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On Exact Foldy–Wouthuysen Transformation of Bozons in an Electromagnetic Field and Reduction of Kemmer–Duffin–Petiau Equation

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Using discrete symmetries of the Kemmer–Duffin–Petiau (KDP) equation the exact Foldy–Wouthuysen transformation (FWT) was found. It is required that the vector-potential of an external field has definite parities. We also described reduction of the KDP equation to uncoupled subsystems which can be solved independently.

The FWT [1] provides several advantages for the understanding and interpretation of the physical properties of the Dirac equation. It permits to reduce of this equation to a two-component equation of the Pauli type. But its main achievement consist in separating of the solution of Dirac equation corresponding to a definite sign of the energy eigenvalues. There are great number of papers are devoted to the construction of FWT for spin-0 [2] and spin-1 [3] particle.

In the presence of interaction the FWT has not, in general, a closed form and one usually uses series expansion methods. There are classes of interaction represented for instance by the static magnetic potentials [4], by the static electric and the pseudo-scalar potentials [5] which admit the exact FWT. The FWT for a two-body equation with oscillator-like interaction [6], for systems composed of one fermion and one boson, and one fermion and one antifermion in the presence of special classes of interactions [7] was also constructed in a closed form.

In this paper we investigate the KDP equation for scalar and vector particle in an electromagnetic field. In order to construct exact FWT we used discrete symmetries of the corresponding equations. The idea to use discrete symmetries (space reflections, time inversion and charge conjugation) for reductions of the Dirac and Schrödinger–Pauli equation to uncoupled subsystems was proposed in [8, 9].

Let us consider KDP equation for scalar ($s = 0$) and vector ($s = 1$) particles minimally interacting with external electromagnetic field. These equations in the Schrödinger form read [10]

$$\begin{aligned} & i\frac{\partial}{\partial t}\Psi(x) = H_1(A_0,\vec{\pi})\Psi(x), \\ & H_1 = \sigma_2 m + (i\sigma_1+\sigma_2)\frac{\pi^2}{2m} + eA_0, \qquad s=0; \end{aligned} \tag{1}$$

$$\begin{aligned} & i\frac{\partial}{\partial t}\Psi(x) = H_2(A_0,\vec{\pi})\Psi(x), \\ & H_2 = \sigma_2 m + (i\sigma_1+\sigma_2)\frac{(\pi^2 - e\vec{S}\cdot\vec{H})}{2m} - i\sigma_1\frac{(\vec{S}\cdot\vec{\pi})^2}{m} + eA_0, \qquad s=1, \end{aligned} \tag{2}$$

where

$$\pi_a = p_a - eA_a, \qquad p_a = -i\frac{\partial}{\partial x_a}, \qquad a = 1, 2, 3,$$

$$\pi^2 = \pi_1^2 + \pi_2^2 + \pi_3^2, \qquad A_0 = A_0(t, \vec{x}), \qquad A_a = A_a(t, \vec{x}),$$

$$\vec{H} = i[\vec{p} \times \vec{A}], \qquad \sigma_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \sigma_2 = i\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

I is a $(2s+1)$-dimensional unit matrix, S_a are 6-dimensional matrices realizing a direct sum of the $AO(3)$-representations $D(1)$, $D(1)$, $\Psi(x)$ is a wave function which has $2(2s+1)$ physical components.

We note that for physical reasons it is preferable to consider another form of (1), (2). It is connected with our consideration by unitary transformation $U = \frac{1}{2}(1 + i\sigma_1)$, $H'_{1,2} = UH_{1,2}U^{-1}$.

In order to construct FWT for Hamiltonians of (1), (2) we will use a method proposed in [11].

