Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 83–89.

On Exact Solutions of the Nonlinear Heat Conduction Equation with Source

Leonid F. BARANNYK

Institute of Math., Pedagogical University, 22b Arciszewskiego Str., 76-200 Słupsk, Poland
E-mail: Barannyk@wsp.slupsk.pl

The symmetry reduction of the equation $u_0 = \nabla\left[u^\mu \nabla u\right] + \delta u$ to ordinary differential equations with respect to all subalgebras of rank three of the invariance algebra of this equation is performed. Some exact solutions of this equation are obtained.

1 Introduction

Symmetry reduction of nonlinear heat conduction equations without a source is investigated in references [1–7]. In this paper, we investigate the equation

$$\frac{\partial u}{\partial x_0} = \nabla\left[u^\mu \nabla u\right] + \delta u, \tag{1}$$

where $u = u(x_0, x_1, x_2, x_3)$, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$; μ, δ are real numbers, $\mu \neq 0$ and $|\delta| = 1$.

The substitution $u = v^{\frac{1}{\mu}}$ transforms equation (1) into the equation

$$\frac{\partial v}{\partial x_0} = v\Delta v + \frac{1}{\mu}(\nabla v)^2 + \delta\mu v. \tag{2}$$

Let L be the maximal invariance algebra of equation (2). If $\mu \neq -\frac{4}{5}$, then L is the direct sum of the extended Euclidean algebras $A\tilde{E}(1) = \langle P_0, D_1\rangle$ and $A\tilde{E}(3) = \langle P_a, J_{ab}, D_2 :\ a, b = 1, 2, 3\rangle$, generated by the vector fields [8]:

$$\begin{gathered} P_0 = e^{-\delta\mu x_0}\left(\frac{\partial}{\partial x_0} + \delta\mu v\frac{\partial}{\partial v}\right), \qquad D_1 = \frac{1}{\delta\mu}\frac{\partial}{\partial x_0}, \qquad P_a = \frac{\partial}{\partial x_a}, \\ J_{ab} = x_a\frac{\partial}{\partial x_b} - x_b\frac{\partial}{\partial x_a}, \qquad D_2 = x_a\frac{\partial}{\partial x_a} + 2v\frac{\partial}{\partial v} \end{gathered} \tag{3}$$

with $a, b = 1, 2, 3$. If $\mu = -\frac{4}{5}$, then L decomposes [8] into the direct sum of $A\tilde{E}(1)$ and the conformal algebra $AC(3) = \langle P_a, K_a, J_{ab}, D_2 : a, b = 1, 2, 3\rangle$, where P_0, P_a, J_{ab}, D_2 are vector fields (3), and

$$K_a = \left(x_1^2 + x_2^2 + x_3^2\right)\frac{\partial}{\partial x_a} - 2x_a D_2, \qquad a = 1, 2, 3.$$

In this paper, the symmetry reduction of equation (2) is performed with respect to all subalgebras of rank three of the algebra L, up to conjugacy with respect to the group $\operatorname{Ad} L$ of inner automorphisms.

Let $u = f(x_1, x_2, x_3)$ be a solution of equation (1). If $\mu + 1 \neq 0$, then $\Delta u^{\mu+1} + \delta(\mu+1)u = 0$, and if $\mu + 1 = 0$, then $\Delta \ln u + \delta u = 0$. Hence, the search for stationary solutions to equation (1) is reduced to a search for relevant solutions of the d'Alembert equation or Liouville equation. Let $u = u(x_0, x_1, x_2, x_3)$ be a solution of equation (1) invariant under P_0. In this case, if $\mu + 1 \neq 0$, then $u = e^{\delta x_0}\varphi(x_1, x_2, x_3)^{\frac{1}{\mu+1}}$, where $\Delta\varphi = 0$. If $\mu + 1 = 0$, then

$$u = e^{\delta x_0 + \psi(x_1,x_2,x_3)},$$

where $\Delta\psi = 0$. In this connection, let us restrict ourselves to those subalgebras of L that do not contain P_0 and D_1. The list of I-maximal subalgebras of rank 3 is obtained in [4, 6, 7].

