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Weyl-Type Quantization Rules and N-Particle Canonical Realization of the Poincaré Algebra in Two-Dimensional Space-Time

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Quantization of canonical realization of the Poincaré algebra $\mathfrak{p}(1,1)$ corresponding to N-particle interacting system in the two-dimensional space-time $\mathbb{M}_2$ in the front form of dynamics is considered. Hermitian operators corresponding to the Lie algebra of the group $\mathcal{P}(1,1)$ are obtained by means of the set of Weyl-type quantization rules. The requirement of preservation of the Lie algebra of this group restricts the set of quantization rules but does not by itself remove the ambiguity of the quantization procedure. The partition of the set of quantizations into equivalence classes is proposed. The quantization rules from the same equivalence class give the same mass spectrum, and the same evolution of the quantized system.

1 Introduction

Quantization – the problem of construction of the quantum description on the basis of classical theory – occupies a prominent place in the theoretical physics in 20th century.

The basic structure of the classical Hamiltonian mechanics for an unconstrained system is a $2N$-dimensional phase space $\mathbb{P} \simeq \mathbb{R}^{2N}$ (in general case a symplectic manifold) with symplectic form ω. The state of a classical system is described by a point in $\mathbb{P}$. Observable quantities are identified with smooth functions on $\mathbb{P}$. They form the space $C^\infty(\mathbb{P})$. Symplectic form determines on $C^\infty(\mathbb{P})$ the structure of Lie algebra (Poisson algebra) by means of the Poisson bracket [1]. In the quantum mechanics a state is described by a vector $|\psi\rangle$ in some Hilbert space $\mathcal{H}$ and physical observables are self-adjoint operators in $\mathcal{H}$. Correspondence between the classical and quantum pictures is established within the framework of certain quantization procedure which is meant as a linear map $\mathcal{Q} : f \mapsto \hat{f}$ of the Poisson algebra into the set of self-adjoint operators in the Hilbert space $\mathcal{H}$ [2, 3].

For every symmetry group, which is some Lie group G, the classical Hamiltonian description provides a canonical realization of this group. It is well known that quantization procedure can violate commutation relations of the Lie algebra of G [2]. Thus, we cannot *a priori* be sure that any classical symmetry leads after quantization to the quantum one. Moreover, different quantization rules may preserve some types of symmetries and break out other ones. It is natural to demand the preservation of physically important symmetries. Therefore, we shall require for the quantization procedure the fulfilment of the condition $\mathcal{Q}(\{f,g\}) = i[\hat{f},\hat{g}]$ only for some subalgebra of the Poisson algebra. It is clear that canonical generators corresponding to physically important symmetries have to belong to this subalgebra.

In the relativistic mechanics the main algebraic structure is the Lie algebra $\mathfrak{p}(1,3)$ of the Poincaré group $\mathcal{P}(1,3)$, and the description of a system of N interacting particles must be Poincaré invariant in the classical case as well as in the quantum one. Therefore, after quantization canonical generators of the Poincaré group have to be transformed into Hermitian operators which satisfy commutation relations of $\mathfrak{p}(1,3)$. In the relativistic case, the quantization problem is of special interest because Poincaré invariance conditions lead to the complicated dependence of interaction potentials on canonical coordinates and momenta. In most cases classical relativistic Hamiltonians depend on the products of non-commutative (in terms the of the Poisson bracket) quantities. This raises the question of symmetrization of non-commutative operators in the quantum description. Different ordering methods may result in different expressions for physical observable quantities [4]. Starting from certain classical system different quantization procedures may result in non-equivalent quantum systems.

