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Symmetry-Breaking Bifurcations in a Single-Mode Class-A Gas Laser

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The invariance properties of equations of motion and symmetric bifurcations of stationary and periodic solutions of codimensionality 1 and 2 corresponding to polarization symmetry breaking and restoration were analyzed in a single-mode standing-wave class-A gas laser with linear phase anisotropy of the cavity at $j \to j+1$ transition between the working levels.

1 Introduction

The account of vectorial nature of electromagnetic field in nonlinear dynamics of laser systems, besides additional equations compared to the scalar field approach, assumes the appearance of radically new properties of dynamical systems inherent in polarized radiation. One of these properties is invariance (symmetry) of the system with respect to the transformation of the state of polarization. In spite of the fact that in optics transformations of polarization accompanied by polarization symmetry breaking are known (see, for example, [1]) their mathematical description in the language of singular (symmetric) bifurcations is absent.

The aim of the present work is to study the invariance properties of the equations of motion of a single-mode class-A gas laser with linear phase anisotropy of the cavity, operating at $j_b = 1 \to j_a = 2$ transition and to analyze symmetric bifurcations of codimension 1 and 2 of stationary and periodic solutions resulting in polarization symmetry breaking and restoration phenomena.

2 Theoretical model

The theoretical analysis is based on the model of a single-mode (two-frequency) anisotropic-cavity gas laser with a longitudinal magnetic field on the active medium, derived and explored in [2, 3]. In the case of linear phase anisotropy of the cavity at the line center tuning the equations of motion can be reduced to a system of three ODEs which takes the following form [3]:

$$\frac{dI_1}{d\tau} = 2I_1 \left\{ \frac{P_1}{P} + \frac{\Delta W'}{P} \tanh 2\beta_1 - q \left(1 - \frac{\cos 2\Phi_1}{\cosh 2\beta_1} \right) - I_1 \left(\theta_1' + \theta_2'' \tanh^2 2\beta_1 \right) \right\}, \tag{1}$$

$$\frac{d\Phi_1}{d\tau} = -(qS_1 + rS_2) - \frac{\Delta W''}{P} + \theta_2'' \tanh 2\beta_1 I_1, \tag{2}$$

$$\frac{d\beta_1}{d\tau} = rS_1 - qS_2 + \frac{\Delta W'}{P} - \theta_2' \tanh 2\beta_1 I_1. \tag{3}$$

Here $I_1 = I_1'/k_0 l P$ is the dimensionless intensity, $f_1 = \Phi_1 + i\beta_1$, $\xi_1 = \tanh \beta_1$ is the ellipticity; Φ_1 is the azimuth of the wave 1, the characteristics of the waves 1 and 2 are interrelated: $I_2 = I_1$, $\Phi_2 = \Phi_1 + \pi/2$, $\xi_2 = -\xi_1$; $W(x \pm \Delta, y) = U(x \pm \Delta, y) + iV(x \pm \Delta, y)$ is the complex error function, $\bar{W} = \bar{U} + i\bar{V} = [W(x-\Delta, y) + W(x+\Delta, y)]/2$, $\Delta W = [W(x-\Delta, y) - W(x+\Delta, y)]/2$; $x_1 \pm \Delta = (\omega_1 - \omega_0 \pm g\mu_B H)/Ku$ is the detuning of the lasing frequency ω_1 from the line center ω_0, relative to Ku; $\theta_1 = b_1 + a_{12} - b_{12}$, $\theta_2 = d_1 - d_{12} + b_{12}$; a, b, d are the coefficients of nonlinear interaction which as well as the other parameters are determined in [3]; $S_1 = \sin 2\Phi_1 \cosh 2\beta_1$, $S_2 = \cos 2\Phi_1 \sinh 2\beta_1$, $q = (1 - \cos 4\psi)/2\tau_0$, $r = \sin 4\psi/2\tau_0$, $\Delta W = \Delta W' + i\Delta W''$, $\Delta = \Delta_0 + \Delta_1 \sin f\tau$, $\omega_f/2\pi(KHz) = 10^3 f\tau_0 c/2\pi L$, $\Delta = g\mu_B H/Ku$, $\Delta_0 = g\mu_B H_0/Ku$, $\Delta_1 = g\mu_B H_1/Ku$, H_0 and H_1 are the strength of constant and sinusoidal magnetic field, respectively 2ψ is the linear phase anisotropy of the cavity;

$$\Delta W = \Delta W' + i\Delta W'' \approx 2\Delta\left\{\delta x\left(1 - \frac{4y}{\sqrt{\pi}}\right) + i\left[y - \frac{1}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}\left(\frac{\Delta^2}{3} - y^2 + \delta x^2 + \frac{\delta xy}{\Delta}\right)\right]\right\}, \tag{4}$$

$\delta x = (x_1 - x_2)/2$.

