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Quasipotential Approach to Solitary Wave Solutions in Nonlinear Plasma

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Sagdeev's Quasipotential approach is extremely suitable for studying large amplitude solitary waves in plasma. One can derive all the one soliton results of perturbation methods and can compare it with the exact results obtained by the Quasipotential (also called the pseudopotential) method. However comparatively fewer works in relativistic plasma and plasma with trapped electrons have used this method. In this paper the pseudopotential is derived for a relativistic plasma with non-isothermal electrons and finite temperature ions. Expanding the quasipotential different types of solitons are obtained which agree with the perturbation results. Also the relativistic effect and finite ion temperature appear to restrict the region of existence of solitary waves.

1 Introduction

Theoretical studies on soliton dynamics were made very early in the frame work of Korteweg-de Vries (K-dV) equation using the reductive perturbative method in fluid dynamics. It was later extended to plasma dynamics [1, 2]. However the pertubation methods were mainly valid for small amplitude solitary waves. Sagdeev's pseudopotential approach [3] is appropriate for studying large amplitude solitary waves. Though this approach was rather widely used in obtaining travelling solitary waves solutions in simple non-relativistic plasmas, the applications to relativistic plasmas or plasmas with trapped electron are few and far between. But relativistic effects play an important part in the formation of solitary waves for particles with very high velocities which are comparable to that of light (for experimental and other details see references [4–11]).

Again most of the studies concerning solitary waves in both relativistic and non-relativistic plasmas did not consider the resonant particles which interact strongly with the wave during its evolution. These particles have to be treated in a way differant from what is done in the case of the free particles. Schamel [12, 13] made a theoretical study on ion-acoustic waves due to resonant electrons in a frame work of KdV and M KdV equations.

In this paper our aim is to study large amplitude solitary waves in a relativistic plasma with warm ions and with two differant distribution function for the electrons, one for the trapped and another for the free electrons. In this case the electron density is defined from the Vlasov equations consisting of free and trapped electrons as

$$n_e(\phi) = k_0 \left[e^{\phi} \operatorname{erfc}(\phi)^{1/2} + |\beta|^{-1/2} \left\{ \begin{array}{l} \exp(\beta\phi) \operatorname{erf}(\beta\phi)^{1/2} \\ \dfrac{1}{\sqrt{2}} w\left(-\beta\phi^{1/2}\right) \end{array} \right\} \begin{array}{l} -\beta \geq 0 \\ \beta < 0 \end{array} \right], \tag{1}$$

where k_0 is some constant and

$$\beta = T_{\rm ef}/T_{\rm et}, \tag{2}$$

$T_{\rm ef}$, $T_{\rm et}$ being the temperatures for the free electrons and the trapped electron respectively, where $\operatorname{erf}(x)$ is given by $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

In the present paper the case $\beta \geq 0$ will be considered and the case $\beta < 0$ which gives a dip in the distribution function can be treated in a similar manner. The organization of the paper is as follows.

In Section 2 the exact pseudopotential is derived from the basic equations. In Section 3 solitary wave solutions are discussed. Small amplitude-approximations are derived in Section 4.

2 Basic equations and derivation of Sagdeev's potential

The basic system of equations governing the ion motion in plasma dynamics in unidirectional propagation is given by

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0, \tag{3}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\gamma u + \frac{\sigma}{n}\frac{\partial p}{\partial x} = -\frac{\partial \phi}{\partial x}, \tag{4}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)p + 3p\frac{\partial u}{\partial x} = 0, \tag{5}$$

where n, u, p denote the density, velocity and pressure respectively for the ion species. γ is given by $\gamma = \sqrt{1 - u^2/c^2}$, c being the speed of light.

The above equations are supplemented by the Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} + n - n_e = 0, \tag{6}$$

where we take

$$n_e = e^{\phi}\operatorname{erfc}(\phi^{1/2}) + \beta^{-1/2}e^{\beta\phi}\operatorname{erf}(\beta\phi)^{1/2} \tag{7}$$

with $\beta > 0$. β and $\operatorname{erf}(x)$ are defined earlier.

Also $\sigma = T_i/T_{\rm eff}$, T_i being the ion-temperature and $T_{\rm eff}$ is defined below. The above equations are normalized in the following way.

The velocities are normalized to the ion-acoustic speed $c_s = \left(\frac{kT_{\rm eff}}{m}\right)^{1/2}$, k being the Boltzmann constant and m_i the ion mass.