In the two-dimensional space-time $\mathbb{M}_2$ the front form of relativistic dynamics [5, 6] corresponds to the foliation of $\mathbb{M}_2$ by isotropic hyperplanes [7]: $x^0 + x = t$. The Poincaré group $\mathcal{P}(1,1)$ is the automorphism group of this foliation. Only one generator of $\mathfrak{p}(1,1)$ contains an interaction and mechanical description is in some sense similar to the nonrelativistic one. The two-dimensional variant of the front form permits the construction of the number of exactly solvable classical and quantum relativistic models [8, 9, 10, 11, 12]. Due to the certain simplicity of the relativistic description in the front form in $\mathbb{M}_2$, we are able to elucidate the peculiarities of the quantization procedure in the relativistic case [8, 9, 10, 11].

The aim of this article is quantization of the canonical realization of the Poincaré algebra $\mathfrak{p}(1,1)$ corresponding to N-particle relativistic system with an interaction (Section 2) within the framework of the two-dimensional variant of the front form of dynamics. Using the set of Weyl-type quantization rules we construct in Section 3 symmetric operators satisfying quantum commutation relations of $\mathfrak{p}(1,1)$. We study the influence of different quantization rules on quantized system and propose some classification method of non-equivalent quantizations of the canonical realization of the Lie algebra of $\mathcal{P}(1,1)$. We demonstrane the obtained results by the example of N-particle relativistic system with oscillator-like interaction.

2 Hamiltonian description in the front form of dynamics in $\mathbb{M}_2$

The classical Hamiltonian description of the system of N structureless particles with masses m_a $(a = \overline{1,N})$ in the two-dimensional Minkowski space $\mathbb{M}_2$ in the framework of the front form of dynamics leads to the canonical realization of the Lie algebra of $\mathcal{P}(1,1)$ with generators H, P, K [7]. They correspond to energy, momentum, and boost integral. Due to the positiveness of the momentum variables $(p_a > 0)$ [6, 7] in the front form of dynamics, the phase space of N-particle Hamiltonian system is $\mathbb{P} = \mathbb{R}^N_+ \times \mathbb{R}^N$ with standard Poisson bracket

$$\{f,g\} = \sum_{a=1}^{N} \left(\partial f/\partial x_a \partial g/p_a - \partial g/\partial x_a \partial f/\partial p_a\right).$$

The generators $P_\pm = H \pm P$ satisfy the following Poisson bracket relations of the Poincaré algebra $\mathfrak{p}(1,1)$

$$\{P_+, P_-\} = 0, \qquad \{K, P_\pm\} = \pm P_\pm. \tag{2.1}$$

They are determined in terms of particle canonical variables x_a, p_a [7] as follows:

$$P_+ = \sum_{a=1}^{N} p_a, \qquad K = \sum_{a=1}^{N} x_a p_a, \qquad P_- = \sum_{a=1}^{N} \frac{m_a^2}{p_a} + \frac{1}{P_+} V(rp_b, r_{1c}/r). \tag{2.2}$$

Only one generator, namely P_-, depends on interaction. The Poincaré-invariant function V describes the particles interaction and depends on $2N-1$ indicated arguments, where $r_{ac} =$

$x_a - x_c$; $r = r_{12}$; $a, b = \overline{1, N}$, $c = \overline{2, N}$. Generators (2.2), determine the square of the mass function of the system

$$M^2 = P_+ P_- = P_+ \sum_{a=1}^{N} \frac{m_a^2}{p_a} + V(rp_b, r_{1c}/r). \tag{2.3}$$

The description of the motion of a system as a whole may be performed by choosing P_+ and $Q = K/P_+$ as new (external) variables. There exist a lot of possibilities of the choice of inner variables. One of the possible choices of inner canonical variables is [9]:

$$\eta_a = (P_{a+} - p_{a+1})/(2P_{(a+1)+}), \qquad q_a = P_{(a+1)+}(Q_a - x_{a+1}); \tag{2.4}$$

where $a, b = \overline{1, N-1}$ and we use the following notations $P_{a+} = \sum_{i=1}^{a} p_i$, $Q_a = P_{a+}^{-1} \sum_{i=1}^{a} x_i p_i$, $P_{N+} = P_+$, $Q_N = Q$. In the two-particle case variables (2.4) coincide with the variables proposed in Ref. [6].

