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Quantum Integrability of the Generalized Euler's Top with Symmetries

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We prove integrability, i.e the existence of the full set of commuting integrals, of the quantized generalized rigid body in the case when inertia tensor possesses additional symmetry.

1 Introduction

In the present paper we deal with the quantum systems that are direct higher-rank generalization of the standard $so(3)$ Eulers top.

Integrability of their classical counterparts was originally proved by Manakov [1] for the case of $so(n)$ and by Mishchenko and Fomenko [2, 3] for the case of arbitrary semisimple Lie algebras. They constructed the algebra of the mutually commuting with respect to the Lie–Poisson brackets integrals of these systems, which we will call Mishchenko–Fomenko algebra, with the help of the so-called procedure of the "shift of the argument". This procedure has the Lie-algebraic explanation that was given by Reyman and Semenov-Tian-Shansky [4, 5] in the framework of the so called Kostant–Adler scheme [6]. But in the quantum case the analogous scheme fails. This fact has again purely algebraic nature. Indeed, Kostant–Adler scheme, that was used by Reyman and Semenov-Tian-Shansky involves loop algebras and their invariant functions. In the quantum case corresponding invariant operators (symmetrized invariant functions) are badly defined due to infinite-dimensionality of the loop algebra.

Nevertheless Vinberg proved [8] that a subalgebra of the Mishchenko–Fomenko algebra consisting of the elements of the second order in the generators of semisimple Lie algebra is commutative also on the quantum level (i.e. in the universal enveloping algebra). This fact indicates that Mishchenko–Fomenko algebra should have commutative quantum counterpart.

In the presented paper we made one more step in proving this conjecture. We prove commutativity in the universal enveloping of the other subalgebra of Mishchenko–Fomenko algebra. Contrary to the case of Vinberg our subalgebra $L_A \subset \mathfrak{A}(\mathfrak{g}_A)$ is not homogeneous in the coordinates of the underlying Lie algebra, but is of the order not higher than one in the coefficients of the inertia tensor A. Although this is not enough for proving the integrability of the quantum Euler top in the case of the inertia tensor of the general position, but we show, that if the inertia tensor possesses additional symmetries one could construct the full set of "quantum integrals" using the symmetry algebra. Indeed it is known [10], that if the inertia tensor is symmetric with the symmetry group G_A and the symmetry algebra $\mathfrak{g}_A$ then Mishchenko–Fomenko algebra $MF_A \subset P(\mathfrak{g}^*)$ is centralized by $\mathfrak{g}_A$. From the Chevalley isomorphism between $P(\mathfrak{g}^*)$ and $\mathfrak{A}(\mathfrak{g})$ as $\mathfrak{g}$ modules follows that the same fact holds true also for the quantum case. So, for the set of commuting quantum integrals one could take independent integrals of the Mishchenko–Fomenko algebra along with some commutative elements from $\mathfrak{A}(\mathfrak{g}_A)$. Taking into account the number of the independent operators in the algebra L_A (equal to $2\operatorname{rank}\mathfrak{g} - 1$) and the maximal possible number of the independent commuting operators in $\mathfrak{A}(\mathfrak{g}_A)$ (equal to $1/2(\operatorname{ind}\mathfrak{g}_A + \dim\mathfrak{g}_A)$)

one could verify that for obtaining with their help a complete set of commuting quantum integrals one should have the following restrictions on the degeneracy of the matrix A. In the case when the underlying Lie algebra $\mathfrak{g}$ is equal to $gl(n)$, $so(n)$ or $sp(n)$ algebra $\mathfrak{g}_A$ should contain subalgebra $gl(n-2)$, $so(n-2)$ or $sp(n-1)$ correspondingly[1].

Using the "duality" in the dependence of the generators of Mishchenko–Fomenko algebra in the generators of $\mathfrak{g}$ and parameters of the "shift" A along with the result of Vinberg [8] we also prove the integrability of quantum systems that correspond to some strongly degenerated orbits in $\mathfrak{g}^*$ which are characterized as such that their stabilizers include Lie groups $Gl(n-2)$, $SO(n-2)$ or $Sp(n-1)$.