The distance and time t are normalized to the Deby length $\left(\frac{\epsilon_0 kT_{\rm eff}}{n_0\ e^2}\right)^{1/2}$ and ion plasma period $\left(\frac{\epsilon_0 m}{n_0 r^2}\right)^{1/2}$ respectively, ϵ_0 being the dielectric constant. The ion pressure is normalized to $(n_0 kT_i)^{-1}$ and the electrostatic potential ϕ is normalized to $\frac{kT_{\rm eff}}{e}$, e being the electron charge. Here $T_{\rm eff}$ is given by $T_{\rm eff} = T_{\rm ef}\ T_{\rm et}/(n_{\rm ef}T_{\rm ef} + n_{\rm et}T_{\rm et})$, $n_{\rm ef}$, $n_{\rm et}$ being the initial densities of the free and trapped electrons respectively and $n_{\rm ef} + n_{\rm et} = 1$.

To obtain the solitary wave solution we make the dependent variables depend on a single independent variable $\xi = x - Vt$, where V is the velocity of the solitary wave.

Equations (3)–(6) can now be written as

$$-V\frac{dn}{d\xi} + \frac{d}{d\xi}(nu) = 0, \tag{8}$$

$$-V\frac{d}{d\xi}(\gamma u)+u\frac{d}{d\xi}(\gamma u)+\frac{\sigma}{n}\frac{dp}{d\xi}=-\frac{d\phi}{d\xi}, \tag{9}$$

$$-V\frac{dp}{d\xi}+u\frac{dp}{d\xi}+3p\frac{du}{d\xi}=0, \tag{10}$$

$$\frac{d^2\phi}{d\xi^2}=n_e-n. \tag{11}$$

Equation (10) is consistent with

$$p=n^3p_0, \tag{12}$$

i.e. we consider the adibatic case and hence forth we shall take $p_0=1$.

From the above equations one can eliminate, n, n_e, u and p to obtain a differential equation involving ϕ which can be written as Newton's equation in the following way

$$\frac{d^2\phi}{d\xi^2}=-\frac{\partial\psi}{\partial\phi}, \tag{13}$$

where ψ is the so called Sagdeev's potential which is in general a transcendental function of ϕ. The exact form of $\psi(\phi)$ is given by

$$\psi(\phi)=\psi_e(\phi)+\psi_i(\phi), \tag{14}$$

where

$$\psi_e(\phi)=e^{\phi}\,\mathrm{erfc}\left(\sqrt{\phi}\right)+\frac{1}{\beta\sqrt{\beta}}e^{\beta\phi}\,\mathrm{erf}\left(\sqrt{\beta\phi}\right)+\frac{2}{\beta\sqrt{\pi}}\phi^{1/2}(\beta-1) \tag{15}$$

and

$$\psi_i(\phi) \ = \ Vu\gamma-Vu_0\gamma_0+\sigma v^3\left[\frac{1}{(V-u_0)^3}-\frac{1}{(V-u)^3}\right]. \tag{16}$$

In deriving equations (15) and (16) the following boundary conditions were used. As $\xi\to\infty$, $\phi\to 0$, $u\to u_0$, $p\to 1$, $n\to 1$. Also the relation between ϕ and u is given by

$$\phi=\left(vu-c^2\right)\gamma-\left(vu_0-c^2\right)\gamma^0+\frac{3\sigma}{2}V^2\left[\frac{1}{(V-u_0)^2}-\frac{1}{(V-u)^2}\right], \tag{17}$$

where

$$\gamma_0=\frac{1}{\sqrt{1-u_0^2/c^2}}.$$

3 Solitary waves solution

The form of the pseudopotential would determine whether soliton like solutions of equation (13) may exist or not.

The condition for the existence of solitary waves are the following

(i) $\left.\dfrac{d^2\psi}{d\phi^2}\right|_{\phi=0}<0.$

This is the condition for the existence of potential well another conditions

(ii) $\psi(\phi_m) > 0$,

where ϕ_m is the maximum (magnitude wise) value of ϕ beyond which ψ becomes imaginary. In this case ψ crosses the ϕ axis from below at the point $\phi = \phi_m$.

In Fig. 1 $\psi(\phi)$ is plotted against ϕ for different values of β ranging from 0.03 to 0.2 the ohter parameters are $V = 1.5$, $\sigma = 0.001$, $u_0 = 0$, $c/c_s = 100$.

