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Symmetric Properties of First-Order Equations of Motion for $N = 2$ Super Yang–Mills Theory

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We find 4-parameter non-hermitean $N = 1$ transformations, under which the first-order equations of motion for $N = 2$ supersymmetric Yang–Mills (YM) theory in $N = 1$ superspace are invariant.

One of the essential features of non-Abelian gauge theories is the existence of classical solutions with non-trivial topological properties (monopoles, instantons). Their importance encourages mathematical and physical community to further investigations in this domain. But searching the solutions of YM equations is very difficult task because of their non-linearity. As usual, to gain the goal one tries by finding the solutions of simpler equations, for example, first-order equations, which satisfy the second-order equations of motion. Thus the problem of finding corresponding first-order equations arises.

In pure YM theory they usually deal with self-duality equation. In YM theories with scalar fields the first-order equations are mainly generalizations of self-duality equation. One of such generalizations, *quasi-self-duality equation,* was introduced by V.A. Yatsun [1]. He proposed the additional term in the equation of self-duality have to be added, which is properly chosen combination of scalar fields. In the case of vanishing scalar fields the quasi-self-duality equation boils down to self-duality equation. The quasi-self-duality equation together with constraints on scalar fields form *the system of quasi-self-duality equations.*

In this report first we deal with $N = 2$ YM theory in $N = 1$ superspace with the Lagrangian [2]:

$$\begin{aligned} L = \mathrm{Tr}\bigg\{ & -\frac{1}{4}F_{mn}F^{mn} - i\bar{\lambda}\bar{\sigma}^m\mathcal{D}_m\lambda - \frac{1}{2}(\mathcal{D}_mA)^2 - \frac{1}{2}(\mathcal{D}_mB)^2 - i\bar{\psi}\bar{\sigma}^m\mathcal{D}_m\psi \\ & + ig(A+iB)\{\lambda^\alpha, \psi_\alpha\} + ig(A-iB)\{\bar{\lambda}_{\dot{\alpha}}, \bar{\psi}^{\dot{\alpha}}\} + igD[A,B] + \frac{1}{2}D^2 + \frac{1}{2}F^2 + \frac{1}{2}G^2 \bigg\}. \end{aligned} \tag{1}$$

The theory (1) is invariant under $N = 1$ supersymmetry transformations:

$$\begin{aligned} & \delta_\xi(A-iB) = 2\xi\psi, \qquad \delta_\xi(A+iB) = 2\bar{\xi}\bar{\psi}, \qquad \delta_\xi V_{\alpha\dot{\alpha}} = -2i(\xi_\alpha\bar{\lambda}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}}\lambda_\alpha), \\ & \delta_\xi D = -\xi^\alpha\mathcal{D}_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}}\mathcal{D}^{\alpha\dot{\alpha}}\lambda_\alpha, \qquad \delta_\xi(F+iG) = 2i\bar{\xi}_{\dot{\beta}}\left(\mathcal{D}^{\alpha\dot{\beta}}\psi_\alpha + g[\bar{\lambda}^{\dot{\beta}}, A-iB]\right), \\ & \delta_\xi(F-iG) = 2i\xi^\alpha\left(\mathcal{D}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}} + g[\lambda_\alpha, A+iB]\right), \qquad \delta_\xi\lambda_\alpha = \frac{1}{2}\xi^\beta(f_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}D), \\ & \delta_\xi\bar{\lambda}_{\dot{\alpha}} = \frac{1}{2}\bar{\xi}^{\dot{\beta}}(f_{\dot{\alpha}\dot{\beta}} - 2i\varepsilon_{\dot{\alpha}\dot{\beta}}D), \qquad \delta_\xi\psi_\alpha = i\bar{\xi}^{\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}(A-iB) + \xi_\alpha(F+iG), \\ & \delta_\xi\bar{\psi}_{\dot{\alpha}} = -i\xi^\alpha\mathcal{D}_{\alpha\dot{\alpha}}(A+iB) + \bar{\xi}_{\dot{\alpha}}(F-iG), \end{aligned} \tag{2}$$

where ξ_α, $\bar{\xi}_{\dot{\alpha}}$ are 4 parameters of the transformations.