Stability analysis of the stationary and periodic solutions was carried out on the basis of the numerical methods of the theory of bifurcation [4, 5].

3 Pitchfork bifurcation of the stationary solution

Assuming the amplitude of the longitudinal magnetic field on the active medium equal to zero: ($H_0 = H_1 = 0$), let us consider the transformation of the state of polarization of the emitted field at transition from isotropic ($\psi = 0$), to anisotropic cavity, when $\psi \neq 0$.

Under the assumptions that the changes in intensity with time can be neglected ($I_1 = I_2 = I_0$) the stationary solutions to equations (2), (3) for polarization characteristics can be found analytically (see, for example, [3]):

$$\Phi_1 = 0, \pm\pi/2, \qquad \xi_1 = 0, \tag{5}$$

$$\Phi_1 = \pm\pi/4, \qquad \sinh 2\beta_1 = \pm\left\{-\alpha/2r' - \left(\alpha^2/4r'^2 - 1\right)^{1/2}\right\}, \tag{6}$$

$$\Phi_1 = \pm\pi/4, \qquad \sinh 2\beta_1 = \pm\left\{-\alpha/2r' + \left(\alpha^2/4r'^2 - 1\right)^{1/2}\right\}, \tag{7}$$

where $\alpha = 2\tau_0\theta_2 I_0$, $r' = \sin 4\psi$.

Stability analysis of these solutions, carried out numerically in [6], showed that for $j \to j$ transitions for all values of ψ the two orthogonal linearly polarized waves, described by (5), are stable. For $j \to j+1$ transitions in a region of $\psi : \alpha/4 < \sin 2\psi < (\alpha/4)^{1/2}$, a steady-state regime with periodic oscillations of the intensity, ellipticity and azimuth of the emitted field is found. The limit cycle appears at $\psi^* = 1/2 \arcsin(\alpha/4)^{1/2}$ due to the Hopf bifurcation and is destroyed at point $\psi^{**} = 1/2 \arcsin(\alpha/4)$ due to the appearance of a saddle-node point (see, for example, [5]).

The expressions (6) describe the stable solutions (the state of equilibrium is the node), the expressions (7) describe the unstable solutions (the state of equilibrium is the saddle), existing in the region $\psi^{**} < 1/2 \arcsin(\alpha/4)$. The detailed bifurcation analysis of the system under consideration requires more complicated model, it was carried out in [3]. Here we consider only the symmetry-breaking bifurcations.

At $\psi = 0$ (isotropic cavity) two steady-state solutions (6), corresponding to the ($\pm$) signs, coincide. Each of them describes two waves with orthogonal circular states of polarization and zero frequency difference, which give one wave with linear polarization. This result is in agreement with previous studies (see, for example, [6]).

When the cavity anisotropy ψ is infinitesimal so that the isotropic cavity is transformed into an anisotropic cavity, the solutions (6) give two different two-frequency regimes. Each of these regimes is represented by two orthogonal elliptically polarized waves with high ellipticities (practically circular) and different frequencies. A particular regime of lasing is determined by the initial conditions.

This bistability reflects the invariance properties of equations (1)–(3) with respect to the transformation of variables:

$$G = \{I_1, \Phi_1, \xi_1\} \to \{I_1, -\Phi_1, -\xi_1\} \qquad (8)$$

and corresponds to the symmetric pitchfork bifurcation (see, for example, [8]) which takes place in the vicinity of the point $\psi = 0$. As a result of this bifurcation the initial solution (one linearly polarized wave) loses its stability, and two new stationary two-frequency elliptically polarized solutions with high ellipticities (practically circular) appear. These new solutions can be obtained from each other by the transformation G. Pitchfork bifurcation of the stationary solution and polarization symmetry breaking phenomenon is shown schematically in Fig. 1.