Let us define an unitary involution operator I anticommuting with H of (1), (2):

$$I^+I = II^+ = I^2 = 1, \qquad IH + HI \equiv [I, H]_+. \tag{3}$$

We seek the involution I in the form

$$I = MD, \tag{4}$$

where M is a numeric matrix, D are operators of the discrete transformation:

$$D = \langle R_a, T, R_aT\rangle, \qquad a = 1, 2, 3, 12, 23, 31, 123,$$

$$R_a\Psi(t, \vec{x}) = r_a\Psi(t, r_a\vec{x}), \qquad r_a : x_a \to -x_a,$$

$$T\Psi(t, \vec{x}) = r_0\Psi(r_0t, \vec{x}), \qquad r_0 : t \to -t,$$

$$r_a = \pm 1, \qquad r_0 = \pm 1, \qquad R_{123} \equiv R, \qquad r_{123} \equiv r,$$

$$r_ar_b : \ x_a \to -x_a, \quad x_b \to -x_b, \quad a \neq b,$$

$$r_ar_br_c : \ x_a \to -x_a, \quad x_b \to -x_b, \quad x_c \to -x_c, \quad a \neq b, \ b \neq c, \ a \neq c.$$

Theorem 1. *I. All possible involutions (up to equivalence) of form (4) anticommuting with (1) have the form*

$$1. \quad I_1 = \sigma_3 R, \tag{5.1}$$

$$2. \quad I_2 = \sigma_3 T, \tag{5.2}$$

$$3. \quad I_3 = \sigma_3 RT, \tag{5.3}$$

$$4. \quad I_4 = \sigma_3 R_a, \qquad a = 1, 2, 3, \tag{5.4}$$

$$5. \quad I_5 = \sigma_3 R_{ab}, \qquad a \neq b, \tag{5.5}$$

$$6. \quad I_6 = \sigma_3 R_aT, \tag{5.6}$$

$$7. \quad I_7 = \sigma_3 R_{ab}T \tag{5.7}$$

if the corresponding parities of vector-potential $A_\mu(t, \vec{x})$ $(\mu = 0, 1, 2, 3)$ are given by the relations

$$1. \quad \begin{aligned} &A_0(t, \vec{x}) = -A_0(-t, \vec{x}), \\ &A_a(t, \vec{x}) = -A_a(t, r\vec{x}). \end{aligned} \tag{6.1}$$

$$\begin{aligned} 2.\quad & A_0(t,\vec{x}) = -A_0(t,r\vec{x}), \\ & A_a(t,\vec{x}) = A_a(-t,\vec{x}). \end{aligned} \tag{6.2}$$

3. *(there are four subcases of parities of A_μ):*

$$\begin{aligned} a)\quad & A_0(t,\vec{x}) = -A_0(-t,\vec{x}), \quad A_0(t,\vec{x}) = A_0(t,r\vec{x}), \\ & A_a(t,\vec{x}) = A_a(-t,\vec{x}), \quad A_a(t,\vec{x}) = -A_a(t,r\vec{x}); \\ b)\quad & A_0(t,\vec{x}) = -A_0(-t,\vec{x}), \quad A_0(t,\vec{x}) = A_0(t,r\vec{x}), \\ & A_a(t,\vec{x}) = -A_a(-t,\vec{x}), \quad A_a(t,\vec{x}) = A_a(t,r\vec{x}); \\ c)\quad & A_0(t,\vec{x}) = A_0(-t,\vec{x}), \quad A_0(t,\vec{x}) = -A_0(t,r\vec{x}), \\ & A_a(t,\vec{x}) = -A_a(-t,\vec{x}), \quad A_a(t,\vec{x}) = A_a(t,r\vec{x}); \\ d)\quad & A_0(t,\vec{x}) = A_0(-t,\vec{x}), \quad A_0(t,\vec{x}) = -A_0(t,r\vec{x}), \\ & A_a(t,\vec{x}) = A_a(-t,\vec{x}), \quad A_a(t,\vec{x}) = -A_a(t,r\vec{x}). \end{aligned} \tag{6.3}$$

