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Separation of Variables and Construction of Exact Solutions of Nonlinear Wave Equations

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An approach to construction of exact solutions of nonlinear equations on the basis of separated variables is proposed.

1 Introduction

To construct the exact solutions of nonlinear equations in mathematical physics the following ansatz is commonly used

$$u(x) = f(x)\varphi(\omega) + g(x), \tag{1}$$

where $f(x)$, $g(x)$, $\omega = \omega(x,u)$ are certain functions, and functions $\varphi(\omega)$ are undetermined. If the explicit form of variables $\omega = \omega(x,u)$ and functions $f(x)$, $g(x)$ is determined on the basis of subalgebra of invariance algebra of this equation, then ansatz (1) is called as a symmetry or Lie one. Not all ansatzes are symmetry ones.

In [1–4] a definition of conditional invariance of this differential equation was introduced. If the explicit form of new variables $\omega = \omega(x,u)$ and functions $f(x)$, $g(x)$ are determined on the basis of conditional symmetry operators then ansatz (1) is called an arbitrary invariant or non-Lie one. By means of arbitrary invariant ansatzes new classes (types) of exact solutions of many nonlinear equations in mathematical physics were constructed. Let us note an effective algorithm for finding of arbitrary symmetry operators is not found yet.

In this paper an approach to the construction of exact solutions of nonlinear equations is proposed. It is based on the method of separated variables and has a great advantage in view of its simplicity and possibility to be unchanged for construction of exact solutions for many-dimensional equations. We will consider this approach using the Boussinesq equation.

2 Exact solutions of the Boussinesq equation $u_0 = \lambda(\nabla u)^2 + \lambda u \Delta u$

Let us consider the Boussinesq equation

$$u_0 = \lambda(\nabla u)^2 + \lambda u \Delta u, \tag{2}$$

where λ is an arbitrary constant, $u = u(x_0, x_1, \ldots, x_n)$, $u_0 = \dfrac{\partial u}{\partial x_0}$, and

$$(\nabla u)^2 = \left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2, \qquad \Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n}.$$

Certain partial solutions of Eq.(2) for two variables x_0, x have been obtained in [5, 6], and for many variables in [1, 7].

Now let us consider the one-dimensional Boussinesq equation

$$u_0 = \lambda \left(\frac{\partial u}{\partial x_1} \right)^2 + \lambda u \frac{\partial^2 u}{\partial x_1^2}. \tag{3}$$

2.1. We seek for a solution of Eq.(3) in the form $u = a(x_0)b(x_1)$, where functions $a(x_0)$ and $b(x_1)$ are not constants. Substituting into Eq.(3) we have

$$\lambda a^2 bb'' + \lambda a^2 b'^2 - a'b = 0. \tag{4}$$

It follows from (4) that the functions a^2, a' are linearly dependent. Consequently $a' = \alpha a^2$ for a real number α and Eq.(3) has the form $(\lambda bb'' + \lambda b'^2)a^2 - \alpha a^2 b = 0$. We find from this equation $\lambda bb'' + \lambda b'^2 - \alpha b = 0$. Notice that the substitution $a' = \alpha a$ suggests $\alpha = 1$. Thus we will consider the equation

$$\lambda bb'' + \lambda b'^2 - b = 0. \tag{5}$$

The general solution of Eq.(5) has the form

$$\int \frac{b\,db}{\sqrt{c + b^3}} = \pm\sqrt{\frac{2}{3\lambda}}(x_1 + c_1), \tag{6}$$

where c, c_1 are arbitrary constants. If, for example, $c = 0$, then $b = \frac{1}{6\lambda}(x_1 + c_1)^2$, and we obtain the solution of (3)

$$u = -\frac{(x_1 + c_1)^2}{6\lambda(x_0 + c_2)},$$

which is transformed into

$$u = -\frac{x_1^2}{6\lambda x_0}. \tag{7}$$

The solution (7) is a partial case for

$$u = -\frac{x_1^2}{6\lambda x_0} + f(x_0, x_1).$$

Substituting into Eq.(3), we find

$$f_0 = -\frac{2x_1 f_1}{3x_0} - \frac{x_1^2}{6x_0} f_{11} - \frac{1}{3x_0} f + \lambda f_1^2 + \lambda f f_{11}. \tag{8}$$

