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Quantum Mechanism of Generation of the $SU(N)$ Gauge Fields

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A generation mechanism for non-Abelian gauge fields in the $SU(N)$ gauge theory is studied. We show that $SU(N)$ gauge fields ensuring the local invariance of the theory are generated at the quantum level only. It is demonstrated that the generation of these fields is related to nonsmoothness of the scalar phases of the fundamental spinor fields, but not to the simple requirement of gauge symmetry locality. The expressions for the gauge fields are obtained in terms of the nonsmooth scalar phases. From the viewpoint of the described scheme of the gauge field generation, the gauge principle is an "automatic" consequence of field trajectory nonsmoothness in Feynman path integral.

All known fundamental interactions possess the property of local gauge invariance. The principle central to quantum field theory is the gauge principle. This principle stakes that the fundamental fields involved in Lagrangian allow the local transformations which do not modify Lagrangian. The gauge principle was first used by Weyl [1] who discovered the local $U(1)$ gauge symmetry in quantum electrodynamics. The non-Abelian local $SU(2)$ gauge symmetry and corresponding gauge fields were introduced by Yang and Mills [2]. Based on this approach, later on the structure of weak and strong interactions was established [3, 4]. Einstein's General Relativity can also be considered as the gauge theory with Lorentz or Poincaré gauge groups [5, 6].

It is generally agreed that the existence of gauge fields must necessarily be a consequence of the requirement of the gauge symmetry locality. However, this statement is not quite correct. Ogievetski and Polubarinov [7] showed that within the framework of classical field theory, the local gauge invariance can be ensured without introduction of nontrivial gauge fields, i.e., vector fields with nonzero field strengths. It suffices to introduce only gradient vector field $\partial_\mu B(x)$, as a "compensative field", with zero strength $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)B(x) = 0$. Such field does not contribute to dynamics [7]. From the viewpoint of the classification of fields by spin, the scalar field $B(x)$ corresponds to spin of zero and gradient vector field $\partial_\mu B(x)$ is longitudinal. True vector gauge fields A_μ are transversal fields corresponding to spin of unity. Gauge invariance of theory means that the longitudinal part of vector gauge fields does not contribute to dynamics.

If so, what is the real cause of the existence of gauge fields and interactions? Early in Ref. [8] the "quantum gauge principle" was formulated in the context of quantum electrodynamics. This principle states that the Abelian $U(1)$ gauge fields are generated at the quantum level only and the generation of these fields is related to nonsmoothness of the field trajectories in the Feynman path integrals, by which the field quantization is determined. In this paper, we investigate the generation mechanism for non-Abelian $SU(N)$ gauge fields. It is shown that the non-Abelian nontrivial vector fields are generated due to nonsmoothness of the field trajectories for the scalar phases of the spinor fields in the $SU(N)$ gauge theory.

Let us consider a Lagrangian for free spinor fields

$$L = i\overline{\psi}^j\gamma^\mu\partial_\mu\psi^j - m\overline{\psi}^j\psi^j, \tag{1}$$

where $j = 1, 2, \ldots, N$. In what follows the index j will be omitted.

The Lagrangian (1) is invariant under global non-Abelian $SU(N)$-transformations

$$\psi'(x) = e^{it^a\omega_a}\psi(x), \qquad \overline{\psi}'(x) = \overline{\psi}(x)e^{-it^a\omega_a}, \tag{2}$$

where t^a are $SU(N)$ group generators, $\omega_a = \text{const}$, $a = 1, 2, \dots, N^2-1$. This invariance generates the conserved currents J_a^μ:

$$J_a^\mu = -\overline{\psi}\gamma^\mu t_a\psi, \qquad \partial_\mu J_a^\mu = 0. \tag{3}$$