$$\begin{aligned} 4.\quad & A_0(t,\vec{x}) = -A_0(t,r_a\vec{x}), \\ & A_a(t,\vec{x}) = -A_a(t,r_a\vec{x}) \qquad \textit{(no sum over a)}, \\ & A_a(t,\vec{x}) = A_a(t,r_b\vec{x}), \qquad a \neq b. \end{aligned} \tag{6.4}$$

$$\begin{aligned} 5.\quad & A_0(t,\vec{x}) = -A_0(t,r_a r_b\vec{x}), \qquad a \neq b, \\ & A_a(t,\vec{x}) = -A_a(t,r_a r_b\vec{x}) \qquad a \neq b, \\ & A_a(t,\vec{x}) = A_a(t,r_b r c\vec{x}), \qquad a \neq b, \quad b \neq c, \quad c \neq a. \end{aligned} \tag{6.5}$$

6. In this case the parities of A_μ are the same as in the case 3 (formula (6.3)) up to the replacement of r by r_a. In addition, A_μ should satisfy the following relations:

$$\left.\begin{aligned} & A_a(t,\vec{x}) = -A_a(t,r_a\vec{x}), \\ & A_a(t,\vec{x}) = A_a(t,r_b\vec{x}), \quad a \neq b, \end{aligned}\right\} \textit{for a) and d)}; \qquad \left.\begin{aligned} & A_a(t,\vec{x}) = A_a(t,r_a\vec{x}), \\ & A_a(t,\vec{x}) = -A_a(t,r_b\vec{x}), \quad a \neq b, \end{aligned}\right\} \textit{for b) and c)}. \tag{6.6}$$

7. In this case the parities of A_μ are the same as in the case 3 (formula (6.3)) up to the replacement of r by $r_b r_c$, $b \neq c$, $a \neq b$, $a \neq c$. In addition, A_μ should satisfy the following relations ($b \neq c$, $a \neq b$, $a \neq c$)

$$\begin{aligned} & A_a(t,\vec{x}) = A_a(t,r_b r_c\vec{x}), \qquad \textit{for a) and d)}, \\ & A_a(t,\vec{x}) = -A_a(t,r_b r_c\vec{x}), \qquad \textit{for b) and c)}. \end{aligned}$$

II. All possible involutions (up to equivalence) of form (4) anticommuting with (2) have the form (5.1), (5.2), (5.3) if the vector-potential A_μ has the parities (6.1), (6.2), (6.3), correspondingly.

Proof. Requiring the anticommutativity of operator (4) with the Hamiltonian of (1), we obtain the following conditions for M and D:

$$[\sigma_2, M]_+ = 0, \qquad [\sigma_1, M]_+ = 0, \tag{7.1}$$

$$[A_0, D]_+ = 0, \qquad [\pi^2, D] = 0. \tag{7.2}$$

It follows from (7.1) that

$$M = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Conditions (7.2) lead us to (5), (6). In a similar way we can find the involutions for Hamiltonian of (2). Theorem is proved.

Theorem 2. *The exact FWT for Hamiltonians of (1), (2) which have no zero eigenvalues have the form*

$$U = \frac{1}{2}(1 + \sigma_1 I_a)(1 + I_a \varepsilon), \qquad U^+ = U^{-1}, \qquad \varepsilon = \frac{H}{\sqrt{H^2}}, \tag{8}$$

$a = \overline{1,7}$ *for Hamiltonian of (1),* $a = \overline{1,3}$ *for Hamiltonian of (2).*