The solution of Eq.(8) can be found in the form $f = a(x_0)b(x_1)$ and we have

$$a'b = \frac{a}{x_0}\left(-\frac{2}{3}x_1 b' - \frac{x_1^2}{6}b'' - \frac{1}{3}b\right) + a^2\left(\lambda b'^2 + \lambda bb''\right).$$

Let $a' = \alpha \frac{a}{x_0}$, where α is a real number. Hence, $a = cx_0^\alpha$. To determine the function $b(x_1)$ we find the system of equations:

$$x_1^2 b'' + 4x_1 b' + (2 + 6\alpha)b = 0, \qquad b'^2 + bb'' = 0.$$

Thus, the Boussinesq equation possesses the following solution

$$u = cx_0^{-5/8}x_1^{1/2} - \frac{x_1^2}{6\lambda x_0}.$$

If the function f in (8) depends on x_0 only, then we obtain $f_0 = -\frac{1}{3x_0}f$. Thus, Eq.(3) has a solution

$$u = -\frac{x_1^2}{6\lambda x_0} + cx_0^{-1/3}.$$

2.2. Now let us consider Eq.(2) for the case $n > 1$. We shall look for solution of (2) in the form $u = a(x_0)b(x_1, \ldots, x_k)$, where the functions $a(x_0)$ and $b(x_1, \ldots, x_k)$ are not constant. Substituting this expression into (2) we find

$$\lambda a^2 \left[(\nabla b)^2 + b\Delta b\right] - a'b = 0. \tag{9}$$

It follows from (9) that functions a^2, a' are linearly dependent, thus $a' = \alpha a^2$ and Eq.(9) has a form

$$\left(\lambda b\Delta b + \lambda(\nabla b)^2\right) a^2 - \lambda a^2 b = 0.$$

It can be obtained from this equation that

$$\lambda b\Delta b + \lambda(\nabla b)^2 - \alpha b = 0. \tag{10}$$

The function $b = \varphi(\omega)$, $\omega = x_1^2 + \cdots + x_k^2$, $b \leq n$ satisfies Eq.(10) iff

$$4\lambda\omega\varphi\varphi'' + 2k\lambda\varphi\varphi' + 4\lambda\omega\varphi'^2 - \alpha\varphi = 0. \tag{11}$$

If $\alpha = 2\lambda(k+2)$ then a particular solution of Eq.(11) is the function $\varphi = \omega$. Since the equation $a' = \alpha a^2$ possesses the solution $a = -\frac{1}{\alpha x_0}$, then Eq.(2) has a solution of the form

$$u = -\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0}. \tag{12}$$

The solution (12) is a particular case of

$$u = -\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0} + f(x_0, \ldots, x_k).$$

Substituting this expression into Eq.(2) we obtain

$$\begin{aligned} f_0 = &-\frac{2x_1 f_1}{\lambda(k+2)x_0} - \cdots - \frac{2x_k f_k}{\lambda(k+2)x_0} + \lambda\left(f_1^2 + \cdots + f_k^2 + f_{k+1}^2 + \cdots + f_n^2\right) \\ &+\lambda\left(-\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0} + f\right)(f_{11} + \cdots + f_{nn}) - \frac{k}{(k+2)x_0}f. \end{aligned} \tag{13}$$

Let the function f be independent of $x_1, \ldots, x_k$, then

$$f_0 = \lambda\left(f_{k+1}^2 + \cdots + f_n^2\right) + \lambda\left(-\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0} + f\right)(f_{k+1,k+1} + \cdots + f_{nn}) - \frac{k}{(k+2)x_0}f.$$

Thus,

$$(f_{k+1,k+1} + \cdots + f_{nn}) = 0, \qquad f_0 = \lambda\left(f_{k+1}^2 + \cdots + f_n^2\right) - \frac{k}{(k+2)x_0}f. \tag{14}$$

The solution of (14) can be found in the form

$$f = \mu_{k+1}x_{k+1} + \cdots + \mu_n x_n + \nu,$$

where $\mu_{k+1}, \ldots, \mu_n$, ν are functions dependent on x_0 only. Substituting this expression into the second equation of (14) we have

$$\frac{\partial\mu_{k+1}}{\partial x_0}x_{k+1} + \cdots + \frac{\partial\mu_n}{\partial x_0}x_n + \frac{\partial\nu}{\partial x_0}$$
$$= \lambda\left(\mu_{k+1}^2 + \cdots + \mu_n^2\right) - \frac{k}{(k+2)x_0}\left(\mu_{k+1}x_{k+1} + \cdots + \mu_n x_n + \nu\right).$$

