitude and *differ* for the AVV-amplitude. This reflects the different kind of *discrete symmetries* (DS) of these amplitudes for $m \neq 0$ and $m = 0$. The AAA-amplitude has the DS of equilaterial triangle both for $m \neq 0$ and for $m = 0$, in contrast to the AVV-amplitude having the DS of isosceles triangle for $m \neq 0$ which at $m = 0$ enlarges abruptly to the DS of equilaterial triangle.

14. For the complex Clifford algebra $Cl(g)_{\mathbf{C}}$, the matrix γ_* in Eq.(2) may be always redefined as $\gamma_* := i^{(1-\varepsilon(g))/2}\gamma_1\gamma_2\cdots\gamma_n$ and, hence, $\gamma_*^2 = I_g$. Therefore, from the Table 1 it follows that the QC's to "left-handed" WI's are zero in the chiral case. This may give some insight into why just the left-handed neutrino exists in Nature. This also requires a revision of the conventional viewpoint about an impact of anomalies on the renormalizability of unified field theories in which gauge fields are coupled to left-handed fermions.

The presence of a mass spectrum of constituent fermions in general QC's (see Eqs.(30)–(32)) increases the predictive power of formulas (which involves the axial-vector anomaly) widely used in the low energy phenomenological physics, e.g., for describing the elementary particle decays [1, 2].

Table 1. The chiral behaviour of amplitudes appearing in the regular analogs of the CWI's (29) for AVV- and AAA-cases, $n = 4$; here $\left(R^{\nu+1}P_0^{[2,3]}\right) \equiv \left(R^{\nu+1}P_3^{[2,3]}\right)$.

Feynman amplitudes	$(\dot{\mathrm{A}}\mathrm{VV})$	$(\mathrm{A}\dot{\mathrm{V}}\mathrm{V})$	$(\mathrm{AV}\dot{\mathrm{V}})$	$(\dot{\mathrm{A}}\mathrm{AA})$	$(\mathrm{A}\dot{\mathrm{A}}\mathrm{A})$	$(\mathrm{AA}\dot{\mathrm{A}})$
$\underline{(\epsilon, m)-\lim \equiv \text{chiral limit:}}$						
$\left(R^{\nu+1}D_i^{[2,3]}\right)^{(\cdots)}(m,k) =$	1	0	0	1/3	−1/3	1/3
$= (-1)^{i-1}\Big[\left(R^{\nu+1}P_i^{[2,3]}\right)^{(\cdots)}(m,k) + \left(R^{\nu+1}P_{i-1}^{[2,3]}\right)^{(\cdots)}(m,k)\Big] +$	0	0	0	0	0	0
$+\begin{bmatrix} (m_3+m_1) \\ (m_1 \mp m_2) \\ (m_2 \mp m_3) \end{bmatrix} (R^{\nu+1}I^{[2,3]})^{(\cdots)}(m,k)$	1	0	0	1/3	−1/3	1/3
$\underline{(m, \epsilon)-\lim \equiv \text{chiral case:}}$						
$\left(R^{\nu+1}D_i^{[2,3]}\right)^{(\cdots)}(m,k) =$	1/3	−1/3	1/3	1/3	−1/3	1/3
$= (-1)^{i-1}\Big[\left(R^{\nu+1}P_i^{[2,3]}\right)^{(\cdots)}(m,k) + \left(R^{\nu+1}P_{i-1}^{[2,3]}\right)^{(\cdots)}(m,k)\Big] +$	1/3	−1/3	1/3	1/3	−1/3	1/3
$+\begin{bmatrix} (m_3+m_1) \\ (m_1 \mp m_2) \\ (m_2 \mp m_3) \end{bmatrix} (R^{\nu+1}I^{[2,3]})^{(\cdots)}(m,k)$	0	0	0	0	0	0

Acknowledgements

The author is sincerely grateful to the colleagues of the Department of the Mathematical Methods in Theoretical Physics for valuable discussions, comments, and stimulating criticism. This research was supported by the Ukrainian State Foundation of Fundamental Researches Grant No. 1.4/206.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 501–506.

Quantum Mechanism of Generation of the $SU(N)$ Gauge Fields

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A generation mechanism for non-Abelian gauge fields in the $SU(N)$ gauge theory is studied. We show that $SU(N)$ gauge fields ensuring the local invariance of the theory are generated at the quantum level only. It is demonstrated that the generation of these fields is related to nonsmoothness of the scalar phases of the fundamental spinor fields, but not to the simple requirement of gauge symmetry locality. The expressions for the gauge fields are obtained in terms of the nonsmooth scalar phases. From the viewpoint of the described scheme of the gauge field generation, the gauge principle is an "automatic" consequence of field trajectory nonsmoothness in Feynman path integral.

All known fundamental interactions possess the property of local gauge invariance. The principle central to quantum field theory is the gauge principle. This principle stakes that the fundamental fields involved in Lagrangian allow the local transformations which do not modify Lagrangian. The gauge principle was first used by Weyl [1] who discovered the local $U(1)$ gauge symmetry in quantum electrodynamics. The non-Abelian local $SU(2)$ gauge symmetry and corresponding gauge fields were introduced by Yang and Mills [2]. Based on this approach, later on the structure of weak and strong interactions was established [3, 4]. Einstein's General Relativity can also be considered as the gauge theory with Lorentz or Poincaré gauge groups [5, 6].

It is generally agreed that the existence of gauge fields must necessarily be a consequence of the requirement of the gauge symmetry locality. However, this statement is not quite correct. Ogievetski and Polubarinov [7] showed that within the framework of classical field theory, the local gauge invariance can be ensured without introduction of nontrivial gauge fields, i.e., vector fields with nonzero field strengths. It suffices to introduce only gradient vector field $\partial_\mu B(x)$, as a "compensative field", with zero strength $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)B(x) = 0$. Such field does not contribute to dynamics [7]. From the viewpoint of the classification of fields by spin, the scalar field $B(x)$ corresponds to spin of zero and gradient vector field $\partial_\mu B(x)$ is longitudinal. True vector gauge fields A_μ are transversal fields corresponding to spin of unity. Gauge invariance of theory means that the longitudinal part of vector gauge fields does not contribute to dynamics.

If so, what is the real cause of the existence of gauge fields and interactions? Early in Ref. [8] the "quantum gauge principle" was formulated in the context of quantum electrodynamics. This principle states that the Abelian $U(1)$ gauge fields are generated at the quantum level only and the generation of these fields is related to nonsmoothness of the field trajectories in the Feynman path integrals, by which the field quantization is determined. In this paper, we investigate the generation mechanism for non-Abelian $SU(N)$ gauge fields. It is shown that the non-Abelian nontrivial vector fields are generated due to nonsmoothness of the field trajectories for the scalar phases of the spinor fields in the $SU(N)$ gauge theory.

Let us consider a Lagrangian for free spinor fields

$$L = i\overline{\psi}^j \gamma^\mu \partial_\mu \psi^j - m\overline{\psi}^j \psi^j, \tag{1}$$

where $j = 1, 2, \ldots, N$. In what follows the index j will be omitted.

The Lagrangian (1) is invariant under global non-Abelian $SU(N)$-transformations

$$\psi'(x) = e^{it^a\omega_a}\psi(x), \qquad \overline{\psi}\,'(x) = \overline{\psi}(x)e^{-it^a\omega_a}, \tag{2}$$

where t^a are $SU(N)$ group generators, $\omega_a = \text{const}$, $a = 1, 2, \ldots, N^2-1$. This invariance generates the conserved currents J^μ_a:

$$J^\mu_a = -\overline{\psi}\gamma^\mu t_a\psi, \qquad \partial_\mu J^\mu_a = 0. \tag{3}$$

In the framework of classical field theory, physical fields are known to be described by sufficiently smooth functions. Considering a smooth local infinitesimal $SU(N)$-transformation at the classical level

$$\psi'(x) = (I + it^a\omega_a(x))\psi(x), \qquad \overline{\psi}\,'(x) = \overline{\psi}(x)(I - it^a\omega_a(x)), \tag{4}$$

we obtain that the transformed Lagrangian differs from the original one by the term:

$$\triangle L = J^\mu_a\partial_\mu\omega_a(x). \tag{5}$$

In consequence of the conservation of currents (3) the term (5) reduces to 4-divergence and does not contribute to dynamics. In the case of local non-infinitesimal $SU(N)$-transformations, it was shown [7] that the local gauge invariance of the transformed Lagrangian can be ensured by introducing scalar fields $B_a(x)$. In other words, the Lagrangian

$$L = i\overline{\psi}\gamma^\mu\partial_\mu\psi + i\overline{\psi}(x)e^{-it^aB_a(x)}\gamma^\mu(\partial_\mu e^{it^bB_b(x)})\psi(x) - m\overline{\psi}\psi$$

is invariant under the local non-infinitesimal $SU(N)$-transformations provided the fields $B_a(x)$ transform as:

$$e^{it^aB'_a(x)} = e^{it^aB_a(x)}e^{-it^b\omega_b(x)}.$$

The introduced scalar fields $B_a(x)$ do not contribute to dynamics, since they do not give rise to nonzero strengths and can be eliminated by means of the smooth point transformations of the field variables $\psi \to \exp(it^aB_a)\psi$ [7]. Thus we need not compensate the term (5) by introducing nontrivial vector fields A^a_μ that do not reduce to gradients of scalar functions.

The situation changes in the quantum approach. In the Feynman formulation of quantum field theory the transition amplitudes are expressed by the path integrals that are centered on nonsmooth field trajectories [9]:

$$\langle\Phi_2, t_2|\Phi_1, t_1\rangle = N\int\limits_{\Phi_1}^{\Phi_2}(D\Phi)\exp\left[\frac{i}{\hbar}\int\limits_{t_1}^{t_2}d^4xL(\Phi, \partial\Phi)\right].$$

In this context the Lagrangian (1) and its symmetries are determined on the class of nonsmooth functions $\psi(x)$, corresponding to nonsmooth trajectories in path integrals. In the strict sense, the derivatives involved in the Lagrangian (1) are discontinuous functions. From physics standpoint, field trajectory nonsmoothness is related to fluctuations of the local fields. Feynman integrals, as a rule, are additionally specified by the implicit switch to “smoothed-out” approximations [10]. In this case the degrees of freedom corresponding to gauge vector fields are lost. Here we show that, as in quantum electrodynamics [8], in the non-Abelian $SU(N)$ gauge theory these degrees of freedom can be explicitly taken into account when “smoothing” of nonsmooth fields is more carefully carried out.

Let us approximate nonsmooth functions $\theta^a(x)$ by smooth functions $\omega^a(x)$:

$$\theta^a(x) = \omega^a(x) + \cdots .$$

In order to write down the next term of the "smoothed-out" representation of the nonsmooth functions $\theta^a(x)$ it is necessary to consider the behaviour of the first derivatives of $\theta^a(x)$. The derivatives $\partial_\mu \theta^a(x)$ at nonsmoothness points of $\theta^a(x)$ are discontinuous functions. Since the derivatives $\partial_\mu \omega^a(x)$ are continuous functions, they badly approximate the behaviour of the derivatives of the "smoothed-out" $\theta^a(x)$. Let us denote a difference between them by $\theta^a_\mu(x)$ and write $\partial_\mu \theta^a(x)$ as follows:

$$\partial_\mu \theta^a(x) = \partial_\mu \omega^a(x) + \theta^a_\mu(x). \tag{6}$$

Since the nonsmooth fields $\theta^a_\mu(x)$ do not reduce to gradients of smooth scalar fields, they are the nontrivial vector fields that give rise to nonzero field strengths:

$$\partial_\mu \theta^a_\nu(x) - \partial_\nu \theta^a_\mu(x) \neq 0.$$

Therefore the fields $\partial_\mu \theta^a(x)$ involve the additional degrees of freedom which are related to nonsmoothness of the $\theta^a(x)$. It should be noted that the fields $\theta^a_\mu(x)$ are ambiguously determined due to ambiguity of choice of $\omega^a(x)$.

Let us now consider $\theta^a(x)$ as scalar phases of the spinor fields $\psi(x)$ realizing the fundamental representation of the $SU(N)$ gauge group and separate out these phase degrees of freedom in an explicit form:

$$\psi(x) = e^{it^a \theta_a(x)} \psi_0(x), \tag{7}$$

where the spinor fields ψ_0 are representatives of the class of gauge-equivalent fields [11], $e^{it^a\theta_a}$ is a unitary $N \times N$ matrix. Then, provided the Lagrangian (1) is determined on the class of nonsmooth functions $\psi(x)$, using Eq.(7) we obtain:

$$L = i\overline{\psi}_0 \gamma^\mu \partial_\mu \psi_0 + i\overline{\psi}_0 e^{-it^a\theta_a} \gamma^\mu (\partial_\mu e^{it^b\theta_b}) \psi_0 - m\overline{\psi}_0 \psi_0. \tag{8}$$

Represent the matrix $e^{it^a\theta_a}$ as a superposition of the unit matrix I and $SU(N)$ group generators t^a:

$$e^{it^a\theta_a} = CI + iS_a t^a. \tag{9}$$

Since t^a are traceless matrices normalized by $\mathrm{Tr}\,(t^a t^b) = \frac{1}{2}\delta^{ab}$, the coefficients C and S_a in Eq.(9) are given by:

$$C = \frac{1}{N}\,\mathrm{Tr}(e^{it^a\theta_a}), \qquad S_a = -2i\,\mathrm{Tr}(t^a e^{it^b\theta_b}). \tag{10}$$

It is easy to verify that $\mathrm{Tr}(e^{-it^a\theta_a}\partial_\mu e^{it^b\theta_b}) = 0$. Then taking into account the commutation rules for $SU(N)$ group generators [12] we can write down:

$$e^{-it^a\theta_a}\partial_\mu e^{it^b\theta_b} = it^a A^a_\mu, \tag{11}$$

$$A^a_\mu = \bar{C}\partial_\mu S^a - \bar{S}^a \partial_\mu C + (f^{abc} - id^{abc})\bar{S}_b \partial_\mu S_c, \tag{12}$$

where d_{abc} (f_{abc}) are totally symmetric (antisymmetric) structural constants of $SU(N)$-group, the overline denotes complex conjugation.

Since the matrix $e^{it^a\theta_a}$ is unitary, the following equation is valid:

$$\bar{C}S_a - \bar{S}_a C + (f_{abc} - id_{abc})\bar{S}^b S^c = 0. \tag{13}$$

Differentiating the left and right sides of Eq.(13) and using the property of antisymmetry of f_{abc} we derive:

$$A^a_\mu - \bar{A}^a_\mu = 0,$$

whence it follows that the expression (12) is a real function. Thus A^a_μ can be identified with the gauge fields. Unlike the gauge field in electrodynamics [6], these fields are nonlinear functions of $\theta^a(x)$. As a consequence of nonsmoothness of the phases $\theta^a(x)$ the fields A^a_μ are also not smooth. If we take into account only the first term in the right hand side of relation (6) we obtain that the fields A^a_μ do not contribute to the dynamics, as in classical field theory [5], and the degrees of freedom corresponding to gauge vector fields are lost. The account of $\theta^a_\mu(x)$ enables us to interpret the fields A^a_μ as nontrivial vector fields that give rise to nonzero field strenghths:

$$\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) \neq 0.$$

By the way of illustration let us consider the Yang–Mills $SU(2)$ gauge group. In consequence of anti-commutativity of the $SU(2)$ group generators the coefficients C and S_a (see Eq.(10)) are given by:

$$C = \cos(\theta/2), \qquad S_a = 2n_a \sin(\theta/2), \tag{14}$$

where

$$\theta = \sqrt{\theta_a\theta^a}, \qquad n_a = \theta_a/\theta, \qquad a = 1, 2, 3. \tag{15}$$

From Eqs.(14) and (15) it follows that the gauge fields A^a_μ can be written as:

$$A^a_\mu = n^a\partial_\mu\theta + \sin\theta(\partial_\mu n^a) + \sin^2(\theta/2)[\mathbf{n} \times \partial_\mu\mathbf{n}]^a. \tag{16}$$

Expression (16) demonstrates explicitly the relation between the Yang–Mills gauge fields and the nonsmooth scalar phases of the spinor fields.

Let us obtain the transformation law for the vector fields (12). For this purpose we consider the infinitesimal smooth local transformations for the spinor fields:

$$\psi'_0(x) = e^{it^a\omega_a(x)}\psi_0(x), \qquad \overline{\psi}'_0(x) = \overline{\psi}_0(x)e^{-it^a\omega_a(x)}. \tag{17}$$

Then the Lagrangian (8) can be written as:

$$L = i\overline{\psi}'_0\gamma^\mu\partial_\mu\psi'_0 + i\overline{\psi}'_0 e^{it^a\omega_a}e^{-it^b\theta_b}\gamma^\mu\partial_\mu(e^{it^c\theta_c}e^{-it^l\omega_l})\psi'_0 - m\overline{\psi}'_0\psi'_0. \tag{18}$$

Defining the gauge fields $A^{a\prime}_\mu(x)$ similarly to Eqs.(11) and (12) by the following equation:

$$it_a A^{a\prime}_\mu(x) = e^{it^a\omega_a}e^{-it^b\theta_b}\partial_\mu(e^{it^c\theta_c}e^{-it^l\omega_l}), \tag{19}$$

we find that the transformed gauge fields $A^{a\prime}_\mu(x)$ are related to the fields (12) as follows:

$$A^{a\prime}_\mu(x) = A^a_\mu(x) - \partial_\mu\omega^a(x) - f_{abc}\omega^b(x)A^c_\mu(x). \tag{20}$$

Hence, in the framework of considered scheme of the gauge field generation we derive the usual transformation law for the $SU(N)$ gauge fields, with the local gauge invariance of the Lagrangian (8) being not necessary.

Using Eqs.(11) and (12) we obtain that the Lagrangian (8) takes the form:

$$L = i\overline{\psi}_0\gamma^\mu \hat{D}_\mu \psi_0 - m\overline{\psi}_0\psi_0, \tag{21}$$

where $\hat{D}_\mu \equiv \partial_\mu + iA^a_\mu t_a$ is the covariant derivative. It is easy to verify that the Lagrangian (21) is invariant under the transformations (17) and (20).

Therefore the gauge fields A^a_μ ensuring the local $SU(N)$ gauge invariance of the Lagrangian (21) are generated because of nonsmoothness of the field trajectories in Feynman path integral. The nonsmoothness of the fields A^a_μ corresponds to their quantum nature and means that these fields should also be quantized, i.e., continual integration is to be carried out over the variables $A^a_\mu(x)$. However the fields A^a_μ in the Lagrangian (21) do not exhibit all the properties of physical fields since they cannot propagate in space because of the absence of the kinetic term.

An expression similar to the kinetic term can be obtained by the calculation of the effective action for the spinor fields described by the Lagrangian (21). Using the results of the calculations performed in Ref.[13], we find the following expression for the kinetic term in the one-loop approximation

$$L_{\text{eff}} = \kappa \ln \frac{\Lambda}{\mu_0} \operatorname{tr} \hat{F}^2_{\mu\nu}, \qquad \hat{F}_{\mu\nu} = [\hat{D}_\mu, \hat{D}_\nu], \tag{22}$$

where Λ and μ_0 are the momentum of the ultraviolet and infrared cut-off respectively; κ is the numerical coefficient.

The formula (22) takes the usual form [12]

$$L_{\text{eff}} = \frac{\hbar c}{8g^2} \operatorname{tr} F^2_{\mu\nu}$$

upon identifying

$$g^2 = \frac{\hbar c}{8\kappa \ln \frac{\Lambda}{\mu_0}}. \tag{23}$$

The last equation relates the charge g with the parameters Λ and μ_0 as well as with the universal constants $\hbar$ and c, and thus demonstrates explicitly quantum origin of the charge.

Let us discuss the results obtained. We show that the “compensating” gauge fields need not be artificially introduced for the local gauge invariance of the theory to be ensured. As a result of conservation of currents (3), the Lagrangian for classical spinor fields is invariant under local $SU(N)$ gauge transformations. The generation of gauge fields is purely quantum phenomenon. The vector gauge fields are generated through nonsmoothness of the scalar phases of the fundamental spinor fields. From the viewpoint of the described scheme of the gauge field generation, the gauge principle is an “automatic” consequence of field trajectory nonsmoothness in Feynman path integral.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 507–509.

Symmetric Properties of First-Order Equations of Motion for $N=2$ Super Yang–Mills Theory

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We find 4-parameter non-hermitean $N=1$ transformations, under which the first-order equations of motion for $N=2$ supersymmetric Yang–Mills (YM) theory in $N=1$ superspace are invariant.

One of the essential features of non-Abelian gauge theories is the existence of classical solutions with non-trivial topological properties (monopoles, instantons). Their importance encourages mathematical and physical community to further investigations in this domain. But searching the solutions of YM equations is very difficult task because of their non-linearity. As usual, to gain the goal one tries by finding the solutions of simpler equations, for example, first-order equations, which satisfy the second-order equations of motion. Thus the problem of finding corresponding first-order equations arises.