Proof. Let us consider the case $a = 1$. Then

$$U = \frac{1}{2}(1 - i\sigma_2 R)(1 + \sigma_3 R \varepsilon). \tag{9}$$

The straightforward computation yields

$$\begin{aligned} H_1' &= U H_1 U^{-1} = \sigma_3 \sqrt{H_0^2} \equiv \\ &\equiv \sigma_3 \left(m^2 + \pi^2 (1 + \sigma_1 R) + e^2 A_0^2 + \frac{ie}{2m}[A_0, \pi^2]_+ (R + \sigma_1) + \frac{\pi^4}{m^2}(1 + \sigma_1 R) \right)^{1/2}, \end{aligned} \tag{10}$$

$$\begin{aligned} H_2' = U H_2 U^{-1} = \sigma_3 \Bigg(& H_0^2 + \frac{2}{m^2}\left\{ 2(\vec{S}\cdot\vec{\pi})^4 + (\vec{S}\cdot\vec{H})^2 + [(\vec{S}\cdot\vec{\pi})^2, \vec{S}\cdot\vec{H}]_+ \right\} \\ & + 2\vec{S}\cdot\vec{H} R + 2\sigma_1 \left[2(\vec{S}\cdot\vec{\pi})^2 + \vec{S}\cdot\vec{H} \right] - \frac{ie\sigma_1 R}{m}\left[A_0, 2(\vec{S}\cdot\vec{\pi})^2 + \vec{S}\cdot\vec{H} \right]_+ \\ & + \frac{ie}{m}[A_0, \vec{S}\cdot\vec{H}]_+ + \frac{2(\sigma_1 + R)}{m^2}\left\{ 2\pi^2 (\vec{S}\cdot\vec{\pi})^2 + [\pi^2, \vec{S}\cdot\vec{H}]_+ \right\} \\ & + \frac{2\sigma_1 R}{m^2}\left\{ (\vec{S}\cdot\vec{H})^2 + [(\vec{S}\cdot\vec{\pi})^2, \vec{S}\cdot\vec{H}]_+ \right\} \Bigg)^{1/2}. \end{aligned} \tag{11}$$

We can see that transformation (9) reduces Hamiltonians of (1), (2) to the diagonal form (10), (11). Theorem is proved.

Finally, let us consider relativistic KDP equation for spin-1 particle with minimal and anomalous interaction with electromagnetic field [12]:

$$\left[\beta^\mu \pi_\mu - m + \frac{e}{2m}\left(1 - \beta_5^2\right) S_{\mu\nu} F^{\mu\nu} \right] \Psi(x) = 0, \tag{12}$$

$$\begin{aligned} & S_{\mu\nu} = i[\beta_\mu, \beta_\nu], \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}, \\ & \beta_0 = i(e_{1,7} + e_{2,8} + e_{3,9} - e_{7,1} - e_{8,2} - e_{9,3}), \\ & \beta_1 = -i(e_{1,10} - e_{5,9} + e_{6,8} + e_{8,6} - e_{9,5} + e_{10,1}), \\ & \beta_2 = -i(e_{2,10} + e_{4,9} - e_{6,7} - e_{7,6} + e_{9,4} + e_{10,2}), \\ & \beta_3 = -i(e_{3,10} - e_{4,8} + e_{5,7} + e_{7,5} - e_{8,4} + e_{10,3}), \\ & \beta_5 = i(e_{4,1} + e_{5,2} + e_{6,3} - e_{1,4} - e_{2,5} - e_{3,6}), \end{aligned} \tag{13}$$

where we use the notations $e_{i,j}$ for 10 by 10 matrices, whose only nonzero elements are ones at the intersection of the i-th line and j-th column which are equal to unity.

Substituting the explicit form of β_μ-matrices (13) into (12) and expressing nonphysical components $(1-\beta_0^2)\Psi$ via $2(2s+1)$ physical components $\beta_0^2\Psi$, we come to the equation in Schrödinger form $i\partial_t\Psi = H\Psi$, where Ψ is a 6-component wave function and the Hamiltonian has the form