In the framework of classical field theory, physical fields are known to be described by sufficiently smooth functions. Considering a smooth local infinitesimal $SU(N)$-transformation at the classical level

$$\psi'(x) = (I + it^a\omega_a(x))\psi(x), \qquad \overline{\psi}'(x) = \overline{\psi}(x)(I - it^a\omega_a(x)), \tag{4}$$

we obtain that the transformed Lagrangian differs from the original one by the term:

$$\triangle L = J_a^\mu \partial_\mu \omega_a(x). \tag{5}$$

In consequence of the conservation of currents (3) the term (5) reduces to 4-divergence and does not contribute to dynamics. In the case of local non-infinitesimal $SU(N)$-transformations, it was shown [7] that the local gauge invariance of the transformed Lagrangian can be ensured by introducing scalar fields $B_a(x)$. In other words, the Lagrangian

$$L = i\overline{\psi}\gamma^\mu\partial_\mu\psi + i\overline{\psi}(x)e^{-it^aB_a(x)}\gamma^\mu(\partial_\mu e^{it^bB_b(x)})\psi(x) - m\overline{\psi}\psi$$

is invariant under the local non-infinitesimal $SU(N)$-transformations provided the fields $B_a(x)$ transform as:

$$e^{it^aB'_a(x)} = e^{it^aB_a(x)}e^{-it^b\omega_b(x)}.$$

The introduced scalar fields $B_a(x)$ do not contribute to dynamics, since they do not give rise to nonzero strengths and can be eliminated by means of the smooth point transformations of the field variables $\psi \to \exp(it^aB_a)\psi$ [7]. Thus we need not compensate the term (5) by introducing nontrivial vector fields A_μ^a that do not reduce to gradients of scalar functions.

The situation changes in the quantum approach. In the Feynman formulation of quantum field theory the transition amplitudes are expressed by the path integrals that are centered on nonsmooth field trajectories [9]:

$$\langle\Phi_2, t_2|\Phi_1, t_1\rangle = N\int_{\Phi_1}^{\Phi_2}(D\Phi)\exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2} d^4x L(\Phi, \partial\Phi)\right].$$

In this context the Lagrangian (1) and its symmetries are determined on the class of nonsmooth functions $\psi(x)$, corresponding to nonsmooth trajectories in path integrals. In the strict sense, the derivatives involved in the Lagrangian (1) are discontinuous functions. From physics standpoint, field trajectory nonsmoothness is related to fluctuations of the local fields. Feynman integrals, as a rule, are additionally specified by the implicit switch to "smoothed-out" approximations [10]. In this case the degrees of freedom corresponding to gauge vector fields are lost. Here we show that, as in quantum electrodynamics [8], in the non-Abelian $SU(N)$ gauge theory these degrees of freedom can be explicitly taken into account when "smoothing" of nonsmooth fields is more carefully carried out.

Let us approximate nonsmooth functions $\theta^a(x)$ by smooth functions $\omega^a(x)$:

$$\theta^a(x) = \omega^a(x) + \cdots.$$

In order to write down the next term of the "smoothed-out" representation of the nonsmooth functions $\theta^a(x)$ it is necessary to consider the behaviour of the first derivatives of $\theta^a(x)$. The derivatives $\partial_\mu \theta^a(x)$ at nonsmoothness points of $\theta^a(x)$ are discontinuous functions. Since the derivatives $\partial_\mu \omega^a(x)$ are continuous functions, they badly approximate the behaviour of the derivatives of the "smoothed-out" $\theta^a(x)$. Let us denote a difference between them by $\theta^a_\mu(x)$ and write $\partial_\mu \theta^a(x)$ as follows:

$$\partial_\mu \theta^a(x) = \partial_\mu \omega^a(x) + \theta^a_\mu(x). \tag{6}$$

Since the nonsmooth fields $\theta^a_\mu(x)$ do not reduce to gradients of smooth scalar fields, they are the nontrivial vector fields that give rise to nonzero field strengths:

$$\partial_\mu \theta^a_\nu(x) - \partial_\nu \theta^a_\mu(x) \neq 0.$$

Therefore the fields $\partial_\mu \theta^a(x)$ involve the additional degrees of freedom which are related to nonsmoothness of the $\theta^a(x)$. It should be noted that the fields $\theta^a_\mu(x)$ are ambiguously determined due to ambiguity of choice of $\omega^a(x)$.

