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On a Class of Equations for Particles with Arbitrary Spin

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The relativistic wave equations proposed by Moshinsky and Smirnov [1] are analysed. It is proved that these equations are causal. A simple algorithm for solving of these equations for particles interacting with a constant magnetic fields proposed.

1 Introduction

Let us consider the wave equation for particle with arbitrary spins proposed by Moshinsky and Smirnov [1]. For $s = 1$ these equations have the form

$$p_0\psi = H\psi, \qquad H = \left(\bar{\alpha}^{(1)} + \bar{\alpha}^{(2)}\right)\bar{p} + \left(\beta^{(1)} + \beta^{(2)}\right)m, \tag{1}$$

where $\bar{\alpha}^{(1)}$, $\bar{\alpha}^{(2)}$, $\beta^{(1)}$, $\beta^{(2)}$ are 16×16 matrices which can be written in the form

$$\alpha_a^{(i)} = \gamma_0^{(i)}\gamma_a^{(i)}, \qquad \beta^{(i)} = \gamma_0^{(i)}, \qquad i = 1,2, \quad a = 1,2,3 \tag{2}$$

and the matrices $\gamma_\mu^{(1)}$, $\gamma_\mu^{(2)}$ are connected to Dirac matrices γ_μ as

$$\gamma_\mu^{(1)} = \gamma_\mu \otimes I_4, \qquad \gamma_\mu^{(2)} = I_4 \otimes \gamma_\mu. \tag{3}$$

The matrices $\beta_\mu = \frac{1}{2}(\gamma_\mu^{(1)} + \gamma_\mu^{(2)})$ satisfy the Kemmer–Duffin relations

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = g_{\mu\nu}\beta_\lambda + g_{\nu\lambda}\beta_\mu. \tag{4}$$

Using (4), equation (1) can be rewritten as

$$p_0\psi = \bar{H}\psi, \qquad \bar{H} = [\beta_0,\beta_a]p_a + \beta_0 m. \tag{5}$$

2 Transformation to the quasidiagonal form

Now let us show that equation (5) can be reduced to the system of three uncoupled equations. To do this, we transform β_μ using the unitary transformation $\beta_\mu \to \hat{\beta}_\mu = U\beta_\mu U^\dagger$, where [3]

$$\begin{aligned} U = \frac{1-i}{2}(&e_{1,1} + e_{1,13} + e_{2,2} + e_{2,14} + e_{3,3} + e_{3,15} - e_{10,8} + e_{10,12} - e_{11,4} - e_{11,16} \\ &+ e_{13,15} - e_{13,9} + e_{14,6} - e_{14,10} + e_{15,7} - e_{15,11}) \\ + \frac{1+i}{2}(&-e_{4,5} - e_{4,9} - e_{5,6} - e_{5,10} - e_{6,7} - e_{6,11} - e_{7,1} + e_{7,13} - e_{8,2} + e_{8,14} \\ &- e_{9,3} + e_{9,15} - e_{12,4} + e_{12,16} + e_{16,8} + e_{16,12}). \end{aligned} \tag{6}$$

Here $e_{i,j}$ stand for the square matrices, whose only nonzery entry, equal to unity, is located at the intersection of i-th row and j-th column.

As a result we obtain

$$\hat{\beta}_\mu = \begin{pmatrix} \beta_\mu^{(10)} & \cdot & \cdot \\ \cdot & \beta_\mu^{(5)} & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}, \qquad \mu = 0, 1, 2, 3, \tag{7}$$

where $\beta_\mu^{(10)}$, $\beta_\mu^{(5)}$ are Kemmer–Duffin 5×5 and 10×10 matrices correspondingly:

$$\begin{aligned}
&\beta_0^{(10)} = i(e_{1,7} + e_{2,8} + e_{3,9} - e_{7,1} - e_{8,2} - e_{9,3}), && \beta_0^{(5)} = -i(e_{1,2} - e_{2,1}),\\
&\beta_1^{(10)} = -i(e_{1,10} - e_{5,9} + e_{6,8} + e_{8,6} - e_{9,5} + e_{10,1}), && \beta_1^{(5)} = i(e_{1,3} + e_{3,1}),\\
&\beta_2^{(10)} = -i(e_{2,10} + e_{4,9} - e_{6,7} - e_{7,6} + e_{9,4} + e_{10,2}), && \beta_2^{(5)} = i(e_{1,4} + e_{4,1}),\\
&\beta_3^{(10)} = -i(e_{3,10} - e_{4,8} + e_{5,7} + e_{7,5} - e_{8,4} + e_{10,3}), && \beta_3^{(5)} = i(e_{1,5} + e_{5,1}).
\end{aligned} \tag{8}$$

Then equation (5) will be reduced to the system of three equations

$$p_0\psi_{(10)} = \left([\beta_0^{(10)}, \beta_a^{(10)}]p_a + \beta_0^{(10)} m\right)\psi_{(10)}, \tag{9.a}$$

$$p_0\psi_{(5)} = \left([\beta_0^{(5)}, \beta_a^{(5)}]p_a + \beta_0^{(5)} m\right)\psi_{(5)}, \tag{9.b}$$

$$p_0\psi_{(1)} = 0. \tag{9.c}$$

Equation (9.c) is not hyperbolic and hence system (5) is not causal.

Equations (9.a), (9.b) can be represented in the form

$$p_0\psi_{(k)} = \hat{H}\psi_{(k)}, \qquad \hat{H} = \frac{1}{n_1}(S_{a4}p_a + S_{45}m), \qquad k = 5, 10, \tag{13}$$

where $S_{0a} = [\beta_0, \beta_a]$, $S_{ab} = i[\beta_a, \beta_b]$, $S_{45} = \beta_4$, $S_{4a} = -i\beta_a$ are matrices which belong to algebra $AO(5)$. The irreducible representations of $AO(5)$ are labelled by pairs of numbers (n_1, n_2) (simultaneously integer or half-integer). Any of equations proposed in [1] can be reduced to a system of uncoupled equations, corresponding to these irreducible representations.

