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Symmetries, their Breaking and Anomalies in the Self-Consistent Renormalization. Discrete Symmetry Shadow in Chiral Anomalies

Valentyn I. KUCHERYAVY

Bogolyubov Institute for Theoretical Physics, National Academy of Sciences,
Metrologichna Street 14-b, Kyiv-143, Ukraine
E-mail: mmtpitp@bitp.kiev.ua

Using the self-consistent renormalization (SCR), a careful study of complicated tangle of problems associated with renormalizations, symmetries conservation, their breaking and anomalies is performed for some set of UV-divergent Feynman amplitudes (FA's) connected with mass-anysotropic AVV- and AAA-triangles in the space-time $n = 4$. Most general quantum corrections (QC's) to the canonical Ward identities (WI's) and some nontrivial "daughter reduction identities" (DRI's) are obtained. The results are new both for a nondegenerate case and for the chiral case. For a nondegenerate case ($m_1 \neq m_2 \neq m_3$, $m_l \neq 0$), the QC's are the zero degree homogeneous functions of masses and are expressed in terms of the Appel hypergeometric functions F_1. For the chiral case ($m = 0$) and the chiral limit ($m \to 0$) the behaviour of the AVV- and AAA-amplitudes depends crucially on the discrete symmetry of these amplitudes in the cases $m = 0$ and $m \to 0$. In the chiral case the QC's to "left-handed" WI's vanish. This may give some insight into why just the left-handed neutrino exists in Nature.

1. Symmetries of quantum field theories (QFT) often manifest themselves via certain formal relations between UV-divergent FA's known as the canonical WI's (CWI's). Anomalies of QFT's occur as breakdown of the CWI's at the level of regular (finite) values of FA's [1–3]. We hope to clarify some obscure points in these violations by employing the SCR [4] to spinor triangle FA's, as the most important subject in such investigations, and to illustrate possibilities of the SCR. Recall that the SCR is an effective realization of the Bogoliubov–Parasiuk R-operation [5] which is complemented with recurrence, compatibility and differential relations fixing a renormalization arbitrariness of the R-operation in some universal way based on mathematical properties of FA's only.

2. The main Feynman amplitude corresponding to the triangle spinor graph of the most general kind (different masses, arbitrary Clifford structure of vertices, the n-dimensional space-time with the (q,p)-signature) looks as follows:

$$I^{\gamma_1\gamma_2\gamma_3}(m,k) := \int_{-\infty}^{\infty} (d^n p)\ \delta(p,k) \frac{\operatorname{tr}[\gamma_1(m_1+\hat{p}_1)\gamma_2(m_2+\hat{p}_2)\gamma_3(m_3+\hat{p}_3)]}{(m_1^2-p_1^2-i\epsilon_1)\,(m_2^2-p_2^2-i\epsilon_2)\,(m_3^2-p_3^2-i\epsilon_3)},$$
$$(d^np) := d^np_1 d^np_2 d^np_3, \qquad \hat{p}_l := \gamma^\mu p_{l\mu}, \qquad m := (m_1,m_2,m_3), \qquad k := (k_1,k_2,k_3),$$
$$\delta(p,k) := \delta(-k_1+p_3-p_1)\,\delta(-k_2+p_1-p_2)\,\delta(-k_3+p_2-p_3). \tag{1}$$

The matrices γ_i, γ_μ, I_g act in the N_g-dimensional space of the faithful representation π_g of lowest dimension for the Clifford algebra $Cl(g)_{\mathbf{K}}$, $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, with $\gamma_\mu \in \Lambda_1(g)$, $\mu = 1,\ldots,n$, being the generating elements of the $Cl(g)_{\mathbf{K}}$-algebra in its matrix representation π_g; also $\gamma_i \in \Lambda_k(g)$,

$i = 1, 2, 3$, are some k-degree ($k = 0, 1, \ldots, n$) homogeneous elements of the $Cl(g)_{\mathbf{K}}$-algebra in the π_g-representation; I_g is N_g-dimensional unit matrix. The natural analog of the Dirac γ^5-matrix is $\gamma_* := \gamma_1\gamma_2\cdots\gamma_n \in \Lambda_n(g)$, with properties

$$\gamma_\mu\gamma_* = (-1)^{n+1}\gamma_*\gamma_\mu, \quad \mu = 1, \ldots, n, \quad \gamma_*^2 = \varepsilon(g)I_g, \quad \varepsilon(g) := (-1)^q(-1)^{n(n-1)/2}. \tag{2}$$

3. The UV-divergent FA's (1) satisfy formally the *canonical* WI's (CWI's):

$$\begin{aligned}
&k_{1\mu} I^{(\gamma^\mu\gamma)\gamma_2\gamma_3}(m,k) = D_1^{\dot\gamma\gamma_2\gamma_3}(m,k) = \\
&\quad = (-1)^{\pi_1} P_1^{\gamma\gamma_2\gamma_3}(m,k) - P_3^{\gamma\gamma_2\gamma_3}(m,k) + (m_3 - (-1)^{\pi_1} m_1)\, I^{\gamma\gamma_2\gamma_3}(m,k), \\
&k_{2\alpha} I^{\gamma_1(\gamma^\alpha\gamma)\gamma_3}(m,k) = D_2^{\gamma_1\dot\gamma\gamma_3}(m,k) = \\
&\quad = (-1)^{\pi_2} P_2^{\gamma_1\gamma\gamma_3}(m,k) - P_1^{\gamma_1\gamma\gamma_3}(m,k) + (m_1 - (-1)^{\pi_2} m_2)\, I^{\gamma_1\gamma\gamma_3}(m,k), \\
&k_{3\beta} I^{\gamma_1\gamma_2(\gamma^\beta\gamma)}(m,k) = D_3^{\gamma_1\gamma_2\dot\gamma}(m,k) = \\
&\quad = (-1)^{\pi_3} P_3^{\gamma_1\gamma_2\gamma}(m,k) - P_2^{\gamma_1\gamma_2\gamma}(m,k) + (m_2 - (-1)^{\pi_3} m_3)\, I^{\gamma_1\gamma_2\gamma}(m,k).
\end{aligned} \tag{3}$$