In pure YM theory they usually deal with self-duality equation. In YM theories with scalar fields the first-order equations are mainly generalizations of self-duality equation. One of such generalizations, *quasi-self-duality equation,* was introduced by V.A. Yatsun [1]. He proposed the additional term in the equation of self-duality have to be added, which is properly chosen combination of scalar fields. In the case of vanishing scalar fields the quasi-self-duality equation boils down to self-duality equation. The quasi-self-duality equation together with constraints on scalar fields form *the system of quasi-self-duality equations.*

In this report first we deal with $N=2$ YM theory in $N=1$ superspace with the Lagrangian [2]:

$$\begin{aligned} L = \mathrm{Tr}\biggl\{ &-\frac{1}{4}F_{mn}F^{mn} - i\bar\lambda\bar\sigma^m\mathcal{D}_m\lambda - \frac{1}{2}(\mathcal{D}_mA)^2 - \frac{1}{2}(\mathcal{D}_mB)^2 - i\bar\psi\bar\sigma^m\mathcal{D}_m\psi \\ &+ ig(A+iB)\{\lambda^\alpha,\psi_\alpha\} + ig(A-iB)\{\bar\lambda_{\dot\alpha},\bar\psi^{\dot\alpha}\} + igD[A,B] + \frac{1}{2}D^2 + \frac{1}{2}F^2 + \frac{1}{2}G^2\biggr\}. \end{aligned} \tag{1}$$

The theory (1) is invariant under $N=1$ supersymmetry transformations:

$$\begin{aligned} &\delta_\xi(A-iB) = 2\xi\psi, \qquad \delta_\xi(A+iB) = 2\bar\xi\bar\psi, \qquad \delta_\xi V_{\alpha\dot\alpha} = -2i(\xi_\alpha\bar\lambda_{\dot\alpha} + \bar\xi_{\dot\alpha}\lambda_\alpha), \\ &\delta_\xi D = -\xi^\alpha\mathcal{D}_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha} + \bar\xi_{\dot\alpha}\mathcal{D}^{\alpha\dot\alpha}\lambda_\alpha, \qquad \delta_\xi(F+iG) = 2i\bar\xi_{\dot\beta}\left(\mathcal{D}^{\alpha\dot\beta}\psi_\alpha + g[\bar\lambda^{\dot\beta}, A-iB]\right), \\ &\delta_\xi(F-iG) = 2i\xi^\alpha\left(\mathcal{D}_{\alpha\dot\beta}\bar\psi^{\dot\beta} + g[\lambda_\alpha, A+iB]\right), \qquad \delta_\xi\lambda_\alpha = \frac{1}{2}\xi^\beta(f_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}D), \\ &\delta_\xi\bar\lambda_{\dot\alpha} = \frac{1}{2}\bar\xi^{\dot\beta}(f_{\dot\alpha\dot\beta} - 2i\varepsilon_{\dot\alpha\dot\beta}D), \qquad \delta_\xi\psi_\alpha = i\bar\xi^{\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}(A-iB) + \xi_\alpha(F+iG), \\ &\delta_\xi\bar\psi_{\dot\alpha} = -i\xi^\alpha\mathcal{D}_{\alpha\dot\alpha}(A+iB) + \bar\xi_{\dot\alpha}(F-iG), \end{aligned} \tag{2}$$

where ξ_α, $\bar\xi_{\dot\alpha}$ are 4 parameters of the transformations.

It was found that the equations of motion, corresponding to Lagrangian (1), are satisfied by the following system of first-order equations [3] (they are first–order quasi-self-duality equations):

$$
\begin{aligned}
&f_{\alpha\beta} = 2gc_{\alpha\beta}[A,B], \qquad (c_{\alpha\beta}-\varepsilon_{\alpha\beta})\mathcal{D}^{\beta\dot\beta}(A-iB)=0, \qquad (c_{\alpha\beta}+\varepsilon_{\alpha\beta})\mathcal{D}^{\beta\dot\beta}(A+iB)=0,\\
&D+ig[A,B]=0, \qquad F=G=0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta}-g[\psi_\alpha, A+iB]=0,\\
&\mathcal{D}^{\alpha\dot\beta}\psi_\alpha + g[\bar\lambda^{\dot\beta}, A-iB]=0, \qquad (c_{\alpha\beta}-\varepsilon_{\alpha\beta})\psi^\beta=0, \qquad \lambda_\alpha=\bar\psi_{\dot\alpha}=0,
\end{aligned}
\tag{3}
$$

where $c_{\alpha\beta}$ are complex constant coefficients, satisfying the conditions

$$
c_{\alpha\beta}=c_{\beta\alpha}, \qquad \det\|c_{\alpha\beta}\| \equiv c_{11}\cdot c_{22}-c_{12}^2 \equiv \frac{1}{2}c^{\alpha\beta}c_{\alpha\beta}=-1. \tag{4}
$$

The system (3) is not invariant under transformations (2). But it is invariant when the following constraints on the parameters of these transformations are imposed

$$
(c_{\alpha\beta}+\varepsilon_{\alpha\beta})\xi^\beta=0. \tag{5}
$$

In other words, the system (3) is invariant in 3-dimensional subspace of 4-dimensional space of parameters of transformations (2). Our aim is to find such $N=1$ transformations depending on four parametrs under which the system (3) to be invariant.

The $N=2$ supersymmetric YM theory in $N=2$ superspace, given by the Lagrangian [4]

$$
\begin{aligned}
L = \mathrm{Tr}\Big(&-\frac{1}{4}F_{mn}F^{mn} - i\bar\lambda_{\dot\alpha i}\bar\sigma^{m\dot\alpha\beta}\mathcal{D}_m\lambda^i_\beta - 2\mathcal{D}_mC\mathcal{D}^mC^* - \frac{1}{2}\vec{C}^2\\
&+igC\{\bar\lambda_{\dot\alpha i},\bar\lambda^{\dot\alpha i}\} + igC^*\{\lambda^i_\alpha,\lambda^\alpha_i\} + 4g^2C[C,C^*]C^*\Big),
\end{aligned}
\tag{6}
$$

is invariant under $N=2$ supersymmetry transformations [5]:

$$
\begin{aligned}
&\delta_\xi C = -\xi^\alpha_i\lambda^i_\alpha, \qquad \delta_\xi C^* = -\bar\xi_{\dot\alpha i}\bar\lambda^{\dot\alpha i}, \qquad \delta_\xi V_{\alpha\dot\alpha} = 2i(\xi^i_\alpha\bar\lambda_{\dot\alpha i}+\bar\xi_{\dot\alpha i}\lambda^i_\alpha),\\
&\delta_\xi\lambda^i_\alpha = -\frac{1}{2}\xi^{\beta i}f_{\alpha\beta} + 2ig\xi^i_\alpha[\,C\,,\,C^*\,] - \xi_{\alpha j}\vec{C}\vec\tau^{ij} + 2i\bar\xi^{\dot\alpha i}\mathcal{D}_{\alpha\dot\alpha}C,\\
&\delta_\xi\bar\lambda_{\dot\alpha i} = -\frac{1}{2}\bar\xi^{\dot\beta}_i f_{\dot\alpha\dot\beta} - 2ig\bar\xi_{\dot\alpha i}[C,C^*] + \bar\xi^j_{\dot\alpha}\vec{C}\vec\tau_{ij} + 2i\xi^\alpha_i\mathcal{D}_{\alpha\dot\alpha}C^*,\\
&\delta_\xi\vec{C} = -i\xi^{\alpha i}(\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta j}+2g[\lambda^j_\alpha,C^*])\vec\tau_{ij} + \bar\xi^i_{\dot\alpha}(\mathcal{D}^{\alpha\dot\beta}\lambda^j_\alpha - 2g[\bar\lambda^{\dot\beta j},C])\vec\tau_{ij},
\end{aligned}
\tag{7}
$$

where ξ^i_α, $\bar\xi_{\dot\alpha i}$ are 8 parameters of $N=2$ supersymmetry transformations, and $\vec\tau^j_i$ are Pauli matrices.

The theory (1) can be obtained from (6) by the replacement

$$
\begin{aligned}
&C=\frac{1}{2}(A-iB), \qquad C^*=\frac{1}{2}(A+iB), \qquad g[C,C^*]=-\frac{1}{2}D,\\
&C_1=-iG, \qquad C_2=-iF, \qquad C_3=0,\\
&\lambda^1_\alpha=\lambda_\alpha, \qquad \lambda^2_\alpha=\psi_\alpha, \qquad \bar\lambda_{\dot\alpha 1}=\bar\lambda_{\dot\alpha}, \qquad \bar\lambda_{\dot\alpha 2}=\bar\psi_{\dot\alpha}.
\end{aligned}
\tag{8}
$$

Using (8) we rewrite the system (3) in terms of $N=2$ fields

$$
\begin{aligned}
&f_{\alpha\beta}=4igc_{\alpha\beta}[C^*,C], \qquad (c_{\alpha\beta}-\varepsilon_{\alpha\beta})\mathcal{D}^{\beta\dot\beta}C=0, \qquad (c_{\alpha\beta}+\varepsilon_{\alpha\beta})\mathcal{D}^{\beta\dot\beta}C^*=0,\\
&\vec{C}=0, \qquad \mathcal{D}^{\alpha\dot\beta}\lambda^2_\alpha - 2g[\bar\lambda^{\dot\beta 2},C]=0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta 2}+2g[\lambda^2_\alpha,C^*]=0,\\
&(c_{\alpha\beta}-\varepsilon_{\alpha\beta})\lambda^\alpha_{i=1}=0, \qquad \lambda^1_\alpha=\bar\lambda_{\dot\alpha 2}=0.
\end{aligned}
\tag{9}
$$

The system (9) is invariant under $N = 2$ transformations (7) only in 4-dimensional subspace of 8-dimensional space of parameters of transformations. This subspace is defined by

$$\xi_1^{i=1} = 0, \qquad \xi_2^{i=1} = 0, \qquad \bar{\xi}_{\dot\alpha 2} = 0. \tag{10}$$

We can identify these four $N = 2$ parameters that survived with four $N = 1$ parameters

$$\xi_1 = \xi_1^{i=2}, \qquad \xi_2 = \xi_2^{i=2}, \qquad \bar{\xi}_{\dot\alpha} = \bar{\xi}_{\dot\alpha 1}. \tag{11}$$

Now in transformations (7) we make the substitution of parameters (10), (11) and the substitution of component fields (8) and obtain

$$\begin{aligned}
&\delta_\xi(A - iB) = 2\xi\lambda, \qquad \delta_\xi(A + iB) = 2\bar{\xi}\bar{\psi},\\
&\delta_\xi V_{\alpha\dot\alpha} = 2i(\xi_\alpha\bar{\psi}_{\dot\alpha} - \bar{\xi}_{\dot\alpha}\lambda_\alpha), \qquad \delta_\xi D = \xi^\alpha \mathcal{D}_{\alpha\dot\alpha}\bar{\psi}^{\dot\alpha} + \bar{\xi}_{\dot\alpha}\mathcal{D}^{\alpha\dot\alpha}\lambda_\alpha,\\
&\delta_\xi(F + iG) = 2i\xi^\alpha\Big(\mathcal{D}_{\alpha\dot\beta}\bar{\lambda}^{\dot\beta} - g[\psi_\alpha, A + iB]\Big) + 2i\bar{\xi}_{\dot\beta}\Big(\mathcal{D}^{\alpha\dot\beta}\psi_\alpha + g[\bar{\lambda}^{\dot\beta}, A - iB]\Big),\\
&\delta_\xi(F - iG) = 0, \qquad \delta_\xi\lambda_\alpha = \xi_\alpha(F - iG),\\
&\delta_\xi\bar{\lambda}_{\dot\alpha} = \frac{1}{2}\bar{\xi}^{\dot\beta}(f_{\dot\alpha\dot\beta} - 2i\varepsilon_{\dot\alpha\dot\beta}D) - i\xi^\alpha\mathcal{D}_{\alpha\dot\alpha}(A + iB),\\
&\delta_\xi\psi_\alpha = -\frac{1}{2}\xi^\beta(f_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}D) + i\bar{\xi}^{\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}(A - iB), \qquad \delta_\xi\bar{\psi}_{\dot\alpha} = \xi_{\dot\alpha}(F - iG).
\end{aligned} \tag{12}$$

The transformations (12) are non-hermitean transformations depending on four $N = 1$ parameters. These transformations form Lie algebra. The equations of motion of the theory (1) as well as the first-order equations of motion (3) are invariant under transformations (12).

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 510–515.

Quasipotential Approach to Solitary Wave Solutions in Nonlinear Plasma

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Sagdeev's Quasipotential approach is extremely suitable for studying large amplitude solitary waves in plasma. One can derive all the one soliton results of perturbation methods and can compare it with the exact results obtained by the Quasipotential (also called the pseudopotential) method. However comparatively fewer works in relativistic plasma and plasma with trapped electrons have used this method. In this paper the pseudopotential is derived for a relativistic plasma with non-isothermal electrons and finite temperature ions. Expanding the quasipotential different types of solitons are obtained which agree with the perturbation results. Also the relativistic effect and finite ion temperature appear to restrict the region of existence of solitary waves.

1 Introduction

Theoretical studies on soliton dynamics were made very early in the frame work of Korteweg-de Vries (K-dV) equation using the reductive perturbative method in fluid dynamics. It was later extended to plasma dynamics [1, 2]. However the pertubation methods were mainly valid for small amplitude solitary waves. Sagdeev's pseudopotential approach [3] is appropriate for studying large amplitude solitary waves. Though this approach was rather widely used in obtaining travelling solitary waves solutions in simple non-relativistic plasmas, the applications to relativistic plasmas or plasmas with trapped electron are few and far between. But relativistic effects play an important part in the formation of solitary waves for particles with very high velocities which are comparable to that of light (for experimental and other details see references [4–11]).

Again most of the studies concerning solitary waves in both relativistic and non-relativistic plasmas did not consider the resonant particles which interact strongly with the wave during its evolution. These particles have to be treated in a way differant from what is done in the case of the free particles. Schamel [12, 13] made a theoretical study on ion-acoustic waves due to resonant electrons in a frame work of KdV and M KdV equations.

In this paper our aim is to study large amplitude solitary waves in a relativistic plasma with warm ions and with two differant distribution function for the electrons, one for the trapped and another for the free electrons. In this case the electron density is defined from the Vlasov equations consisting of free and trapped electrons as

$$n_e(\phi) = k_0 \left[e^{\phi} \operatorname{erfc}(\phi)^{1/2} + |\beta|^{-1/2} \left\{ \begin{array}{l} \exp(\beta\phi) \operatorname{erf}(\beta\phi)^{1/2} \\ \dfrac{1}{\sqrt{2}} w\left(-\beta\phi^{1/2}\right) \end{array} \right\} \begin{array}{l} -\beta \geq 0 \\ \beta < 0 \end{array} \right], \tag{1}$$

where k_0 is some constant and

$$\beta = T_{\rm ef}/T_{\rm et}, \tag{2}$$

$T_{\rm ef}$, $T_{\rm et}$ being the temperatures for the free electrons and the trapped electron respectively, where $\operatorname{erf}(x)$ is given by $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}dt$.

In the present paper the case $\beta \geq 0$ will be considered and the case $\beta < 0$ which gives a dip in the distribution function can be treated in a similar manner. The organization of the paper is as follows.

In Section 2 the exact pseudopotential is derived from the basic equations. In Section 3 solitary wave solutions are discussed. Small amplitude-approximations are derived in Section 4.

2 Basic equations and derivation of Sagdeev's potential

The basic system of equations governing the ion motion in plasma dynamics in unidirectional propagation is given by

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0, \tag{3}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\gamma u + \frac{\sigma}{n}\frac{\partial p}{\partial x} = -\frac{\partial \phi}{\partial x}, \tag{4}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)p + 3p\frac{\partial u}{\partial x} = 0, \tag{5}$$

where n, u, p denote the density, velocity and pressure respectively for the ion species. γ is given by $\gamma = \sqrt{1 - u^2/c^2}$, c being the speed of light.

The above equations are supplemented by the Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} + n - n_e = 0, \tag{6}$$

where we take

$$n_e = e^{\phi}\operatorname{erfc}(\phi^{1/2}) + \beta^{-1/2}e^{\beta\phi}\operatorname{erf}(\beta\phi)^{1/2} \tag{7}$$

with $\beta > 0$. β and $\operatorname{erf}(x)$ are defined earlier.

Also $\sigma = T_i/T_{\rm eff}$, T_i being the ion-temperature and $T_{\rm eff}$ is defined below. The above equations are normalized in the following way.

The velocities are normalized to the ion-acoustic speed $c_s = \left(\frac{kT_{\rm eff}}{m}\right)^{1/2}$, k being the Boltzmann constant and m_i the ion mass.

The distance and time t are normalized to the Deby length $\left(\frac{\epsilon_0 kT_{\rm eff}}{n_0\ e^2}\right)^{1/2}$ and ion plasma period $\left(\frac{\epsilon_0 m}{n_0 r^2}\right)^{1/2}$ respectively, ϵ_0 being the dielectric constant. The ion pressure is normalized to $(n_0 kT_i)^{-1}$ and the electrostatic potential ϕ is normalized to $\frac{kT_{\rm eff}}{e}$, e being the electron charge. Here $T_{\rm eff}$ is given by $T_{\rm eff} = T_{\rm ef}\ T_{\rm et}/(n_{\rm ef}T_{\rm ef} + n_{\rm et}T_{\rm et})$, $n_{\rm ef}$, $n_{\rm et}$ being the initial densities of the free and trapped electrons respectively and $n_{\rm ef} + n_{\rm et} = 1$.

To obtain the solitary wave solution we make the dependent variables depend on a single independent variable $\xi = x - Vt$, where V is the velocity of the solitary wave.

Equations (3)–(6) can now be written as

$$-V\frac{dn}{d\xi} + \frac{d}{d\xi}(nu) = 0, \tag{8}$$

$$-V\frac{d}{d\xi}(\gamma u)+u\frac{d}{d\xi}(\gamma u)+\frac{\sigma}{n}\frac{dp}{d\xi}=-\frac{d\phi}{d\xi}, \tag{9}$$

$$-V\frac{dp}{d\xi}+u\frac{dp}{d\xi}+3p\frac{du}{d\xi}=0, \tag{10}$$

$$\frac{d^2\phi}{d\xi^2}=n_e-n. \tag{11}$$

Equation (10) is consistent with

$$p=n^3p_0, \tag{12}$$

i.e. we consider the adibatic case and hence forth we shall take $p_0=1$.

From the above equations one can eliminate, n, n_e, u and p to obtain a differential equation involving ϕ which can be written as Newton's equation in the following way

$$\frac{d^2\phi}{d\xi^2}=-\frac{\partial\psi}{\partial\phi}, \tag{13}$$

where ψ is the so called Sagdeev's potential which is in general a transcendental function of ϕ. The exact form of $\psi(\phi)$ is given by

$$\psi(\phi)=\psi_e(\phi)+\psi_i(\phi), \tag{14}$$

where

$$\psi_e(\phi)=e^{\phi}\operatorname{erfc}\left(\sqrt{\phi}\right)+\frac{1}{\beta\sqrt{\beta}}e^{\beta\phi}\operatorname{erf}\left(\sqrt{\beta\phi}\right)+\frac{2}{\beta\sqrt{\pi}}\phi^{1/2}(\beta-1) \tag{15}$$

and

$$\psi_i(\phi) \;=\; Vu\gamma-Vu_0\gamma_0+\sigma v^3\left[\frac{1}{(V-u_0)^3}-\frac{1}{(V-u)^3}\right]. \tag{16}$$

In deriving equations (15) and (16) the following boundary conditions were used. As $\xi\to\infty$, $\phi\to 0$, $u\to u_0$, $p\to 1$, $n\to 1$. Also the relation between ϕ and u is given by

$$\phi=\left(vu-c^2\right)\gamma-\left(vu_0-c^2\right)\gamma^0+\frac{3\sigma}{2}V^2\left[\frac{1}{(V-u_0)^2}-\frac{1}{(V-u)^2}\right], \tag{17}$$

where

$$\gamma_0=\frac{1}{\sqrt{1-u_0^2/c^2}}.$$

3 Solitary waves solution

The form of the pseudopotential would determine whether soliton like solutions of equation (13) may exist or not.

The condition for the existence of solitary waves are the following

(i) $\left.\dfrac{d^2\psi}{d\phi^2}\right|_{\phi=0}<0.$

This is the condition for the existence of potential well another conditions

(ii) $\psi(\phi_m) > 0$,

where ϕ_m is the maximum (magnitude wise) value of ϕ beyond which ψ becomes imaginary. In this case ψ crosses the ϕ axis from below at the point $\phi = \phi_m$.

In Fig. 1 $\psi(\phi)$ is plotted against ϕ for different values of β ranging from 0.03 to 0.2 the ohter parameters are $V = 1.5$, $\sigma = 0.001$, $u_0 = 0$, $c/c_s = 100$.

It is seen that for $\beta \geq 0.2$ the amplitude of the soliton becomes very small and for a much larger value of β soliton solutions will disappear. Again for values of $\beta < 0.044$ solutions would cease to exist. In Fig. 2 the solitory wave solution $\phi(\xi)$ is plotted against ϕ for $\beta = 0.045$ and $\beta = 0.1$ other parameters are same as those in Fig. 1. It is found that both the height and of the width of the soliton decrease as β increases.