3 Quantization of canonical realization of the Poincaré algebra in $\mathbb{M}_2$

To quantize the classical generators we have first to determine quantum operators corresponding to the particular canonical variables x_a, p_a. Then for a given set of classical observables $a = a(x, p)$ we construct corresponding quantum operators $\hat{A}$. Let $\hat{x}_a$, $\hat{p}_a$ be Hermitian operators corresponding to the classical particle coordinates and momenta with the following commutation relations: $[\hat{x}_a, \hat{p}_b] = i\delta_{ab}$. The original Weyl application [13] is a basis for the whole set of quantization rules $W_{\mathcal{F}} : a \mapsto \hat{A}$, which map bijectively a family of classical real functions $a(x, p) \in C^\infty(\mathbb{P})$ to a family of Hermitian operators $\hat{A}$ in some Hilbert space $\mathcal{H}$. For $\mathbb{P} \approx \mathbb{R}^{2N}$, the formal definition is given in the explicit form [14] as follows

$$\hat{A} = \int (dk)(ds) \tilde{a}(k, s) \mathcal{F}(k, s) \exp\left[i \sum_a (k_a \hat{x}_a + s_a \hat{p}_a) \right], \tag{3.1}$$

where $\tilde{a}(k, s)$ is the Fourier transform of the function $a(p, q)$. Function $\mathcal{F}(k, s)$ determines the type of quantization. Different choices of $\mathcal{F}(k, s)$ correspond to different ordering conventions. We shall call the elements of the family of quantizations (3.1) Weyl-type quantization rules. For the original Weyl quantization $\mathcal{F}(k, s) = 1$. Let us restrict ourselves to real functions $\mathcal{F}(k, s) \in C^\infty(\mathbb{R}^{2N})$, i.e. $\mathcal{F}(k, s) = \mathcal{F}^*(k, s)$. Every quantization rule must obey the following condition: $\mathcal{Q}(1) = \hat{1}$. As a result, for the family of quantizations (3.1) we obtain $\mathcal{F}(0, 0) = 1$. Hermiticity condition means: $\mathcal{F}(k, s) = \mathcal{F}(-k, -s)$.

In the momentum representation the wave functions $\psi(p) = \langle p | \psi \rangle$ describing the physical (normalized) states in the front form of dynamics constitute the Hilbert space $\mathcal{H}_N^F = \mathcal{L}^2(\mathbb{R}_+^N, d\mu_N^F)$ with the inner product [8]

$$(\psi_1, \psi) = \int d\mu_N^F(p) \psi_1^*(p) \psi(p), \qquad d\mu_N^F(p) = \prod_{a=1}^{N} \frac{dp_a}{2p_a} \Theta(p_a), \tag{3.2}$$

where $d\mu_N^F(p)$ is the Poincaré-invariant measure and $\Theta(p_a)$ is Heaviside step function. Operators act on wave functions $\psi(p) \in \mathcal{H}_N^F$ as integral operators:

$$(\hat{A}\psi)(p) = \int d\mu_N^F(p') \widetilde{A}(p, p') \psi(p'). \tag{3.3}$$

The kernel corresponding to operator (3.1) has the form

$$\widetilde{A}(p,p') = \frac{1}{(2\pi)^N}\int (dx)(dz)\exp\left(i\sum_{a=1}^{N}\left(p'_a - p_a\right)x_a\right) \times \left(\prod_{a=1}^{N}\delta\left(z_a - \frac{p_a+p'_a}{2}\right)2\sqrt{p_a p'_a}\right)\mathcal{F}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial z}\right)a(x,z). \tag{3.4}$$

Now let us consider the quantization procedure of classical canonical generators (2.2) of $\mathfrak{p}(1,1)$. Substituting expressions (2.2) of the generators K, P_+ into (3.4) we obtain the following operators

$$\hat{P}_+ = P_+, \qquad \hat{K} = i\sum_{a=1}^{N} p_a\frac{\partial}{\partial p_a} - \sum_{a=1}^{N}\frac{\partial^2\mathcal{F}(0,0)}{\partial k_a \partial s_a}. \tag{3.5}$$