Let us now consider $\theta^a(x)$ as scalar phases of the spinor fields $\psi(x)$ realizing the fundamental representation of the $SU(N)$ gauge group and separate out these phase degrees of freedom in an explicit form:

$$\psi(x) = e^{it^a\theta_a(x)}\psi_0(x), \tag{7}$$

where the spinor fields ψ_0 are representatives of the class of gauge-equivalent fields [11], $e^{it^a\theta_a}$ is a unitary $N \times N$ matrix. Then, provided the Lagrangian (1) is determined on the class of nonsmooth functions $\psi(x)$, using Eq.(7) we obtain:

$$L = i\overline{\psi}_0\gamma^\mu\partial_\mu\psi_0 + i\overline{\psi}_0 e^{-it^a\theta_a}\gamma^\mu(\partial_\mu e^{it^b\theta_b})\psi_0 - m\overline{\psi}_0\psi_0. \tag{8}$$

Represent the matrix $e^{it^a\theta_a}$ as a superposition of the unit matrix I and $SU(N)$ group generators t^a:

$$e^{it^a\theta_a} = CI + iS_a t^a. \tag{9}$$

Since t^a are traceless matrices normalized by $\mathrm{Tr}\,(t^a t^b) = \frac{1}{2}\delta^{ab}$, the coefficients C and S_a in Eq.(9) are given by:

$$C = \frac{1}{N}\,\mathrm{Tr}(e^{it^a\theta_a}), \qquad S_a = -2i\,\mathrm{Tr}(t^a e^{it^b\theta_b}). \tag{10}$$

It is easy to verify that $\mathrm{Tr}(e^{-it^a\theta_a}\partial_\mu e^{it^b\theta_b}) = 0$. Then taking into account the commutation rules for $SU(N)$ group generators [12] we can write down:

$$e^{-it^a\theta_a}\partial_\mu e^{it^b\theta_b} = it^a A^a_\mu, \tag{11}$$

$$A^a_\mu = \bar{C}\partial_\mu S^a - \bar{S}^a\partial_\mu C + (f^{abc} - id^{abc})\bar{S}_b\partial_\mu S_c, \tag{12}$$

where d_{abc} (f_{abc}) are totally symmetric (antisymmetric) structural constants of $SU(N)$-group, the overline denotes complex conjugation.

Since the matrix $e^{it^a\theta_a}$ is unitary, the following equation is valid:

$$\bar{C}S_a - \bar{S}_a C + (f_{abc} - id_{abc})\bar{S}^b S^c = 0. \tag{13}$$

Differentiating the left and right sides of Eq.(13) and using the property of antisymmetry of f_{abc} we derive:

$$A^a_\mu - \bar{A}^a_\mu = 0,$$

whence it follows that the expression (12) is a real function. Thus A^a_μ can be identified with the gauge fields. Unlike the gauge field in electrodynamics [6], these fields are nonlinear functions of $\theta^a(x)$. As a consequence of nonsmoothness of the phases $\theta^a(x)$ the fields A^a_μ are also not smooth. If we take into account only the first term in the right hand side of relation (6) we obtain that the fields A^a_μ do not contribute to the dynamics, as in classical field theory [5], and the degrees of freedom corresponding to gauge vector fields are lost. The account of $\theta^a_\mu(x)$ enables us to interpret the fields A^a_μ as nontrivial vector fields that give rise to nonzero field strenghths:

$$\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) \neq 0.$$

By the way of illustration let us consider the Yang–Mills $SU(2)$ gauge group. In consequence of anti-commutativity of the $SU(2)$ group generators the coefficients C and S_a (see Eq.(10)) are given by:

$$C = \cos(\theta/2), \qquad S_a = 2n_a \sin(\theta/2), \tag{14}$$

where

$$\theta = \sqrt{\theta_a\theta^a}, \qquad n_a = \theta_a/\theta, \qquad a = 1, 2, 3. \tag{15}$$

From Eqs.(14) and (15) it follows that the gauge fields A^a_μ can be written as:

$$A^a_\mu = n^a\partial_\mu\theta + \sin\theta(\partial_\mu n^a) + \sin^2(\theta/2)[\mathbf{n}\times\partial_\mu\mathbf{n}]^a. \tag{16}$$

Expression (16) demonstrates explicitly the relation between the Yang–Mills gauge fields and the nonsmooth scalar phases of the spinor fields.