2 Reduction of equation (2) for an arbitrary μ to ordinary differential equations

Up to the conjugacy under the group of inner automorphisms, the algebra $A\tilde{E}(1) \oplus A\tilde{E}(3)$ has 12 I-maximal subalgebras of rank three, which do not contain P_0 and D_1 [4, 7]:

$$L_1 = \langle P_1, P_2, P_3, J_{12}, J_{13}, J_{23}\rangle;$$

$$L_2 = \langle P_0 + P_1, P_2, P_3, J_{23}\rangle;$$

$$L_3 = \langle P_2, P_3, J_{23}, D_1 + \alpha D_2\rangle \quad (\alpha \in \mathbb{R},\ \alpha \neq 0);$$

$$L_4 = \langle P_0 + P_1, P_3, D_1 + D_2\rangle;$$

$$L_5 = \langle P_3, J_{12}, D_1 + \alpha D_2\rangle \ (\alpha \in \mathbb{R},\ \alpha \neq 0);$$

$$L_6 = \langle P_0 + P_3, J_{12}, D_1 + D_2\rangle;$$

$$L_7 = \langle P_3, J_{12} + \alpha P_0, D_2 + \beta P_0\rangle \quad (\alpha = 1, \beta \in \mathbb{R} \text{ or } \alpha = 0 \text{ and } \beta = 0, \pm 1);$$

$$L_8 = \langle P_2, P_3, J_{23}, D_1 + P_1\rangle;$$

$$L_9 = \langle P_2, P_3, J_{23}, D_2 + \alpha P_0\rangle \quad (\alpha = 0, \pm 1);$$

$$L_{10} = \langle P_3, D_1 + \alpha J_{12}, D_2 + \beta J_{12}\rangle \quad (\alpha, \beta \in \mathbb{R} \text{ and } \alpha > 0);$$

$$L_{11} = \langle J_{12}, J_{13}, J_{23}, D_1 + \alpha D_2\rangle \quad (\alpha \in \mathbb{R},\ \alpha \neq 0);$$

$$L_{12} = \langle J_{12}, J_{13}, J_{23}, D_2 + \alpha P_0\rangle \quad (\alpha = 0, \pm 1).$$

For each of the subalgebras $L_1, \ldots, L_{12}$ we indicate the corresponding ansatz $\omega' = \varphi(\omega)$ solved for v, where ω and ω' are functionally independent invariants of a subalgebra, as well as the reduced equation which is obtained by means of this ansatz. In cases when the reduced equation can be solved, we indicate the corresponding invariant solutions of equation (2).

2.1. $L_1 : v = \varphi(\omega),\ \omega = x_0,\ \dot{\varphi} = \delta\mu\varphi.$

In this case

$$v = Ce^{\delta\mu x_0},$$

where C is an arbitrary constant.

2.2. $L_2 : v = e^{\delta\mu x_0}\varphi(\omega),\ \omega = \dfrac{1}{\delta\mu}e^{\delta\mu x_0} - x_1,\ \varphi\ddot{\varphi} + \dfrac{1}{\mu}\dot{\varphi}^2 - \dot{\varphi} = 0.$

Integrating the reduced equation, we obtain $\varphi = C'$ or

$$\int \frac{d\varphi}{\mu + C|\varphi|^{-\frac{1}{\mu}}} = \omega + C',$$

where C, C' are arbitrary constants and $C \neq 0$. Corresponding invariant solutions to equation (2) are of the form

$$v = C^{-1}e^{-\delta x_0}[1 + \tilde{C}\ \exp(-\delta C e^{-\delta x_0} - Cx_1)],\ \text{if}\ \mu = -1;$$

$$v = Ae^{-\frac{1}{2}\delta x_0}\tan\left(\frac{\delta}{A}e^{-\frac{1}{2}\delta x_0} + \frac{x_1}{2A} + B\right),\ \text{if}\ \mu = -\frac{1}{2},\ C = -\frac{1}{2A^2};$$

$$v = Ae^{-\frac{1}{2}\delta x_0}\tanh\left(\frac{\delta}{A}e^{-\frac{1}{2}\delta x_0} + \frac{x_1}{2A} + B\right)$$

and

$$v = Ae^{-\frac{1}{2}\delta x_0}\coth\left(\frac{\delta}{A}e^{-\frac{1}{2}\delta x_0} + \frac{x_1}{2A} + B\right),\ \text{if}\ \mu = -\frac{1}{2},\ C = \frac{1}{2A^2}.$$