Here the quantities $D_1^{\dot\gamma\gamma_2\gamma_3}(m,k)$, $D_2^{\gamma_1\dot\gamma\gamma_3}(m,k)$, $D_3^{\gamma_1\gamma_2\dot\gamma}(m,k)$, $P_l^{\gamma_1\gamma_2\gamma_3}(m,k)$ are similar to the main amplitude $I^{\gamma_1\gamma_2\gamma_3}$ and differ from it only in polynomials of the integrand:

$$\begin{aligned}
&D_1^{\dot\gamma\gamma_2\gamma_3}(m,k) \longleftrightarrow (p_3 - p_1)_\mu \operatorname{tr}[\gamma^\mu\gamma(m_1 + \hat p_1)\gamma_2(m_2 + \hat p_2)\gamma_3(m_3 + \hat p_3)], \\
&D_2^{\gamma_1\dot\gamma\gamma_3}(m,k) \longleftrightarrow (p_1 - p_2)_\alpha \operatorname{tr}[\gamma_1(m_1 + \hat p_1)\gamma^\alpha\gamma(m_2 + \hat p_2)\gamma_3(m_3 + \hat p_3)], \\
&D_3^{\gamma_1\gamma_2\dot\gamma}(m,k) \longleftrightarrow (p_2 - p_3)_\beta \operatorname{tr}[\gamma_1(m_1 + \hat p_1)\gamma_2(m_2 + \hat p_2)\gamma^\beta\gamma(m_3 + \hat p_3)];
\end{aligned} \tag{4}$$

$$\begin{aligned}
&P_1^{\gamma_1\gamma_2\gamma_3}(m,k) \longleftrightarrow \operatorname{tr}[\gamma_1(m_1^2 - p_1^2)\gamma_2(m_2 + \hat p_2)\gamma_3(m_3 + \hat p_3)], \\
&P_2^{\gamma_1\gamma_2\gamma_3}(m,k) \longleftrightarrow \operatorname{tr}[\gamma_1(m_1 + \hat p_1)\gamma_2(m_2^2 - p_2^2)\gamma_3(m_3 + \hat p_3)], \\
&P_3^{\gamma_1\gamma_2\gamma_3}(m,k) \longleftrightarrow \operatorname{tr}[\gamma_1(m_1 + \hat p_1)\gamma_2(m_2 + \hat p_2)\gamma_3(m_3^2 - p_3^2)].
\end{aligned} \tag{5}$$

In Eqs.(3) the vector CWI's ($\gamma = I_g$) and the axial-vector CWI's ($\gamma = \gamma^*$) are represented in the uniform manner. The factors $(-1)^{\pi_i}$ stem from the commutation relations $\gamma^\sigma\gamma = (-1)^\pi\gamma\gamma^\sigma$, $s = 1, \ldots, n$, and are equal: $(-1)^{\pi_i} = 1$ if $\gamma = I_g$, $\forall n$, or $\gamma = \gamma^*$, $n = 2r+1$; $(-1)^{\pi_i} = -1$ if $\gamma = \gamma^*$, $n = 2r$.

4. The *reduction identities* (RI's) is a name given to the obvious identities:

$$\begin{aligned}
&P_{l\epsilon}^{\gamma_1\gamma_2\gamma_3}(m,k) = \overline{P}_{l\epsilon}^{\gamma_1\gamma_2\gamma_3}(\overline{m}_{(l)},k), \qquad l = 1,2,3, \\
&\overline{m}_{(1)} \equiv (m_2, m_3), \qquad \overline{m}_{(2)} \equiv (m_1, m_3), \qquad \overline{m}_{(3)} \equiv (m_1, m_2),
\end{aligned} \tag{6}$$

in which we use the simple idea of cancelling the equal factors in factorized polynomials in numerators and the denominator of integrands. For example, for $l = 1$,

$$P_{1\epsilon}^{\gamma_1\gamma_2\gamma_3}(m,k) := \int_{-\infty}^{\infty} (d^n p)\ \delta(p,k) \frac{\operatorname{tr}\left[\gamma_1\left(m_1^2 - p_1^2 - i\epsilon_1\right)\gamma_2\left(m_2 + \hat p_2\right)\gamma_3\left(m_3 + \hat p_3\right)\right]}{\left(m_1^2 - p_1^2 - i\epsilon_1\right)\left(m_2^2 - p_2^2 - i\epsilon_2\right)\left(m_3^2 - p_3^2 - i\epsilon_3\right)}, \tag{7}$$

$$\overline{P}_{1\epsilon}^{\gamma_1\gamma_2\gamma_3}(m_2,m_3,k) := \int_{-\infty}^{\infty} (d^n p)\ \delta(p,k) \frac{\operatorname{tr}\left[\gamma_1\gamma_2\left(m_2 + \hat p_2\right)\gamma_3\left(m_3 + \hat p_3\right)\right]}{\left(m_2^2 - p_2^2 - i\epsilon_2\right)\left(m_3^2 - p_3^2 - i\epsilon_3\right)}. \tag{8}$$

The RI's (6) naturally induce primitive *daughter* RI's (DRI's) via decompositions involving: i) the Clifford tensors $\operatorname{tr}(\gamma_1\gamma_2 m_2\gamma_3 m_3)$, $\operatorname{tr}(\gamma_1\gamma_2 m_2\gamma_3\gamma_\sigma)$, $\operatorname{tr}(\gamma_1\gamma_2\gamma_\sigma\gamma_3 m_3)$, $\operatorname{tr}(\gamma_1\gamma_2\gamma_\sigma\gamma_3\gamma_\tau)$, for