$$
\begin{aligned}
H &= mI_3\otimes\sigma_2 + \frac{\pi^2}{2m}I_3\otimes(\sigma_2+i\sigma_1) - \frac{i}{m}\sum_{a,b=1}^{3}\pi_a\pi_b(S_aS_b\otimes\sigma_1)\\
&+ \frac{e^2}{2m^3}\sum_{a=1}^{3}(F_{0a})^2 I_3\otimes(\sigma_2-i\sigma_1) - \frac{e^2}{2m^3}\sum_{a,b=1}^{3}F_{0a}F_{0b}S_aS_b\otimes(\sigma_2-i\sigma_1)\\
&+ \frac{ie}{m^2}\sum_{a,b=1}^{3}F_{0a}\pi_bS_aS_b\otimes\sigma_3 - \frac{ie}{m^2}\sum_{a=1}^{3}(F_{0a}\pi_a)^2 I_3\otimes\sigma_3 + \frac{ie}{2m^2}M\otimes(1-\sigma_3)\\
&+ \frac{e}{2m}(S_1F_{23}-S_2F_{31}+S_3F_{12})\otimes(\sigma_2-i\sigma_1) + eA_0,
\end{aligned}
\tag{14}
$$

where we refer to direct products between 3 by 3 unit I_3 and S_a $(a=1,2,3)$ matrices belonging to the $D(1)$-representation of $AO(3)$ with the usual Pauli matrices and M is a matrix with matrix elements $m_{ab} = -i\frac{\partial F_{0b}}{\partial x_a}$.

Requiring that (14) and (4) satisfy (3) we find the involutions of Hamiltonian (14):

$$\tilde{I}_1 = \left[(2S_1^2-1)\otimes\sigma_3\right]P_{23}T, \tag{15.1}$$

$$\tilde{I}_2 = \left[(2S_2^2-1)\otimes\sigma_3\right]P_{31}T, \tag{15.2}$$

$$\tilde{I}_3 = \left[(2S_3^2-1)\otimes\sigma_3\right]P_{12}T \tag{15.3}$$

and the following conditions for A_μ and E_a (E_a are components of electric field strength):

$$
\begin{aligned}
& A_\mu(t,\vec{x}) = A_\mu(-t,\vec{x}), \qquad A_0(t,\vec{x}) = -A_0(t,r_ar_b\vec{x}),\\
& A_1(t,\vec{x}) = \alpha_1A_1(t,r_ar_b\vec{x}), \qquad A_2(t,\vec{x}) = \alpha_2A_2(t,r_ar_b\vec{x}),\\
& A_3(t,\vec{x}) = \alpha_3A_3(t,r_ar_b\vec{x}), \qquad \frac{\partial E_a}{\partial x_a} = 0, \qquad \text{no sum over } a,\\
& \alpha_1 = -\alpha_2 = -\alpha_3 = 1, \quad a=2, \quad b=3 \quad \text{for} \quad (15.1),\\
& -\alpha_1 = \alpha_2 = -\alpha_3 = 1, \quad a=1, \quad b=3 \quad \text{for} \quad (15.2),\\
& -\alpha_1 = -\alpha_2 = \alpha_3 = 1, \quad a=1, \quad b=2 \quad \text{for} \quad (15.3).
\end{aligned}
$$

The exact FWT of (14) has the form:

$$
\begin{aligned}
U_1 &= \frac{1}{2}\left(1+S_2\otimes\hat{I}_2\cdot\tilde{I}_1\right)\left(1+\tilde{I}_1\varepsilon\right),\\
U_2 &= \frac{1}{2}\left(1+S_3\otimes\hat{I}_2\cdot\tilde{I}_2\right)\left(1+\tilde{I}_2\varepsilon\right),\\
U_3 &= \frac{1}{2}\left(1+S_1\otimes\hat{I}_2\cdot\tilde{I}_3\right)\left(1+\tilde{I}_3\varepsilon\right),
\end{aligned}
$$

$\hat{I}_2$ is a 2×2 unit matrix.

Another problem that we explore in this note is a reduction of KDP equation to uncoupled subsystems. Let us show how it is possible to make such reduction using discrete symmetries of corresponding equations.