It was found that the equations of motion, corresponding to Lagrangian (1), are satisfied by the following system of first-order equations [3] (they are first–order quasi-self-duality equations):

$$\begin{aligned} &f_{\alpha\beta} = 2gc_{\alpha\beta}[A,B], \qquad (c_{\alpha\beta}-\varepsilon_{\alpha\beta})\mathcal{D}^{\beta\dot\beta}(A-iB)=0, \qquad (c_{\alpha\beta}+\varepsilon_{\alpha\beta})\mathcal{D}^{\beta\dot\beta}(A+iB)=0,\\ &D+ig[A,B]=0, \qquad F=G=0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta}-g[\psi_\alpha, A+iB]=0,\\ &\mathcal{D}^{\alpha\dot\beta}\psi_\alpha+g[\bar\lambda^{\dot\beta}, A-iB]=0, \qquad (c_{\alpha\beta}-\varepsilon_{\alpha\beta})\psi^\beta=0, \qquad \lambda_\alpha=\bar\psi_{\dot\alpha}=0, \end{aligned} \tag{3}$$

where $c_{\alpha\beta}$ are complex constant coefficients, satisfying the conditions

$$c_{\alpha\beta}=c_{\beta\alpha}, \qquad \det\|c_{\alpha\beta}\| \equiv c_{11}\cdot c_{22}-c_{12}^2 \equiv \frac{1}{2}c^{\alpha\beta}c_{\alpha\beta}=-1. \tag{4}$$

The system (3) is not invariant under transformations (2). But it is invariant when the following constraints on the parameters of these transformations are imposed

$$(c_{\alpha\beta}+\varepsilon_{\alpha\beta})\xi^\beta=0. \tag{5}$$

In other words, the system (3) is invariant in 3-dimensional subspace of 4-dimensional space of parameters of transformations (2). Our aim is to find such $N=1$ transformations depending on four parametrs under which the system (3) to be invariant.

The $N=2$ supersymmetric YM theory in $N=2$ superspace, given by the Lagrangian [4]

$$\begin{aligned} L = \mathrm{Tr}\Big(&-\frac{1}{4}F_{mn}F^{mn}-i\bar\lambda_{\dot\alpha i}\overline{\sigma}^{m\dot\alpha\beta}\mathcal{D}_m\lambda^i_\beta-2\mathcal{D}_mC\mathcal{D}^mC^*-\frac{1}{2}\vec{C}^2\\ &+igC\{\bar\lambda_{\dot\alpha i},\bar\lambda^{\dot\alpha i}\}+igC^*\{\lambda^i_\alpha,\lambda^\alpha_i\}+4g^2C[C,C^*]C^*\Big), \end{aligned} \tag{6}$$

is invariant under $N=2$ supersymmetry transformations [5]:

$$\begin{aligned} &\delta_\xi C=-\xi^\alpha_i\lambda^i_\alpha, \qquad \delta_\xi C^*=-\bar\xi_{\dot\alpha i}\bar\lambda^{\dot\alpha i}, \qquad \delta_\xi V_{\alpha\dot\alpha}=2i(\xi^i_\alpha\bar\lambda_{\dot\alpha i}+\bar\xi_{\dot\alpha i}\lambda^i_\alpha),\\ &\delta_\xi\lambda^i_\alpha=-\frac{1}{2}\xi^{\beta i}f_{\alpha\beta}+2ig\xi^i_\alpha[\,C\,,\,C^*\,]-\xi_{\alpha j}\vec{C}\vec\tau^{ij}+2i\bar\xi^{\dot\alpha i}\mathcal{D}_{\alpha\dot\alpha}C,\\ &\delta_\xi\bar\lambda_{\dot\alpha i}=-\frac{1}{2}\bar\xi^{\dot\beta}_i f_{\dot\alpha\dot\beta}-2ig\bar\xi_{\dot\alpha i}[C,C^*]+\bar\xi^j_{\dot\alpha}\vec{C}\vec\tau_{ij}+2i\xi^\alpha_i\mathcal{D}_{\alpha\dot\alpha}C^*,\\ &\delta_\xi\vec{C}=-i\xi^{\alpha i}(\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta j}+2g[\lambda^j_\alpha,C^*])\vec\tau_{ij}+\bar\xi^i_{\dot\alpha}(\mathcal{D}^{\alpha\dot\beta}\lambda^j_\alpha-2g[\bar\lambda^{\dot\beta j},C])\vec\tau_{ij}, \end{aligned} \tag{7}$$

where ξ^i_α, $\bar\xi_{\dot\alpha i}$ are 8 parameters of $N=2$ supersymmetry transformations, and $\vec\tau^{\,j}_i$ are Pauli matrices.