$l = 1$; ii) the symmetric and antisymmetric parts $\frac{1}{2}(p_a^\sigma p_b^\tau \pm p_b^\sigma p_a^\tau)$ of the $p_a^\sigma p_b^\tau$; iii) the tensor structures 1, k_i^σ, $g^{\sigma\tau}$, $(k_i, k_j)^{\sigma\tau} := (k_i^\sigma k_j^\tau + k_j^\sigma k_i^\tau)$, $[\,k_i, k_j\,]^{\sigma\tau} := (k_i^\sigma k_j^\tau - k_j^\sigma k_i^\tau)$, with independent external momenta (e.g., k_2, k_3, or k_1, k_2, or k_1, k_3). There are 10 primitive DRI's, $\forall\, l = 1, 2, 3$. The difference between $P_l^{\gamma_1\gamma_2\gamma_3}(m, k)$ involved in Eqs.(3) and $P_{l\epsilon}^{\gamma_1\gamma_2\gamma_3}(m, k)$ results from $i\epsilon_l$-terms in polynomials of numerators.

5. The amplitude $I^{\gamma_1\gamma_2\gamma_3}(m, k)$ has the divergence index $\nu = n - 3$, whereas the amplitudes $D_1^{\dot{\gamma}\gamma_2\gamma_3}(m, k)$, $D_2^{\gamma_1\dot{\gamma}\gamma_3}(m, k)$, $D_3^{\gamma_1\gamma_2\dot{\gamma}}(m, k)$, $P_l^{\gamma_1\gamma_2\gamma_3}(m, k)$, $P_{l\epsilon}^{\gamma_1\gamma_2\gamma_3}(m, k)$, $\overline{P}_{l\epsilon}^{\gamma_1\gamma_2\gamma_3}(\overline{m}_{(l)}, k)$, $l = 1, 2, 3$, have the divergence index $\nu + 1 = n - 2$. The regular values for all of them are obtained according to [4] and are given in [6] in the most general form (for arbitrary Clifford structure of vertices and for n-dimensional space-time with the (q, p)-signature). It turns out that so calculated regular values satisfy the identities:

$$
\begin{aligned}
&k_{1\mu}(R^\nu I)^{(\gamma^\mu\gamma)\gamma_2\gamma_3}(m, k) = (R^{\nu+1}D_1)^{\dot{\gamma}\gamma_2\gamma_3}(m, k) \\
&\quad = (-1)^{\pi_1}(R^{\nu+1}P_1)^{\gamma\gamma_2\gamma_3} - (R^{\nu+1}P_3)^{\gamma\gamma_2\gamma_3} + (m_3 - (-1)^{\pi_1}m_1)\,(R^{\nu+1}I)^{\gamma\gamma_2\gamma_3}, \\
&k_{2\alpha}(R^\nu I)^{\gamma_1(\gamma^\alpha\gamma)\gamma_3}(m, k) = (R^{\nu+1}D_2)^{\gamma_1\dot{\gamma}\gamma_3}(m, k) \\
&\quad = (-1)^{\pi_2}(R^{\nu+1}P_2)^{\gamma_1\gamma\gamma_3} - (R^{\nu+1}P_1)^{\gamma_1\gamma\gamma_3} + (m_1 - (-1)^{\pi_2}m_2)\,(R^{\nu+1}I)^{\gamma_1\gamma\gamma_3}, \\
&k_{3\beta}(R^\nu I)^{\gamma_1\gamma_2(\gamma^\beta\gamma)}(m, k) = (R^{\nu+1}D_3)^{\gamma_1\gamma_2\dot{\gamma}}(m, k) \\
&\quad = (-1)^{\pi_3}(R^{\nu+1}P_3)^{\gamma_1\gamma_2\gamma} - (R^{\nu+1}P_2)^{\gamma_1\gamma_2\gamma} + (m_2 - (-1)^{\pi_3}m_3)\,(R^{\nu+1}I)^{\gamma_1\gamma_2\gamma},
\end{aligned}
\tag{9}
$$

which are referred to as the *regular analog* (RA) of the CWI's [6]. It is important to note that the last terms in the identities (9) are calculated by the renormalization index $\nu + 1$, although their proper divergence index is ν. It is this peculiarity that permits the RA of the CWI's (9) both to imitate the CWI's (3) and to differ from them simultaneously. It is this peculiarity that permits to obtain some effective formulae for calculating of the *quantum corrections* (QC's) to the CWI's in the most general nonchiral case [6].

6. The primitive DRI's stemming from tensors $\mathrm{tr}(\gamma_1\gamma_2\gamma_\sigma\gamma_3\gamma_\tau)$, $\mathrm{tr}(\gamma_1\gamma_\sigma\gamma_2\gamma_3\gamma_\tau)$, $\mathrm{tr}(\gamma_1\gamma_\sigma\gamma_2\gamma_\tau\gamma_3)$, $\frac{1}{2}(p_a^\sigma p_b^\tau - p_b^\sigma p_a^\tau)$, and $\frac{1}{2}[k_2, k_3]^{\sigma\tau}$ are as follows:

$$
\left(R^{\nu+1}P_{l\epsilon[a,b]}\right)^{\sigma\tau}(m, k) = \left(R^{\nu+1}\overline{P}_{l\epsilon[a,b]}\right)^{\sigma\tau}(\overline{m}_{(l)}, k), \qquad a, b \neq l, \quad a < b, \quad l = 1, 2, 3, \tag{10}
$$

$$
\left(R^{\nu+1}P_{l\epsilon[a,b]}\right)^{\sigma\tau}(m, k) = (2\pi)^n\delta(k)b(g)\mathrm{tr}(\cdot)\frac{1}{2}[k_2, k_3]^{\sigma\tau}(-1)^{l-1}\left(R^{\nu+1}P_{l\epsilon}^{[2,3]}\right)(m, k), \tag{11}
$$