2 Generalized Euler top

In this section we briefly remind several facts from the theory of classical finite-dimensional integrable systems.

As it is known, equation of the motion of rigid body could be written in the form of Puanso [7]:

$$I_1\dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3, \qquad I_2\dot{\Omega}_2 = (I_3 - I_1)\Omega_1\Omega_3, \qquad I_3\dot{\Omega}_3 = (I_1 - I_2)\Omega_1\Omega_2,$$

where $\vec{\Omega}$ is a vector of the angular velocity. Making the replacement of variables $M_i = I_i\Omega_i$ we will have Poinsot equations in the form:

$$\dot{M}_i = \epsilon_{ijk}M_k\frac{\partial H}{\partial M_j},$$

where $H = \sum\limits_{i=1,3} M_i^2/I_i$. They also could be rewritten as:

$$\dot{M}_i = \{M_i, H\},$$

where $\{\ ,\ \}$ is Lie Poisson brackets defined in the following way:

$$\{M_i, M_j\} = \epsilon_{ijk}M_k.$$

The key observation that was made by Arnold [7] is that this equations could be generalized to arbitrary Lie algebra. In this general case they will have the following form:

$$\dot{M}_i = C_{ij}^k M_k\frac{\partial H}{\partial M_j}, \tag{1}$$

where C_{ij}^k are the structure constants of some Lie algebra $\mathfrak{g}$ and M_i-coordinate functions on the dual space $\mathfrak{g}^*$. These equation are so called Euler–Arnold equations. Of course not for every function $H \in \mathfrak{g}^*$ these equations are integrable. Mishchenko and Fomenko [2] found quadratic hamiltonian that provides integrability of equation (1) for arbitrary semisimple Lie algebra. It has the following form:

$$H = (M, \mathrm{ad}_A^{-1}\,\mathrm{ad}_B M),$$

[1]Note, that without knowledge of the commutativity of subalgebra $L_A \subset \mathfrak{A}(\mathfrak{g})$ one can only state integrability of quantum Eulers top in the cases of the Lie algebras $gl(n)$ and $so(n)$ when inertia tensor A is symmetric with respect to the Lie subalgebra $gl(n-1)$ and $so(n-1)$ correspondingly. In this cases the algebra of commuting quantum integrals will simply coincide with the well-known Gelfand–Tsetlin algebra.

where $A, B \in \mathfrak{g}$ are any constant covectors, $(\ ,\ )$ is Killing–Kartan form. Corresponding Euler–Arnold equations are:

$$\frac{dM}{dt} = [M, (\mathrm{ad}_A)^{-1}(\mathrm{ad}_B)M]. \tag{2}$$

It is evident, that these equations are nonlinear. But, as it was shown in [2], they are integrable with the algebra of integrals constructed by the method of the shift of the argument. Let $\{C_{m_k}(M) \subset P(\mathfrak{g}^*)\}$, where m_k is exponents of $\mathfrak{g}$, be a full set of the independent polynomial generators of $I^G(\mathfrak{g}^*)$. Then functions $C^A_{l,m_k}(M)$ obtained from the decomposition:

$$C_{m_k}(M + \lambda A) = \sum_{l=0}^{m_k} \lambda^l C^A_{l,m_k}(M)$$

are mutually commuting integrals of equations (2). Moreover, in the case of the generic covector A they form a full set of the independent integrals of Euler–Arnold equations. If the covector A is nongeneric, then, in order to obtain complete set of mutually commuting integrals one should take along with the set $\{C^A_{l,m_k}(M)\}$ any complete set of commuting functions in $P(\mathfrak{g}^*_A)$ [10]. Here $\mathfrak{g}_A$ is centralizer of A in $\mathfrak{g}$.