The generator P_- is transformed into integral operator (3.3) with the kernel

$$\widetilde{P}_-(p,p') = \frac{1}{(2\pi)^N}\int (dx)(dz)\exp\left(i\sum_{a=1}^{N}\left(p'_a - p_a\right)x_a\right) \times \left(\prod_{a=1}^{N}\delta\left(z_a - \frac{p_a+p'_a}{2}\right)2\sqrt{p_a p'_a}\right)\mathcal{F}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial z}\right)\left(\sum_{a=1}^{N}\frac{m_a^2}{z_a} + \frac{V(rz_b, r_{1c}/r)}{\sum_{a=1}^{N} z_a}\right). \tag{3.6}$$

To obtain a unitary representation of the group $\mathcal{P}(1,1)$, we must construct first and foremost such symmetric operators that satisfy the quantum commutation relations of $\mathfrak{p}(1,1)$

$$[\hat{P}_+, \hat{P}_-] = 0, \qquad [\hat{K}, \hat{P}_\pm] = \pm i\hat{P}_\pm. \tag{3.7}$$

The second task is the construction of self-adjoint extensions (if they exist). Here we consider only the first part of the problem.

The last term in the expression (3.5) of the boost operator $\hat{K}$ has no influence on commutation relations (3.7). Thus, the quantization problem reduces in fact to the construction of quantum operator $\hat{P}_-$. That in its turn determines the form of the function $\mathcal{F}$.

Proposition 1. *So that operators (3.5), (3.6) could satisfy the commutation relations (3.7), the function $\mathcal{F}$ has to be of the following form:*

$$\mathcal{F} = \mathcal{F}(ks), \tag{3.8}$$

where the function $\mathcal{F}$ on the right-hand side depends on the all possible products of arguments: $k_1 s_1, \ldots, k_1 s_N, k_2 s_1, \ldots, k_2 s_N, \ldots$.

Proof. In order to satisfy relations (3.7) the kernel $\widetilde{P}_-(p,p')$ must be homogeneous function of the order -1. To satisfy this condition the function $\mathcal{F}$ must obey the following homogeneity equation: $\mathcal{F}(\beta k, \beta^{-1}s) = \mathcal{F}(k,s)$. The only possibility to satisfy this equation is (3.8).

In the classical case the square of total mass function M^2 is an invariant of the group $\mathcal{P}(1,1)$. Thus, to obtain in the quantum case the algebraic structure which is most closely related to the classical one, the quantum Kasimir operator $\hat{M}^2 = \hat{P}_+\hat{P}_-$ should be a quantization result of the classical function $M^2 = P_+P_-$. Unfortunately not every Weyl-type quantization rule with the arbitrary function $\mathcal{F}$ of the form (3.8) will transform the product $P_+P_- = M^2$ ($\{P_+, P_-\} = 0$) of classical functions into the corresponding product of quantum (commutating) operators $\hat{P}_+\hat{P}_- = \hat{M}^2$. This means that not every quantization rule $W_{\mathcal{F}}$, preserving the structure of Lie algebra of the group $\mathcal{P}(1,1)$, preserves commutability of the following diagram

$$\begin{array}{ccc} P_+,\ P_- & \xrightarrow{M^2=P_+P_-} & M^2 \\ \Big\downarrow W_{\mathcal{F}} & & \Big\downarrow W_{\mathcal{F}} \\ \hat{P}_+,\ \hat{P}_- & \xrightarrow{\hat{M}^2=\hat{P}_+\hat{P}_-} & \hat{M}^2\ . \end{array} \tag{3.9}$$

Proposition 2. *If the function $\mathcal{F}$ has the following form*

$$\mathcal{F}=\mathcal{F}(\Delta_1, \Delta_2), \qquad \Delta_1=\sum_{a=1}^{N} k_a s_a, \qquad \Delta_2=\sum_{\substack{a=1\\a\neq b}}^{N}\sum_{b=1}^{N} k_a s_b,$$

then diagram (3.9) *is commutative.*

Proof. The proposition follows from the translation invariance of P_-.