$$
\begin{aligned}
\left(R^{\nu+1}P_{l\epsilon}^{[2,3]}\right)(m, k) := \int_{\Sigma^2}\frac{d\mu(\alpha)}{\Delta^{n/2}}\Big\{&\frac{\alpha_l}{\Delta}(m_l^2 - i\epsilon_l)\left(R^{\nu+1}\mathcal{F}\right)_{20} - \frac{\alpha_l}{\Delta}Y_l^2\left(R^{\nu+1}\mathcal{F}\right)_{40} \\
&+ \left[\frac{\alpha_l}{\Delta}\left(\frac{n}{2} + 1\right) - 1\right]\Delta^{-1}\left(R^{\nu+1}\mathcal{F}\right)_{41}\Big\} = 0.
\end{aligned}
\tag{12}
$$

The zero result in Eq.(12) is due to $\left(R^{\nu+1}\overline{P}_{l\epsilon[a,b]}\right)^{\sigma\tau}(\overline{m}_{(l)}, k) = 0$, which in turn follows from the antisymmetry of the $\frac{1}{2}\left(p_a^\sigma p_b^\tau - p_b^\sigma p_a^\tau\right)$ and from the special external momentum dependence in this case (independent momenta are: k_3 or $k_1 + k_2$ for $l = 1$, etc.). Hereafter the integration measure is $d\mu(\alpha) := \delta\left(1 - \sum_{l=1}^{3}\alpha_l\right)d\alpha_1\,d\alpha_2\,d\alpha_3$, the integration region is $\Sigma^2 := \left\{\alpha_l \,|\, \alpha_l \geq 0,\ l = 1, 2, 3,\ \sum_{l=1}^{3}\alpha_l = 1\right\}$, overall δ-function is $\delta(k) := \delta(-k_1 - k_2 - k_3)$, and the metric dependent constant is $b(g) := \left(\pi^{n/2}i^p\right)/(2\pi)^n$, where p is the number of positive squares in the space-time metric g. The basic functions $(R^{\nu+1}\mathcal{F})_{sj}$ and the determining numbers ν^1_{sj}, λ^1_{sj}, and ω appearing in them are defined as:

$$\begin{aligned}
&\left(R^{\nu+1}\mathcal{F}\right)_{sj} := M_\epsilon^{\omega+j}\, Z_\epsilon^{1+\nu^1_{sj}}\Gamma(\lambda^1_{sj})/\Gamma(2+\nu^1_{sj})\, {}_2F_1\left(1,\lambda^1_{sj};\, 2+\nu^1_{sj};\, Z_\epsilon\right),\\
&\nu^1_{sj} := [(\nu+1-s)/2]+j, \qquad \lambda^1_{sj} := 1+\nu^1_{sj}-\omega-j, \qquad \omega := n/2-3.
\end{aligned} \tag{13}$$

The $[(\nu+1-s)/2]$ in Eqs.(13) is the integral part of the number $(\nu+1-s)/2$. The α-parametric functions Z_ϵ, M_ϵ, A, Δ, Y_l, involved in Eqs.(11)–(13) have the form:

$$\begin{aligned}
&Z_\epsilon := A/M_\epsilon, \qquad M_\epsilon := \sum_{l=1}^{3}\alpha_l(m_l^2-i\epsilon_l), \qquad \Delta := \alpha_1+\alpha_2+\alpha_3,\\
&A := \Delta^{-1}\left[\alpha_1(\alpha_2+\alpha_3)k_2^2+\alpha_3(\alpha_2+\alpha_1)k_3^2+2\alpha_1\alpha_3(k_2\cdot k_3)\right],\\
&Y_1 := \Delta^{-1}[(\alpha_2+\alpha_3)k_2+\alpha_3k_3], \qquad Y_2 := \Delta^{-1}[-\alpha_1k_2+\alpha_3k_3],\\
&Y_3 := \Delta^{-1}[-\alpha_1k_2-(\alpha_1+\alpha_2)k_3].
\end{aligned} \tag{14}$$

7. Now let us consider the AVV ($\gamma_1=\gamma^\mu\gamma^*$, $\gamma_2=\gamma^\alpha$, $\gamma_3=\gamma^\beta$) and the AAA ($\gamma_1=\gamma^\mu\gamma^*$, $\gamma_2=\gamma^\alpha\gamma^*$, $\gamma_3=\gamma^\beta\gamma^*$) spinor amplitudes for $n=4$ [7]. There is the relation

$$I^{\mu\alpha\beta(AAA)}(m_1,m_2,m_3,k)=\varepsilon(g)I^{\mu\alpha\beta(AVV)}(m_1,-m_2,m_3,k) \tag{15}$$

between them. Therefore, in the chiral case ($m_l=0$, $\forall\, l$) they may differ only by the sign.

Using Eqs.(12) and the compatibility relations $(R^\nu\mathcal{F})_{sj}=\left(R^{\nu+1}\mathcal{F}\right)_{s+1,j}$ one finds that the regular values of the main triangle amplitudes (1) after calculating nonzero traces have the followimg representation (here $\nu=1$ and $\omega=-1$):

$$\begin{aligned}
(R^\nu I)^{\mu\alpha\beta(\cdots)}(m_1,m_2,m_3,k) &= (2\pi)^4\delta(k)C^{(\cdots)}(g)\int_{\Sigma^2}\frac{d\mu(\alpha)}{\Delta^2}\\
&\times\Big\{\varepsilon^{\mu\alpha\beta\tau}k_{2\tau}\,(R^\nu\mathcal{I}_1)^{(\cdots)}(m,\alpha,k)+\varepsilon^{\mu\alpha\beta\tau}k_{3\tau}\,(R^\nu\mathcal{I}_2)^{(\cdots)}(m,\alpha,k)\\
&+\varepsilon^{\mu\alpha\sigma\tau}k_{2\sigma}k_{3\tau}(R^\nu\mathcal{I}_3)^\beta(m,\alpha,k)+\varepsilon^{\mu\beta\sigma\tau}k_{2\sigma}k_{3\tau}(R^\nu\mathcal{I}_4)^\alpha(m,\alpha,k)\Big\},
\end{aligned} \tag{16}$$