Thus,

$$\begin{aligned}
&\frac{\partial\mu_{k+1}}{\partial x_0} = -\frac{k}{(k+2)x_0}\mu_{k+1},\\
&\ldots\ldots\ldots\ldots\ldots\\
&\frac{\partial\mu_n}{\partial x_0} = -\frac{k}{(k+2)x_0}\mu_n,\\
&\frac{\partial\nu}{\partial x_0} = \lambda\left(\mu_{k+1}^2 + \cdots + \mu_n^2\right) - \frac{k}{(k+2)x_0}\nu.
\end{aligned} \tag{15}$$

The general solution of (15) has the following form:

$$\mu_{k+1} = c_{k+1}x_0^{-\frac{k}{k+2}}, \quad \ldots, \quad \mu_n = c_n x_0^{-\frac{k}{k+2}},$$
$$\nu = \frac{\lambda(k+2)}{2}\left(c_{k+1}^2 + \cdots + c_n^2\right)x_0^{\frac{-k+2}{k+2}} + c x_0^{-\frac{k}{k+2}},$$

where $c, c_{k+1}, \ldots, c_n$ are arbitrary constants.

Thus, we obtain the multiparameter set of solutions of Eq.(2)

$$\begin{aligned}
u = &-\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0} + \left(c_{k+1}x_{k+1} + \cdots + c_n x_n + c\right)x_0^{-\frac{k}{k+2}}\\
&+\frac{\lambda(k+2)}{2}\left(c_{k+1}^2 + \cdots + c_n^2\right)x_0^{\frac{-k+2}{k+2}}.
\end{aligned} \tag{16}$$

Moreover, if $k = 1$, $n = 3$ then solution (16) takes the form

$$u = -\frac{x_1^2}{6\lambda x_0} + \left(c_2x_2 + c_3x_3\right)x_0^{-1/3} + \frac{3\lambda}{2}\left(c_2^2 + c_3^2\right)x_0^{1/3}.$$

If $k = 2$, $n = 3$ then solution (16) has a form

$$u = -\frac{x_1^2 + x_2^2}{8\lambda x_0} + c_3x_3x_0^{-1/2} + 2\lambda c_3^2.$$

If the function f in (13) does not depend on $x_1, \ldots, x_k$, then we have

$$f_0 = -\frac{k}{(k+2)x_0}f.$$

Thus,

$$f = c x_0^{-\frac{k}{k+2}}.$$

And the Boussinesq equation (2) has also the following solution

$$u = -\frac{x_1^2 + \cdots + x_k^2}{2\lambda(k+2)x_0} + cx_0^{-\frac{k}{k+2}}.$$

If, for example, $k = 2$ then we have

$$u = -\frac{x_1^2 + x_2^2}{8\lambda x_0} + cx_0^{-1/2}.$$

In the case of $k = 3$ we have

$$u = -\frac{x_1^2 + x_2^2 + x_3^2}{10\lambda x_0} + cx_0^{-3/5}.$$

3 Exact solutions of the Boussinesq equation $u_{00} + (\nabla u)^2 + u\Delta u + \Delta(\Delta u) = 0$

Let us consider the Boussinesq equation

$$u_{00} + uu_{11} + u_1^2 + u_{1111} = 0, \tag{17}$$

where

$$u = u(x), \qquad x = (x_0, x_1), \qquad u_1 = \frac{\partial u}{\partial x_1}, \qquad u_{11} = \frac{\partial^2 u}{\partial x_1^2}, \qquad u_{1111} = \frac{\partial^4 u}{\partial x_1^4}.$$

It is invariant with respect to the algebra with operators [8]

$$P_0 = \frac{\partial}{\partial x_0}, \qquad P_1 = \frac{\partial}{\partial x_1}, \qquad D = 2x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} - 2u\frac{\partial}{\partial u}.$$