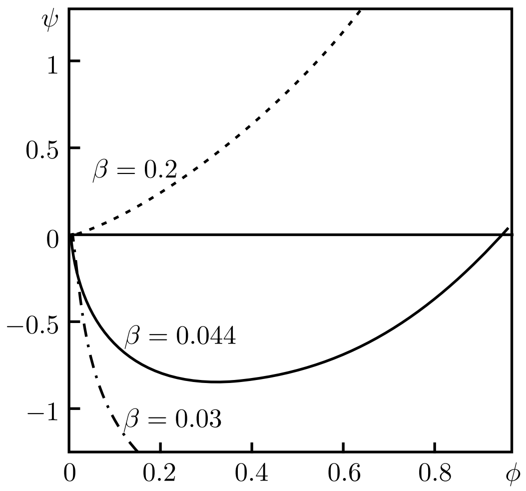

Figure 1

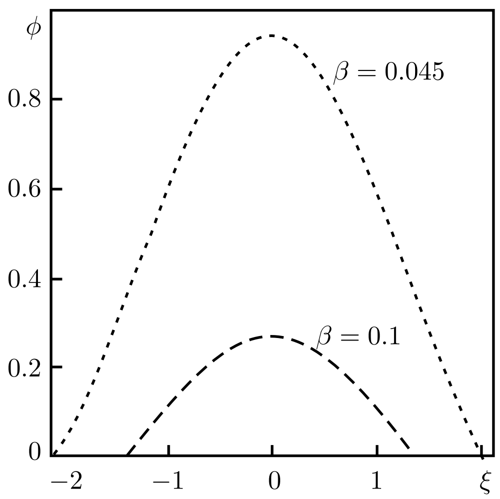

Figure 2

4 Small amplitude approximation

To obtain KdV (Korteweg de Vries) type soliton we obtain here small amplitude approximation of $\psi(\phi)$.

Expanding $\psi(\phi)$ from (15) and (16) we have

$$\frac{d^2\phi}{d\xi} = -\frac{\partial\psi}{\partial\phi} = A_1\phi - A_2\phi^{3/2} + A_3\phi^2 - A_4\phi^{5/2} + \cdots, \tag{18}$$

where

$$\begin{aligned} A_1 = 1 &- \frac{3\sigma}{(V-u_0)^4} - \frac{1}{(V-u_0)^2}\left[1 - \frac{3u_0^2}{2c^2}\right] \\ &+ \frac{3\sigma}{16c^2(V-u_0)^2}\left[\frac{40\left(3u_0^4 - 4Vu_0^3\right)}{(V-u_0)^4} - \frac{15V^4}{(V-u_0)^4} + \frac{18V^2}{(V-u_0)^2} - 3\right], \end{aligned} \tag{19}$$

$$A_2 = \frac{4b_1}{3}, \tag{20}$$

and

$$A_3 = \frac{1}{2}\left[1 - \frac{30\sigma}{(V-u_0)^6} - \frac{3}{(V-u_0)^4}\left(1 + \frac{40(V+2u_0)}{c^2}\right)\right.$$
$$\left. + \frac{9\sigma}{16c^2(V-u_0)^4}\left(-\frac{630\left(3u_0^4 - 4Vu_0^3\right)}{(V-u_0)^4}\right) + \frac{35V^4}{(V-u_0)^4} - \frac{30V^2}{(V-u_0)^2} + 3\right], \tag{21}$$

while

$$A_4 = \frac{8}{15}b_2. \tag{22}$$

Neglecting A_3 and A_4 solution of (20) is

$$\phi(\xi) = \left(\frac{5A_1}{4A_2}\right)^2 \operatorname{sech}^4\left(\frac{\xi}{\partial}\right), \qquad \text{where} \quad \partial = \frac{4}{\sqrt{A_1}}. \tag{23}$$

To get a shock wave solution we include A_3 term and put $\phi = Y^2$ to get

$$2\left(\frac{dY}{d\xi}\right)^2 = \frac{A_1}{2}Y^2 - \frac{2A_2}{5}Y^3 + \frac{A_3}{3}Y^4. \tag{24}$$

For a shock wave like solution $\frac{dY}{d\xi}$ should vanish both at $Y = 0$ and at a value $Y = Y_m$, Y_m being the amplitude of the solitory wave type solution. (24) can then be written as

$$\frac{dY}{d\xi} = kY\left(Y_m - Y\right), \tag{25}$$

where we take

$$Y_m = \frac{3}{5}\frac{A_2}{A_3}, \qquad 25A_1A_3 = 6A_2^2 \qquad \text{and} \qquad k = \pm\left(\frac{A_3}{6}\right)^{1/2}, \tag{26}$$

putting

$$\partial = \left(\frac{k}{2}\phi_m\right)^{-1} \tag{27}$$

the final solution becomes

$$\phi = \frac{\phi_m}{4}\left(1 \pm \tanh\frac{\xi}{\partial}\right)^2,$$

where

$$\phi_m = Y_m^2. \tag{28}$$

Other types of solitons viz, spiky type solitary waves collapsible waves etc. can be obtained by taking higher order terms and using the so called 'tanh' method [14, 15].

Since the expression for $\psi(\phi)$ derived in equations (14), (15) and (16) is exact, one can expand it up to any order in ϕ and obtain all the different types of solitary waves depending on the non-isothermality parameter β, obtained by perturbation methods.

For example if we include the A_4 term and write

$$\frac{d^2\psi(\phi)}{d\xi^2} = A_1\phi - A_2\phi^{3/2} + A_3\phi^2 - A_4\phi^{3/2}. \tag{29}$$

Equation (29) for the spiky type solitary wave can be studied by transforming the equation as

$$\left(\frac{d\Phi}{d\eta}\right)^2 = a_1\Phi^2\left(\phi_0 - \Phi\right)^3, \qquad \text{where} \qquad \phi = \Phi^2 \tag{30}$$

given below Eq.(20) and we take $a_1 = \frac{A_4}{7}$, $\phi_0 = \frac{7}{18}\frac{A_3}{A_4}$, $A_2 = \frac{35}{108}\frac{A_3^2}{A_4}$, and $A_1\ A_4 = \frac{7}{270}A_2A_3$.

The Eq.(30) can be solved for soliton profile and the solution $\phi_S(\eta)$ can be obtained only as an implicit function of η in the following way.

$$\phi_S(\eta) = \phi_0^2\,\mathrm{sech}^4\left[\left(\frac{\phi_0}{\phi_0 - \sqrt{\phi_S(\eta)}}\right)^{1/2} \pm \frac{1}{2}\sqrt{a_1\phi_0^3}(\eta - \eta_0) - C_1\right], \tag{31}$$

where $C_1 = \left(\frac{\phi_0}{\phi_0 - \sqrt{\phi_m}}\right)^{1/2} - \mathrm{sech}^{-1}\left(\frac{\sqrt{\phi_m}}{\phi_0}\right)^{1/2}$ and ϕ_m is the optimal amplitude of the acoustic mode. Note that $\phi_S(\eta)$ occurs on both left and right hand sides of Eq.(31). The solution (Eq.(31)) gives a profile of spiky solitary wave defined in the region $0 < \phi(\eta) < \sqrt{\phi_0}$. While for other region defined as $\phi < 0$, the soliton solution can be obtained in a similar manner and is given by

$$\phi_E(\eta) = \phi_0^2\,\mathrm{cosech}^4\left[\left(\frac{\phi_0}{\phi_0 - \sqrt{\phi_E(\eta)}}\right)^{1/2} \pm \frac{1}{2}\sqrt{a_1\phi_0^3}(\eta - \eta_0) - C_2\right], \tag{32}$$

where $C_2 = \left(\frac{\phi_0}{\phi_0 - \sqrt{\phi_m}}\right)^{1/2} - \mathrm{cosech}^{-1}\left(\frac{\sqrt{\phi_m}}{\phi_0}\right)^{1/2}$, and this is to be recognised as the explosive solitary wave in the plasma-acoustic dynamics. Thus one can proceed taking the nonlinear term to any order in ϕ and could derive different natures of the solitary waves under different approximations.

Acknowledgements

The author is grateful to the Department of Science and Technology (DST), Government of India for financial help. The author is also grateful to Dr. P. Chatterjee for drawing the diagrams.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 516–523.

Weyl-Type Quantization Rules and N-Particle Canonical Realization of the Poincaré Algebra in Two-Dimensional Space-Time

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Quantization of canonical realization of the Poincaré algebra $\mathfrak{p}(1,1)$ corresponding to N-particle interacting system in the two-dimensional space-time $\mathbb{M}_2$ in the front form of dynamics is considered. Hermitian operators corresponding to the Lie algebra of the group $\mathcal{P}(1,1)$ are obtained by means of the set of Weyl-type quantization rules. The requirement of preservation of the Lie algebra of this group restricts the set of quantization rules but does not by itself remove the ambiguity of the quantization procedure. The partition of the set of quantizations into equivalence classes is proposed. The quantization rules from the same equivalence class give the same mass spectrum, and the same evolution of the quantized system.

1 Introduction

Quantization – the problem of construction of the quantum description on the basis of classical theory – occupies a prominent place in the theoretical physics in 20th century.

The basic structure of the classical Hamiltonian mechanics for an unconstrained system is a $2N$-dimensional phase space $\mathbb{P} \simeq \mathbb{R}^{2N}$ (in general case a symplectic manifold) with symplectic form ω. The state of a classical system is described by a point in $\mathbb{P}$. Observable quantities are identified with smooth functions on $\mathbb{P}$. They form the space $C^\infty(\mathbb{P})$. Symplectic form determines on $C^\infty(\mathbb{P})$ the structure of Lie algebra (Poisson algebra) by means of the Poisson bracket [1]. In the quantum mechanics a state is described by a vector $|\psi\rangle$ in some Hilbert space $\mathcal{H}$ and physical observables are self-adjoint operators in $\mathcal{H}$. Correspondence between the classical and quantum pictures is established within the framework of certain quantization procedure which is meant as a linear map $\mathcal{Q} : f \mapsto \hat{f}$ of the Poisson algebra into the set of self-adjoint operators in the Hilbert space $\mathcal{H}$ [2, 3].

For every symmetry group, which is some Lie group G, the classical Hamiltonian description provides a canonical realization of this group. It is well known that quantization procedure can violate commutation relations of the Lie algebra of G [2]. Thus, we cannot *a priori* be sure that any classical symmetry leads after quantization to the quantum one. Moreover, different quantization rules may preserve some types of symmetries and break out other ones. It is natural to demand the preservation of physically important symmetries. Therefore, we shall require for the quantization procedure the fulfilment of the condition $\mathcal{Q}(\{f,g\}) = i[\hat{f},\hat{g}]$ only for some subalgebra of the Poisson algebra. It is clear that canonical generators corresponding to physically important symmetries have to belong to this subalgebra.

In the relativistic mechanics the main algebraic structure is the Lie algebra $\mathfrak{p}(1,3)$ of the Poincaré group $\mathcal{P}(1,3)$, and the description of a system of N interacting particles must be Poincaré invariant in the classical case as well as in the quantum one. Therefore, after quantization canonical generators of the Poincaré group have to be transformed into Hermitian operators which satisfy commutation relations of $\mathfrak{p}(1,3)$. In the relativistic case, the quantization problem is of special interest because Poincaré invariance conditions lead to the complicated dependence of interaction potentials on canonical coordinates and momenta. In most cases classical relativistic Hamiltonians depend on the products of non-commutative (in terms the of the Poisson bracket) quantities. This raises the question of symmetrization of non-commutative operators in the quantum description. Different ordering methods may result in different expressions for physical observable quantities [4]. Starting from certain classical system different quantization procedures may result in non-equivalent quantum systems.

In the two-dimensional space-time $\mathbb{M}_2$ the front form of relativistic dynamics [5, 6] corresponds to the foliation of $\mathbb{M}_2$ by isotropic hyperplanes [7]: $x^0+x=t$. The Poincaré group $\mathcal{P}(1,1)$ is the automorphism group of this foliation. Only one generator of $\mathfrak{p}(1,1)$ contains an interaction and mechanical description is in some sense similar to the nonrelativistic one. The two-dimensional variant of the front form permits the construction of the number of exactly solvable classical and quantum relativistic models [8, 9, 10, 11, 12]. Due to the certain simplicity of the relativistic description in the front form in $\mathbb{M}_2$, we are able to elucidate the peculiarities of the quantization procedure in the relativistic case [8, 9, 10, 11].

The aim of this article is quantization of the canonical realization of the Poincaré algebra $\mathfrak{p}(1,1)$ corresponding to N-particle relativistic system with an interaction (Section 2) within the framework of the two-dimensional variant of the front form of dynamics. Using the set of Weyl-type quantization rules we construct in Section 3 symmetric operators satisfying quantum commutation relations of $\mathfrak{p}(1,1)$. We study the influence of different quantization rules on quantized system and propose some classification method of non-equivalent quantizations of the canonical realization of the Lie algebra of $\mathcal{P}(1,1)$. We demonstrane the obtained results by the example of N-particle relativistic system with oscillator-like interaction.

2 Hamiltonian description in the front form of dynamics in $\mathbb{M}_2$

The classical Hamiltonian description of the system of N structureless particles with masses m_a $(a=\overline{1,N})$ in the two-dimensional Minkowski space $\mathbb{M}_2$ in the framework of the front form of dynamics leads to the canonical realization of the Lie algebra of $\mathcal{P}(1,1)$ with generators H, P, K [7]. They correspond to energy, momentum, and boost integral. Due to the positiveness of the momentum variables ($p_a>0$) [6, 7] in the front form of dynamics, the phase space of N-particle Hamiltonian system is $\mathbb{P}=\mathbb{R}_+^N\times\mathbb{R}^N$ with standard Poisson bracket

$$\{f,g\}=\sum_{a=1}^{N}\left(\partial f/\partial x_a\partial g/p_a-\partial g/\partial x_a\partial f/\partial p_a\right).$$

The generators $P_\pm=H\pm P$ satisfy the following Poisson bracket relations of the Poincaré algebra $\mathfrak{p}(1,1)$

$$\{P_+,P_-\}=0,\qquad \{K,P_\pm\}=\pm P_\pm. \tag{2.1}$$

They are determined in terms of particle canonical variables x_a, p_a [7] as follows:

$$P_+=\sum_{a=1}^{N}p_a,\qquad K=\sum_{a=1}^{N}x_ap_a,\qquad P_-=\sum_{a=1}^{N}\frac{m_a^2}{p_a}+\frac{1}{P_+}V(rp_b,r_{1c}/r). \tag{2.2}$$

Only one generator, namely P_-, depends on interaction. The Poincaré-invariant function V describes the particles interaction and depends on $2N-1$ indicated arguments, where $r_{ac}=$

$x_a - x_c$; $r = r_{12}$; $a, b = \overline{1, N}$, $c = \overline{2, N}$. Generators (2.2), determine the square of the mass function of the system

$$M^2 = P_+ P_- = P_+ \sum_{a=1}^{N} \frac{m_a^2}{p_a} + V(rp_b, r_{1c}/r). \tag{2.3}$$

The description of the motion of a system as a whole may be performed by choosing P_+ and $Q = K/P_+$ as new (external) variables. There exist a lot of possibilities of the choice of inner variables. One of the possible choices of inner canonical variables is [9]:

$$\eta_a = (P_{a+} - p_{a+1})/(2P_{(a+1)+}), \qquad q_a = P_{(a+1)+}(Q_a - x_{a+1}); \tag{2.4}$$

where $a, b = \overline{1, N-1}$ and we use the following notations $P_{a+} = \sum\limits_{i=1}^{a} p_i$, $Q_a = P_{a+}^{-1} \sum\limits_{i=1}^{a} x_i p_i$, $P_{N+} = P_+$, $Q_N = Q$. In the two-particle case variables (2.4) coincide with the variables proposed in Ref. [6].

3 Quantization of canonical realization of the Poincaré algebra in $\mathbb{M}_2$

To quantize the classical generators we have first to determine quantum operators corresponding to the particular canonical variables x_a, p_a. Then for a given set of classical observables $a = a(x, p)$ we construct corresponding quantum operators $\hat{A}$. Let $\hat{x}_a$, $\hat{p}_a$ be Hermitian operators corresponding to the classical particle coordinates and momenta with the following commutation relations: $[\hat{x}_a, \hat{p}_b] = i\delta_{ab}$. The original Weyl application [13] is a basis for the whole set of quantization rules $W_{\mathcal{F}} : a \mapsto \hat{A}$, which map bijectively a family of classical real functions $a(x, p) \in C^\infty(\mathbb{P})$ to a family of Hermitian operators $\hat{A}$ in some Hilbert space $\mathcal{H}$. For $\mathbb{P} \approx \mathbb{R}^{2N}$, the formal definition is given in the explicit form [14] as follows

$$\hat{A} = \int (dk)(ds)\tilde{a}(k, s)\mathcal{F}(k, s) \exp\left[i \sum_a (k_a \hat{x}_a + s_a \hat{p}_a)\right], \tag{3.1}$$

where $\tilde{a}(k, s)$ is the Fourier transform of the function $a(p, q)$. Function $\mathcal{F}(k, s)$ determines the type of quantization. Different choices of $\mathcal{F}(k, s)$ correspond to different ordering conventions. We shall call the elements of the family of quantizations (3.1) Weyl-type quantization rules. For the original Weyl quantization $\mathcal{F}(k, s) = 1$. Let us restrict ourselves to real functions $\mathcal{F}(k, s) \in C^\infty(\mathbb{R}^{2N})$, i.e. $\mathcal{F}(k, s) = \mathcal{F}^*(k, s)$. Every quantization rule must obey the following condition: $\mathcal{Q}(1) = \hat{1}$. As a result, for the family of quantizations (3.1) we obtain $\mathcal{F}(0, 0) = 1$. Hermiticity condition means: $\mathcal{F}(k, s) = \mathcal{F}(-k, -s)$.

In the momentum representation the wave functions $\psi(p) = \langle p|\psi\rangle$ describing the physical (normalized) states in the front form of dynamics constitute the Hilbert space $\mathcal{H}_N^F = \mathcal{L}^2(\mathbb{R}_+^N, d\mu_N^F)$ with the inner product [8]

$$(\psi_1, \psi) = \int d\mu_N^F(p)\psi_1^*(p)\psi(p), \qquad d\mu_N^F(p) = \prod_{a=1}^{N} \frac{dp_a}{2p_a}\Theta(p_a), \tag{3.2}$$

where $d\mu_N^F(p)$ is the Poincaré-invariant measure and $\Theta(p_a)$ is Heaviside step function. Operators act on wave functions $\psi(p) \in \mathcal{H}_N^F$ as integral operators:

$$(\hat{A}\psi)(p) = \int d\mu_N^F(p')\widetilde{A}(p, p')\psi(p'). \tag{3.3}$$

The kernel corresponding to operator (3.1) has the form

$$\widetilde{A}(p,p') = \frac{1}{(2\pi)^N}\int (dx)(dz)\exp\left(i\sum_{a=1}^{N}\left(p'_a - p_a\right)x_a\right) \times\left(\prod_{a=1}^{N}\delta\left(z_a - \frac{p_a+p'_a}{2}\right)2\sqrt{p_a p'_a}\right)\mathcal{F}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial z}\right)a(x,z). \tag{3.4}$$

Now let us consider the quantization procedure of classical canonical generators (2.2) of $\mathfrak{p}(1,1)$. Substituting expressions (2.2) of the generators K, P_+ into (3.4) we obtain the following operators

$$\hat{P}_+ = P_+, \qquad \hat{K} = i\sum_{a=1}^{N} p_a\frac{\partial}{\partial p_a} - \sum_{a=1}^{N}\frac{\partial^2\mathcal{F}(0,0)}{\partial k_a\partial s_a}. \tag{3.5}$$

The generator P_- is transformed into integral operator (3.3) with the kernel

$$\widetilde{P}_-(p,p') = \frac{1}{(2\pi)^N}\int (dx)(dz)\exp\left(i\sum_{a=1}^{N}\left(p'_a - p_a\right)x_a\right) \times\left(\prod_{a=1}^{N}\delta\left(z_a - \frac{p_a+p'_a}{2}\right)2\sqrt{p_a p'_a}\right)\mathcal{F}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial z}\right)\left(\sum_{a=1}^{N}\frac{m_a^2}{z_a} + \frac{V(rz_b, r_{1c}/r)}{\sum\limits_{a=1}^{N} z_a}\right). \tag{3.6}$$

To obtain a unitary representation of the group $\mathcal{P}(1,1)$, we must construct first and foremost such symmetric operators that satisfy the quantum commutation relations of $\mathfrak{p}(1,1)$

$$[\hat{P}_+, \hat{P}_-] = 0, \qquad [\hat{K}, \hat{P}_\pm] = \pm i\hat{P}_\pm. \tag{3.7}$$

The second task is the construction of self-adjoint extensions (if they exist). Here we consider only the first part of the problem.