3 Analysis of hyperbolicity

The Hamiltonian $\hat{H}$ can be reduced to the diagonal form using operator

$$U_1 = \exp\left(i\frac{S_{i5}p_i}{p}\arctan\frac{p}{m}\right) = \exp(iA), \qquad \text{where} \qquad p = \sqrt{p_1^2 + p_2^2 + p_3^2}. \tag{14}$$

Taking into account the Campbell–Hausdorff formula

$$\exp(-iA)B\exp(iA) = B - i[A, B] - \frac{1}{2!}[A, [A, B]] + \cdots$$

and the commutation relations of $AO(4)$

$$[S_{\mu\nu}, S_{\rho\sigma}] = i(\delta_{\mu\rho}S_{\nu\sigma} + \delta_{\nu\sigma}S_{\mu\rho} - \delta_{\mu\sigma}S_{\nu\rho} - \delta_{\nu\rho}S_{\mu\sigma}),$$

we find $\bar{H} = U_1 \hat{H} U_1^{-1} = \frac{1}{n_1} S_{45}\sqrt{p^2+m^2}$. Thus we come to the equations

$$p_0\psi' = \bar{H}\psi', \qquad \bar{H} = \frac{1}{n_1} S_{45}\sqrt{p^2+m^2}, \tag{15}$$

where $\psi'_{(k)} = e^{iA}\psi_{(k)}$ $(k = 5, 10)$.

Form (15) is suitable for studying of hyperbolicity of equation (1). Let us take the matrix S_{45} in the diagonal form and consider two cases: n_1, n_2 are half-integer and n_1, n_2 are integer.

For the first case the reduction of $O(5)$ to $O(4)$ and $O(4)$ to $O(3)$ shows that system (15) is hyperbolical.

Indeed, bearing in mind that the eigenvalues of the matrix S_{45} are

$$s = \frac{n_1+n_2}{2}, \frac{n_1+n_2-1}{2}, \frac{n_1+n_2-2}{2}, \ldots, 0$$

and their multiplicity is given by the formula [2]

$$M_s = \begin{cases} (n_1-n_2+1)(n_1+n_2+1-2s), & s \geq (n_1-n_2)/2, \\ (2n_1+1)(2s+1), & s < (n_1-n_2)/2, \end{cases}$$

we find that Ψ' satisfies the equation

$$\prod_s \left(p_0^2 - \frac{1}{s}c^2p^2\right)^{M_s} \Psi' = 0. \tag{16}$$

It follows from (16) that system (15) is hyperbolical and the velocity of waves described by this system can have different values $\tilde{c}_s$ and $\tilde{c}_s \leq c$. So the velocity of propagation does not exceed the velocity of light and the causality is not broken.

For n_1, n_2 integer we have equation (9.c), and hence the causality is broken.

4 Equation for particle interacting with constant magnetic field

Consider the generalized equation (13) which describes a particle interacting with constant magnetic field which is directed along x_3.

In accordance with the principle of minimal coupling we change

$$p_\mu \to \pi_\mu = p_\mu - eA_\mu, \tag{17}$$

where $A_0 = A_2 = A_3 = 0, A_1 = eHx_2$.

It is straightforward to check that H satisfies the following relations

$$\hat{H}^3 - \hat{H}\xi \mp MH = 0, \tag{18}$$

where $\xi = \pi_1^2 + \pi_2^2 - 2S_{12}H + M^2$, $M^2 = m^2 + p_3^2$.

Inasmuch as the operators H, $S_{12}H$, $\pi^2 - 2S_{12}H$ commute, relations (17) can be replaced by the relations for eigenvalues of these operators

$$E^3 - E\left((2n+1)\omega \pm \omega + m^2 + p_3^2\right) \mp \left(m^2+p_3^2\right) H = 0, \tag{19}$$

where $\omega = (eH)^{1/2}$.

In [1] Moshinsky and Smirnov obtained sixth order algebraic equations for the eigenvalues of H. We show that these equations can be factorized into the product of two third order equations.

For the case of spin 3/2 the authors of [1] found tenth order algebraic equations for the eigenvalues of H. Using our approach, it is possible to show that in fact these equations also can be factorized into the product of sixth and fourth order equations.

5 Problem of complex energies

Analysing equation (19), we see that for

$$H > \frac{M^2}{4} \tag{20}$$

the eigenvalues of Hamiltonian become complex.

The appearance of complex energies in the problem of interaction of a particle with the constant magnetic field is typical for the equations with higher spins [3]. Thus, the equations which were proposed in [1] also have nonphysical solutions corresponding to complex energies.

Let us note that the magnetic field, satisfying (20), is exteremely strong one, and hardly can be encountered in practice. Therefore, the appearance of complex energy eigenvalues should not be considered as an obstacle for application of the equation in question.

6 Conclusions

Analysing the equations for arbitrary spin given above, we found that

1. For the case of half-integer spin these equations are hyperbolical.
2. For the case of integer spin the hyperbolicity is broken in view of appearing of the solutions which correspond to zero energies and these solutions must be rejected.
3. Hyperbolical solutions of equation (13) describe waves having velocity less than c.
4. The equation for energy eigenvalues of (13) with magnetic field can be reduced to algebraic equations, whose order is less than that of the equations given in paper [1].
5. This equation, like other equations with higher spins, has solutions with complex energies.

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