3 Quantization and integrability

3.1 Generalities

Quantization is the map from set of coordinates in the phase space into the set of Hermitian operators in some Hilbert space $\mathcal{H}$, so that the following relations holds:

$$\widehat{\{M_i, M_j\}} = \frac{\hbar}{i}[\hat{M}_i, \hat{M}_j].$$

In other words quantization is the homomorphism from the Lie algebra $\mathfrak{g}$, realized as Lie subalgebra in $P(\mathfrak{g}^*)$ (with respect to the Lie–Poisson brackets) into the Lie algebra $\mathfrak{g}$, realized as the subalgebra in the Lie algebra of Hermitian operators in some Hilbert space $\mathcal{H}$. Due to the well known fact that every representation of the arbitrary Lie algebra could be lifted to the representation of its universal enveloping algebra, one could present quantization as the map:

$$P(\mathfrak{g}^*) \hookleftarrow \mathfrak{g} \hat{\rightarrow} \mathfrak{A}(\mathfrak{g}).$$

This map could not be extended to the isomorphism of the algebras $(P(\mathfrak{g}^*), \{\ ,\ \})$ and $(\mathfrak{A}(\mathfrak{g}), [\ ,\ ])$.

From the point of view of integrable systems the latter fact means that, generally speaking, quantum counterparts of the Poisson-commuting classical polynomial integrals are not necessary commutative operators:

$$[\widehat{I}_i, \widehat{I}_j] \neq \widehat{\{I_i, I_j\}}.$$

By other words, proof of the quantum integrability of the classically integrable hamiltonian systems is additional, separated from the process of quantization problem.

3.2 Quantum Euler tops

Let us at last consider the problem of quantum integrability of the generalized Eulers tops. In the standard $so(3)$ case one has only two independent integrals – Hamiltonian and the square of a vector $\vec{M}$: $H = \sum\limits_{i=1,3} M_i^2/I_i$, $M^2 = \sum\limits_{i=1,3} M_i^2$.

Due to the simple fact that M^2 is an invariant function no problems with the quantum integrability arises in the $so(3)$ case. Indeed, operator $\widehat{M^2}$ is a second order Casimir operator, which commutes with the whole Lie algebra $\mathfrak{A}(so(3))$. Hence, evidently:

$$[\widehat{H}, \widehat{M^2}] = 0.$$

In the case of the Lie algebras of the higher rank situation is more complicated. Indeed to prove quantum integrability of the described above systems one should prove that

$$[\widehat{C^A_{p,m_k}(M)}, \widehat{C^A_{s,m_n}(M)}] = 0.$$

Due to the fact that operators $\widehat{C^A_{0,m_k}(M)} \equiv \widehat{C_{m_k}(M)}$, $k = 1, \ldots, \operatorname{rank}\mathfrak{g}$ are Casimir operators (i.e. analogs of the square of a vector $\vec{M}$) one obtains that

$$[\widehat{C^A_{0,m_k}(M)}, \widehat{C^A_{s,m_n}(M)}] = 0$$

for every s, m_n. By other words Casimir operators are always "quantum integrals".

Let us consider other subalgebra in the Mishchenko–Fomenko algebra, namely the algebra generated by the integrals $\{C^A_{1,m_k}(M),\ k = 2, \ldots, \operatorname{rank}\mathfrak{g}\}$.

Next theorem states the commutativity of this algebra in $\mathfrak{A}(\mathfrak{g})$.

Theorem 3.1. *Let $\mathfrak{g}$ be a classical simple Lie algebra over the field $\mathbf{K}$, with the basis $\{\widehat{M_i}\}$. Let $\mathfrak{A}(\mathfrak{g})$ be its universal enveloping algebra, $\mathfrak{Z}(\mathfrak{A}(\mathfrak{g}))$ its center. Let $\{\widehat{C_{m_k}(M)},\ k = 1, \ldots, \operatorname{rank}\mathfrak{g}\}$ be a full set of the generators of $\mathfrak{Z}(\mathfrak{A}(\mathfrak{g}))$:*