Operators P_0, P_1 and D give rise to the one-parameter symmetry group of equations:

$$\begin{aligned} &G_0 : (x_0, x_1, u) \to (x_0 + \varepsilon, x_1, u), \\ &G_1 : (x_0, x_1, u) \to (x_0, x_1 + \varepsilon, u), \\ &G_2 : (x_0, x_1, u) \to \left(e^{2\varepsilon}x_0, e^{\varepsilon}x_1, e^{-2\varepsilon}u\right). \end{aligned} \tag{18}$$

Eq.(17) is also invariant under the discrete transformations

$$\begin{aligned} &(x_0, x_1, u) \to (-x_0, x_1, u), \\ &(x_0, x_1, u) \to (x_0, -x_1, u), \\ &(x_0, x_1, u) \to (-x_0, -x_1, u). \end{aligned} \tag{19}$$

One-parameter subgroups (18) and discrete transformations (19) give rise to the group G of Eq.(17). Therefore, the most general solution obtained from $u = f(x_0, x_1)$ by means of the transformations of the group G has the form

$$u = \alpha^2 f\left(\alpha^2 x_0 + \beta_0, \alpha x_1 + \beta_1\right),$$

where α, β_0, β_1 are arbitrary real numbers.

The derivation of exact solutions of Eq.(17) is discussed in [1–4]. A new method of invariant reduction of the Boussinesq equation is proposed in [2]. Exact solutions of Eq.(17) on the basis of the conditional symmetry concept are obtained in [3–4].

3.1. We seek a solution of Eq.(17) in the form $u = a(x_0) + b(x_1)$, where the functions $a(x_0)$ and $b(x_1)$ are not constant. Substituting this expression into Eq.(17) we have

$$a'' + ab'' + \left(bb' + b'^2 + b''''\right) = 0. \tag{20}$$

Since b is independent of x_0, it is clear from Eq.(20) that $a'' = \alpha + \beta a$ for real α and β. Therefore, we obtain from (20) that $a(\beta + b'') + (\alpha + bb'' + b'^2 + b'''') = 0$, i.e.

$$b'' + \beta = 0, \qquad bb'' + b'^2 + b'''' + \alpha = 0. \tag{21}$$

If $\beta = 0$ then the system of Eqs.(21) possesses a solution $b = \gamma x_1 + \delta$, where $\gamma^2 = -\alpha$. The function $b(x_1)$ can be transformed into $2x_1$ by means of a transformation from the group G. Then $\alpha = -4$ and $a = -2x_0^2 + \gamma_1 x_0 + \delta_1$, where γ_1, δ_1 are real numbers. Since a can be rewritten as $a = -2(x_0 - \gamma_1/2)^2 + \delta_1 - \gamma_1^2/4$, this solution can be transformed with the help of the group G to become

$$u = 2\left(x_1 - x_0^2\right). \tag{22}$$

Let us construct another type of solutions to Eq.(17) with partial solution (22) to the Boussinesq equation. A partial solution of Eq.(17) can be found in the form

$$u = 2\left(x_1 - x_0^2\right) + f(x_0, x_1). \tag{23}$$

Ansatz (23) reduces Eq.(17) to the form

$$f_{00} + ff_{11} + f_1^2 + f_{1111} + 2\left(x_1 - x_0^2\right) f_{11} + 4f_1 = 0. \tag{24}$$

Ansatz $\varphi = \varphi(\omega)$, $\omega = x_1 + x_0^2$ reduces Eq.(24) to the ordinary differential equation

$$\varphi\varphi'' + \varphi'^2 + \varphi'''' + 2\omega\varphi'' + 6\varphi' = 0. \tag{25}$$

A partial solution of Eq.(25) we find in the form $\varphi = t\omega^s$, $s \neq 1$. Substituting it into (25) we obtain $s = -2$, $t = -12$. Thus, the function

$$u = 2\left(x_1 - x_0^2\right) - 12\left(x_1 + x_0^2\right)^{-2} \tag{26}$$

is a solution of Eq.(17).