It is seen that for $\beta \geq 0.2$ the amplitude of the soliton becomes very small and for a much larger value of β soliton solutions will disappear. Again for values of $\beta < 0.044$ solutions would cease to exist. In Fig. 2 the solitory wave solution $\phi(\xi)$ is plotted against ϕ for $\beta = 0.045$ and $\beta = 0.1$ other parameters are same as those in Fig. 1. It is found that both the height and of the width of the soliton decrease as β increases.

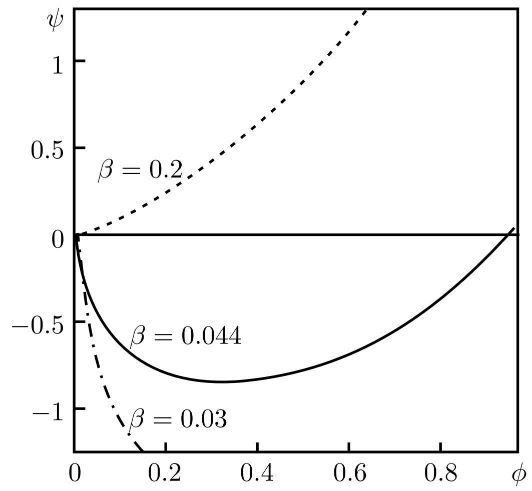

Figure 1

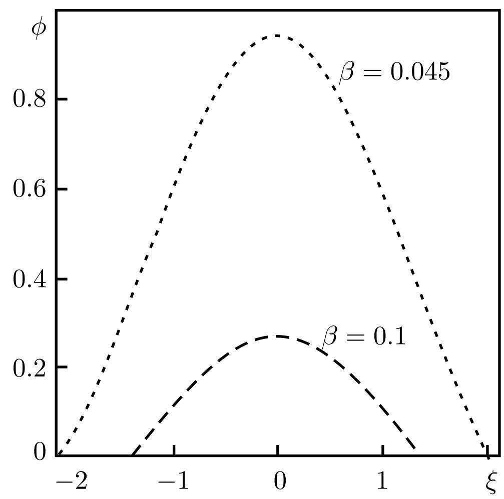

Figure 2

4 Small amplitude approximation

To obtain KdV (Korteweg de Vries) type soliton we obtain here small amplitude approximation of $\psi(\phi)$.

Expanding $\psi(\phi)$ from (15) and (16) we have

$$\frac{d^2\phi}{d\xi} = -\frac{\partial\psi}{\partial\phi} = A_1\phi - A_2\phi^{3/2} + A_3\phi^2 - A_4\phi^{5/2} + \cdots, \tag{18}$$

where

$$A_1 = 1 - \frac{3\sigma}{(V-u_0)^4} - \frac{1}{(V-u_0)^2}\left[1 - \frac{3u_0^2}{2c^2}\right] + \frac{3\sigma}{16c^2(V-u_0)^2}\left[\frac{40\left(3u_0^4 - 4Vu_0^3\right)}{(V-u_0)^4} - \frac{15V^4}{(V-u_0)^4} + \frac{18V^2}{(V-u_0)^2} - 3\right], \tag{19}$$

$$A_2 = \frac{4b_1}{3}, \tag{20}$$

and

$$A_3 = \frac{1}{2}\left[1 - \frac{30\sigma}{(V-u_0)^6} - \frac{3}{(V-u_0)^4}\left(1 + \frac{40(V+2u_0)}{c^2}\right)\right.$$
$$\left. + \frac{9\sigma}{16c^2(V-u_0)^4}\left(-\frac{630\left(3u_0^4 - 4Vu_0^3\right)}{(V-u_0)^4}\right) + \frac{35V^4}{(V-u_0)^4} - \frac{30V^2}{(V-u_0)^2} + 3\right], \tag{21}$$

while

$$A_4 = \frac{8}{15}b_2. \tag{22}$$

Neglecting A_3 and A_4 solution of (20) is

$$\phi(\xi) = \left(\frac{5A_1}{4A_2}\right)^2 \operatorname{sech}^4\left(\frac{\xi}{\partial}\right), \qquad \text{where} \quad \partial = \frac{4}{\sqrt{A_1}}. \tag{23}$$