2.3. $L_3 : v = {x_1}^2\varphi(\omega),\ \omega = \alpha\delta\mu x_0 - \ln x_1,$

$$\varphi\ddot{\varphi} + \frac{1}{\mu}\dot{\varphi}^2 - \frac{3\mu + 4}{\mu}\varphi\dot{\varphi} - \alpha\delta\mu\dot{\varphi} + \frac{2\mu + 4}{\mu}\varphi^2 + \delta\mu\varphi = 0.$$

If $\mu = -2$, $\alpha = -\frac{1}{2}$, then $\varphi = -2\delta\omega + C$ is a solution of the reduced equation. By means of φ we obtain the exact solution

$$v = x_1^2(-2x_0 + 2\delta\ln x_1 + C)$$

of equation (2).

2.4. $L_4 : v = x_2 e^{\delta\mu x_0}\varphi(\omega),\ \omega = \dfrac{\delta\mu x_1 - e^{\delta\mu x_0}}{x_2},$

$$\mu(\mu^2 + \omega)\varphi\ddot{\varphi} + \mu^2\dot{\varphi}^2 + (\varphi - \omega\dot{\varphi})^2 + \delta\mu^2\dot{\varphi} = 0.$$

The function $\varphi = A\omega + B$, where A and B are constants, satisfies this reduced equation if and only if $B^2 = -(\delta A + A^2)\mu^2$. The corresponding invariant solution of equation (2) is of the form

$$v = A\left(\delta\mu x_1 e^{\delta\mu x_0} - e^{2\delta\mu x_0}\right) + Bx_2 e^{\delta\mu x_0}.$$

2.5. $L_5 : v = \left(x_1^2 + x_2^2\right)\varphi(\omega),\ \omega = \alpha\delta\mu x_0 - \dfrac{1}{2}\ln\left(x_1^2 + x_2^2\right),$

$$\varphi\ddot{\varphi} + \frac{\mu + 1}{\mu}(2\varphi - \dot{\varphi})^2 - \dot{\varphi}^2 - \alpha\delta\mu\dot{\varphi} + \delta\mu\varphi = 0.$$

2.6. $L_6 : v = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}} e^{\delta\mu x_0}\varphi(\omega),\ \omega = \left(\delta\mu x_3 - e^{\delta\mu x_0}\right)\left(x_1^2 + x_2^2\right)^{-\frac{1}{2}},$

$$\left(\omega^2 + \mu^2\right)\varphi\ddot{\varphi} - \left(1 + \frac{2}{\mu}\right)\omega\varphi\dot{\varphi} + \left(\mu + \frac{\omega^2}{\mu}\right)\dot{\varphi}^2 + \delta\mu\dot{\varphi} + \left(1 + \frac{1}{\mu}\right)\varphi^2 = 0.$$

For $\mu = -1$, this equation has the solution $\varphi = -\delta\omega$. In this case

$$v = \delta e^{-2\delta x_0} + x_3 e^{-\delta x_0}$$

is the corresponding solution of (2).

2.7. $L_7 : v = e^{\delta\mu x_0}\left(x_1^2 + x_2^2\right)\varphi(\omega),\ \omega = \alpha\delta\mu\arctan\dfrac{x_1}{x_2} - \dfrac{\beta\delta\mu}{2}\ln\left(x_1^2 + x_2^2\right) + e^{\delta\mu x_0},$

$$\left(\alpha^2 + \beta^2\right)\mu^2\varphi\ddot{\varphi} + \left(\alpha^2 + \beta^2\right)\mu\dot{\varphi}^2 - 4\beta\delta(1 + \mu)\varphi\dot{\varphi} - \delta\mu\dot{\varphi} + \frac{4(\mu + 1)}{\mu}\varphi^2 = 0. \quad (4)$$

For $\alpha = \beta = 0$, $\mu \neq -1$, the reduced equation takes the form

$$-\delta\mu\dot{\varphi} + \frac{4(\mu+1)}{\mu}\varphi^2 = 0.$$

It's general solution is

$$\varphi = \frac{\mu^2}{C - 4\delta(\mu+1)\omega},$$

and the corresponding invariant solution of equation (2) is

$$v = \mu^2\left(x_1^2 + x_2^2\right)e^{\delta\mu x_0}\left[C - 4\delta(\mu+1)e^{\delta\mu x_0}\right]^{-1}.$$