where integrands $(R^\nu\mathcal{I}_l)^{(\cdots)}(m,\alpha,k)$, etc., and constants $C^{(\cdots)}(g)$ are given as follows:

$$\begin{aligned}
(R^\nu\mathcal{I}_1)^{(\cdots)} :=& -\left[\pm m_2m_3\frac{\alpha_2+\alpha_3}{\Delta}+(m_3\mp m_2)m_1\frac{\alpha_1}{\Delta}+\mu_1\frac{\alpha_1}{\Delta}\right](R^\nu\mathcal{F})_{10}\\
&+\left[k_2^2\frac{\alpha_1(\alpha_2+\alpha_3)}{\Delta^2}-k_3^2\frac{\alpha_3(\alpha_2+\alpha_1)}{\Delta^2}\right](R^\nu\mathcal{F})_{30},\\
(R^\nu\mathcal{I}_2)^{(\cdots)} :=& \left[\pm m_2m_1\frac{\alpha_2+\alpha_1}{\Delta}+(m_1\mp m_2)m_3\frac{\alpha_3}{\Delta}+\mu_3\frac{\alpha_3}{\Delta}\right](R^\nu\mathcal{F})_{10}\\
&+\left[k_2^2\frac{\alpha_1(\alpha_2+\alpha_3)}{\Delta^2}-k_3^2\frac{\alpha_3(\alpha_2+\alpha_1)}{\Delta^2}\right](R^\nu\mathcal{F})_{30},\\
(R^\nu\mathcal{I}_3)^\beta :=& -2\left[k_2^\beta\frac{\alpha_1\alpha_3}{\Delta^2}+k_3^\beta\frac{\alpha_3(\alpha_2+\alpha_1)}{\Delta^2}\right](R^\nu\mathcal{F})_{30},\\
(R^\nu\mathcal{I}_4)^\alpha :=& 2\left[k_2^\alpha\frac{\alpha_1(\alpha_2+\alpha_3)}{\Delta^2}+k_3^\alpha\frac{\alpha_1\alpha_3}{\Delta^2}\right](R^\nu\mathcal{F})_{30}, \qquad \mu_l := (m_l^2-i\epsilon_l),
\end{aligned} \tag{17}$$

$$C^{(AAA)}(g) := \varepsilon(g) C^{(AVV)}(g), \qquad C^{(AVV)}(g) := \varepsilon(g)\, \mathrm{tr}\,(I_g) \left(\pi^2 i^p\right)/(2\pi)^4. \tag{18}$$

The basic functions $(R^\nu \mathcal{F})_{sj}$, along with the determining numbers ν_{sj}, λ_{sj}, are defined as:

$$\begin{aligned} &(R^\nu \mathcal{F})_{sj} := M_\epsilon^{\omega+j}\, Z_\epsilon^{1+\nu_{sj}} \Gamma(\lambda_{sj})/\Gamma(2+\nu_{sj})\, {}_2F_1(1, \lambda_{sj};\, 2+\nu_{sj};\, Z_\epsilon), \\ &\nu_{sj} := [(\nu - s)/2] + j, \qquad \lambda_{sj} := 1 + \nu_{sj} - \omega - j, \qquad \omega := n/2 - 3. \end{aligned} \tag{19}$$

In Eqs.(16)–(17), the notation $(\cdots)$ means (AVV) or (AAA). Hereafter the upper signs in $(\pm)$ or $(\mp)$ correspond to the AVV-amplitudes while the lower ones to the AAA-amplitudes. The relation (15) and its chiral ($m_l = 0$, $\forall\, l$) form are obeyed at the regular level as well.

8. The first and second amplitudes in the first lines of Eqs.(9) take the form:

$$\begin{bmatrix} k_{1\mu}(R^\nu I)^{\mu\alpha\beta(\cdots)}(m,k) \\ k_{2\alpha}(R^\nu I)^{\mu\alpha\beta(\cdots)}(m,k) \\ k_{3\beta}(R^\nu I)^{\mu\alpha\beta(\cdots)}(m,k) \end{bmatrix} = (2\pi)^4 \delta(k) C^{(\cdots)}(g) \begin{bmatrix} \varepsilon^{\alpha\beta\sigma\tau} k_{2\sigma} k_{3\tau} \left(R^{\nu+1} D_1^{[2,3]}\right)^{(:..)} \\ \varepsilon^{\mu\beta\sigma\tau} k_{2\sigma} k_{3\tau} \left(R^{\nu+1} D_2^{[2,3]}\right)^{(.:.)} \\ \varepsilon^{\mu\alpha\sigma\tau} k_{2\sigma} k_{3\tau} \left(R^{\nu+1} D_3^{[2,3]}\right)^{(..:)} \end{bmatrix}, \tag{20}$$

$$\left(R^{\nu+1} D_l^{[2,3]}\right)^{(\cdots)}(m,k) := \int_{\Sigma^2} \frac{d\mu(\alpha)}{\Delta^2} \left(R^{\nu+1} \mathcal{D}_l^{[2,3]}\right)^{(\cdots)}(m,\alpha,k), \qquad l = 1,2,3, \tag{21}$$