Let us obtain the transformation law for the vector fields (12). For this purpose we consider the infinitesimal smooth local transformations for the spinor fields:

$$\psi'_0(x) = e^{it^a\omega_a(x)}\psi_0(x), \qquad \overline{\psi}'_0(x) = \overline{\psi}_0(x)e^{-it^a\omega_a(x)}. \tag{17}$$

Then the Lagrangian (8) can be written as:

$$L = i\overline{\psi}'_0\gamma^\mu\partial_\mu\psi'_0 + i\overline{\psi}'_0 e^{it^a\omega_a}e^{-it^b\theta_b}\gamma^\mu\partial_\mu(e^{it^c\theta_c}e^{-it^l\omega_l})\psi'_0 - m\overline{\psi}'_0\psi'_0. \tag{18}$$

Defining the gauge fields $A^{a\prime}_\mu(x)$ similarly to Eqs.(11) and (12) by the following equation:

$$it_a A^{a\prime}_\mu(x) = e^{it^a\omega_a}e^{-it^b\theta_b}\partial_\mu(e^{it^c\theta_c}e^{-it^l\omega_l}), \tag{19}$$

we find that the transformed gauge fields $A^{a\prime}_\mu(x)$ are related to the fields (12) as follows:

$$A^{a\prime}_\mu(x) = A^a_\mu(x) - \partial_\mu\omega^a(x) - f_{abc}\omega^b(x)A^c_\mu(x). \tag{20}$$

Hence, in the framework of considered scheme of the gauge field generation we derive the usual transformation law for the $SU(N)$ gauge fields, with the local gauge invariance of the Lagrangian (8) being not necessary.

Using Eqs.(11) and (12) we obtain that the Lagrangian (8) takes the form:

$$L = i\overline{\psi}_0\gamma^\mu \hat{D}_\mu\psi_0 - m\overline{\psi}_0\psi_0, \tag{21}$$

where $\hat{D}_\mu \equiv \partial_\mu + iA^a_\mu t_a$ is the covariant derivative. It is easy to verify that the Lagrangian (21) is invariant under the transformations (17) and (20).

Therefore the gauge fields A^a_μ ensuring the local $SU(N)$ gauge invariance of the Lagrangian (21) are generated because of nonsmoothness of the field trajectories in Feynman path integral. The nonsmoothness of the fields A^a_μ corresponds to their quantum nature and means that these fields should also be quantized, i.e., continual integration is to be carried out over the variables $A^a_\mu(x)$. However the fields A^a_μ in the Lagrangian (21) do not exhibit all the properties of physical fields since they cannot propagate in space because of the absence of the kinetic term.

An expression similar to the kinetic term can be obtained by the calculation of the effective action for the spinor fields described by the Lagrangian (21). Using the results of the calculations performed in Ref.[13], we find the following expression for the kinetic term in the one-loop approximation

$$L_{\text{eff}} = \kappa \ln \frac{\Lambda}{\mu_0} \operatorname{tr} \hat{F}^2_{\mu\nu}, \qquad \hat{F}_{\mu\nu} = [\hat{D}_\mu, \hat{D}_\nu], \tag{22}$$

where Λ and μ_0 are the momentum of the ultraviolet and infrared cut-off respectively; κ is the numerical coefficient.

The formula (22) takes the usual form [12]

$$L_{\text{eff}} = \frac{\hbar c}{8g^2} \operatorname{tr} F^2_{\mu\nu}$$

upon identifying

$$g^2 = \frac{\hbar c}{8\kappa \ln \frac{\Lambda}{\mu_0}}. \tag{23}$$

The last equation relates the charge g with the parameters Λ and μ_0 as well as with the universal constants $\hbar$ and c, and thus demonstrates explicitly quantum origin of the charge.

Let us discuss the results obtained. We show that the "compensating" gauge fields need not be artificially introduced for the local gauge invariance of the theory to be ensured. As a result of conservation of currents (3), the Lagrangian for classical spinor fields is invariant under local $SU(N)$ gauge transformations. The generation of gauge fields is purely quantum phenomenon. The vector gauge fields are generated through nonsmoothness of the scalar phases of the fundamental spinor fields. From the viewpoint of the described scheme of the gauge field generation, the gauge principle is an "automatic" consequence of field trajectory nonsmoothness in Feynman path integral.

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