3.2. Now, we look for a solution of Eq.(17) in the form $u = a(x_0)b(x_1)$, where the functions $a(x_0)$ and $b(x_1)$ are not constant. Substituting this expression into (17) we obtain

$$a''b + a^2\left(bb'' + b'^2\right) + ab'''' = 0. \tag{27}$$

In complete analogy with Subsection 3.1 we see that $a'' = \alpha a^2 + \beta a$. Substituting a'' into Eq.(27) and taking into account the functions a and a^2 are linearly independent we obtain the following system to determine the function $b(x_1)$

$$b'''' + \beta b = 0, \qquad bb'' + b'^2 + \alpha b = 0. \tag{28}$$

It may be easily seen from these equations that $\beta = 0$ and $\alpha \neq 0$. We can always set $\alpha = 6$ by multiplying the function a by the number $\alpha/6$ and the function b by $6/\alpha$. Since $\beta = 0$, we see from the first of equations (28) that b is polynomial in x_1 of degree not higher than three. Plugging b in the form of the general polynomial of degree three into the second of equations (28), we see that in fact $b = -x_1^2$. Hence, Eq.(17) possesses the solution

$$u = -x_1^2\mathcal{P}(x_0), \tag{29}$$

$$u = -x_1^2 x_0^{-2}, \tag{30}$$

where $\mathcal{P}(x_0)$ is the Weierstrass function with invariants $g_2 = 0$ and $g_3 = c_1$.

A new class of solutions of Eq.(17) can be constructed using its partial solution (29). We look for these new solutions in the form

$$u = -x_1^2 \mathcal{P}(x_0) + f(x_0, x_1). \tag{31}$$

Ansatz (31) reduces Eq.(17) to

$$\left(f_{00} + f f_{11} + f_1^2 + f_{1111}\right) - \mathcal{P}\left(x_1^2 f_{11} + 4x_1 f_1 + 2f\right) = 0. \tag{32}$$

If the function f is independent of x_1, then we have $f_{00} = 2\mathcal{P}f$. This is the Lamé equation and its solutions are well-known [11]. Thus, the function

$$u = -x_1^2 \mathcal{P}(x_0) + \Lambda(x_0), \qquad \Lambda'' = 2\mathcal{P}\Lambda \tag{33}$$

is a solution of the Boussinesq equation.

If the function f in (32) does not depend on x_0, then we have a system of equations to determine the function f

$$x_1^2 f_{11} + 4x_1 f_1 + 2f = 0, \qquad f f_{11} + f_1^2 + f_{1111} = 0. \tag{34}$$

The first equation of this system is linear and its complementary function is well-known [11]. Hence, $f = -12x_1^{-2}$, and Eq.(17) possesses a solution

$$u = -x_1^2 \mathcal{P}(x_0) - 12x_1^{-2}. \tag{35}$$

We obtain simultaneously that the function

$$u = -12x_1^{-2} \tag{36}$$

is a solution of the Boussinesq equation too.

Then we find a solution of Eq.(32) which is dependent on x_0 and x_1. It can be found in the form $f = a(x_0)b(x_1) + c(x_0)$ where functions $a(x_0)$ and $c(x_0)$ are linearly independent. Substituting into Eq.(17) we obtain

$$c'' + a''b + a^2\left(bb'' + b'^2\right) + acb'' + ab'''' + a\mathcal{P}\left(-x_1^{-2}b'' - 4x_1 b' - 2b\right) - 2\mathcal{P}c = 0. \tag{37}$$

Without going into details let us suppose from the outset that $b'' = 0$. Then $b = \alpha x_1 + \beta$ and consequently $f = \alpha a(x_0)x_1 + (\beta a(x_0) + c(x_0))$. It means that setting $\alpha = 1$, $\beta = 0$ in Eq.(37) we arrive at

$$c'' + \alpha a'' x_1 + a^2 + a\mathcal{P}(-4x_1 - 2x_1) - 2\mathcal{P}c = 0.$$

Thus,

$$a'' - 6\mathcal{P}a = 0, \qquad c'' = -a^2 + 2\mathcal{P}c. \tag{38}$$

The equation $a'' - 6\mathcal{P}a = 0$ is the Lamé equation with a solution $a = \mathcal{P}(x_0)$. Hence the complementary function of the Lamé equation can be written as $a = \gamma_1\mathcal{P}(x_0) + \gamma_2\Lambda(x_0)$, where $\mathcal{P}(x_0)$ and $\Lambda(x_0)$ are linearly independent. The corresponding solution of Eq.(17) has the form

$$u = -\mathcal{P}(x_0)(x_1 - \gamma_1/2)^2 + \gamma_2 x_1 \Lambda(x_0) + \left(c(x_0) + \gamma_1^2/4\mathcal{P}(x_0)\right).$$