If $\mu = -1$, $\alpha^2 + \beta^2 \neq 0$, then equation (4) takes the form

$$\left(\alpha^2 + \beta^2\right)\varphi\ddot{\varphi} - \left(\alpha^2 + \beta^2\right)\dot{\varphi}^2 + \delta\dot{\varphi} = 0.$$

In this case, $\varphi = C'$ or

$$\varphi = \frac{1}{C}\left[C'\exp\left(\frac{C\omega}{\alpha^2+\beta^2}\right) - \delta\right],$$

and, therefore,

$$v = C'e^{-\delta x_0}\left(x_1^2 + x_2^2\right)$$

or

$$v = \frac{1}{C}\left(x_1^2 + x_2^2\right)e^{-\delta x_0}\left\{C'\exp\left[\frac{C}{\alpha^2+\beta^2}\left(-\alpha\delta\arctan\frac{x_1}{x_2} + \frac{\beta\delta}{2}\ln\left(x_1^2+x_2^2\right) + e^{-\delta x_0}\right)\right] - \delta\right\},$$

where C, C' are arbitrary constants and $C \neq 0$.

2.8. $L_8 : v = \varphi(\omega),\ \omega = \delta\mu x_0 - x_1,\ \varphi\ddot{\varphi} + \dfrac{1}{\mu}\dot{\varphi}^2 + \delta\mu(\varphi - \dot{\varphi}) = 0.$

For $\mu = -1$, the reduced equation has the solution $\varphi = Ce^{\omega}$, where C is an arbitrary constant. The corresponding invariant solution of the equation (2) is of the form $v = C\exp(-\delta x_0 - x_1)$.

2.9. $L_9 : v = x_1^2 e^{\delta\mu x_0}\varphi(\omega),\ \omega = \alpha\ln x_1 - \dfrac{1}{\delta\mu}e^{\delta\mu x_0},$

$$\alpha^2\varphi\ddot{\varphi} + \frac{\alpha^2}{\mu}\dot{\varphi}^2 + \left(3\alpha + \frac{4\alpha}{\mu}\right)\varphi\dot{\varphi} + \dot{\varphi} + \left(2 + \frac{4}{\mu}\right)\varphi^2 = 0.$$

For $\alpha = 0$, $\mu \neq -2$, we obtain $\varphi = \mu[(2\mu+4)\omega + \tilde{C}]^{-1}$, therefore,

$$v = \frac{\mu^2 x_1^2 e^{\delta\mu x_0}}{C - \delta(2\mu+4)e^{\delta\mu x_0}}.$$

If $\alpha = 0$, $\mu = -2$, then

$$v = Cx_1^2 e^{-2\delta x_0}.$$

For $\alpha \neq 0$, $\mu = -2$, the reduced equation has the solution $\varphi = \frac{1}{\alpha} + Ce^{-2\alpha\omega}$. The corresponding invariant solution of equation (2) is of the form

$$v = \alpha^{-1} x_1^2 \exp(-2\delta x_0) + C \exp[-2\delta x_0 - \alpha\delta \exp(-2\delta x_0)].$$

2.10. $L_{10} : v = \left(x_1^2 + x_2^2\right)\varphi(\omega),\ \omega = \arctan \dfrac{x_1}{x_2} + \alpha\delta\mu x_0 + \dfrac{\beta}{2} \ln\left(x_1^2 + x_2^2\right),$

$$\left(1+\beta^2\right)\varphi\ddot{\varphi} + \frac{1+\beta^2}{\mu}\dot{\varphi}^2 + 4\beta\left(1+\frac{1}{\mu}\right)\varphi\dot{\varphi} - \alpha\delta\mu\dot{\varphi} + 4\left(1+\frac{1}{\mu}\right)\varphi^2 + \mu\delta\varphi = 0.$$