The last term in the expression (3.5) of the boost operator $\hat{K}$ has no influence on commutation relations (3.7). Thus, the quantization problem reduces in fact to the construction of quantum operator $\hat{P}_-$. That in its turn determines the form of the function $\mathcal{F}$.

Proposition 1. *So that operators (3.5), (3.6) could satisfy the commutation relations (3.7), the function $\mathcal{F}$ has to be of the following form:*

$$\mathcal{F} = \mathcal{F}(ks), \tag{3.8}$$

where the function $\mathcal{F}$ on the right-hand side depends on the all possible products of arguments: $k_1s_1, \ldots, k_1s_N, k_2s_1, \ldots, k_2s_N, \ldots$.

Proof. In order to satisfy relations (3.7) the kernel $\widetilde{P}_-(p,p')$ must be homogeneous function of the order -1. To satisfy this condition the function $\mathcal{F}$ must obey the following homogeneity equation: $\mathcal{F}(\beta k, \beta^{-1}s) = \mathcal{F}(k,s)$. The only possibility to satisfy this equation is (3.8).

In the classical case the square of total mass function M^2 is an invariant of the group $\mathcal{P}(1,1)$. Thus, to obtain in the quantum case the algebraic structure which is most closely related to the classical one, the quantum Kasimir operator $\hat{M}^2 = \hat{P}_+\hat{P}_-$ should be a quantization result of the classical function $M^2 = P_+P_-$. Unfortunately not every Weyl-type quantization rule with the arbitrary function $\mathcal{F}$ of the form (3.8) will transform the product $P_+P_- = M^2$ ($\{P_+, P_-\} = 0$) of classical functions into the corresponding product of quantum (commutating) operators $\hat{P}_+\hat{P}_- = \hat{M}^2$. This means that not every quantization rule $W_{\mathcal{F}}$, preserving the structure of Lie algebra of the group $\mathcal{P}(1,1)$, preserves commutability of the following diagram

$$\begin{tikzcd} P_+,\ P_- \arrow[r, "M^2=P_+P_-"] \arrow[d, "W_{\mathcal{F}}"] & M^2 \arrow[d, "W_{\mathcal{F}}"] \\ \hat{P}_+,\ \hat{P}_- \arrow[r, "\hat{M}^2=\hat{P}_+\hat{P}_-"] & \hat{M}^2\ . \end{tikzcd} \tag{3.9}$$

Proposition 2. *If the function* $\mathcal{F}$ *has the following form*

$$\mathcal{F}=\mathcal{F}(\Delta_1,\Delta_2), \qquad \Delta_1=\sum_{a=1}^{N} k_a s_a, \qquad \Delta_2=\sum_{\substack{a=1\\a\neq b}}^{N}\sum_{b=1}^{N} k_a s_b,$$

then diagram (3.9) *is commutative.*

Proof. The proposition follows from the translation invariance of P_-.

It is obvious that for partial cases with

$$\mathcal{F} = \mathcal{F}(\Delta_1, 0) = \mathcal{F}_1(\Delta_1), \qquad \mathcal{F} = \mathcal{F}(0, \Delta_2) = \mathcal{F}_2(\Delta_2). \tag{3.10}$$

diagram (3.7) is commutative too. $W_{\mathcal{F}_1}$-quantization has been considered, for example, in Ref. [15].

If $\mathcal{F} = \mathcal{F}(\Delta_0) = \mathcal{F}_0$, $\Delta_0 = \Delta_1 + \Delta_2$, then for arbitrary translation invariant function f we have:

$$\mathcal{F}(\hat{\Delta}_0)f = f. \tag{3.11}$$

As follows from (3.11), (3.5), (3.6) the $W_{\mathcal{F}_0}$-quantization leads to the same operators $\hat{P}_-$, $\hat{P}_+$ as well as the original Weyl quantization does. Moreover, quantization rules $W_{\mathcal{F}}$ and $W_{\mathcal{F}\mathcal{F}_0}$ give us the same realization of commutative ideal $\mathfrak{h} = \mathrm{span}\,(\hat{P}_+,\ \hat{P}_-)$. The quantizations $W_{\mathcal{F}}$ and $W_{\mathcal{F}\mathcal{F}_0}$ may lead to different boost operators: $\hat{K}$, $\hat{K}'$. But these operators generate Lorentz transformations which distinguish on phase factor: $\left(e^{-i\lambda\hat{K}'}\psi\right)(p) = e^{i\alpha}\left(e^{-i\lambda\hat{K}}\psi\right)(p)$. Thus, $\exp{(-i\lambda\hat{K}')}\psi(p)$ and $\exp{(-i\lambda\hat{K})}\psi(p)$ belong to the some ray.

In the front form of dynamics the evolution of the quantum system is described by the Schrödinger-type equation

$$i\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \tag{3.12}$$

where $\Psi \in \mathcal{H}_N^F$ and $\hat{H} = (\hat{P}_+ + \hat{P}_-)/2 = (\hat{P}_+ + \hat{M}^2/\hat{P}_+)/2$. Putting $\Psi = \chi(t, P_+)\psi$, where ψ is a function of some Poincaré-invariant inner variables, we obtain the stationary eigenvalue problem for the operator $\hat{M}^2$:

$$\hat{M}^2\psi = \hat{P}_+\hat{P}_-\psi = M^2_{n,\lambda}\psi. \tag{3.13}$$

The ideal $\mathfrak{h}$ generates by means of the Eqs. (3.12), (3.13) the evolution of the system and the mass spectrum. Therefore, it is natural to introduce the following

Definition 1. *Quantizations $W_{\mathcal{F}}$, $W_{\mathcal{F}'}$ which lead to the same realization of the ideal $\mathfrak{h}$ are called equivalent:*

$$W_{\mathcal{F}} \simeq W_{\mathcal{F}'} \ . \tag{3.14}$$

Proposition 3. *Quantization rules $W_{\mathcal{F}}$, $W_{\mathcal{F}'}$ preserving the commutation relations of $\mathfrak{p}(1,1)$, where $\mathcal{F} = \mathcal{F}(ks, \Delta_0)$, $\mathcal{F}' = \mathcal{F}(ks, 0)$, are equivalent:*

$$W_{\mathcal{F}(ks,\Delta_0)} \simeq W_{\mathcal{F}(ks,0)}. \tag{3.15}$$

Proof. This follows immediately from (3.11) and translation invariance of P_-.

Corollary 1.

$$W_{\mathcal{F}(ks)\mathcal{F}_0} \simeq W_{\mathcal{F}(ks)}. \tag{3.16}$$

For the special class of quantization rules which preserve, in addition to the commutation relation of $\mathfrak{p}(1,1)$, the commutability of the diagram (3.9) we have $W_{\mathcal{F}_1(\Delta_1)} \simeq W_{\mathcal{F}_2(-\Delta_2)}$, $W_{\mathcal{F}_2(\Delta_2)} \simeq W_{\mathcal{F}_1(-\Delta_1)}$. Hence, we see that the Weyl-type quantization rules which preserve the commutation relation of the Poincaré algebra $\mathfrak{p}(1,1)$ fall apart into equivalence classes. Rules from different classes can give non-equivalent unitary representations of the group $\mathcal{P}(1,1)$ and may result in different expressions for such important observable quantity as the mass spectrum of the system. We shall demonstrate this fact by the example of N-particle system with oscillator-like interaction.

Let us choose the interaction function V in the following form

$$V = \omega^2 \sum\sum_{a<b} r_{ab}^2 p_a p_b, \qquad \omega^2 > 0. \tag{3.17}$$

The function (3.17) describes N-particle oscillator-like interaction [9]. In the nonrelativistic limit such a system is reduced to the nonrelativistic oscillator system. The system with interaction (3.17) has $N-2$ additional integrals of motion λ_j in involution: $\{\lambda_i, \lambda_k\} = 0$, $i, k = \overline{2, N-1}$. In terms of the variables (2.4) they have the form

$$\begin{aligned} \lambda_{j+1}^2 = {} & \sum_{d=1}^{j} \frac{m_d^2}{1/2 - \eta_{d-1}} \prod_{i=d}^{j} (1/2 + \eta_i)^{-1} + \frac{m_{j+1}^2}{1/2 - \eta_j} \\ & + \omega^2 \sum_{d=1}^{j-1} \left(1/4 - \eta_d^2\right) q_d^2 \prod_{i=d+1}^{j} (1/2 + \eta_i)^{-1} + \omega^2 \left(1/4 - \eta_j^2\right) q_j^2, \end{aligned} \tag{3.18}$$

where $\lambda_N^2 = M^2$, $j = \overline{1, N-1}$.

Quantum mechanical description for the system with interaction (3.17) was constructed by means of the ordinary Weyl quantization in Ref. [9]. Here we consider $W_{\mathcal{F}_1}$-quantization (see (3.10)). One can show that $W_{\mathcal{F}_1}$-quantization transforms the classical integrals into quantum ones ($[\hat{\lambda}_i, \hat{\lambda}_j]$) and we obtain the following mass spectrum of the system:

$$\begin{aligned} M_n^2 = {} & \left[\sum_{a=1}^{N} \sqrt{m_a^2 - (\omega \mathcal{F}_1'(0))^2} + \omega \sum_{b=1}^{N-1} (n_b + 1/2) \right]^2 \\ & + \omega^2 \left[(N-1) \left(\frac{1}{4} - N \mathcal{F}_1''(0) \right) + \left(N \mathcal{F}_1'(0) \right)^2 \right]. \end{aligned} \tag{3.19}$$

The discrete spectrum exists only if $\omega|\mathcal{F}_1'(0)| \leq \min\{m_a\}$, $a = \overline{1,N}$. This gives additional restriction for the type of $W_{\mathcal{F}_1}$-quantization. We see that the mass spectrum depends essentially on the choice of quantization rule. In the case $\mathcal{F}_1 = 1$ we come to the spectrum of the system with the interaction (3.17) which has been obtained by the original Weyl quantization in Ref. [9]. In this work the generalization of the pure oscillator-like interaction has been considered too. This new interaction function contains also the terms which are linear in the coordinates: $V \to \tilde{V} = V + \alpha \sum\sum_{a<b} r_{ab}(p_a - p_b)$. The original Weyl quantization gives the following result (see Ref. [9]):

$$M_n^2 = \left[\sum_{a=1}^{N} \sqrt{m_a^2 - \frac{\alpha^2}{4\omega^2}} + \omega \sum_{b=1}^{N-1} (n_b + 1/2)\right]^2 + \frac{N-1}{4}\omega^2 + \frac{\alpha^2 N^2}{4\omega^2}. \tag{3.20}$$

Comparing the equalities ((3.19)), ((3.20)) we see that the quantizations $W_{\mathcal{F}_1}$, $\mathcal{F}_1'(0) \neq 0$, $\mathcal{F}_1''(0) = 0$ of the classical system with the pure oscillator-like interaction (3.17) gives the terms in the expression for mass spectrum ((3.19)) which one can treat as a presence of the linear interaction with $\alpha = -2\omega^2 \mathcal{F}_1'(0)$. Then such a quantum system is equivalent to those which is obtained from the classical system with the interaction $\tilde{V}$ by means of the original Weyl quantization. Thus, the use of different quantization rules may lead to essentially different quantum results. Moreover different quantizations may lead to quantum systems with physically different interactions!

In the nonrelativistic case all the ambiguities in the mass spectrum ((3.19)) vanish and we obtain well known energy spectrum of nonrelativistic system with the oscillator interaction. But the first relativistic correction to the nonrelativistic energy depends on the type of quantization:

$$\begin{aligned} E \approx \hbar\omega \sum_{b=1}^{N-1} (n_b + 1/2) + \frac{\hbar^2\omega^2}{2c^2} \Bigg\{ \frac{1}{m} \left(\sum_{b=1}^{N-1} (n_b + 1/2) \right)^2 \\ -(\mathcal{F}_1'(0))^2 \sum_{a=1}^{N} \frac{1}{m_a} + \frac{1}{m}\left[(N-1)\left(\frac{1}{4} - N\mathcal{F}_1''(0)\right) + \left(N\mathcal{F}_1'(0)\right)^2 \right] \Bigg\}. \end{aligned} \tag{3.21}$$

Here we renewed the constants $\hbar$, c.

Let us note that for the quantization of the oscillator-like interaction we used only quantizations preserving the commutability of the diagram (3.9). Using the quantization rules $W_{\mathcal{F}}$ (3.8), which preserve only the commutation relations of the Poincaré algebra $\mathfrak{p}(1,1)$, we could obtain more ambiguous results for the mass spectrum.

4 Conclusions

We have considered the problem of the quantization of the classical canonical realization of the Poincaré algebra $\mathfrak{p}(1,1)$ corresponding to N-particle relativistic system with an interaction. It has been demonstrated that for Weyl-type quantization rules (3.1) the requirement of preservation of the Lie algebra $\mathfrak{p}(1,1)$ restricts the set of quantization rules but does not by itself remove the ambiguity of the quantization procedure.

In the classical case the square of total mass function $M^2 = P_+P_-$ is an invariant of the group $\mathcal{P}(1,1)$. To obtain in the quantum case the algebraic structure which is most closely related to the classical one, the quantum Kasimir operator $\hat{M}^2 = \hat{P}_+\hat{P}_-$ must be the quantization result of the classical expression $M^2 = P_+P_-$. This additional requirement imposes additional restriction on the family of the Weyl-type quantization rules. Thus we see, that if one require the quantization to preserve at least some of the associative algebra structure of $C^{\infty}(\mathbb{P})$ then one can restrict abbiguties of quantization procedure. But it does not fully eliminate the ambiguity of the quantization either.

We also demonstrated that the Weyl-type quantization rules are split into equivalence classes. Quantization rules from the same equivalence class lead to the same realization of the ideal $\mathfrak{h}$ and therefore give the same mass spectrum and the evolution of quantized system. The quantizations which belong to different classes lead to non-equivalent quantum systems. We have demonstrated the last fact by the example of the N-particle system with the oscillator-like interaction. Therefore, if we start with the classical description of a mechanical system then quantization rule seems to be an essential part of the definition of the corresponding quantum system.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 524–529.

Quantum Integrability of the Generalized Euler's Top with Symmetries

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We prove integrability, i.e the existence of the full set of commuting integrals, of the quantized generalized rigid body in the case when inertia tensor possesses additional symmetry.

1 Introduction

In the present paper we deal with the quantum systems that are direct higher-rank generalization of the standard $so(3)$ Eulers top.

Integrability of their classical counterparts was originally proved by Manakov [1] for the case of $so(n)$ and by Mishchenko and Fomenko [2, 3] for the case of arbitrary semisimple Lie algebras. They constructed the algebra of the mutually commuting with respect to the Lie–Poisson brackets integrals of these systems, which we will call Mishchenko–Fomenko algebra, with the help of the so-called procedure of the "shift of the argument". This procedure has the Lie-algebraic explanation that was given by Reyman and Semenov-Tian-Shansky [4, 5] in the framework of the so called Kostant–Adler scheme [6]. But in the quantum case the analogous scheme fails. This fact has again purely algebraic nature. Indeed, Kostant–Adler scheme, that was used by Reyman and Semenov-Tian-Shansky involves loop algebras and their invariant functions. In the quantum case corresponding invariant operators (symmetrized invariant functions) are badly defined due to infinite-dimensionality of the loop algebra.

Nevertheless Vinberg proved [8] that a subalgebra of the Mishchenko–Fomenko algebra consisting of the elements of the second order in the generators of semisimple Lie algebra is commutative also on the quantum level (i.e. in the universal enveloping algebra). This fact indicates that Mishchenko–Fomenko algebra should have commutative quantum counterpart.

In the presented paper we made one more step in proving this conjecture. We prove commutativity in the universal enveloping of the other subalgebra of Mishchenko–Fomenko algebra. Contrary to the case of Vinberg our subalgebra $L_A \subset \mathfrak{A}(\mathfrak{g}_A)$ is not homogeneous in the coordinates of the underlying Lie algebra, but is of the order not higher than one in the coefficients of the inertia tensor A. Although this is not enough for proving the integrability of the quantum Euler top in the case of the inertia tensor of the general position, but we show, that if the inertia tensor possesses additional symmetries one could construct the full set of "quantum integrals" using the symmetry algebra. Indeed it is known [10], that if the inertia tensor is symmetric with the symmetry group G_A and the symmetry algebra $\mathfrak{g}_A$ then Mishchenko–Fomenko algebra $MF_A \subset P(\mathfrak{g}^*)$ is centralized by $\mathfrak{g}_A$. From the Chevalley isomorphism between $P(\mathfrak{g}^*)$ and $\mathfrak{A}(\mathfrak{g})$ as $\mathfrak{g}$ modules follows that the same fact holds true also for the quantum case. So, for the set of commuting quantum integrals one could take independent integrals of the Mishchenko–Fomenko algebra along with some commutative elements from $\mathfrak{A}(\mathfrak{g}_A)$. Taking into account the number of the independent operators in the algebra L_A (equal to $2\operatorname{rank}\mathfrak{g} - 1$) and the maximal possible number of the independent commuting operators in $\mathfrak{A}(\mathfrak{g}_A)$ (equal to $1/2(\operatorname{ind}\mathfrak{g}_A + \dim\mathfrak{g}_A)$)

one could verify that for obtaining with their help a complete set of commuting quantum integrals one should have the following restrictions on the degeneracy of the matrix A. In the case when the underlying Lie algebra $\mathfrak{g}$ is equal to $gl(n)$, $so(n)$ or $sp(n)$ algebra $\mathfrak{g}_A$ should contain subalgebra $gl(n-2)$, $so(n-2)$ or $sp(n-1)$ correspondingly[1].

Using the "duality" in the dependence of the generators of Mishchenko–Fomenko algebra in the generators of $\mathfrak{g}$ and parameters of the "shift" A along with the result of Vinberg [8] we also prove the integrability of quantum systems that correspond to some strongly degenerated orbits in $\mathfrak{g}^*$ which are characterized as such that their stabilizers include Lie groups $Gl(n-2)$, $SO(n-2)$ or $Sp(n-1)$.

2 Generalized Euler top

In this section we briefly remind several facts from the theory of classical finite-dimensional integrable systems.

As it is known, equation of the motion of rigid body could be written in the form of Puanso [7]:

$$I_1\dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3, \qquad I_2\dot{\Omega}_2 = (I_3 - I_1)\Omega_1\Omega_3, \qquad I_3\dot{\Omega}_3 = (I_1 - I_2)\Omega_1\Omega_2,$$

where $\vec{\Omega}$ is a vector of the angular velocity. Making the replacement of variables $M_i = I_i\Omega_i$ we will have Poinsot equations in the form:

$$\dot{M}_i = \epsilon_{ijk}M_k\frac{\partial H}{\partial M_j},$$

where $H = \sum\limits_{i=1,3} M_i^2/I_i$. They also could be rewritten as:

$$\dot{M}_i = \{M_i, H\},$$

where $\{\ ,\ \}$ is Lie Poisson brackets defined in the following way:

$$\{M_i, M_j\} = \epsilon_{ijk}M_k.$$

The key observation that was made by Arnold [7] is that this equations could be generalized to arbitrary Lie algebra. In this general case they will have the following form:

$$\dot{M}_i = C_{ij}^k M_k\frac{\partial H}{\partial M_j}, \tag{1}$$

where C_{ij}^k are the structure constants of some Lie algebra $\mathfrak{g}$ and M_i-coordinate functions on the dual space $\mathfrak{g}^*$. These equation are so called Euler–Arnold equations. Of course not for every function $H \in \mathfrak{g}^*$ these equations are integrable. Mishchenko and Fomenko [2] found quadratic hamiltonian that provides integrability of equation (1) for arbitrary semisimple Lie algebra. It has the following form:

$$H = (M, \mathrm{ad}_A^{-1}\,\mathrm{ad}_B M),$$

[1]Note, that without knowledge of the commutativity of subalgebra $L_A \subset \mathfrak{A}(\mathfrak{g})$ one can only state integrability of quantum Eulers top in the cases of the Lie algebras $gl(n)$ and $so(n)$ when inertia tensor A is symmetric with respect to the Lie subalgebra $gl(n-1)$ and $so(n-1)$ correspondingly. In this cases the algebra of commuting quantum integrals will simply coincide with the well-known Gelfand–Tsetlin algebra.

where $A, B \in \mathfrak{g}$ are any constant covectors, (,) is Killing–Kartan form. Corresponding Euler–Arnold equations are:

$$\frac{dM}{dt} = [M, (\mathrm{ad}_A)^{-1}(\mathrm{ad}_B)M]. \tag{2}$$

It is evident, that these equations are nonlinear. But, as it was shown in [2], they are integrable with the algebra of integrals constructed by the method of the shift of the argument. Let $\{C_{m_k}(M) \subset P(\mathfrak{g}^*)\}$, where m_k is exponents of $\mathfrak{g}$, be a full set of the independent polynomial generators of $I^G(\mathfrak{g}^*)$. Then functions $C^A_{l,m_k}(M)$ obtained from the decomposition:

$$C_{m_k}(M + \lambda A) = \sum_{l=0}^{m_k} \lambda^l C^A_{l,m_k}(M)$$

are mutually commuting integrals of equations (2). Moreover, in the case of the generic covector A they form a full set of the independent integrals of Euler–Arnold equations. If the covector A is nongeneric, then, in order to obtain complete set of mutually commuting integrals one should take along with the set $\{C^A_{l,m_k}(M)\}$ any complete set of commuting functions in $P(\mathfrak{g}^*_A)$ [10]. Here $\mathfrak{g}_A$ is centralizer of A in $\mathfrak{g}$.