It is obvious that for partial cases with

$$\mathcal{F} = \mathcal{F}(\Delta_1, 0) = \mathcal{F}_1(\Delta_1), \qquad \mathcal{F} = \mathcal{F}(0, \Delta_2) = \mathcal{F}_2(\Delta_2). \tag{3.10}$$

diagram (3.7) is commutative too. $W_{\mathcal{F}_1}$-quantization has been considered, for example, in Ref. [15].

If $\mathcal{F} = \mathcal{F}(\Delta_0) = \mathcal{F}_0$, $\Delta_0 = \Delta_1 + \Delta_2$, then for arbitrary translation invariant function f we have:

$$\mathcal{F}(\hat{\Delta}_0)f = f. \tag{3.11}$$

As follows from (3.11), (3.5), (3.6) the $W_{\mathcal{F}_0}$-quantization leads to the same operators $\hat{P}_-$, $\hat{P}_+$ as well as the original Weyl quantization does. Moreover, quantization rules $W_{\mathcal{F}}$ and $W_{\mathcal{F}\mathcal{F}_0}$ give us the same realization of commutative ideal $\mathfrak{h} = \operatorname{span}(\hat{P}_+,\ \hat{P}_-)$. The quantizations $W_{\mathcal{F}}$ and $W_{\mathcal{F}\mathcal{F}_0}$ may lead to different boost operators: $\hat{K}$, $\hat{K}'$. But these operators generate Lorentz transformations which distinguish on phase factor: $\left(e^{-i\lambda\hat{K}'}\psi\right)(p) = e^{i\alpha}\left(e^{-i\lambda\hat{K}}\psi\right)(p)$. Thus, $\exp(-i\lambda\hat{K}')\psi(p)$ and $\exp(-i\lambda\hat{K})\psi(p)$ belong to the some ray.

In the front form of dynamics the evolution of the quantum system is described by the Schrödinger-type equation

$$i\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \tag{3.12}$$

where $\Psi \in \mathcal{H}_N^F$ and $\hat{H} = (\hat{P}_+ + \hat{P}_-)/2 = (\hat{P}_+ + \hat{M}^2/\hat{P}_+)/2$. Putting $\Psi = \chi(t, P_+)\psi$, where ψ is a function of some Poincaré-invariant inner variables, we obtain the stationary eigenvalue problem for the operator $\hat{M}^2$:

$$\hat{M}^2\psi = \hat{P}_+\hat{P}_-\psi = M_{n,\lambda}^2\psi. \tag{3.13}$$

The ideal $\mathfrak{h}$ generates by means of the Eqs. (3.12), (3.13) the evolution of the system and the mass spectrum. Therefore, it is natural to introduce the following

Definition 1. *Quantizations $W_{\mathcal{F}}$, $W_{\mathcal{F}'}$ which lead to the same realization of the ideal $\mathfrak{h}$ are called equivalent:*

$$W_{\mathcal{F}} \simeq W_{\mathcal{F}'} \ . \tag{3.14}$$

Proposition 3. *Quantization rules $W_{\mathcal{F}}$, $W_{\mathcal{F}'}$ preserving the commutation relations of $\mathfrak{p}(1,1)$, where $\mathcal{F} = \mathcal{F}(ks, \Delta_0)$, $\mathcal{F}' = \mathcal{F}(ks, 0)$, are equivalent:*

$$W_{\mathcal{F}(ks,\Delta_0)} \simeq W_{\mathcal{F}(ks,0)}. \tag{3.15}$$

Proof. This follows immediately from (3.11) and translation invariance of P_{-}.