$$\begin{aligned} &\left(R^{\nu+1} \mathcal{D}_1^{[2,3]}\right)^{(:..)}(m,\alpha,k) := \left[(m_3 + m_1) m_{20}^{(:..)} + i\left(\epsilon_1 \frac{\alpha_1}{\Delta} + \epsilon_3 \frac{\alpha_3}{\Delta}\right)\right] \left(R^{\nu+1}\mathcal{F}\right)_{20}, \\ &\left(R^{\nu+1} \mathcal{D}_2^{[2,3]}\right)^{(.:.)}(m,\alpha,k) := \left[(m_1 \mp m_2) m_{20}^{(.:.)} - i\left(\epsilon_1 \frac{\alpha_1}{\Delta} + \epsilon_2 \frac{\alpha_2}{\Delta}\right)\right] \left(R^{\nu+1}\mathcal{F}\right)_{20}, \\ &\left(R^{\nu+1} \mathcal{D}_3^{[2,3]}\right)^{(..:)}(m,\alpha,k) := \left[(m_2 \mp m_3) m_{20}^{(..:)} + i\left(\epsilon_2 \frac{\alpha_2}{\Delta} + \epsilon_3 \frac{\alpha_3}{\Delta}\right)\right] \left(R^{\nu+1}\mathcal{F}\right)_{20}, \end{aligned} \tag{22}$$

$$\begin{aligned} &m_{20}^{(:..)}(m,\alpha) := -(\ m_1\alpha_1 \pm m_2\alpha_2 + m_3\alpha_3)\Delta^{-1}, \\ &m_{20}^{(.:.)}(m,\alpha) := -(-m_1\alpha_1 \pm m_2\alpha_2 + m_3\alpha_3)\Delta^{-1}, \\ &m_{20}^{(..:)}(m,\alpha) := -(\pm m_1\alpha_1 + m_2\alpha_2 \mp m_3\alpha_3)\Delta^{-1}. \end{aligned} \tag{23}$$

The notations: (: ..) := ($\dot{\mathrm{A}}$VV) or ($\dot{\mathrm{A}}$AA), (. : .) := (A$\dot{\mathrm{V}}$V) or (A$\dot{\mathrm{A}}$A), (.. :) := (AV$\dot{\mathrm{V}}$) or (AA$\dot{\mathrm{A}}$) are used in Eqs.(20)–(23) and further on.

9. The first and second amplitudes in the second lines of Eqs.(9) are as follows:

$$\begin{aligned} &\begin{bmatrix} \left(R^{\nu+1} P_1\right)^{\alpha\beta(:..)}(m,k) \\ \left(R^{\nu+1} P_3\right)^{\alpha\beta(:..)}(m,k) \end{bmatrix} = (2\pi)^4 \delta(k) C^{(\cdots)}(g) \varepsilon^{\alpha\beta\sigma\tau} k_{2\sigma} k_{3\tau} \begin{bmatrix} -\left(R^{\nu+1} P_1^{[2,3]}\right) \\ -\left(R^{\nu+1} P_3^{[2,3]}\right) \end{bmatrix}, \\ &\begin{bmatrix} \left(R^{\nu+1} P_2\right)^{\mu\beta(.:.)}(m,k) \\ \left(R^{\nu+1} P_1\right)^{\mu\beta(.:.)}(m,k) \end{bmatrix} = (2\pi)^4 \delta(k) C^{(\cdots)}(g) \varepsilon^{\mu\beta\sigma\tau} k_{2\sigma} k_{3\tau} \begin{bmatrix} \mp\left(R^{\nu+1} P_2^{[2,3]}\right) \\ \left(R^{\nu+1} P_1^{[2,3]}\right) \end{bmatrix}, \\ &\begin{bmatrix} \left(R^{\nu+1} P_3\right)^{\mu\alpha(..:)}(m,k) \\ \left(R^{\nu+1} P_2\right)^{\mu\alpha(..:)}(m,k) \end{bmatrix} = (2\pi)^4 \delta(k) C^{(\cdots)}(g) \varepsilon^{\mu\alpha\sigma\tau} k_{2\sigma} k_{3\tau} \begin{bmatrix} \pm\left(R^{\nu+1} P_3^{[2,3]}\right) \\ -\left(R^{\nu+1} P_2^{[2,3]}\right) \end{bmatrix}. \end{aligned} \tag{24}$$

The $\left(R^{\nu+1}P_l^{[2,3]}\right)(m,k)$ in Eqs.(24) are almost the same as the $\left(R^{\nu+1}P_{l\epsilon}^{[2,3]}\right)(m,k)$ in Eq.(12) in which the $(m_l^2 - i\epsilon_l)$ must be replaced by the m_l^2 in the braces. Notice that Eq.(12) is the only nontrivial primitive DRI, $\forall\, l = 1,2,3$, in the AVV- and AAA-cases, $n = 4$. Taking into account the vanishing r.h.s. of Eq.(12), one obtains the important result:

$$\left(R^{\nu+1}P_l^{[2,3]}\right)(m,k) = \int_{\Sigma^2} \frac{d\mu(\alpha)}{\Delta^2} i\epsilon_l \frac{\alpha_l}{\Delta} \left(R^{\nu+1}\mathcal{F}\right)_{20}, \qquad l = 1,2,3. \tag{25}$$

Due to properties of the hypergeometric function ${}_2F_1$ it follows (for $l = 1,2,3$) that

$$\left(R^{\nu+1}P_l^{[2,3]}\right)(m,k) = \begin{cases} 0, & \text{if } (\epsilon_s \to 0,\ m_s \neq 0 \text{ or } m_s = m \to 0,\ \forall\, s); \\ 1/6, & \text{if } (m_s \to 0,\ \epsilon_s = \epsilon \to 0,\ \forall\, s). \end{cases} \tag{26}$$