It is easy to verify that the involutions I_2, I_3, I_6, I_7 (see formulae (5)) are the discrete symmetries of (1) if vector-potential $A_\mu(t,\vec{x})$ has parities (6.2), (6.3), (6.6), (6.7) correspondingly. Indeed, these operators satisfy the invariance condition $[Q,L]\Psi = 0$, where $Q = \langle I_2, I_3, I_6, I_7\rangle$, $L = i\partial_t - H_1$, Ψ is an arbitrary solution of equation $L\Psi(x) = 0$. In analogy with the above we can find that equation (2) admits discrete symmetries I_2 and I_3, (formulae (5.2) and (5.3)) for the vector-potential (6.2) and (6.3) respectively.

In order to reduce (1) and (2) to uncoupled subsystems it suffices to construct unitary operators that diagonalize the discrete symmetries of these equations [8].

Let the vector-potential $A_\mu(t,\vec{x})$ satisfy relations (6.2). In this case equation (1) admits the symmetry $Q_1 = \sigma_3 T$.

Constructing the operator

$$U_1 = (T_+ - i\sigma_2 T_-), \qquad U_1^{-1} = (T_+ + i\sigma_2 T_-), \qquad T_\pm = \frac{1 \pm T}{2} \tag{16}$$

we reduce Q_1 to the block diagonal form

$$U_1 Q_1 U_1^{-1} = \sigma_3.$$

The equation (1) is transformed as

$$\begin{aligned} &L_1'\Psi' = 0,\\ &L_1' = U_1 L U_1^{-1} = U_1(i\partial_t - H_1)U_1^{-1}, \qquad \Psi' = U_1\Psi. \end{aligned} \tag{17}$$

Multiplying L_1' by nonzero matrix $-i\sigma_2$ on the left and by T on the right we obtain

$$\tilde{L}_1 = p_0 - eA_0 - imT - i\frac{\pi^2}{2m}(T + \sigma_3) \tag{18}$$

and the corresponding uncoupled equations

$$\left\{p_0 - eA_0 - imT - i\frac{\pi^2}{2m}(T+1)\right\}\Psi_+ = 0,$$

$$\left\{p_0 - eA_0 - imT - i\frac{\pi^2}{2m}(T-1)\right\}\Psi_- = 0,$$

where $\Psi_\pm$ are one-component functions.

If the vector-potential $A_\mu(t,\vec{x})$ satisfies relations (6.3) then equation (1) admits the symmetry $Q_2 = \sigma_3 RT$. We find diagonalizing operator in the form:

$$U_2 = (T_+ - i\sigma_2 T_-)(R_+ - i\sigma_2 R_-), \qquad U_2^{-1} = (R_+ + i\sigma_2 R_-)(T_+ + i\sigma_2 T_-),$$

$$R_\pm = \frac{1 \pm R}{2}, \qquad U_2 Q_2 U_2^{-1} = \sigma_3.$$

Corresponding reduced equation have the form

$$\tilde{L}_2\Psi' = 0,$$

$$\tilde{L}_2 = p_0 - eA_0 - imT - i\frac{\pi^2}{2m}(\sigma_3 R + T).$$

In analogy with (1) we make a reduction of (2). As a result we obtain

$$\tilde{L}_1\Psi' = 0,$$

$$\tilde{L}_1 = p_0 - eA_0 - imT + i\sigma_3\frac{(\vec{S}\vec{\pi})^2}{m} - i\frac{(\pi^2 - e\vec{S}\cdot\vec{H})}{2m}(T+\sigma_3),$$

where $A_\mu(t,\vec{x})$ satisfy relations (6.2);

$$\tilde{L}_2\Psi' = 0,$$

$$\tilde{L}_2 = p_0 - eA_0 - imT + i\sigma_3\frac{(\vec{S}\vec{\pi})^2}{m}R - i\frac{(\pi^2 - e\vec{S}\cdot\vec{H})}{2m}(T+\sigma_3 R),$$

where $A_\mu(t,\vec{x})$ satisfy relations (6.3).

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