Under transformations from the group G it reduces to

$$u = -x_1^2\mathcal{P}(x_0) + \gamma_2 x_1\Lambda(x_0) + d(x_0), \tag{39}$$

where the function $d(x_0)$ is a solution of the following equation

$$d'' = -\gamma_1^2\Lambda^2 + 2\mathcal{P}d.$$

In a similar manner from (30) a new class of the Boussinesq equation solutions can be constructed

$$u = -x_0^2x_1^2 - 12x_1^{-2}, \tag{40}$$

$$u = -x_0^2x_1^2 + c_1x_0^3x_1 - \frac{c_1^2}{54}x_0^8 + c_2x_0^2 + c_3x_0^{-1}. \tag{41}$$

The solution of Eq.(17) is in the form $u = a(x_0)b(x_1) + c(x_0)$, where functions $a(x_0)$ and $c(x_0)$ are linearly independent. By substituting in Eq.(17) we obtain

$$a''b + c'' + a^2\left(b'^2 + bb''\right) + acb'' + ab'''' = 0.$$

If $c'' = \alpha a^2$, $a'' = 0$, then

$$a^2\left(\alpha + b'^2 + bb''\right) + acb'' + ab'''' = 0.$$

It follows from this equation that

$$b'^2 + bb'' + a = 0, \qquad b'' = 0.$$

The solution of this system up to transformations from the group G is a function $b = x_1$ if $\alpha = -1$. Then with the requirement that $c'' = \alpha a^2$, $a'' = 0$, it is possible to obtain $a = x_0$, $c = -\frac{1}{12}x_0^4 + \gamma x_0 + \delta$. Thus the function

$$u = x_0x_1 - \frac{1}{12}x_0^4 + \gamma x_0 + \delta$$

is the Boussinesq equation solution with arbitrary real numbers γ, δ.

3.3. We go now to the construction of exact solutions of the Boussinesq equation for the case $n > 1$. The generalization of Eq.(17) for arbitrary number of variables $x_0, x_1, \ldots, x_n$ is the equation [10]

$$u_{00} + (\nabla u)^2 + u\Delta u + \Delta(\delta u) = 0, \tag{42}$$

where

$$(\nabla u)^2 = \left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2, \qquad \Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$

The solution of (42) can be found in the form $u = a(x_0)b(x_1, \ldots, x_k)$, $k \leq n$. Substituting this expression into (42) we have

$$a''b + a^2\left[b\Delta b + (\nabla b)^2\right] + a\Delta(\Delta b) = 0.$$

Hence $c'' = \alpha a^2 + \beta a$ and as a result we obtain the following system to determine the function $b(x_1, \ldots, x_k)$

$$\Delta(\Delta b) + \beta b = 0, \qquad b\Delta b + (\nabla b)^2 + \alpha b = 0.$$

If $b = 0$, then $\alpha \neq 0$ and it may be considered that $\alpha = 6$. The system has a solution $b = -\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)$ for these values b and α. Therefore, the Boussinesq equation solutions are functions

$$u = -\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)\mathcal{P}(x_0), \tag{43}$$

$$u = -\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)x_0^{-2}. \tag{44}$$

Let us construct another solution of Eq.(42) from (43). We will look for it in the form

$$u = -\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)\mathcal{P}(x_0) + f(x_0, x_1, \ldots, x_k). \tag{45}$$

Ansatz (45) reduces Eq.(42) to

$$\begin{aligned} &f_{00} + (\nabla f)^2 + f\Delta f + \Delta(\Delta f) \\ &\quad -\mathcal{P}\left[\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)\Delta f + (4x_1 f_1 + \cdots + 4x_k f_k) + 2kf\right] = 0. \end{aligned} \tag{46}$$

If f does not depend on variables $x_1, \ldots, x_k$ in Eq.(46) then $f_{00} = \frac{6k}{k+2}f$ and, therefore, the function

$$u = -\frac{3}{k+2}\left(x_1^2 + \cdots + x_k^2\right)\mathcal{P}(x_0) + \Lambda(x_0), \qquad \Lambda'' = \frac{6k}{k+2}\mathcal{P}\Lambda \tag{47}$$

is a solution of Eq.(42).