$$\widehat{C_{m_k}(M)} = \sum_{i_1,i_2,\ldots,i_k=1}^{\dim\mathfrak{g}} c_{i_1 i_2 \cdots i_k} \widehat{M_{i_1}}\widehat{M_{i_2}}\cdots\widehat{M_{i_k}},$$

where $c_{i_1 i_2 \cdots i_k}$ is some invariant tensor. Let us consider decomposition:

$$C_{m_k}(\widehat{M + \lambda A\mathbf{1}}) = \sum_{i_1,i_2,\ldots,i_k=1}^{\dim\mathfrak{g}} c_{i_1 i_2 \cdots i_k} (\widehat{M_{i_1}} + A_{i_1}\mathbf{1})(\widehat{M_{i_2}} + A_{i_1}\mathbf{1})\cdots(\widehat{M_{i_k}} + A_{i_1}\mathbf{1}) = \sum_{l=0}^{m_k} \lambda^l \widehat{C^A_{l,m_k}},$$

where $A_i \in \mathbf{K}$, $i \in 1, \cdots, \dim\mathfrak{g}$. Let L_A be the subalgebra in $\mathfrak{A}(\mathfrak{g})$ generated by the elements $\{C^A_{0,m_k}(\widehat{M})C^A_{1,m_k}(\widehat{M}),\ k = 1, \ldots, \operatorname{rank}\mathfrak{g}\}$.

Then subalgebra L_A is commutative.

Proof of the theorem follows from the results of paper [11]. Indeed as it is easy to prove, from the parts (i) of the theorems 1 and 2 of paper [11] follows the commutativity of the subalgebra $L_A \subset \mathfrak{A}(\mathfrak{g})$ in the case of the special choice of the Casimir operators. On the other hand L_A does not depend on the choice of the generating set of Casimir operators. Indeed it could be easily proved, using the fact that every other set of Casimir operators could be expressed as a polynomials in the elements $\widehat{C_{m_k}(M)}$ and vice versa. From the latter fact follows that different choice of the set of Casimir elements leads just to the other choice of the generators of L_A. Under such correspondence new set of Casimir elements are expressed by the polynomials in the

$\widehat{C_{m_k}(M)}$. New set of integrals linear in the tensor of inertia A will be expressed polynomially in the $\widehat{C_{m_k}}$ and linearly in $C^A_{1,m_k}(\widehat{M})$, $(k = 1, \ldots, \operatorname{rank}\mathfrak{g})$.

Example. $\mathfrak{g} = gl(n, \mathbf{K})$. Let $\widehat{M}_{i,j}, i, j \in I$, where $I = (1, 2, \ldots, n)$, be the basis in this algebra with the commutation relations:

$$[\widehat{M}_{i,j}, \widehat{M}_{k,l}] = \delta_{k,j}\widehat{M}_{i,l} - \delta_{i,l}\widehat{M}_{k,j}.$$

Universal enveloping algebra $U(gl(n))$ consists of formal polynomials in the elements $\hat{M}_{i,j}$. Let us define the following elements of $U(gl(n))$:

$$(\widehat{M^m})_{i,j} = \sum_{i_1,\ldots,i_{m-1}\in I} \widehat{M}_{i,i_1}\widehat{M}_{i_1,i_2}\cdots\widehat{M}_{i_{m-1},j}.$$

It is known [9], that the elements: $\widehat{C_{m_k}(M)} = (\widehat{M^m}) = \sum_{i\in I}(\widehat{M^m})_{i,i}$, $m \in (1, 2, \ldots, n)$ generate the center of the universal enveloping algebra. For the generalized inertia tensor one can take arbitrary matrix $A \in \operatorname{Mat}(n, \mathbf{K})$. It is not difficult to check, that

$$C^A_{1,m} = \sum_{l=0}^{m-1}(\widehat{M}^l)_{i,j}A_{jk}(\widehat{M}^{m-l})_{k,i}.$$

In this case, instead of the elements $C^A_{1,m}$ one can chose another generators of L_A, which have more simple form:

$$(\widehat{AM^m}) = \sum_{i,j\in I} A_{j,i}(\widehat{M}^m)_{i,j}.$$

Indeed, using the commutation relation one can easily express $C^A_{1,m}$ linear in the terms of $(\widehat{AM^m})$ and $(\widehat{M^n})$ (and vice versa).