Corollary 1.

$$W_{\mathcal{F}(ks)\mathcal{F}_0} \simeq W_{\mathcal{F}(ks)}. \tag{3.16}$$

For the special class of quantization rules which preserve, in addition to the commutation relation of $\mathfrak{p}(1,1)$, the commutability of the diagram (3.9) we have $W_{\mathcal{F}_1(\Delta_1)} \simeq W_{\mathcal{F}_2(-\Delta_2)}$, $W_{\mathcal{F}_2(\Delta_2)} \simeq W_{\mathcal{F}_1(-\Delta_1)}$. Hence, we see that the Weyl-type quantization rules which preserve the commutation relation of the Poincaré algebra $\mathfrak{p}(1,1)$ fall apart into equivalence classes. Rules from different classes can give non-equivalent unitary representations of the group $\mathcal{P}(1,1)$ and may result in different expressions for such important observable quantity as the mass spectrum of the system. We shall demonstrate this fact by the example of N-particle system with oscillator-like interaction.

Let us choose the interaction function V in the following form

$$V = \omega^2 \sum\sum_{a<b} r_{ab}^2 p_a p_b, \qquad \omega^2 > 0. \tag{3.17}$$

The function (3.17) describes N-particle oscillator-like interaction [9]. In the nonrelativistic limit such a system is reduced to the nonrelativistic oscillator system. The system with interaction (3.17) has $N-2$ additional integrals of motion λ_j in involution: $\{\lambda_i, \lambda_k\} = 0$, $i, k = \overline{2, N-1}$. In terms of the variables (2.4) they have the form

$$\begin{aligned} \lambda_{j+1}^2 = {} & \sum_{d=1}^{j} \frac{m_d^2}{1/2 - \eta_{d-1}} \prod_{i=d}^{j} (1/2 + \eta_i)^{-1} + \frac{m_{j+1}^2}{1/2 - \eta_j} \\ & + \omega^2 \sum_{d=1}^{j-1} \left(1/4 - \eta_d^2\right) q_d^2 \prod_{i=d+1}^{j} (1/2 + \eta_i)^{-1} + \omega^2 \left(1/4 - \eta_j^2\right) q_j^2, \end{aligned} \tag{3.18}$$

where $\lambda_N^2 = M^2$, $j = \overline{1, N-1}$.

Quantum mechanical description for the system with interaction (3.17) was constructed by means of the ordinary Weyl quantization in Ref. [9]. Here we consider $W_{\mathcal{F}_1}$-quantization (see (3.10)). One can show that $W_{\mathcal{F}_1}$-quantization transforms the classical integrals into quantum ones ($[\hat{\lambda}_i, \hat{\lambda}_j]$) and we obtain the following mass spectrum of the system:

$$\begin{aligned} M_n^2 = {} & \left[\sum_{a=1}^{N} \sqrt{m_a^2 - (\omega \mathcal{F}_1'(0))^2} + \omega \sum_{b=1}^{N-1} (n_b + 1/2) \right]^2 \\ & + \omega^2 \left[(N-1) \left(\frac{1}{4} - N \mathcal{F}_1''(0) \right) + \left(N \mathcal{F}_1'(0) \right)^2 \right]. \end{aligned} \tag{3.19}$$

The discrete spectrum exists only if $\omega|\mathcal{F}_1'(0)| \le \min\{m_a\}$, $a = \overline{1,N}$. This gives additional restriction for the type of $W_{\mathcal{F}_1}$-quantization. We see that the mass spectrum depends essentially on the choice of quantization rule. In the case $\mathcal{F}_1 = 1$ we come to the spectrum of the system with the interaction (3.17) which has been obtained by the original Weyl quantization in Ref. [9]. In this work the generalization of the pure oscillator-like interaction has been considered too. This new interaction function contains also the terms which are linear in the coordinates: $V \to \tilde{V} = V + \alpha \sum\limits_{a<b}\sum r_{ab}(p_a - p_b)$. The original Weyl quantization gives the following result (see Ref. [9]):

$$M_n^2 = \left[\sum_{a=1}^{N}\sqrt{m_a^2 - \frac{\alpha^2}{4\omega^2}} + \omega\sum_{b=1}^{N-1}(n_b + 1/2)\right]^2 + \frac{N-1}{4}\omega^2 + \frac{\alpha^2 N^2}{4\omega^2}. \tag{3.20}$$