The theory (1) can be obtained from (6) by the replacement

$$\begin{aligned} &C=\frac{1}{2}(A-iB), \qquad C^*=\frac{1}{2}(A+iB), \qquad g[C,C^*]=-\frac{1}{2}D,\\ &C_1=-iG, \qquad C_2=-iF, \qquad C_3=0,\\ &\lambda^1_\alpha=\lambda_\alpha, \qquad \lambda^2_\alpha=\psi_\alpha, \qquad \bar\lambda_{\dot\alpha 1}=\bar\lambda_{\dot\alpha}, \qquad \bar\lambda_{\dot\alpha 2}=\bar\psi_{\dot\alpha}. \end{aligned} \tag{8}$$

Using (8) we rewrite the system (3) in terms of $N=2$ fields

$$\begin{aligned} &f_{\alpha\beta}=4igc_{\alpha\beta}[C^*,C], \qquad (c_{\alpha\beta}-\varepsilon_{\alpha\beta})\mathcal{D}^{\beta\dot\beta}C=0, \qquad (c_{\alpha\beta}+\varepsilon_{\alpha\beta})\mathcal{D}^{\beta\dot\beta}C^*=0,\\ &\vec{C}=0, \qquad \mathcal{D}^{\alpha\dot\beta}\lambda^2_\alpha-2g[\bar\lambda^{\dot\beta 2},C]=0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta 2}+2g[\lambda^2_\alpha,C^*]=0,\\ &(c_{\alpha\beta}-\varepsilon_{\alpha\beta})\lambda^\alpha_{i=1}=0, \qquad \lambda^1_\alpha=\bar\lambda_{\dot\alpha 2}=0. \end{aligned} \tag{9}$$

The system (9) is invariant under $N = 2$ transformations (7) only in 4-dimensional subspace of 8-dimensional space of parameters of transformations. This subspace is defined by

$$\xi_1^{i=1} = 0, \qquad \xi_2^{i=1} = 0, \qquad \bar{\xi}_{\dot{\alpha}2} = 0. \tag{10}$$

We can identify these four $N = 2$ parameters that survived with four $N = 1$ parameters

$$\xi_1 = \xi_1^{i=2}, \qquad \xi_2 = \xi_2^{i=2}, \qquad \bar{\xi}_{\dot{\alpha}} = \bar{\xi}_{\dot{\alpha}1}. \tag{11}$$

Now in transformations (7) we make the substitution of parameters (10), (11) and the substitution of component fields (8) and obtain

$$\begin{aligned}
&\delta_\xi(A - iB) = 2\xi\lambda, \qquad \delta_\xi(A + iB) = 2\bar{\xi}\bar{\psi},\\
&\delta_\xi V_{\alpha\dot{\alpha}} = 2i(\xi_\alpha\bar{\psi}_{\dot{\alpha}} - \bar{\xi}_{\dot{\alpha}}\lambda_\alpha), \qquad \delta_\xi D = \xi^\alpha\mathcal{D}_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}}\mathcal{D}^{\alpha\dot{\alpha}}\lambda_\alpha,\\
&\delta_\xi(F + iG) = 2i\xi^\alpha\Big(\mathcal{D}_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}} - g[\psi_\alpha, A + iB]\Big) + 2i\bar{\xi}_{\dot{\beta}}\Big(\mathcal{D}^{\alpha\dot{\beta}}\psi_\alpha + g[\bar{\lambda}^{\dot{\beta}}, A - iB]\Big),\\
&\delta_\xi(F - iG) = 0, \qquad \delta_\xi\lambda_\alpha = \xi_\alpha(F - iG),\\
&\delta_\xi\bar{\lambda}_{\dot{\alpha}} = \frac{1}{2}\bar{\xi}^{\dot{\beta}}(f_{\dot{\alpha}\dot{\beta}} - 2i\varepsilon_{\dot{\alpha}\dot{\beta}}D) - i\xi^\alpha\mathcal{D}_{\alpha\dot{\alpha}}(A + iB),\\
&\delta_\xi\psi_\alpha = -\frac{1}{2}\xi^\beta(f_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}D) + i\bar{\xi}^{\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}(A - iB), \qquad \delta_\xi\bar{\psi}_{\dot{\alpha}} = \xi_{\dot{\alpha}}(F - iG).
\end{aligned} \tag{12}$$

The transformations (12) are non-hermitean transformations depending on four $N = 1$ parameters. These transformations form Lie algebra. The equations of motion of the theory (1) as well as the first-order equations of motion (3) are invariant under transformations (12).

References

[1] Yatsun V.A., Integrable model of Yang–Mills theory and quasi-instantons, *Lett. Math. Phys.*, 1986, V.11, N 1, 153–159.

[2] Ferrara S. and Zumino B., Supergauge invariant Yang–Mills theories, *Nucl. Phys. B.*, 1974, V.79, N 3, 413–421.

[3] Pavlyuk A.M. and Yatsun V.A., First-order equations of motion in supersymmetric Yang–Mills theory with scalar multiplet, *Ukr. Math. Zh.*, to appear.

[4] Grimm R., Sohnius M. and Wess J., Extended supersymmetry and gauge theories, *Nucl. Phys. B.*, 1978, V.133, N 2, 275–284.

[5] Sohnius M.F., Introducing supersymmetry, *Phys. Rep.*, 1985, V.128, N 2–3, 39–204.