2.11. $L_{11} : v = \left(x_1^2 + x_2^2 + x_3^2\right)\varphi(\omega),\ \omega = \alpha\delta\mu x_0 - \dfrac{1}{2} \ln\left(x_1^2 + x_2^2 + x_3^2\right),$

$$\varphi\ddot{\varphi} - \left(5 + \frac{4}{\mu}\right)\varphi\dot{\varphi} + \frac{1}{\mu}\dot{\varphi}^2 - \alpha\delta\mu\dot{\varphi} + \left(6 + \frac{4}{\mu}\right)\varphi^2 + \delta\mu\varphi = 0.$$

For $\alpha = \frac{3}{2}$, $\mu = -\frac{2}{3}$, the reduced equation has the solution $\varphi = \frac{2}{3}\delta\omega + C$, and the corresponding solution of (2) is of the form

$$v = \left(x_1^2 + x_2^2 + x_3^2\right)\left[C - \frac{2}{3}x_0 - \frac{1}{3}\delta \ln\left(x_1^2 + x_2^2 + x_3^2\right)\right].$$

2.12. $L_{12} : v = \left(x_1^2 + x_2^2 + x_3^2\right) e^{\delta\mu x_0}\varphi(\omega),\ \omega = e^{\delta\mu x_0} - \dfrac{\alpha\delta\mu}{2} \ln\left(x_1^2 + x_2^2 + x_3^2\right),$

$$\alpha^2\mu^2\varphi\ddot{\varphi} - (5\mu + 4)\alpha\delta\varphi\dot{\varphi} + \alpha^2\mu\dot{\varphi}^2 - \delta\mu\dot{\varphi} + \left(6 + \frac{4}{\mu}\right)\varphi^2 = 0.$$

If $\alpha = 0$, then $\varphi = \mu^2[C - (6\mu + 4)\,\delta\omega]^{-1}$, where $C \neq 0$ or $6\mu + 4 \neq 0$. Therefore,

$$v = \frac{\mu^2\left(x_1^2 + x_2^2 + x_3^2\right) e^{\delta\mu x_0}}{C - (6\mu + 4)\delta e^{\delta\mu x_0}}.$$

For $\mu = -\frac{2}{3}$, $\alpha \neq 0$, the reduced equation has the solution

$$\int \frac{d\varphi}{C\varphi^{\frac{3}{2}} - \frac{3}{\alpha\delta}\varphi + \frac{1}{\alpha^2\delta}} = \omega + C'.$$

If $C = 0$, then

$$\varphi = \frac{1}{3\alpha} + Ae^{-\frac{3}{\alpha\delta}\omega},$$

where A is an arbitrary constant. In this case,

$$v = \frac{1}{3\alpha}\left(x_1^2 + x_2^2 + x_3^2\right)\exp\left(-\frac{2}{3}\delta x_0\right) + A\exp\left[-\frac{2}{3}\delta x_0 - \frac{3}{\alpha\delta}\exp\left(-\frac{2}{3}\delta x_0\right)\right].$$

3 Complementary reduction of equation (2) for $\mu = -\frac{4}{5}$ to ordinary differential equations

Let F be an I-maximal subalgebra of rank three of the algebra $A\tilde{E}(1) \oplus AC(3)$ and $P_0, D_1 \notin F$. If a projection of F onto $AC(3)$ is not conjugate to any subalgebra of the algebra $A\tilde{E}(3)$ under the group $\mathrm{Ad}\, AC(3)$, then F is conjugate under the group $\mathrm{Ad}(A\tilde{E}(1) \oplus AC(3))$ to one of the following subalgebras [6, 7]:

$$F_1 = \langle P_1 + K_1, P_2 + K_2, J_{12}, K_3 - P_3\rangle;$$

$$F_2 = \langle P_a + K_a, J_{ab} : a, b = 1, 2, 3\rangle;$$

$$F_3 = \langle P_1 + K_1, P_2 + K_2, J_{12}, K_3 - P_3 + \alpha D_1\rangle \ (\alpha \in \mathbb{R},\ \alpha > 0);$$

$$F_4 = \langle P_1 + K_1, P_2 + K_2, J_{12}, K_3 - P_3 + P_0\rangle;$$

$$F_5 = \langle K_a - P_a, J_{ab} : a, b = 1, 2, 3\rangle.$$

3.1. $F_1 : v = \left[\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2\right] \varphi(\omega),\ \omega = x_0,\ \dot{\varphi} = -4\varphi^2 - \frac{4}{5}\delta\varphi.$