To prove the complete quantum integrability of the generalized Eulers top with the generic inertia tensor A associated with the generic (co)adjoint orbit in the Lie algebra $\mathfrak{g}$ one have to prove the commutativity of the whole Mishchenko–Fomenko algebra in $\mathfrak{A}(\mathfrak{g})$. But, if one consider the case of the nongeneric inertia tensor A or the nongeneric (co)adjoint orbit in $\mathfrak{g}$ then for proving quantum integrability only some commutative subalgebras from the Mishchenko–Fomenko algebra are needed.

Let us consider the case of nongeneric inertia tensor first. We will essentially use the following

Lemma 3.1. *Let $\mathfrak{g}_A$ be a centralizer of matrix A in $\mathfrak{g}$. Then $\mathfrak{g}_A$ centralize L_A in $\mathfrak{g}$.*

Proof. It follows from the parts (ii) of theorems 1 and 2 [11] along with the fact that L_A does not depend on the choice of the full set of independent Casimir operators.

Example. Let $\mathfrak{g} = gl(n)$, $A \in \operatorname{Mat}(n, \mathbf{K})$. Then

$$g_A = \Big\{(B\widehat{M}) = \sum_{i,j\in I} B_{j,i}\widehat{M}_{i,j} \mid B \in \operatorname{Mat}(n, \mathbf{K}), [B, A] = 0\Big\}.$$

This lemma enables us to construct full set of commuting quantum integrals of the generalized Euler's top in the case when inertia tensor is not generic, i.e., possesses additional symmetries. in this case Hamiltonian and all other integrals commute with the generators of this symmetries. That is why one can take for the full set of the integrals (both classical and quantum) some set of independent integrals from the Mishchenko–Fomenko algebra along with some full set of commuting integrals from the $\mathfrak{A}(\mathfrak{g}_A)$. One has only find out under what conditions on the matrix A the number of independent generators of L_A plus the maximal number of the commuting integrals from $\mathfrak{A}(\mathfrak{g}_A)$ is equal to $(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})/2$.

Answer to this question gives the following theorem.

Theorem 3.2. *Let $\mathfrak{g}$ be Lie algebra of the type $gl(n)$, $so(n)$ or $sp(n)$. Let centralizer of the numerical matrix $A \in \mathfrak{g} \subset \mathrm{Mat}\,(n, \mathbf{K})$ contain Lie subalgebra of the type $gl(n-2)$, $so(n-2)$ or $sp(n-1)$ respectively. Then quantum Eulers top with the inertia tensor A is integrable.*

Let us consider the "dual" case when the inertia tensor is generic, but the Euler–Arnold equations are restricted to the symplectic leaf of low dimension – strongly degenerated coadjoint orbit $O_{\deg} \simeq G/K$. Due to the fact that the number of mutually commuting integrals should be equal to the one half of the dimension of the phase space it will be in this case substantially smaller. This enables us to state the following theorem.

Theorem 3.3. *Let $O_{\deg} \simeq G/K$ be the degenerated coadjoint orbit in the Lie algebra of the type $gl(n)$, $so(n)$ or $sp(n)$. Let its stabilizer K contains Lie subgroup of the type $Gl(n-2)$, $SO(n-2)$ or $Sp(n-1)$ respectively. Then quantum Eulers top associated with this orbit is integrable for the arbitrary inertia tensors A.*

Proof. From the results of [12] follows that after restriction of the generators of Mishchenko-Fomenko algebra to the orbits of the described in the theorem type, independent generators could be chosen among the generators of the first and second orders in the coordinates of algebra. Hence the statement of the theorem follows from the results of [8].

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