Comparing the equalities ((3.19)), ((3.20)) we see that the quantizations $W_{\mathcal{F}_1}$, $\mathcal{F}_1'(0) \neq 0$, $\mathcal{F}_1''(0) = 0$ of the classical system with the pure oscillator-like interaction (3.17) gives the terms in the expression for mass spectrum ((3.19)) which one can treat as a presence of the linear interaction with $\alpha = -2\omega^2\mathcal{F}_1'(0)$. Then such a quantum system is equivalent to those which is obtained from the classical system with the interaction $\tilde{V}$ by means of the original Weyl quantization. Thus, the use of different quantization rules may lead to essentially different quantum results. Moreover different quantizations may lead to quantum systems with physically different interactions!

In the nonrelativistic case all the ambiguities in the mass spectrum ((3.19)) vanish and we obtain well known energy spectrum of nonrelativistic system with the oscillator interaction. But the first relativistic correction to the nonrelativistic energy depends on the type of quantization:

$$\begin{aligned} E \approx \hbar\omega\sum_{b=1}^{N-1}(n_b + 1/2) + \frac{\hbar^2\omega^2}{2c^2}\Bigg\{ & \frac{1}{m}\left(\sum_{b=1}^{N-1}(n_b + 1/2)\right)^2 \\ & -(\mathcal{F}_1'(0))^2\sum_{a=1}^{N}\frac{1}{m_a} + \frac{1}{m}\left[(N-1)\left(\frac{1}{4} - N\mathcal{F}_1''(0)\right) + \left(N\mathcal{F}_1'(0)\right)^2\right]\Bigg\}. \end{aligned} \tag{3.21}$$

Here we renewed the constants $\hbar$, c.

Let us note that for the quantization of the oscillator-like interaction we used only quantizations preserving the commutability of the diagram (3.9). Using the quantization rules $W_{\mathcal{F}}$ (3.8), which preserve only the commutation relations of the Poincaré algebra $\mathfrak{p}(1,1)$, we could obtain more ambiguous results for the mass spectrum.

4 Conclusions

We have considered the problem of the quantization of the classical canonical realization of the Poincaré algebra $\mathfrak{p}(1,1)$ corresponding to N-particle relativistic system with an interaction. It has been demonstrated that for Weyl-type quantization rules (3.1) the requirement of preservation of the Lie algebra $\mathfrak{p}(1,1)$ restricts the set of quantization rules but does not by itself remove the ambiguity of the quantization procedure.

In the classical case the square of total mass function $M^2 = P_+P_-$ is an invariant of the group $\mathcal{P}(1,1)$. To obtain in the quantum case the algebraic structure which is most closely related to the classical one, the quantum Kasimir operator $\hat{M}^2 = \hat{P}_+\hat{P}_-$ must be the quantization result of the classical expression $M^2 = P_+P_-$. This additional requirement imposes additional restriction on the family of the Weyl-type quantization rules. Thus we see, that if one require the quantization to preserve at least some of the associative algebra structure of $C^\infty(\mathbb{P})$ then one can restrict abbiguties of quantization procedure. But it does not fully eliminate the ambiguity of the quantization either.

We also demonstrated that the Weyl-type quantization rules are split into equivalence classes. Quantization rules from the same equivalence class lead to the same realization of the ideal $\mathfrak{h}$ and therefore give the same mass spectrum and the evolution of quantized system. The quantizations which belong to different classes lead to non-equivalent quantum systems. We have demonstrated the last fact by the example of the N-particle system with the oscillator-like interaction. Therefore, if we start with the classical description of a mechanical system then quantization rule seems to be an essential part of the definition of the corresponding quantum system.

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