The general solution of the reduced equation is $\varphi = C\delta \left[e^{\frac{4}{5}\delta\omega} - 5C\right]^{-1}$, where C is an arbitrary constant. The corresponding invariant solution of equation (2) is of the form

$$v = C\delta \left[\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2\right] \left(e^{\frac{4}{5}\delta x_0} - 5C\right)^{-1}.$$

3.2. $F_2 : v = \left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 \varphi(\omega),\ \omega = x_0,\ \dot{\varphi} = -12\varphi^2 - \frac{4}{5}\delta\varphi.$

In this case, $\varphi = \frac{C\delta}{15} \left[e^{\frac{4}{5}\delta\omega} - C\right]^{-1}$, therefore,

$$v = \frac{C\delta}{15} \left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 \left(e^{\frac{4}{5}\delta x_0} - C\right)^{-1}.$$

3.3. $F_3 : v = \left[\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2\right] \varphi(\omega),\ \omega = \arctan \dfrac{x_1^2 + x_2^2 + x_3^2 - 1}{2x_3} - \dfrac{8}{5\alpha\delta} x_0,$

$$4\varphi\ddot{\varphi} - 5\dot{\varphi}^2 + \frac{8}{5\alpha\delta}\dot{\varphi} - 4\varphi^2 - \frac{4}{5}\delta\varphi = 0.$$

3.4. $F_4 : v = e^{-\frac{4}{5}\delta x_0} \left[\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2\right] \varphi(\omega),$

$$\omega = \arctan \frac{x_1^2 + x_2^2 + x_3^2 - 1}{2x_3} - \frac{5\delta}{2} e^{-\frac{4}{5}\delta x_0},$$

$$4\varphi\ddot{\varphi} - 5\dot{\varphi}^2 - 2\dot{\varphi} - 4\varphi^2 = 0.$$

3.5. $F_5 : v = \left(x_1^2 + x_2^2 + x_3^2 + 1\right)^2 \varphi(\omega),\ \omega = x_0,\ \dot{\varphi} = 12\varphi^2 - \frac{4}{5}\delta\varphi.$

Integrating this equation, we obtain $\varphi = \delta \left[15 - Ce^{\frac{4}{5}\delta\omega}\right]^{-1}$, and, therefore,

$$v = \delta \left(x_1^2 + x_2^2 + x_3^2 + 1\right)^2 \left(15 - Ce^{\frac{4}{5}\delta x_0}\right)^{-1}.$$

References

[1] Ovsyannikov L.V., Group properties of the non-linear heat conduction equation, *Dokl. AN USSR*, 1959, V.125, N 3, 492–495 (in Russian).

[2] Dorodnitsyn W.A., On invariant solutions of the non-linear heat conduction equation with source, *J. Comput. Math. and Math. Phys.*, 1982, V.22, N 6, 1393–1400 (in Russian).

[3] Fushchych W.I., Shtelen V.M. and Serov N.I., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Kluwer Acad. Publ., Dordrecht, 1993.

[4] Barannik L.F. and Lahno H.O., Symmetry reduction of the Boussinesq equation to ordinary differential equations, *Reports on Math. Phys.*, 1996, V.38, N 1, 1–9.

[5] Barannyk L.F., Kloskowska B. and Mityushev V.V., Invariant solutions of the multidimensional Boussinesq equation, in Proc. of the Second International Conference "Symmetry in Nonlinear Mathematical Physics", Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, 1997, V.1, 62–69.

[6] Barannyk L.F. and Sulewski P., Exact solutions of the nonlinear diffusion equation $u_0 + \nabla[u^{-\frac{4}{5}}\nabla u] = 0$, in Proc. of the Second International Conference "Symmetry in Nonlinear Mathematical Physics", Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, 1997, V.2, 429–436.

[7] Barannyk L.F. and Kloskowska B., On symmetry reduction and invariant solutions to some nonlinear multidimensional heat equations, *Reports on Math. Phys.*, 1999 (accepted for publication).

[8] Dorodnitsyn W.A., Knyazewa I.W. and Svirschewskij S.R., Group properties of the heat conduction equation with source in the two- and three-dimensional cases, *Differential Equations*, 1983, V.19, N 7, 1215–1223 (in Russian).