3 Quantization and integrability

3.1 Generalities

Quantization is the map from set of coordinates in the phase space into the set of Hermitian operators in some Hilbert space $\mathcal{H}$, so that the following relations holds:

$$\widehat{\{M_i, M_j\}} = \frac{\hbar}{i}[\hat{M}_i, \hat{M}_j].$$

In other words quantization is the homomorphism from the Lie algebra $\mathfrak{g}$, realized as Lie subalgebra in $P(\mathfrak{g}^*)$ (with respect to the Lie–Poisson brackets) into the Lie algebra $\mathfrak{g}$, realized as the subalgebra in the Lie algebra of Hermitian operators in some Hilbert space $\mathcal{H}$. Due to the well known fact that every representation of the arbitrary Lie algebra could be lifted to the representation of its universal enveloping algebra, one could present quantization as the map:

$$P(\mathfrak{g}^*) \hookleftarrow \mathfrak{g} \hat{\to} \mathfrak{A}(\mathfrak{g}).$$

This map could not be extended to the isomorphism of the algebras $(P(\mathfrak{g}^*), \{\ ,\ \})$ and $(\mathfrak{A}(\mathfrak{g}), [\ ,\ ])$.

From the point of view of integrable systems the latter fact means that, generally speaking, quantum counterparts of the Poisson-commuting classical polynomial integrals are not necessary commutative operators:

$$[\widehat{I}_i, \widehat{I}_j] \neq \widehat{\{I_i, I_j\}}.$$

By other words, proof of the quantum integrability of the classically integrable hamiltonian systems is additional, separated from the process of quantization problem.

3.2 Quantum Euler tops

Let us at last consider the problem of quantum integrability of the generalized Eulers tops. In the standard $so(3)$ case one has only two independent integrals – Hamiltonian and the square of a vector $\vec{M}$: $H = \sum\limits_{i=1,3} M_i^2/I_i$, $M^2 = \sum\limits_{i=1,3} M_i^2$.

Due to the simple fact that M^2 is an invariant function no problems with the quantum integrability arises in the $so(3)$ case. Indeed, operator $\widehat{M^2}$ is a second order Casimir operator, which commutes with the whole Lie algebra $\mathfrak{A}(so(3))$. Hence, evidently:

$$[\widehat{H}, \widehat{M^2}] = 0.$$

In the case of the Lie algebras of the higher rank situation is more complicated. Indeed to prove quantum integrability of the described above systems one should prove that

$$[\widehat{C^A_{p,m_k}(M)}, \widehat{C^A_{s,m_n}(M)}] = 0.$$

Due to the fact that operators $\widehat{C^A_{0,m_k}(M)} \equiv \widehat{C_{m_k}(M)}$, $k = 1, \dots, \operatorname{rank}\mathfrak{g}$ are Casimir operators (i.e. analogs of the square of a vector $\vec{M}$) one obtains that

$$[\widehat{C^A_{0,m_k}(M)}, \widehat{C^A_{s,m_n}(M)}] = 0$$

for every s, m_n. By other words Casimir operators are always "quantum integrals".

Let us consider other subalgebra in the Mishchenko–Fomenko algebra, namely the algebra generated by the integrals $\{C^A_{1,m_k}(M),\ k = 2, \dots, \operatorname{rank}\mathfrak{g}\}$.

Next theorem states the commutativity of this algebra in $\mathfrak{A}(\mathfrak{g})$.

Theorem 3.1. *Let $\mathfrak{g}$ be a classical simple Lie algebra over the field $\mathbf{K}$, with the basis $\{\widehat{M_i}\}$. Let $\mathfrak{A}(\mathfrak{g})$ be its universal enveloping algebra, $\mathfrak{Z}(\mathfrak{A}(\mathfrak{g}))$ its center. Let $\{\widehat{C_{m_k}(M)},\ k = 1, \dots, \operatorname{rank}\mathfrak{g}\}$ be a full set of the generators of $\mathfrak{Z}(\mathfrak{A}(\mathfrak{g}))$:*

$$\widehat{C_{m_k}(M)} = \sum_{i_1,i_2,\dots,i_k=1}^{\dim\mathfrak{g}} c_{i_1 i_2 \cdots i_k} \widehat{M_{i_1}}\widehat{M_{i_2}} \cdots \widehat{M_{i_k}},$$

where $c_{i_1 i_2 \cdots i_k}$ is some invariant tensor. Let us consider decomposition:

$$C_{m_k}(\widehat{M + \lambda A\mathbf{1}}) = \sum_{i_1,i_2,\dots,i_k=1}^{\dim\mathfrak{g}} c_{i_1 i_2 \cdots i_k} (\widehat{M_{i_1}} + A_{i_1}\mathbf{1})(\widehat{M_{i_2}} + A_{i_1}\mathbf{1}) \cdots (\widehat{M_{i_k}} + A_{i_1}\mathbf{1}) = \sum_{l=0}^{m_k} \lambda^l \widehat{C^A_{l,m_k}},$$

where $A_i \in \mathbf{K}$, $i \in 1, \cdots, \dim\mathfrak{g}$. Let L_A be the subalgebra in $\mathfrak{A}(\mathfrak{g})$ generated by the elements $\{C^A_{0,m_k}(\widehat{M})C^A_{1,m_k}(\widehat{M}),\ k = 1, \dots, \operatorname{rank}\mathfrak{g}\}$.

Then subalgebra L_A is commutative.

Proof of the theorem follows from the results of paper [11]. Indeed as it is easy to prove, from the parts (i) of the theorems 1 and 2 of paper [11] follows the commutativity of the subalgebra $L_A \subset \mathfrak{A}(\mathfrak{g})$ in the case of the special choice of the Casimir operators. On the other hand L_A does not depend on the choice of the generating set of Casimir operators. Indeed it could be easily proved, using the fact that every other set of Casimir operators could be expressed as a polynomials in the elements $\widehat{C_{m_k}(M)}$ and vice versa. From the latter fact follows that different choice of the set of Casimir elements leads just to the other choice of the generators of L_A. Under such correspondence new set of Casimir elements are expressed by the polynomials in the

$\widehat{C_{m_k}(M)}$. New set of integrals linear in the tensor of inertia A will be expressed polynomially in the $\widehat{C_{m_k}}$ and linearly in $C^A_{1,m_k}(\widehat{M})$, $(k = 1, \ldots, \operatorname{rank}\mathfrak{g})$.

Example. $\mathfrak{g} = gl(n, \mathbf{K})$. Let $\widehat{M}_{i,j}, i, j \in I$, where $I = (1, 2, \ldots, n)$, be the basis in this algebra with the commutation relations:

$$[\widehat{M}_{i,j}, \widehat{M}_{k,l}] = \delta_{k,j}\widehat{M}_{i,l} - \delta_{i,l}\widehat{M}_{k,j}.$$

Universal enveloping algebra $U(gl(n))$ consists of formal polynomials in the elements $\hat{M}_{i,j}$. Let us define the following elements of $U(gl(n))$:

$$(\widehat{M^m})_{i,j} = \sum_{i_1,\ldots,i_{m-1}\in I} \widehat{M}_{i,i_1}\widehat{M}_{i_1,i_2}\cdots\widehat{M}_{i_{m-1},j}.$$

It is known [9], that the elements: $\widehat{C_{m_k}(M)} = (\widehat{M^m}) = \sum\limits_{i\in I}(\widehat{M^m})_{i,i}$, $m \in (1, 2, \ldots, n)$ generate the center of the universal enveloping algebra. For the generalized inertia tensor one can take arbitrary matrix $A \in \operatorname{Mat}(n, \mathbf{K})$. It is not difficult to check, that

$$C^A_{1,m} = \sum_{l=0}^{m-1}(\widehat{M}^l)_{i,j}A_{jk}(\widehat{M}^{m-l})_{k,i}.$$

In this case, instead of the elements $C^A_{1,m}$ one can chose another generators of L_A, which have more simple form:

$$(\widehat{AM^m}) = \sum_{i,j\in I} A_{j,i}(\widehat{M}^m)_{i,j}.$$

Indeed, using the commutation relation one can easily express $C^A_{1,m}$ linear in the terms of $(\widehat{AM^m})$ and $(\widehat{M^n})$ (and vice versa).

To prove the complete quantum integrability of the generalized Eulers top with the generic inertia tensor A associated with the generic (co)adjoint orbit in the Lie algebra $\mathfrak{g}$ one have to prove the commutativity of the whole Mishchenko–Fomenko algebra in $\mathfrak{A}(\mathfrak{g})$. But, if one consider the case of the nongeneric inertia tensor A or the nongeneric (co)adjoint orbit in $\mathfrak{g}$ then for proving quantum integrability only some commutative subalgebras from the Mishchenko–Fomenko algebra are needed.

Let us consider the case of nongeneric inertia tensor first. We will essentially use the following

Lemma 3.1. *Let $\mathfrak{g}_A$ be a centralizer of matrix A in $\mathfrak{g}$. Then $\mathfrak{g}_A$ centralize L_A in $\mathfrak{g}$.*

Proof. It follows from the parts (ii) of theorems 1 and 2 [11] along with the fact that L_A does not depend on the choice of the full set of independent Casimir operators.

Example. Let $\mathfrak{g} = gl(n)$, $A \in \operatorname{Mat}(n, \mathbf{K})$. Then

$$g_A = \Big\{(B\widehat{M}) = \sum_{i,j\in I} B_{j,i}\widehat{M}_{i,j} \mid B \in \operatorname{Mat}(n, \mathbf{K}), [B, A] = 0\Big\}.$$

This lemma enables us to construct full set of commuting quantum integrals of the generalized Euler's top in the case when inertia tensor is not generic, i.e., possesses additional symmetries. in this case Hamiltonian and all other integrals commute with the generators of this symmetries. That is why one can take for the full set of the integrals (both classical and quantum) some set of independent integrals from the Mishchenko–Fomenko algebra along with some full set of commuting integrals from the $\mathfrak{A}(\mathfrak{g}_A)$. One has only find out under what conditions on the matrix A the number of independent generators of L_A plus the maximal number of the commuting integrals from $\mathfrak{A}(\mathfrak{g}_A)$ is equal to $(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})/2$.

Answer to this question gives the following theorem.

Theorem 3.2. *Let $\mathfrak{g}$ be Lie algebra of the type $gl(n)$, $so(n)$ or $sp(n)$. Let centralizer of the numerical matrix $A \in \mathfrak{g} \subset \mathrm{Mat}\,(n, \mathbf{K})$ contain Lie subalgebra of the type $gl(n-2)$, $so(n-2)$ or $sp(n-1)$ respectively. Then quantum Eulers top with the inertia tensor A is integrable.*

Let us consider the "dual" case when the inertia tensor is generic, but the Euler–Arnold equations are restricted to the symplectic leaf of low dimension – strongly degenerated coadjoint orbit $O_{\deg} \simeq G/K$. Due to the fact that the number of mutually commuting integrals should be equal to the one half of the dimension of the phase space it will be in this case substantially smaller. This enables us to state the following theorem.

Theorem 3.3. *Let $O_{\deg} \simeq G/K$ be the degenerated coadjoint orbit in the Lie algebra of the type $gl(n)$, $so(n)$ or $sp(n)$. Let its stabilizer K contains Lie subgroup of the type $Gl(n-2)$, $SO(n-2)$ or $Sp(n-1)$ respectively. Then quantum Eulers top associated with this orbit is integrable for the arbitrary inertia tensors A.*

Proof. From the results of [12] follows that after restriction of the generators of Mishchenko-Fomenko algebra to the orbits of the described in the theorem type, independent generators could be chosen among the generators of the first and second orders in the coordinates of algebra. Hence the statement of the theorem follows from the results of [8].

Acknowledgments. The research described in this publication was possible in part by the Award number UP1-309 of the U.S. Civilian Research and Development Foundation (CRDF) for independent states of the former Soviet Union.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 530–536.

Symmetry-Breaking Bifurcations in a Single-Mode Class-A Gas Laser

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The invariance properties of equations of motion and symmetric bifurcations of stationary and periodic solutions of codimensionality 1 and 2 corresponding to polarization symmetry breaking and restoration were analyzed in a single-mode standing-wave class-A gas laser with linear phase anisotropy of the cavity at $j \to j+1$ transition between the working levels.

1 Introduction

The account of vectorial nature of electromagnetic field in nonlinear dynamics of laser systems, besides additional equations compared to the scalar field approach, assumes the appearance of radically new properties of dynamical systems inherent in polarized radiation. One of these properties is invariance (symmetry) of the system with respect to the transformation of the state of polarization. In spite of the fact that in optics transformations of polarization accompanied by polarization symmetry breaking are known (see, for example, [1]) their mathematical description in the language of singular (symmetric) bifurcations is absent.

The aim of the present work is to study the invariance properties of the equations of motion of a single-mode class-A gas laser with linear phase anisotropy of the cavity, operating at $j_b = 1 \to j_a = 2$ transition and to analyze symmetric bifurcations of codimension 1 and 2 of stationary and periodic solutions resulting in polarization symmetry breaking and restoration phenomena.

2 Theoretical model

The theoretical analysis is based on the model of a single-mode (two-frequency) anisotropic-cavity gas laser with a longitudinal magnetic field on the active medium, derived and explored in [2, 3]. In the case of linear phase anisotropy of the cavity at the line center tuning the equations of motion can be reduced to a system of three ODEs which takes the following form [3]:

$$\frac{dI_1}{d\tau} = 2I_1 \left\{ \frac{P_1}{P} + \frac{\Delta W'}{P} \tanh 2\beta_1 - q \left(1 - \frac{\cos 2\Phi_1}{\cosh 2\beta_1} \right) - I_1 \left(\theta_1' + \theta_2'' \tanh^2 2\beta_1 \right) \right\}, \tag{1}$$

$$\frac{d\Phi_1}{d\tau} = -(qS_1 + rS_2) - \frac{\Delta W''}{P} + \theta_2'' \tanh 2\beta_1 I_1, \tag{2}$$

$$\frac{d\beta_1}{d\tau} = rS_1 - qS_2 + \frac{\Delta W'}{P} - \theta_2' \tanh 2\beta_1 I_1. \tag{3}$$

Here $I_1 = I_1'/k_0 l P$ is the dimensionless intensity, $f_1 = \Phi_1 + i\beta_1$, $\xi_1 = \tanh \beta_1$ is the ellipticity; Φ_1 is the azimuth of the wave 1, the characteristics of the waves 1 and 2 are interrelated: $I_2 = I_1$, $\Phi_2 = \Phi_1 + \pi/2$, $\xi_2 = -\xi_1$; $W(x \pm \Delta, y) = U(x \pm \Delta, y) + iV(x \pm \Delta, y)$ is the complex error function, $\bar{W} = \bar{U} + i\bar{V} = [W(x-\Delta, y) + W(x+\Delta, y)]/2$, $\Delta W = [W(x-\Delta, y) - W(x+\Delta, y)]/2$; $x_1 \pm \Delta = (\omega_1 - \omega_0 \pm g\mu_B H)/Ku$ is the detuning of the lasing frequency ω_1 from the line center ω_0, relative to Ku; $\theta_1 = b_1 + a_{12} - b_{12}$, $\theta_2 = d_1 - d_{12} + b_{12}$; a, b, d are the coefficients of nonlinear interaction which as well as the other parameters are determined in [3]; $S_1 = \sin 2\Phi_1 \cosh 2\beta_1$, $S_2 = \cos 2\Phi_1 \sinh 2\beta_1$, $q = (1 - \cos 4\psi)/2\tau_0$, $r = \sin 4\psi/2\tau_0$, $\Delta W = \Delta W' + i\Delta W''$, $\Delta = \Delta_0 + \Delta_1 \sin f\tau$, $\omega_f/2\pi(KHz) = 10^3 f\tau_0 c/2\pi L$, $\Delta = g\mu_B H/Ku$, $\Delta_0 = g\mu_B H_0/Ku$, $\Delta_1 = g\mu_B H_1/Ku$, H_0 and H_1 are the strength of constant and sinusoidal magnetic field, respectively 2ψ is the linear phase anisotropy of the cavity;

$$\begin{aligned} \Delta W &= \Delta W' + i\Delta W'' \\ &\approx 2\Delta\left\{\delta x\left(1 - \frac{4y}{\sqrt{\pi}}\right) + i\left[y - \frac{1}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}\left(\frac{\Delta^2}{3} - y^2 + \delta x^2 + \frac{\delta x y}{\Delta}\right)\right]\right\}, \end{aligned} \tag{4}$$

$\delta x = (x_1 - x_2)/2$.

Stability analysis of the stationary and periodic solutions was carried out on the basis of the numerical methods of the theory of bifurcation [4, 5].

3 Pitchfork bifurcation of the stationary solution

Assuming the amplitude of the longitudinal magnetic field on the active medium equal to zero: $(H_0 = H_1 = 0)$, let us consider the transformation of the state of polarization of the emitted field at transition from isotropic $(\psi = 0)$, to anisotropic cavity, when $\psi \neq 0$.

Under the assumptions that the changes in intensity with time can be neglected $(I_1 = I_2 = I_0)$ the stationary solutions to equations (2), (3) for polarization characteristics can be found analytically (see, for example, [3]):

$$\Phi_1 = 0, \pm\pi/2, \qquad \xi_1 = 0, \tag{5}$$

$$\Phi_1 = \pm\pi/4, \qquad \sinh 2\beta_1 = \pm\left\{-\alpha/2r' - \left(\alpha^2/4r'^2 - 1\right)^{1/2}\right\}, \tag{6}$$

$$\Phi_1 = \pm\pi/4, \qquad \sinh 2\beta_1 = \pm\left\{-\alpha/2r' + \left(\alpha^2/4r'^2 - 1\right)^{1/2}\right\}, \tag{7}$$

where $\alpha = 2\tau_0\theta_2 I_0$, $r' = \sin 4\psi$.

Stability analysis of these solutions, carried out numerically in [6], showed that for $j \to j$ transitions for all values of ψ the two orthogonal linearly polarized waves, described by (5), are stable. For $j \to j+1$ transitions in a region of $\psi : \alpha/4 < \sin 2\psi < (\alpha/4)^{1/2}$, a steady-state regime with periodic oscillations of the intensity, ellipticity and azimuth of the emitted field is found. The limit cycle appears at $\psi^* = 1/2 \arcsin(\alpha/4)^{1/2}$ due to the Hopf bifurcation and is destroyed at point $\psi^{**} = 1/2 \arcsin(\alpha/4)$ due to the appearance of a saddle-node point (see, for example, [5]).

The expressions (6) describe the stable solutions (the state of equilibrium is the node), the expressions (7) describe the unstable solutions (the state of equilibrium is the saddle), existing in the region $\psi^{**} < 1/2 \arcsin(\alpha/4)$. The detailed bifurcation analysis of the system under consideration requires more complicated model, it was carried out in [3]. Here we consider only the symmetry-breaking bifurcations.

At $\psi = 0$ (isotropic cavity) two steady-state solutions (6), corresponding to the $(\pm)$ signs, coincide. Each of them describes two waves with orthogonal circular states of polarization and zero frequency difference, which give one wave with linear polarization. This result is in agreement with previous studies (see, for example, [6]).

When the cavity anisotropy ψ is infinitesimal so that the isotropic cavity is transformed into an anisotropic cavity, the solutions (6) give two different two-frequency regimes. Each of these regimes is represented by two orthogonal elliptically polarized waves with high ellipticities (practically circular) and different frequencies. A particular regime of lasing is determined by the initial conditions.

This bistability reflects the invariance properties of equations (1)–(3) with respect to the transformation of variables:

$$G = \{I_1, \Phi_1, \xi_1\} \to \{I_1, -\Phi_1, -\xi_1\} \tag{8}$$

and corresponds to the symmetric pitchfork bifurcation (see, for example, [8]) which takes place in the vicinity of the point $\psi = 0$. As a result of this bifurcation the initial solution (one linearly polarized wave) loses its stability, and two new stationary two-frequency elliptically polarized solutions with high ellipticities (practically circular) appear. These new solutions can be obtained from each other by the transformation G. Pitchfork bifurcation of the stationary solution and polarization symmetry breaking phenomenon is shown schematically in Fig. 1.