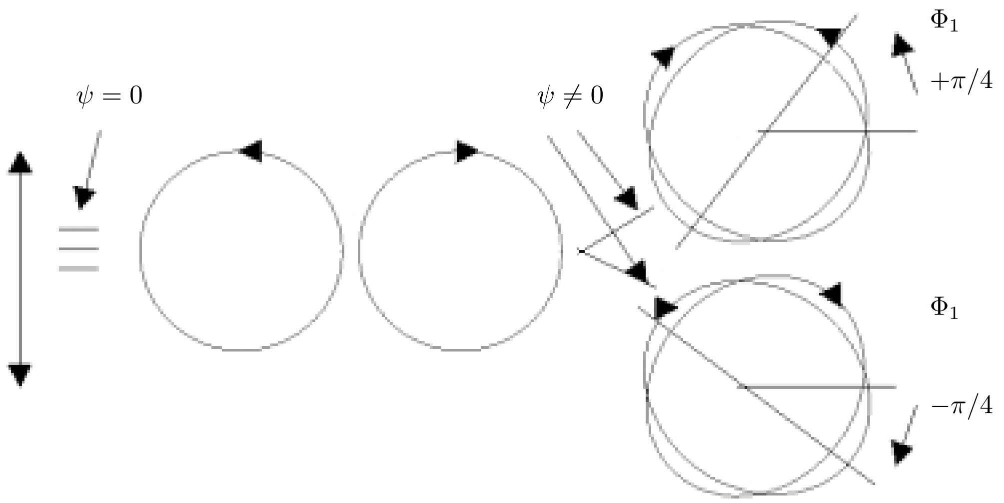

Fig. 1. Pitchfork bifurcation of the stationary solution resulting in spontaneous polarization symmetry breaking at transition from isotropic to anisotropic cavity.

Due to the bistability of these two-frequency regimes, which are connected with each other by the transformation G, and due to the fact that the choice of solution is fully determined by the initial conditions, we can say that the decomposition of one linearly polarized wave into two orthogonal elliptically polarized (with high values of ellipticities, practically circular) waves with different frequencies which occurs at transition from an isotropic to an anisotropic cavity is a spontaneous polarization symmetry breaking.

4 Symmetry-breaking bifurcation of periodic solutions of equations of motion

Let us consider the invariance properties of equations (1)–(3) in the presence of a sinusoidal magnetic field on the active medium ($H_0 = 0, H_1 \neq 0$) and analyze the bifurcations of periodic solutions of codimensionality 1 and 2, reflecting these properties. Let us choose the range of control parameter ψ where in the system a stable limit cycle of the first kind is realized [2, 3].

It is easy to see that when the longitudinal magnetic field is imposed on the medium, the system of equations (1)–(3) is invariant with respect to the following transformation:

$$G = \{I_1, \Phi_1, \xi_1, H_1\} \to \{I_1, -\Phi_1, -\xi_1, -H_1\}. \tag{9}$$

Reversing of the sign of H_1 is equivalent to the shift of the phase of the external force signal on π. Analogous symmetry properties are intrinsic for some dynamical systems, in particular, for those, describing by the Duffing equations (see, for example, [9]). In these systems the periodic solutions are possible whose bifurcations occur by a different way than bifurcations of periodic solutions in systems without symmetry (see, for example, [10]).

Let $X(t)$ be the periodic solution of the system (1)–(3), and $\tilde{X}$ be the trajectory of this solution in the phase space. Then in accordance with classification of symmetric limit cycles (see, for example, [10]), the following solutions are possible:

F-cycle is the solution, invariant with respect to the transformation G:

$$GX(t) = X(t); \tag{10}$$

S-cycle is the solution which is not invariant with respect to the transformation G, but the trajectories of the both cycles coincide, i.e. the phase trajectory of the cycle consists of two congruent parts. The solution is invariant with respect to the transformation G + shift of the time series on a half of period of a cycle T:

$$GX(t) \neq X(t), \qquad GX(t) = \tilde{X}, \qquad GX(t) = X(t + T/2); \tag{11}$$

M-cycle is the asymmetric solution which at the transformation G turns into the second asymmetric solution:

$$GX_1(t) = X_2(t). \tag{12}$$

M-cycles always originate in pair and undergo simultaneously the same sequence of bifurcations, intrinsic for systems without symmetry. In the system under consideration S- and M-cycles were found.

Fig. 2 shows the upper part of the diagram on the plane of parameters H_1, ω_f inside the resonance (1/1), calculated at the following parameters of the $He - Ne$ ($\lambda = 0.63\mu m$) laser: $\psi = 0.001$ rad, $\eta_1 = \eta_2 = 1.9$, $c/L = 612$ MHz, $y_1 = 0.011$, $y_2 = 0.005$, $y = 0.2$, $k_0 l = 0.025$, $Ku = 870$ MHz.

Detailed study of the dynamics of this nonautonomous system has been carried out in [11], where the evolution of solutions in the region of resonance (1/1) at large values of H_1 is shown schematically.