10. The third amplitudes in the second lines of Eqs.(9) calculated by the renormalization index $\nu + 1 = 2$ are as follows:

$$\begin{bmatrix} \left(R^{\nu+1}I\right)^{\alpha\beta(:..)}(m,k) \\ \left(R^{\nu+1}I\right)^{\mu\beta(.:.)}(m,k) \\ \left(R^{\nu+1}I\right)^{\mu\alpha(..:)}(m,k) \end{bmatrix} = (2\pi)^4\delta(k)C^{(\cdots)}(g) \begin{bmatrix} \varepsilon^{\alpha\beta\sigma\tau}k_{2\sigma}k_{3\tau}\left(R^{\nu+1}I^{[2,3]}\right)^{(:..)} \\ \varepsilon^{\mu\beta\sigma\tau}k_{2\sigma}k_{3\tau}\left(R^{\nu+1}I^{[2,3]}\right)^{(.:.)} \\ \varepsilon^{\mu\alpha\sigma\tau}k_{2\sigma}k_{3\tau}\left(R^{\nu+1}I^{[2,3]}\right)^{(..:)} \end{bmatrix}, \tag{27}$$

where

$$\left(R^{\nu+1}I^{[2,3]}\right)^{(:..)}(m,k) := \int_{\Sigma^2} \frac{d\mu(\alpha)}{\Delta^2} m_{20}^{(:..)}(m,\alpha)\left(R^{\nu+1}\mathcal{F}\right)_{20}, \qquad \text{etc.}, \tag{28}$$

and the quantities $m_{20}^{(:..)}(m,\alpha)$, etc., are defined in Eq.(23).

11. As a result the regular analogs of the CWI's (9) take the form:

$$\begin{aligned} \left(R^{\nu+1}D_1^{[2,3]}\right)^{(:..)} &= \left(R^{\nu+1}P_1^{[2,3]}\right) + \left(R^{\nu+1}P_3^{[2,3]}\right) + (m_3 + m_1)\left(R^{\nu+1}I^{[2,3]}\right)^{(:..)}, \\ \left(R^{\nu+1}D_2^{[2,3]}\right)^{(.:.)} &= -\left(R^{\nu+1}P_2^{[2,3]}\right) - \left(R^{\nu+1}P_1^{[2,3]}\right) + (m_1 \mp m_2)\left(R^{\nu+1}I^{[2,3]}\right)^{(.:.)}, \\ \left(R^{\nu+1}D_3^{[2,3]}\right)^{(..:)} &= \left(R^{\nu+1}P_3^{[2,3]}\right) + \left(R^{\nu+1}P_2^{[2,3]}\right) + (m_2 \mp m_3)\left(R^{\nu+1}I^{[2,3]}\right)^{(..:)}. \end{aligned} \tag{29}$$

Limiting values of quantities in Eqs.(29) depend strongly on the limit employed.

12. Let us first consider a *nonchiral* case. Here, due to Eqs.(25)–(26), the r.h.s. of Eqs.(24) and terms in Eqs.(20)–(22) containing ϵ_l are zero for $\epsilon_l \to 0$, $l = 1,2,3$. The *quantum corrections* (anomalous contributions) to the CWI's appear as

$$\begin{bmatrix} a^{\alpha\beta(:..)}(m,k) \\ a^{\mu\beta(.:.)}(m,k) \\ a^{\mu\alpha(..:)}(m,k) \end{bmatrix} = (2\pi)^4\delta(k)C^{(\cdots)}(g) \begin{bmatrix} \varepsilon^{\alpha\beta\sigma\tau}k_{2\sigma}k_{3\tau}a^{(:..)}(m_1,m_2,m_3) \\ \varepsilon^{\mu\beta\sigma\tau}k_{2\sigma}k_{3\tau}a^{(.:.)}(m_1,m_2,m_3) \\ \varepsilon^{\mu\alpha\sigma\tau}k_{2\sigma}k_{3\tau}a^{(..:)}(m_1,m_2,m_3) \end{bmatrix}, \tag{30}$$

where the mass functions $a^{(\cdots)}(m_1,m_2,m_3)$ have the integral representation:

$$a^{(:..)}(m_1,m_2,m_3) := \int_{\Sigma^2} \frac{d\mu(\alpha)}{\Delta^2} m_{20}^{(:..)}(m,\alpha)\left[(R^{\nu+1}\mathcal{F})_{20} - (R^{\nu}\mathcal{F})_{20}\right], \qquad \text{etc.}, \tag{31}$$

$$\left[(R^{\nu+1}\mathcal{F})_{20} - (R^{\nu}\mathcal{F})_{20}\right] = -M_\epsilon^{-1}, \quad \text{as for } n = 4, \qquad \nu_{20} = -1, \quad \lambda_{20} = 1, \quad \omega = -1;$$

for a *nonchiral nondegenerate* case they are expressed in terms of the Appel hypergeometric functions F_1 of two variables (e.g., $x := m_1/m_2$, $y := m_3/m_2$ if $m_2 \neq 0$) [6, 7]:

$$\begin{aligned} a^{(\dot{\cdot}\cdot\cdot)}(m_1,m_2,m_3) &= \frac{y+x}{6}\Big[xF_1\left(1,2,1;4;1-x^2,1-y^2\right) \\ &\pm F_1\left(1,1,1;4;1-x^2,1-y^2\right) + yF_1\left(1,1,2;4;1-x^2,1-y^2\right)\Big], \\ a^{(\cdot\dot{\cdot}\cdot)}(m_1,m_2,m_3) &= \frac{x\mp 1}{6}\Big[-xF_1\left(1,2,1;4;1-x^2,1-y^2\right) \\ &\pm F_1\left(1,1,1;4;1-x^2,1-y^2\right) + yF_1\left(1,1,2;4;1-x^2,1-y^2\right)\Big], \\ a^{(\cdot\cdot\dot{\cdot})}(m_1,m_2,m_3) &= \frac{1\mp y}{6}\Big[\pm xF_1\left(1,2,1;4;1-x^2,1-y^2\right) \\ &+ F_1\left(1,1,1;4;1-x^2,1-y^2\right) \mp yF_1\left(1,1,2;4;1-x^2,1-y^2\right)\Big]. \end{aligned} \tag{32}$$

This confirms the Frampton's conjecture [8] about a possibility of a mass dependence of the axial-vector anomaly. But the nature of such a dependence revealed here is strongly different from the Frampton's one. Actually it is closely tied with a mass spectrum of fermions and with flavor current structures producing non-conserved vector currents. The Frampton's mechanism appeals to prorerties of the dimensional regularization.