To get a shock wave solution we include A_3 term and put $\phi = Y^2$ to get

$$2\left(\frac{dY}{d\xi}\right)^2 = \frac{A_1}{2}Y^2 - \frac{2A_2}{5}Y^3 + \frac{A_3}{3}Y^4. \tag{24}$$

For a shock wave like solution $\frac{dY}{d\xi}$ should vanish both at $Y = 0$ and at a value $Y = Y_m$, Y_m being the amplitude of the solitory wave type solution. (24) can then be written as

$$\frac{dY}{d\xi} = kY\left(Y_m - Y\right), \tag{25}$$

where we take

$$Y_m = \frac{3}{5}\frac{A_2}{A_3}, \qquad 25A_1A_3 = 6A_2^2 \qquad \text{and} \qquad k = \pm\left(\frac{A_3}{6}\right)^{1/2}, \tag{26}$$

putting

$$\partial = \left(\frac{k}{2}\phi_m\right)^{-1} \tag{27}$$

the final solution becomes

$$\phi = \frac{\phi_m}{4}\left(1 \pm \tanh\frac{\xi}{\partial}\right)^2,$$

where

$$\phi_m = Y_m^2. \tag{28}$$

Other types of solitons viz, spiky type solitary waves collapsible waves etc. can be obtained by taking higher order terms and using the so called 'tanh' method [14, 15].

Since the expression for $\psi(\phi)$ derived in equations (14), (15) and (16) is exact, one can expand it up to any order in ϕ and obtain all the different types of solitary waves depending on the non-isothermality parameter β, obtained by perturbation methods.

For example if we include the A_4 term and write

$$\frac{d^2\psi(\phi)}{d\xi^2} = A_1\phi - A_2\phi^{3/2} + A_3\phi^2 - A_4\phi^{3/2}. \tag{29}$$

Equation (29) for the spiky type solitary wave can be studied by transforming the equation as

$$\left(\frac{d\Phi}{d\eta}\right)^2 = a_1\Phi^2\left(\phi_0 - \Phi\right)^3, \qquad \text{where} \qquad \phi = \Phi^2 \tag{30}$$

given below Eq.(20) and we take $a_1 = \frac{A_4}{7}$, $\phi_0 = \frac{7}{18}\frac{A_3}{A_4}$, $A_2 = \frac{35}{108}\frac{A_3^2}{A_4}$, and $A_1\ A_4 = \frac{7}{270}A_2A_3$.

The Eq.(30) can be solved for soliton profile and the solution $\phi_S(\eta)$ can be obtained only as an implicit function of η in the following way.

$$\phi_S(\eta) = \phi_0^2\,\mathrm{sech}^4\left[\left(\frac{\phi_0}{\phi_0 - \sqrt{\phi_S(\eta)}}\right)^{1/2} \pm \frac{1}{2}\sqrt{a_1\phi_0^3}(\eta - \eta_0) - C_1\right], \tag{31}$$

where $C_1 = \left(\frac{\phi_0}{\phi_0 - \sqrt{\phi_m}}\right)^{1/2} - \mathrm{sech}^{-1}\left(\frac{\sqrt{\phi_m}}{\phi_0}\right)^{1/2}$ and ϕ_m is the optimal amplitude of the acoustic mode. Note that $\phi_S(\eta)$ occurs on both left and right hand sides of Eq.(31). The solution (Eq.(31)) gives a profile of spiky solitary wave defined in the region $0 < \phi(\eta) < \sqrt{\phi_0}$. While for other region defined as $\phi < 0$, the soliton solution can be obtained in a similar manner and is given by

$$\phi_E(\eta) = \phi_0^2\,\mathrm{cosech}^4\left[\left(\frac{\phi_0}{\phi_0 - \sqrt{\phi_E(\eta)}}\right)^{1/2} \pm \frac{1}{2}\sqrt{a_1\phi_0^3}(\eta - \eta_0) - C_2\right], \tag{32}$$

where $C_2 = \left(\frac{\phi_0}{\phi_0 - \sqrt{\phi_m}}\right)^{1/2} - \mathrm{cosech}^{-1}\left(\frac{\sqrt{\phi_m}}{\phi_0}\right)^{1/2}$, and this is to be recognised as the explosive solitary wave in the plasma-acoustic dynamics. Thus one can proceed taking the nonlinear term to any order in ϕ and could derive different natures of the solitary waves under different approximations.

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