In the region 1, bounded by the lines l_0, on which the Neimark–Sacker bifurcation takes place (a pair of complex-conjugate multipliers crosses the unit circle: $|\mu_1| = |\mu_2| = 1$), the resonance S-cycle exists. On going out of the resonance region across these lines, softly, with zero amplitude on the second frequency in the power spectrum, a two-dimensional S-torus is originated, which exists in the region 2.

With increasing parameter H_1 the lines l_0 are ended at the points F of codimension 2, where both of the multipliers of the cycle 1 become unit: $\mu_1 = \mu_2 = +1$), which corresponds to the strong resonance condition (1/1) [12]. Through points F the line l_{11} is passing on which the resonance S-cycle loses its stability as a result of the saddle-node bifurcation for a system with symmetric properties and instead of it a pair of stable asymmetric M-cycles with period of the driving force T originates.

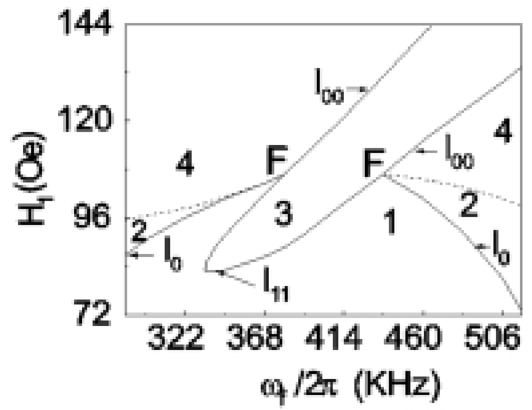

Fig. 2. A part of the bifurcation diagram on the plane of parameters (H_1, ω_f) in the region of the resonance (1/1).

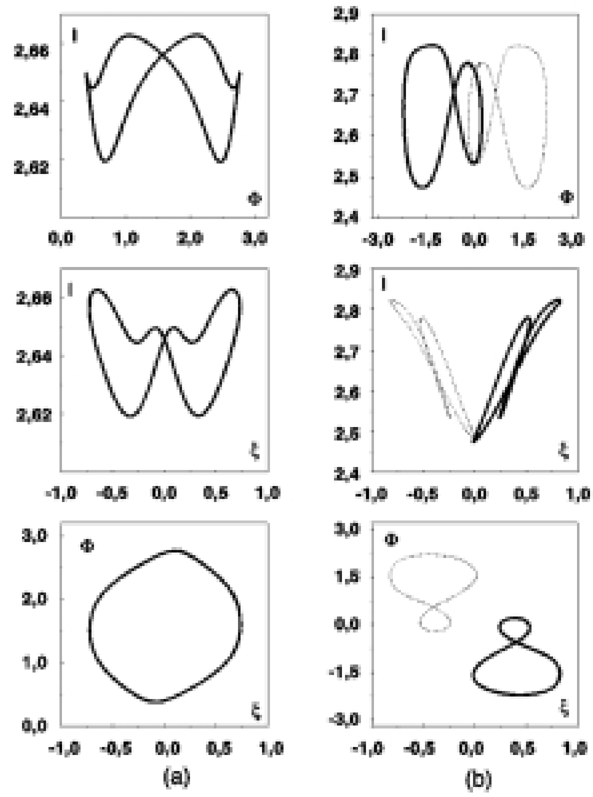

Fig. 3. Pitchfork bifurcation of the limit cycle. Phase projections reflecting the invariance with respect to the transformation G of symmetric S-cycle calculated at $H_1 = 0.88.4$ Oe, $\omega_f = 381$ KHz (a) and asymmetric M-cycles calculated at $H_1 = 90.8$ Oe, $\omega_f = 381$ KHz (b).

In the region 3 one of the two possible asymmetric solutions is realized; the switch to the other solution occurs at exchange of the signs of the initial conditions and magnetic field strength. Fig. 3 reflects changes of the phase projections of the S (a) and M (b) cycles with respect to the transformation G.

As it can be seen from Fig. 3, as a result of the transformation G, the phase trajectory of the S-cycle remains unaltered, while the M-cycles are transformed into each other. Thus, on the line l_{11} polarization symmetry breaking phenomenon for periodic solutions of the equations of motion takes place. This bifurcation is an analog to the pitchfork bifurcation of the stationary solutions.