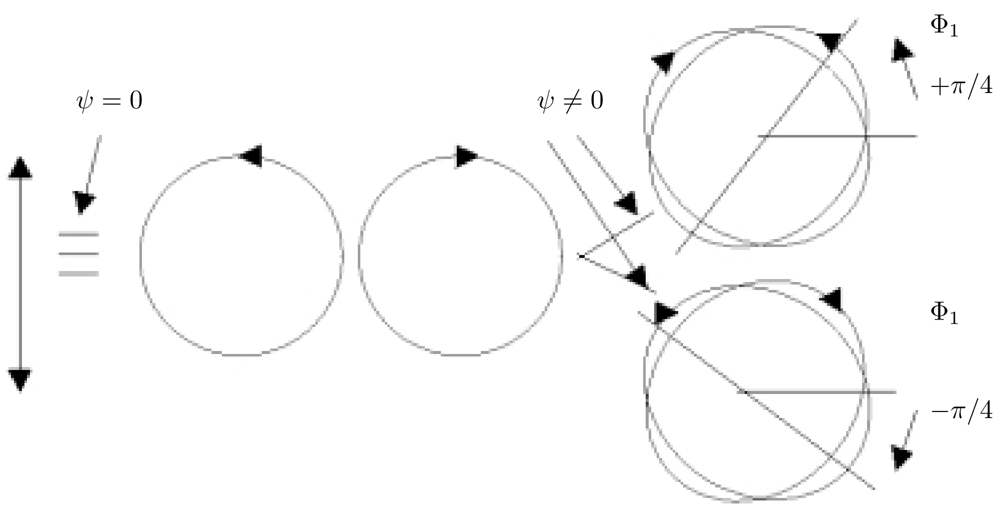

Fig. 1. Pitchfork bifurcation of the stationary solution resulting in spontaneous polarization symmetry breaking at transition from isotropic to anisotropic cavity.

Due to the bistability of these two-frequency regimes, which are connected with each other by the transformation G, and due to the fact that the choice of solution is fully determined by the initial conditions, we can say that the decomposition of one linearly polarized wave into two orthogonal elliptically polarized (with high values of ellipticities, practically circular) waves with different frequencies which occurs at transition from an isotropic to an anisotropic cavity is a spontaneous polarization symmetry breaking.

4 Symmetry-breaking bifurcation of periodic solutions of equations of motion

Let us consider the invariance properties of equations (1)–(3) in the presence of a sinusoidal magnetic field on the active medium ($H_0 = 0, H_1 \neq 0$) and analyze the bifurcations of periodic solutions of codimensionality 1 and 2, reflecting these properties. Let us choose the range of control parameter ψ where in the system a stable limit cycle of the first kind is realized [2, 3].

It is easy to see that when the longitudinal magnetic field is imposed on the medium, the system of equations (1)–(3) is invariant with respect to the following transformation:

$$G = \{I_1, \Phi_1, \xi_1, H_1\} \to \{I_1, -\Phi_1, -\xi_1, -H_1\}. \tag{9}$$

Reversing of the sign of H_1 is equivalent to the shift of the phase of the external force signal on π. Analogous symmetry properties are intrinsic for some dynamical systems, in particular, for those, describing by the Duffing equations (see, for example, [9]). In these systems the periodic solutions are possible whose bifurcations occur by a different way than bifurcations of periodic solutions in systems without symmetry (see, for example, [10]).

Let $X(t)$ be the periodic solution of the system (1)–(3), and $\tilde{X}$ be the trajectory of this solution in the phase space. Then in accordance with classification of symmetric limit cycles (see, for example, [10]), the following solutions are possible:

F-cycle is the solution, invariant with respect to the transformation G:

$$GX(t) = X(t); \tag{10}$$

S-cycle is the solution which is not invariant with respect to the transformation G, but the trajectories of the both cycles coincide, i.e. the phase trajectory of the cycle consists of two congruent parts. The solution is invariant with respect to the transformation G + shift of the time series on a half of period of a cycle T:

$$GX(t) \neq X(t), \qquad GX(t) = \tilde{X}, \qquad GX(t) = X(t + T/2); \tag{11}$$

M-cycle is the asymmetric solution which at the transformation G turns into the second asymmetric solution:

$$GX_1(t) = X_2(t). \tag{12}$$

M-cycles always originate in pair and undergo simultaneously the same sequence of bifurcations, intrinsic for systems without symmetry. In the system under consideration S- and M-cycles were found.

Fig. 2 shows the upper part of the diagram on the plane of parameters H_1, ω_f inside the resonance (1/1), calculated at the following parameters of the $He - Ne$ ($\lambda = 0.63\mu m$) laser: $\psi = 0.001$ rad, $\eta_1 = \eta_2 = 1.9$, $c/L = 612$ MHz, $y_1 = 0.011$, $y_2 = 0.005$, $y = 0.2$, $k_0 l = 0.025$, $Ku = 870$ MHz.

Detailed study of the dynamics of this nonautonomous system has been carried out in [11], where the evolution of solutions in the region of resonance (1/1) at large values of H_1 is shown schematically.

In the region 1, bounded by the lines l_0, on which the Neimark–Sacker bifurcation takes place (a pair of complex-conjugate multipliers crosses the unit circle: $|\mu_1| = |\mu_2| = 1$), the resonance S-cycle exists. On going out of the resonance region across these lines, softly, with zero amplitude on the second frequency in the power spectrum, a two-dimensional S-torus is originated, which exists in the region 2.

With increasing parameter H_1 the lines l_0 are ended at the points F of codimension 2, where both of the multipliers of the cycle 1 become unit: $\mu_1 = \mu_2 = +1$), which corresponds to the strong resonance condition (1/1) [12]. Through points F the line l_{11} is passing on which the resonance S-cycle loses its stability as a result of the saddle-node bifurcation for a system with symmetric properties and instead of it a pair of stable asymmetric M-cycles with period of the driving force T originates.

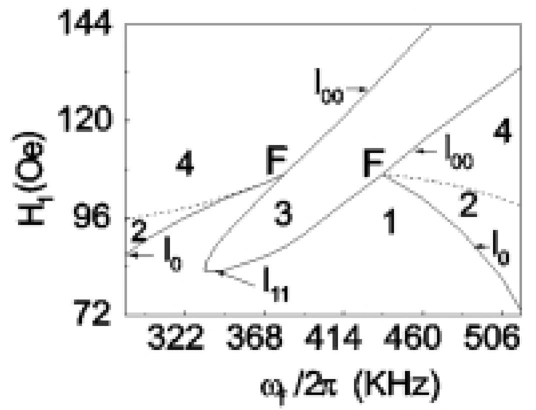

Fig. 2. A part of the bifurcation diagram on the plane of parameters (H_1, ω_f) in the region of the resonance (1/1).

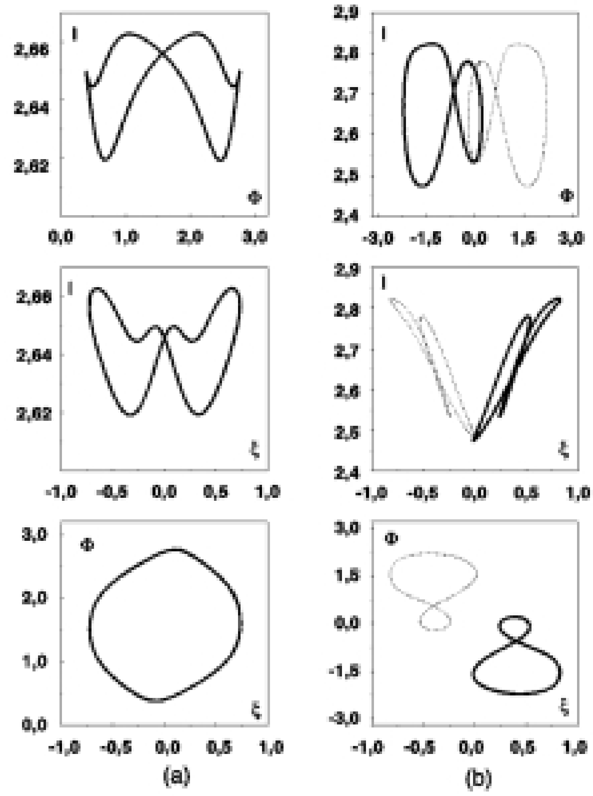

Fig. 3. Pitchfork bifurcation of the limit cycle. Phase projections reflecting the invariance with respect to the transformation G of symmetric S-cycle calculated at $H_1 = 0.88.4$ Oe, $\omega_f = 381$ KHz (a) and asymmetric M-cycles calculated at $H_1 = 90.8$ Oe, $\omega_f = 381$ KHz (b).

In the region 3 one of the two possible asymmetric solutions is realized; the switch to the other solution occurs at exchange of the signs of the initial conditions and magnetic field strength. Fig. 3 reflects changes of the phase projections of the S (a) and M (b) cycles with respect to the transformation G.

As it can be seen from Fig. 3, as a result of the transformation G, the phase trajectory of the S-cycle remains unaltered, while the M-cycles are transformed into each other. Thus, on the line l_{11} polarization symmetry breaking phenomenon for periodic solutions of the equations of motion takes place. This bifurcation is an analog to the pitchfork bifurcation of the stationary solutions.

Above the points F the lines of formation of a new two-dimensional S-torus with complicated form of oscillations are fixed. At so doing, both of the asymmertic M-cycles simultaneously lose their stability as a result of the saddle-node bifurcation and form the symmetric long-period oscillation on torus, which exists in the region 4. Complication of the form and increase of the period of oscillation on torus in the region 4, originated inside of the region of synchronization (1/1), is due to appearance of the high order ($p > 5$) resonances [12]. Inside these resonances symmetric periodic oscillations are fixed whose period is p times larger than the period of the driving force T. In the system under consideration long-periodic oscillations with period $7T$ and $9T$ have been found [11]. The dashed lines in the diagram separate approximately the region of the torus 2, originated from the initial cycle and torus 4, originated from the long-periodic cycle inside the locking zone. When these lines intersect, a torus with long-periodic oscillations appears as a result of a long-term intermediate process, so in the Poincare section it is possible to observe simultaneously the destruction of the old and the creation of a new torus. The behavior of the dynamical system in the vicinity of the points F of codimension 2 corresponds to the suggestion $5S$ [10], namely, this is the case of resonance (1/2) in a system without symmetry. To elucidate this statement one should take into account that the behavior of symmetric S-cycles is characterized not by the eigenstates of the linearization matrix in the Poincare map P, which describes the shift of the solution for the period of cycle T, but by the eigenstates of the matrix $Q = P^{1/2}$, describing the transformation of the periodic solution during one half of the period T.

At the strong resonance conditions (1/1) for matrix P, depending on the symmetry properties of the matrix Q, its eigenstates can be multiple and equal to $+1$ (resonance (1/1) in systems without symmetry), multiple and equal to -1 (resonance (1/2)), as well as equal to ± 1. As mentioned above, in the system under consideration the resonance (1/2) conditions for system without symmetry are realized.

When constant longitudinal magnetic is added, the phase space become cylindrical [11]. Equations (1)–(3) are invariant with respect to the transformation G given by (9), where H_1 should be replaced by $H = H_0 + H_1$. This invariance results in the bistability of M-cycles of the first and the second kind and in the experimentally observed effect of sign reversal of azimuth rotation. Bistability of the asymmetric M-cycles with period of the external force T is shown in Fig. 4.

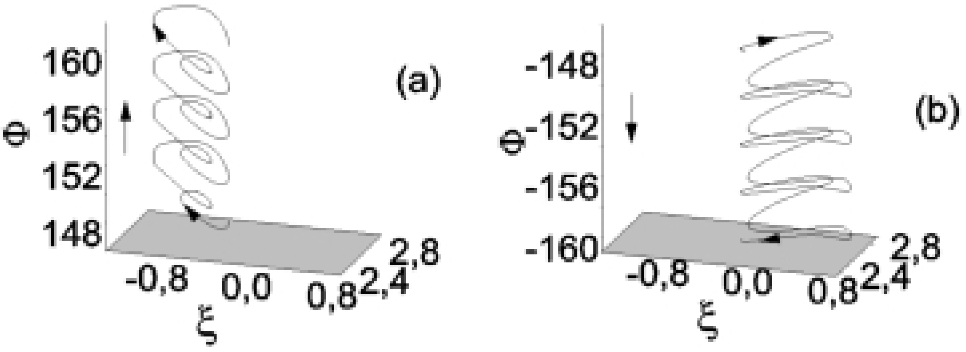

Fig. 4. Bistability of the asymmetric M-cycles of the second kind; the arrows mark the direction of azimuth rotation: right rotation (a), left rotation (b).

Conclusions

Analysis of the invariance properties of the equations of motion of a single-mode standing-wave class-A gas laser with linear phase anisotropy of the cavity at the $j \to j+1$ transition between the working levels has revealed a series of polarization symmetry breaking phenomena. Spontaneous polarization symmetry breaking, corresponding to pitchfork bifurcation of stationary solution, occurs at transition from an isotropic to an anisotropic cavity and destruction of the laser modes degeneracy. In the vicinity of the bifurcation point one wave with linear state of polarization is decomposed into two elliptically polarized waves with high values of ellipticity, and bistability of these two-wave solutions arises. In the presence of a sinusoidal magnetic field on the active medium polarization symmetry breaking has been found which corresponds to pitchfork bifurcation of periodic solution: symmetric S-cycle is decomposed into two asymmetric M-cycles. Then S-type symmetry is restored through appearance of S-torus which undergoes high-order resonances.

In the presence of a constant longitudinal magnetic field on the active medium bistability has been found of asymmetric M-cycles of the first and the second kind resulting in the experimentally observed effect of sign reversal of azimuth rotation.

At present the polarization symmetry breaking and the restoration phenomena are becoming the subject of intensive studies due to their possible application in the optical processing of information for making devices based on novel physical principles.

Acknowledgments

The author wishes to thank Ye.V. Nikolaev for fruitful discussions.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 537–543.

On Exact Foldy–Wouthuysen Transformation of Bozons in an Electromagnetic Field and Reduction of Kemmer–Duffin–Petiau Equation

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Using discrete symmetries of the Kemmer–Duffin–Petiau (KDP) equation the exact Foldy–Wouthuysen transformation (FWT) was found. It is required that the vector-potential of an external field has definite parities. We also described reduction of the KDP equation to uncoupled subsystems which can be solved independently.

The FWT [1] provides several advantages for the understanding and interpretation of the physical properties of the Dirac equation. It permits to reduce of this equation to a two-component equation of the Pauli type. But its main achievement consist in separating of the solution of Dirac equation corresponding to a definite sign of the energy eigenvalues. There are great number of papers are devoted to the construction of FWT for spin-0 [2] and spin-1 [3] particle.

In the presence of interaction the FWT has not, in general, a closed form and one usually uses series expansion methods. There are classes of interaction represented for instance by the static magnetic potentials [4], by the static electric and the pseudo-scalar potentials [5] which admit the exact FWT. The FWT for a two-body equation with oscillator-like interaction [6], for systems composed of one fermion and one boson, and one fermion and one antifermion in the presence of special classes of interactions [7] was also constructed in a closed form.

In this paper we investigate the KDP equation for scalar and vector particle in an electromagnetic field. In order to construct exact FWT we used discrete symmetries of the corresponding equations. The idea to use discrete symmetries (space reflections, time inversion and charge conjugation) for reductions of the Dirac and Schrödinger–Pauli equation to uncoupled subsystems was proposed in [8, 9].

Let us consider KDP equation for scalar ($s = 0$) and vector ($s = 1$) particles minimally interacting with external electromagnetic field. These equations in the Schrödinger form read [10]

$$\begin{aligned} & i\frac{\partial}{\partial t}\Psi(x) = H_1(A_0, \vec{\pi})\Psi(x), \\ & H_1 = \sigma_2 m + (i\sigma_1 + \sigma_2)\frac{\pi^2}{2m} + eA_0, \qquad s = 0; \end{aligned} \tag{1}$$

$$\begin{aligned} & i\frac{\partial}{\partial t}\Psi(x) = H_2(A_0, \vec{\pi})\Psi(x), \\ & H_2 = \sigma_2 m + (i\sigma_1 + \sigma_2)\frac{(\pi^2 - e\vec{S}\cdot\vec{H})}{2m} - i\sigma_1\frac{(\vec{S}\cdot\vec{\pi})^2}{m} + eA_0, \qquad s = 1, \end{aligned} \tag{2}$$

where

$$\pi_a = p_a - eA_a, \qquad p_a = -i\frac{\partial}{\partial x_a}, \qquad a = 1, 2, 3,$$

$$\pi^2 = \pi_1^2 + \pi_2^2 + \pi_3^2, \qquad A_0 = A_0(t, \vec{x}), \qquad A_a = A_a(t, \vec{x}),$$

$$\vec{H} = i[\vec{p} \times \vec{A}], \qquad \sigma_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \sigma_2 = i\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

I is a $(2s+1)$-dimensional unit matrix, S_a are 6-dimensional matrices realizing a direct sum of the $AO(3)$-representations $D(1)$, $D(1)$, $\Psi(x)$ is a wave function which has $2(2s+1)$ physical components.

We note that for physical reasons it is preferable to consider another form of (1), (2). It is connected with our consideration by unitary transformation $U = \frac{1}{2}(1 + i\sigma_1)$, $H'_{1,2} = UH_{1,2}U^{-1}$.

In order to construct FWT for Hamiltonians of (1), (2) we will use a method proposed in [11].

Let us define an unitary involution operator I anticommuting with H of (1), (2):

$$I^+I = II^+ = I^2 = 1, \qquad IH + HI \equiv [I, H]_+. \tag{3}$$

We seek the involution I in the form

$$I = MD, \tag{4}$$

where M is a numeric matrix, D are operators of the discrete transformation:

$$D = \langle R_a, T, R_aT\rangle, \qquad a = 1, 2, 3, 12, 23, 31, 123,$$

$$R_a\Psi(t, \vec{x}) = r_a\Psi(t, r_a\vec{x}), \qquad r_a : x_a \to -x_a,$$

$$T\Psi(t, \vec{x}) = r_0\Psi(r_0t, \vec{x}), \qquad r_0 : t \to -t,$$

$$r_a = \pm 1, \qquad r_0 = \pm 1, \qquad R_{123} \equiv R, \qquad r_{123} \equiv r,$$

$$r_ar_b: \ x_a \to -x_a, \quad x_b \to -x_b, \quad a \neq b,$$

$$r_ar_br_c: \ x_a \to -x_a, \quad x_b \to -x_b, \quad x_c \to -x_c, \quad a \neq b, \ b \neq c, \ a \neq c.$$

Theorem 1. *I. All possible involutions (up to equivalence) of form (4) anticommuting with (1) have the form*

1. $I_1 = \sigma_3 R,$ (5.1)

2. $I_2 = \sigma_3 T,$ (5.2)

3. $I_3 = \sigma_3 RT,$ (5.3)

4. $I_4 = \sigma_3 R_a, \qquad a = 1, 2, 3,$ (5.4)

5. $I_5 = \sigma_3 R_{ab}, \qquad a \neq b,$ (5.5)

6. $I_6 = \sigma_3 R_aT,$ (5.6)

7. $I_7 = \sigma_3 R_{ab}T$ (5.7)

if the corresponding parities of vector-potential $A_\mu(t, \vec{x})$ $(\mu = 0, 1, 2, 3)$ *are given by the relations*

1. $A_0(t, \vec{x}) = -A_0(-t, \vec{x}),$
   $A_a(t, \vec{x}) = -A_a(t, r\vec{x}).$ (6.1)

$$\begin{aligned} 2.\quad & A_0(t,\vec{x}) = -A_0(t, r\vec{x}), \\ & A_a(t,\vec{x}) = A_a(-t,\vec{x}). \end{aligned} \tag{6.2}$$

3. *(there are four subcases of parities of* A_μ*):*

$$\begin{aligned} a)\quad & A_0(t,\vec{x}) = -A_0(-t,\vec{x}), \quad A_0(t,\vec{x}) = A_0(t,r\vec{x}), \\ & A_a(t,\vec{x}) = A_a(-t,\vec{x}), \qquad A_a(t,\vec{x}) = -A_a(t,r\vec{x}); \\ b)\quad & A_0(t,\vec{x}) = -A_0(-t,\vec{x}), \quad A_0(t,\vec{x}) = A_0(t,r\vec{x}), \\ & A_a(t,\vec{x}) = -A_a(-t,\vec{x}), \quad A_a(t,\vec{x}) = A_a(t,r\vec{x}); \\ c)\quad & A_0(t,\vec{x}) = A_0(-t,\vec{x}), \qquad A_0(t,\vec{x}) = -A_0(t,r\vec{x}), \\ & A_a(t,\vec{x}) = -A_a(-t,\vec{x}), \quad A_a(t,\vec{x}) = A_a(t,r\vec{x}); \\ d)\quad & A_0(t,\vec{x}) = A_0(-t,\vec{x}), \qquad A_0(t,\vec{x}) = -A_0(t,r\vec{x}), \\ & A_a(t,\vec{x}) = A_a(-t,\vec{x}), \qquad A_a(t,\vec{x}) = -A_a(t,r\vec{x}). \end{aligned} \tag{6.3}$$

$$\begin{aligned} 4.\quad & A_0(t,\vec{x}) = -A_0(t, r_a\vec{x}), \\ & A_a(t,\vec{x}) = -A_a(t, r_a\vec{x}) \qquad \textit{(no sum over } a\textit{)}, \\ & A_a(t,\vec{x}) = A_a(t, r_b\vec{x}), \qquad a \neq b. \end{aligned} \tag{6.4}$$

$$\begin{aligned} 5.\quad & A_0(t,\vec{x}) = -A_0(t, r_a r_b\vec{x}), \qquad a \neq b, \\ & A_a(t,\vec{x}) = -A_a(t, r_a r_b\vec{x}) \qquad a \neq b, \\ & A_a(t,\vec{x}) = A_a(t, r_b r c\vec{x}), \qquad a \neq b, \quad b \neq c, \quad c \neq a. \end{aligned} \tag{6.5}$$

6. In this case the parities of A_μ *are the same as in the case 3 (formula (6.3)) up to the replacement of* r *by* r_a*. In addition,* A_μ *should satisfy the following relations:*

$$\begin{aligned} & \left.\begin{aligned} & A_a(t,\vec{x}) = -A_a(t, r_a\vec{x}), \\ & A_a(t,\vec{x}) = A_a(t, r_b\vec{x}), \quad a \neq b, \end{aligned}\right\} \quad \textit{for } a) \textit{ and } d); \\ & \left.\begin{aligned} & A_a(t,\vec{x}) = A_a(t, r_a\vec{x}), \\ & A_a(t,\vec{x}) = -A_a(t, r_b\vec{x}), \quad a \neq b, \end{aligned}\right\} \quad \textit{for } b) \textit{ and } c). \end{aligned} \tag{6.6}$$

7. In this case the parities of A_μ *are the same as in the case 3 (formula (6.3)) up to the replacement of* r *by* $r_b r_c$*,* $b \neq c$*,* $a \neq b$*,* $a \neq c$*. In addition,* A_μ *should satisfy the following relations (*$b \neq c$*,* $a \neq b$*,* $a \neq c$*)*

$$\begin{aligned} & A_a(t,\vec{x}) = A_a(t, r_b r_c\vec{x}), \qquad \textit{for } a) \textit{ and } d), \\ & A_a(t,\vec{x}) = -A_a(t, r_b r_c\vec{x}), \qquad \textit{for } b) \textit{ and } c). \end{aligned}$$

II. All possible involutions (up to equivalence) of form (4) anticommuting with (2) have the form (5.1), (5.2), (5.3) if the vector-potential A_μ *has the parities (6.1), (6.2), (6.3), correspondingly.*

Proof. Requiring the anticommutativity of operator (4) with the Hamiltonian of (1), we obtain the following conditions for M and D:

$$[\sigma_2, M]_+ = 0, \qquad [\sigma_1, M]_+ = 0, \tag{7.1}$$

$$[A_0, D]_+ = 0, \qquad [\pi^2, D] = 0. \tag{7.2}$$

It follows from (7.1) that

$$M = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Conditions (7.2) lead us to (5), (6). In a similar way we can find the involutions for Hamiltonian of (2). Theorem is proved.