If the function f depends on variables $x_0, x_1, \ldots, x_k$ in (46) then the solution of Eq.(42) can be obtained in the following form

$$u = -\frac{3}{k+2}\left(x_1^2 + \ldots + x_k^2\right)\mathcal{P}(x_0) + \alpha x_1 \Lambda(x_0) + c(x_0), \tag{48}$$

where $\mathcal{P}^{11} = 6\mathcal{P}^2$, $\Lambda'' = (4 + 2k)\mathcal{P}\Lambda$, $c'' = -\alpha^2\Lambda^2 + 2\mathcal{P}c$.

Similarly we find a solution of Eq.(42) from (44):

$$u = -\frac{3}{k+2}\left(x_1^2 + \ldots + x_k^2\right)x_0^{-2} + c_1 x_0^3 x_1 - \frac{k+2}{50k+112}x_0^8 + c_2 x_0^{\frac{1+\sqrt{\frac{25k+2}{k+2}}}{2}} + c_3 x_0^{\frac{1-\sqrt{\frac{25k+2}{k+2}}}{2}},$$

where c_1, c_2, c_3, c_4 are arbitrary real numbers; $k = 1, \ldots, n$.

A new type of solutions of Eq.(42) can be constructed using

$$u = -x_1^2\mathcal{P}(x_0) + f(x_0, x_2, x_3). \tag{49}$$

Substituting anzats (49) into (42) we have

$$f_{00} + (\nabla f)^2 + f(\Delta f) + \Delta(\Delta f) - x_1^2\mathcal{P}(\Delta f) - 2\mathcal{P}f = 0.$$

Since the function f does not depend on x_1, then $\Delta f = 0$ and we obtain the following system of equations to determine the function f:

$$f_{00} + f_2^2 + f_3^2 - 2\mathcal{P}f = 0, \qquad f_{22} + f_{33} = 0. \tag{50}$$

We will seek now a solution of Eqs.(50) in the form $f = a(x_0)x_2 + b(x_0)x_3 + c(x_0)$. Substitution of f into the first equation of (50) gives

$$a''x_2 + b''x_3 + c'' + a^2 + b^2 - 2\mathcal{P}(ax_2 + bx_3 + c) = 0.$$

It follows from this equation that

$$a'' = 2\mathcal{P}a, \qquad b^{11} = 2\mathcal{P}b, \qquad c'' = -a^2 - b^2 + 2\mathcal{P}c. \tag{51}$$

Solving Eq.(51) we find the explicit form of functions $a(x_0)$, $b(x_0)$, $c(x_0)$ and the solution of Eq.(42) too.

If we use the ansatz

$$u = -x_1^2 x_0^{-2} + f(x_0, x_2, x_3),$$

we construct by analogy with the above the following solution of Eq.(42):

$$u = -x_1^2 x_0^{-2} + \left(c_1 x_0^2 + c_4 x_0^{-1}\right) x_2 + \left(c_3 x_0^2 + c_4 x_0^{-1}\right) x_3$$
$$-\frac{c_1^2 + c_3^2}{28} x_0^6 - \frac{c_1 c_2 + c_3 c_4}{2} x_0^3 + \frac{c_2^2 + c_3^2}{28}.$$

And using the ansatz

$$u = -\frac{1}{2}\left(x_1^2 + x_2^2\right)\mathcal{P}(x_0) + f(x_0, x_3)$$

another solution of Eq.(42) can be obtained

$$u = -\frac{1}{2}\left(x_1^2 + x_2^2\right)\mathcal{P}(x_0) + \Lambda(x_0)x_3 + c(x_0),$$

where $\Lambda'' = 2\mathcal{P}\Lambda$, $c'' = -\Lambda^2 + 2\mathcal{P}c$.

Making use of

$$u = -\frac{1}{2}\left(x_1^2 + x_2^2\right) x_0^{-2} + f(x_0, x_3)$$

we find a solution of Eq.(42) in the form

$$u = -\frac{1}{2}\left(x_1^2 + x_2^2\right) x_0^{-2} + \left(c_1 x_0^2 + c_2 x_0^{-1}\right) x_3 + \frac{c_1^2}{28} x_0^6 - \frac{c_1 c_2}{2} x_0^3 + \frac{c_2^2}{2}.$$

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