Above the points F the lines of formation of a new two-dimensional S-torus with complicated form of oscillations are fixed. At so doing, both of the asymmertic M-cycles simultaneously lose their stability as a result of the saddle-node bifurcation and form the symmetric long-period oscillation on torus, which exists in the region 4. Complication of the form and increase of the period of oscillation on torus in the region 4, originated inside of the region of synchronization (1/1), is due to appearance of the high order ($p > 5$) resonances [12]. Inside these resonances symmetric periodic oscillations are fixed whose period is p times larger than the period of the driving force T. In the system under consideration long-periodic oscillations with period $7T$ and $9T$ have been found [11]. The dashed lines in the diagram separate approximately the region of the torus 2, originated from the initial cycle and torus 4, originated from the long-periodic cycle inside the locking zone. When these lines intersect, a torus with long-periodic oscillations appears as a result of a long-term intermediate process, so in the Poincare section it is possible to observe simultaneously the destruction of the old and the creation of a new torus. The behavior of the dynamical system in the vicinity of the points F of codimension 2 corresponds to the suggestion $5S$ [10], namely, this is the case of resonance (1/2) in a system without symmetry. To elucidate this statement one should take into account that the behavior of symmetric S-cycles is characterized not by the eigenstates of the linearization matrix in the Poincare map P, which describes the shift of the solution for the period of cycle T, but by the eigenstates of the matrix $Q = P^{1/2}$, describing the transformation of the periodic solution during one half of the period T.

At the strong resonance conditions (1/1) for matrix P, depending on the symmetry properties of the matrix Q, its eigenstates can be multiple and equal to $+1$ (resonance (1/1) in systems without symmetry), multiple and equal to -1 (resonance (1/2)), as well as equal to ± 1. As mentioned above, in the system under consideration the resonance (1/2) conditions for system without symmetry are realized.

When constant longitudinal magnetic is added, the phase space become cylindrical [11]. Equations (1)–(3) are invariant with respect to the transformation G given by (9), where H_1 should be replaced by $H = H_0 + H_1$. This invariance results in the bistability of M-cycles of the first and the second kind and in the experimentally observed effect of sign reversal of azimuth rotation. Bistability of the asymmetric M-cycles with period of the external force T is shown in Fig. 4.

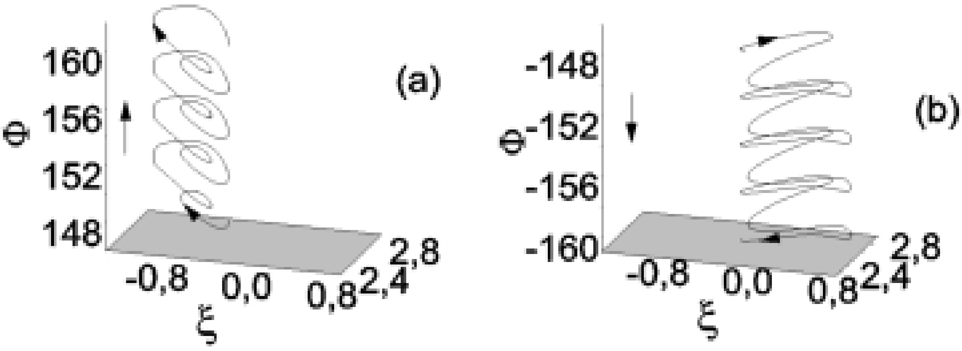

Fig. 4. Bistability of the asymmetric M-cycles of the second kind; the arrows mark the direction of azimuth rotation: right rotation (a), left rotation (b).

Conclusions

Analysis of the invariance properties of the equations of motion of a single-mode standing-wave class-A gas laser with linear phase anisotropy of the cavity at the $j \to j+1$ transition between the working levels has revealed a series of polarization symmetry breaking phenomena. Spontaneous polarization symmetry breaking, corresponding to pitchfork bifurcation of stationary solution, occurs at transition from an isotropic to an anisotropic cavity and destruction of the laser modes degeneracy. In the vicinity of the bifurcation point one wave with linear state of polarization is decomposed into two elliptically polarized waves with high values of ellipticity, and bistability of these two-wave solutions arises. In the presence of a sinusoidal magnetic field on the active medium polarization symmetry breaking has been found which corresponds to pitchfork bifurcation of periodic solution: symmetric S-cycle is decomposed into two asymmetric M-cycles. Then S-type symmetry is restored through appearance of S-torus which undergoes high-order resonances.

In the presence of a constant longitudinal magnetic field on the active medium bistability has been found of asymmetric M-cycles of the first and the second kind resulting in the experimentally observed effect of sign reversal of azimuth rotation.

At present the polarization symmetry breaking and the restoration phenomena are becoming the subject of intensive studies due to their possible application in the optical processing of information for making devices based on novel physical principles.

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