Theorem 2. *The exact FWT for Hamiltonians of (1), (2) which have no zero eigenvalues have the form*

$$U = \frac{1}{2}(1 + \sigma_1 I_a)(1 + I_a \varepsilon), \qquad U^+ = U^{-1}, \qquad \varepsilon = \frac{H}{\sqrt{H^2}}, \tag{8}$$

$a = \overline{1,7}$ *for Hamiltonian of (1),* $a = \overline{1,3}$ *for Hamiltonian of (2).*

Proof. Let us consider the case $a = 1$. Then

$$U = \frac{1}{2}(1 - i\sigma_2 R)(1 + \sigma_3 R\varepsilon). \tag{9}$$

The straightforward computation yields

$$\begin{aligned} H_1' &= UH_1U^{-1} = \sigma_3\sqrt{H_0^2} \equiv \\ &\equiv \sigma_3\left(m^2 + \pi^2(1+\sigma_1 R) + e^2A_0^2 + \frac{ie}{2m}[A_0, \pi^2]_+(R+\sigma_1) + \frac{\pi^4}{m^2}(1+\sigma_1 R)\right)^{1/2}, \end{aligned} \tag{10}$$

$$\begin{aligned} H_2' = UH_2U^{-1} = \sigma_3\Bigg(& H_0^2 + \frac{2}{m^2}\left\{2(\vec{S}\cdot\vec{\pi})^4 + (\vec{S}\cdot\vec{H})^2 + [(\vec{S}\cdot\vec{\pi})^2, \vec{S}\cdot\vec{H}]_+\right\} \\ & + 2\vec{S}\cdot\vec{H}R + 2\sigma_1\left[2(\vec{S}\cdot\vec{\pi})^2 + \vec{S}\cdot\vec{H}\right] - \frac{ie\sigma_1 R}{m}\left[A_0, 2(\vec{S}\cdot\vec{\pi})^2 + \vec{S}\cdot\vec{H}\right]_+ \\ & + \frac{ie}{m}[A_0, \vec{S}\cdot\vec{H}]_+ + \frac{2(\sigma_1 + R)}{m^2}\left\{2\pi^2(\vec{S}\cdot\vec{\pi})^2 + [\pi^2, \vec{S}\cdot\vec{H}]_+\right\} \\ & + \frac{2\sigma_1 R}{m^2}\left\{(\vec{S}\cdot\vec{H})^2 + [(\vec{S}\cdot\vec{\pi})^2, \vec{S}\cdot\vec{H}]_+\right\}\Bigg)^{1/2}. \end{aligned} \tag{11}$$

We can see that transformation (9) reduces Hamiltonians of (1), (2) to the diagonal form (10), (11). Theorem is proved.

Finally, let us consider relativistic KDP equation for spin-1 particle with minimal and anomalous interaction with electromagnetic field [12]:

$$\left[\beta^\mu \pi_\mu - m + \frac{e}{2m}\left(1 - \beta_5^2\right) S_{\mu\nu}F^{\mu\nu}\right]\Psi(x) = 0, \tag{12}$$

$$\begin{aligned} & S_{\mu\nu} = i[\beta_\mu, \beta_\nu], \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}, \\ & \beta_0 = i(e_{1,7} + e_{2,8} + e_{3,9} - e_{7,1} - e_{8,2} - e_{9,3}), \\ & \beta_1 = -i(e_{1,10} - e_{5,9} + e_{6,8} + e_{8,6} - e_{9,5} + e_{10,1}), \\ & \beta_2 = -i(e_{2,10} + e_{4,9} - e_{6,7} - e_{7,6} + e_{9,4} + e_{10,2}), \\ & \beta_3 = -i(e_{3,10} - e_{4,8} + e_{5,7} + e_{7,5} - e_{8,4} + e_{10,3}), \\ & \beta_5 = i(e_{4,1} + e_{5,2} + e_{6,3} - e_{1,4} - e_{2,5} - e_{3,6}), \end{aligned} \tag{13}$$

where we use the notations $e_{i,j}$ for 10 by 10 matrices, whose only nonzero elements are ones at the intersection of the i-th line and j-th column which are equal to unity.

Substituting the explicit form of β_μ-matrices (13) into (12) and expressing nonphysical components $(1-\beta_0^2)\Psi$ via $2(2s+1)$ physical components $\beta_0^2\Psi$, we come to the equation in Schrödinger form $i\partial_t\Psi = H\Psi$, where Ψ is a 6-component wave function and the Hamiltonian has the form

$$\begin{aligned} H &= mI_3 \otimes \sigma_2 + \frac{\pi^2}{2m} I_3 \otimes (\sigma_2 + i\sigma_1) - \frac{i}{m}\sum_{a,b=1}^{3} \pi_a\pi_b (S_aS_b \otimes \sigma_1) \\ &+ \frac{e^2}{2m^3}\sum_{a=1}^{3}(F_{0a})^2 I_3 \otimes (\sigma_2 - i\sigma_1) - \frac{e^2}{2m^3}\sum_{a,b=1}^{3} F_{0a}F_{0b}S_aS_b \otimes (\sigma_2 - i\sigma_1) \\ &+ \frac{ie}{m^2}\sum_{a,b=1}^{3} F_{0a}\pi_b S_aS_b \otimes \sigma_3 - \frac{ie}{m^2}\sum_{a=1}^{3}(F_{0a}\pi_a)^2 I_3 \otimes \sigma_3 + \frac{ie}{2m^2} M \otimes (1 - \sigma_3) \\ &+ \frac{e}{2m}(S_1F_{23} - S_2F_{31} + S_3F_{12}) \otimes (\sigma_2 - i\sigma_1) + eA_0, \end{aligned} \tag{14}$$

where we refer to direct products between 3 by 3 unit I_3 and S_a $(a = 1, 2, 3)$ matrices belonging to the $D(1)$-representation of $AO(3)$ with the usual Pauli matrices and M is a matrix with matrix elements $m_{ab} = -i\frac{\partial F_{0b}}{\partial x_a}$.

Requiring that (14) and (4) satisfy (3) we find the involutions of Hamiltonian (14):

$$\tilde{I}_1 = \left[(2S_1^2 - 1) \otimes \sigma_3\right] P_{23}T, \tag{15.1}$$

$$\tilde{I}_2 = \left[(2S_2^2 - 1) \otimes \sigma_3\right] P_{31}T, \tag{15.2}$$

$$\tilde{I}_3 = \left[(2S_3^2 - 1) \otimes \sigma_3\right] P_{12}T \tag{15.3}$$

and the following conditions for A_μ and E_a (E_a are components of electric field strength):

$$\begin{aligned} &A_\mu(t,\vec{x}) = A_\mu(-t,\vec{x}), \qquad A_0(t,\vec{x}) = -A_0(t, r_ar_b\vec{x}), \\ &A_1(t,\vec{x}) = \alpha_1 A_1(t, r_ar_b\vec{x}), \qquad A_2(t,\vec{x}) = \alpha_2 A_2(t, r_ar_b\vec{x}), \\ &A_3(t,\vec{x}) = \alpha_3 A_3(t, r_ar_b\vec{x}), \qquad \frac{\partial E_a}{\partial x_a} = 0, \qquad \text{no sum over } a, \\ &\alpha_1 = -\alpha_2 = -\alpha_3 = 1, \quad a = 2, \quad b = 3 \quad \text{for} \quad (15.1), \\ &-\alpha_1 = \alpha_2 = -\alpha_3 = 1, \quad a = 1, \quad b = 3 \quad \text{for} \quad (15.2), \\ &-\alpha_1 = -\alpha_2 = \alpha_3 = 1, \quad a = 1, \quad b = 2 \quad \text{for} \quad (15.3). \end{aligned}$$

The exact FWT of (14) has the form:

$$\begin{aligned} U_1 &= \frac{1}{2}\left(1 + S_2 \otimes \hat{I}_2 \cdot \tilde{I}_1\right)\left(1 + \tilde{I}_1\varepsilon\right), \\ U_2 &= \frac{1}{2}\left(1 + S_3 \otimes \hat{I}_2 \cdot \tilde{I}_2\right)\left(1 + \tilde{I}_2\varepsilon\right), \\ U_3 &= \frac{1}{2}\left(1 + S_1 \otimes \hat{I}_2 \cdot \tilde{I}_3\right)\left(1 + \tilde{I}_3\varepsilon\right), \end{aligned}$$

$\hat{I}_2$ is a 2×2 unit matrix.

Another problem that we explore in this note is a reduction of KDP equation to uncoupled subsystems. Let us show how it is possible to make such reduction using discrete symmetries of corresponding equations.

It is easy to verify that the involutions I_2, I_3, I_6, I_7 (see formulae (5)) are the discrete symmetries of (1) if vector-potential $A_\mu(t,\vec{x})$ has parities (6.2), (6.3), (6.6), (6.7) correspondingly. Indeed, these operators satisfy the invariance condition $[Q,L]\Psi = 0$, where $Q = \langle I_2, I_3, I_6, I_7\rangle$, $L = i\partial_t - H_1$, Ψ is an arbitrary solution of equation $L\Psi(x) = 0$. In analogy with the above we can find that equation (2) admits discrete symmetries I_2 and I_3, (formulae (5.2) and (5.3)) for the vector-potential (6.2) and (6.3) respectively.

In order to reduce (1) and (2) to uncoupled subsystems it suffices to construct unitary operators that diagonalize the discrete symmetries of these equations [8].

Let the vector-potential $A_\mu(t,\vec{x})$ satisfy relations (6.2). In this case equation (1) admits the symmetry $Q_1 = \sigma_3 T$.

Constructing the operator

$$U_1 = (T_+ - i\sigma_2 T_-), \qquad U_1^{-1} = (T_+ + i\sigma_2 T_-), \qquad T_\pm = \frac{1 \pm T}{2} \tag{16}$$

we reduce Q_1 to the block diagonal form

$$U_1 Q_1 U_1^{-1} = \sigma_3.$$

The equation (1) is transformed as

$$\begin{aligned} & L_1' \Psi' = 0, \\ & L_1' = U_1 L U_1^{-1} = U_1(i\partial_t - H_1)U_1^{-1}, \qquad \Psi' = U_1\Psi. \end{aligned} \tag{17}$$

Multiplying L_1' by nonzero matrix $-i\sigma_2$ on the left and by T on the right we obtain

$$\tilde{L}_1 = p_0 - eA_0 - imT - i\frac{\pi^2}{2m}(T + \sigma_3) \tag{18}$$

and the corresponding uncoupled equations

$$\left\{ p_0 - eA_0 - imT - i\frac{\pi^2}{2m}(T+1) \right\} \Psi_+ = 0,$$
$$\left\{ p_0 - eA_0 - imT - i\frac{\pi^2}{2m}(T-1) \right\} \Psi_- = 0,$$

where $\Psi_\pm$ are one-component functions.

If the vector-potential $A_\mu(t,\vec{x})$ satisfies relations (6.3) then equation (1) admits the symmetry $Q_2 = \sigma_3 RT$. We find diagonalizing operator in the form:

$$U_2 = (T_+ - i\sigma_2 T_-)(R_+ - i\sigma_2 R_-), \qquad U_2^{-1} = (R_+ + i\sigma_2 R_-)(T_+ + i\sigma_2 T_-),$$
$$R_\pm = \frac{1 \pm R}{2}, \qquad U_2 Q_2 U_2^{-1} = \sigma_3.$$

Corresponding reduced equation have the form

$$\tilde{L}_2 \Psi' = 0,$$
$$\tilde{L}_2 = p_0 - eA_0 - imT - i\frac{\pi^2}{2m}(\sigma_3 R + T).$$

In analogy with (1) we make a reduction of (2). As a result we obtain

$$\tilde{L}_1\Psi' = 0,$$

$$\tilde{L}_1 = p_0 - eA_0 - imT + i\sigma_3\frac{(\vec{S}\vec{\pi})^2}{m} - i\frac{(\pi^2 - e\vec{S}\cdot\vec{H})}{2m}(T + \sigma_3),$$

where $A_\mu(t,\vec{x})$ satisfy relations (6.2);

$$\tilde{L}_2\Psi' = 0,$$

$$\tilde{L}_2 = p_0 - eA_0 - imT + i\sigma_3\frac{(\vec{S}\vec{\pi})^2}{m}R - i\frac{(\pi^2 - e\vec{S}\cdot\vec{H})}{2m}(T + \sigma_3 R),$$

where $A_\mu(t,\vec{x})$ satisfy relations (6.3).

Acknowledgments

The author wishes to thank Professor A.G. Nikitin for fruitful discussions and Professor J. Beckers for useful remarks on the subject of this note.

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Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 544–550.

The Tangent Groups of a Lie Group and Gauge Invariance in Lagrangian Dynamics

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A tangent Lie group with elements and group operations which are tangent prolongations of those corresponding to another Lie group is examined. An action of such extended Lie group on differentiable manifold and its tangent bundle is defined by using contact transformations. It turned out that a tangent Lie symmetrical Lagrangian describes a dynamical system with first-class constraints. Geometrical aspects of the reduction procedure are considered.

Introduction

We propose to analyse dynamical systems with first-class constraints by using of the tangent Lie groups [1]. In our opinion, this approach explains the genesis of Dirac systems of this type and offers the conceptual clarity.

1 Second-order tangent group of a Lie group

Let G be an $\mathcal{R}$-dimensional Lie group. The tangent bundle TG and the second tangent bundle T^2G are also the Lie groups with elements and group operations which are tangent prolongations of those corresponding to the original Lie group G [1, 2, 3].

Let us consider the curves $\nu : I \to G$ and $\lambda : I \to G$ where I is open neighbourhood of the zero point $0 \in \mathbb{R}$. We introduce the coordinate system (U, g) in G. Henceforth Greek symbols $\nu_\alpha(a) := (g_\alpha \circ \nu)(0)$, $\lambda_\beta(b) := (g_\beta \circ \lambda)(0)$ etc. denote the local coordinates of group elements a, b etc. in manifold G. These coordinates are chosen so that $\varepsilon_\kappa(e) = 0$, $\kappa = 1, \ldots, \mathcal{R} := \overline{1, \mathcal{R}}$, for identity element e. We also use the induced coordinate systems (U^1, g^1) and (U^2, g^2) on the 1-st and 2-nd order tangent bundles TG and T^2G, respectively. Greek indices are meant to run from 1 to $\mathcal{R}$ throughout the paper; the summation convention is used for dummy indices. We denote t^1a, t^2a etc. the elements $t\nu(0)$ and $t^2\nu(0)$ of tangent Lie groups TG and T^2G, respectively. Their coordinates are $\left(\nu_\alpha, \nu^1_\alpha\right)$ and $\left(\nu_\alpha, \nu^1_\alpha, \nu^2_\alpha\right)$ where $\nu^i_\alpha = d^i(g_\alpha \circ \nu)/dt^i\big|_0$, $i = 0, 1, 2$.

Starting with a group multiplication

$$\mu : G \times G \to G, \tag{1.1}$$

we construct the multiplication law for TG [1, 2]

$$T\mu : TG \times TG \to TG$$

defined by

$$T\mu(t\lambda(0), t\nu(0)) = t(\mu(\lambda, \nu))(0).$$

We obtain $\mathcal{R}$ expressions in local coordinates

$$\eta^1_\alpha = d_T\mu_\alpha(\lambda_\beta(b), \nu_\gamma(a)) = \lambda^1_\beta \frac{\partial\mu_\alpha(b,a)}{\partial\lambda_\beta(b)} + \nu^1_\gamma \frac{\partial\mu_\alpha(b,a)}{\partial\nu_\gamma(a)}, \tag{1.2}$$

in addition to the relations

$$\eta_\alpha = \mu_\alpha(\lambda_\beta, \nu_\gamma), \tag{1.3}$$

which illustrate the law (1.1) locally. Here d_T is the Tulczyjew differential operator [4].

As it follows from eqs.(1.2), the identity element t^1e has zero-valued derivative coordinates: $\varepsilon^1_\kappa\left(t^1e\right) = 0$ for all $\kappa = \overline{1,\mathcal{R}}$.

In analogy with $T\mu$ we construct the second order prolongation $T^2\mu : T^2G \times T^2G \to T^2G$ of the multiplication law (1.1):

$$T^2\mu\left(t^2\lambda(0), t^2\nu(0)\right) = t^2(\mu(\lambda,\nu))(0),$$

where on the right side is the second tangent prolongation of the curve $\eta = \mu(\lambda,\nu) : I \to G$ taken at zero point. On the local level we have $\mathcal{R}$ relations

$$\eta^2_\alpha = d^2_T\mu_\alpha(\lambda_\beta, \nu_\gamma), \tag{1.4}$$

in addition to eqs.(1.3) and (1.2).

Now we consider the embedding $\iota_1 : G \to TG$, locally given by $(\nu_\alpha) \mapsto (\nu_\alpha, 0)$. The submanifold $\iota_1(G) \subset TG$ is a slice [5] of the coordinate system $\left(U^1, g^1\right)$. According to ref.[1], ι_1 is a group homomorphism and (G, ι_1) is a closed subgroup of a Lie group TG. Similarly we construct a closed subgroup $\iota_2(G) \subset T^2G$ where the inclusion map $\iota_2 : G \to T^2G$ is the embedding locally written as $(\nu_\alpha) \mapsto (\nu_\alpha, 0, 0)$.

Note that the bundle projections $\tau_G : TG \to G$ and $\tau^2_G : T^2G \to G$ are also the group homomorphisms. The projection $\tau^{2,1}_G$ is the homomorphism from group T^2G to group TG.

Therefore, an original Lie group is a Lie subgroup and a submanifold of its own first- and second-order tangent groups [1]. (More exactly, we consider the slices $\iota_1(G) \subset TG$ and $\iota_2(G) \subset T^2G$ on which all the derivative coordinates are equal to zero.) Moreover, the constants of structure of these tangent Lie groups are determined by the structure constants of G. To demonstrate it we study the involutive distribution $\mathcal{X}_L(T^2G)$ of all left invariant vector fields on T^2G and its dual space $\mathcal{X}^*_L(T^2G)$ of all left invariant one-forms.