For the *degenerate nonchiral* case ($m_1 = m_2 = m_3 \equiv m \neq 0$), the Eqs.(32) display the famous mass-independent Adler–Bell–Jackiw result [1, 2]:

$$\begin{aligned} &a^{(\dot{A}VV)}(m,m,m) = 1, \qquad a^{(A\dot{V}V)}(m,m,m) = a^{(AV\dot{V})}(m,m,m) = 0, \\ &a^{(\dot{A}AA)}(m,m,m) = -a^{(A\dot{A}A)}(m,m,m) = a^{(AA\dot{A})}(m,m,m) = 1/3, \end{aligned} \tag{33}$$

about the axial-vector anomaly (trivial QC's to the CWI's in our terminology).

13. Now we turn to the *chiral* behaviour. Let us consider two ways leading to the chiral state in renormalized amplitudes at hand: (i) the (ϵ, m)-limit, when first $\epsilon_l \to 0$ and then $m_l = m \to 0$, $l = 1, 2, 3$; (ii) the (m, ϵ)-limit, when first $m_l \to 0$ and then $\epsilon_l = \epsilon \to 0$, $l = 1, 2, 3$. In the (ϵ, m)-limit, all the amplitudes for AVV- and AAA-cases *inherit* the behaviour of those in the degenerate nonchiral case considered in [6]; the QC's to CWI's are the same as in Eqs.(33). In the (m, ϵ)-limit all amplitudes for the AVV- and AAA-cases *coincide with each other* (apart from the factor $\varepsilon(g) = (-1)^q$ of course). Here the QC's to CWI's are caused by the nonzero contributions of the amplitudes $\left(R^{\nu+1}P_l^{[2,3]}\right)^{(\cdots)}(m,k)$. The results are summarized in Table 1.

Thus, the chiral limit ($m \to 0$) and the chiral case ($m = 0$) are *equivalent* for the AAA-amplitude and *differ* for the AVV-amplitude. This reflects the different kind of *discrete symmetries* (DS) of these amplitudes for $m \neq 0$ and $m = 0$. The AAA-amplitude has the DS of equilaterial triangle both for $m \neq 0$ and for $m = 0$, in contrast to the AVV-amplitude having the DS of isosceles triangle for $m \neq 0$ which at $m = 0$ enlarges abruptly to the DS of equilaterial triangle.

14. For the complex Clifford algebra $Cl(g)_{\mathbf{C}}$, the matrix γ_* in Eq.(2) may be always redefined as $\gamma_* := i^{(1-\varepsilon(g))/2}\gamma_1\gamma_2\cdots\gamma_n$ and, hence, $\gamma_*^2 = I_g$. Therefore, from the Table 1 it follows that the QC's to "left-handed" WI's are zero in the chiral case. This may give some insight into why just the left-handed neutrino exists in Nature. This also requires a revision of the conventional viewpoint about an impact of anomalies on the renormalizability of unified field theories in which gauge fields are coupled to left-handed fermions.

The presence of a mass spectrum of constituent fermions in general QC's (see Eqs.(30)–(32)) increases the predictive power of formulas (which involves the axial-vector anomaly) widely used in the low energy phenomenological physics, e.g., for describing the elementary particle decays [1, 2].

Table 1. The chiral behaviour of amplitudes appearing in the regular analogs of the CWI's (29) for AVV- and AAA-cases, $n = 4$; here $\left(R^{\nu+1}P_0^{[2,3]}\right) \equiv \left(R^{\nu+1}P_3^{[2,3]}\right)$.

Feynman amplitudes	$(\dot{A}VV)$	$(A\dot{V}V)$	$(AV\dot{V})$	$(\dot{A}AA)$	$(A\dot{A}A)$	$(AA\dot{A})$
$\underline{(\epsilon, m)-\lim \equiv \text{chiral limit:}}$						
$\left(R^{\nu+1}D_i^{[2,3]}\right)^{(\cdots)}(m,k) =$	1	0	0	1/3	−1/3	1/3
$= (-1)^{i-1}\Big[\left(R^{\nu+1}P_i^{[2,3]}\right)^{(\cdots)}(m,k) + \left(R^{\nu+1}P_{i-1}^{[2,3]}\right)^{(\cdots)}(m,k)\Big]+$	0	0	0	0	0	0
$+\begin{bmatrix}(m_3+m_1)\\(m_1\mp m_2)\\(m_2\mp m_3)\end{bmatrix}(R^{\nu+1}I^{[2,3]})^{(\cdots)}(m,k)$	1	0	0	1/3	−1/3	1/3
$\underline{(m, \epsilon)-\lim \equiv \text{chiral case:}}$						
$\left(R^{\nu+1}D_i^{[2,3]}\right)^{(\cdots)}(m,k) =$	1/3	−1/3	1/3	1/3	−1/3	1/3
$= (-1)^{i-1}\Big[\left(R^{\nu+1}P_i^{[2,3]}\right)^{(\cdots)}(m,k) + \left(R^{\nu+1}P_{i-1}^{[2,3]}\right)^{(\cdots)}(m,k)\Big]+$	1/3	−1/3	1/3	1/3	−1/3	1/3
$+\begin{bmatrix}(m_3+m_1)\\(m_1\mp m_2)\\(m_2\mp m_3)\end{bmatrix}(R^{\nu+1}I^{[2,3]})^{(\cdots)}(m,k)$	0	0	0	0	0	0

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