Taking into account an exclusive role of Tulczyjew differential operator in prolongation algorithm (see eqs.(1.2) and (1.4)), we deal with $\mathcal{X}^*_L(T^2G)$. We write the local expressions for canonical left invariant one-forms [5, 6] which constitute the basis for $\mathcal{X}^*_L(T^2G)$ at a point t^2a:

$$\theta^{\gamma k} = \left[\frac{\partial}{\partial\nu^i_\alpha(t^2a)} d^k_T\mu_\gamma(b,a)\right]_{t^2b = t^2a^{-1}} d\nu^i_\alpha. \tag{1.5}$$

Small roman indices run from 0 to 2. The exterior derivatives of the $\theta^{\gamma k}$ are given by the Maurer–Cartan equation

$$d\theta^{\gamma k} = -\frac{1}{2} C^\Gamma_{\mathrm{AB}} \theta^{\alpha i} \wedge \theta^{\beta j}. \tag{1.6}$$

We use multi-index notation in structure constants $\{C^\Gamma_{\mathrm{AB}}\}$ of T^2G, where multi-indices A, B and Γ are the 2-tuples of natural numbers, e.g. $\mathrm{A} = (\alpha i)$. Particularly, for subgroup $\iota_2(G) \subset T^2G$ we have

$$d\theta^\gamma = -\frac{1}{2}\, c^\gamma_{\alpha\beta} \theta^\alpha \wedge \theta^\beta,$$

where $\{c^{\gamma}_{\alpha\beta}\}$ are structure constants of the original Lie group G (zero-valued roman indices are omitted). Left-invariant one-forms θ^{α} are given by eqs.(1.5) if integer k is equal to zero.

Tulczyjew operator d_T is the derivation of type d_* of zero degree which acts on the 0-forms as a total time derivative [4]. Having used the commutation $d \circ d_T = d_T \circ d$ and the expressions $d_T {\nu_\beta}^i = {\nu_\beta}^{i+1}$, after short calculations we establish the following relations between the "higher-order" one-forms (1.5) and the original ones:

$$\theta^{\gamma 1} = d_T\theta^{\gamma}, \qquad \theta^{\gamma 2} = d_T^2\theta^{\gamma}.$$

Thanks to commutation of Tulczyjew operator with an exterior derivative and positively signed Leibniz' rule for wedge product [3] we arrive at

$$d\theta^{\gamma 1} = -\frac{1}{2}c^{\gamma}_{\alpha\beta}\theta^{\alpha 1}\wedge\theta^{\beta} - \frac{1}{2}c^{\gamma}_{\alpha\beta}\theta^{\alpha}\wedge\theta^{\beta 1},$$

$$d\theta^{\gamma 2} = -\frac{1}{2}c^{\gamma}_{\alpha\beta}\theta^{\alpha 2}\wedge\theta^{\beta} - c^{\gamma}_{\alpha\beta}\theta^{\alpha 1}\wedge\theta^{\beta 1} - \frac{1}{2}c^{\gamma}_{\alpha\beta}\theta^{\alpha}\wedge\theta^{\beta 2}.$$

When comparing these expressions with the Maurer–Cartan equation (1.6) we deduce the constants of structure $\{C^{\Gamma}_{\mathrm{AB}}\}$. It is convenient to write them as the following block matrices:

$$\hat{C}^{(\gamma 0)} = \begin{bmatrix} \hat{c}^{\gamma} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \hat{C}^{(\gamma 1)} = \begin{bmatrix} 0 & \hat{c}^{\gamma} & 0 \\ \hat{c}^{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \hat{C}^{(\gamma 2)} = \begin{bmatrix} 0 & 0 & \hat{c}^{\gamma} \\ 0 & 2\hat{c}^{\gamma} & 0 \\ \hat{c}^{\gamma} & 0 & 0 \end{bmatrix}.$$

Here symbol $\hat{c}^{\gamma}$ denotes the skew-symmetric matrix $\|c^{\gamma}_{\alpha\beta}\|$ with fixed integer γ.

Similarly we obtain the structure constants of a Lie group TG:

$$\hat{C}^{(\gamma 0)} = \begin{bmatrix} \hat{c}^{\gamma} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{C}^{(\gamma 1)} = \begin{bmatrix} 0 & \hat{c}^{\gamma} \\ \hat{c}^{\gamma} & 0 \end{bmatrix}.$$

The basis for $\mathcal{X}_L(T^2G)$ consists of the left invariant vector fields [2, 5, 6], say $X^{(2)}_B$, locally given by

$$X^{(2)}_{(\beta 0)} = L^{\alpha}_{\beta}\frac{\partial}{\partial g_{\alpha}} + d_T(L^{\alpha}_{\beta})\frac{\partial}{\partial {g_{\alpha}}^1} + d_T^2(L^{\alpha}_{\beta})\frac{\partial}{\partial {g_{\alpha}}^2},$$

$$X^{(2)}_{(\beta 1)} = L^{\alpha}_{\beta}\frac{\partial}{\partial {g_{\alpha}}^1} + 2d_T(L^{\alpha}_{\beta})\frac{\partial}{\partial {g_{\alpha}}^2},$$

$$X^{(2)}_{(\beta 2)} = L^{\alpha}_{\beta}\frac{\partial}{\partial {g_{\alpha}}^2}.$$

Here $L^{\alpha}_{\beta}(g)$ are the components of the left invariant vector fields X_{β} which form the basis for $\mathcal{X}_L(G)$.

Vector field $X^{(2)}_{(\beta 0)}$ is the 2-lift of corresponding one X_{β} to tangent bundle T^2G, i.e. $X^{(2)}_{(\beta 0)} = X^{(2,2)}_{\beta}$ (see refs. [1, 9]). The former belongs to the basis of the sub distribution $\mathcal{X}_L(\iota_2(G)) \subset \mathcal{X}_L(T^2G)$. The others $X^{(2)}_{(\beta 1)}$ and $X^{(2)}_{(\beta 2)}$ are intimately connected with the 1-st and 0-th lifts [1, 9] of X_{β} to T^2G, respectively. Namely, we have $X^{(2)}_{(\beta 1)} = J_1 X^{(2,2)}_{\beta}$ and $X^{(2)}_{(\beta 2)} = (1/2)(J_1)^2 X^{(2,2)}_{\beta}$, where J_1 is the canonical almost tangent structure [10] of order 2 on T^2G.

Let $X^{(1)}_{(\beta i)}$, $i = 0, 1$, be the canonical left-invariant vector fields on $\mathcal{X}_L(TG)$. If $\tau^2_1 : T^2G \to TG$ is the canonical projection, then $X^{(2)}_{(\beta i)}$ and $X^{(1)}_{(\beta i)}$ are τ^2_1-related, i.e. $T\tau^2_1(X^{(2)}_{(\beta i)}) = X^{(1)}_{(\beta i)}$. Each of homomorphism of groups, mentioned in this Section, corresponds the Lie algebra homomorphism which describes its effect on left invariant vector fields, as well as the mapping which relates the dual algebras. The former are then nothing but the differential of originating group homomorphism and the latter is precisely the transpose of this differential [5].

2 An action of the tangent Lie group on a smooth manifold

When considering the group parameters as constants, an action $r : Q \times G \to Q$ of a Lie group G on an $\mathcal{N}$-dimensional smooth manifold Q lifts to an action $r^1 : TQ \times G \to TQ$ of G on the tangent bundle TQ as follows [6, 11, 9]: $\left(r^1\right)_a : TQ \to TQ$, where $\left(r^1\right)_a = Tr_a$ for any fixed $a \in G$. Treatment of group parameters via the time-dependent variables makes the notion of lift of a group action quite different from mentioned above. The desired map is $Tr : TQ \times TG \to TQ$ [1]. It defines the transformation $(Tr)_{t^1a} : TQ \to TQ$ for any fixed $t^1a \in TG$.

We introduce the coordinate system (V, q) in Q and we also use the induced charts $\left(V^1, q^1\right)$ in TQ. The action of TG on TQ induces a Lie algebra homomorphism of $\mathrm{Lie}\,(TG) := T_{t^1e}(TG)$ into vector space $\mathcal{X}(TQ)$. To each vector field $\xi^{(1)}_{(\alpha i)} := X^{(1)}_{\alpha i}\left(t^1e\right)$, $i = 0, 1$, we assign the vector field $Y^{(1)}_{(\alpha i)}$ on TQ:

$$Y^{(1)}_{(\alpha 0)} = Y^b_\alpha \frac{\partial}{\partial q_b} + d_T(Y^b_\alpha)\frac{\partial}{\partial {q_b}^1}, \qquad Y^{(1)}_{(\alpha 1)} = Y^b_\alpha \frac{\partial}{\partial {q_b}^1}. \tag{2.1}$$

Symbol $Y^b_\alpha(q_a)$, $a, b = \overline{1, \mathcal{N}}$, denotes the component of the fundamental vector field Y_α corresponding to $\xi_\alpha \in \mathrm{Lie}\,(G)$. Actually $\left\{Y^{(1)}_{(\alpha i)} | \alpha = \overline{1, \mathcal{R}}; i = 0, 1\right\}$ is a Lie subalgebra of the set $\mathcal{X}(TQ)$ of all vector fields on TQ.

Let us compare these results with standard situation where coordinates of the group elements are meant to be constants. In such a case the transformations of TQ are generated by fundamental vector fields which are complete lifts of their prototypes, acting on Q [9]. Since $\dim TG = 2 \dim G$, we have double number of infinitesimal generators, namely $Y^{(1)}_{(\alpha 0)}$ and $Y^{(1)}_{(\alpha 1)}$, which are then nothing but the complete and vertical lifts [7] of the original one Y_α. Therefore, it is reasonable to say that we deal with the *total* 1-st lift of an action of G on Q.

We may lift an action of TG on Q to the action T^2G on TQ in similar circumstances. We introduce a smooth map

$$\begin{aligned} r^{(0,1)} :\ & Q \times TG \to Q, \\ & (y(0), t\nu(0)) \mapsto x(0), \end{aligned} \tag{2.2}$$

which is an action of a Lie group TG on manifold Q on the right [6]. The bracketed and separated by comma integers $(0, 1)$ up to letter r are associated with the orders of tangent bundles over Q and G, respectively. The curve $y : \mathbb{R} \to Q$ runs across a point $y(0) \in V$ with coordinates $\left\{y_a | a = \overline{1, \mathcal{N}}\right\}$ and the curve $x : \mathbb{R} \to Q$ passes through a point $x(0) \in V$ with coordinates $\left\{x_a | a = \overline{1, \mathcal{N}}\right\}$. In local coordinates (2.2) is written as

$$x_a = f_a(y_b, \nu^i_\alpha(t^1a)).$$

An action of TG on Q induces a Lie algebra homomorphism of the Lie algebra $\mathrm{Lie}\,(TG)$ into vector space $\mathcal{X}(Q)$ [6]. To each vector field $\xi^{(1)}_{(\alpha i)} \in \mathrm{Lie}\,(TG)$, $i = 0, 1$, we assign the following fundamental vector field on Q:

$$Y^{(0,1)}_{(\alpha i)} = \left.\frac{\partial f_a(q_b, t^1a)}{\partial {\nu_\alpha}^i}\right|_{t^1a = t^1e} \frac{\partial}{\partial q_a}. \tag{2.3}$$

Each of them is the infinitesimal generator of an 1-parameter group of transformations of Q.

The map (2.2) lifts to the right action $r^{(1,2)} : TQ \times T^2G \to TQ$ of group T^2G on tangent bundle TQ by composition of the tangent mapping $Tr^{(1,0)} : T(Q\times TG) \to TQ$ with the canonical embedding

$$
\begin{aligned}
i_{1,0}:\quad & TQ \times T^2G \to T(Q \times TG),\\
& (ty(0), t^2\nu(0)) \mapsto t(y, t\nu)(0).
\end{aligned}
$$

The tangent prolongation $ty(0)$ is represented in TV by $(y_b, {y_b}^1)$, where $y_b = (q_b \circ y)(0)$, and ${y_b}^1 = d(q_b \circ y)/dt|_0$. In local coordinates we obtain the following transformational law for first-order derivative coordinates: ${x_a}^1 = d_T f_a$, where ${x_a}^1 = d(q_a \circ x)/dt|_0$.

The fundamental vector fields which correspond to $\xi^{(2)}_{(\alpha i)} \in \mathrm{Lie}\,(T^2G)$ may be expressed in terms of both complete and vertical lifts [7] of vector fields (2.3):

$$
Y^{(1,2)}_{(\alpha 0)} = \left(Y^{(0,1)}_{(\alpha 0)}\right)^c, \qquad Y^{(1,2)}_{(\alpha 1)} = \left(Y^{(0,1)}_{(\alpha 1)}\right)^c + \left(Y^{(0,1)}_{(\alpha 0)}\right)^v, \qquad Y^{(1,2)}_{(\alpha 2)} = \left(Y^{(0,1)}_{(\alpha 1)}\right)^v. \tag{2.4}
$$

They constitute an involutive distribution [2] on TQ.

3 Dynamical system with first-class constraints

In ref. [8] the degenerate Lagrangian system was examined in which the action integral is invariant with respect to the so-called gauge transformation. (By this is meant that the coordinate transformation of the configuration manifold is specified by some time-dependent parameters.) The theorem was proved that a necessary and sufficient condition of such invariance is the existence of relations linking the expressions for Euler–Lagrange equations together. It was shown that a Hamiltonian system with first-class constraints is derived from this degenerate Lagrangian. Conversely, a Dirac system with first-class constraints admits the symmetry of this type.

In geometrical approach [12, 13] a first-class constraint set C is the co-isotropic submanifold of phase space P. The symplectic polar $T^{\P}C \subset TC$ is the integrable distribution on C which is called the *characteristic distribution of* C. This distribution induces the *characteristic foliation* of C. The necessary and sufficient condition for the Dirac system to possess the solutions is that the Hamiltonian $H : C \to \mathbb{R}$ takes a constant value on leaves of the characteristic foliation of C (see refs. [12, 13, Theorem 1]). This theorem can be coordinated with the results obtained in [8] as follows.

Let $P = T^*Q$ and let Q admits the foliation caused by an integrable distribution E. A vector field $Y \in E$ induces the transformation $Fl^Y_t : Q \to Q$ of the configuration manifold. This flow should be identified with the gauge transformation introduced in [8]. The parameters of gauge transformation distinguish the points of an individual leaf of foliation. We interpret this transformation as the invertible contact transformation [14]:

$$
\begin{aligned}
y'_a &= f_a(y_b, \nu_\alpha, {\nu_\alpha}^1),\\
\nu'_\alpha &= \nu_\alpha\,, \qquad \alpha = \overline{1, \mathcal{R}},
\end{aligned} \tag{3.1}
$$

which leaves a Lagrangian $L : TQ \to \mathbb{R}$ invariant. Since the transformed Lagrangian does not depend on the variables ν_α and their time derivatives ${\nu_\alpha}^1$, corresponding Jacobi–Ostrogradski momenta $\hat\eta_{\alpha,0}$ and $\hat\eta_{\alpha,1}$ [10]

$$\hat\eta_{\alpha,1} = \frac{\partial(L\circ f^1)}{\partial\nu_\alpha^2} = \left(\frac{\partial L}{\partial \dot y_a'}\circ f^1\right)\frac{\partial f_a}{\partial \nu_\alpha^1},$$

$$\hat\eta_{\alpha,0} = \frac{\partial(L\circ f^1)}{\partial\nu_\alpha^1} - d_T\hat\eta_{\alpha,1} = \left(\frac{\partial L}{\partial \dot y_a'}\circ f^1\right)\frac{\partial f_a}{\partial \nu_\alpha},$$

are equal to zero. Symbol f^1 denotes the first holonomic prolongation [14] of the map (3.1). Taking the $\nu_\alpha \to 0$ limits we obtain the constraint manifold

$$C = \left\{(y_c,\pi_c)\in T^*Q;\ \pi_b\, Y^b_{\alpha k}(y_c) = 0\right\}, \tag{3.2}$$

where the index $k = 1, 0$ enumerates steps of iterative constraint algorithm [13]. So, the initial constraint set is

$$C^0 = \left\{(y_c,\pi_c)\in T^*Q;\ \pi_b\, Y^b_{\alpha 1}(y_c) = 0\right\}.$$

Note that $Y^b_{\alpha k}(y_c)$ are the components of vector fields $Y_{\alpha k}\in E$ which generate the gauge transformations (3.1).

The characteristic distribution $T^\P C$ of the constraint manifold (3.2) is spanned by the vector fields

$$Y^*_{\alpha k} = Y^b_{\alpha k}(y_c)\frac{\partial}{\partial y_b} - \pi_b\frac{\partial Y^b_{\alpha k}}{\partial y_c}\frac{\partial}{\partial \pi_c},$$

which are complete lifts of vector fields from the distribution E to the phase manifold T^*Q [9]. The tangent manifold TQ admits the foliation caused by the distribution E^c involving both the complete, Y^c, and the vertical, Y^v, lifts [7] of the original vector fields $Y\in E$:

$$\begin{aligned}
\dot Y_{\alpha 0} &= Y^b_{\alpha 0}(y_c)\frac{\partial}{\partial y_b} + d_T\left(Y^b_{\alpha 0}(y_c)\right)\frac{\partial}{\partial {y_b}^1} = Y^c_{\alpha 0},\\
\dot Y_{\alpha 1} &= Y^b_{\alpha 1}(y_c)\frac{\partial}{\partial y_b} + \left(Y^b_{\alpha 0}(y_c) + d_T\left(Y^b_{\alpha 1}(y_c)\right)\right)\frac{\partial}{\partial {y_b}^1} = Y^c_{\alpha 1} + Y^v_{\alpha 0},\\
\dot Y_{\alpha 2} &= Y^b_{\alpha 1}(y_c)\frac{\partial}{\partial {y_b}^1} = Y^v_{\alpha 1},
\end{aligned}\tag{3.3}$$

(cf. eqs.(2.4)). From the theorem about the local structure of foliation [2] we see that an invariant Lagrangian (Hamiltonian) depends on the coordinates of points of plaque and their derivatives (conjugates) only. The momentum canonically conjugated to leaf's coordinate variable is equal to zero. All the momenta of this type constitute the first-class constraint set, i.e. the co-isotropic submanifold of T^*Q. Thus, the distinguished chart [2] for the distribution (3.3) is the key to the reduction procedure here.

The *primary* first-class constraints only hold the independent degrees of freedom [8]. Whence the reduction procedure leads to the constrained Hamiltonian system which does not involve the secondary ones. This Dirac system is derived from a degenerate Lagrangian which does not depend explicitly on some variables.

Concluding remarks

The requirement of invariance of a Lagrangian function under the action of a tangent Lie group leads to degeneracy of this Lagrangian. Fundamental vector fields corresponding to such group constitute the involutive distributions which are important particular cases of the integrable distributions associated with the characteristic distribution of constraint manifold. Distinguished charts for the foliations induced by these involutive distributions may help to explain the geometrical essence of gauge theories.

Acknowledgments

The author would like to thank Prof. V. Tretyak, Dr. A. Duviryak, and Dr. V. Shpytko for very helpful discussions and comments, and constant encouragement to pursue this work.

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Conclusion

Only now, when you are holding in your hands the last volume of the Proceedings of the Conference, we may say that we did everything possible to make the Conference happen.

We wish to thank the Director of the Institute of Mathematics Academician A.M. Samoilenko and the Rector of the M. Dragomanov National Pedagogical University Academician M.I. Shkil whithout help of whom the conference would be impossible. We are indebted to the members of the Organizing Committee and especially to the members of the Local Organizing Committee for their huge work which made the Conference successful. We thank the staff of the M. Dragomanov National Pedagogical University for invaluable help in solving all problems during the Conference.

Unfortunately, not all participants were able to present their papers for publication in the Proceedings. Here are the titles of their talks given at the Conference which were not submitted to the Proceedings

1. *P. Basarab-Horwath*, "Aspects of Conformal Symmetry"
2. *O. Batsula*, "Principle of Spontaneous Symmetry Breaking, Hidden Symmetry and Algebraic Geometry"
3. *E. Belokolos*, "Algebraically Integrable Hamiltonian Systems, Their Symmetry Reduction and Averaging"
4. *A. Cheikhi*, "Generalisation of the Preliminary Classification by Use of Nonclassical Equivalence Transformations"
5. *H. Grundling*, "Dynamics Constraining in Quantum Mechanics"
6. *P. Holod*, "Duality for the Nonlinear Integrable Hierarchies in Finite-Gap Sector"
7. *Yu. Kondratiev*, "Diffeomorphism Groups and Current Algebras in Models of Quantum Field Theory"
8. *I. Korneva*, "The Asymptotic Solution of Systems of Nonlinear PDEs in the Representation Spaces of Finite-Dimensional Lie Groups"
9. *R. Leandre*, "A Stochastic Approach to Some Objects of Conformal Field Theory"
10. *A. Lopatin*, "Group-Theoretical Methods in the Problem of Study of Qualitative Behaviour of Solutions of Differential Equations"
11. *M. Marvan*, "On the Horizontal Gauge Cohomology"
12. *V. Ostrovskii*, "KMS-States and Type III Factor Representations of Cunz Algebras"
13. *Z. Popowicz*, "Computer Algebra, Supersymmetry and Solitons"

Finally, we would like to express our sincere gratitude to all participants of the Conference "Symmetry in Nonlinear Mathematical Physics". And also we invite everybody to participate in the next Conference planned for July, 2001.

Anatoly NIKITIN,
December, 1999.

Наукове видання

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Підп. до друку 21.12.1999. Формат 60x90/8. Папір офсетний. Друк цифровий.
Обл.-вид. арк. 25,7. Ум. друк. арк. 33,6. Тираж 200 пр.

Підготовано до друку в Інституті математики НАН України
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